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Resolutions as directed colimits

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Abstract – A general principle suggests that "anything flat is a directed colimit of countably presentable flats". In this paper we consider resolutions and coresolutions of modules over a countably coherent ring R (e.g., any coherent ring or any countably Noetherian ring). We show that any R-module of flat dimension n is a directed colimit of countably presentable R-modules of flat dimension at most n, and any flatly coresolved R-module is a directed colimit of countably presentable flatly coresolved R-modules. If R is a countably coherent ring with a dualizing complex, then any F-totally acyclic complex of flat R-modules is a directed colimit of F-totally acyclic complexes of countably presentable flat R-modules. The proofs are applications of an even more general category-theoretic principle going back to an unpublished 1977 preprint of Ulmer. Our proof of the assertion that every Gorenstein-flat module over a countably coherent ring is a directed colimit of countably presentable Gorenstein-flat modules uses a different technique, based on results of Šaroch and Šťovíček. We also discuss totally acyclic complexes of injectives and Gorenstein-injective modules, obtaining various cardinality estimates for the accessibility rank under various assumptions.

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Keywords – flat modules, injective modules, resolutions, direct limits, flat and injective dimensions, Gorenstein-flat modules, Gorenstein-injective modules, countably presented modules, countably coherent rings, dualizing complexes, accessible categories.

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Contents

Introduction

The classical Govorov–Lazard theorem [16, 27] states that all flat modules are directed colimits of finitely generated projective modules. What about modules of flat dimension n, for a fixed $n \geq 1$?

Any module of projective dimension 1, over an arbitrary ring, is a directed colimit of finitely presentable modules of projective dimension at most 1. Indeed, such a module is a cokernel of an injective morphism of free modules, and such a morphism is a directed union of injective morphisms of finitely generated free modules. This fact is mentioned, and a generalization to projective dimension n is discussed, in [5, Section 4]; see specifically [5, Theorem 4.6].

According to [3, Theorem 3.5], any module of flat dimension 1 over a commutative integral domain is a directed colimit of finitely presentable modules of projective dimension at most 1. Nevertheless, over commutative Noetherian local rings, a module of flat dimension 1 need not be a directed colimit of (finitely generated) modules of projective dimension at most 1 [5, Example 8.5], [20, Theorem B]. A further discussion can be found in [15, Section 9.2].

In this paper we show that, over any right countably coherent ring, any right module of flat dimension n is a directed colimit of countably presentable modules of flat dimension at most n (see Corollary 5.2). Here, a ring R is said to be *right countably coherent* if every finitely generated right ideal in R is countably presentable as a right R-module. In other words, in the language of [1, Chapter 2], the category of right R-modules of flat dimension $\le n$ is $\aleph_1$*-accessible*, and its $\aleph_1$*-presentable objects* are the countably presentable modules of flat dimension $\le n$ (i.e., countably presentable as objects of the category of all modules). Why is this interesting?

From our perspective, the significance of flat modules with small cardinalities of generators and relations lies in the fact that such modules have finite projective dimensions. In fact, any countably presentable flat module has projective dimension at most 1 [15, Corollary 2.23]. More generally, any flat module with less than $\aleph_m$ generators and relations has projective dimension at most m [24, Proposition 5.3] (see also our Corollary 2.4).

There has been a stream of research about *deconstructibility* properties of various classes of modules and abelian/exact category objects, using the *Hill lemma* [18, Theorem 1], [13, Theorem 2.1], [43, Theorem 6], [15, Theorem 7.10], [42, Theorem 2.1] as one of the main technical tools. The importance of deconstructibility is explained by the Eklof lemma [10, Lemma 1], [15, Lemma 6.2] and the Eklof–Trlifaj theorem [10, Theorems 2 and 10], [15, Theorem 6.11 and Corollary 6.14].

In this context, a natural question is whether every module of projective dimension n is a direct summand of a module filtered by modules of projective dimension at most n admitting a resolution by finitely generated projective modules. The answer is already negative for $n = 1$ and commutative Noetherian local rings [5, Theorem 8.6 or Lemma 9.1], [21, Theorem 3.14], [20, Theorem A]. So a module of projective dimension 1 need not be a direct summand of a module filtered by finitely generated modules of projective dimension 1.

On the other hand, over a right countably Noetherian ring, any right module of projective dimension n is filtered by countably generated modules of projective dimension at most n [39, Corollaire II.3.2.5], [2, Proposition 4.1]. Here, a ring R is called *right countably Noetherian* if every right ideal in R is countably generated. A far-reaching generalization of this result can be found in [41, Theorem 3.4].

An approach to accessibility based on deconstructibility is possible, but its results may be suboptimal. Let κ be a regular cardinal and S be a set of κ-presentable modules over a ring R. Then it follows from the Hill lemma that every module filtered by S is a κ-directed union of κ-presentable modules filtered by S. Arguing in this way, and using the mentioned result from [41] together with purification considerations [6, Lemma 1 and Proposition 2], one can show that any R-module of flat dimension n is a directed colimit of κ-presentable R-modules of flat dimension at most n, where κ is an uncountable regular cardinal greater than the cardinality of R. This is not as good as the result of our Corollary 5.2, which gives $\kappa = \aleph_1$ for modules over countably coherent rings.

The main techniques presented in this paper are category theoretic in nature, based on a general principle going back to an unpublished 1977 preprint of Ulmer [44] and exemplified by the pseudopullback theorem of Raptis and Rosický [7, Proposition 3.1], [38, Theorem 2.2]. Another exposition can be found in the recent preprint [35] by the present author.

With these methods, we can and do treat coresolutions on a par with resolutions. In particular, one says that an R-module M is *flatly coresolved* if there exists an exact sequence of R-modules $0 \to M \to F^0 \to F^1 \to F^2 \to \cdots$ with flat R-modules F^i. We show that, over a countably right coherent ring R, any flatly coresolved right R-module is a directed colimit of countably presentable flatly coresolved right R-modules (see Corollary 6.2).

The latter result brings us closer to the presently popular subject of Gorenstein homological algebra. Approaching it with our category-theoretic methods, we show that, over a right countably coherent ring R with a dualizing complex, any F-totally acyclic complex of flat right R-modules (in the sense of [12, Section 2]) is an $\aleph_1$-directed colimit of F-totally acyclic complexes of countably presentable flat modules (see Theorem 8.3). Consequently, any Gorenstein-flat right R-module is a directed colimit of countably presentable Gorenstein-flat modules. Furthermore, any countably presentable Gorenstein-flat right R-module is a direct summand of a (Gorenstein-flat) module admitting an F-totally acyclic two-sided resolution by countably presentable flat R-modules (Corollary 8.5). The same results apply to commutative Noetherian rings with at most countable spectrum (instead of a dualizing complex).

A more powerful approach is based on difficult results of the paper [40], or specifically [40, Theorems 4.9 and 4.11 (4)]. It allows us to prove that, over any right countably coherent ring, any Gorenstein-flat right module is an $\aleph_1$-directed colimit of countably presentable Gorenstein-flat modules (Theorem 10.2).

We also consider Gorenstein-injective modules over a left Noetherian ring R with a dualizing complex. Here we prove that, if the cardinality of R is smaller than κ, then any totally acyclic complex of injective left R-modules is a κ-directed colimit of totally acyclic complexes of injective modules of cardinality less than κ (see Theorem 15.1). Consequently, every Gorenstein-injective left R-module is a directed colimit of Gorenstein-injective modules of cardinality less than κ. Furthermore, any Gorenstein-injective left R-module of cardinality less than κ is a direct summand of a (Gorenstein-injective) module admitting a totally acyclic two-sided resolution by injective R-modules of cardinality less than κ (Corollary 15.5).

More generally, let us say that an R-module M is *injectively resolved* if there exists an exact sequence of R-modules $\cdots \to J_2 \to J_1 \to J_0 \to M \to 0$ with injective

R-modules J_i. We show that, over a left Noetherian ring R, any injectively resolved left R-module is a directed colimit of injectively resolved R-modules of cardinality not exceeding $\aleph_0$ plus the cardinality of R.

Let us say a few more words about the proofs. The main arguments in this paper are based on a very general category-theoretic principle going back to [44] and rediscovered in [35]. The pseudopullback theorem [7, Proposition 3.1], [38, Theorem 2.2] is one of the instances of this principle, and it is essentially sufficient for our purposes. In full generality, the principle involves a regular cardinal κ and a smaller infinite cardinal $\lambda < \kappa$, and the claim is that the class of all κ-accessible categories with directed colimits of λ-indexed chains is stable under many category-theoretic constructions, including the pseudopullbacks, the inserters, and the equifiers. Moreover, there is a good control over the full subcategories of κ-presentable objects in the categories arising under such constructions; this is crucial for applications.

A general principle that *anything flat is a directed colimit of countably presentable flats* is mentioned in the abstract. This is illustrated by the results of the preprints [35, Sections 10.2, 10.4, and 10.5], [37, Sections 2–4], [33, Sections 3, 4, 10, and 11], and [34, Sections 4, 6, and 7]. This is also confirmed by the results of the present paper, such as Corollaries 5.2 and 6.2, and Theorem 10.2.

Sections 1–3 contain various kinds of preliminary material and preliminary discussions. In Sections 4–6 we discuss modules with two-sided resolutions by modules from accessible classes and two particular cases, modules of bounded flat dimension and flatly coresolved modules. In the subsequent Sections 7–10 we study F-totally acyclic complexes and Gorenstein-flat modules. In the next Sections 11–12 we construct modules of bounded injective dimension and injectively resolved modules as directed colimits. The final Sections 13–16 are dedicated to accessibility properties of totally acyclic complexes of injectives and Gorenstein-injective modules.

1. Accessible categories and acyclic complexes

We use the book [1] as the main reference source on accessible categories. In particular, we refer to [1, Definition 1.4, Theorem and Corollary 1.5, Definition 1.13 (1), and Remark 1.21] for the discussion of *κ-directed posets* vs. *κ-filtered small categories* and, accordingly, *κ-directed* vs. *κ-filtered colimits*. Here, κ denotes a regular cardinal. Let us just mention that a poset Ξ is said to be *κ-directed* if every subset of cardinality less than κ has an upper bound in Ξ.

Let K be a category with κ-directed (equivalently, κ-filtered) colimits. An object $S \in \mathsf{K}$ is said to be *κ-presentable* [1, Definition 1.13(2)] (or "$<\kappa$-presented" in the traditional module-theoretic terminology) if the functor $\mathrm{Hom}_{\mathsf{K}}(S, -)\colon \mathsf{K} \to \mathsf{Sets}$ preserves

κ-directed colimits. We will denote the full subcategory of κ-presentable objects of K by $\mathsf{K}_{<\kappa} \subset \mathsf{K}$.

The category K is called *κ-accessible* [1, Definition 2.1] if there is a *set* of κ-presentable objects $\mathsf{S} \subset \mathsf{K}$ such that every object of K is a κ-directed colimit of objects from S. If this is the case, then the κ-presentable objects of K are precisely all the retracts of the objects from S. A κ-accessible category K is called *locally κ-presentable* if all colimits exist in K [1, Definition 1.17 and Theorem 1.20].

In the standard terminology, $\aleph_0$-presentable objects are called *finitely presentable* [1, Definition 1.1], and we will call $\aleph_1$-presentable objects *countably presentable*. Similarly, $\aleph_0$-accessible categories are called *finitely accessible* [1, Remark 2.2 (1)], and locally $\aleph_0$-presentable categories are called *locally finitely presentable* [1, Definition 1.9 and Theorem 1.11].

Given a category K with κ-directed colimits and a class of objects $\mathsf{T} \subset \mathsf{K}$, we denote by $\varinjlim_{(\kappa)} \mathsf{T} \subset \mathsf{K}$ the class of all κ-directed colimits of objects from T. In the case of $\kappa = \aleph_0$, we will simply write $\varinjlim \mathsf{T}$ instead of $\varinjlim_{(\aleph_0)} \mathsf{T}$.

The following proposition is essentially well known.

Proposition 1.1. *Let κ be a regular cardinal and K be a κ-accessible category. Let $\mathsf{T} \subset \mathsf{K}$ be a set of κ-presentable objects. Then the full subcategory $\varinjlim_{(\kappa)} \mathsf{T} \subset \mathsf{K}$ is closed under κ-directed colimits in K. The category $\varinjlim_{(\kappa)} \mathsf{T}$ is κ-accessible, and the κ-presentable objects of $\varinjlim_{(\kappa)} \mathsf{T}$ are precisely all the retracts of objects from T. An object $K \in \mathsf{K}$ belongs to $\varinjlim_{(\kappa)} \mathsf{T}$ if and only if, for every object $S \in \mathsf{K}_{<\kappa}$, any morphism $S \to K$ in K factorizes through an object from T.*

Proof. In the particular case of finitely accessible additive categories, versions of this result were discussed in [28, Proposition 2.1], [9, Section 4.1], and [26, Proposition 5.11]. (The terminology "finitely presented categories" was used in [9, 26] for what we call finitely accessible categories.) The key step is to prove the "if" part of the last assertion; then the remaining arguments are easy. ■

Proposition 1.2. *Let κ be a regular cardinal, and let $(\mathsf{K}_\xi)_{\xi\in\Xi}$ be a family of κ-accessible categories, indexed by a set Ξ of cardinality less than κ. Then the Cartesian product category $\mathsf{K} = \prod_{\xi\in\Xi} \mathsf{K}_\xi$ is also κ-accessible. The κ-presentable objects of K are precisely all the collections of objects $(S_\xi \in \mathsf{K}_\xi)_{\xi\in\Xi}$ such that the object S_ξ is κ-presentable in K_ξ for every $\xi \in \Xi$.*

Proof. This is a corrected version of [1, proof of Proposition 2.67]. We refer to [35, Proposition 2.1] for the details. ■

In the next two theorems we consider a regular cardinal κ and a smaller infinite cardinal $\lambda < \kappa$ (so κ is necessarily uncountable). The cardinal λ is viewed as an ordinal, and directed colimits of λ-indexed chains are considered. Here, a *λ-indexed chain* in K is a directed diagram of the form $(K_i \to K_j)_{0 \le i < j < \lambda}$.

Let K_1, K_2, and L be three categories, and let $\Phi_1\colon \mathsf{K}_1 \to \mathsf{L}$ and $\Phi_2\colon \mathsf{K}_2 \to \mathsf{L}$ be two functors. The *pseudopullback* C of the pair of functors Φ_1 and Φ_2 is defined as the category of triples (K_1, K_2, θ), where $K_1 \in \mathsf{K}_1$ and $K_2 \in \mathsf{K}_2$ are two objects, and $\theta\colon \Phi_1(K_1) \simeq \Phi_2(K_2)$ is an isomorphism in L.

Theorem 1.3. *Let κ be a regular cardinal and $\lambda < \kappa$ be a smaller infinite cardinal. Let K_1, K_2, and L be three κ-accessible categories where colimits of λ-indexed chains exist. Assume that two functors $\Phi_1\colon \mathsf{K}_1 \to \mathsf{L}$ and $\Phi_2\colon \mathsf{K}_2 \to \mathsf{L}$ preserve κ-directed colimits and colimits of λ-indexed chains, and take κ-presentable objects to κ-presentable objects. Then the pseudopullback C is a κ-accessible category. The κ-presentable objects of C are precisely all the triples $(S_1, S_2, \theta) \in \mathsf{C}$ such that the object S_1 is κ-presentable in K_1 and the object S_2 is κ-presentable in K_2.*

Proof. This assertion, going back to [44, Remark 3.2 (I), Theorem 3.8, Corollary 3.9, and Remark 3.11 (II)], can be found in [38, Pseudopullback Theorem 2.2] with the proof in [7, Proposition 3.1]. See also [35, Corollary 5.1]. ∎

Let K and L be two categories, and let $\Phi_1, \Phi_2\colon \mathsf{K} \rightrightarrows \mathsf{L}$ be two parallel functors. The *isomorpher* C of the pair of functors Φ_1 and Φ_2 is defined as the category of pairs (K, θ), where $K \in \mathsf{K}$ is an object and $\theta\colon \Phi_1(K) \simeq \Phi_2(K)$ is an isomorphism in L.

Theorem 1.4. *Let κ be a regular cardinal and $\lambda < \kappa$ be a smaller infinite cardinal. Let K and L be two κ-accessible categories where colimits of λ-indexed chains exist. Assume that two functors Φ_1 and $\Phi_2\colon \mathsf{K} \rightrightarrows \mathsf{L}$ preserve κ-directed colimits and colimits of λ-indexed chains, and take κ-presentable objects to κ-presentable objects. Then the isomorpher C is a κ-accessible category. The κ-presentable objects of C are precisely all the pairs $(S, \theta) \in \mathsf{C}$ such that the object S is κ-presentable in K.*

Proof. This is essentially an equivalent version of Theorem 1.3, as explained in [35, Remark 5.2]. The reference to [44, Remark 3.2 (I), Theorem 3.8, Corollary 3.9, and Remark 3.11 (II)] is also applicable. ∎

Let R be an associative ring. We will denote by Mod–R the category of right R-modules and by R–Mod the category of left R-modules. The abelian category Mod–R is locally finitely presentable, hence locally κ-presentable for every regular cardinal κ [1, Remark 1.20]. The κ-presentable objects of Mod–R are precisely all the

R-modules with less than κ generators and less than κ relations, i.e., in other words, the cokernels of morphisms of free R-modules with less than κ generators.

For any additive category A, we denote by $\mathsf{Com}(\mathsf{A})$ the category of (unbounded) complexes in A.

Lemma 1.5. *For any associative ring R, the following assertions hold:*

(a) *The abelian category $\mathsf{Com}(\mathsf{Mod}\text{–}R)$ of complexes of right R-modules is locally finitely presentable. Consequently, this category is locally κ-presentable for any regular cardinal κ.*

(b) *The finitely presentable objects of $\mathsf{Com}(\mathsf{Mod}\text{–}R)$ are precisely all the* bounded *complexes of finitely presentable R-modules.*

(c) *For every uncountable regular cardinal κ, the κ-presentable objects of the category $\mathsf{Com}(\mathsf{Mod}\text{–}R)$ are precisely all the (unbounded) complexes of κ-presentable R-modules.*

Proof. Essentially, complexes are modules over a suitable "ring with many objects" (viz., the objects are indexed by the integers $n \in \mathbb{Z}$). As usual, the results about modules over rings are applicable to modules over rings with many objects, and provide the assertions of the lemma. Alternatively, part (c) can be deduced by applying an additive version of [17, Theorem 1.2] or [35, Theorem 6.2]. ■

The *category of epimorphisms of right R-modules* has epimorphisms of right R-modules $L \to M$ as objects and commutative squares $L' \to M' \to M''$, $L' \to L'' \to M''$ (with epimorphisms $L' \to M'$ and $L'' \to M''$) as morphisms.

Lemma 1.6. *For any ring R and every regular cardinal κ, the category of epimorphisms of right R-modules is κ-accessible. The κ-presentable objects of this category are the epimorphisms of κ-presentable right R-modules.*

Proof. This is a particular case of [35, Lemma 10.7]. ■

We will say that an associative ring R is *right $<\kappa$-coherent* if, for every κ-presentable right R-module S, any submodule in S having less than κ generators is κ-presentable. Equivalently, R is right $<\kappa$-coherent if and only if every right ideal in R with less than κ generators is κ-presentable as a right R-module, and if and only if every finitely generated right ideal in R is κ-presentable. We will call right $<\aleph_1$-coherent rings *right countably coherent*.

Corollary 1.7. *For any regular cardinal κ and any right $<\kappa$-coherent ring R, the category of short exact sequences of right R-modules is κ-accessible. The*

κ-presentable objects of this category are the short exact sequences of κ-presentable right R-modules.

Proof. The first assertion is a restatement of the first assertion of Lemma 1.6 and holds for any ring R; but one needs the $<\kappa$-coherence assumption in order to obtain the second assertion of the corollary from the second assertion of the lemma (cf. [35, Corollary 10.13]). ■

Proposition 1.8. *For any uncountable regular cardinal κ and any right $<\kappa$-coherent ring R, the category of (unbounded) acyclic complexes of right R-modules is κ-accessible. The κ-presentable objects of this category are the acyclic complexes of κ-presentable right R-modules.*

Proof. The argument is similar to [35, proof of Corollary 10.14]. Notice that an acyclic complex of modules $C^\bullet$ is the same thing as a collection of short exact sequences of modules $0 \to K^n \to C^n \to M^n \to 0$, $n \in \mathbb{Z}$, together with an isomorphism of modules $M^n \simeq K^{n+1}$ for every $n \in \mathbb{Z}$. This observation allows us to construct the category of acyclic complexes of R-modules from the category of short exact sequences of R-modules using Cartesian products and the isomorpher construction.

Specifically, put $\mathsf{L} = \prod_{n\in\mathbb{Z}} \mathsf{Mod}\text{–}R$, and let K be the Cartesian product of the categories of short exact sequences of right R-modules taken over all $n \in \mathbb{Z}$. Let $\Phi_1 : \mathsf{K} \to \mathsf{L}$ be the functor assigning to a family of short exact sequences $(0 \to K^n \to L^n \to M^n \to 0)_{n\in\mathbb{Z}}$ the family of modules $(M^n)_{n\in\mathbb{Z}}$, and let Φ_2 be the functor assigning to the same family of short exact sequences the family of modules $(K^{n+1})_{n\in\mathbb{Z}}$. Then the isomorpher category C is equivalent to the desired category of acyclic complexes of right R-modules.

The categories K and L are κ-accessible by Corollary 1.7 and Proposition 1.2, and Theorem 1.4 is applicable (for $\lambda = \aleph_0$). The theorem tells us that the category C is κ-accessible, and provides the desired description of its full subcategory of κ-presentable objects. ■

2. Modules of small presentability rank as small directed colimits

The aim of this section is to prove the following proposition, which is intended to complement the main results of this paper. Recall that an R-module is said to be *κ-presentable* in our (category-theoretic) terminology if it is the cokernel of a morphism of free R-modules with *less than κ* generators.

Proposition 2.1. *Let R be an associative ring and S be a set of finitely presentable right R-modules. Let $C \in \mathsf{Mod}\text{–}R$ be an $\aleph_m$-presentable R-module belonging to*

$\varinjlim \mathsf{S} \subset \mathsf{Mod}\text{–}R$ *(where $m \ge 0$ is an integer). Let $D \in \mathsf{S}^{\perp_{\ge 1}}$ be a right R-module such that* $\operatorname{Ext}_R^i(S, D) = 0$ *for all $S \in \mathsf{S}$ and $i > 0$. Then* $\operatorname{Ext}_R^i(C, D) = 0$ *for all $i > m$.*

Lemma 2.2. *Let R be an associative ring and S be a set of finitely presentable R-modules. Let κ be a regular cardinal and C be a κ-presentable R-module belonging to $\varinjlim \mathsf{S}$. Then C is a direct summand of a directed colimit of modules from S indexed by a directed poset of cardinality less than κ.*

Proof. Let T denote the class of all directed colimits of modules from S indexed by directed posets of cardinality less than κ. Then, following [1, proof of Theorem 2.11 (iv) ⇒ (i) and Example 2.13 (1)] (with $\lambda = \aleph_0$ and $\mu = \kappa$), every module from $\varinjlim \mathsf{S}$ is a κ-directed colimit of modules from T. For a κ-presentable module $C \in \varinjlim \mathsf{S}$, it follows that C is a direct summand of a module from T. ■

The following lemma can be found in [24, Théorème 4.2].

Lemma 2.3. *Let Ξ be a directed poset, $(S_\xi)_{\xi \in \Xi}$ be a Ξ-indexed diagram of right R-modules, and D be a right R-module. Let $\varprojlim^n_{\xi \in \Xi}$ denote the derived functors of Ξ-indexed limit of abelian groups. Then there is a spectral sequence*

$$E_2^{pq} = \varprojlim\nolimits^p_{\xi \in \Xi} \operatorname{Ext}_R^q(S_\xi, D) \Longrightarrow E_\infty^{pq} = \operatorname{gr}^p \operatorname{Ext}_R^{p+q}(\varinjlim\nolimits_{\xi \in \Xi} S_\xi,\ D).$$

Proof. Let $J^\bullet$ be an injective coresolution of the R-module D, and let $B_\bullet$ be the bar-complex of the diagram $(S_\xi)_{\xi \in \Xi}$. Consider the bicomplex of abelian groups $A^{pq} = \operatorname{Hom}_R(B_p, J^q)$. Let $T^\bullet$ denote the total complex of the bicomplex $A^{\bullet,\bullet}$.

Then one has $H_0(B_\bullet) \simeq \varinjlim_{\xi \in \Xi} S_\xi$ and $H_p(B_\bullet) = 0$ for all $p > 0$ (because the directed colimit functors are exact). Consequently, $H^0(A^{\bullet,q}) = H^0(\operatorname{Hom}_R(B_\bullet, J^q)) \simeq \operatorname{Hom}_R(\varinjlim_{\xi \in \Xi} S_\xi,\ J^q)$ and $H^p(A^{\bullet,q}) = H^p(\operatorname{Hom}_R(B_\bullet, J^q)) = 0$ for all $q \ge 0$ and $p > 0$ (since J^q is an injective R-module). Hence a natural isomorphism $H^n(T^\bullet) \simeq H^n(\operatorname{Hom}_R(\varinjlim_{\xi \in \Xi} S_\xi,\ J^\bullet)) = \operatorname{Ext}_R^n(\varinjlim_{\xi \in \Xi} S_\xi,\ D)$ for all $n \ge 0$.

On the other hand, for every $q \ge 0$, one has $H^q(A^{p,\bullet}) = \operatorname{Ext}_R^q(B_p, D)$. The complex $H^q(A^{\bullet,\bullet})$ is the cobar-complex computing the derived functor of Ξ-indexed limit $\varprojlim^*_{\xi \in \Xi} \operatorname{Ext}_R^q(S_\xi, D)$, so one has $H^p H^q(A^{\bullet,\bullet}) = \varprojlim^p_{\xi \in \Xi} \operatorname{Ext}_R^q(S_\xi, D)$. Thus the spectral sequence $E_2^{pq} = H^p H^q(A^{\bullet,\bullet}) \Rightarrow E_\infty^{pq} = \operatorname{gr}^p H^{p+q}(T^\bullet)$ is the desired one. ■

Proof of Proposition 2.1. In view of Lemma 2.2, one can assume without loss of generality that $C = \varinjlim_{\xi \in \Xi} S_\xi$, where Ξ is a directed poset of cardinality smaller than $\aleph_m$ and $S_\xi \in \mathsf{S}$ for all $\xi \in \Xi$. Then the spectral sequence from Lemma 2.3 degenerates to a natural isomorphism $\operatorname{Ext}_R^n(C, D) \simeq \varprojlim^n_{\xi \in \Xi} \operatorname{Hom}_R(S_\xi, D)$ for all

$n \geq 0$. It remains to recall that the derived functor of Ξ-indexed limit in the category of abelian groups has cohomological dimension at most m [31]. ■

The following corollary can be found in [24, Proposition 5.3]. It is a partial generalization of [15, Corollary 2.23].

Corollary 2.4. *Let R be an associative ring and F be an $\aleph_m$-presentable flat R-module. Then the projective dimension of F does not exceed m.*

Proof. Let S be the set of all finitely generated projective (or free) R-modules, D be an arbitrary R-module, and apply Proposition 2.1. ■

3. Deconstructibility and directed colimits

In this section we discuss the deconstructibility-based approach to accessibility. Both the deconstructible classes and the right Ext^1-orthogonal classes to deconstructible classes are considered.

Let R be an associative ring, F be an R-module, and α be an ordinal. An *α-indexed filtration* of F is a family of submodules $F_\beta \subset F$, indexed by the ordinals $0 \leq \beta \leq \alpha$, satisfying the following conditions:

- $F_0 = 0$ and $F_\alpha = F$;
- one has $F_\gamma \subset F_\beta$ for all $0 \leq \gamma \leq \beta \leq \alpha$;
- one has $F_\beta = \bigcup_{\gamma<\beta} F_\gamma$ for all limit ordinals $\beta \leq \alpha$.

An R-module F endowed with an α-indexed filtration $(F_\beta)_{0\leq\beta\leq\alpha}$ is said to be *filtered by* the quotient modules $F_{\beta+1}/F_\beta$, $0 \leq \beta < \alpha$. Given a class of R-modules $\mathsf{S} \subset \mathsf{Mod}\text{–}R$, one says that an R-module F is *filtered by* S if there exist an ordinal α and an α-indexed filtration on F such that the quotient module $F_{\beta+1}/F_\beta$ is isomorphic to a module from S for every $0 \leq \beta < \alpha$.

The class of all R-modules filtered by S is denoted by $\mathsf{Fil}(\mathsf{S}) \subset \mathsf{Mod}\text{–}R$. A class of R-modules F is said to be *κ-deconstructible* (for a regular cardinal κ) if $\mathsf{F} = \mathsf{Fil}(\mathsf{S})$ for a set of κ-presentable R-modules S.

Proposition 3.1. *Let R be an associative ring, κ be a regular cardinal, and S be a set of κ-presentable R-modules. Then any R-module filtered by S is a κ-directed colimit of κ-presentable R-modules filtered by S. In other words, for any κ-deconstructible class of modules F, all modules from F are κ-directed colimits (in fact, κ-directed unions) of κ-presentable modules from F.*

Proof. This is a direct corollary of the Hill lemma [43, Theorem 6], [15, Theorem 7.10], [42, Theorem 2.1]. Let F be a module filtered by modules from S. Then the Hill lemma provides a complete lattice of submodules in F such that every subset of cardinality less than κ in F is contained in a κ-presentable submodule of F belonging to this complete lattice, and every module belonging to the lattice is filtered by S. The family of all κ-presentable submodules of F belonging to the lattice is κ-directed by inclusion, and F is the directed union of these submodules. ■

An R-module is said to be *$<\kappa$-generated* if it has a set of generators of cardinality less than κ. An associative ring R is said to be *right $<\kappa$-Noetherian* if every submodule of a $<\kappa$-generated right R-module is $<\kappa$-generated, or equivalently, every right ideal in R is $<\kappa$-generated. *Left $<\kappa$-Noetherian rings* are defined similarly. The $<\aleph_1$-generated modules are called *countably generated*, and the right $<\aleph_1$-Noetherian rings are called *right countably Noetherian*.

Proposition 3.2. *Let κ be an uncountable regular cardinal and R be a right $<\kappa$-Noetherian associative ring. Then, for every integer $m \geq 0$, the class of all right R-modules of projective dimension at most m is κ-deconstructible.*

Proof. This result goes back to [39, Corollaire II.3.2.5] (for $\kappa = \aleph_1$) and [2, Proposition 4.1]. For all successor cardinals κ, it is a particular case of [41, Theorem 3.4], and the general case is similar. ■

For any set X, we denote by $|X|$ the cardinality of X. The successor cardinal of a cardinal ν is denoted by ν^+. We will use the notation $\rho = |R| + \aleph_0$ for the minimal infinite cardinal greater than or equal to the cardinality of the ring R. Notice that, for any given cardinal $\nu \geq \rho$, an R-module is ν^+-presentable if and only if it has cardinality at most ν.

Proposition 3.3. *Let R be an associative ring; put $\rho = |R| + \aleph_0$. Then*

(a) *the class of all flat R-modules is ρ^+-deconstructible;*

(b) *for every integer $m \geq 0$, the class of all R-modules of flat dimension at most m is ρ^+-deconstructible.*

Proof. Part (a) is [6, Lemma 1 and Proposition 2]. Part (b) follows from part (a) by virtue of [41, Theorem 3.4]. ■

Given a class of R-modules $\mathsf{S} \subset \mathsf{Mod}\text{–}R$, one denotes by $\mathsf{S}^{\perp_1} \subset \mathsf{Mod}\text{–}R$ the class of all modules $M \in \mathsf{Mod}\text{–}R$ such that $\mathrm{Ext}^1_R(S, M) = 0$ for all $S \in \mathsf{S}$. The following result is known as the *Eklof lemma* [10, Lemma 1], [15, Lemma 6.2].

Lemma 3.4. *For any class of modules* $\mathsf{S} \subset \mathsf{Mod}\text{–}R$ *one has* $\mathsf{S}^{\perp_1} = \mathsf{Fil}(\mathsf{S})^{\perp_1}$. ∎

Given two cardinals ν and λ, one denotes by $\nu^{<\lambda}$ the supremum of the cardinals ν^μ taken over all the cardinals $\mu < \lambda$. The following lemma is a generalization of [15, Lemma 10.5]; it can be found in [8, Lemma 4.1].

Lemma 3.5. *Let R be a ring; put $\rho = |R| + \aleph_0$. Let M be an R-module, λ be a regular cardinal, and ν be a cardinal such that $\rho \le \nu$ and $\nu^{<\lambda} = \nu$. Then for every subset $X \subset M$ of cardinality at most ν there exists a submodule $N \subset M$ of cardinality at most ν such that $X \subset N$ and the following property holds: every system of less than λ nonhomogeneous R-linear equations in less than λ variables, with parameters from N, has a solution in N provided that it has a solution in M.*

Proof. The argument is similar to the one in [15, Lemma 10.5]. Notice that one has $\lambda \le \nu$. The submodule $N \subset M$ is constructed as the union of an increasing chain of submodules $(N_i \subset M)_{0 \le i < \lambda}$, with the cardinality of N_i not exceeding ν for every i. Let N_0 be the submodule spanned by X in M.

For a successor ordinal $j = i + 1 < \lambda$, we define N_{i+1} by adjoining to N one solution of every system of less than λ nonhomogeneous R-linear equations in less than λ variables with parameters from N_i that has a solution in M. As such a system of equations has less than λ coefficients from R and less than λ parameters from N_i, the cardinality of the set of all such systems of equations is not greater than ν. For a limit ordinal $j < \lambda$, we put $N_j = \bigcup_{i<j} N_i$.

Now, if a system of less than λ nonhomogeneous R-linear equations in less than λ variables has parameters in $N = \bigcup_{i<\lambda} N_i$, then all these parameters belong to N_i for some $i < \lambda$. Hence such a system of equations has a solution in N_{i+1}. ∎

An R-module S is said to be *FP_2* [15, Section 5.2] if there exists an exact sequence $P_2 \to P_1 \to P_0 \to S \to 0$ with finitely generated projective R-modules P_0, P_1, and P_2. So, over a right coherent ring, all finitely presentable right modules are FP_2. Similarly, let us say that an R-module S is *λ-P_2* if there exists an exact sequence $P_2 \to P_1 \to P_0 \to S \to 0$, where P_0, P_1, and P_2 are projective R-modules with less than λ generators.

Proposition 3.6. *Let R be a ring; put $\rho = |R| + \aleph_0$. Let λ be a regular cardinal, ν be a cardinal such that $\rho \le \nu$ and $\nu^{<\lambda} = \nu$, and S be a set of λ-P_2 right R-modules. Then the full subcategory $\mathsf{S}^{\perp_1} \subset \mathsf{Mod}\text{–}R$ is closed under λ-directed colimits, and every module $M \in \mathsf{S}^{\perp_1}$ is a ν^+-directed union of the ν^+-directed poset of all the ν^+-presentable submodules of M belonging to $\mathsf{S}^{\perp_1}$. Consequently, the category $\mathsf{S}^{\perp_1}$ is ν^+-accessible with directed colimits of λ-indexed chains, and the ν^+-presentable objects of $\mathsf{S}^{\perp_1}$ are precisely all the ν^+-presentable right R-modules belonging to $\mathsf{S}^{\perp_1}$.*

Proof. One can easily see that for any λ-P_2 module S there exists an exact sequence $P_2 \to P_1 \to P_0 \to S \to 0$ such that P_0, P_1, and P_2 are free modules with less than λ generators. Then, for any R-module M, the group $\mathrm{Ext}^1_R(S, M)$ can be computed as the middle cohomology group of the complex $\mathrm{Hom}_R(P_0, M) \to \mathrm{Hom}_R(P_1, M) \to \mathrm{Hom}_R(P_2, M)$. This proves the first assertion of the proposition. Furthermore, the property of an R-module M to belong to the class $\{S\}^{\perp_1} \subset \mathsf{Mod}\text{–}R$ is expressed by solvability of certain systems of less than λ nonhomogeneous R-linear equations in less than λ variables with parameters in M. Hence the second assertion of the proposition follows from Lemma 3.5. Notice that an R-module is ν^+-presentable if and only if its cardinality does not exceed ν. ∎

Corollary 3.7. *Let λ be a regular cardinal and R be a right $<\lambda$-Noetherian ring. Let ν be a cardinal such that $\rho = |R| + \aleph_0 \le \nu$ and $\nu^{<\lambda} = \nu$. Then the full subcategory of injective R-modules is closed under λ-directed colimits in $\mathsf{Mod}\text{–}R$. The category of injective right R-modules is ν^+-accessible with directed colimits of λ-indexed chains, and the ν^+-presentable objects of this category are precisely all the injective R-modules that are ν^+-presentable in $\mathsf{Mod}\text{–}R$ (i.e., injective R-modules of cardinality at most ν). Moreover, every injective right R-module is a ν^+-directed union of ν^+-presentable injective R-modules.*

Proof. Take S to be the set of all cyclic right R-modules R/I (where I ranges over all the right ideals in R), and apply Proposition 3.6. ∎

4. Two-sided resolutions by accessible classes

In this section we begin our discussion of accessibility of categories of (co)resolutions based on the techniques described in Section 1. We start with a general abstract formulation before passing to a finite/countable special case.

Proposition 4.1. *Let κ be a regular cardinal and $\lambda < \kappa$ be a smaller infinite cardinal. Let R be a right $<\kappa$-coherent ring, and let $(\mathsf{T}_n)_{n\in\mathbb{Z}}$ be a sequence of classes of κ-presentable right R-modules, $\mathsf{T}_n \subset (\mathsf{Mod}\text{–}R)_{<\kappa}$. Assume that, for every $n \in \mathbb{Z}$, the class of R-modules T_n is closed under direct summands and the class of R-modules $\varinjlim_{(\kappa)} \mathsf{T}_n$ is closed under colimits of λ-indexed chains in $\mathsf{Mod}\text{–}R$. Then the category C of all acyclic complexes of right R-modules $C^\bullet$ with $C^n \in \varinjlim_{(\kappa)} \mathsf{T}_n$ for every $n \in \mathbb{Z}$ is κ-accessible. The κ-presentable objects of the category C are all the acyclic complexes of R-modules $T^\bullet$ with $T^n \in \mathsf{T}_n$ for every $n \in \mathbb{Z}$. Consequently, every acyclic complex of R-modules $C^\bullet$ with the terms $C^n \in \varinjlim_{(\kappa)} \mathsf{T}_n$ is a κ-directed colimit of acyclic complexes $T^\bullet$ with the terms $T^n \in \mathsf{T}_n$.*

Proof. This is an application of Theorem 1.3 together with Propositions 1.1, 1.2, and 1.8. Denote by K_1 the category of acyclic complexes of right R-modules from Proposition 1.8, and let $\mathsf{K}_2 = \prod_{n\in\mathbb{Z}}(\varinjlim_{(\kappa)} \mathsf{T}_n)$ be the Cartesian product of the full subcategories $\varinjlim_{(\kappa)} \mathsf{T}_n \subset \mathsf{Mod}\text{–}R$, taken over all the integers $n \in \mathbb{Z}$.

Let $\mathsf{L} = \prod_{n\in\mathbb{Z}} \mathsf{Mod}\text{–}R$ be the Cartesian product of $\mathbb{Z}$ copies of the abelian category of right R-modules. Consider the following functors $\Phi_1 \colon \mathsf{K}_1 \to \mathsf{L}$ and $\Phi_2 \colon \mathsf{K}_2 \to \mathsf{L}$: The functor Φ_1 takes an acyclic complex of right R-modules $A^\bullet$ to the collection of modules $(A^n)_{n\in\mathbb{Z}}$. The functor Φ_2 is the Cartesian product of the identity inclusion functors $\varinjlim_{(\kappa)} \mathsf{T}_n \to \mathsf{Mod}\text{–}R$, taken over all $n \in \mathbb{Z}$. Then the pseudopullback of the pair of functors Φ_1 and Φ_2 is equivalent to the desired category C.

The category K_1 is κ-accessible by Proposition 1.8. The categories K_2 and L are κ-accessible by Propositions 1.1 and 1.2. Theorem 1.3 is applicable; it tells us that the category C is κ-accessible and provides the desired description of the full subcategory of κ-presentable objects in C. ■

Theorem 4.2. *Let R be a right countably coherent ring, and let $(\mathsf{S}_n)_{n\in\mathbb{Z}}$ be a sequence of sets of finitely presentable right R-modules, $\mathsf{S}_n \subset (\mathsf{Mod}\text{–}R)_{<\aleph_0}$. Then the category C of all acyclic complexes of right R-modules $C^\bullet$ with $C^n \in \varinjlim \mathsf{S}_n$ for every $n \in \mathbb{Z}$ is $\aleph_1$-accessible. The $\aleph_1$-presentable objects of the category C are all the acyclic complexes $T^\bullet$ such that $T^n \in \varinjlim \mathsf{S}_n$ and T^n is a countably presentable R-module for every $n \in \mathbb{Z}$. Consequently, every acyclic complex of R-modules $C^\bullet$ with the terms $C^n \in \varinjlim \mathsf{S}_n$ is an $\aleph_1$-directed colimit of acyclic complexes of countably presentable R-modules $T^\bullet$ with the terms $T^n \in \varinjlim \mathsf{S}_n$.*

Proof. This is a particular case of Proposition 4.1. Put $\lambda = \aleph_0$, $\kappa = \aleph_1$, and denote by $\mathsf{T}_n \subset \mathsf{Mod}\text{–}R$ the class of all direct summands of countable directed colimits of modules from S_n, for every $n \in \mathbb{Z}$. Then $\varinjlim \mathsf{S}_n = \varinjlim_{(\aleph_1)} \mathsf{T}_n$, as per the proof of Lemma 2.2. The class $\varinjlim \mathsf{S}_n$ is closed under directed colimits in $\mathsf{Mod}\text{–}R$ by Proposition 1.1. ■

Given an acyclic complex of R-modules $C^\bullet$, we denote by $Z^0(C^\bullet)$ the module of degree 0 cocycles in $C^\bullet$. So $Z^0(C^\bullet)$ is the kernel of the differential $d^0 \colon C^0 \to C^1$, or equivalently, the cokernel of the differential $d^{-2} \colon C^{-2} \to C^{-1}$.

Corollary 4.3. *Let R be a right countably coherent ring, and let $(\mathsf{S}_n)_{n\in\mathbb{Z}}$ be a sequence of sets of finitely presentable right R-modules, $\mathsf{S}_n \subset (\mathsf{Mod}\text{–}R)_{<\aleph_0}$. Denote by M the class of all right R-modules M of the form $M = Z^0(C^\bullet)$, where $C^\bullet$ is an acyclic complex of R-modules with the terms $C^n \in \varinjlim \mathsf{S}_n$. Then every module from M is an $\aleph_1$-directed colimit of countably presentable modules from M.*

Proof. Let us emphasize that there is *no* claim about closedness of the class M under $\aleph_1$-directed colimits in this corollary. The assertion of the corollary follows directly from the last assertion of Theorem 4.2. One needs to observe that, in any acyclic complex of countably presentable modules, the modules of cocycles are also countably presentable. ■

Corollary 4.4. *In the context of Corollary 4.3, let M be a countably presentable R-module belonging to the class M. Then M is a direct summand of an R-module $N = Z^0(T^\bullet)$, where $T^\bullet$ is an acyclic complex of countably presentable R-modules with the terms $T^n \in \varinjlim \mathsf{S}_n$.*

Proof. It is clear from Theorem 4.2 that every R-module $M \in \mathsf{M}$ is the colimit of an $\aleph_1$-directed diagram of R-modules $(N_\xi)_{\xi\in\Xi}$ such that $N_\xi = Z^0(T^\bullet_\xi)$ for some acyclic complexes of countably presentable modules $T^\bullet_\xi$ with the terms $T^n_\xi \in \varinjlim \mathsf{S}_n$. Now, if M is countably presentable, then it follows that there exists $\xi \in \Xi$ such that M is a direct summand of N_ξ. ■

5. Modules of finite flat dimension

Recall from the introduction that, even over commutative Noetherian local rings R of Krull dimension 2, a module of flat dimension 1 need not be a directed colimit of modules of projective dimension at most 1 [5, Example 8.5], [20, Theorem B]. Any finitely generated R-module of flat dimension ≤ 1 would, of course, have projective dimension ≤ 1 (since finitely presentable flat modules are always projective). So a module of flat dimension 1 *need not* be a directed colimit of finitely generated flat modules of flat dimension at most 1.

By contrast, in this section we show that any module of flat dimension m is a directed colimit of countably presentable modules of flat dimension $\le m$.

Theorem 5.1. *Let R be a right countably coherent ring and $m \ge 0$ be an integer. Then the category of all exact sequences of flat right R-modules $0 \to F_m \to F_{m-1} \to \dots \to F_0$ is $\aleph_1$-accessible. The $\aleph_1$-presentable objects of this category are precisely all the exact sequences $0 \to T_m \to T_{m-1} \to \dots \to T_0$ with countably presentable flat right R-modules T_n, $0 \le n \le m$. Consequently, every finite flat resolution $0 \to F_m \to F_{m-1} \to \dots \to F_0$ in $\mathsf{Mod}\text{–}R$ is an $\aleph_1$-directed colimit of finite flat resolutions $0 \to T_m \to T_{m-1} \to \dots \to T_0$ of the same length m with countably presentable flat modules T_n, $0 \le n \le m$.*

Proof. This is a particular case of Theorem 4.2. Take S_n to be the set of all finitely generated projective (or free) right R-modules for all $-m-1 \le n \le -1$, S_0 to be the set of all finitely presentable right R-modules, and $\mathsf{S}_n = \{0\}$ for all $n \le -m-2$ and all $n \ge 1$. Then $\varinjlim \mathsf{S}_n$ is the class of all flat right R-modules for all $-m-1 \le n \le -1$, $\varinjlim \mathsf{S}_0 = \mathsf{Mod}\text{–}R$, and $\varinjlim \mathsf{S}_n = \{0\}$ for all $n \le -m-2$ and all $n \ge 1$. So the category C from Theorem 4.2 is equivalent to the category of finite flat resolutions we are interested in. ∎

Corollary 5.2. *Let R be a right countably coherent ring and $m \ge 0$ be an integer. Denote by $\mathsf{F}_m \subset \mathsf{Mod}\text{–}R$ the full subcategory of all right R-modules of flat dimension at most m. Then the category F_m is $\aleph_1$-accessible. The $\aleph_1$-presentable objects of F_m are precisely all the countably presentable right R-modules of flat dimension $\le m$ (i.e., those modules from F_m that are countably presentable in $\mathsf{Mod}\text{–}R$). So every right R-module of flat dimension m is an $\aleph_1$-directed colimit of countably presentable R-modules of flat dimension at most m.*

Proof. It is easy to see that the full subcategory F_m is closed under directed colimits in $\mathsf{Mod}\text{–}R$ (since the functor Tor preserves directed colimits). In view of Proposition 1.1, it suffices to check the last assertion of the corollary, which follows immediately from Theorem 5.1. Notice that the cokernel of any morphism of countably presentable modules is countably presentable. ∎

Corollary 5.3. *Let R be a right countably coherent ring and $m \ge 0$ be an integer. Then every right R-module of flat dimension m is an $\aleph_1$-directed colimit of countably presentable right R-modules of projective dimension at most $m+1$.*

Proof. This is a corollary of Corollary 5.2. The point is that any countably presentable right R-module of flat dimension $\le m$ has projective dimension $\le m+1$, since any countably presentable flat module has projective dimension at most 1 (by [15, Corollary 2.23] or Corollary 2.4 above). ∎

Notice that Corollary 5.2 (or more generally, Proposition 4.1) gives a better cardinality estimate for the accessibility rank of the category of R-modules of flat dimension $\le m$ than Proposition 3.1 combined with Proposition 3.3 (b).

6. Flatly coresolved modules

By a *flat coresolution* we mean an exact sequence $F^0 \to F^1 \to F^2 \to \cdots$, where F^n are flat modules for all $n \ge 0$. An R-module M is said to be *flatly coresolved* if there exists an exact sequence of R-modules $0 \to M \to F^0 \to F^1 \to F^2 \to \cdots$ with flat R-modules F^n.

Theorem 6.1. *Let R be a right countably coherent ring. Then the category of flat coresolutions $F^0 \to F^1 \to F^2 \to \cdots$ in $\mathsf{Mod}\text{–}R$ is $\aleph_1$-accessible. The $\aleph_1$-presentable objects of this category are precisely all the flat coresolutions $T^0 \to T^1 \to T^2 \to \cdots$ with countably presentable flat right R-modules T^n, $n \ge 0$. Consequently, every flat coresolution $F^0 \to F^1 \to F^2 \to \cdots$ in $\mathsf{Mod}\text{–}R$ is an $\aleph_1$-directed colimit of flat coresolutions $T^0 \to T^1 \to T^2 \to \cdots$ with countably presentable flat R-modules T^n, $n \ge 0$.*

Proof. This is another particular case of Theorem 4.2. Take S_n to be the set of all finitely generated projective (or free) right R-modules for all $n \ge 0$, S_{-1} to be the set of all finitely presentable right R-modules, and $\mathsf{S}_n = \{0\}$ for all $n \le -2$. Then $\varinjlim \mathsf{S}_n$ is the class of all flat right R-modules for all $n \ge 0$, $\varinjlim \mathsf{S}_{-1} = \mathsf{Mod}\text{–}R$, and $\varinjlim \mathsf{S}_n = 0$ for all $n \le -2$. So the category C from Theorem 4.2 is equivalent to the category of flat coresolutions we are interested in. ■

Corollary 6.2. *Let R be a right countably coherent ring. Then any flatly coresolved right R-module is an $\aleph_1$-directed colimit of countably presentable flatly coresolved right R-modules.*

Proof. This is a corollary of Theorem 6.1 and a particular case of Corollary 4.3. Let us emphasize once again that there is *no* claim about the class of flatly coresolved modules being closed under $\aleph_1$-directed colimits in this corollary. To deduce the assertion of the corollary from Theorem 6.1, one needs to observe that the kernel of any morphism of countably presentable right modules over a right countably coherent ring R is countably presentable. ■

Corollary 6.3. *Let R be a right countably coherent ring and M be a countably presentable flatly coresolved right R-module. Then M is a direct summand of an R-module N admitting a flat coresolution $0 \to N \to T^0 \to T^1 \to T^2 \to \cdots$ with countably presentable flat right R-modules T^n, $n \ge 0$.*

Proof. This is a corollary of Theorem 6.1 and a particular case of Corollary 4.4. ■

7. Dualizing complexes and F-totally acyclic complexes

An acyclic complex of flat right R-modules $F^\bullet$ is said to be *F-totally acyclic* [12, Section 2] if the complex of abelian groups $F^\bullet \otimes_R J$ is acyclic for every injective left R-module J. A right R-module M is said to be *Gorenstein-flat* if there exists an F-totally acyclic complex of flat right R-modules $F^\bullet$ such that $M \simeq Z^0(F^\bullet)$ is its module of cocycles.

The aim of the following four sections, Sections 7–10, is to establish some accessibility properties of the categories of F-totally acyclic complexes and Gorenstein-flat modules, under suitable assumptions. In particular, the purpose of the present section is to prepare the ground for the next one.

We recall that a right R-module K is said to be *fp-injective* (or "absolutely pure") if $\mathrm{Ext}^1_R(T, K) = 0$ for all finitely presentable right R-modules T. The definition of an fp-injective left module is similar.

Given an abelian category A, we denote by $\mathsf{D}^{\mathsf{b}}(\mathsf{A})$ the bounded derived category of A. A bounded complex of left R-modules $K^\bullet$ is said to have *finite fp-injective dimension* if it is isomorphic, as an object of the derived category of left R-modules $\mathsf{D}^{\mathsf{b}}(R\text{–}\mathsf{Mod})$, to a bounded complex of fp-injective left R-modules.

Let R and S be two associative rings. Assume that the ring S is right coherent. In the following definition we list some properties of a complex of R-S-bimodules $D^\bullet$ that are relevant for the purposes of this section. For comparison, see the definition of a *dualizing complex* in [32, Section 4].

We will say that a bounded complex of R-S-bimodules $D^\bullet$ is a *right dualizing complex* for the rings R and S if it satisfies the following conditions:

(i) the terms of the complex $D^\bullet$ are fp-injective as right S-modules, and the whole complex $D^\bullet$ has finite fp-injective dimension as a complex of left R-modules;

(ii) the R-S-bimodules of cohomology of the complex $D^\bullet$ are finitely presentable as right S-modules;

(iii) the homothety map $R \to \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{–}S)}(D^\bullet, D^\bullet[*])$ is an isomorphism of graded rings.

The following proposition is the main result of this section. It is our version of [25, Lemma 1.7].

Proposition 7.1. *Let R be a ring, S be a right coherent ring, and $D^\bullet$ be a right dualizing complex of R-S-bimodules. Then an acyclic complex of flat right R-modules $F^\bullet$ is F-totally acyclic if and only if the complex of right S-modules $F^\bullet \otimes_R D^\bullet$ is acyclic.*

The proof of Proposition 7.1 is based on a sequence of lemmas.

Lemma 7.2. *Let $F^\bullet$ be a complex of flat right R-modules and $L^\bullet$ be a bounded acyclic complex of left R-modules. Then the complex of abelian groups $F^\bullet \otimes_R L^\bullet$ is acyclic.*

Proof. The point is that the complex of abelian groups $F^n \otimes_R L^\bullet$ is acyclic for every $n \in \mathbb{Z}$. The total complex of any bounded acyclic complex of complexes of abelian groups is acyclic. ∎

Lemma 7.3. *Let $F^\bullet$ be a complex of right R-modules such that the complex of abelian groups $F^\bullet \otimes_R J$ is acyclic for every injective left R-module J. Then the complex of abelian groups $F^\bullet \otimes_R K$ is acyclic for every fp-injective left R-module K.*

Proof. This is [29, proof of Lemma 2.8 (1) ⇒ (2)]. Let an fp-injective R-module K be a submodule in an injective R-module J; then K is a pure submodule in J (see, e.g., [15, Definition 2.6 and Lemma 2.19]). The point is that whenever some left R-module M is a pure submodule in a left R-module L, and $F^\bullet$ is a complex of right R-modules such that the complex of abelian groups $F^\bullet \otimes_R L$ is acyclic, it follows that the complex of abelian groups $F^\bullet \otimes_R M$ is acyclic as well. One can prove this by observing that $\mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is a split epimorphism of right R-modules; hence the complex $\mathrm{Hom}_{\mathbb{Z}}(F^\bullet \otimes_R M,\ \mathbb{Q}/\mathbb{Z})$ is a direct summand of the complex $\mathrm{Hom}_{\mathbb{Z}}(F^\bullet \otimes_R L,\ \mathbb{Q}/\mathbb{Z})$. ∎

Corollary 7.4. *Let $F^\bullet$ be an F-totally acyclic complex of flat right R-modules and $K^\bullet$ be a bounded complex of left R-modules. Assume that the complex $K^\bullet$ has finite fp-injective dimension. Then the complex of abelian groups $F^\bullet \otimes_R K^\bullet$ is acyclic.*

Proof. Combine Lemmas 7.2 and 7.3. ∎

Lemma 7.5. *Let S be a ring, $H^\bullet$ be an acyclic complex of right S-modules, and $G^\bullet$ be a bounded complex of flat left S-modules. Then the complex of abelian groups $H^\bullet \otimes_S G^\bullet$ is acyclic.*

Proof. The point is that the total complex of any bounded complex of acyclic complexes of abelian groups is acyclic. ∎

Lemma 7.6. *Let R be a ring, S be a right coherent ring, E be an R-S-bimodule that is fp-injective as a right S-module, and J be an injective left R-module. Then the left S-module $\mathrm{Hom}_R(E, J)$ is flat.*

Proof. This is [32, Lemma 4.1 (b)]. The point is that the functor $N \mapsto N \otimes_S \mathrm{Hom}_R(E, J) \simeq \mathrm{Hom}_R(\mathrm{Hom}_S(N, E), J)$ is exact on the abelian category of finitely presentable right S-modules N. ∎

Lemma 7.7. *Let R be a ring, S be a right coherent ring, and $D^\bullet$ be a right dualizing complex of R-S-bimodules. Then the natural (evaluation) morphism of complexes of* *left R-modules $D^\bullet \otimes_S \operatorname{Hom}_R(D^\bullet, J) \to J$ is a quasi-isomorphism for every injective left R-module J.*

Proof. This lemma only uses conditions (ii)–(iii) and the first part of condition (i) from the definition of a right dualizing complex above. See [32, Lemma 4.2 (b)]. ■

Proof of Proposition 7.1. The "only if" implication is provided by Corollary 7.4. To prove the "if", let J be an injective left R-module. By Lemma 7.7, we have a quasi-isomorphism of bounded complexes of left R-modules $D^\bullet \otimes_S \operatorname{Hom}_R(D^\bullet, J) \to J$. In view of Lemma 7.2, in order to show that the complex $F^\bullet \otimes_R J$ is acyclic, it suffices to check that so is the complex $F^\bullet \otimes_R D^\bullet \otimes_S \operatorname{Hom}_R(D^\bullet, J)$. Now, by assumption, the complex of right S-modules $F^\bullet \otimes_R D^\bullet$ is acyclic. By Lemma 7.6, $\operatorname{Hom}_R(D^\bullet, J)$ is a bounded complex of flat left S-modules. It remains to refer to Lemma 7.5. ■

Remark 7.8. The concepts of Gorenstein homological algebra become less complicated over a ring which is itself Gorenstein in a suitable sense. Notice that, for any acyclic complex of flat right R-modules $F^\bullet$ and any left R-module G of finite flat dimension, the complex of abelian groups $F^\bullet \otimes_R G$ is acyclic. This follows from Lemmas 7.2 and 7.5. Now, if every injective left R-module has finite flat dimension, then every acyclic complex of flat right R-modules is F-totally acyclic, and every flatly coresolved right R-module is Gorenstein-flat. Assuming additionally that the ring R is right countably coherent, the results of Theorem 4.2, Theorem 6.1, Corollary 6.2, and Corollary 6.3 become directly applicable (essentially) as properties of F-totally acyclic complexes and Gorenstein-flat modules.

8. F-totally acyclic complexes as directed colimits

We start with a very general result of Šaroch and Šťovíček.

Proposition 8.1. *Let R be an associative ring. Then*

(a) *the class of Gorenstein-flat modules is closed under directed colimits in* $\mathsf{Mod}\text{–}R$*;*

(b) *the class of Gorenstein-flat R-modules is ρ^+-deconstructible, where $\rho = |R| + \aleph_0$.*

Proof. These assertions are parts of [40, Corollary 4.12]. ■

We will need to assume another condition in addition to the conditions (i)–(iii) in the definition of a right dualizing complex in Section 7. We will say that a right dualizing complex of R-S-bimodules $D^\bullet$ has *right countable type* if

(iv) there exist a right countably coherent ring S' and a bounded complex of R-S'-bimodules $\widetilde{D}^\bullet$ whose terms are countably presentable as right S'-modules such that the complexes $D^\bullet$ and $\widetilde{D}^\bullet$ are isomorphic as objects of the bounded derived category of left R-modules $\mathsf{D}^{\mathrm{b}}(R\text{–}\mathsf{Mod})$.

Lemma 8.2. *Let R and S be two rings, T be a right R-module, and E be an R-S-bimodule. Assume that the right R-module T is κ-presentable and the right S-module E is κ-presentable, for a given regular cardinal κ. Then the right S-module $T \otimes_R E$ is also κ-presentable.*

Proof. Represent T as the cokernel of a morphism of free R-modules with less than κ generators, and use the fact that the class of all κ-presentable S-modules is closed under colimits indexed by categories with less than κ morphisms [1, Proposition 1.16]. ∎

Theorem 8.3. *Let R be a right countably coherent ring, S be a right coherent ring, and $D^\bullet$ be a right dualizing complex of R-S-bimodules. Assume that $D^\bullet$ is a right dualizing complex of right countable type. Then the category of F-totally acyclic complexes of flat right R-modules is $\aleph_1$-accessible. The $\aleph_1$-presentable objects of this category are precisely all the F-totally acyclic complexes of countably presentable flat right R-modules. Consequently, every F-totally acyclic complex of flat right R-modules is an $\aleph_1$-directed colimit of F-totally acyclic complexes of countably presentable flat right R-modules.*

Proof. This is an application of Theorem 1.3 together with Theorem 4.2 and Proposition 7.1. Notice first of all that the full subcategory of F-totally acyclic complexes of flat modules is obviously closed under directed colimits in the ambient abelian category of all complexes of modules.

Now let $\widetilde{D}^\bullet$ be a bounded complex of S'-countably presentable R-S'-bimodules provided by the definition of a right dualizing complex of right countable type (item (iv)) above. It is clear from Lemma 7.2 that, for any complex of flat right R-modules $F^\bullet$, the complex of right S-modules $F^\bullet \otimes_R D^\bullet$ is acyclic if and only if the complex of right S'-modules $F^\bullet \otimes_R \widetilde{D}^\bullet$ is acyclic.

Denote by K_1 the category of acyclic complexes of flat right R-modules. Applying Theorem 4.2 for S_n being the set of all finitely generated projective (or free) R-modules, for every $n \in \mathbb{Z}$, we see that K_1 is an $\aleph_1$-accessible category, and obtain a description of its $\aleph_1$-presentable objects (cf. Theorem 6.1).

Furthermore, denote by K_2 the category of acyclic complexes of right S'-modules. Proposition 1.8 states that K_2 is an $\aleph_1$-accessible category and provides a description of its full subcategory of $\aleph_1$-presentable objects. Finally, let L be the abelian category of arbitrary complexes of right S'-modules. Lemma 1.5 (a), (c) states that L is a locally $\aleph_1$-presentable category and describes its full subcategory of $\aleph_1$-presentable objects.

Let the functor $\Phi_1\colon \mathsf{K}_1 \to \mathsf{L}$ take any acyclic complex of flat right R-modules $F^\bullet$ to the complex of right S'-modules $F^\bullet \otimes_R \widetilde{D}^\bullet$. Let $\Phi_2\colon \mathsf{K}_2 \to \mathsf{L}$ be the identity inclusion of the category of acyclic complexes of right S'-modules into the category of all complexes of right S'-modules. Then the pseudopullback C of the two functors Φ_1 and Φ_2 is the category of all acyclic complexes of flat right R-modules $F^\bullet$ for which the complex of right S'-modules $F^\bullet \otimes_R \widetilde{D}^\bullet$ is acyclic. By Proposition 7.1, and in view of the argument with Lemma 7.2 above, the category C is the desired category of F-totally acyclic complexes of flat right R-modules.

Finally, by Lemma 8.2, the functor Φ_1 takes $\aleph_1$-presentable objects to $\aleph_1$-presentable objects. All the other assumptions of Theorem 1.3 (for $\kappa = \aleph_1$ and $\lambda = \aleph_0$) are clearly satisfied. Theorem 1.3 tells us that C is an $\aleph_1$-accessible category, and provides the desired description of its full subcategory of $\aleph_1$-presentable objects. ■

Corollary 8.4. *Let R be a right countably coherent ring, S be a right coherent ring, and $D^\bullet$ be a right dualizing complex of R-S-bimodules. Assume that $D^\bullet$ is a right dualizing complex of right countable type. Then the category of Gorenstein-flat right R-modules GF is $\aleph_1$-accessible. The $\aleph_1$-presentable objects of GF are precisely all the Gorenstein-flat right R-modules that are countably presentable in Mod–R. So every Gorenstein-flat right R-module is an $\aleph_1$-directed colimit of countably presentable Gorenstein-flat right R-modules.*

Proof. The full subcategory $\mathsf{GF} \subset \mathsf{Mod}$–$R$ is closed under directed colimits in Mod–R by Proposition 8.1 (a). On the other hand, it is obvious from Theorem 8.3 that any Gorenstein-flat right R-module is an $\aleph_1$-directed colimit of Gorenstein-flat right R-modules that are countably presentable in Mod–R. In view of Proposition 1.1, all the assertions of the corollary follow. ■

For a far-reaching generalization of Corollary 8.4, see Theorem 10.2 below.

Corollary 8.5. *Let R be a right countably coherent ring, S be a right coherent ring, and $D^\bullet$ be a right dualizing complex of R-S-bimodules. Assume that $D^\bullet$ is a right dualizing complex of right countable type. Let M be a countably presentable Gorenstein-flat right R-module. Then M is a direct summand of an R-module N admitting an F-totally acyclic two-sided resolution $T^\bullet$ with countably presentable flat right R-modules T^n, $n \in \mathbb{Z}$.*

Proof. The argument is similar to the proof of Corollary 4.4. It is clear from Theorem 8.3 that every Gorenstein-flat right R-module M is the colimit of an $\aleph_1$-directed diagram of R-modules $(N_\xi)_{\xi\in\Xi}$ such that $N_\xi = Z^0(T^\bullet_\xi)$ for some F-totally acyclic complexes of countably presentable flat right R-modules $T_\xi^\bullet$. Now if M is countably presentable, then it follows that M is a direct summand of N_ξ for some $\xi \in \Xi$. ∎

9. Commutative Noetherian rings with countable spectrum

In this section we prove results similar to those of Section 8, but under a different set of assumptions. Instead of existence of a dualizing complex, we assume the ring R to be commutative Noetherian with small cardinality of the spectrum.

For any R-module M, we denote by $E_R(M)$ an injective envelope of M. For a prime ideal $\mathfrak{p}$ in a commutative ring R, we denote by $R_\mathfrak{p}$ the local ring $(R \setminus \mathfrak{p})^{-1}R$. The following proposition is standard commutative algebra material:

Proposition 9.1. *Let R be a commutative Noetherian ring. In this setting, we have the following:*

(a) *All injective R-modules are direct sums of indecomposable injective R-modules.*

(b) *The indecomposable injective R-modules, viewed up to isomorphism, correspond bijectively to prime ideals of R. For every prime ideal $\mathfrak{p} \in \operatorname{Spec} R$, the corresponding indecomposable injective R-module is the injective envelope $E_R(R/\mathfrak{p})$ of the R-module $R/\mathfrak{p}$.*

(c) *For every prime ideal $\mathfrak{p} \in \operatorname{Spec} R$, the R-module $E_R(R/\mathfrak{p})$ is a module over the local ring $R_\mathfrak{p}$. The $R_\mathfrak{p}$-module $E_R(R/\mathfrak{p})$ is (at most) countably generated.*

Proof. All these results are due to Matlis [30]. Part (a) is [30, Theorem 2.5]. Part (b) is [30, Proposition 3.1]. Part (c) is [30, Theorems 3.6 and 3.11]. ∎

Theorem 9.2. *Let κ be an uncountable regular cardinal and R be a commutative Noetherian ring with the cardinality of the spectrum $|\operatorname{Spec} R| < \kappa$. Then the category of F-totally acyclic complexes of flat R-modules is κ-accessible. The κ-presentable objects of this category are precisely all the F-totally acyclic complexes of $<\kappa$-generated flat R-modules. Consequently, every F-totally acyclic complex of flat R-modules is a κ-directed colimit of F-totally acyclic complexes of $<\kappa$-generated flat R-modules.*

Proof. This is an application of Theorem 1.3 together with Propositions 1.2 and 4.1. The argument is somewhat similar to the proof of Theorem 8.3.

Denote by K_1 the category of acyclic complexes of flat R-modules. Applying Proposition 4.1 for T_n being the class of all $<\kappa$-generated flat R-modules for every $n \in \mathbb{Z}$ (cf. [35, Proposition 10.2]) and $\lambda = \aleph_0$, we see that K_1 is a κ-accessible category and obtain a description of its κ-presentable objects.

Furthermore, denote by K_2 the Cartesian product of the categories of acyclic complexes of $R_{\mathfrak{p}}$-modules, taken over all the prime ideals $\mathfrak{p} \in \operatorname{Spec} R$. Proposition 1.8 together with Proposition 1.2 tells us that K_2 is a κ-accessible category and provides a description of its full subcategory of κ-presentable objects. Finally, denote by L the Cartesian product of the abelian categories of arbitrary complexes of $R_{\mathfrak{p}}$-modules, taken over all the spectrum points $\mathfrak{p} \in \operatorname{Spec} R$. Lemma 1.5 (a), (c) together with Proposition 1.2 tells us that L is a locally κ-presentable category and describes its full subcategory of κ-presentable objects.

Let the functor $\Phi_1 : \mathsf{K}_1 \to \mathsf{L}$ take any acyclic complex of flat R-modules $F^\bullet$ to the collection of all complexes of $R_{\mathfrak{p}}$-modules $F^\bullet \otimes_R E_R(R/\mathfrak{p})$. Let $\Phi_2 : \mathsf{K}_2 \to \mathsf{L}$ be the Cartesian product of the identity inclusions of the categories of acyclic complexes of $R_{\mathfrak{p}}$-modules into the respective categories of all complexes of $R_{\mathfrak{p}}$-modules. Then the pseudopullback C of the two functors Φ_1 and Φ_2 is the category of all F-totally acyclic complexes of flat R-modules (in view of Proposition 9.1 (a), (b)).

Finally, it follows from Proposition 9.1 (c) and Lemma 8.2 (for $S = R_{\mathfrak{p}}$) that the functor Φ_1 takes κ-presentable objects to κ-presentable objects. All the other assumptions of Theorem 1.3 (for the given cardinal κ and $\lambda = \aleph_0$) are clearly satisfied. Theorem 1.3 tells us that C is a κ-accessible category, and provides the desired description of its full subcategory of κ-presentable objects. ■

Corollary 9.3. *Let κ be an uncountable regular cardinal and R be a commutative Noetherian ring with the cardinality of the spectrum $|\operatorname{Spec} R| < \kappa$. Then the category GF of Gorenstein-flat R-modules is κ-accessible. The κ-presentable objects of GF are precisely all the $<\kappa$-generated Gorenstein-flat R-modules. So every Gorenstein-flat R-module is a κ-directed colimit of $<\kappa$-generated Gorenstein-flat R-modules.*

Proof. Similar to the proof of Corollary 8.4 and based on Theorem 9.2 together with Proposition 8.1 (a). ■

For a far-reaching generalization of Corollary 9.3, see Theorem 10.2 below.

Corollary 9.4. *Let κ be an uncountable regular cardinal and R be a commutative Noetherian ring with the cardinality of the spectrum $|\operatorname{Spec} R| < \kappa$. Let M be a $<\kappa$-generated Gorenstein-flat R-module. Then M is a direct summand of an R-module N admitting an F-totally acyclic two-sided resolution $T^\bullet$ with $<\kappa$-generated flat R-modules T^n, $n \in \mathbb{Z}$.*

Proof. Similar to the proof of Corollary 8.5 and based on Theorem 9.2. ■

In particular, if the cardinality of Spec R is at most countable, then Theorem 9.2 and Corollaries 9.3–9.4 are applicable for $\kappa = \aleph_1$, providing a description of F-totally acyclic complexes of flat R-modules as $\aleph_1$-directed colimits of F-totally acyclic complexes of countably generated flat R-modules.

10. Gorenstein-flat modules as directed colimits

In this section we use powerful and difficult results of the paper [40] in order to deduce a common generalization of Corollaries 8.4 and 9.3 applicable to all right countably coherent rings R.

LEMMA 10.1. *Let R be an associative ring, κ be a regular cardinal, and $\lambda < \kappa$ be a smaller infinite cardinal. Let S and T be two classes of κ-presentable right R-modules. Assume that the class of R-modules T is closed under direct summands, the class of R-modules $\varinjlim_{(\kappa)} \mathsf{T}$ is closed under colimits of λ-indexed chains in $\mathsf{Mod}\text{–}R$, and the class S contains all the κ-presentable R-modules belonging to $\mathsf{Fil}(\mathsf{S})$. Let $F \in \varinjlim_{(\kappa)} \mathsf{T}$ and $N \in \mathsf{Fil}(\mathsf{S})$ be two R-modules, and let $f\colon F \to N$ be a surjective R-module morphism. Then the morphism f is a κ-directed colimit of surjective R-module morphisms $t\colon T \to S$ with $T \in \mathsf{T}$ and $S \in \mathsf{S}$.*

PROOF. This is another application of Theorem 1.3 together with Lemma 1.6 and the Hill lemma. Let K_1 be the category of epimorphisms of right R-modules. Lemma 1.6 states that the category K_1 is κ-accessible and provides a description of its full subcategory of κ-presentable objects. Let $\mathsf{L} = \mathsf{Mod}\text{–}R \times \mathsf{Mod}\text{–}R$ be the Cartesian square of the abelian category of right R-modules. The category L is locally κ-presentable, and Proposition 1.2 provides a description of its full subcategory of κ-presentable objects.

Finally, let $\mathsf{K}_2 = \mathsf{F} \times \mathsf{N}$ be the Cartesian product of the following two categories. The category F is the class of modules $\mathsf{F} = \varinjlim_{(\kappa)} \mathsf{T}$, viewed as a full subcategory in $\mathsf{Mod}\text{–}R$. To construct the category N, we need to recall the Hill lemma [43, Theorem 6], [15, Theorem 7.10], [42, Theorem 2.1]. In application to the R-module N with its given filtration by modules from S, the Hill lemma produces a certain complete lattice of submodules of N. The category N is this complete lattice of submodules in N, viewed as a poset, and this poset is viewed as a category. (Notice that the category N is *not* additive.)

As any complete lattice interpreted as a category, the category N has all limits and colimits. In particular, the colimits in N are just the joins in the lattice. Furthermore, in the situation at hand it follows from the Hill lemma that N is a locally κ-presentable category, and the κ-presentable objects of N are precisely all the κ-presentable submodules of N belonging to the lattice. On the other hand, the category F is κ-accessible, and

$\mathsf{T} \subset \mathsf{F}$ is its full subcategory of κ-presentable objects, by Proposition 1.1. Once again, Proposition 1.2 tells us that the category $\mathsf{K}_2 = \mathsf{F} \times \mathsf{N}$ is κ-accessible, and describes its full subcategory of κ-presentable objects.

The functor $\Phi_1 : \mathsf{K}_1 \to \mathsf{L}$ assigns to an epimorphism of R-modules $A \to B$ the pair of R-modules $(A, B) \in \mathsf{L}$. The functor $\Phi_2 : \mathsf{K}_2 \to \mathsf{L}$ assigns to a pair of objects (G, M), where $G \in \mathsf{F}$ and $M \in \mathsf{N}$, the pair of R-modules $(G, M) \in \mathsf{Mod}\text{–}R \times \mathsf{Mod}\text{–}R = \mathsf{L}$. In other words, Φ_2 is the Cartesian product of two functors $\mathsf{F} \to \mathsf{Mod}\text{–}R$ and $\mathsf{N} \to \mathsf{Mod}\text{–}R$. The functor $\mathsf{F} \to \mathsf{Mod}\text{–}R$ is the identity inclusion $\mathsf{F} = \varinjlim_{(\kappa)} \mathsf{T} \to \mathsf{Mod}\text{–}R$. The functor $\mathsf{N} \to \mathsf{Mod}\text{–}R$ assigns the R-module M to an element $M \subset N$ of the Hill lattice of submodules in N.

The pseudopullback category C of the pair of functors Φ_1 and Φ_2 is the category of R-module epimorphisms $G \to M$, where $G \in \mathsf{F}$ and $M \subset N$, $M \in \mathsf{N}$. In particular, $f : F \to N$ is an object of C. Theorem 1.3 tells us that the category C is κ-accessible, and provides a description of its full subcategory of κ-presentable objects. It follows that the object $f \in \mathsf{C}$ is a κ-directed colimit of κ-presentable objects of C, as desired. ■

THEOREM 10.2. *Let R be a right countably coherent ring. Then the category of Gorenstein-flat right R-modules* GF *is $\aleph_1$-accessible. The $\aleph_1$-presentable objects of* GF *are precisely all the Gorenstein-flat right R-modules that are countably presentable in* $\mathsf{Mod}\text{–}R$. *So every Gorenstein-flat right R-module is an $\aleph_1$-directed colimit of countably presentable Gorenstein-flat right R-modules.*

PROOF. The full subcategory $\mathsf{GF} \subset \mathsf{Mod}\text{–}R$ is closed under directed colimits in $\mathsf{Mod}\text{–}R$ by Proposition 8.1 (a). In view of Proposition 1.1, it remains to show that any Gorenstein-flat right R-module is an $\aleph_1$-directed colimit of countably presentable Gorenstein-flat right R-modules.

We use the description of Gorenstein-flat modules provided by [40, Theorem 4.11 (4)], which states that the Gorenstein-flat modules are precisely all the kernels of surjective morphisms from flat modules onto *projectively coresolved Gorenstein-flat modules*. Furthermore, by [40, Theorem 4.9], every projectively coresolved Gorenstein-flat right R-module is filtered by countably presentable projectively coresolved Gorenstein-flat R-modules.

We apply Lemma 10.1 for $\kappa = \aleph_1$ and $\lambda = \aleph_0$. Let T be the set of all countably presentable flat right R-modules, and let S be the set of all countably presentable projectively coresolved Gorenstein-flat right R-modules. Then $\varinjlim_{(\aleph_1)} \mathsf{T}$ is the class of all flat right R-modules by [35, Proposition 10.2], while $\mathsf{Fil}(\mathsf{S})$ is the class of all projectively coresolved Gorenstein-flat right R-modules.

So the lemma is applicable, and it tells us that every surjective morphism from a flat right R-module onto a projectively coresolved Gorenstein-flat right R-module is an $\aleph_1$-directed colimit of surjective morphisms from countably presentable flat R-modules onto countably presentable projectively coresolved Gorenstein-flat R-modules. It remains to pass to the kernels. ■

Notice that Theorem 10.2 gives a better cardinality estimate for the accessibility rank of the category of Gorenstein-flat R-modules than Proposition 3.1 combined with Proposition 8.1 (b).

11. Modules of finite injective dimension

In this section we discuss the accessibility rank of the category of modules of injective dimension $\le m$, where m is a fixed integer. In this context, our category-theoretic approach based on Theorems 1.3 and 1.4 via Proposition 4.1 gives the same result as the deconstructibility-based approach of Proposition 3.6.

Theorem 11.1. *Let λ be a regular cardinal and R be a right $<\lambda$-Noetherian ring. Let ν be a cardinal such that $\rho = |R| + \aleph_0 \le \nu$ and $\nu^{<\lambda} = \nu$. Let $m \ge 0$ be an integer. Then the category of all exact sequences of injective right R-modules $J^0 \to J^1 \to \dots \to J^m \to 0$ is ν^+-accessible. The ν^+-presentable objects of this category are precisely all the exact sequences $T^0 \to T^1 \to \dots \to T^m \to 0$ with injective right R-modules T^n, $0 \le n \le m$, of cardinality at most ν. Consequently, every finite injective coresolution $J^0 \to J^1 \to \dots \to J^m \to 0$ in $\mathsf{Mod}{-}R$ is a ν^+-directed colimit of finite injective coresolutions $T^0 \to T^1 \to \dots \to T^m \to 0$ of the same length m with injective R-modules T^n, $0 \le n \le m$, of cardinality at most ν.*

Proof. Recall first of all that the full subcategory of injective R-modules is closed under λ-directed colimits in $\mathsf{Mod}{-}R$ by Corollary 3.7. The same corollary states that the category of injective right R-modules is ν^+-accessible, and describes its full subcategory of ν^+-presentable objects as consisting precisely of all the injective R-modules of cardinality at most ν.

Now we apply Proposition 4.1 for $\kappa = \nu^+$ (notice that $\lambda \le \nu < \kappa$). Take T_n to be the class of all injective right R-modules of cardinality at most ν for all $0 \le n \le m$, T_{-1} to be the class of all right R-modules of cardinality at most ν, and $\mathsf{T}_n = \{0\}$ for all $n \ge m+1$ and $n \le -2$. Then $\varinjlim_{(\kappa)} \mathsf{T}_n$ is the class of all injective right R-modules for all $0 \le n \le m$, $\varinjlim_{(\kappa)} \mathsf{T}_{-1} = \mathsf{Mod}{-}R$, and $\varinjlim_{(\kappa)} \mathsf{T}_n = 0$ for all $n \ge m+1$ and $n \le -2$. So the category C from Proposition 4.1 is equivalent to the category of finite injective coresolutions we are interested in. ■

Corollary 11.2. *Let λ be a regular cardinal and R be a right $<\lambda$-Noetherian ring. Let ν be a cardinal such that $\rho = |R| + \aleph_0 \le \nu$ and $\nu^{<\lambda} = \nu$. Let $m \ge 0$ be an integer. Denote by $\mathsf{I}_m \subset \mathsf{Mod}\text{–}R$ the full subcategory of all right R-modules of injective dimension at most m. Then the category I_m is ν^+-accessible. The ν^+-presentable objects of I_m are precisely all the right R-modules of cardinality $\le \nu$ and of injective dimension $\le m$. So every right R-module of injective dimension m is a ν^+-directed colimit of right R-modules of cardinality at most ν and of injective dimension at most m.*

Proof. A right R-module K has injective dimension at most m if and only if $\operatorname{Ext}_R^{m+1}(R/I, K) = 0$ for all right ideals $I \subset R$. Over a right $<\lambda$-Noetherian ring R, the functor $\operatorname{Ext}_R^*(M, -)$ preserves λ-directed colimits for every $<\lambda$-generated right R-module M. Consequently, the full subcategory I_m is closed under λ-directed colimits in $\mathsf{Mod}\text{–}R$. In view of Proposition 1.1, it remains to check the last assertion of the corollary, which follows immediately from Theorem 11.1. ∎

Remark 11.3. The full subcategory $\mathsf{I}_m \subset \mathsf{Mod}\text{–}R$ can also be described as $\mathsf{I}_m = \mathsf{S}^{\perp_1}$, where S denotes the set of all ($<\lambda$-generated) mth syzygy modules of $<\lambda$-generated (or just cyclic) right R-modules. This allows one to obtain the assertion of Corollary 11.2 directly as a particular case of Proposition 3.6.

12. Injectively resolved modules

By an *injective resolution* we mean an exact sequence $\cdots \to J_2 \to J_1 \to J_0$, where J_n are injective modules for all $n \ge 0$. An R-module M is said to be *injectively resolved* if there exists an exact sequence of R-modules $\cdots \to J_2 \to J_1 \to J_0 \to M \to 0$ with injective R-modules J_n.

In this section, as in the previous one, our category-theoretic approach based on Theorems 1.3 and 1.4 via Proposition 4.1 produces results which can also be obtained with the deconstructibility-based approach using a suitable version of Proposition 3.6.

Theorem 12.1. *Let λ be a regular cardinal and R be a right $<\lambda$-Noetherian ring. Let ν be a cardinal such that $\rho = |R| + \aleph_0 \le \nu$ and $\nu^{<\lambda} = \nu$. Then the category of injective resolutions $\cdots \to J_2 \to J_1 \to J_0$ in $\mathsf{Mod}\text{–}R$ is ν^+-accessible. The ν^+-presentable objects of this category are precisely all the injective resolutions $\cdots \to T_2 \to T_1 \to T_0$ with injective right R-modules T_n, $n \ge 0$, of cardinality at most ν. Consequently, every injective resolution $\cdots \to J_2 \to J_1 \to J_0$ in $\mathsf{Mod}\text{–}R$ is a ν^+-directed colimit of injective resolutions $\cdots \to T_2 \to T_1 \to T_0$ with injective R-modules T_n, $n \ge 0$, of cardinality at most ν.*

Proof. Similarly to the proof of Theorem 11.1, the argument in based on Corollary 3.7 and Proposition 4.1 for $\kappa = \nu^+$. Take T_n to be the class of all injective right R-modules of cardinality at most ν for all $n \le -1$, T_0 to be the class of all right R-modules of cardinality at most ν, and $\mathsf{T}_n = \{0\}$ for all $n \ge 1$. Then $\varinjlim_{(\kappa)} \mathsf{T}_n$ is the class of all injective right R-modules for all $n \le -1$, $\varinjlim_{(\kappa)} \mathsf{T}_0 = \mathsf{Mod}\text{–}R$, and $\varinjlim_{(\kappa)} \mathsf{T}_n = 0$ for all $n \ge 1$. So the category C from Proposition 4.1 is equivalent to the category of injective resolutions we are interested in. ■

Corollary 12.2. *Let λ be a regular cardinal and R be a right $<\lambda$-Noetherian ring. Let ν be a cardinal such that $\rho = |R| + \aleph_0 \le \nu$ and $\nu^{<\lambda} = \nu$. Then any injectively resolved right R-module is a ν^+-directed colimit of injectively resolved right R-modules of cardinality at most ν.*

Proof. Let us emphasize that there is *no* claim about the class of injectively resolved modules being closed under μ-directed colimits for any cardinal μ in this corollary. The assertion of the corollary follows immediately from Theorem 12.1. ■

Corollary 12.3. *Let λ be a regular cardinal and R be a right $<\lambda$-Noetherian ring. Let ν be a cardinal such that $\rho = |R| + \aleph_0 \le \nu$ and $\nu^{<\lambda} = \nu$. Let M be an injectively resolved right R-module of cardinality $\le \nu$. Then M is a direct summand of an R-module N admitting an injective resolution $\cdots \to T_2 \to T_1 \to T_0$ with injective R-modules T_n, $n \ge 0$, of cardinality at most ν.*

Proof. This is a corollary of Theorem 12.1 provable similarly to Corollary 4.4. It is clear from the theorem that every injectively resolved right R-module M is the colimit of a ν^+-directed diagram of right R-modules $(N_\xi)_{\xi\in\Xi}$ such that N_ξ admits an injective resolution $\cdots \to T_{2,\xi} \to T_{1,\xi} \to T_{0,\xi} \to N_\xi \to 0$ with injective R-modules $T_{n,\xi}$, $n \ge 0$, of cardinality at most ν for every $\xi \in \Xi$. Now, if M has cardinality at most ν, then M is ν^+-presentable, and it follows that there exists $\xi \in \Xi$ such that M is a direct summand of N_ξ. ■

Remark 12.4. The full subcategory of acyclic complexes of injective modules in $\mathsf{Com}(\mathsf{Mod}\text{–}R)$ can also be described as the right Ext^1-orthogonal full subcategory $\mathsf{S}^{\perp_1} \subset \mathsf{Com}(\mathsf{Mod}\text{–}R)$, where $\mathsf{S} \subset \mathsf{Com}(\mathsf{Mod}\text{–}R)$ denotes the set of all contractible two-term complexes of cyclic R-modules $\cdots \to 0 \to R/I \xrightarrow{=} R/I \to 0 \to \cdots$ (placed in various cohomological degrees) and one-term complexes of free R-modules with one generator $\cdots \to 0 \to R \to 0 \to \cdots$ (placed in various cohomological degrees). Here, I ranges over all the right ideals of R, and the construction of the right Ext^1-orthogonal class of objects $\mathsf{S}^{\perp_1}$ is applied in the abelian category of complexes $\mathsf{Com}(\mathsf{Mod}\text{–}R)$. This assertion is a special case of [14, Proposition 4.6]. Therefore, Theorem 12.1 can also be obtained as a particular case of a suitable version of Proposition 3.6 for the category of complexes $\mathsf{Com}(\mathsf{Mod}\text{–}R)$.

13. Totally acyclic complexes of injectives over a left Noetherian ring

An acyclic complex of injective left R-modules $I^\bullet$ is said to be *totally acyclic* [23, Section 5.2] if the complex of abelian groups $\mathrm{Hom}_R(J, I^\bullet)$ is acyclic for every injective left R-module J. A left R-module M is said to be *Gorenstein-injective* if there exists a totally acyclic complex of injective left R-modules $I^\bullet$ such that $M \simeq Z^0(I^\bullet)$ is its module of cocycles. The definitions for right modules are similar.

In the following four sections, Sections 13–16, we study accessibility properties of the categories of totally acyclic complexes of injective modules and of Gorenstein-injective modules. In the present section, the assumptions are more restrictive than in Section 16, but we obtain a better cardinality estimate.

We start with a lemma and a corollary suggested to the author by Jan Šťovíček.

Lemma 13.1. *For any left Noetherian ring R, there exists a functor assigning to every Gorenstein-injective left R-module M one of its totally acyclic two-sided injective resolutions, i.e., a totally acyclic complex of injective left R-modules $I^\bullet$ together with an isomorphism $M \simeq Z^0(I^\bullet)$.*

Proof. First of all we observe that, for any Gorenstein-injective R-module M and an arbitrary injective R-module coresolution $0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$ of M, the complex $0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$ remains exact after applying the functor $\mathrm{Hom}_R(J, -)$ for any injective R-module J. Furthermore, it is well known that there exists a functor $M \mapsto (M \hookrightarrow I(M))$ assigning to an R-module M its embedding into an injective R-module $I(M)$. One can use a functorial version of the small object argument as in [10, Theorem 2], or simply let $I(M)$ be the direct product of copies of the R-module $\mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ indexed over all the abelian group maps $M \to \mathbb{Q}/\mathbb{Z}$. These observations allow us to construct the positive cohomological degree part of the desired functorial two-sided resolution of M. This part of the argument does not use the Noetherianity assumption on R.

The construction of the negative cohomological degree part of the functorial two-sided resolution is based on the results of [36, Lemma 2.1 and Example 6.4]. By [30, Theorem 2.5], every injective left R-module is a direct sum of indecomposable injective R-modules. It is easy to see that there is only a set of isomorphism classes of indecomposable injectives. Therefore, there exists an injective left R-module J such that all the injective left R-modules are direct summands of direct sums of copies of J. Starting with a Gorenstein-injective left R-module M, we denote by $I^{-1}(M)$

the direct sum of copies of J indexed over all the R-module maps $J \to M$, that is, $I^{-1}(M) = J^{(\mathrm{Hom}_R(J,M))}$. Clearly, the natural R-module map $I^{-1}(M) \to M$ is surjective (since M is Gorenstein-injective). By [36, Lemma 2.1 (c)] (for $\mathsf{A} = R\text{–}\mathsf{Mod}$ and $T = J$), the kernel of this map is a Gorenstein-injective left R-module again, and so the inductive process can be continued indefinitely, producing the desired negative part of a totally acyclic two-sided injective resolution in a functorial way. ■

Corollary 13.2. *For any left Noetherian ring R, the class of all Gorenstein-injective left R-modules is closed under ρ^+-directed colimits in $R\text{–}\mathsf{Mod}$, where $\rho = |R| + \aleph_0$. Furthermore, the class of all totally acyclic complexes of injective left R-modules is closed under ρ^+-directed colimits in $\mathsf{Com}(R\text{–}\mathsf{Mod})$ as well.*

Proof. Let $(M_\xi)_{\xi\in\Xi}$ be a diagram of Gorenstein-injective left R-modules, indexed by some poset Ξ. By Lemma 13.1, one can construct a Ξ-indexed diagram of totally acyclic two-sided injective resolutions $(I_\xi^\bullet)_{\xi\in\Xi}$ for the given diagram of R-modules $(M_\xi)_{\xi\in\Xi}$. So the first assertion of the corollary follows from the second one.

In order to prove the second assertion, notice first of all that the class of injective left R-modules is closed under directed colimits in $R\text{–}\mathsf{Mod}$ (e.g., by the left-right opposite version of Corollary 3.7 for $\lambda = \aleph_0$ and $\nu = \rho$). Acyclicity of complexes of modules is obviously preserved by directed colimits.

To show that the class of totally acyclic complexes of injective left R-modules is closed under ρ^+-directed colimits, notice that in the definition of a totally acyclic complex of injectives it suffices to check the preservation of acyclicity by the functors $\mathrm{Hom}_R(J,-)$, where J ranges over *indecomposable* injective left R-modules. Then it remains to observe that each indecomposable injective left R-module has cardinality at most ρ (again by Corollary 3.7 for the same λ and ν). ■

We refer to the recent preprint [22] for a general discussion of directed colimit-closedness properties of Gorenstein-injective modules. For a version of Corollary 13.2 with more restrictive assumptions and a stronger conclusion, see Corollary 15.2 below.

Theorem 13.3. *Let R be a left Noetherian ring; put $\rho = |R| + \aleph_0$. Let ν be an infinite cardinal such that $\nu^\rho = \nu$. Then the category of totally acyclic complexes of injective left R-modules is ν^+-accessible. The ν^+-presentable objects of this category are precisely all the totally acyclic complexes of injective left R-modules of cardinality at most ν. Consequently, every totally acyclic complex of injective left R-modules is a ν^+-directed colimit of totally acyclic complexes of injective left R-modules of cardinality at most ν.*

Proof. This is an application of Theorem 1.3 together with Proposition 4.1. For a comparable or somewhat similar argument, see the proof of Theorem 15.1 below.

Let K_1 denote the category of acyclic complexes of injective left R-modules. By the left-right opposite versions of Corollary 3.7 and Proposition 4.1 for $\lambda = \aleph_0$, $\kappa = \nu^+$, and T_n being the set of all injective R-modules of cardinality at most ν, for every $n \in \mathbb{Z}$, we know that K_1 is a ν^+-accessible category, and have a description of its ν^+-accessible objects (cf. Theorem 12.1).

Furthermore, let K_2 denote the Cartesian product of copies of the category of acyclic complexes of abelian groups, taken over a set of representatives of isomorphism classes of indecomposable injective left R-modules J. Notice that every indecomposable injective left R-module is an injective envelope of a cyclic R-module R/I, where I is a left ideal in R, and there are at most ρ of these. Clearly, $\rho < \nu < \nu^+$. Proposition 1.8 for $\kappa = \nu^+$ and the ring of integers $\mathbb{Z}$ in the role of R, together with Proposition 1.2, tells us that K_2 is a ν^+-accessible category and provides a description of its full subcategory of ν^+-presentable objects.

Finally, let L be the Cartesian product of copies of the abelian category of arbitrary complexes of abelian groups, taken over the same set of indecomposable injectives J. Lemma 1.5 (a), (c) together with Proposition 1.2 tells us that L is a locally ν^+-presentable category and describes its full subcategory of ν^+-presentable objects.

Let $\Phi_1\colon \mathsf{K}_1 \to \mathsf{L}$ be the functor taking any acyclic complex of injective left R-modules $I^\bullet$ to the collection of complexes of abelian groups $\operatorname{Hom}_R(J, I^\bullet)$, indexed over the indecomposable injective left R-modules J. Let $\Phi_2\colon \mathsf{K}_2 \to \mathsf{L}$ be the Cartesian product of the identity inclusions of the category of acyclic complexes of abelian groups into the category of all complexes of abelian groups. Then the pseudopullback C of the two functors Φ_1 and Φ_2 is the desired category of totally acyclic complexes of injective left R-modules.

Recall that any indecomposable injective left R-module has cardinality at most ρ (as explained in the proof of Corollary 13.2). Therefore, the functor Φ_1 takes ν^+-presentable objects to ν^+-presentable objects (since $\nu^\rho = \nu$) and preserves ρ^+-directed colimits. Put $\lambda = \rho^+$ and $\kappa = \nu^+$; then $\lambda < \kappa$.

All the other assumptions of Theorem 1.3 are clearly satisfied. Theorem 1.3 tells us that C is a ν^+-accessible category, and provides the desired description of its full subcategory of ν^+-presentable objects. ∎

Corollary 13.4. *Let R be a left Noetherian ring; put $\rho = |R| + \aleph_0$. Let ν be an infinite cardinal such that $\nu^\rho = \nu$. Then the category of Gorenstein-injective left R-modules GI is ν^+-accessible. The ν^+-presentable objects of GI are precisely all the Gorenstein-injective left R-modules of cardinality at most ν. So every Gorenstein-injective left R-module is a ν^+-directed colimit of Gorenstein-injective left R-modules of cardinality at most ν.*

Proof. The full subcategory $\mathsf{GI} \subset R\text{–}\mathsf{Mod}$ is closed under ρ^+-directed colimits in $R\text{–}\mathsf{Mod}$ by Corollary 13.2. Since any R-module of cardinality at most ν is ν^+-presentable in $R\text{–}\mathsf{Mod}$, it follows that any Gorenstein-injective left R-module of cardinality at most ν is ν^+-presentable in the category of Gorenstein-injective left R-modules. In view of Proposition 1.1, it remains to check the last assertion of the corollary, which follows immediately from Theorem 13.3. ∎

For a version of Corollary 13.4 with more restrictive assumptions and a better cardinality estimate, see Corollary 15.4 below.

Corollary 13.5. *Let R be a left Noetherian ring; put $\rho = |R| + \aleph_0$. Let ν be an infinite cardinal such that $\nu^\rho = \nu$. Let M be a Gorenstein-injective left R-module of cardinality at most ν. Then M is a direct summand of an R-module N admitting a two-sided totally acyclic injective resolution $T^\bullet$ with injective left R-modules T^n, $n \in \mathbb{Z}$, of cardinality at most ν.*

Proof. This is a corollary of Theorem 13.3. The argument is similar to the proofs of Corollaries 8.5 and 12.3. ∎

14. Dualizing complexes and totally acyclic complexes of injectives

The present section prepares ground for the next one. As in Section 7 we consider two associative rings R and S, and assume that the ring S is right coherent. We will say that a bounded complex of left R-modules $K^\bullet$ has *finite injective dimension* if it is isomorphic, as an object of the derived category $\mathsf{D}^{\mathsf{b}}(R\text{–}\mathsf{Mod})$, to a bounded complex of injective R-modules.

By a *strong right dualizing complex of R-S-bimodules $D^\bullet$* we mean a bounded complex of R-S-bimodules satisfying conditions (ii), (iii) from Section 7 together with the following stronger version of condition (i):

(i′) the terms of the complex $D^\bullet$ are fp-injective as right S-modules, and the whole complex $D^\bullet$ has finite injective dimension as a complex of left R-modules.

For a left Noetherian ring R, all fp-injective left R-modules are injective, and there is no difference between right dualizing complexes and strong right dualizing complexes.

The following proposition is the main result of this section. It is dual-analogous to Proposition 7.1. It is also another version of [25, Lemma 1.7].

Proposition 14.1. *Let R be a ring, S be a right coherent ring, and $D^\bullet$ be a strong right dualizing complex of R-S-bimodules. Then an acyclic complex of injective left R-modules $I^\bullet$ is totally acyclic if and only if the complex of left S-modules $\mathrm{Hom}_R(D^\bullet, I^\bullet)$ is acyclic.*

The proof of Proposition 14.1 is based on a sequence of lemmas.

LEMMA 14.2. *Let $L^\bullet$ be a bounded acyclic complex of left R-modules and $I^\bullet$ be a complex of injective left R-modules. Then the complex of abelian groups* $\mathrm{Hom}_R(L^\bullet, I^\bullet)$ *is acyclic.*

PROOF. This is dual-analogous to Lemma 7.2. The point is that the complex of abelian groups $\mathrm{Hom}_R(L^\bullet, I^n)$ is acyclic for every $n \in \mathbb{Z}$. ■

COROLLARY 14.3. *Let $I^\bullet$ be a totally acyclic complex of injective left R-modules and $K^\bullet$ be a bounded complex of left R-modules. Assume that the complex $K^\bullet$ has finite injective dimension. Then the complex of abelian groups* $\mathrm{Hom}_R(K^\bullet, I^\bullet)$ *is acyclic.*

PROOF. Follows from Lemma 14.2. ■

A left R-module C is said to be (*Enochs*) *cotorsion* [11, Section 2] if $\mathrm{Ext}^1_R(F, C) = 0$ for all flat left R-modules F. One can easily see that $\mathrm{Ext}^n_R(F, C) = 0$ for all flat R-modules F, all cotorsion R-modules C, and all $n \geq 1$.

LEMMA 14.4. *Let R and S be two rings, E be an R-S-bimodule, and I be an injective left R-module. Then the left S-module* $\mathrm{Hom}_R(E, I)$ *is cotorsion.*

PROOF. One can immediately see that the functor $F \mapsto \mathrm{Hom}_S(F, \mathrm{Hom}_R(E, I)) \simeq \mathrm{Hom}_R(E \otimes_S F, I)$ takes short exact sequences of flat left S-modules F to short exact sequences of abelian groups. ■

The following lemma is a (much more nontrivial) dual-analogous version of Lemma 7.5.

LEMMA 14.5. *Let S be a ring, $C^\bullet$ be an acyclic complex of cotorsion left S-modules, and $G^\bullet$ be a bounded complex of flat left S-modules. Then the complex of abelian groups* $\mathrm{Hom}_R(G^\bullet, C^\bullet)$ *is acyclic.*

PROOF. The *cotorsion periodicity theorem* [4, Theorem 5.1 (2)] (see also [37, Corollary 8.2] or [33, Theorem 5.3]) claims that, in any acyclic complex of cotorsion modules, the modules of cocycles are also cotorsion. Now it is clear that, for every $n \in \mathbb{Z}$, the complex $\mathrm{Hom}_R(G^n, C^\bullet)$ is acyclic. ■

PROOF OF PROPOSITION 14.1. The "only if" implication is provided by Corollary 14.3. To prove the "if", let J be an injective R-module. By Lemma 7.7, the natural morphism of bounded complexes of left R-modules $D^\bullet \otimes_S \mathrm{Hom}_R(D^\bullet, J) \to J$ is a quasi-isomorphism. In view of Lemma 14.2, in order to show that the complex $\operatorname{Hom}_R(J, I^\bullet)$ is acyclic, it suffices to check that so is the complex

$$\operatorname{Hom}_R(D^\bullet \otimes_S \operatorname{Hom}_R(D^\bullet, J),\ I^\bullet) \simeq \operatorname{Hom}_S(\operatorname{Hom}_R(D^\bullet, J), \operatorname{Hom}_R(D^\bullet, I^\bullet)).$$

Now, by assumption, the complex of left S-modules $\operatorname{Hom}_R(D^\bullet, I^\bullet)$ is acyclic. By Lemma 14.4, $\operatorname{Hom}_R(D^\bullet, I^\bullet)$ is a complex of cotorsion left S-modules. By Lemma 7.6, $\operatorname{Hom}_R(D^\bullet, J)$ is a bounded complex of flat left S-modules. It remains to refer to Lemma 14.5. ■

Remark 14.6. Once again, the theory simplifies for rings that are Gorenstein in a suitable sense. Similarly to Remark 7.8, one can see that, over any ring R, for any acyclic complex of injective left R-modules $I^\bullet$ and any left R-module G of finite flat dimension, the complex of abelian groups $\operatorname{Hom}_R(G, I^\bullet)$ is acyclic by Lemmas 14.2 and 14.5. Now, if every injective left R-module has finite flat dimension, then it follows that every acyclic complex of injective left R-modules is totally acyclic, and every injectively resolved left R-module is Gorenstein-injective. In this case, the results of Proposition 4.1, Theorem 12.1, and Corollaries 12.2 and 12.3 become directly applicable (essentially) as properties of totally acyclic complexes of injective modules and Gorenstein-injective modules.

15. Totally acyclic complexes of injectives as directed colimits

In the present section, the assumptions are somewhat restrictive, as they include existence of a (strong) right dualizing complex; but the cardinality estimates are even better than in Section 13.

In addition to conditions (i′) and (ii), (iii) in the definition of a strong right dualizing complex in Sections 7 and 14, we will need to assume another condition (which should be compared to condition (iv) from Section 8). We will say that a strong right dualizing complex of R-S-bimodules $D^\bullet$ has *left type* $<\lambda$ if

(iv′) $D^\bullet$ is isomorphic, as an object of the bounded derived category of left R-modules $\mathsf{D}^{\mathsf{b}}(R\text{–}\mathsf{Mod})$, to a bounded complex of $<\lambda$-generated left R-modules $\widetilde{D}^\bullet$.

Condition (iv′) will be used for left $<\lambda$-Noetherian rings R, so the classes of $<\lambda$-generated and λ-presentable left R-modules coincide. In particular, in the case of the cardinal $\lambda = \aleph_0$, we will speak about dualizing complexes of *left finite type*.

Theorem 15.1. *Let λ be a regular cardinal and R be a left $<\lambda$-Noetherian ring. Let ν be a cardinal such that $\rho = |R| + \aleph_0 \leq \nu$ and $\nu^{<\lambda} = \nu$. Let S be a right coherent ring and $D^\bullet$ be a strong right dualizing complex of R-S-bimodules. Assume that*

$D^\bullet$ is a strong right dualizing complex of left type $<\lambda$. Then the category of totally acyclic complexes of injective left R-modules is ν^+-accessible. The ν^+-presentable objects of this category are precisely all the totally acyclic complexes of injective left R-modules of cardinality at most ν. Consequently, every totally acyclic complex of injective left R-modules is a ν^+-directed colimit of totally acyclic complexes of injective left R-modules of cardinality at most ν.

Proof. This is an application of Theorem 1.3 together with Propositions 4.1 and 14.1. The argument bears some similarity to the proof of Theorem 13.3. First of all, let us show that the full subcategory of totally acyclic complexes of injective left R-modules is closed under λ-directed colimits in the ambient abelian category $\mathsf{Com}(R\text{–}\mathsf{Mod})$.

By Corollary 3.7, the full subcategory of injective left R-modules is closed under λ-directed colimits in $R\text{–}\mathsf{Mod}$. It is also clear that acyclicity of complexes is preserved by all directed colimits. Now let $I^\bullet$ be an acyclic complex of injective left R-modules, and let $\widetilde{D}^\bullet$ be a bounded complex of $<\lambda$-generated left R-modules provided by the definition of a strong right dualizing complex of left type $<\lambda$ (item (iv$'$)) above. It is clear from Lemma 14.2 that the complex of left S-modules $\mathrm{Hom}_R(D^\bullet, I^\bullet)$ is acyclic if and only if the complex of abelian groups $\mathrm{Hom}_R(\widetilde{D}^\bullet, I^\bullet)$ is acyclic. By Proposition 14.1, we can conclude that the acyclic complex of injective R-modules $I^\bullet$ is totally acyclic if and only if the complex of abelian groups $\mathrm{Hom}_R(\widetilde{D}^\bullet, I^\bullet)$ is acyclic. The latter condition is obviously preserved by λ-directed colimits of the complexes $I^\bullet$.

Now denote by K_1 the category of acyclic complexes of injective left R-modules. Applying the left-right opposite version of Proposition 4.1 for $\kappa = \nu^+$ and T_n being the set of all injective R-modules of cardinality at most ν, for every $n \in \mathbb{Z}$, and using Corollary 3.7 again, we see that K_1 is a ν^+-accessible category, and obtain a description of its ν^+-accessible objects (cf. Theorem 12.1). Notice that $\lambda \le \nu$, so the ring R is left $<\nu^+$-coherent.

Furthermore, let K_2 denote the category of acyclic complexes of abelian groups. Proposition 1.8 for $\kappa = \nu^+$ and the ring of integers $\mathbb{Z}$ in the role of R states that K_2 is a ν^+-accessible category and provides a description of its full subcategory of ν^+-presentable objects. Finally, let L be the abelian category of arbitrary complexes of abelian groups. Lemma 1.5 (a), (c) states that L is a locally ν^+-presentable category and describes its full subcategory of ν^+-presentable objects as the category of complexes of abelian groups of cardinality at most ν.

Let the functor $\Phi_1\colon \mathsf{K}_1 \to \mathsf{L}$ take any acyclic complex of injective left R-modules $I^\bullet$ to the complex of abelian groups $\mathrm{Hom}_R(\widetilde{D}^\bullet, I^\bullet)$. Let $\Phi_2\colon \mathsf{K}_2 \to \mathsf{L}$ be the identity inclusion of the category of acyclic complexes of abelian groups into the category of all complexes of abelian groups. Then the pseudopullback C of the two functors

Φ_1 and Φ_2 is the category of all acyclic complexes of injective left R-modules $I^\bullet$ for which the complex of abelian groups $\mathrm{Hom}_R(\widetilde{D}^\bullet, I^\bullet)$ is acyclic. As we have seen in the second paragraph of this proof, the category C is the desired category of totally acyclic complexes of injective left R-modules.

Clearly, the functor Φ_1 takes ν^+-presentable objects to ν^+-presentable objects (since $\nu^{<\lambda} = \nu$), and preserves λ-directed colimits. All the other assumptions of Theorem 1.3 (for $\kappa = \nu^+$) are also satisfied. Theorem 1.3 tells us that C is a ν^+-accessible category, and provides the desired description of its full subcategory of ν^+-presentable objects. ∎

Corollary 15.2. *Let R be a left Noetherian ring, S be a right coherent ring, and $D^\bullet$ be a right dualizing complex of R-S-bimodules. Assume that $D^\bullet$ is a (strong) right dualizing complex of left finite type. Then the class of all Gorenstein-injective left R-modules is closed under directed colimits in R–Mod. Furthermore, the class of all totally acyclic complexes of injective left R-modules is closed under directed colimits in $\mathsf{Com}(R$–$\mathsf{Mod})$ as well.*

Proof. In view of the first paragraph of the proof of Corollary 13.2, it suffices to prove the second assertion, which is a particular case of the first two paragraphs of the proof of Theorem 15.1 (for $\lambda = \aleph_0$). ∎

In the case of a commutative Noetherian ring R with a dualizing complex, the first assertion of Corollary 15.2 can be obtained by combining [22, Theorem 2] with [19, Lemma 2.5 (b)].

Corollary 15.3. *Let λ be a regular cardinal and R be a left $<\lambda$-Noetherian ring. Let ν be a cardinal such that $\rho = |R| + \aleph_0 \le \nu$ and $\nu^{<\lambda} = \nu$. Let S be a right coherent ring and $D^\bullet$ be a strong right dualizing complex of R-S-bimodules. Assume that $D^\bullet$ is a strong right dualizing complex of left type $<\lambda$. Then every Gorenstein-injective left R-module is a ν^+-directed colimit of Gorenstein-injective left R-modules of cardinality at most ν.*

Proof. Follows immediately from the last assertion of Theorem 15.1. ∎

Corollary 15.4. *Let R be a left Noetherian ring; put $\rho = |R| + \aleph_0$. Let S be a right coherent ring and $D^\bullet$ be a right dualizing complex of R-S-bimodules. Assume that $D^\bullet$ is a (strong) right dualizing complex of left finite type. Then the category of Gorenstein-injective left R-modules GI is ρ^+-accessible. The ρ^+-presentable objects of GI are precisely all the Gorenstein-injective left R-modules of cardinality at most ρ. So every Gorenstein-injective left R-module is a ρ^+-directed colimit of Gorenstein-injective left R-modules of cardinality at most ρ.*

Proof. The full subcategory $\mathsf{GI} \subset R\text{–}\mathsf{Mod}$ is closed under directed colimits in $R\text{–}\mathsf{Mod}$ by Corollary 15.2. Since any R-module of cardinality at most ρ is ρ^+-presentable in $R\text{–}\mathsf{Mod}$, it follows that any Gorenstein-injective left R-module of cardinality at most ρ is ρ^+-presentable in the category of Gorenstein-injective left R-modules. In view of Proposition 1.1, it remains to check the last assertion of the corollary, which is provided by Corollary 15.3 (for $\lambda = \aleph_0$ and $\nu = \rho$). ■

Corollary 15.5. *Let λ be a regular cardinal and R be a left $<\lambda$-Noetherian ring. Let ν be a cardinal such that $\rho = |R| + \aleph_0 \le \nu$ and $\nu^{<\lambda} = \nu$. Let S be a right coherent ring and $D^\bullet$ be a strong right dualizing complex of R-S-bimodules. Assume that $D^\bullet$ is a strong right dualizing complex of left type $<\lambda$. Let M be a Gorenstein-injective left R-module of cardinality at most ν. Then M is a direct summand of an R-module N admitting a two-sided totally acyclic injective resolution $T^\bullet$ with injective left R-modules T^n, $n \in \mathbb{Z}$, of cardinality at most ν.*

Proof. This is a corollary of Theorem 15.1. The argument is similar to the proofs of Corollaries 8.5 and 12.3. ■

In particular, if the ring R in Corollary 15.4 is (at most) countable, then $\rho^+ = \aleph_1$; so every Gorenstein-injective left R-module is an $\aleph_1$-directed colimit of at most countable Gorenstein-injective left R-modules. Furthermore, every Gorenstein-injective left R-module of cardinality ν is a direct summand of an R-module admitting a two-sided totally acyclic injective resolution by injective left R-modules of cardinality at most ν. This holds for any infinite cardinal ν.

16. Totally acyclic complexes of injectives in full generality

The results of this section, based on [40, Section 5], are an example of the deconstructibility-based approach to accessibility. They are applicable in much greater generality than the results of Sections 13 and 15, but the cardinality estimate is not as good.

Proposition 16.1. *Let R be an associative ring; put $\rho = |R| + \aleph_0$. Let μ be an infinite cardinal such that $\mu^\rho = \mu$, and let ν be an infinite cardinal such that $\nu^\mu = \nu$. Then the category of totally acyclic complexes of injective right R-modules is ν^+-accessible. The ν^+-presentable objects of this category are precisely all the totally acyclic complexes of injective R-modules of cardinality at most ν. So every totally acyclic complex of injective R-modules is a ν^+-directed colimit (in fact, a ν^+-directed union) of totally acyclic complexes of injective R-modules of cardinality at most ν.*

Proof. By [40, Proposition 5.5], there is a set S of μ^+-presentable objects in the abelian category of complexes $\mathsf{Com}(\mathsf{Mod}\text{–}R)$ such that the class of all totally acyclic complexes of injective right R-modules is the right Ext^1-orthogonal class $\mathsf{S}^{\perp_1}$ to S in $\mathsf{Com}(\mathsf{Mod}\text{–}R)$. The assertion of the proposition is provable by applying a suitable version of Proposition 3.6 for the category of complexes $\mathsf{Com}(\mathsf{Mod}\text{–}R)$. ∎

Proposition 16.2. *Let R be an associative ring; put $\rho = |R| + \aleph_0$. Let μ be an infinite cardinal such that $\mu^\rho = \mu$, and let ν be an infinite cardinal such that $\nu^\mu = \nu$. Then the category of Gorenstein-injective right R-modules GI is ν^+-accessible. The ν^+-presentable objects of GI are precisely all the Gorenstein-injective R-modules of cardinality at most ν. So every Gorenstein-injective R-module is a ν^+-directed colimit (in fact, a ν^+-directed union) of Gorenstein-injective R-modules of cardinality at most ν.*

Proof. Once again, by [40, Theorem 5.6], there is a set S of μ^+-presentable R-modules such that $\mathsf{GI} = \mathsf{S}^{\perp_1} \subset \mathsf{Mod}\text{–}R$ is the right Ext^1-orthogonal class to S in $\mathsf{Mod}\text{–}R$. It remains to apply Proposition 3.6. ∎

Corollary 16.3. *Let R be an associative ring; put $\rho = |R| + \aleph_0$. Let μ be an infinite cardinal such that $\mu^\rho = \mu$, and let ν be an infinite cardinal such that $\nu^\mu = \nu$. Let M be a Gorenstein-injective R-module of cardinality at most ν. Then M is a direct summand of an R-module N admitting a two-sided totally acyclic injective resolution $T^\bullet$ with injective R-modules T^n, $n \in \mathbb{Z}$, of cardinality at most ν.*

Proof. This is a corollary of Proposition 16.1. The argument is similar to the proofs of Corollaries 8.5, 13.5, and 15.5. ∎

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