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# Minimal cones and a problem of Euler

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Abstract – The minimal cones $C = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m; |x| \leq |y|\}$ are shown to minimize the weighted perimeter $P_\alpha(E) = \int |x|^\alpha |D\varphi_E|$, $\alpha \in \mathbb{R}$, $E \subset \mathbb{R}^m \times \mathbb{R}^m$, whenever $m + \alpha \geq 4$. This completes recent results of Dierkes and Huisken [Math. Ann. (2023)].

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## 1. Introduction

Here we consider the $n$-dimensional analogue of a problem already investigated by Leonhard Euler [5], namely the variational integral

$$\mathcal{E}_\alpha(M) := \int_M |x|^\alpha \, d\mathcal{H}_{n-1}, \quad \alpha \in \mathbb{R},$$

where $M \subset \mathbb{R}^n$ denotes some smooth hypersurface and $\mathcal{H}_k$ stands for the $k$-dimensional Hausdorff measure. The hypersurface $M$ is called *stationary with respect to* $\mathcal{E}_\alpha$ or simply $\alpha$-*stationary*, if the first variation $\delta\mathcal{E}_\alpha(M, X)$ vanishes, and a stationary surface $M$ is called $\alpha$-*stable* if the second variation $\delta^2\mathcal{E}_\alpha(M, X)$ is nonnegative for suitable variations $X$ of $M$.

Standard computations show that $M$ is $\alpha$-stationary iff the mean curvature $H(x)$ and the unit normal $\nu = \nu(x)$ of $M$ at $x \in M$ respectively satisfy

$$H(x) = \alpha|x|^{-2}\langle x, \nu\rangle, \quad x \neq 0,$$

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while in addition $M$ is also $\alpha$-stable if for all $\xi \in C_c^1(M, \mathbb{R})$ we have

$$\int_M |x|^\alpha \Big\{\frac{2}{\alpha} H^2 + |A|^2\Big\} \xi^2 \, d\mathcal{H}_{n-1} \le \int_M |x|^\alpha \big\{\alpha |x|^{-2} \xi^2 + |\nabla \xi|^2\big\} \, d\mathcal{H}_{n-1}, \tag{1.1}$$

where $|A|$ denotes the length of the second fundamental form $A$ of $M$; cf. [4, Proposition 1.1 and 1.4].

In particular, if $C$ = closure $M$ is a cone in $\mathbb{R}^n$ with the only singularity at zero and if $M := C - \{0\}$ is $\alpha$-stationary, then $C$ is called $\alpha$-stable if (1.1) holds for all $\xi \in C_c^1(M, \mathbb{R})$. Obviously, every area-minimal cone $C$ in $\mathbb{R}^n$ (i.e. $H = 0$ on $C - \{0\}$) such that $0 \in C$ and $C - \{0\}$ is a regular hypersurface is $\alpha$-stationary and we have the following stability result:

Theorem ([4]). *Let $\alpha > 3 - n$ and suppose $C \subset \mathbb{R}^n$ is an $\alpha$-stationary cone with vertex at the origin and such that $(n - 3 + \alpha)^2 \ge 4|x|^2|A|^2 - 4\alpha$. Then $C$ is also $\alpha$-stable.*

(Note that the dimension of the cone $C$ here is $(n - 1)$ rather than $n$ in [4].)

In particular, the cone over the Clifford torus $S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2$ is stable for all $\alpha \ge 1$. (Here $n = 4$ and $|A|^2 = 2|x|^{-2}$.) Furthermore, the 7-dimensional cones over products of spheres $S^1 \times S^5$, $S^2 \times S^4$ and $S^3 \times S^3$ are all $\alpha$-stable, whenever $\alpha \ge \sqrt{48} - 7$; cf. the list in [4, Corollary 2.2].

Moreover, it could be shown in [4, Theorem 3.1] that all cones

$$C_m = \big\{(x_1, \dots, x_{2m}) \in \mathbb{R}^m \times \mathbb{R}^m;\ x_1^2 + \cdots + x_m^2 \le x_{m+1}^2 + \cdots + x_{2m}^2\big\}$$

minimize the integral $\mathcal{E}_\alpha$ in a suitable sense, if $1 \le \alpha \le 2(m - 1)$. So in particular the cone over the Clifford torus $S^1 \times S^1$ minimizes $\mathcal{E}_\alpha$ if $1 \le \alpha \le 2$.

The proof of these results uses a Weierstraß–Schwarz-foliation type of argument, as introduced in the celebrated paper by Bombieri, De Giorgi and Giusti [1]. In fact, under the prescribed conditions on $\alpha$ and $m$ the cones can be embedded in a *field* or *calibration* consisting of $\alpha$-stationary surfaces. On the other hand, it has long been known that a much easier device, known as *subcalibration*, is applicable in the case of classical area-minimal cones and we refer to Lawson [6], Miranda [8], Massari–Miranda [7], Morgan [9], Davini [2] and in particular to De Philippis–Paolini [3] for more pertinent information.

We show here that many of the $\alpha$-stable cones can be subcalibrated with respect to the functional $\mathcal{E}_\alpha$ and are hence also minimizers for $\mathcal{E}_\alpha$ in a very general sense. The simplified proof which will be presented here follows the approach by De Philippis–Paolini [3] and hence we may omit some of the details, referring instead to their paper.

## 2. $\alpha$-subminimal sets and minimal cones

Let $E \subset \mathbb{R}^n$ be a measurable set and $\Omega \subset \mathbb{R}^n$ be an open set. We define the *$\alpha$-perimeter* of $E$ in $\Omega$ as

$$P_\alpha(E,\Omega) := \sup\bigl\{\textstyle\int_E \operatorname{div}\{|x|^\alpha g\}\,dx;\ g \in C_c^1(\Omega,\mathbb{R}^n) \text{ with } |g(x)| \le 1\bigr\}.$$

Note that $P_\alpha(E,\Omega)$ is well defined (and possibly infinite) for all $\alpha \in \mathbb{R}$ and measurable sets $E$. However we have the following proposition:

Proposition 1. *Let $\alpha + n > 1$ and suppose $\partial E$ is of class $C^2$. Then $P_\alpha(E,\Omega) = \int_{\partial E\cap\Omega} |x|^\alpha \, d\mathcal{H}_{n-1} < \infty$, for every bounded, open set $\Omega \subset \mathbb{R}^n$.*

Remark. Clearly, if $0 \notin \partial E$ and $\Omega$ is bounded, then $\int_{\partial E\cap\Omega} |x|^\alpha \, d\mathcal{H}_{n-1}$ is always finite, independent of the value of $\alpha \in \mathbb{R}$.

Proof of Proposition 1. Let $g \in C_c^1(\Omega,\mathbb{R}^n)$, $|g(x)| \le 1$ be arbitrary; then

$$\operatorname{div}(|x|^\alpha g) = \alpha|x|^{\alpha-2}(x\cdot g) + |x|^\alpha \operatorname{div} g$$

which is a function of Lebesgue class $L_1(\Omega)$, if $\alpha + n > 1$. Denoting by $\varphi_E$ the characteristic function of $E$, we obtain for arbitrary $\varepsilon > 0$ and ball $B_\varepsilon = B_\varepsilon(0) \subset \mathbb{R}^n$ with center zero,

$$\begin{aligned}
\int_E \operatorname{div}(|x|^\alpha g)\,dx &= \int_\Omega \varphi_E \operatorname{div}(|x|^\alpha g)\,dx \\
&= \int_{\Omega - B_\varepsilon} \varphi_E \operatorname{div}(|x|^\alpha g)\,dx + \int_{B_\varepsilon} \varphi_E \operatorname{div}(|x|^\alpha g)\,dx \\
&= \int_{\partial E - B_\varepsilon} |x|^\alpha (g\cdot\nu)\,d\mathcal{H}_{n-1} + \int_{\partial B_\varepsilon \cap E} |x|^\alpha (g\cdot\nu_\varepsilon)\,d\mathcal{H}_{n-1} \\
&\quad + \int_{B_\varepsilon} \varphi_E \operatorname{div}(|x|^\alpha g)\,dx,
\end{aligned}$$

where $\nu$ and $\nu_\varepsilon$ stand for the exterior and interior unit normals of $\partial E$ and $\partial B_\varepsilon$ respectively. By assumption, $\alpha + n > 1$, thus the last two integrals tend to zero, as $\varepsilon \searrow 0$, whence

$$P_\alpha(E,\Omega) \le \int_{\partial E} |x|^\alpha \, d\mathcal{H}_{n-1}.$$

On the other hand, since by assumption $\nu$ is of class $C^1$ on the boundary $\partial E$, we may extend $\nu = \nu(x)$ to some function $N \in C^1(\mathbb{R}^n,\mathbb{R}^n)$ such that $|N(x)| \le 1$ for all $x \in \mathbb{R}^n$. Take some function $\eta \in C_c^1(\Omega, \mathbb{R})$ with $|\eta(x)| \le 1$ for all $x \in \Omega$ and put $g := \eta N \in C_c^1(\Omega, \mathbb{R}^n)$; then we have for every $\varepsilon > 0$ the relation

$$\begin{aligned}\int_E \operatorname{div}(|x|^\alpha g)\,dx &= \int_\Omega \varphi_E \operatorname{div}(|x|^\alpha g)\,dx \\ &= \int_{\Omega - B_\varepsilon} \varphi_E \operatorname{div}(|x|^\alpha g)\,dx + \int_{B_\varepsilon} \varphi_E \operatorname{div}(|x|^\alpha g)\,dx \\ &= \int_{\partial E - B_\varepsilon} |x|^\alpha \eta\, d\mathcal{H}_{n-1} + \int_{\partial B_\varepsilon \cap E} |x|^\alpha \eta (N \cdot \nu_\varepsilon)\, d\mathcal{H}_{n-1} \\ &\quad + \int_{B_\varepsilon} \varphi_E \operatorname{div}(|x|^\alpha g)\,dx.\end{aligned}$$

By virtue of $\alpha + n > 1$ and upon letting $\varepsilon \searrow 0$ we find

$$\int_E \operatorname{div}(|x|^\alpha g)\,dx = \int_{\partial E} |x|^\alpha \eta\, d\mathcal{H}_{n-1},$$

whence also

$$P_\alpha(E, \Omega) \ge \int_{\partial E} |x|^\alpha\, d\mathcal{H}_{n-1}.$$

To see the finiteness of both integrals we assume without loss of generality that – locally near zero – $\partial E$ is described by some $C^2$-function $x_n = \psi(x_1, \dots, x_{n-1})$ and that $\alpha < 0$. Then we have

$$\begin{aligned}\int_{\partial E \cap B_R(0)} |x|^\alpha\, d\mathcal{H}_{n-1} &\le \int_{B_R(0)} (x_1^2 + \cdots + x_{n-1}^2)^{\alpha/2} \sqrt{1 + |D\psi|^2}\, dx_1 \dots dx_{n-1} \\ &\le \text{const.} \int_0^R r^{\alpha+n-2}\, dr < \infty, \quad \text{since } \alpha + n > 1. \qquad \blacksquare\end{aligned}$$

Definition 1. A measurable set $E \subset \mathbb{R}^n$ is a *local minimum for* $P_\alpha(\cdot)$ in $\Omega \subset \mathbb{R}^n$, or simply *$\alpha$-minimal in* $\Omega$ if for all open, bounded sets $A \subset \Omega$,

$$P_\alpha(E, A) \le P_\alpha(F, A)$$

for every measurable set $F$, such that the symmetric difference $E \Delta F \subset\subset A$ (i.e. the closure $\overline{E \Delta F} \subset A$).

The set $E$ is called *$\alpha$-subminimal in* $\Omega$ if for all bounded, open $A \subset \Omega$ we have

$$P_\alpha(E, A) \le P_\alpha(F, A)$$

for every measurable set $F \subset E$, such that $E \setminus F \subset\subset A$.

Theorem 1. *The cones* $C_m = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m;\ |x| \le |y|\}$ *are $\alpha$-minimal in* $\mathbb{R}^n$, *$n = 2m$, provided* $m + \alpha \ge 4$.

REMARK. Combining [4, Theorem 3.1] and Theorem 1 we conclude that the cone

$$C_2 = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2;\ |x| \leq |y|\}$$

is $\alpha$-minimal in $\mathbb{R}^4$ for all $\alpha \geq 1$; in other words, the cone over the Clifford torus $S^1 \times S^1$ in $\mathbb{R}^4$ minimizes $\mathcal{E}_\alpha$ for any $\alpha \geq 1$. Conversely, there are no nontrivial stable cones in $\mathbb{R}^4$ with vertex at zero for every $\alpha < 1$, according to [4, Theorem 2.3].

For the proof of the theorem we present three propositions, which follow immediately from [3].

PROPOSITION 2. *Let $\alpha + n > 1$ and suppose $E \subset \mathbb{R}^n$ and its complement $E^c := \Omega \setminus E$ are both $\alpha$-subminimal in $\Omega$; then $E$ is $\alpha$-minimal in $\Omega$.*

PROOF. We first claim that, since $\alpha + n > 1$, it follows that $P_\alpha(E, \Omega) = P_\alpha(E^c, \Omega)$. Indeed, arguing as in Proposition 1 and because of $\operatorname{div}(|x|^\alpha g) \in L_1(\Omega)$ for any $g \in C_c^1(\Omega, \mathbb{R}^n)$, $|g(x)| \leq 1$, $\alpha + n > 1$, we obtain

$$\int_\Omega \operatorname{div}(|x|^\alpha g)\, dx = 0.$$

Therefore,

$$\int_\Omega \varphi_E \operatorname{div}(|x|^\alpha g)\, dx = \int_\Omega (\varphi_E - 1) \operatorname{div}(|x|^\alpha g)\, dx = -\int_\Omega \varphi_{E^c} \operatorname{div}(|x|^\alpha g)\, dx,$$

where $\varphi_{E^c}$ stands for the characteristic function of the complement $E^c = \Omega \setminus E$. Hence we have the equality $P_\alpha(E, \Omega) = P_\alpha(E^c, \Omega)$ and the proof can be completed as in [3, Proposition 1.2]. ■

PROPOSITION 3. *Let $\alpha + n > 1$ and suppose $E_k \subset E$, $k \in \mathbb{N}$, are measurable, $\alpha$-subminimal sets in $\Omega \subset \mathbb{R}^n$ for all $k \in \mathbb{N}$. In addition, assume that $E_k \to E$ in $L_{1,\mathrm{loc}}(\Omega)$ as $k \to \infty$; then also $E$ is $\alpha$-subminimal in $\Omega$.*

PROOF. We only show the lower semicontinuity of the $\alpha$-perimeter, since the rest of the proof follows as in [3, Proposition 1.3] To this end suppose $\varphi_{E_k} \to \varphi_E$ in $L_{1,\mathrm{loc}}(\Omega)$.

For $g \in C_c^1(\Omega, \mathbb{R}^n)$, $|g(x)| \leq 1$ and $\alpha + n > 1$ we first have $\operatorname{div}(|x|^\alpha g) \in L_1(\Omega)$ and

$$\int_\Omega (\varphi_{E_k} - \varphi_E) \operatorname{div}(|x|^\alpha g)\, dx = \int_{E_k - E} \operatorname{div}(|x|^\alpha g)\, dx - \int_{E - E_k} \operatorname{div}(|x|^\alpha g)\, dx \to 0,$$

as $k \to \infty$, whence

$$\begin{aligned} \int_\Omega \varphi_E \operatorname{div}(|x|^\alpha g)\, dx &= \lim_{k\to\infty} \int_\Omega \varphi_{E_k} \operatorname{div}(|x|^\alpha g)\, dx \\ &\leq \liminf_{k\to\infty} P_\alpha(E_k, \Omega), \end{aligned}$$

and the semicontinuity of $P_\alpha$ follows. Now we can argue as in [3, Proposition 1.3] We skip the details. ∎

Definition 2. Let $\Omega \subset \mathbb{R}^n - \{0\}$ be open and $E \subset \Omega$ be measurable with boundary $\partial E$ of class $C^2$. A vector field $\xi \in C^1(\Omega, \mathbb{R}^n)$ is an $\alpha$*-subcalibration* of $E$ (or $\partial E$) in $\Omega$, if we have

(i) for all $x \in \partial E$, $\xi(x) = \nu(x) =$ exterior unit normal of $\partial E$ at $x$;

(ii) $\operatorname{div}(|x|^\alpha \xi(x)) \le 0$ for all $x \in E \cap \Omega$;

(iii) $|\xi(x)| \le 1$ for all $x \in \Omega$.

Proposition 4. *Suppose $E \subset \Omega$ has boundary of class $C^2$ and admits an $\alpha$-subcalibration in $\Omega \subset R^n - \{0\}$. Then $E$ is $\alpha$-subminimal in $\Omega$, i.e. $P_\alpha(E, A) \le P_\alpha(F, A)$ for every bounded, open set $A \subset \Omega$ and all measurable $F \subset E$ with $E \setminus F \subset\subset A$.*

Proof. Analogous to [3, Theorem 1.5] with perimeter replaced by $\alpha$-perimeter. ∎

We now turn to the proof of the main result.

Proof of Theorem 1. The idea is to approximate the cone $C_m$ with a sequence of smooth sets $E_k$ which admit $\alpha$-subcalibrations. By Propositions 4 and 3 it follows that $E$ itself is $\alpha$-subminimal. Again, the same reasoning applies to the complement $C_m^c = \mathbb{R}^n - C_m$ and hence the cone $C_m$ is $\alpha$-minimal by Proposition 2.

To this end, consider the function $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, $(x, y) \mapsto f(x, y) := \frac{|x|^4 - |y|^4}{4}$, $x, y \in \mathbb{R}^m$. Obviously,

$$C_m = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m;\ f(x, y) \le 0\},$$

and for every $k \in \mathbb{N}$ we put

$$E_k := \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m;\ f(x, y) \le -1/k\}$$

and

$$F_k := \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m;\ f(x, y) \le 1/k\}.$$

Clearly, we have for all $k \in \mathbb{N}$,

$$E_k \subset C_m \subset F_k,$$

and $E_k \to C_m$, $F_k \to C_m$, both in $L_{1,\mathrm{loc}}(\mathbb{R}^n)$ as $k \to \infty$, and the complement $F_k^c = \mathbb{R}^n \setminus F_k \subset C_m^c$ and $F_k^c \to \mathbb{R}^n - C_m = C_m^c$ in $L_{1,\mathrm{loc}}(\mathbb{R}^n)$ as $k \to \infty$.

Additionally, all boundaries in $\partial E_k$, $\partial F_k^c$ are smooth hypersurfaces in $\mathbb{R}^n$ and are all $\alpha$-subminimal in $\mathbb{R}^n$. Indeed, we claim that the vector field

$$\xi(x,y) := \pm\frac{Df(x,y)}{|Df(x,y)|}$$

defines an $\alpha$-subcalibration for both $E_k$ and $F_k^c$ in $\mathbb{R}^n - \{0\}$ respectively, i.e. we have (i), (ii) and (iii) of Definition 2 for all $E_k$ and $F_k^c$, $k \in \mathbb{N}$.

While (i) and (iii) are obviously fulfilled in both cases, (ii) needs a simple calculation: if $f(x,y) = \frac{|x|^4-|y|^4}{4}$ we find successively

$$\begin{gathered} f_x = |x|^2 x, \quad f_y = -|y|^2 y, \quad |Df|^2 = |x|^6 + |y|^6, \\ f_{x_i x_j} = 2x_i x_j + \delta_{ij}|x|^2, \quad f_{y_i y_j} = -2y_i y_j - \delta_{ij}|y|^2, \\ f_{x_i y_j} = 0, \end{gathered}$$

for all indices $i, j = 1, \dots, m$. Also, for $(x,y) \neq (0,0)$ we have

$$\begin{aligned} |(x,y)|^\alpha &= (|x|^2+|y|^2)^{\alpha/2}, \qquad & \frac{Df}{|Df|} &= \frac{(|x|^2x, -|y|^2y)}{(|x|^6+|y|^6)^{1/2}}, \\ D|(x,y)|^\alpha &= \alpha|(x,y)|^{\alpha-2}(x,y), \end{aligned}$$

$$|Df|^3 \operatorname{div}\Big(\frac{Df}{|Df|}\Big) = (|x|^4-|y|^4)\big\{(m-1)|x|^4 - (m+2)|x|^2|y|^2 + (m-1)|y|^4\big\},$$

whence

$$\begin{aligned} &|Df|^3 \operatorname{div}\Big[|(x,y)|^\alpha \frac{Df}{|Df|}\Big] \\ &\quad = \alpha|Df|^2|(x,y)|^{\alpha-2}[|x|^4-|y|^4] \\ &\qquad + (|x|^2+|y|^2)^{\alpha/2}\{(|x|^4-|y|^4)[(m-1)|x|^4-(m+2)|x|^2|y|^2 \\ &\qquad\qquad + (m-1)|y|^4]\} \\ &\quad = \alpha|(x,y)|^{\alpha-2}[|x|^4-|y|^4](|x|^6+|y|^6) \\ &\qquad + |(x,y)|^\alpha(|x|^4-|y|^4)[(m-1)|x|^4-(m+2)|x|^2|y|^2+(m-1)|y|^4] \\ &\quad = |(x,y)|^{\alpha-2}(|x|^4-|y|^4)\big\{(m-1+\alpha)|x|^6 - 3|x|^4|y|^2 - 3|x|^2|y|^4 \\ &\qquad\qquad + (m-1+\alpha)|y|^6\big\}. \end{aligned}$$

Upon putting $t := \frac{|x|^2}{|y|^2}$ we see that the sign of $\operatorname{div}[|(x,y)|^\alpha \frac{Df}{|Df|}]$ is the same as that of

$$f(x,y) = \frac{|x|^4-|y|^4}{4}$$

provided the polynomial

$$p_{m,\alpha}(t) := (m-1+\alpha)t^3 - 3t^2 - 3t + (m-1+\alpha)$$

is nonnegative for all $t \geq 0$. Since for every $t \geq 0$ we have $3t^3 - 3t^2 - 3t + 3 \geq 0$ (with zero minimum for $t \geq 0$ at $t = 1$), it also follows that $p_{m,\alpha}(t) \geq 0$ for all $t \geq 0$, if we assume that $(m - 1 + \alpha) \geq 3$.

Concluding, we have shown that the vector field $\xi(x, y) = \frac{Df(x,y)}{|Df(x,y)|}$ defines an $\alpha$-subcalibration for the sets $E_k$ in $\mathbb{R}^n - \{0\}$, $k \in \mathbb{N}$, while $-\xi(x, y)$ furnishes an $\alpha$-subcalibration for the sets $F_k^c = \mathbb{R}^n - F_k$ in $\mathbb{R}^n - \{0\}$. Proposition 4 yields the $\alpha$-subminimality of the sets $E_k$ and $F_k^c$ in $\mathbb{R}^n - \{0\}$ . However, $0 \notin E_k$ or $F_k^c$ and we have

$$P_\alpha(E_k, A) = P_\alpha(E_k, A - \{0\}) \leq P_\alpha(F, A - \{0\}) \leq P_\alpha(F, A)$$

for every bounded, open set $A \subset \mathbb{R}^n$ and arbitrary $F \subset E_k$ with $E_k \setminus F \subset\subset A$, which shows that all sets $E_k$ are also $\alpha$-subminimal in all of $\mathbb{R}^n$ (rather than just in $\mathbb{R}^n - \{0\}$). A similar argument implies the $\alpha$-subminimality of $F_k^c$ in $\mathbb{R}^n$ and since $E_k \to C_m$ or $F_k^c \to \mathbb{R}^n - C_m$ both in $L_{1,\mathrm{loc}}(\mathbb{R}^n)$ as $k \to \infty$, we infer from Proposition 3 that $C_m$, as well as its complement $C_m^c = \mathbb{R}^n - C_m$, is $\alpha$-subminimal in $\mathbb{R}^n$. An application of Proposition 2 concludes the proof of Theorem 1. ■

Remark. Using the same type of argument as in the proof of Theorem 1, one can also deal with the minimal cones

$$C_{m,k} = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^k;\ (k - 1)|x|^2 \leq (m - 1)|y|^2\}.$$

A subcalibration might then be determined by the normalized gradient of the function

$$f : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}, \quad f(x, y) := \frac{1}{4}((k - 1)^2|x|^4 - (m - 1)^2|y|^4)$$

under suitable conditions on $m$ and $k$; however, we shall not dwell on this.

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