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Global smooth axisymmetric solutions of the damped Boussinesq equations with zero thermal diffusion

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Abstract – In this paper we study the Cauchy problem for the three-dimensional damped Boussinesq system with zero thermal diffusion. We show that the solution of this system with $1 \leq \beta \leq \frac{7}{3}$ is globally well posed if the initial data is axisymmetric without swirl.

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1. Introduction

We study the following three-dimensional incompressible damped Boussinesq equations with zero thermal diffusion:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + |u|^{\beta-1} u + \nabla P = \rho e_3, \quad x \in \mathbb{R}^3,\ t > 0, \\ \partial_t \rho + u \cdot \nabla \rho = 0, \\ \operatorname{div} u = 0, \\ u(x,0) = u_0(x), \quad \rho(x,0) = \rho_0(x), \end{cases} \tag{1.1}$$

where $u = (u_1(x,t), u_2(x,t), u_3(x,t)) \in \mathbb{R}^3$ is the velocity of the fluid, $P = P(x,t) \in \mathbb{R}$ is the scalar pressure, $\rho = \rho(x,t) \in \mathbb{R}$ is the temperature and $e_3 = (0,0,1)$.

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Here, $\beta \geq 1$ and $|u|^{\beta-1}u$ is the so-called damping term, which is from the resistance to the motion of the flow; see [4] and references cited therein. The system (1.1) is widely used to model the dynamics of the ocean or the atmosphere; see [19–21, 25].

When the damping term is absent, (1.1) becomes the well-known three-dimensional Boussinesq system with zero thermal diffusion. It is well known that local existence and uniqueness of smooth solutions for the three-dimensional Boussinesq system with zero thermal diffusion have been proved; see [5, 6, 17]. However, the problem of the global existence of the unique smooth local solution still remains unsolved. Therefore, in connection with this point, many works are concerned with the study of solutions with some special structures such as axisymmetric flows. We first introduce the definition of axisymmetric vector fields. A vector field f is axisymmetric, that is,

$$f(t,x) = f^r(t,r,z)e_r + f^\theta(t,r,z)e_\theta + f^z(t,r,z)e_z,$$

where $x = (x_1, x_2, z)$, $r = \sqrt{x_1^2 + x_2^2}$ and

$$e_r = \Big(\frac{x_1}{r}, \frac{x_2}{r}, 0\Big), \quad e_\theta = \Big(-\frac{x_2}{r}, \frac{x_1}{r}, 0\Big), \quad e_z = (0,0,1).$$

We call f^θ the swirl component and f is axisymmetric without swirl if $f^\theta = 0$. Abidi, Hmidi and Keraani [2] showed the global existence and uniqueness result for the axisymmetric Boussinesq system without swirl. They use as a key ingredient the quadratic growth estimate

$$\Big\|\frac{\rho}{r}(t)\Big\|_{L^2} \leq \Big\|\frac{\rho_0}{r}\Big\|_{L^2} + C_0\Big\|\frac{u^r}{r}\Big\|_{L_t^1L^\infty}(1 + \|u\|_{L_t^1L^\infty}),$$

under the assumption that the support of the initial temperature does not intersect the axis $r = 0$. Later on, this result was improved in [9] by removing the assumption on the support of the temperature. When the swirl component presents, the global regularity with large initial data is still open. See Fang–Le–Zhang [8], Li–Pan [16] and Wang–Guo [23] for the regularity criterion of the axisymmetric Boussinesq system with zero thermal diffusion. Other interesting results on the Boussinesq system can be found in [1, 10, 14, 18, 24].

When the temperature ρ vanishes, system (1.1) reduces to the three-dimensional incompressible damped Navier–Stokes equations, which were studied first by Cai–Jiu [4]. The authors in [4] obtained that for such a system there exist a global weak solution for any $\beta \geq 1$, and global strong solutions for any $\beta \geq \frac{7}{2}$. Moreover, the strong solution is unique for any $\frac{7}{2} \leq \beta \leq 5$. Zhou [30] proved the strong solution exists globally for $\beta \geq 3$. However, since the good effect of the damping becomes weaker in the case $1 < \beta < 3$, the global existence of a strong solution to the damped Navier–Stokes equations for $1 < \beta < 3$ is still unknown. Recently, for the axisymmetric without swirl solutions of the damped Navier–Stokes system, Yu [26] established the global existence and uniqueness of a strong solution with large initial data. For more related results, see [27–29] and references therein.

Wen and Ye [25] gave the global existence and the uniqueness for the three-dimensional incompressible damped Boussinesq equations (1.1) with $\beta \geq 3$. Li, Liu and Zhou [15] proved regularity criteria of the three-dimensional damped Boussinesq system (1.1) with $1 \leq \beta < 3$ via two components of the velocity or the gradient of velocity involving Lorentz spaces in both time and spatial directions. Motivated by [25, 26], we investigate in this paper the global well-posedness of (1.1) with $1 \leq \beta \leq \frac{7}{3}$, corresponding to large axisymmetric data without swirl. Our main result is as follows.

THEOREM 1.1. *Assume that u_0 and ρ_0 are axisymmetric and u_0 is a divergence-free vector field with $u_0^\theta = 0$. Let $u_0 \in H^2(\mathbb{R}^3)$, and $\rho_0 \in H^2(\mathbb{R}^3)$ such that* $\operatorname{supp}\rho_0$ *does not intersect the z-axis and the projection of* $\operatorname{supp}\rho_0$ *to the z-axis is compact. Then there exists a unique global solution (u, ρ) to system (1.1) with $1 \leq \beta \leq \frac{7}{3}$ satisfying*

$$\begin{aligned} &u \in L^\infty(0, T; H^2(\mathbb{R}^3)) \cap L^2(0, T; H^3(\mathbb{R}^3)),\\ &\rho \in L^\infty(0, T; H^2(\mathbb{R}^3)), \end{aligned}$$

for any $0 < T < \infty$.

REMARK 1. When $\rho = 0$, system (1.1) reduces to the incompressible Navier–Stokes equations with damping in $\mathbb{R}^3$, and Theorem 1.1 generalizes a previous result by Yu [26].

REMARK 2. As pointed out in [2], we can relax the assumption that $\operatorname{supp}\rho_0$ is away from the z-axis by assuming that ρ is a constant c_0 near the z-axis, by taking a change of variable $\bar{\rho} = \rho - c_0$ and $\overline{P} = P - c_0 z$. More detail can be founded in [2].

REMARK 3. It is worth pointing out that we only prove the global regularity when $1 \leq \beta \leq \frac{7}{3}$. For $\frac{7}{3} < \beta < 3$, due to the strong nonlinearity of $|u|^{\beta-1}u$, it is insufficient to control the nonlinear term using only the dissipative term. In the future, we will further study how to use the good effect of the damping term to extend the exponent to $1 \leq \beta < 3$.

To prove Theorem 1.1, we need to deeply use the special structure of system (1.1) in the axisymmetric without swirl case. In contrast with the case in [26], the singular term $\frac{\partial_r \rho}{r}$ causes difficulties in estimating $\|\frac{w^\theta}{r}\|_{L^2}$, due to the appearance of the temperature ρ. To do this, inspired by [2], we assume that $\operatorname{supp}\rho_0$ is away from the z-axis and its projection to the z-axis is compact. Taking advantage of the local-in-space estimate for solutions and some interpolation inequalities, we derive the $L^\infty(0,T;L^2(\mathbb{R}^3))\cap L^2(0,T;H^1(\mathbb{R}^3))$ estimate of $\frac{w^\theta}{r}$; see (3.8) below. Moreover, we establish the $L^1(0,T;L^\infty(\mathbb{R}^3))$ estimate of ∇u by making good use of maximal regularity results of the heat flow, and then the $L^\infty(0,T;L^6(\mathbb{R}^3))$ estimate of $\nabla\rho$ follows; see Proposition 3.3 below. Based on this preparation, combining with commutator estimates allows us to obtain the H^2 estimate of the solution.

An outline of this paper is as follows. In Section 2, we introduce system (1.1) in cylindrical coordinates and review some basic facts. Section 3 is devoted to showing some a priori estimates, and completing the proof of Theorem 1.1.

Throughout the paper, we always denote $\int\cdot\,dx := \int_{\mathbb{R}^3}\cdot\,dx$, $\|\cdot\|_{L^p} := \|\cdot\|_{L^p(\mathbb{R}^3)}$ and $\nabla_h := (\partial_1,\partial_2)$. The letter C stands for some real positive constant, which may vary from line to line. In addition, we use the notation Π_z for the orthogonal projector over the axis (Oz).

2. Preliminaries

Let us now introduce system (1.1) in cylindrical coordinates. For system (1.1), it is easy to show the following local existence and uniqueness result.

Lemma 2.1. *Let* $(u_0,\rho_0)\in H^2(\mathbb{R}^3)$ *be axisymmetric and* u_0 *be divergence-free. Then there exist* $T>0$ *and a unique axisymmetric solution* (u,ρ) *on* $[0,T)$ *for system* (1.1) *with* $\beta\geq 1$ *such that*

$$\begin{aligned} &u\in L^\infty(0,T;H^2(\mathbb{R}^3))\cap L^2(0,T;H^3(\mathbb{R}^3)),\\ &\rho\in L^\infty(0,T;H^2(\mathbb{R}^3)). \end{aligned}$$

The proof is quite similar to that local existence and uniqueness result for the incompressible Navier–Stokes equations and so is omitted. We refer the reader to [12]. From the uniqueness of solutions, we see that $u_0^\theta=0$ implies $u^\theta=0$ for all later time. In this case, system (1.1) can be written as

$$\begin{cases} \partial_t u^r + u\cdot\nabla u^r + |u|^{\beta-1}u^r - \Big(\Delta-\dfrac{1}{r^2}\Big)u^r + \partial_r P = 0,\\ \partial_t u^z + u\cdot\nabla u^z + |u|^{\beta-1}u^z - \Delta u^z + \partial_z P = \rho,\\ \partial_t\rho + u\cdot\nabla\rho = 0,\\ \partial_r u^r + \dfrac{u^r}{r} + \partial_z u^z = 0, \end{cases}$$

where $u\cdot\nabla = u^r\partial_r + u^z\partial_z$ and $\Delta = \partial_r^2+\partial_z^2+\frac{1}{r}\partial_r$.

Direct computations show us that the vorticity of the swirl-free axisymmetric velocity takes the form

$$w = \nabla \times u = w^\theta e_\theta = (\partial_r u^r - \partial_r u^z) e_\theta,$$

and satisfies

$$\begin{aligned}\partial_t \omega^\theta + u \cdot \nabla \omega^\theta - \Delta \omega^\theta + \frac{1}{r^2}\omega^\theta - \frac{1}{r}\omega^\theta u^r + |u|^{\beta-1}\omega^\theta \\ + (\beta-1)|u|^{\beta-3}((u^r)^2 \partial_z u^r + u^r u^z \partial_z u^z - u^r u^z \partial_r u^r - (u_z)^2 \partial_r u^z) \\ = -\partial_r \rho. \end{aligned} \tag{2.1}$$

Moreover, letting $\Gamma := \frac{\omega^\theta}{r}$, the above system gives

$$\begin{aligned}\partial_t \Gamma + u \cdot \nabla \Gamma - \Big(\Delta + \frac{2}{r}\partial_r\Big)\Gamma + |u|^{\beta-1}\Gamma + (\beta-1)|u|^{\beta-3}(u^z)^2\Gamma \\ + (\beta-1)|u|^{\beta-3}\Big((u^r)^2 \partial_z\Big(\frac{u^r}{r}\Big) - (u^z)^2 \partial_z\Big(\frac{u^r}{r}\Big) \\ - 2u^r u^z \partial_r\Big(\frac{u^r}{r}\Big) - 3\Big(\frac{u^r}{r}\Big)^2 u^z\Big) \\ = -\frac{\partial_r \rho}{r}. \end{aligned} \tag{2.2}$$

To ensure the reasonableness of the calculation, we recall the following two lemmas.

LEMMA 2.2 ([12]). *Let u be a smooth axisymmetric vector field with $u \in L^\infty(0,T; H^2(\mathbb{R}^3)) \cap L^2(0,T; H^3(\mathbb{R}^3))$ and $\omega = \omega^\theta e_\theta$ its curl. Then*

(i) $\frac{\omega^\theta}{r^{2-\varepsilon}}$ *and* $\frac{1}{r^{1-\varepsilon}}\frac{\partial \omega^\theta}{\partial r}$ *belong to* $L^2(0,T; L^2(\mathbb{R}^3))$ *for all* $\varepsilon > 0$*;*

(ii) *letting*

$$g_1(\eta) = \int_{-\infty}^{\infty} \Big(\eta^\delta \Big|\frac{\omega^\theta}{\eta}\Big|^2\Big)(\eta, z)\, dz, \quad g_2(\eta) = \int_{-\infty}^{\infty} \Big(\eta^\delta \Big|\frac{\partial \omega^\theta}{\partial r}\Big|^2\Big)(\eta, z)\, dz,$$

then g_1 and g_2 are bounded for any $\delta \in (0,2)$.

LEMMA 2.3 ([12]). *For any $\varepsilon > 0$, there holds*

$$\lim_{\eta \to 0^+} \int_{-\infty}^{\infty} \frac{\partial \omega^\theta}{\partial r} \frac{\omega^\theta}{\eta^{1-\varepsilon}}(\eta, z)\, dz = 0.$$

We end up in this section by citing the following lemma, which is the cornerstone for establishing the estimate $\|\frac{\omega^\theta}{r}\|_{L^2}$.

Lemma 2.4 ([2]). *Let u be a smooth axisymmetric divergence-free vector field, and ρ be a solution of the transport equation*

$$\partial_t \rho + u \cdot \nabla \rho = 0$$

with initial data $\rho_0 \in L^2 \cap L^\infty$. In addition, assume that

$$r_0 := d(\operatorname{supp}\rho_0, \{Oz\}) > 0 \quad and \quad d_0 := \operatorname{diam}(\Pi_z(\operatorname{supp}\rho_0)) < \infty.$$

Then there holds

$$\int_{\mathbb{R}^3} \frac{\rho^2(t,x)}{r^2}\,dx \le \frac{1}{r_0^2}\|\rho_0\|_{L^2}^2 + 2\pi\|\rho_0\|_{L^\infty}^2 \int_0^t \left\|\frac{u^r}{r}(\tau)\right\|_{L^\infty} d\tau\left(d_0 + 2\int_0^t \|u(\tau)\|_{L^\infty}\,d\tau\right).$$

3. Proof of Theorem 1.1

This section is devoted to deriving some useful a priori estimates and then completing the proof of Theorem 1.1. We first give the basic estimate for solutions of system (1.1), which do not need the axisymmetric assumption.

Proposition 3.1. *Suppose $1 \le \beta \le \frac{7}{3}$. Let $(u_0, \rho_0) \in H^2(\mathbb{R}^3)$. Then for every smooth solution (u,ρ), it holds that*

$$\|\rho(t)\|_{L^p} \le \|\rho_0\|_{L^p}, \quad \forall 2 \le p \le \infty, \tag{3.1}$$

and

$$\|u(t)\|_{L^2}^2 + \int_0^T \|\nabla u(t)\|_{L^2}^2\,dt + \int_0^T \|u(t)\|_{L^{\beta+1}}^{\beta+1}\,dt \le C. \tag{3.2}$$

Here, C depends on $\|u_0\|_{H^2}$ and $\|\rho_0\|_{H^2}$.

Proof. The estimate in (3.1) is classical and can be found in [25], so we omit the details here. For the estimate in (3.2), by taking the L^2 inner product of the first equation of system (1.1) with u we get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \int |u|^{\beta+1}\,dx \le \|u\|_{L^2}\|\rho\|_{L^2},$$

which along with (3.1) and Gronwall's inequality leads to the desired result. ■

Next, our target is to show the $L^\infty_T L^2 \cap L^2_T H^1$ estimate of ∇u, which plays the key role in our proof.

Proposition 3.2. *Suppose* $1 \le \beta \le \frac{7}{3}$. *Let* (u, ρ) *be the smooth solution of system* (1.1) *with* $(u_0, \rho_0) \in H^2(\mathbb{R}^3)$ *satisfying the assumptions of Theorem* 1.1. *Then one has*

$$\sup_{0\le t\le T} \|\nabla u(t)\|_{L^2}^2 + \int_0^T \|\nabla^2 u(t)\|_{L^2}^2 \, dt \le C(T).$$

Proof. For small ε, multiplying equation (2.2) by $\frac{\omega^\theta}{r^{1-\varepsilon}}$ and integrating the resulting equation over $\mathbb{R}^3$,we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big\|\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\Big\|_{L^2}^2 + \Big\|\nabla\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\Big\|_{L^2}^2 + \Big(\varepsilon-\frac{\varepsilon^2}{4}\Big)\Big\|\frac{\omega^\theta}{r^{2-\frac{\varepsilon}{2}}}\Big\|_{L^2}^2 \\
&\quad + \int |u|^{\beta-1}\Big|\frac{\omega^\theta}{r^{1-\varepsilon}}\Big|^2 dx + (\beta-1)\int |u|^{\beta-3}(u^z)^2\Big|\frac{\omega^\theta}{r^{1-\varepsilon}}\Big|^2 dx \\
&= \frac{\varepsilon}{2}\int \frac{u^r}{r}\frac{(\omega^\theta)^2}{r^{2-\varepsilon}}\, dx - \int \frac{\partial_r\rho}{\rho}\frac{\omega^\theta}{r^{1-\varepsilon}}\, dx \\
&\quad - (\beta-1)\int |u|^{\beta-3}\Big((u^r)^2\partial_z\Big(\frac{u^r}{r}\Big) - (u^z)^2\partial_z\Big(\frac{u^r}{r}\Big) \\
&\qquad\qquad - 2u^r u^z\partial_r\Big(\frac{u^r}{r}\Big) - 3\Big(\frac{u^r}{r}\Big)^2 u^z\Big)\frac{\omega^\theta}{r^{1-\varepsilon}}\, dx
\end{aligned}$$

$$:= \sum_{i=1}^{6} I_i, \tag{3.3}$$

where we have used Lemmas 2.2 and 2.3.

Motivated by the method in [12, 26], we now estimate the terms I_1 to I_6. Using Hölder's inequality, Young's inequality and the interpolation estimate

$$\|f\|_{L^\infty} \le C\|\nabla f\|_{L^2}^{\frac{1}{2}}\|\nabla^2 f\|_{L^2}^{\frac{1}{2}},$$

we get

$$\begin{aligned}
I_1 &\le \frac{\varepsilon}{2}\int |u^r|\Big|\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\Big|\Big|\frac{\omega^\theta}{r^{2-\frac{\varepsilon}{2}}}\Big|\, dx \\
&\le \frac{\varepsilon}{2}\|u^r\|_{L^\infty}\Big\|\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\Big\|_{L^2}\Big\|\frac{\omega^\theta}{r^{2-\frac{\varepsilon}{2}}}\Big\|_{L^2} \\
&\le \frac{\varepsilon}{4}\Big\|\frac{\omega^\theta}{r^{2-\frac{\varepsilon}{2}}}\Big\|_{L^2}^2 + \frac{\varepsilon}{2}\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\Big\|\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\Big\|_{L^2}^2.
\end{aligned}$$

For I_2, integrating by parts and using Hölder's inequality, we have

$$\begin{aligned}
I_2 &= -\int \frac{\partial_r\rho}{\rho}\frac{\omega^\theta}{r^{1-\varepsilon}}\,dx = -\int \frac{\partial_r\rho}{r}\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\frac{1}{r^{-\frac{\varepsilon}{2}}}\,dx\\
&= -2\pi\int_{-\infty}^{\infty}\int_0^{\infty}\partial_r\rho\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\frac{1}{r^{-\frac{\varepsilon}{2}}}\,dr\,dz\\
&= 2\pi\int_{-\infty}^{\infty}\int_0^{\infty}\rho\partial_r\Big(\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\frac{1}{r^{-\frac{\varepsilon}{2}}}\Big)\,dr\,dz\\
&= 2\pi\int_{-\infty}^{\infty}\int_0^{\infty}\rho\frac{1}{r^{-\frac{\varepsilon}{2}}}\partial_r\Big(\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\Big)\,dr\,dz + 2\pi\int_{-\infty}^{\infty}\int_0^{\infty}\rho\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\partial_r\Big(\frac{1}{r^{-\frac{\varepsilon}{2}}}\Big)\,dr\,dz\\
&= \int\frac{\rho}{r^{1-\frac{\varepsilon}{2}}}\partial_r\Big(\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\Big)\,dx + \frac{\varepsilon}{2}\int\frac{\rho}{r}\frac{\omega^\theta}{r^{2-\varepsilon}}\,dx\\
&\le C\Big\|\frac{\rho}{r^{1-\frac{\varepsilon}{2}}}\Big\|_{L^2}\Big\|\nabla\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\Big\|_{L^2} + \frac{\varepsilon}{2}\Big\|\frac{\rho}{r}\Big\|_{L^2}\Big\|\frac{\omega^\theta}{r^{2-\varepsilon}}\Big\|_{L^2}\\
&\le C\Big\|\frac{\rho}{r^{1-\frac{\varepsilon}{2}}}\Big\|_{L^2}^2 + \frac{1}{2}\Big\|\nabla\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\Big\|_{L^2}^2 + \frac{\varepsilon}{4}\Big\|\frac{\rho}{r}\Big\|_{L^2}^2 + \frac{\varepsilon}{4}\Big\|\frac{\omega^\theta}{r^{2-\varepsilon}}\Big\|_{L^2}^2.
\end{aligned}$$

Although the estimates of the terms I_3 to I_6 basically follow from that of [26, Lemma 2.6], we briefly present the proof of these estimates for the sake of completeness. For I_3, it follows from Hölder's inequality, Young's inequality and the interpolation estimate

$$\|f\|_{L^{\frac{4}{3-\beta}}} \le \|f\|_{L^2}^{\frac{7-3\beta}{4}}\|\nabla f\|_{L^2}^{\frac{3\beta-3}{4}}, \quad \forall 1\le\beta\le\frac{7}{3}$$

that

$$\begin{aligned}
|I_3| &\le C\Big(\int_{r\le 1} + \int_{|u|\le 1, r>1} + \int_{|u|>1, r>1}\Big)|u|^{\beta-1}\Big|\partial_z\frac{u^r}{r}\Big|\Big|\frac{\omega^\theta}{r^{1-\varepsilon}}\Big|\,dx\\
&\le C\|\,|u|^{\beta-1}\|_{L^{\frac{2}{\beta-1}}}\Big\|\partial_z\frac{u^r}{r}\Big\|_{L^{\frac{4}{3-\beta}}}\Big\|\frac{\omega^\theta}{r}\Big\|_{L^{\frac{4}{3-\beta}}} + C\Big\|\partial_z\frac{u^r}{r}\Big\|_{L^2}\|\omega^\theta\|_{L^2}\\
&\quad + C\|\,|u|^{\frac{4}{3}}\|_{L^3}\Big\|\partial_z\frac{u^r}{r}\Big\|_{L^6}\|\omega^\theta\|_{L^2}\\
&\le C\|u\|_{L^2}^{\beta-1}\|\Gamma\|_{L^2}^{\frac{7-3\beta}{2}}\|\nabla\Gamma\|_{L^2}^{\frac{3\beta-3}{2}} + C\|\nabla u\|_{L^2}\|\Gamma\|_{L^2}\\
&\quad + C\|u\|_{L^2}^{\frac{1}{3}}\|\nabla u\|_{L^2}\|\nabla\Gamma\|_{L^2}\|\omega^\theta\|_{L^2}\\
&\le \frac{1}{10}\|\nabla\Gamma\|_{L^2}^2 + C\|u\|_{L^2}^{\frac{4(\beta-1)}{7-3\beta}}\|\Gamma\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\\
&\quad + C\|\Gamma\|_{L^2}^2 + C\|u\|_{L^2}^{\frac{2}{3}}\|\nabla u\|_{L^2}^2\|\omega^\theta\|_{L^2}^2.
\end{aligned}$$

This together with (3.2) ensures

$$|I_3| \le \frac{1}{10}\|\nabla\Gamma\|_{L^2}^2 + C\|\Gamma\|_{L^2}^2 + C(1+\|\omega^\theta\|_{L^2}^2)\|\nabla u\|_{L^2}^2.$$

By the same argument as the estimate of I_3, one deduces that

$$|I_4| + |I_5| \le \frac{1}{10}\|\nabla\Gamma\|_{L^2}^2 + C\|\Gamma\|_{L^2}^2 + C(1 + \|\omega^\theta\|_{L^2}^2)\|\nabla u\|_{L^2}^2.$$

Applying Hölder's inequality again, the term I_6 can be bounded by

$$\begin{aligned}
|I_6| &\le C\Big(\int_{|u|>1,r>1} + \int_{|u|>1,r\le 1} + \int_{|u|\le 1,r>1} + \int_{|u|\le 1,r\le 1}\Big)|u|^{\beta-2}\Big|\frac{u^r}{r}\Big|^2\Big|\frac{\omega^\theta}{r^{1-\varepsilon}}\Big|\,dx \\
&\le C\int_{|u|>1,r>1}|u|^{\frac{4}{3}}\Big|\frac{u^r}{r}\Big||\omega^\theta|\,dx + C\int_{|u|>1,r\le 1}|u|^{\frac{1}{3}}\Big|\frac{u^r}{r}\Big|^2\Big|\frac{\omega^\theta}{r}\Big|\,dx \\
&\quad + C\int_{|u|\le 1,r>1}|u|^{\beta-1}\Big|\frac{u^r}{r}\Big||\omega^\theta|\,dx \\
&\quad + C\int_{-\infty}^{+\infty}\Big(\int_{|u|\le 1,r\le 1}\Big|\frac{u^r}{r}\Big|\Big|\frac{\omega^\theta}{r}\Big|\frac{1}{r}\,dx_1\,dx_2\Big)\,dx_3 \\
&\le C\Big\|\frac{u^r}{r}\Big\|_{L^\infty}\|u\|_{L^{\frac{8}{3}}}^{\frac{4}{3}}\|\omega^\theta\|_{L^2} + C\|u\|_{L^2}^{\frac{1}{3}}\Big\|\frac{u^r}{r}\Big\|_{L^\infty}\Big\|\frac{u^r}{r}\Big\|_{L^2}\Big\|\frac{\omega^\theta}{r}\Big\|_{L^3} \\
&\quad + C\Big\|\frac{u^r}{r}\Big\|_{L^2}\|\omega^\theta\|_{L^2} \\
&\quad + C\int_{-\infty}^{+\infty}\Big\|\frac{u^r}{r}\Big\|_{L^6(\mathbb{R}^2)}\Big\|\frac{\omega^\theta}{r}\Big\|_{L^6(\mathbb{R}^2)}\Big\|\frac{1}{r}\Big\|_{L^{\frac{3}{2}}(\mathbb{R}^2)\cap\{r\le 1\}}\,dx_3 \\
&\le C\Big\|\frac{\omega^\theta}{r}\Big\|_{L^2}^{\frac{1}{2}}\Big\|\nabla\frac{\omega^\theta}{r}\Big\|_{L^2}^{\frac{1}{2}}\|u\|_{L^2}^{\frac{5}{6}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\omega^\theta\|_{L^2} \\
&\quad + C\|u\|_{L^2}^{\frac{1}{3}}\Big\|\frac{\omega^\theta}{r}\Big\|_{L^2}\Big\|\nabla\frac{\omega^\theta}{r}\Big\|_{L^2}\Big\|\frac{u^r}{r}\Big\|_{L^2} + C\|\nabla u\|_{L^2}^2 \\
&\quad + C\int_{-\infty}^{+\infty}\Big\|\frac{u^r}{r}\Big\|_{L^2(\mathbb{R}^2)}^{\frac{1}{3}}\Big\|\nabla_h\frac{u^r}{r}\Big\|_{L^2(\mathbb{R}^2)}^{\frac{2}{3}}\Big\|\frac{\omega^\theta}{r}\Big\|_{L^2(\mathbb{R}^2)}^{\frac{1}{3}}\Big\|\nabla_h\frac{\omega^\theta}{r}\Big\|_{L^2(\mathbb{R}^2)}^{\frac{2}{3}}\,dx_3 \\
&\le C\|u\|_{L^2}^{\frac{5}{6}}\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\Gamma\|_{L^2}^{\frac{1}{2}}\|\nabla\Gamma\|_{L^2}^{\frac{1}{2}} + C\|u\|_{L^2}^{\frac{1}{3}}\|\Gamma\|_{L^2}\|\nabla\Gamma\|_{L^2}\|\nabla u\|_{L^2} \\
&\quad + C\|\nabla u\|_{L^2}^2 + C\Big\|\frac{u^r}{r}\Big\|_{L^2}^{\frac{1}{3}}\Big\|\nabla_h\frac{u^r}{r}\Big\|_{L^2}^{\frac{2}{3}}\|\Gamma\|_{L^2}^{\frac{1}{3}}\|\nabla_h\Gamma\|_{L^2}^{\frac{2}{3}} \\
&\le \frac{1}{10}\|\nabla\Gamma\|_{L^2}^2 + C\|u\|_{L^2}^{\frac{10}{9}}\|\nabla u\|_{L^2}^2\|\Gamma\|_{L^2}^{\frac{2}{3}} + C\|u\|_{L^2}^{\frac{2}{3}}\|\Gamma\|_{L^2}^2\|\nabla u\|_{L^2}^2 \\
&\quad + C\|\nabla u\|_{L^2}^2 + C\Big\|\frac{u^r}{r}\Big\|_{L^2}^{\frac{1}{2}}\|\Gamma\|_{L^2}^{\frac{3}{2}},
\end{aligned}$$

which along with (3.2) leads to

$$|I_6| \le \frac{1}{10}\|\nabla\Gamma\|_{L^2}^2 + C\|\Gamma\|_{L^2}^2(1 + \|\nabla u\|_{L^2}^2) + C\|\nabla u\|_{L^2}^2.$$

Putting all the above estimates into (3.3), we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big\|\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\Big\|_{L^2}^2+\Big\|\nabla(\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}})\Big\|_{L^2}^2+\Big(\frac{3}{2}\varepsilon-\frac{\varepsilon^2}{2}\Big)\Big\|\frac{\omega^\theta}{r^{2-\frac{\varepsilon}{2}}}\Big\|_{L^2}^2\\
&\quad\le C\varepsilon\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\Big\|\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\Big\|_{L^2}^2+\frac{3}{5}\|\nabla\Gamma\|_{L^2}^2+C\|\Gamma\|_{L^2}^2\\
&\qquad+C(1+\|\nabla u\|_{L^2}^2)(\|\Gamma\|_{L^2}^2+\|\omega^\theta\|_{L^2}^2)\\
&\qquad+C\|\nabla u\|_{L^2}^2+C\Big\|\frac{\rho}{r^{1-\frac{\varepsilon}{2}}}\Big\|_{L^2}^2+\frac{\varepsilon}{2}\Big\|\frac{\rho}{r}\Big\|_{L^2}^2+\frac{\varepsilon}{2}\Big\|\frac{\omega^\theta}{r^{2-\varepsilon}}\Big\|_{L^2}^2.
\end{aligned}$$

Integrating over $[0,T]$, one then has

$$\begin{aligned}
&\Big\|\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\Big\|_{L^2}^2+\int_0^T\Big\|\nabla(\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}})\Big\|_{L^2}^2\\
&\quad\le\Big\|\frac{\omega_0^\theta}{r^{1-\frac{\varepsilon}{2}}}\Big\|_{L^2}^2+C\varepsilon\int_0^T\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\Big\|\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\Big\|_{L^2}^2\,dt\\
&\qquad+\frac{3}{5}\int_0^T\|\nabla\Gamma\|_{L^2}^2\,dt\\
&\qquad+C\int_0^T\|\Gamma\|_{L^2}^2+C\int_0^T(1+\|\nabla u\|_{L^2}^2)(\|\Gamma\|_{L^2}^2+\|\omega^\theta\|_{L^2}^2)\,dt\\
&\qquad+C\int_0^T\|\nabla u\|_{L^2}^2\,dt+C\int_0^T\Big\|\frac{\rho}{r^{1-\frac{\varepsilon}{2}}}\Big\|_{L^2}^2\,dt\\
&\qquad+\frac{\varepsilon}{2}\int_0^T\Big\|\frac{\rho}{r}\Big\|_{L^2}^2\,dt+\frac{\varepsilon}{2}\int_0^T\Big\|\frac{\omega^\theta}{r^{2-\varepsilon}}\Big\|_{L^2}^2\,dt.
\end{aligned}\tag{3.4}$$

Noticing that

$$\Big|\frac{f}{r^{1-\frac{\varepsilon}{2}}}\Big|\le C\Big(\Big|\frac{f}{r}\Big|+|f|\Big),$$

we get

$$\Big\|\frac{\omega^\theta}{r^{1-\frac{\varepsilon}{2}}}\Big\|_{L^2}\le C\Big(\Big\|\frac{\omega^\theta}{r}\Big\|_{L^2}+\|\omega^\theta\|_{L^2}\Big),\quad\Big\|\frac{\rho}{r^{1-\frac{\varepsilon}{2}}}\Big\|_{L^2}\le C\Big(\Big\|\frac{\rho}{r}\Big\|_{L^2}+\|\rho\|_{L^2}\Big).$$

This together with Lemmas 2.1, 2.2 and 2.4 ensures we can pass to the limit as $\varepsilon\to 0$ on the right-hand side of (3.4). Passing $\varepsilon\to 0$ in (3.4) and using the Lebesgue dominated convergence theorem, one has

$$\begin{aligned}
\|\Gamma\|_{L^2}^2+\frac{2}{5}\int_0^T\|\nabla\Gamma\|_{L^2}^2\,dt&\le C\|u_0\|_{H^2}+C\int_0^T(1+\|\nabla u\|_{L^2}^2)(\|\Gamma\|_{L^2}^2+\|\omega^\theta\|_{L^2}^2)\,dt\\
&\quad+C\int_0^T\Big\|\frac{\rho}{r}\Big\|_{L^2}^2\,dt.
\end{aligned}\tag{3.5}$$

To estimate the last term of the above inequality, we apply Lemma 2.4 and (3.1) to get

$$\begin{aligned}
&\int_0^T \Big\|\frac{\rho}{r}\Big\|_{L^2}^2\,dt\\
&\quad\le CT + CT\int_0^T \Big\|\frac{u^r}{r}\Big\|_{L^\infty}\,dt + CT\int_0^T \Big\|\frac{u^r}{r}\Big\|_{L^\infty}\,dt\int_0^T \|u\|_{L^\infty}\,dt\\
&\quad\le CT + CT\int_0^T \Big\|\frac{u^r}{r}\Big\|_{L^\infty}\,dt + CT^{\frac{3}{2}}\int_0^T \Big\|\frac{u^r}{r}\Big\|_{L^\infty}\,dt\Big(\int_0^T \|u\|_{L^\infty}^2\,dt\Big)^{\frac{1}{2}}\\
&\quad\le CT + CT\int_0^T \Big\|\frac{u^r}{r}\Big\|_{L^\infty}\,dt + CT^3\Big(\int_0^T \Big\|\frac{u^r}{r}\Big\|_{L^\infty}\,dt\Big)^2 + C\int_0^T \|u\|_{L^\infty}^2\,dt.
\end{aligned}$$

Recall the following estimate (see [7, inequality (2.3)]):

$$\Big\|\frac{u^r}{r}\Big\|_{L^\infty} \le C\Big\|\frac{\omega^\theta}{r}\Big\|_{L^2}^{\frac{1}{2}}\Big\|\nabla\frac{\omega^\theta}{r}\Big\|_{L^2}^{\frac{1}{2}}.$$

This together with Hölder's inequality ensures

$$\begin{aligned}
T^\alpha\int_0^T \Big\|\frac{u^r}{r}\Big\|_{L^\infty}\,dt &\le CT^\alpha\int_0^T \|\Gamma\|_{L^2}^{\frac{1}{2}}\|\nabla\Gamma\|_{L^2}^{\frac{1}{2}}\,dt\\
&\le CT^{\frac{1}{2}+\alpha}\Big(\int_0^T \|\Gamma\|_{L^2}^2\,dt\Big)^{\frac{1}{4}}\Big(\int_0^T \|\nabla\Gamma\|_{L^2}^2\,dt\Big)^{\frac{1}{4}},
\end{aligned}$$

which implies

$$\begin{aligned}
T\int_0^T \Big\|\frac{u^r}{r}\Big\|_{L^\infty}\,dt &\le CT^{\frac{3}{2}}\Big(\int_0^T \|\Gamma\|_{L^2}^2\,dt\Big)^{\frac{1}{4}}\Big(\int_0^T \|\nabla\Gamma\|_{L^2}^2\,dt\Big)^{\frac{1}{4}}\\
&\le CT^2\Big(\int_0^T \|\Gamma\|_{L^2}^2\,dt\Big)^{\frac{1}{3}} + \frac{1}{10}\int_0^T \|\nabla\Gamma\|_{L^2}^2\,dt\\
&\le C + CT^6\int_0^T \|\Gamma\|_{L^2}^2\,dt + \frac{1}{10}\int_0^T \|\nabla\Gamma\|_{L^2}^2\,dt,
\end{aligned}$$

and

$$\begin{aligned}
T^3\Big(\int_0^T \Big\|\frac{u^r}{r}\Big\|_{L^\infty}\,dt\Big)^2 &\le CT^4\Big(\int_0^T \|\Gamma\|_{L^2}^2\,dt\Big)^{\frac{1}{2}}\Big(\int_0^T \|\nabla\Gamma\|_{L^2}^2\,dt\Big)^{\frac{1}{2}}\\
&\le CT^8\int_0^T \|\Gamma\|_{L^2}^2\,dt + \frac{1}{10}\int_0^T \|\nabla\Gamma\|_{L^2}^2\,dt.
\end{aligned}$$

Moreover, we use the following estimate (see [2, Proposition 4.2]):

$$\|u\|_{L^\infty} \le C\|\omega^\theta\|_{L^2}^{\frac{1}{2}}\|\nabla\omega^\theta\|_{L^2}^{\frac{1}{2}}$$

to get

$$C\int_0^T \|u\|_{L^\infty}^2\,dt \le C\int_0^T \|\omega^\theta\|_{L^2}\|\nabla\omega^\theta\|_{L^2}\,dt$$
$$\le C\int_0^T \|\omega^\theta\|_{L^2}^2\,dt + \frac{1}{2}\int_0^T \|\nabla\omega^\theta\|_{L^2}^2\,dt.$$

Therefore, we obtain

$$\int_0^T \left\|\frac{\rho}{r}\right\|_{L^2}^2 dt \le C + CT^8\int_0^T \|\Gamma\|_{L^2}^2\,dt + \frac{1}{5}\int_0^T \|\nabla\Gamma\|_{L^2}^2\,dt$$
$$+ C\int_0^T \|\omega^\theta\|_{L^2}^2\,dt + \frac{1}{2}\int_0^T \|\nabla\omega^\theta\|_{L^2}^2\,dt.$$

Inserting the above estimate into (3.5), we infer

$$\|\Gamma\|_{L^2}^2 + \int_0^T \|\nabla\Gamma\|_{L^2}^2\,dt$$
$$\le C\|u_0\|_{H^2} + C(1+T^8)\int_0^T (1+\|\nabla u\|_{L^2}^2)(\|\Gamma\|_{L^2}^2 + \|\omega^\theta\|_{L^2}^2)\,dt$$
$$+ \frac{1}{2}\int_0^T \|\nabla\omega^\theta\|_{L^2}^2\,dt. \tag{3.6}$$

On the other hand, by taking the L^2 inner product of equation (2.1) with ω^θ we get

$$\frac{1}{2}\frac{d}{dt}\|\omega^\theta\|_{L^2}^2 + \|\nabla\omega^\theta\|_{L^2}^2 + \left\|\frac{\omega^\theta}{r}\right\|_{L^2}^2 + \int |u|^{\beta-1}(\omega^\theta)^2\,dx$$
$$= \int \frac{u^r}{r}(\omega^\theta)^2\,dx - \int \partial_r\rho\,\omega^\theta\,dx$$
$$- (\beta-1)\int |u|^{\beta-3}((u^r)^2\partial_z u^r - (u^z)^2\partial_r u^z - u^r u^z\partial_r u^r$$
$$- u^r u^z\partial_z u^z)\omega^\theta\,dx. \tag{3.7}$$

Before proceeding, let us recall the following well-known Biot–Savart law (see [3, inequality (3.8)]):

$$\|\nabla u\|_{L^p} \le C\frac{p^2}{p-1}\|\omega\|_{L^p}, \quad \forall 1 < p < +\infty.$$

From this along with Hölder's inequality and Young's inequality, we have

$$(\beta-1)\int |u|^{\beta-3}((u^r)^2\partial_z u^r - (u^z)^2\partial_r u^z - u^r u^z\partial_r u^r - u^r u^z\partial_z u^z)\omega^\theta\,dx$$
$$\le (\beta-1)\int |u|^{\beta-1}|\nabla u|\,|\omega^\theta|\,dx$$

$$
\begin{aligned}
&\le C\| |u|^{\beta-1}\|_{L^{\frac{\beta+1}{\beta-1}}}\|\nabla u\|_{L^{\beta+1}}\|\omega^\theta\|_{L^{\beta+1}}\\
&\le C\|u\|_{L^{\beta+1}}^{\beta-1}\|\omega^\theta\|_{L^{\beta+1}}^2\\
&\le C\|u\|_{L^{\beta+1}}^{\beta-1}\|\omega^\theta\|_{L^2}^{\frac{5-\beta}{\beta+1}}\|\nabla\omega^\theta\|_{L^2}^{\frac{3(\beta-1)}{\beta+1}}\\
&\le \frac{1}{10}\|\nabla\omega^\theta\|_{L^2}^2 + C\|\omega^\theta\|_{L^2}^2\|u\|_{L^{\beta+1}}^{\frac{2(\beta-1)(\beta+1)}{5-\beta}}
\end{aligned}
$$

and

$$
\begin{aligned}
\int \frac{u^r}{r}(\omega^\theta)^2\,dx &\le C\|u^r\|_{L^3}\Big\|\frac{\omega^\theta}{r}\Big\|_{L^2}\|\omega^\theta\|_{L^6}\\
&\le C\|u^r\|_{L^2}^{\frac12}\|\nabla u^r\|_{L^2}^{\frac12}\Big\|\frac{\omega^\theta}{r}\Big\|_{L^2}\|\nabla\omega^\theta\|_{L^2}\\
&\le C\|u\|_{L^2}\|\nabla u\|_{L^2}\Big\|\frac{\omega^\theta}{r}\Big\|_{L^2}^2 + \frac{1}{10}\|\nabla\omega^\theta\|_{L^2}^2.
\end{aligned}
$$

Applying integration by parts and Hölder's inequality again yields

$$
\begin{aligned}
-\int \partial_r\rho\,\omega^\theta\,dx &= -2\pi\int_{-\infty}^{\infty}\int_0^\infty \partial_r\rho\omega^\theta r\,dr\,dz = 2\pi\int_{-\infty}^{\infty}\int_0^\infty \rho\partial_r(\omega^\theta r)\,dr\,dz\\
&= 2\pi\int_{-\infty}^{\infty}\int_0^\infty \rho\partial_r\omega^\theta r\,dr\,dz + 2\pi\int_{-\infty}^{\infty}\int_0^\infty \rho\frac{\omega^\theta}{r}r\,dr\,dz\\
&= \int\Big(\rho\partial_r\omega^\theta + \rho\frac{\omega^\theta}{r}\Big)\,dx\\
&\le C\|\rho\|_{L^2}\|\nabla\omega^\theta\|_{L^2} + C\|\rho\|_{L^2}\Big\|\frac{\omega^\theta}{r}\Big\|_{L^2}\\
&\le C\|\rho\|_{L^2}^2 + \frac{1}{10}\Big(\|\nabla\omega^\theta\|_{L^2}^2 + \Big\|\frac{\omega^\theta}{r}\Big\|_{L^2}^2\Big).
\end{aligned}
$$

Putting the above estimate into (3.7), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|\omega^\theta\|_{L^2}^2 + \frac{7}{5}\|\nabla\omega^\theta\|_{L^2}^2 + \frac{9}{5}\Big\|\frac{\omega^\theta}{r}\Big\|_{L^2}^2 + 2\int |u|^{\beta-1}(\omega^\theta)^2\,dx\\
&\quad\le C\|\omega^\theta\|_{L^2}^2\|u\|_{L^{\beta+1}}^{\frac{2(\beta-1)(\beta+1)}{5-\beta}} + C\|u\|_{L^2}\|\nabla u\|_{L^2}\Big\|\frac{\omega^\theta}{r}\Big\|_{L^2}^2 + C\|\rho\|_{L^2}^2.
\end{aligned}
$$

Integrating the above inequality with respect to time, we get from Proposition 3.1 that

$$
\begin{aligned}
&\|\omega^\theta\|_{L^2}^2 + \frac{7}{5}\int_0^T \|\nabla\omega^\theta\|_{L^2}^2\,dt + \frac{9}{5}\int_0^T \Big\|\frac{\omega^\theta}{r}\Big\|_{L^2}^2\,dt\\
&\quad\le C\|u_0\|_{H^1} + C\int_0^T \|\omega^\theta\|_{L^2}^2\|u\|_{L^{\beta+1}}^{\frac{2(\beta-1)(\beta+1)}{5-\beta}} + C\int_0^T \|\nabla u\|_{L^2}\Big\|\frac{\omega^\theta}{r}\Big\|_{L^2}^2\,dt\\
&\quad\quad + C\|\rho_0\|_{L^2}^2 T.
\end{aligned}
$$

From this, together with (3.6), we deduce that

$$
\begin{aligned}
&\|\omega^\theta\|_{L^2}^2 + \|\Gamma\|_{L^2}^2 + \int_0^T \|\nabla\omega^\theta\|_{L^2}^2\,dt + \int_0^T \Big\|\frac{\omega^\theta}{r}\Big\|_{L^2}^2\,dt + \int_0^T \|\nabla\Gamma\|_{L^2}^2\,dt \\
&\quad \le C\|u_0\|_{H^2} + CT \\
&\qquad + C(1+T^8)\int_0^T \big(1 + \|u\|_{L^{\beta+1}}^{\frac{2(\beta-1)(\beta+1)}{5-\beta}} + \|\nabla u\|_{L^2}^2\big)(\|\Gamma\|_{L^2}^2 + \|\omega^\theta\|_{L^2}^2)\,dt.
\end{aligned}
$$

Noticing that $\frac{2(\beta-1)(\beta+1)}{5-\beta} \le \beta + 1$ when $1 \le \beta \le \frac{7}{3}$, it follows from the integral Gronwall inequality and (3.2) that

$$
\begin{aligned}
&\|\omega^\theta\|_{L^2}^2 + \|\Gamma\|_{L^2}^2 + \int_0^T \|\nabla\omega^\theta\|_{L^2}^2\,dt \\
&\quad + \int_0^T \Big\|\frac{\omega^\theta}{r}\Big\|_{L^2}^2\,dt + \int_0^T \|\nabla\Gamma\|_{L^2}^2\,dt \le C(T).
\end{aligned} \tag{3.8}
$$

Thanks to the Biot–Savart law, we get

$$
\|\nabla u\|_{L^2} \le C\|\omega^\theta\|_{L^2} \quad \text{and} \quad \|\nabla^2 u\|_{L^2} \le C\Big(\|\nabla\omega^\theta\|_{L^2} + \Big\|\frac{\omega^\theta}{r}\Big\|_{L^2}\Big),
$$

which along with (3.8) implies the desired result. This ends the proof. ∎

To obtain the H^2 estimate for (u,ρ) , we need to establish the following proposition.

Proposition 3.3. *Suppose $1 \le \beta \le \frac{7}{3}$. Let (u,ρ) be the smooth solution of* (1.1) *with $(u_0,\rho_0) \in H^2(\mathbb{R}^3)$ satisfying the assumptions of Theorem* 1.1. *Then there holds*

$$
\int_0^T \|\nabla u\|_{L^\infty}\,dt + \sup_{0\le t\le T} \|\nabla\rho\|_{L^p} \le C(T), \quad \forall\, 2 \le p \le 6.
$$

Proof. Recalling the equation for vorticity $\omega = \nabla\times u$, we have

$$
\partial_t\omega - \Delta\omega = -\nabla\times(u\cdot\nabla u) - \nabla\times(|u|^{\beta-1}u) + \nabla\times(\rho e_3).
$$

In order to obtain the estimate $\nabla\omega \in L^{\frac{4}{3}}(0,T;L^4(\mathbb{R}^3))$, we need to establish the $L_T^{\frac{4}{3}}L^4$ estimates of the terms $u\cdot\nabla u$, $|u|^{\beta-1}u$ and ρe_3. From the interpolation estimate $\|f\|_{L^\infty} \le C\|\nabla f\|_{L^2}^{\frac{1}{2}}\|\nabla^2 f\|_{L^2}^{\frac{1}{2}}$ and Proposition 3.2, one has

$$
\begin{aligned}
\int_0^T \|u\|_{L^\infty}^4\,dt &\le C\int_0^T \|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2\,dt \\
&\le C\sup_{0\le t\le T}\|\nabla u\|_{L^2}^2 \int_0^T \|\nabla^2 u\|_{L^2}^2\,dt \le C(T).
\end{aligned} \tag{3.9}
$$

Applying Hölder's inequality and (3.1) ensures that

$$\begin{aligned}\int_0^T \|u\cdot\nabla u\|_{L^4}^{\frac{4}{3}}\,dt &\le \int_0^T \|u\|_{L^\infty}^{\frac{4}{3}}\|\nabla u\|_{L^4}^{\frac{4}{3}}\,dt\\ &\le \left(\int_0^T \|u\|_{L^\infty}^4\,dt\right)^{\frac{1}{3}}\left(\int_0^T \|\nabla u\|_{L^4}^2\,dt\right)^{\frac{2}{3}}\\ &\le C\left(\int_0^T \|u\|_{L^\infty}^4\,dt\right)^{\frac{1}{3}}\left(\int_0^T \|\nabla^2 u\|_{L^2}^2\,dt\right)^{\frac{2}{3}}\le C(T),\end{aligned}$$

and

$$\int_0^T \|\rho e_3\|_{L^4}^{\frac{4}{3}} \le C\int_0^T \|\rho_0\|_{L^4}^{\frac{4}{3}}\,dt \le C.$$

Since $1\le\beta\le\frac{7}{3}$, we can obtain the following interpolation estimate:

$$\|f\|_{L^{4\beta}} \le C\|f\|_{L^4}^{\frac{1}{\beta}}\|f\|_{L^\infty}^{1-\frac{1}{\beta}}.$$

This together with Proposition 3.2 leads to

$$\begin{aligned}\int_0^T \||u|^{\beta-1}u\|_{L^4}^{\frac{4}{3}}\,dt &\le \int_0^T \|u\|_{L^{4\beta}}^{\frac{4}{3}\beta}\,dt\\ &\le C\int_0^T \|u\|_{L^4}^{\frac{4}{3}}\|u\|_{L^\infty}^{\frac{4}{3}\beta-\frac{4}{3}}\,dt\\ &\le C\sup_{0\le t\le T}\|\nabla u\|_{L^2}^{\frac{4}{3}}\int_0^T \|u\|_{L^\infty}^{\frac{4}{3}\beta-\frac{4}{3}}\,dt\\ &\le C\sup_{0\le t\le T}\|\nabla u\|_{L^2}^{\frac{4}{3}}\left(\int_0^T \|u\|_{L^\infty}^4\,dt\right)^{\frac{\beta-1}{3}}T^{\frac{4-\beta}{3}}\le C(T).\end{aligned}$$

With the above estimates in hand, we apply the $L_T^qL^p$ $(1<q,p<+\infty)$ estimates for the parabolic equation of singular integral and potentials (see [13, 22]) with $q=\frac{4}{3}$ and $p=4$ to get

$$\nabla\omega\in L^{\frac{4}{3}}(0,T;L^4(\mathbb{R}^3)).$$

By using the interpolation estimate $\|f\|_{L^\infty}\le C\|f\|_{L^2}^{\frac{1}{7}}\|\nabla f\|_{L^4}^{\frac{6}{7}}$ and Proposition 3.2, we infer

$$\begin{aligned}\int_0^T \|\nabla u\|_{L^\infty}\,dt &\le C\int_0^T \|\nabla u\|_{L^2}^{\frac{1}{7}}\|\nabla^2 u\|_{L^4}^{\frac{6}{7}}\,dt\\ &\le C\sup_{0\le t\le T}\|\nabla u\|_{L^2}^{\frac{1}{7}}\int_0^T \|\nabla\omega\|_{L^4}^{\frac{6}{7}}\,dt\end{aligned}$$

$$\leq C \sup_{0\leq t\leq T} \|\nabla u\|_{L^2}^{\frac{1}{7}} \Big(\int_0^T \|\nabla \omega\|_{L^4}^{\frac{4}{3}}\,dt\Big)^{\frac{9}{14}} T^{\frac{5}{14}}$$
$$\leq C(T).$$

On the other hand, applying the operator ∇ to the ρ equation of (1.1) gives

$$\partial_t \nabla\rho + u\cdot\nabla\nabla\rho = -\nabla u\cdot\nabla\rho.$$

For $2 \leq p \leq 6$, multiplying the above equation by $|\nabla\rho|^{p-2}\nabla\rho$ and integrating over $\mathbb{R}^3$, we obtain

$$\frac{1}{p}\frac{d}{dt}\|\nabla\rho\|_{L^p}^p = -\int \nabla u\cdot\nabla\rho|\nabla\rho|^{p-2}\nabla\rho\,dx \leq \|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^p}^p,$$

which implies

$$\frac{d}{dt}\|\nabla\rho\|_{L^p} \leq \|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^p}.$$

Thanks to Gronwall's inequality, we have

$$\|\nabla\rho\|_{L^p} \leq \|\nabla\rho_0\|_{L^p}\exp\Big(\int_0^T \|\nabla u\|_{L^\infty}\,dt\Big) \leq C(T).$$

This ends the proof of Proposition 3.3. ■

Finally, our target is to show the H^2 estimate for (u,ρ) and then complete the proof of Theorem 1.1.

PROPOSITION 3.4. *Suppose* $1 \leq \beta \leq \frac{7}{3}$. *Let* (u,ρ) *be the smooth solution of* (1.1) *with* $(u_0,\rho_0) \in H^2(\mathbb{R}^3)$, *which satisfies the conditions of Theorem* 1.1. *Then there holds*

$$\sup_{0\leq t\leq T}(\|u\|_{H^2}^2 + \|\rho\|_{H^2}^2) + \int_0^T \|u\|_{H^2}^2\,dt \leq C(T). \tag{3.10}$$

PROOF. Applying the operator ∇^2 to the u and ρ equations in system (1.1), we get

$$\begin{cases} \partial_t\nabla^2 u + u\cdot\nabla\nabla^2 u + \nabla\nabla^2 P - \Delta\nabla^2 u \\ \qquad = -\nabla^2(|u|^{\beta-1}u) - [\nabla^2, u\cdot\nabla]u + \nabla^2(\rho e_3), \\ \partial_t\nabla^2\rho + u\cdot\nabla\nabla^2\rho = -[\nabla^2, u\cdot\nabla]\rho, \end{cases} \tag{3.11}$$

where $[\mathcal{A},\mathcal{B}] := \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$ denotes the commutator between $\mathcal{A}$ and $\mathcal{B}$. Next we recall the following commutator estimate (see [11, Lemma 2.10]):

$$\begin{aligned} \|[\Lambda^s, f]g\|_{L^p} &= \|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \\ &\leq C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}), \end{aligned} \tag{3.12}$$

where $\Lambda := (-\Delta)^{\frac{1}{2}}$ and $1 < p_2, p_3 < \infty$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$.

Performing the standard L^2 estimate of the first equation of (3.11), we see

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla^2 u\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2 &= -\int \nabla^2(|u|^{\beta-1}u)\cdot\nabla^2 u\,dx - \int [\nabla^2, u\cdot\nabla]u\cdot\nabla^2 u\,dx \\ &\quad + \int \nabla^2(\rho e_3)\cdot\nabla^2 u\,dx \\ &:= \sum_{i=1}^{3} \mathrm{II}_i. \end{aligned}\tag{3.13}$$

For II_1, applying Hölder's inequality and Young's inequality leads to

$$\begin{aligned}\mathrm{II}_1 &= -\int \nabla^2|u|^{\beta-1}u\cdot\nabla^2 u\,dx - 2\int \nabla|u|^{\beta-1}\nabla u\cdot\nabla^2 u\,dx \\ &\quad - \int |u|^{\beta-1}\nabla^2 u\cdot\nabla^2 u\,dx \\ &= \int (\nabla|u|^{\beta-1}\nabla u\cdot\nabla^2 u + \nabla|u|^{\beta-1}u\cdot\nabla^3 u)\,dx \\ &\quad + 2\int |u|^{\beta-1}(\nabla^2 u\cdot\nabla^2 u + \nabla u\cdot\nabla^3 u)\,dx - \int |u|^{\beta-1}\nabla^2 u\cdot\nabla^2 u\,dx \\ &= -\int |u|^{\beta-1}\nabla^2 u\cdot\nabla^2 u\,dx - \int |u|^{\beta-1}\nabla u\cdot\nabla^3 u\,dx + \int \nabla|u|^{\beta-1}u\cdot\nabla^3 u\,dx \\ &\quad + 2\int |u|^{\beta-1}(\nabla^2 u\cdot\nabla^2 u + \nabla u\cdot\nabla^3 u)\,dx - \int |u|^{\beta-1}\nabla^2 u\cdot\nabla^2 u\,dx \\ &\le \int |u|^{\beta-1}|\nabla u|\,|\nabla^3 u|\,dx + (\beta-1)\int |u|^{\beta-2}|\nabla u|\,|u|\,|\nabla^3 u|\,dx \\ &\le C\|u\|_{L^\infty}^{\beta-1}\|\nabla u\|_{L^2}\|\nabla^3 u\|_{L^2} \\ &\le C\|u\|_{L^\infty}^{2(\beta-1)}\|\nabla u\|_{L^2}^2 + \frac{1}{8}\|\nabla^3 u\|_{L^2}^2 \\ &\le C\|u\|_{L^\infty}^4 + C\|\nabla u\|_{L^2}^{\frac{4}{3-\beta}} + \frac{1}{8}\|\nabla^3 u\|_{L^2}^2.\end{aligned}$$

We invoke (3.12) to deduce

$$|\mathrm{II}_2| \le \|[\nabla^2, u\cdot\nabla]u\|_{L^2}\|\nabla^2 u\|_{L^2} \le C\|\nabla u\|_{L^\infty}\|\nabla^2 u\|_{L^2}^2.$$

For II_3 , we have

$$|\mathrm{II}_3| = \left|\int \nabla(\rho e_3)\cdot\nabla^3 u\,dx\right| \le \|\nabla\rho\|_{L^2}\|\nabla^3 u\|_{L^2} \le C\|\nabla\rho\|_{L^2}^2 + \frac{1}{8}\|\nabla^3 u\|_{L^2}^2.$$

Inserting the above estimates into (3.13) implies

$$\begin{aligned}&\frac{d}{dt}\|\nabla^2 u\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2\\&\quad\le C\|u\|_{L^\infty}^4 + C\|\nabla u\|_{L^2}^{\frac{4}{3-\beta}} + C\|\nabla u\|_{L^\infty}\|\nabla^2 u\|_{L^2}^2 + C\|\nabla\rho\|_{L^2}^2,\end{aligned}$$

which together with Gronwall's inequality, Propositions 3.2 and 3.3 and (3.9) gives

$$\begin{aligned}\|\nabla^2 u\|_{L^2}^2 &+ \int_0^T \|\nabla^3 u(t)\|_{L^2}^2\,dt\\&\le C\left(\|\nabla^2 u_0\|_{L^2}^2 + \int_0^T (\|u\|_{L^\infty}^4 + \|\nabla u\|_{L^2}^{\frac{4}{3-\beta}} + \|\nabla\rho\|_{L^2}^2)\,dt\right)\\&\qquad\times\exp\left(C\int_0^T \|\nabla u\|_{L^\infty}\,dt\right)\\&\le C(T).\end{aligned}\tag{3.14}$$

Moreover, we give the estimate of $\nabla^2\rho$. Applying the standard L^2 estimate of the second equation of (3.11), we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla^2\rho\|_{L^2}^2 = -\int[\nabla^2, u\cdot\nabla]\rho\cdot\nabla^2\rho\,dx.$$

It follows from (3.12) that

$$\begin{aligned}&-\int[\nabla^2, u\cdot\nabla]\rho\cdot\nabla^2\rho\,dx\\&\quad\le \|[\nabla^2, u\cdot\nabla]\rho\|_{L^2}\|\nabla^2\rho\|_{L^2}\\&\quad\le C(\|\nabla u\|_{L^\infty}\|\nabla^2\rho\|_{L^2} + \|\nabla^2 u\|_{L^3}\|\nabla\rho\|_{L^6})\|\nabla^2\rho\|_{L^2}\\&\quad\le C(\|\nabla u\|_{L^\infty}\|\nabla^2\rho\|_{L^2} + \|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla^3 u\|_{L^2}^{\frac{1}{2}}\|\nabla\rho\|_{L^2})\|\nabla^2\rho\|_{L^2}\\&\quad\le C\|\nabla u\|_{L^\infty}\|\nabla^2\rho\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla^3 u\|_{L^2}^{\frac{1}{2}}\|\nabla^2\rho\|_{L^2}^2\\&\quad\le C\|\nabla u\|_{L^\infty}\|\nabla^2\rho\|_{L^2}^2 + C(\|\nabla^2 u\|_{L^2}^{\frac{2}{3}} + \|\nabla^3 u\|_{L^2}^2)\|\nabla^2\rho\|_{L^2}^2.\end{aligned}$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\|\nabla^2\rho\|_{L^2}^2 \le C(\|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_{L^2}^{\frac{2}{3}} + \|\nabla^3 u\|_{L^2}^2)\|\nabla^2\rho\|_{L^2}^2,$$

which along with Gronwall's inequality, Proposition 3.3 and (3.14) ensures

$$\begin{aligned}\|\nabla^2\rho\|_{L^2} &\le \|\nabla^2\rho_0\|_{L^2}\exp\left(C\int_0^T (\|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_{L^2}^{\frac{2}{3}} + \|\nabla^3 u\|_{L^2}^2)\,dt\right)\\&\le C(T).\end{aligned}\tag{3.15}$$

Consequently, combining Proposition 3.1, (3.14) and (3.15) yields the desired result (3.10), which completes the proof of Theorem 1.1. ∎

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References

[1] H. Abidi – G. Gui, Inviscid limit for axisymmetric Navier–Stokes–Boussinesq system. *Commun. Math. Sci.* **17** (2019), no. 6, 1625–1652. Zbl 1428.76027 MR 4052753

[2] H. Abidi – T. Hmidi – S. Keraani, On the global regularity of axisymmetric Navier–Stokes–Boussinesq system. *Discrete Contin. Dyn. Syst.* **29** (2011), no. 3, 737–756. Zbl 1208.35107 MR 2773149

[3] X. Ai – Z. Li, Global smooth solutions to the 3D non-resistive MHD equations with low regularity axisymmetric data. *Commun. Math. Sci.* **20** (2022), no. 7, 1979–1994. Zbl 1499.35492 MR 4504019

[4] X. Cai – Q. Jiu, Weak and strong solutions for the incompressible Navier–Stokes equations with damping. *J. Math. Anal. Appl.* **343** (2008), no. 2, 799–809. Zbl 1143.35349 MR 2401535

[5] D. Chae, Local existence and blow-up criterion for the Euler equations in the Besov spaces. *Asymptot. Anal.* **38** (2004), no. 3-4, 339–358. Zbl 1068.35097 MR 2072064

[6] D. Chae – H.-S. Nam, Local existence and blow-up criterion for the Boussinesq equations. *Proc. Roy. Soc. Edinburgh Sect. A* **127** (1997), no. 5, 935–946. Zbl 0882.35096 MR 1475638

[7] H. Chen – D. Fang – T. Zhang, Global axisymmetric solutions of three dimensional inhomogeneous incompressible Navier–Stokes system with nonzero swirl. *Arch. Ration. Mech. Anal.* **223** (2017), no. 2, 817–843. Zbl 1359.35127 MR 3590666

[8] D. Fang – W. Le – T. Zhang, Global solutions of 3D axisymmetric Boussinesq equations with nonzero swirl. *Nonlinear Anal.* **166** (2018), 48–86. Zbl 1379.35239 MR 3729546

[9] T. Hmidi – F. Rousset, Global well-posedness for the Navier–Stokes–Boussinesq system with axisymmetric data. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **27** (2010), no. 5, 1227–1246. Zbl 1200.35229 MR 2683758

[10] H. Houamed – M. Zerguine, On the global solvability of the axisymmetric Boussinesq system with critical regularity. *Nonlinear Anal.* **200** (2020), article no. 112003. Zbl 1455.76030 MR 4108437

[11] C. E. Kenig – G. Ponce – L. Vega, Well-posedness of the initial value problem for the Korteweg–de Vries equation. *J. Amer. Math. Soc.* **4** (1991), no. 2, 323–347. Zbl 0737.35102 MR 1086966

[12] S. Leonardi – J. Málek – J. Nečas – M. Pokorný, On axially symmetric flows in $\mathbf{R}^3$. *Z. Anal. Anwendungen* **18** (1999), no. 3, 639–649. Zbl 0943.35066 MR 1718156

[13] J. E. Lewis, Mixed estimates for singular integrals and an application to initial value problems in parabolic differential equations. In *Singular Integrals (Proc. Sympos. Pure Math., Chicago, Ill., 1966)*, pp. 218–231, American Mathematical Society, Providence, RI, 1967. Zbl 0177.38604 MR 0234130

[14] Z. Li, Global well-posedness of the 2D Euler–Boussinesq system with stratification effects. *Math. Methods Appl. Sci.* **40** (2017), no. 14, 5212–5221. Zbl 1387.35478 MR 3689259

[15] Z. Li – W. Liu – Q. Zhou, Conditional regularity for the 3D damped Boussinesq equations with zero thermal diffusion. *Bull. Braz. Math. Soc. (N.S.)* **55** (2024), no. 3, article no. 36. MR 4769154

[16] Z. Li – X. Pan, One component regularity criteria for the axially symmetric MHD–Boussinesq system. *Discrete Contin. Dyn. Syst.* **42** (2022), no. 5, 2333–2353. Zbl 1490.35337 MR 4405270

[17] A. J. Majda – A. L. Bertozzi, *Vorticity and incompressible flow*. Cambridge Texts Appl. Math. 27, Cambridge University Press, Cambridge, 2002. Zbl 0983.76001 MR 1867882

[18] C. Miao – X. Zheng, On the global well-posedness for the Boussinesq system with horizontal dissipation. *Comm. Math. Phys.* **321** (2013), no. 1, 33–67. Zbl 1307.35233 MR 3089663

[19] J. Pedloski, *Geophysical fluid dynamics*. Springer, New York, 1987. Zbl 0713.76005

[20] O. Sawada – Y. Taniuchi, On the Boussinesq flow with nondecaying initial data. *Funkcial. Ekvac.* **47** (2004), no. 2, 225–250. Zbl 1118.35037 MR 2108674

[21] E. S. Titi – S. Trabelsi, Global well-posedness of a three-dimensional Brinkman–Forchheimer–Bénard convection model in porous media. *Discrete Contin. Dyn. Syst. Ser. S* **17** (2024), no. 5-6, 1857–1875. Zbl 1540.35296 MR 4762564

[22] W. von Wahl, The equation $u' + A(t)u = f$ in a Hilbert space and L^p-estimates for parabolic equations. *J. London Math. Soc. (2)* **25** (1982), no. 3, 483–497. Zbl 0493.35050 MR 0657505

[23] P. Wang – Z. Guo, A regularity criterion for the 3D axisymmetric Boussinesq equations with non-zero swirl. *J. Math. Phys.* **64** (2023), no. 5, article no. 051502. Zbl 1512.35491 MR 4584657

[24] S. Wang – Y. Wang – J. Liu, Regularity criteria to the incompressible axisymmetric Boussinesq equations. *Appl. Math. Lett.* **112** (2021), article no. 106800. Zbl 1454.35304 MR 4155257

[25] Z. Wen – Z. Ye, On the global existence of strong solution to the 3D damped Boussinesq equations with zero thermal diffusion. *Z. Anal. Anwend.* **37** (2018), no. 3, 341–348. Zbl 1397.35232 MR 3826620

[26] H. Yu, Axisymmetric solutions to the convective Brinkman–Forchheimer equations. *J. Math. Anal. Appl.* **520** (2023), no. 2, article no. 126892. Zbl 1509.76091 MR 4514793

[27] Z. Zhang – X. Wu – M. Lu, On the uniqueness of strong solution to the incompressible Navier–Stokes equations with damping. *J. Math. Anal. Appl.* **377** (2011), no. 1, 414–419. Zbl 1210.35181 MR 2754840

[28] X. Zhao, Decay estimates for three-dimensional Navier–Stokes equations with damping. *Nonlinear Anal. Real World Appl.* **60** (2021), article no. 103282. Zbl 1472.35274 MR 4198296

[29] X. Zhong, Global well-posedness to the incompressible Navier–Stokes equations with damping. *Electron. J. Qual. Theory Differ. Equ.* (2017), article no. 62. Zbl 1413.35388 MR 3702503

[30] Y. Zhou, Regularity and uniqueness for the 3D incompressible Navier–Stokes equations with damping. *Appl. Math. Lett.* **25** (2012), no. 11, 1822–1825. Zbl 1426.76095 MR 2957760

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