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Action on the circle at infinity of foliations of $\mathbb{R}^2$

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Abstract. This paper provides a canonical compactification of the plane $\mathbb{R}^2$ by adding a circle at infinity associated to a countable family of singular foliations or laminations (under some hypotheses), generalizing an idea by Mather (1982). Moreover, any homeomorphism of $\mathbb{R}^2$ preserving the foliations extends on the circle at infinity. Then, this paper provides conditions ensuring the minimality of the action on the circle at infinity induced by an action on $\mathbb{R}^2$ preserving one foliation or two transverse foliations. In particular, the action on the circle at infinity associated to an Anosov flow X on a closed 3-manifold is minimal if and only if X is non-$\mathbb{R}$-covered.

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Contents

1. Introduction

1.1. General presentation. There are many ways to compactify the plane $\mathbb{R}^2$, the simplest one being the Alexandrov compactification by a point at infinity, and $\mathbb{R}^2 \cup \{\infty\}$ is the topological sphere $\mathbb{S}^2$. This compactification is canonical and does not depend on any extra structure on $\mathbb{R}^2$. That is its strength, but also its weakness as it does not bring any information on any structure we endow $\mathbb{R}^2$ with.

Another very natural and usual compactification of $\mathbb{R}^2$ is by adding a circle at infinity so that $\mathbb{R}^2 \cup \mathbb{S}^1$ is the disc $\mathbb{D}^2$. This compactification is not canonical: it consists in a homeomorphism $h\colon \mathbb{R}^2 \to \mathring{\mathbb{D}}^2$, where $\mathring{\mathbb{D}}^2$ is the open disc. Two homeomorphisms h_1, h_2 define the same compactification if $h_2 \circ h_1^{-1}\colon \mathring{\mathbb{D}}^2 \to \mathring{\mathbb{D}}^2$ extends on $\mathbb{S}^1 = \partial\mathbb{D}^2$ as a homeomorphism of $\mathbb{D}^2$. There are uncountably many such compactifications. Many authors have built such a compactification associated to a foliation of the plane $\mathbb{R}^2$, the first being by Wilfred Kaplan in 1941 [11].

Here, we start by recalling Mather [13] canonical compactification of the plane $\mathbb{R}^2$, endowed with a foliation $\mathcal{F}$, by a circle at infinity $\mathbb{S}^1_{\mathcal{F}}$. Then, we explore the flexibility of this construction for extension to more general objects. Thus, we provide an elementary (nothing sophisticated), simple (nothing too complicated), and unified construction which associates a compactification $\mathbb{D}^2_{\mathcal{F}}$ of the plane $\mathbb{R}^2$ by the disc $\mathbb{D}^2$ to a countable family $\mathcal{F} = \{\mathcal{F}_i\}$ of foliations, non-singular or with singular points of saddle type, which are pairwise transverse or at least have some kind of weak transversality condition at infinity; see the precise statements below. The boundary $\partial\mathbb{D}^2_{\mathcal{F}}$ is called *the circle at infinity* of $\mathcal{F}$ and is denoted by $\mathbb{S}^1_{\mathcal{F}}$. This compactification is unique, in the sense that the identity on $\mathbb{R}^2$ extends as a homeomorphism on the circles at infinity of two such compactifications.

To give a concrete example, Corollary 5.3 builds this canonical compactification $\mathbb{D}^2_{\mathcal{F}}$ associated to any countable family $\mathcal{F} = \{\mathcal{F}_i\}$ of singular foliations, where each $\mathcal{F}_i$ is directed by a polynomial vector field on $\mathbb{R}^2$ whose singular points are hyperbolic saddles.

The uniqueness of the compactification implies that any homeomorphism of $\mathbb{R}^2$ preserving $\mathcal{F}$ (that is, permuting the $\mathcal{F}_i$) extends as a homeomorphism of the compactification $\mathbb{D}^2_{\mathcal{F}}$, inducing a homeomorphism of the circle at infinity $\mathbb{S}^1_{\mathcal{F}}$.

1.2. Mather idea for building the circle at infinity. The common setting for this unified construction is families of *rays*, where a ray is a proper topological embedding of $[0, +\infty)$ on $\mathbb{R}^2$. We require that the *germs of the rays* in the family be pairwise disjoint, meaning that the intersection between any two distinct rays is compact. The key idea is that a set of rays in $\mathbb{R}^2$ whose germs are pairwise disjoint is *totally cyclically ordered*, and we will use this cyclic order for building the circle at infinity.

The key technical result (essentially due to [13]) is as follows.

Theorem 1.1. *Let $\mathcal{R}$ be a family of rays in $\mathbb{R}^2$ whose germs are pairwise disjoint. Let $\mathcal{E} \subset \mathcal{R}$ be a countable subset which is* separating for the cyclic order*; that is, any non-degenerate interval contains a point in $\mathcal{E}$ (see Definition 2.2).*

Then, there is a compactification of $\mathbb{R}^2$ by the disc $\mathbb{D}^2$ so that

- *any ray of $\mathcal{R}$ tends to a point of the circle at infinity $\partial\mathbb{D}^2 = \mathbb{S}^1$,*
- *any two distinct rays of $\mathcal{R}$ tend to distinct points of $\mathbb{S}^1$,*
- *the points of $\mathbb{S}^1$ which are the limit point of a ray in $\mathcal{R}$ are dense in $\mathbb{S}^1$.*

Furthermore, this compactification is unique up to a homeomorphism of $\mathbb{D}^2$ and does not depend on the separating countable set $\mathcal{E}$.

Furthermore, Theorem 2.1 provides such a canonical compactification for a countable union $\mathcal{R} = \bigcup \mathcal{R}_i, i \in I \subset \mathbb{N}$, of families $\mathcal{R}_i$ of rays, assuming that

- the germs of rays in $\mathcal{R}$ are pairwise disjoint,
- each family $\mathcal{R}_i$ admits a countable separating subset $\mathcal{E}_i$.

The difficulty here is that $\mathcal{R}$ by itself may not admit any separating family. The idea for solving this problem consists in considering a natural equivalence relation on $\mathcal{R}$, identifying the rays which cannot be separated.

1.3. Countable families of transverse foliations. A natural setting where we will apply this general construction is (at most countable) families of transverse foliations on the plane $\mathbb{R}^2$. Notice that any *half-leaf* of a (non-singular) foliation of $\mathbb{R}^2$ is a ray. An *end of leaf* is the germ at infinity of a half-leaf. In this setting, we get the following theorem.

Theorem 1.2. *Let $\mathcal{F} = \{\mathcal{F}_i\}_{i \in I \subset \mathbb{N}}$ be an at most countable family of pairwise transverse foliations on the plane $\mathbb{R}^2$.*

There is a compactification $\mathbb{D}^2_{\mathcal{F}} \simeq \mathbb{D}^2$ of $\mathbb{R}^2$ by adding a circle $\mathbb{S}^1_{\mathcal{F}} = \partial\mathbb{D}^2_{\mathcal{F}}$ with the following properties.

- *Any end of leaf tends to a point of the circle at infinity $\mathbb{S}^1_{\mathcal{F}}$.*
- *The set of ends of leaves tending to the same point of $\mathbb{S}^1_{\mathcal{F}}$ is at most countable.*
- *For any non-empty open subset $O \subset \mathbb{S}^1_{\mathcal{F}}$, the set of ends of leaves having their limit in O is uncountable.*

This compactification with these three properties is unique, up to a homeomorphism of $\mathbb{D}^2_{\mathcal{F}}$.

The circle $\mathbb{S}^1_{\mathcal{F}}$ is called *the circle at infinity* of the family $\mathcal{F} = \{\mathcal{F}_i\}_{i \in I \subset \mathbb{N}}$.

Remark 1.1. The countability of the set of ends tending to the same point implies that

- the two ends of a given leaf always have distinct limits on $\mathbb{S}^1_{\mathcal{F}}$,
- if two leaves L_1, L_2 of the same foliation $\mathcal{F}_i$ have the same pair of limits of ends, they are equal (see Lemma 4.6).

Recall that foliations of $\mathbb{R}^2$ may have leaves which are *not separated* from each other. The leaves which are separated from any other leaves are called *regular leaves*. At most countably many leaves are not regular (see here Lemma 3.3).

Proposition 1.2. *Let $\mathcal{F} = \{\mathcal{F}_i\}_{i\in I\subset\mathbb{N}}$ be an at most countable family of pairwise transverse foliations on the plane $\mathbb{R}^2$. Any two distinct ends of regular leaves of the same foliation $\mathcal{F}_i$ tend to two distinct points of $\mathbb{S}^1_{\mathcal{F}}$.*

Now, in the setting of Theorem 1.2, we can apply this theorem to each foliation $\mathcal{F}_i$, $i \in I$, so that we get a family of compactifications $\mathbb{D}^2_{\mathcal{F}_i}$. In fact, we get a compactification $\mathbb{D}^2_J$ for any subfamily $J \subset I$ leading to an uncountable set of (maybe distinct) compactifications of $\mathbb{R}^2$ by the disc $\mathbb{D}^2$. (Example 2.18 provides a simple example where these compactifications $\mathbb{D}^2_J$, for $J \subset I$, are pairwise distinct and uncountably many.)

These compactifications are easily related: for any subfamily $J \subset I$ the identity map on $\mathbb{R}^2$ extends in a unique way by continuity as a projection

$$\Pi_{I,J}\colon \mathbb{D}^2_{\mathcal{F}} = \mathbb{D}^2_I \to \mathbb{D}^2_J,$$

which simply collapses the intervals in $\mathbb{S}^1_I$ which do not contain any limit of an end of a regular leaf of a foliation $\mathcal{F}_j$, $j \in J$.

We will also see in a simple example that the assumption of *at most countability* of the family I of foliations cannot be erased: for instance, the conclusion in Theorem 1.2 is false for the family of all affine foliations (by parallel straight lines) of $\mathbb{R}^2$, parametrized by $\mathbb{RP}^1$ (see Example 2.17).

Example 4.18 and Lemma 4.19 present a simple example where generic points (i.e., points in a residual set) of the circle at infinity $\mathbb{S}^1_{\mathcal{F}}$ of a foliation $\mathcal{F}$ are not the limit of any end of leaf of $\mathcal{F}$. In this example, there are also points in a dense subset of $\mathbb{S}^1_{\mathcal{F}}$ which are limits of 2 distinct ends of leaves.

Lemmas 4.8 and 4.9 characterize the points p on the circle at infinity $\mathbb{S}^1_{\mathcal{F}}$, where $\mathcal{F}$ is a foliation of $\mathbb{R}^2$, which are limits of several ends of leaves: the rays arriving at p are ordered as an interval of $\mathbb{Z}$ and two successive ends bound a hyperbolic sector.

Corollary 5.10 generalizes Lemmas 4.8 and 4.9 to the case of a countable family $\mathcal{F} = \{\mathcal{F}_i\}$ of transverse foliations and gives a complete description of the points in $\mathbb{S}^1_{\mathcal{F}}$ which are limits of several ends of leaves of the same $\mathcal{F}_i$.

1.4. Countable families of non-transverse or singular foliations. This construction can be generalized easily to the setting of families of non-transverse or singular foliations. Let us present the most general setting we consider here.

The foliations we consider admit singular points which are *saddle points with k-separatrices* (also called *k-prongs singularity*), $k > 1$, the case $k = 2$ corresponding to non-singular points.

In this setting, an *end of leaf* is a ray of $\mathbb{R}^2$ disjoint from the singular points and contained in a leaf.

Theorem 1.3. *Let $\mathcal{F} = \{\mathcal{F}_i\}$, $i \in I \subset \mathbb{N}$, be a family of singular foliations of $\mathbb{R}^2$ whose singular points are each a saddle with k-separatrices with $k > 2$. We assume that, given any two ends L_1, L_2 of leaves,*

- *either the germs of L_1 and L_2 are disjoint*
- *or the germs of L_1 and L_2 coincide.*

Then, there is a compactification $\mathbb{D}^2_{\mathcal{F}} \simeq \mathbb{D}^2$ of $\mathbb{R}^2$ by adding a circle $\mathbb{S}^1_{\mathcal{F}} = \partial\mathbb{D}^2_{\mathcal{F}}$ with the following properties.

- *Any end of leaf tends to a point of the circle at infinity $\mathbb{S}^1_{\mathcal{F}}$.*
- *The set of ends of leaves tending to a same point of $\mathbb{S}^1_{\mathcal{F}}$ is at most countable.*
- *For any non-empty open subset $O \subset \mathbb{S}^1_{\mathcal{F}}$, the set of ends of leaves having their limit in O is uncountable.*

This compactification with these three properties is unique, up to a homeomorphism of $\mathbb{D}^2_{\mathcal{F}}$.

The hypothesis that the germs of ends of leaves are either equal or disjoint means that if the intersection of two leaves is not bounded, then these two leaves coincide on a half-leaf. One easily checks that transverse foliations satisfy this hypothesis so that Theorem 1.2 is a straightforward corollary of Theorem 1.3.

As a simple and natural example, we will see that any countable family

$$\mathcal{F} = \{\mathcal{F}_i\}$$

of singular foliations, directed by polynomial vector fields on $\mathbb{R}^2$ whose singular points are hyperbolic saddles, satisfies the hypotheses of Theorem 1.3: this will prove Corollary 5.3 already mentioned above.

Another natural setting where Theorem 1.3 applies is provided by pseudo-Anosov flows on closed 3-manifold M (see Section 5.2).

1.5. Laminations. The construction of the circle at infinity for foliations cannot be extended without hypotheses to the case of laminations of $\mathbb{R}^2$, as half-leaves of laminations may fail to be rays, and can even be recurrent; see, for instance, Example 6.1.

Theorems 6.1 and 6.3 provide a generalization of this construction to closed orientable laminations with no compact leaves and with uncountably many leaves. This generalization is not as satisfactory as in the case of foliations, and we discuss some of the issues in Section 6. In particular, Theorem 6.2 provides another canonical compactification, which holds also for countable oriented laminations with no compact leaves.

1.6. Minimality of the action on the circle at infinity. We now consider group actions $H \subset \mathrm{Homeo}(\mathbb{R}^2)$ on $\mathbb{R}^2$ preserving 1 or 2 transverse foliations $\mathcal{F}_i$. The action of H extends canonically on the circle at infinity, and we will consider the following question.

Question 1.3. Under what conditions on H and on the foliations $\mathcal{F}_i$ can we ensure that the action induced on $\mathbb{S}^1_{\{\mathcal{F}_i\}}$ is minimal?

Reference [5] provides some answer to this question but with a totally distinct point of view.

Our main result, for the case of 1 foliation, is the following theorem.

Theorem 1.4. *Let $\mathcal{F}$ be a foliation of $\mathbb{R}^2$ and $H \subset \mathrm{Homeo}(\mathbb{R}^2)$ a group of homeomorphisms preserving $\mathcal{F}$. We assume that, for any leaf L, the union of its images $H(L)$ is dense in $\mathbb{R}^2$.*

Then, the following two properties are equivalent:

(1) *the action induced by H on the circle at infinity is minimal;*

(2) *there are pairs of distinct leaves (L_1, L_2) and (L_3, L_4) so that L_1 and L_2 are not separated from above and L_3 and L_4 are not separated from below.*

We will also generalize Theorem 1.4 for families of transverse foliations.

1.7. Action on the circle at infinity of an Anosov flow. Finally, we will consider the setting of an *Anosov flow* X on a closed 3-manifold M.

Remark 1.4. In this setting, it is known that $\pi_1(M)$ acts on $\mathbb{S}^1$ by orientation-preserving homeomorphisms; see work of Calegari Dunfield [5] inspired by an unpublished work of Thurston [14]. This work follows distinct ideas that those presented here.

Another construction of this circle at infinity (called *ideal circle boundary*) is given in [6] for pseudo-Anosov flows.

Barbot and Fenley [1, 8] show that the lift $\tilde{X}$ of X is conjugated to the constant vector field $\frac{\partial}{\partial x}$ on $\mathbb{R}^3$ so that the $\tilde{X}$-orbit space is a plane

$$\mathcal{P}_X \simeq \mathbb{R}^2.$$

This plane $\mathcal{P}_X$ is endowed with two transverse foliations F^s, F^u which are the projections of the stable and unstable foliations of X lifted on $\mathbb{R}^3$. Thus, $(\mathcal{P}_X, F^s, F^u)$ is the *bifoliated plane* associated to X. Furthermore, the fundamental group $\pi_1(M)$ acts on $\mathcal{P}_X$ and its action preserves both foliations F^s and F^u. This action induces a natural action of $\pi_1(M)$ on the circles at infinity $\mathbb{S}^1_{F^s}$, $\mathbb{S}^1_{F^u}$, and $\mathbb{S}^1_{F^s,F^u}$.

A folklore conjecture asserts that two Anosov flows are orbitally equivalent if and only if they induce the same action on the circle at infinity of $\{F^s, F^u\}$; see [1] for a result in this direction. This conjecture has been recently announced to be proved for transitive flows in [2].

References [1, 8] show that every leaf of F^s is regular if and only if every leaf of F^u is regular, in which case the Anosov flow X is called $\mathbb{R}$-*covered*. Our main result in that setting is the following theorem.

Theorem 1.5. *Let X be an Anosov flow on a closed 3-manifold and $(\mathcal{P}_X, F^s, F^u)$ its bifoliated plane. Let $\mathbb{D}^2_{F^s,F^u}$, $\mathbb{D}^2_{F^s}$, and $\mathbb{D}^2_{F^u}$ be the compactifications associated to, respectively, the pair of foliations F^s, F^u, the foliation F^s, and the foliation F^u. Then,*

(1) $\mathbb{D}^2_{F^s,F^u} = \mathbb{D}^2_{F^s} = \mathbb{D}^2_{F^u}$ *unless X is orbitally equivalent to the suspension of an Anosov diffeomorphism of the torus $\mathbb{T}^2$,*

(2) *the action of $\pi_1(M)$ on the circles at infinity $\mathbb{S}^1_{F^s,F^u}$ (or equivalently $\mathbb{S}^1_{F^s}$ or $\mathbb{S}^1_{F^u}$) is minimal if and only if X is not $\mathbb{R}$-covered.*

When X is assumed to be transitive, this result is a simple consequence of Theorem 1.4 above and a result by Fenley [9] ensuring that, assuming X is non-$\mathbb{R}$-covered, then F^s and F^u admit non-separated leaves from above and non-separated leaves from below. The proof of Theorem 1.5, when X is not assumed to be transitive, is certainly the most technically difficult argument of the paper and is based on a description of hyperbolic basic sets for flows on 3-manifolds.

Theorem 1.5 implies that the minimality of the action on the circle at infinity is not related with the transitivity of the flow. However, in the case of transitive Anosov flows, according to [2], the action on the circle at infinity characterizes the dynamics of the flow. This leads to the following question.

Question 1.5. What property of the action of $\pi_1(M)$ on the circle at infinity $\mathbb{S}^1_{F^s,F^u}$ implies the transitivity of X?

Can we find the transverse tori by looking at the action of $\pi_1(M)$ on the circle at infinity?

Remark 1.6. I have no doubt that the proof of Theorem 1.5 extends to pseudo-Anosov flows, showing the minimality of the action at infinity unless the flow was a non-singular $\mathbb{R}$-covered Anosov flow, but showing this would require one to adapt each step to this new setting.

1.8. Organization of the paper. Section 2 first proves Theorem 1.1: the compactification is obtained by adding an abstract circle to $\mathbb{R}^2$, where this circle is a completion of the natural cyclic order associated to any family of pairwise disjoint rays on $\mathbb{R}^2$. Then, it considers a countable family of families of rays for proving Theorem 2.1. The trick here is to define the suitable equivalence relation between rays that cannot be separated and which therefore will go to the same point at infinity.

Section 3 recalls basic properties and definitions on foliations and singular foliations on the plane.

Section 4 builds the compactification $\mathbb{D}^2_{\mathcal{F}}$ associated to a singular foliation whose singular points are saddles. Then, in the case of a non-singular foliation, it provides a complete description of a neighborhood of the points of the circle at infinity which are the limits of several ends of leaves. It also provides several examples, for instance, where generic points of the circle at infinity are not the limit of any end of leaf.

Section 5 proves Theorem 1.3 (and therefore Theorem 1.2) by applying Theorem 2.1 to the rays in a family of singular foliations.

Section 6 considers the case of laminations of $\mathbb{R}^2$. After providing examples of laminations for which the ends of leaves are not rays (and therefore the construction cannot apply), this section focuses on orientable laminations without compact leaves, for which the ends of leaves are rays. Then, it provides two compactifications associated to such laminations, each of them being unique for a choice of rules: one may take into account, or not, the countable set of isolated leaves.

Section 7 considers actions on $\mathbb{R}^2$ preserving a foliation or a pair of transverse foliations and provides conditions on this action for ensuring the minimality of the induced action on the circle at infinity.

Section 8 proves the minimality of the action of the circle at infinity associated to a transitive non-$\mathbb{R}$-covered Anosov flow (Theorem 1.5 in the transitive case) after recalling some classical background on Anosov flows of closed 3-manifolds.

Finally, Section 9 ends the proof of Theorem 1.5 by considering non-transitive Anosov flows. The minimality of the action on the circle at infinity is proved after recalling basic notions and classical properties of non-transitive Anosov flows.

2. Circles at infinity for families of rays on the plane

2.1. Cyclic order. Let X be a set. A *total cyclic order* on X is a map $\theta\colon X^3 \to \{-1, 0, +1\}$ with the following properties.

- $\theta(x, y, z) = 0$ if and only if $x = y$ or $y = z$ or $x = z$.
- $\theta(x, y, z) = -\theta(y, x, z) = -\theta(x, z, y)$ for every (x, y, z).

- For every $x \in X$, the relation on $X \setminus \{x\}$ defined by

$$y < z \Leftrightarrow \theta(x, y, z) = +1$$

is a total order.

The emblematic example is the following.

Example 2.1. The oriented circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is totally cyclically ordered by the relation θ defined as follows:

$$\theta(x, y, z) = +1$$

if and only if, y belongs to the interior of the positively oriented simple arc staring at x and ending at z.

If θ is a total cyclic order, then for $x \neq z$ we define the interval (x, z) by

$$(x, z) = \{y \mid \theta(x, y, z) = 1\}.$$

We define the semi-closed and closed intervals $[x, z)$, $(x, z]$, and $[x, z]$ by adding the corresponding extremities x or z to the interval (x, z).

We say that y is *between* x and z if $y \in (x, z)$.

The following notion of *separating set* will be fundamental all along this work.

Definition 2.2. Let X be a set endowed with a total cyclic order. A subset $\mathcal{E} \subset X$ is said to be *separating* if given any distinct $x, z \in X$ there is $y \in \mathcal{E}$ between x and z.

We will use the following easy exercise in the topology of $\mathbb{R}$ and $\mathbb{S}^1$.

Proposition 2.3. *Let X be a set endowed with a total cyclic order. Assume that there is a countable subset $\mathcal{E} \subset X$ which is separating.*

Then, there is a bijection φ of X with a dense subset $Y \subset \mathbb{S}^1$ which is strictly increasing for the cyclic orders of X and of $\mathbb{S}^1$. Furthermore, this bijection is unique up to composition by a homeomorphism of $\mathbb{S}^1$.

The argument is classical but short and beautiful, and I have no references for this precise statement. So, let me present it.

Proof. One builds a bijection ϕ of $\mathcal{E}$ to a countable dense subset $\mathcal{D} \subset \mathbb{S}^1$ by induction, as follows: one chooses an indexation of $\mathcal{E} = \{e_i, i \in \mathbb{N}\}$ and of $\mathcal{D} = \{d_i, i \in \mathbb{N}\}$. One defines

- $\phi(e_0) = d_0$, $\phi(e_1) = d_1$, $i(0) = j(0) = 0$, and $i(1) = j(1) = 1$.
- Consider e_2; it belongs either in (e_0, e_1) or in (e_1, e_0) and we choose $\phi(e_2)$ being $d_{j(2)}$, where $j(2)$ is the infimum of the d_i in the corresponding interval (d_0, d_1) or (d_1, d_0). One denotes $i(2) = 2$.

- Consider now $j(3) = \inf \mathbb{N} \setminus \{0, 1, j(2)\}$ and define $\phi^{-1}(d_{j(3)}) = e_{i(3)}$, where $i(3)$ is the infimum of the $i \notin \{0, 1, 2\}$ so that the position of $e_{i(3)}$ with respect to e_0, e_1, e_2 is the same as the position of $d_{j(3)}$ with respect to $d_0, d_1, d_{j(2)}$.
- ...
- Choose $i(2n) = \inf \mathbb{N} \setminus \{i(k) \mid k < 2n\}$ and $\phi(e_{i(2n)})$ is $d_{j(2n)}$, where $j(2n)$ is the infimum of the j so that d_j has the same position with respect to the $d_{j(k)}, k < 2n$, as $e_{i(2n)}$ with respect to the $e_{i(k)}$.
- Choose $j(2n+1) = \inf \mathbb{N} \setminus \{j(k) \mid k < 2n+1\}$ and $\phi^{-1}(d_{j(2n+1)})$ is $e_{i(2n+1)}$, where $i(2n+1)$ is the infimum of the i so that e_i has the same position with respect to the $e_{i(k)}, k < 2n+1$, as $d_{j(2n+1)}$ with respect to the $d_{j(k)}$.

At each step of this construction, one uses the separation property of $\mathcal{E}$ and $\mathcal{D}$ for ensuring the existence of the point in the specified position.

Once we built ϕ on $\mathcal{E}$, it extends in a unique increasing way on X. Then, the separation property of $\mathcal{E}$ implies that this extension is injective. ∎

Remark 2.4. Assume that Z, θ is a set endowed with a total cyclic order, and $\mathcal{E} \subset X \subset Z$ are subsets so that $\mathcal{E}$ is separating for X, θ.

Let $\varphi: X \to Y$ be the map given by Proposition 2.3. Then, ϕ extends in a unique way as an (not strictly) increasing map $\Phi: Z \to \mathbb{S}^1$: $\Phi(y)$ is between $\Phi(x)$ and $\Phi(z)$ only if y is between x and z.

The non-injectivity of the map Φ is determined as follows. Consider distinct points $x \neq y$ of Z; then $\Phi(x) = \Phi(y)$ if and only if either (x, y) or (y, x) contains no more than 1 element of X

2.2. Cyclic order on families of rays. A *line* is a proper embedding of $\mathbb{R}$ in $\mathbb{R}^2$. A line L cuts $\mathbb{R}^2$ in two half-planes. If L is oriented, then there is an orientation-preserving homeomorphism h of $\mathbb{R}^2$ mapping L to the oriented x-axis of $\mathbb{R}^2$ (endowed with the coordinates (x, y)). This allows us to define the upper and lower half-planes Δ_L^+ and Δ_L^- as the pre-images by h of $\{y \geq 0\}$ and $\{y \leq 0\}$, respectively.

A *ray* is a proper embedding of $[0, +\infty)$ in $\mathbb{R}^2$. Two rays define the same *germ of ray* if their images coincide out of a compact ball. Two germs of rays are said to be disjoint if they admit disjoint realisations.

Example 2.5. (1) If $\mathcal{F}$ is a foliation of $\mathbb{R}^2$, every leaf defines two germs of rays called the *ends of the leaf*. By fixing an orientation of $\mathcal{F}$, we will speak of the *right and left ends* of a leaf.

(2) If $\{\mathcal{F}_i\}_{i \in \mathcal{I}}$ is a family of pairwise transverse foliations of $\mathbb{R}^2$, then the set of all ends of leaves of the foliations $\mathcal{F}_i$ is a family of pairwise disjoint germs of rays.

(3) Consider the set $\mathcal{S}$ of all germs of rays γ which are contained in an orbit of an affine (polynomial of degree = 1) vector field of saddle type. Then, $\mathcal{S}$ is a family of pairwise disjoint germs of rays.

The next lemmas are simple exercises in plane topology.

Lemma 2.6. *Let γ_0, γ_1, γ_2 be three disjoint rays.*

Assume that C_1 and C_2 are simple closed curves on the plane $\mathbb{R}^2$ so that $\gamma_i \cap C_j$ is a unique point $p_{i,j}$, $i \in \{0,1,2\}$, $j \in \{1,2\}$. We endow C_i with the boundary-orientation corresponding to the compact disc bounded by C_i. Then, the cyclic order of the 3 points $p_{0,1}$, $p_{1,1}$, $p_{2,1}$ for the orientation of C_1 is the same as the cyclic order of the 3 points $p_{0,2}$, $p_{1,2}$, $p_{2,2}$ for the orientation of C_2.

We call it the cyclic order on the rays γ_0, γ_1, γ_2.

Lemma 2.7. *The cyclic order on three disjoint germs of rays R_0, R_1, R_2 does not depend on the choice of disjoint rays γ_0, γ_1, γ_2 realizing the germs R_0, R_1, R_2.*

Corollary 2.8. *Let γ_0, γ_1, γ_2 be three disjoint rays and C any simple close curve, oriented as the boundary of the compact disc bounded by C, and having a non-empty intersection with every γ_i.*

Let p_i be the last point of γ_i in C. Then, the cyclic order of the γ_i coincides with the cyclic order of the p_i for the orientation of C.

Corollary 2.9. *Let R_0, R_1, R_2 be three disjoint germs of rays. Let L be an oriented line whose right end is R_0 and whose left end is R_2. Then, R_1 is between R_0 and R_2 for the cyclic order defined above (we denote $R_1 \in (R_0, R_2)$) if and only if it admits a realization contained in the upper half-plane Δ_L^+ bounded by L.*

The next proposition summarizes what we obtain with this sequence of easy lemmas.

Proposition 2.10. *Consider $\mathcal{R}$ a family of pairwise disjoint germs of rays. Then, $\mathcal{R}$ is totally cyclically ordered by the following relation.*

Given three distinct germs of rays $R_0, R_1, R_2 \in \mathcal{R}$, the germ R_2 is between R_1 and R_3 if it admits a realization contained in the upper-half plane Δ_L^+, where L is an oriented line whose right end is R_0 and whose left end is R_3.

2.3. Compactification of a family of rays by a circle at infinity. In this paper, *a compactification of the plane $\mathbb{R}^2$ by the disc $\mathbb{D}^2$* is by definition a homeomorphism between $\mathbb{R}^2$ and the open disc $\mathring{\mathbb{D}}^2$.

The aim of this section is the proof of Theorem 1.1 which builds a canonical compactification of $\mathbb{R}^2$ associated to a family $\mathcal{R}$ of rays, assuming that it admits a countable separating (for the cyclic order) subset $\mathcal{E} \subset \mathcal{R}$. One of the main ingredients for the proof of Theorem 1.1 is the following lemma which is an easy exercise in plane topology.

Lemma 2.11. *Let $\gamma_0, \gamma_1, \dots, \gamma_n$ be n disjoint rays, $n > 0$, and $K \subset \mathbb{R}^2$ a compact set. Then, there is a simple closed curve C so that*

- *the curve C bounds a compact disc D containing K in its interior,*
- *the intersection $C \cap \gamma_i$ consists in a unique point p_i, $i \in \{0, \dots, n\}$.*

Proof. Just notice that there is a homeomorphism of $\mathbb{R}^2$ mapping γ_i, $i \in \{1, \dots, n\}$ to radial (half-straight lines) rays. Then, the proof is trivial. ■

Sketch of proof of Theorem 1.1. We consider the set of rays endowed with the cyclic order and we embed it in the circle $\mathbb{S}^1$ by Proposition 2.3. We denote by $E \subset \mathbb{S}^1$ the dense countable subset corresponding to $\mathcal{E}$. We define a topology on $\mathbb{R}^2 \amalg \mathbb{S}^1$ by choosing a basis of neighborhoods of the points in $\mathbb{S}^1$ which are the union of

- the half planes bounded by lines L whose both ends are rays R_-, R_+ in $\mathcal{E}$ (each half plane corresponds to an interval (R_-, R_+) or (R_+, R_-) in $\mathbb{S}^1 \setminus \{R_-, R_+\}$)
- and the corresponding interval (R_-, R_+) or (R_+, R_-).

This topology does not depend of the choice of the countable separating subset $\mathcal{E}$: if $\widetilde{\mathcal{E}}$ is another countable separating subset, each neighborhood of a point of $\mathbb{S}^1$ obtained by using one family contains a neighborhood obtained by using the other family.

Now, one builds a map from $\mathbb{R}^2$ to the interior of $\mathbb{D}^2$ as follows.

(1) One considers the circles $C_n, n \geq 1$, of radius $\rho_n = 1 - \frac{1}{n+1}$ (that is, $C_n = \rho_n \cdot \mathbb{S}^1$) endowed with the finite set of point $\rho_n \cdot x_1, \dots, \rho_n \cdot x_n$, where $E = \{x_n \mid n \geq 1\}$ is a choice of indexation of the countable set E.

(2) One chooses by induction a realisation R_n of the rays in $\mathcal{E}$ and a family of simple closed loops γ_n with the following properties.
   - γ_n is the boundary of a compact disc D_n containing D_{n-1} in its interior and containing the disc of radius n of $\mathbb{R}^2$. In particular, $\bigcup_n D_n = \mathbb{R}^2$.
   - γ_n cuts the rays R_m, $m < n$ in a unique point.
   - One chooses a representative of R_n, disjoint from R_m, $m < n$, with origin on γ_n and with no other intersection point with γ_n.

   Then, by definition of the cyclic order on the rays, the points $\gamma_n \cap R_i$, $i \leq n$, are cyclically ordered on γ_n as the points $\rho_n \cdot x_1, \dots, \rho_n \cdot x_n$ on C_n.

(3) This allows us to choose a homeomorphism of $\mathbb{R}^2$ to the interior of $\mathbb{D}^2$ sending the loops γ_n on the circles C_n and the rays R_n on the segments $[\rho_n, 1) \cdot x_n$.

This homeomorphism extends on the circle at infinity $\mathbb{S}^1$ to $\partial\mathbb{D}^2$.

For the uniqueness, assume that $\varphi\colon \mathbb{R}^2 \to \mathbb{D}^2$ is another compactification satisfying the conclusions of Theorem 1.1. The map which associates its end point on $\partial\mathbb{D}^2$ to each ray is an increasing bijection on a dense subset of the circle. Thus, according to Proposition 2.3, there is a unique increasing homeomorphism ϕ of the circle mapping the end of each ray for one compactification to the end of the same ray for the other compactification. Then, the images of the rays in $\mathcal{E}$ still tend to pairwise distinct points on $\partial\mathbb{D}^2$ which are still dense in $\mathbb{S}^1$. Now, given a line L containing two distinct rays R_+, R_- of $\mathcal{E}$, the half plane bounded by $\varphi(L)$ corresponds now on the circle to the interval $\phi((R_+, R_-))$ or $\phi((R_-, R_+))$. Thus, φ extends by continuity on the circle at infinity as the homeomorphism ϕ, inducing a homeomorphism of the closed disc, as announced. ■

2.4. Union of countably many families of rays: The circle.

Proposition 2.12. *Let $\{X_i \mid i \in I\}$, $I \subset \mathbb{N}$ be a finite or countable family of sets so that $\bigcup_i X_i$ is endowed with a total cyclic order. Assume that, for every i, there exists a countable separating subset $E_i \subset X_i$.*

On the union $X = \bigcup_i X_i$, we consider the relation

$$x \sim y \Leftrightarrow ([x, y] \cap E_i \text{ is finite for every } i, \text{ or } [y, x] \cap E_i \text{ is finite for every } i).$$

In other words, $x \sim y$ if one of the two segments (for the cyclic order) bounded by x and y meets each family E_i in at most finitely many points.

Then, $\sim$ is an equivalence relation and every class contains at most 1 point in each X_i.

Consider the projection

$$\pi\colon X \to \mathcal{X} = \bigcup_i X_i/\sim.$$

We denote by $\mathcal{E}$ the projection $\pi(E)$ of $E = \bigcup E_i$ on $\mathcal{X}$.

Then, the cyclic order on X passes to a complete cyclic order on $\mathcal{X}$ and $\mathcal{E}$ is a countable separating subset of this quotient order.

Proof. The fact that $\sim$ is an equivalence relation is quite easy, as the union of two intervals meeting X_i on finite sets meets X_i on a finite set.

Note that, assuming $x \sim y$, the interval $[x, y]$ or $[y, x]$ (meeting every E_i in finitely many points) is contained in the class of x and y. Thus, the class $[x]_\sim$ is a (proper) interval for the cyclic order.

Consider $x, y \in X$, and assume that $[x, y] \cap E_j$ is finite for every j. Assume that there is an i and distinct $z, t \in [x, y] \cap X_i$. Then, the separating property of E_i for X_i

ensures that $[x, y] \cap E_i$ is infinite contradicting the choice of the interval $[x, y]$. We deduce that every class meets every X_i at most at 1 point.

Notice that this implies that the projection of E_i on $\mathcal{X}$ is injective.

Consider $x, y, z \in X$ whose classes are distinct, and assume $z \in (x, y)$. Consider now $a \sim x$, $b \sim y$ and $c \sim z$. Let I_a, I_b, I_c be the intervals $[x, a]$ or $[a, x]$, $[y, b]$ or $[b, y]$, $[z, c]$ or $[c, z]$ with finite intersections with the E_i, respectively. Then, these intervals are disjoint as they are contained in disjoint equivalence classes. Thus, the cyclic order for a point in I_a, I_b, I_c does not depend on the point in I_a, I_b, I_c and thus $c \in (a, b)$.

This shows that the quotient $\mathcal{X}$ is endowed with a total cyclic order.

Consider now two distinct classes $[x]_\sim, [y]_\sim \in \mathcal{X}$ of points $x, y \in X$. Thus, there is i so that $[x, y] \cap X_i$ is infinite. Now, the separating property of E_i implies that $[x, y] \cap E_i$ is infinite.

As π is injective on E_i, one gets that $([x]_\sim, [y]_\sim) \cap \pi(E_i)$ is infinite and thus $([x]_\sim, [y]_\sim) \cap \mathcal{E}$ is infinite. One proved that $\mathcal{E}$ is separating for $\mathcal{X}$, ending the proof. ∎

2.5. Union of countably many families of rays: The compactification.

Theorem 2.1. *Let $\mathcal{R} = \coprod_{i \in I} \mathcal{R}_i$, $I \subset \mathbb{N}$, be a family of rays in $\mathbb{R}^2$ whose germs are pairwise disjoint. Assume that for every $i \in I$ there is a countable subset $E_i \subset \mathcal{R}_i$ which is separating for $\mathcal{R}_i$.*

Then, there is a compactification of $\mathbb{R}^2$ by the disc $\mathbb{D}^2$ so that

- *any ray of $\mathcal{R}$ tends to a point of the circle at infinity $\partial\mathbb{D}^2 = \mathbb{S}^1$,*
- *for every i, any two distinct rays of $\mathcal{R}_i$ tend to distinct points of $\mathbb{S}^1$,*
- *for any non-empty open interval $J \subset \mathbb{S}^1$, there is $i \in I$ so that at least 2 rays in $\mathcal{R}_i$ have their limit points in J.*

Furthermore, this compactification is unique up to a homeomorphism of $\mathbb{D}^2$ and does not depend on the separating countable sets E_i.

Let us discuss item (3), whose formulation may be surprising.

Remark 2.13. (1) The third item implies that the points of $\mathbb{S}^1$ which are the limit points of a ray in $\bigcup E_i$ are dense in $\mathbb{S}^1$.

(2) If I is finite, item (3) is equivalent to the density of points in $\mathbb{S}^1$ which are limit points of rays.

(3) Item (3) is equivalent, by the separating property, to requiring that J contains infinitely many points in $\mathcal{R}_i$ or even in E_i.

(4) Item (3) is necessary when there is an uncountable set of equivalence classes $[c]$, (for the equivalence relation $\sim$ defined in Section 2.4), each of which is infinite (necessarily countable) and contains a set $\mathcal{C}([c])$ which is separating for the cyclic

order. In that case, the uniqueness property announced in Theorem 2.1 would be wrong if we replace item (3) by the density of points in $\mathbb{S}^1$ which are limit points of rays. Example 2.16 provides a simple illustration of this trouble.

Proof of Theorem 2.1. Let us denote $X = \mathcal{R}$, $X_i = \mathcal{R}_i$, and $E = \bigcup E_i$. Let $\sim$ be the equivalence relation defined in Proposition 2.12 on $X = \mathcal{R}$, and let π denote the projection $\pi\colon X \to \mathcal{X} = X/\sim$. We choose a subset $Y \subset X$ with the following properties:

- each equivalence class $[\gamma]$ for $\sim$ contains exactly 1 point y_γ in Y,
- if a class for $\sim$ contains a point in E, then $y_\gamma \in E$.

The existence of such subset Y is certainly implied by the axiom of choice, but this existence does not require this axiom. For instance, we can choose the subset Y as follows.

- If $[\gamma]_\sim \in \pi(\bigcup \mathcal{E}_i)$, then y_γ is the unique point in E_i in the $[\gamma]_\sim$, where i is the smallest index for which $[\gamma]_\sim \in \pi(\mathcal{E}_i)$.
- If $[\gamma]_\sim \notin \pi(E)$, then y_γ is the unique point in $X_i \cap [\gamma]_\sim$, where i is the smallest index for which $[\gamma]_\sim \cap X_i \neq \emptyset$.

We denote $F = Y \cap E$. Notice that $\pi(F) = \pi(E)$ by construction.

Now, the projection $\pi\colon Y \to \mathcal{X}$ is a bijection which is strictly increasing for the cyclic order. As $\mathcal{E} = \pi(E)$ is separating for $\mathcal{X}$ (see Proposition 2.12), one gets that F is separating for Y.

We can now apply Theorem 1.1 to the set of rays Y and the countable separating subset F. One gets a compactification of $\mathbb{R}^2$ as a disc $\mathbb{D}^2$ so that every ray in Y tends to a point at the circle at infinity, two distinct rays in Y tend to two distinct points, and the set of points at infinity which are limits of rays in F is dense in the circle at infinity.

Let us check now that every ray $\gamma \in \mathcal{R}$ tends to a point at infinity. By construction of Y there is $\sigma \in Y$ so that $\sigma \sim \gamma$. We will prove that γ tends to the limit point $s \in \mathbb{S}^1$ of σ. For that, we recall that a basis of neighborhood of s is given by the half planes Δ_n bounded by lines L_n whose both ends are $\sigma_n^-, \sigma_n^+ \in Y$ so that $\sigma \in (\sigma_n^-, \sigma_n^+)$. Note that both intervals (σ, σ_n^+) and (σ_n^-, σ) are infinite when one of the intervals (σ, γ) or (γ, σ) is finite. One deduces that $\gamma \in (\sigma_n^-, \sigma_n^+)$ and thus the end of γ is contained in Δ_n. Thus, γ tends to s. Notice that this argument implies that two equivalent rays tend to the same point at infinity.

Now, consider two distinct rays $\gamma, \gamma' \in \mathcal{R}_i$. As every class for $\sim$ contains at most 1 point of $\mathcal{R}_i$, the classes of γ and γ' are distinct. Thus, there are $\sigma \neq \sigma' \in Y$ which are equivalent to γ and γ', respectively. The limit points of γ and γ' are those of σ and σ', respectively, which are distinct. We just checked that distinct rays in $\mathcal{R}_i$ tend to a distinct point at infinity.

This already proves that the compactification satisfies the two first items.

Consider now a non-empty open interval J of the circle at infinity. We will show that there is an i for which J contains at least 2 limits of rays in $\mathcal{R}_i$. Recall that, according to Theorem 1.1, the points in J which are limits of rays in Y are dense in J. Thus, there are at least 2 distinct points in J which are limit of rays R_1, R_2 in Y. This implies that, up to exchanging R_1 and R_2, any ray in $\mathcal{R}$ between R_1 and R_2 tends to a point in J. Now, the rays R_1, R_2 are not equivalent for $\sim$, as we have seen that equivalent rays tend to the same point at infinity. By definition of $\sim$, there is i so that there are infinitely many rays in $\mathcal{R}_i$ between R_1 and R_2. This proves item (3) of Theorem 2.1.

It remains to prove the uniqueness.

Claim 2.14. Let $\varphi\colon \mathbb{R}^2 \to \mathbb{D}^2$ be a compactification satisfying the announced properties. Then, 2 rays in $\mathcal{R}$ tend to the same point of the circle at infinity of the compactification if and only if they are equivalent for $\sim$.

Proof. If two rays a, b are not equivalent, then each of (a,b) and (b,a) contains infinitely many rays in some set $\mathcal{R}_i$, by definition of $\sim$. As the limits of distinct rays in the same R_i are different, one deduces that the limits of a and b are distinct.

Conversely, if a and b have different limit points r and s in $\mathbb{S}^1$ for the compactification, then item (3) implies that there is $i \in I$ (resp., $j \in I$) so that (a,b) (resp., (b,a)) contains the ends of at least 2 rays in $\mathcal{R}_i$ (resp., $\mathcal{R}_j$). As $\mathcal{R}_i$ and $\mathcal{R}_j$ admit separating subsets, this implies that both $(a,b) \cap \mathcal{R}_i$ and $(b,a) \cap \mathcal{R}_j$ are infinite so that $a \nsim b$. ∎

Assume now that one has another compactification $\psi\colon \mathbb{R}^2 \to \mathbb{D}^2$ satisfying also the announced properties. One deduces from Claim 2.14 the fact that the images by ψ of two distinct rays in the set Y (that we used for building the first compactification φ) have two distinct limit points and that the limit points of the image by ψ of rays in Y are dense in $\mathbb{S}^1$. Thus, this new compactification satisfies the same property on the set of rays Y as the one we built. Now, Theorem 1.1 asserts that these compactifications differ from φ by a homeomorphism of $\mathbb{D}^2$, concluding the proof. ∎

Lemma 2.15. *Assume that $\mathcal{R}$ satisfies the hypotheses of Theorem 2.1, and let $\widetilde{\mathcal{R}}$ be a set of rays so that the germs of rays in $\mathcal{R} \cup \widetilde{\mathcal{R}}$ are pairwise disjoint.*

Let $\mathbb{R}^2 \hookrightarrow \mathbb{D}^2$ be a compactification given by Theorem 2.1 applied to $\mathcal{R}$. Then, any ray $\tilde{\gamma}$ in $\widetilde{\mathcal{R}}$ tends to a point at infinity.

Proof. The candidate for the limit is the intersection of all the closed intervals in $\mathbb{S}^1$, bounded by limits of rays $a, b \in \mathcal{R}$, so that $\tilde{\gamma} \in (a,b)$. The basis of neighborhood of this point that we exhibit implies that indeed $\tilde{\gamma}$ tends to that point at infinity. ∎

2.6. An example with uncountably many compactifications. The example below shows that, in the case of an infinite countable family $\mathcal{R} = \{\mathcal{R}_i\}, i \in \mathbb{N}$, the compactification announced by Theorem 2.1 would not be unique if we replace the item (3) of the conclusion by the density in $\mathbb{S}^1_{\mathcal{R}}$ of the limits of the rays in $\mathcal{R}$. Consider the example below.

Example 2.16. Let $B \subset \mathbb{R}^2$ be the open strip $\{(x, y) \in \mathbb{R}^2 \mid |x - y| < 1\}$. Let $I \subset \mathbb{RP}^1$ be the (countable) set of linear lines with a rational inclination $\neq 1$. For any $i \in I$, let $\mathcal{F}_i$ be the restriction to B of the trivial foliation by parallel straight lines directed by i. For any i, let $\mathcal{R}_i$ be the set of ends of leaves of $\mathcal{F}_i$. Each $\mathcal{R}_i$ admits a countable separating subset. Thus, $\mathcal{R} = \bigcup_{i \in I} \mathcal{R}_i$ satisfies the hypotheses of Theorem 2.1.

Then, there are uncountably many distinct compactifications of $\mathbb{R}^2$ for which

- any ray of $\mathcal{R}$ tends to a point of the circle at infinity $\partial\mathbb{D}^2 = \mathbb{S}^1$,
- for every i, any two distinct rays of $\mathcal{R}_i$ tend to distinct points of $\mathbb{S}^1$,
- the points of $\mathbb{S}^1$ which are the limit point of a ray in $\bigcup E_i$ are dense in $\mathbb{S}^1$.

Proof. Let $\mathbb{D}^2_{\mathcal{R}}$ be the compactification of $B \simeq \mathbb{R}^2$ by adding the circle at infinity $\mathbb{S}^1_{\mathcal{R}}$.

Every class C for $\sim$ contains exactly 1 ray in $\mathcal{R}_i$ for any $i \in I$. The rays in C are ordered, for the cyclic order, as the points of I in $\mathbb{RP}_1$. So, C is a separating set for itself.

By construction of $\mathbb{S}^1_{\mathcal{R}}$, the class C corresponds to a point $c \in \mathbb{S}^1_{\mathbb{R}}$. We can build another circle $\mathbb{S}^1_{\mathcal{R},C}$ by opening the point c in a segment I_C. Then, we can build a compactification $\mathbb{D}^2_{\mathcal{R},C}$ so that the rays in C tend to distinct points dense in I_C. In particular, I_C contains exactly 1 limit point of a ray in $\mathcal{R}_i$ for any i.

We can repeat this argument opening not just a point in $\mathbb{S}^1_{\mathcal{R}}$ but a countable subset $\mathcal{C}$ of classes for $\sim$: we build a compactification $\mathbb{D}^2_{\mathcal{R},\mathcal{C}}$, where the circle at infinity contains disjoint intervals $I_C, C \in \mathcal{C}$, so that each I_C contains exactly 1 limit point of a ray in $\mathcal{R}_i$ for any i, and these points are dense in I_C.

As there are uncountably many such countable subsets $\mathcal{C}$, this provides an uncountable family of pairwise distinct compactifications of B satisfying the 2 first items and the density of the limit points of rays. ∎

This shows that the uniqueness part of Theorem 2.1 becomes wrong if we replace item (3) by the density in $\mathbb{S}^1$ of the set of limits of rays.

2.7. Uncountable families of families of rays. Theorem 2.1 is wrong for the union of an uncountable family of sets of rays, as shown by Example 2.17 below.

Example 2.17. We consider $\mathbb{R}^2$ endowed with all constant foliations $\mathcal{F}_\theta, \theta \in \mathbb{RP}^1$, where $\mathcal{F}_\theta$ is the foliation whose leaves are the straight lines parallel to θ.

Then, given any compactification of $\mathbb{R}^2$ by $\mathbb{D}^2$ for which every end of leaf tends to a point at infinity, then for all but a countable set of θ the right ends of the leaves of $\mathcal{F}_\theta$ tend to the same point at infinity.

Proof. The ends of leaves F_θ^+ at the right, and those at the left F_θ^- of the foliation $\mathcal{F}_\theta$, are disjoint intervals J_θ^+ and J_θ^- depending on the uncountable parameter θ. On the circle, at most countably many disjoint intervals can be non-trivial, ending the proof. ∎

2.8. Projection on the compactifications associated to each family. Let us start with a very easy example, showing that the circles at infinity associated to the subsets of a countable family of transverse foliations may lead to uncountably many distinct compactifications. (All these compactifications are quotient of the compactification associated to the whole family.)

Example 2.18. Consider an infinite countable subset $I \subset \mathbb{RP}^1$ and consider the family $\mathcal{R}_I$ of the leaves of the constant foliations $\mathcal{F}_\theta, \theta \in I$ on $\mathbb{R}^2$ as already considered in Example 2.17.

Now, the set of ends of leaves of each foliation $\mathcal{F}_\theta$ corresponds to 2 (because each leaf has 2 ends) non-empty open intervals in $\mathbb{S}^1_I$, and these intervals do not contain any end of leaf of any other foliation.

Thus, if $J, K \subset I$ are distinct subsets, the circles at infinity $\mathbb{S}^1_J$ and $\mathbb{S}^1_K$ are obtained by collapsing distinct intervals of $\mathbb{S}^1_I$ and they are different.

As the set $\mathcal{P}(I)$ of all subsets of I is uncountable, this leads to an uncountable family of compactifications $\{\mathbb{D}^2_J\}_{J\in\mathcal{P}(I)}$ of $\mathbb{R}^2$ by a circle at infinity.

This situation is quite general.

Let $\mathcal{R} = \bigcup \mathcal{R}_i, i \in I \subset \mathbb{N}$ be a family of rays in $\mathbb{R}^2$ whose germs are pairwise disjoint. Assume that for every $i \in I$ there is a countable subset $E_i \subset \mathcal{R}_i$ which is separating.

Thus, for every subset $J \subset I$, Theorem 2.1 provides a compactification $\mathbb{D}^2_I$ of $\mathbb{R}^2$, by the circle at infinity corresponding to the rays in $\mathcal{R}_j, j \in J$.

Proposition 2.19. *If $J \subset I$, then the identity map on $\mathbb{R}^2$ extends by continuity as a projection $\Pi_{I,J}\colon \mathbb{D}^2_I \to \mathbb{D}^2_J$. This projection consists in collapsing intervals of $\mathbb{S}^1_I = \partial\mathbb{D}^2_I$ which contains at most 1 limit point of ray in $\mathcal{R}_j$ for each $j \in J$.*

Furthermore, if $K \subset J$, then

$$\Pi_{I,K} = P_{J,K} \circ P_{I,J}.$$

Proof. Two rays a, b in $\mathcal{R}_J = \bigcup_{j\in J} \mathcal{R}_j$ tend to the same point in $\mathbb{S}^1_I$ (resp., $\mathbb{S}^1_J$) if and only if they are equivalent for the equivalence relation $\sim_I$ (reps., $\sim_J$) defined in

Proposition 2.12 for $\mathcal{R}_I$ (resp., $\mathcal{R}_j$): in other words, if there is an interval (a,b) or (b,a) (for the cyclic order) meeting every $\mathcal{R}_i$, $i \in I$, (resp., $\mathcal{R}_j$, $j \in J$), in at most one point.

Thus, $\sim_I$ is more restrictive than $\sim_J$ for $a, b \in \mathcal{R}_i$:

$$a \sim_I b \Rightarrow a \sim_J b.$$

There is therefore a natural projection for $\phi_{I,J}\colon \mathcal{R}_J/\sim_I \to \mathcal{R}_J/\sim_J$, which is (non-strictly) increasing for the cyclic order. Notice that $\mathcal{R}_J/\sim_I$ and $\mathcal{R}_J/\sim_J$ are in increasing bijection with the sets of $\mathbb{S}^1_I$ and $\mathbb{S}^1_J$, respectively, that consist of limits of rays in $\mathcal{R}_J$. Furthermore, these points are dense in $\mathbb{S}^1_J$. Thus, $\pi_{I,J}$ extends in a unique way in a (non-strictly) increasing continuous map $\pi_{I,J}\colon \mathbb{S}^1_I \to \mathbb{S}^1_J$ of topological degree equal to 1.

Let us understand the injectivity defect of $\pi_{I,J}$. The inverse image of a point $x \in \mathbb{S}^1_J$ is an interval. All rays in $\mathcal{R}_J$ tending to points in the interval $\pi^{-1}_{I,J}(x) \subset \mathbb{S}^1_I$ are equivalent for $\sim_J$: in other words, this interval contains at most 1 limit point of ray in $\mathcal{R}_j$ for each $j \in J$, which is the announced characterization.

For ending the proof, we will check that $\pi_{I,J}$ is the extension by continuity of the identity map of $\mathbb{R}^2$ to the circles at infinity.

Recall that we defined a basis of neighborhood of each point of the circle at infinity $\mathbb{S}^1_I$ (resp., $\mathbb{S}^1_J$) as the half-planes Δ^+_L bounded by lines whose both ends are in $\mathcal{R}_I$ (resp., $\mathcal{R}_J$). In particular, as $J \subset I$, the neighborhoods of points at infinity in $\mathbb{D}^2_J$ are still neighborhoods of points at infinity for $\mathbb{D}^2_I$, proving that the map which is the identity from $\mathbb{R}^2 = \mathring{\mathbb{D}}^2_I$ to $\mathbb{R}^2 = \mathring{\mathbb{D}}^2_J$ and is $\pi_{I,J}$ from $\mathbb{S}^1_I$ to $\mathbb{S}^1_J$ is continuous. This ends the proof. ∎

3. Background on foliations: Regular leaves and non-separated leaves

3.1. Non-singular foliations. Let $\mathcal{F}$ be a foliation of $\mathbb{R}^2$. Then, the following hold.

(1) As $\mathbb{R}^2$ is simply connected, $\mathcal{F}$ is orientable and admits a transverse orientation. Let us fix an orientation of $\mathcal{F}$ and a transverse orientation.

(2) Every leaf is *a line* (i.e., a proper embedding of $\mathbb{R}$ in $\mathbb{R}^2$).

(3) A basis of neighborhoods of a leaf L is obtained by considering the union of leaves through a transverse segment σ through a point of L.

Definition 3.1.
- Two leaves L_1, L_2 are *not separated* from each other if they do not admit disjoint neighborhood.
- A leaf L is called *not separated* or *not regular* if there is a leaf L' which is not separated from L.

- A leaf is called *regular* if it is separated from any other leaf.

We will need some times to be somewhat more specific.

Let L_1 and L_2 be distinct leaves of $\mathcal{F}$. Consider two segments $\sigma_i\colon[-1,1]$ transverse to $\mathcal{F}$, positively oriented for the transverse orientation of $\mathcal{F}$, so that $\sigma_i(0)\in L_i$, $i=1,2$. Then, L_1 being not separated from L_2 means that there are sequences t_n^i, $i=1,2$, tending to 0 as $n\to+\infty$ so that $\sigma_1(t_n^1)$ and $\sigma_2(t_n^2)$ belong to the same leaf L^n. Then, the following hold.

- As $\mathcal{F}$ is transversely oriented and the σ_i are positively oriented, one gets that t_n^1 has the same sign as t_n^2 for every n. Furthermore, all the t_n^i have the same sign (because $\mathbb{R}^2$ is simply connected).
  One says that L_1 and L_2 are *not separated from above* (resp., *from below*) if the t_n^i are positive (resp., negative).
- By shrinking the segments σ_i if necessary, one may assume that they are disjoint. Now, up to exchanging L_1 with L_2, we may assume that $\sigma_1(t_n)$ is at the left of $\sigma_2(t_n)$ in the oriented leaf L^n. We say that L_1 (resp., L_2) is *not separated from L_2 at its right* (resp., *at its left*).

Consider a leaf L and $\sigma\colon[-1,1]\to\mathbb{R}^2$ a transverse segment (positively oriented) with $\sigma(0)\in L$. Let L_t be the leaf through $\sigma(t)$. Let U_t, $t\in(0,1)$, be the closure of the connected component of $\mathbb{R}^2\setminus(L_t\cup L_{-t})$ containing L. Then, we have the following lemma.

Lemma 3.2. *The leaf L is regular if and only if*

$$\bigcap_t U_t = L.$$

The intersection $\bigcap_t U_t$ does not depend on the segment σ and is denoted $\mathfrak{U}(L)$.

If L is not regular, $\mathfrak{U}(L)$ has non-empty interior, and the leaves which are not separated from L are precisely the leaves in the boundary of $\mathfrak{U}(L)$.

Proof. A leaf $\widetilde{L}$ not separated from L is contained in every U_t and is accumulated by leaves L_{t_n} in the boundary of U_{t_n}. Thus, $\widetilde{L}$ is contained in the boundary of $\mathfrak{U}(L)=\bigcap_t U_t$. Furthermore, one of the two half planes bounded by $\widetilde{L}$ is contained in U_t and therefore in $\mathfrak{U}(L)$.

Conversely, $\bigcap_t U_t$ consists of entire leaves of $\mathcal{F}$, and so does its boundary. Now, any transverse segment through a leaf in the boundary of $\bigcap_t U_t$ crosses the boundary $L_t\cup L_{-t}$ of U_t for t small: that is the definition of being not separated from L. ∎

Lemma 3.3. *Let $\mathcal{F}$ be a foliation of $\mathbb{R}^2$. The set of not separated leaves is at most countable.*

Proof. We consider a countable family of transverse lines Σ whose union cuts every leaf of $\mathcal{F}$. It is enough to prove that such a transverse line Σ cuts at most a countable set of non-regular leaves L admitting a non-separated leaf $\widetilde{L}$.

For that just notice that the sets $\mathfrak{U}(L)$, for $L \cap \Sigma \neq \emptyset$, are pairwise disjoint. Thus, there are at most countably many of them with non-empty interior, ending the proof. ∎

Note that L cuts the strip U_t, $t \in (0, 1]$ in two strips U_t^+ and U_t^- bounded, respectively, by $L_t \cup L$ and by $L_{-t} \cup L$, and we denote

$$\mathfrak{U}^+(L) = \bigcap_t U_t^+ \quad \text{and} \quad \mathfrak{U}^-(L) = \bigcap_t U_t^-.$$

Then, we have the following lemma.

Lemma 3.4. *L is non-separated from above (resp., from below) if and only if $\mathfrak{U}^+(L) \neq L$ (resp., $\mathfrak{U}^-(L) \neq L$) and if and only if $\mathfrak{U}^+(L)$ (resp., $\mathfrak{U}^-(L)$) has non-empty interior.*

In the same spirit, σ cuts the strip U_t in two half-strips U_t^{left} and U_t^{right} according to the orientation of $\mathcal{F}$. Then, one says that the *right end* L^+ (resp., *left end* L^-) *of L is regular* if

$$\mathfrak{U}_{\text{right}}(L) = \bigcap_t U_t^{\text{right}} = L^+ (\text{resp., } \mathfrak{U}_{\text{left}}(L) = \bigcap_t U_t^{\text{left}} = L^-).$$

We can be even more precise by considering the 4 quadrants $U_t^{+,\text{right}}$, $U_t^{+,\text{left}}$, $U_t^{-,\text{right}}$, $U_t^{-,\text{left}}$ obtained by considering the intersections of U_t^+ and U_t^- with U_t^{right} and U_t^{left}. This allows us to speak of right or left ends of leaves non-separated from above or from below in the obvious way.

3.2. Singular foliations: Saddles with k-separatrices. A singular foliation $\mathcal{F}$ on $\mathbb{R}^2$ is a foliation on $\mathbb{R}^2 \setminus \mathrm{Sing}(\mathcal{F})$, where $\mathrm{Sing}(\mathcal{F})$ is a closed subset of $\mathbb{R}^2$. A *leaf of* $\mathcal{F}$ is a leaf of the restriction of $\mathcal{F}$ to $\mathbb{R}^2 \setminus \mathrm{Sing}(\mathcal{F})$. Let us now recall the notion of saddles with k-separatrices, also called k-prongs singularities.

We denote by A_0 the quotient of $[-1, 1]^2$ by the involution

$$(x, y) \mapsto (-x, -y).$$

The projection of $(0, 0)$ on A_0 is still called $(0, 0)$. Note that the horizontal foliation (whose leaves are the segments $[-1, 1] \times \{t\}$) is invariant by the involution $(x, y) \mapsto (-x, -y)$ and therefore passes to the quotient on $A_0 \setminus (0, 0)$, and we denote by $\mathcal{H}_1$ the induced foliation on $A_0 \setminus \{(0, 0)\}$.

A 1*-prong singular point p* of $\mathcal{F}$ is an isolated point of $\mathrm{Sing}(\mathcal{F})$ which admits a neighborhood U and a homeomorphism h from U to A_0 so that $h(p) = (0, 0)$ and h maps $\mathcal{F}$ on $\mathcal{H}_1$.

We denote by A_k the k-fold cyclic ramified cover of A_0 at the point $(0,0)$ and by $\mathcal{H}_k$ the lift of $\mathcal{H}_1$ on A_k. *A separatrix* of $(0,0)$ for $\mathcal{K}_k$ is a connected component of the lift of the leaf $]0,1] \times \{0\}$ of $\mathcal{H}_1$. There are k separatrices.

A *k-prongs singular point p*, equivalently a *saddle point with k separatrices* of $\mathcal{F}$, is a singular point admitting a homeomorphism of a neighborhood onto A_k mapping p to $(0,0)$ and $\mathcal{F}$ to $\mathcal{H}_k$. *A separatrix* (also called *a prong*) of the saddle point p is the leaf of $\mathcal{F}$ containing a separatrix of $(0,0)$ for $\mathcal{H}_k$.

Remark 3.5.
- If p is a 2-prongs singular point of $\mathcal{F}$, then the foliation $\mathcal{F}$ can be extended on p so that p is not singular.
- The Poincaré–Hopf index of a k-prongs singular point is $1 - \frac{k}{2}$.

A *foliation with singularities of saddle type* on $\mathbb{R}^2$ is a singular foliation for which each singular point is a saddle with k separatrices, $k > 2$.

3.3. Leaves of singular foliations.

Lemma 3.6. *Let $\mathcal{F}$ be a foliation on $\mathbb{R}^2$ with singular points of saddle type. Let*

$$\sigma\colon [0,1] \to \mathbb{R}^2 \setminus \mathrm{Sing}(\mathcal{F})$$

be a segment transverse to $\mathcal{F}$. Then, for every leaf γ, one has

$$\#\sigma \cap \gamma \leq 1,$$

where # denotes the cardinal.

Proof. Assume (arguing by contradiction) that $\#\sigma \cap \gamma \geq 2$. Let x, y be two successive (for the parametrization of γ) intersection points with σ. The concatenation of the segments $[x,y]_\gamma$ and $[y,x]_\sigma$ is a simple closed curve c in $\mathbb{R}^2 \setminus \mathrm{Sing}(X)$. By Jordan theorem, the curve c bounds a disc D in $\mathbb{R}^2$. The Poincaré Hopf index of $\mathcal{F}$ on D is either equal to 1, if γ cuts σ with the same orientation at x and y, or $\frac{1}{2}$ otherwise: anyway this index is strictly positive. However, this index is the sum of the Poincaré–Hopf indices of the singular points of $\mathcal{F}$ contained in D. As each of them is negative, that is a contradiction, ending the proof. ∎

The same argument gives the following lemma.

Lemma 3.7. *Let $\mathcal{F}$ be a foliation on $\mathbb{R}^2$ with singular points of saddle type. Then, $\mathcal{F}$ has no compact leaves.*

Proof. The index of $\mathcal{F}$ on the disc bounded by a compact leaf would be 1, which is impossible with singular points with negative index. ∎

Corollary 3.8. *Let $\mathcal{F}$ be a singular foliation of $\mathbb{R}^2$ with singularities of saddle type. Then, every half-leaf of $\mathcal{F}$ either is a ray or tends to a singular point p of $\mathcal{F}$ (in that case it is contained in a separatrix of p).*

Proof. Consider the Alexandrov compactification of $\mathbb{R}^2$ by a point at infinity. Consider a leaf γ and choose a parametrization $\gamma(t)$. Consider

$$\limsup_{t\to+\infty} \gamma(t) = \bigcap_{t>0} \overline{\gamma([t,+\infty))},$$

where the closure is considered in $\mathbb{R}^2 \cup \{\infty\}$. It is a decreasing intersection of connected compact sets, and hence, it is a non-empty connected compact set.

If $\limsup_{t\to+\infty} \gamma(t)$ is not just a point, it contains a regular point x of $\mathcal{F}$; hence, it cuts infinitely many times any transverse segment through x, which is forbidden by Lemma 3.6.

Now, $\limsup_{t\to+\infty} \gamma(t)$ either is the point ∞ or is a singular point of $\mathcal{F}$, which is the announced alternative. ∎

3.4. Regular leaves of singular foliations. Let $\mathcal{F}$ be a foliation with singular points of saddle type, L_0 a leaf of $\mathcal{F}$, and σ a transverse segment through the point $\sigma(0) \in L_0$.

The set of t so that $\sigma(t)$ is contained in a separatrix of a singular point is at most countable. For any t so that $\sigma(t)$ and $\sigma(-t)$ are not in a separatrix of a singular point, the leaves L_t and L_{-t} through $\sigma(t)$ and $\sigma(-t)$ are disjoint lines and therefore cut $\mathbb{R}^2$ in 3 connected components. We denote by U_t the closure of the connected component of $\mathbb{R}^2 \setminus (L_t \cup L_{-t})$ containing L_0. Notice that U_t is a strip (homeomorphic to $\mathbb{R} \times [-1,1]$) bounded by $L_t \cup L_{-t}$ and saturated for $\mathcal{F}$.

Lemma 3.9. *With the notation above, $\bigcap_t U_t$ is a non-empty closed subset of $\mathbb{R}^2$ saturated for $\mathcal{F}$, and we have the following alternative:*

- *either $\bigcap_t U_t = L_0$ and L_0 is a non-singular leaf of $\mathcal{F}$,*
- *or $\bigcap_t U_t$ has non-empty interior.*

Furthermore, $\bigcap_t U_t$ does not depend on the choice of the transverse segment σ through L_0 and is denoted $\mathfrak{U}(L_0)$.

Proof. $\mathfrak{U}(L_0) \setminus L_0$ is saturated for $\mathcal{F}$. If it contains a non-singular leaf, it contains one of the half planes bounded by this leaf. If it contains a singular leaf, it contains the corresponding singular point, and then it contains at least one of the sectors bounded by the separatrices. ∎

Definition 3.10. With the notation above, the leaf L_0 is called *regular* if $\mathfrak{U}(L_0) = L_0$, and will be called *non-regular* otherwise.

Remark 3.11. If L_0 is a separatrix[1] of a singular point, then it is non-regular.

As in the case of non-singular foliations, we have the following proposition.

Proposition 3.12. *Let $\mathcal{F}$ be a foliation with singular points of saddle type. Then, the set of non-regular leaves is at most countable.*

Proof. For any transverse segment σ, let us denote by L_t the leaf through σ_t. Then, by construction, the closed sets $\mathfrak{U}(L_t)$ are pairwise disjoint. Thus, at most countably many of them may have non-empty interior; that is, at most countably many of leaves L_t are non-regular. We conclude the proof by noticing that $\mathcal{F}$ admits a countable family of transverse segment $\sigma_n, n \in \mathbb{N}$, so that every leaf of $\mathcal{F}$ cuts at least 1 segment σ_n. ∎

The leaves of a foliation have two ends, and the notion of regular leaves can be made more precise, looking at each of its ends.

More precisely, let $L_{0,+}$ be a half-leaf of $\mathcal{F}$, and let σ be a transverse segment so that $\sigma(0)$ is the initial point of $L_{0,+}$. For any t so that $\sigma(t)$ and $\sigma(-t)$ do not belong to a separatrix of a singular point, we consider $L_{t,+}$ and $L_{t,-}$ the half leaves starting at $\sigma(t)$ and $\sigma(-t)$ in the same side of σ as $L_{0,+}$. We denote by $U_t(L_{0,+}) \subset \mathbb{R}^2$ the closed half plane containing $L_{0,+}$ and bounded by the line of $\mathbb{R}^2$ obtained by concatenation of $L_{t,+}$, $\sigma([-t,t])$ and $L_{t,-}$. We denote $\mathfrak{U}(L_{0,+}) = \bigcap_t U_t(L_{0,+})$. Then,

- either $\mathfrak{U}(L_{0,+}) = L_{0,+}$ and one says that the half-leaf $L_{0,+}$ (or equivalently, the end of L_0 corresponding to $L_{0,+}$) is *regular*,
- or $\mathfrak{U}(L_{0,+}) \neq L_{0,+}$ is a closed subset with non-empty interior.

A leaf is regular if and only if its two ends are regular, and the set of non-regular ends of leaves is at most countable.

3.5. Orientations. A foliation with singular points of saddle type is locally orientable (and transversely orientable) in a neighborhood of a singular point x if and only if the number of separatrices of x is even.

Thus, a foliation of $\mathbb{R}^2$ whose singular points are saddles with even numbers of separatrices is locally orientable and transversely orientable, and therefore is globally orientable and transversely orientable, as $\mathbb{R}^2$ is simply connected.

Let $\mathcal{F}$ be a foliation with singular points of saddle type with even numbers of separatrices, and fix an orientation and transverse orientation of $\mathcal{F}$.

[1]Recall that, in this paper, a leaf of a singular foliation $\mathcal{F}$ is by definition a leaf of the restriction of $\mathcal{F}$ to $\mathbb{R}^2 \setminus \mathrm{Sing}(\mathcal{F})$. In this terminology, a separatrix is a leaf. This terminology comes from foliations theory. Other schools prefer to consider singular leaves, union of the singular point and of all its separatrices, but that is not the choice here.

Thus, every leaf L has a right and left end. We define $\mathfrak{U}^{\text{right}}(L)$ and $\mathfrak{U}^{\text{left}}(L)$ so that L can be regular at the right or at the left.

If L_0 is a leaf which is not a separatrix and σ be a transverse segment with $\sigma(0) \in L_0$. One defines in the same way the notions of being regular from above and from below, for L_0 or for each of its two ends.

For instance, L_0^{right} is regular from above if $\mathfrak{U}_+(L_0^{\text{right}}) = \bigcap_t U_{t,+}(L_0^{\text{right}}) = L_0^{\text{right}}$, where $U_{t,+}(L^{\text{right}})$ is bounded by L_0^{right}, $\sigma([0,t])$ and L_t^{right}.

4. The circle at infinity of a singular foliation

4.1. The circle at infinity of a foliation of $\mathbb{R}^2$: Statement. The aim of this section is to recall the following result essentially due to [13] and to present a short proof of it.

Theorem 4.1. *Let $\mathcal{F}$ be a foliation of the plane $\mathbb{R}^2$, possibly with singularities of saddle type. Then, there is a compactification $\mathbb{D}^2_{\mathcal{F}}$ of $\mathbb{R}^2$ by adding a circle at infinity $\mathbb{S}^1_{\mathcal{F}} = \partial\mathbb{D}^2_{\mathcal{F}}$ with the following property.*

- *Any half-leaf tends either to a saddle point or to a point at infinity.*
- *Given a point $\theta \in \mathbb{S}^1_{\mathcal{F}}$, the set of ends of leaves tending to θ is at most countable.*
- *The subset of $\mathbb{S}^1_{\mathcal{F}}$ corresponding to limits of regular ends of leaves is dense in $\mathbb{S}^1_{\mathcal{F}}$.*

Furthermore, this compactification of $\mathbb{R}^2$ by $\mathbb{D}^2$ with these three properties is unique, up to a homeomorphisms of the disc $\mathbb{D}^2$.

Remark 4.1. If $L_1^+ \neq L_2^+$ are two ends of leaves[2] tending to the same point $\theta \in \mathbb{S}^1_{\mathcal{F}}$, then $L_2^+ \subset \mathfrak{U}(L_1^+)$. In particular, the ends L_1^+ and L_2^+ are not regular.

Remark 4.2. The third item is equivalent to the following:

- every non-empty open interval of $\mathbb{S}^1_{\mathcal{F}}$ contains the limit points of an uncountable set of ends of leaves.

Corollary 4.3. *If a homeomorphisms f of the plane $\mathbb{R}^2$ preserves the foliation $\mathcal{F}$, then it extends in a unique way as a homeomorphism F of the compactification $\mathbb{D}^2_{\mathcal{F}}$.*

Furthermore, the restriction of F to $\mathbb{S}^1_{\mathcal{F}}$ is the identity map if and only if f preserves every leaf of $\mathcal{F}$ and preserves the orientation on each leaf.

Proof. The first part is, as already noted, a straightforward consequence of the uniqueness of the compactification.

[2]The notation L_i^+ only indicates that they are half leaves but has no orientation meaning.

If f preserves every leaf and preserves the orientation of the leaves, then it preserves every end of leaf. Thus, the extension F fixes every point of $\mathbb{S}^1_{\mathcal{F}}$ which is limit of an end of leaf. As the limit points of end of leaves are dense in $\mathbb{S}^1_{\mathcal{F}}$, one deduces that the restriction of F to $\mathbb{S}^1_{\mathcal{F}}$ is the identity map.

Conversely, assume that F is the identity map on $\mathbb{S}^1_{\mathcal{F}}$. If $\theta \in \mathbb{S}^1_{\mathcal{F}}$ is the limit of an unique end of leaf L_+, then L_+ is preserved by f.

Thus, f preserves every regular end of leaf. As the regular leaves are dense in $\mathbb{R}^2$, one deduces that f preserves every oriented leaf, concluding the proof. ∎

4.2. Proof of Theorem 4.1. We denote by $Reg(\mathcal{F})$ the set of regular leaves of $\mathcal{F}$ and by $\mathcal{R}(\mathcal{F})$ the set of ends of regular leaves. (Any non-singular leaf and, in particular, any regular leaf have two ends.) Recall that $\mathcal{R}(\mathcal{F})$ is a family of disjoint rays of $\mathbb{R}^2$ and therefore is cyclically ordered.

Lemma 4.4. *If D is a family of regular leaves whose union in dense in $\mathbb{R}^2$, then the set $\mathcal{D}$ of ends of the leaves in D is a separating family for the set of ends of regular leaves $\mathcal{R}(\mathcal{F})$.*

Proof. Let L_0 be a regular leaf of $\mathcal{F}$, $\sigma\colon[-1,1] \to \mathbb{R}^2$ a segment transverse to $\mathcal{F}$ with $\sigma(0) \in L_0$, and U_t the family of neighborhoods of L_0 associated to the transverse segment σ. Our assumption implies that for a dense subset of $t \in [-1,1]$, the leaf L_t belongs to $\mathcal{D}$. Consider a sequence $t_n \in [-1,1]$, $n \in \mathbb{Z}$ so that

- $L_n = L_{t_n} \in \mathcal{D}$,
- $t_n \to 0$ as $|n| \to \infty$,
- t_n has the same sign as $n \in \mathbb{Z}$.

Let L_n^+ and L_n^- be the half leaves of L_n (for the orientation given by the transverse orientation induced by σ). As L_0 is regular, one gets $\mathfrak{U}(L_0^+) = L_0^+$. This implies that L_0^+ (resp., L_0^-) is the intersection of the intervals (for the cyclic order) $[L_{-n}^+, L_n^+]$ (resp., $[L_n^-, L_{-n}^-]$) for $n > 0$. In other words, the rays L_{-n}^+, L_n^+ (resp., L_n^-, L_{-n}^-) are separating the ray L_0^+ (resp., L_0^-) from any other ray in $\mathcal{R}(\mathcal{F})$ (and indeed from any other ray of leaf, regular or not), concluding the proof. ∎

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. We choose a countable set E of regular leaves whose union is dense in $\mathbb{R}^2$. According to Lemma 4.4, the set $\mathcal{E}$ of ends of leaves in E is a countable separating subset of $\mathcal{R}(\mathcal{F})$. Thus, we may apply Theorem 1.1.

One gets a compactification of $\mathbb{R}^2$ by the disc $\mathbb{D}^2_{\mathcal{F}} \simeq \mathbb{D}^2$ so that every two distinct ends of regular leaves tend to two distinct points at the circle at infinity $\mathbb{S}^1_{\mathcal{F}}$ and these points are dense on the circle and this compactification does not depend on the choice of the family. This proves items (2) and (3) of the theorem, and also proves that these two items are enough for the uniqueness of this compactification.

It remains to prove the first item, that is, to show that the rays contained in non-regular leaves also tend to points on $\mathbb{S}^1_{\mathcal{F}}$. That is done by Lemma 2.15. ■

Remark 4.5. Let $\mathcal{F}$ be a foliation (possibly with saddles). Then, every line L transverse to $\mathcal{F}$ has 2 distinct limit points at infinity corresponding to its 2 ends.

Proof. The two ends of L are rays disjoint from the ends in $\mathcal{R}(\mathcal{F})$ (that is, of the ends of leaves of $\mathcal{F}$), as any transverse segment intersects any leaf in at most 1 point. Now, Lemma 2.15 implies that the ends of L tend to points on $\mathbb{S}^1_{\mathcal{F}}$. These points are distinct because the regular half leaves through L are between these two ends. ■

Lemma 4.6. *Let $\mathcal{F}$ be a foliation (possibly with saddles). Given any two (non-singular) leaves L_1, L_2, if the ends of L_1 and L_2 tend to the same 2 points in $\mathbb{S}^1_{\mathcal{F}}$, then*

$$L_1 = L_2.$$

Proof. Assume that $L_1 \neq L_2$ share the same end points. Then, the leaves in the strip bounded by $L_1 \cup L_2$ would have their ends on the same points in $\mathbb{S}^1_{\mathcal{F}}$ contradicting the fact that at most countably many ends of leaves share the same end point on $\mathbb{S}^1_{\mathcal{F}}$. ■

As a by-product of the proof of Lemma 4.4, we get the following lemma.

Lemma 4.7. *Let $\mathcal{F}$ be a foliation (maybe with saddle-like singular points), and let*

$$\sigma\colon [-1, 1] \to \mathbb{R}^2 \setminus \mathrm{Sing}(\mathcal{F})$$

be a transverse segment. Let $\{L_t^+\}$ and $\{L_t^-\}$ be the half leaves starting at $\sigma(t)$. Consider the map associating to $t \in (-1, 1)$, for which L_t^+ does not lie on a singular leaf, the limit point of L_t^+ on $\mathbb{S}^1_{\mathcal{F}}$. Then, t is a continuous point of this map if and only if L_t^+ is a regular end.

4.3. Points at $\mathbb{S}^1_{\mathcal{F}}$ which are limits of several ends of leaves: Hyperbolic sectors.

Lemma 4.8. *Let A and B be distinct ends of leaves. Then, the following properties are equivalent.*

- *There are no ends of regular leaves between A and B.*
- *The set of ends of leaves between A and B is at most countable.*
- *The set of ends of leaves between A and B is finite.*

Proof. First, assume that there is an end L^+ of a regular leaf L between A and B. We will prove that the interval (A, B) is uncountable.

Consider the neighborhood U_t of L associated to a transverse segment σ with $\sigma(0) \in L$. As L is regular, one gets that

$$\mathfrak{U}(L) = \bigcap_t U_t = L.$$

As a consequence, there is t so that A and B are out of U_t.

First, assume that A and B are in the same connected component of $\mathbb{R}^2 \setminus U_t$. Then, there is a line L whose left end is B and whose right end is A and which is disjoint from U_t. One deduces that one of the intervals (A, B) and (B, A) contains no end of leaf in U_t (this cannot be (A, B) which contains L^+ by assumption) and the other contains all ends of leaves in U_t, so (A, B) contains uncountably many ends of leaves as announced.

Now, assume that A and B are in distinct connected components of $\mathbb{R}^2 \setminus U_t$. Then, there is a line Γ whose left end is B, whose right end is A, and whose intersection with U_t is $\sigma([-t, t])$. As L^+ is in the interval (A, B) so that $L^+ \subset \Delta^+_\Gamma$, one deduces that all the positive half leaves L^+_r, $r \in [-t, t]$ are contained in the upper half plane Δ^+_Γ and therefore are between A and B. So, the interval (A, B) (and also (B, A)) is uncountable which is what we announced.

Conversely, if there are uncountably many ends in (A, B), one of them is the end of a regular leaf as non-regular leaves are countably many.

This proves the equivalence of the two first items. The third item implies trivially the second, so we now prove that the second implies the third.

Let A and B be two distinct ends of leaves so that (A, B) is at most countable. We consider a line δ with the following properties:

- A and B are the right and left ends of δ, respectively,
- $\delta \setminus (B \cup A)$ is a segment σ, consisting in finitely many transverse segments $a_0, \dots, a_k$ and finitely many leaf segments $b_1, \dots, b_k$, with $a_0(0) \in B$ and $a_k(1) \in A$.

Let $\Delta = \Delta^+(\delta)$ be the upper half plane bounded by δ and corresponding to the interval (A, B).

Notice that no entire leaf may be contained in Δ; otherwise, there would be uncountably many ends between A and B.

We consider the half leaves $L^+_{0,t}$ that enter Δ through $a_0(t)$. As there are only countably many ends between A and B, there is a sequence of $t_n \to 0$ so that L^+_{0,t_n} goes out of Δ through a point $\sigma(s_n)$.

Note that

- this point $\sigma(s_n)$ cannot belong to the transverse segment a_0 as a transverse segment meets a leaf in at most a point,

- the half leaves $L^+_{0,t}$, $t \in [t_{n+1}, t_n]$ need to go out of Δ.

Thus, every $L^+_{0,t}$, $t \leq t_0$, goes out of Δ at a point $\sigma(s(t))$, where $t \mapsto s(t)$ is a decreasing function. Let s_0 be the limit

$$s_0 = \lim_{t \to 0} s(t).$$

Notice that a half-leaf entering Δ through a_0 cannot go out Δ through a_0 because a transverse segment cuts a leaf in at most a point. Thus, we deduce that s_0 belongs to some a_i, $i > 0$.

We consider the compact segments $I_t \subset L^+_{0,t}$ joining $a_0(t)$ to $\sigma(s(t))$. We consider

$$\limsup_{t \to 0} I_t.$$

It is a closed subset of $\mathbb{R}^2$ consisting of B and of whole leaves contained in Δ and of a half-leaf $\widetilde{B}_1$ ending at $\sigma(s_0)$. We already noticed that no entire leaves may be contained in Δ. Thus, this limit consists in $B \cup \widetilde{B}_1$. As a consequence, the ends B and $\widetilde{B}_1$ are successive ends, $\widetilde{B}_1 \in (A, B)$, and thus $(A, \widetilde{B}_1)$ is at most countable too.

We consider $B_1 \subset \widetilde{B}_1$ the half-leaf starting at the last intersection point of $\widetilde{B}_1$ with σ. Note that B_1 starts at a point of some segment a_i, with $i > 0$.

Thus, if $B_1 \neq A$, one may iterate the argument, getting successive half leaves B_i starting at points of some transverse segment $a_{j(i)}$, where $i \mapsto j(i)$ is strictly increasing. As there are finitely many segments a_i, one gets that this inductive argument needs to stop. In other words, there is i with $B_i = A$, ending the proof:

$$[A, B] = A = B_i, B_{i-1}, \dots, B_1, B.$$

This proves that the second item is equivalent to the third. ■

The proof of Lemma 4.8 gives, as a by-product, the following lemma.

Lemma 4.9. *Assume that A and B are successive ends of leaves; that is, the interval (A, B) is empty. Then, there is an embedding of $\psi\colon [-1, 1] \times [0, 1] \to \mathbb{D}^2_{\mathcal{F}}$ so that*

- *the segments $\psi([-1, 1] \times \{t\})$, $0 \leq t < 1$, are leaf segments,*
- *$A = \psi([-1, 0) \times \{1\})$ and $B = \psi((0, 1] \times \{1\})$,*
- *the point $\psi(0, 1)$ is the point is $\mathbb{S}^1_{\mathcal{F}}$ that is the limit of both A and B.*

Definition 4.10. The embedding $\psi\colon [-1, 1] \times [0, 1] \to \mathbb{D}^2_{\mathcal{F}}$ is called a *hyperbolic sector*.

We say that two half leaves A, B are *asymptotic* if $[A, B]$ or $[B, A]$ does not contain any end of regular leaf. We already proved the next lemma.

Lemma 4.11. *To be asymptotic is an equivalence relation in the set of ends of leaves of* $\mathcal{F}$.

Each equivalence class is either finite or countable and is, as an ordered set, isomorphic to an interval of $(\mathbb{Z}, <)$.

There are at most countably many non-trivial classes.

We also already proved the following lemma.

Lemma 4.12. *Let* $\mathcal{F}$ *be a foliation (possibly with singular points of saddle type). Then, two half leaves tend to the same point* $\theta \in \mathbb{S}^1_{\mathcal{F}}$ *if and only if they are asymptotic, and every half-leaf arriving to* θ *belongs to their asymptotic class.*

In particular, if a point of $\mathbb{S}^1_{\mathcal{F}}$ is the limit of a regular end of leaf, it is the limit of a unique end of leaf.

Notice that a point at infinity which is the limit of a unique end of leaf may be the limit of a non-separated end of leaf, as shown in the next example.

Example 4.13. Let $K \subset \mathbb{R}$ be a Cantor set, and consider

$$\mathcal{P}_K = \mathbb{R}^2 \setminus (K \times [0, +\infty)).$$

Thus, $\mathcal{P}_K$ is homeomorphic to $\mathbb{R}^2$.

Let $\mathcal{F}_K$ be the restriction to $\mathcal{P}_K$ of the horizontal foliation on $\mathbb{R}^2$ (whose leaves are the $\mathbb{R} \times \{y\}$). Thus, all the leaves of the form $I \times \{0\}$, where I is a connected component of $\mathbb{R} \setminus K$, are pairwise not separated from below.

However, any two distinct ends of leaves of $\mathcal{F}_K$ tend to distinct points in $\mathbb{S}^1_{\mathcal{F}_K}$.

Remark 4.14. Assume that $\mathcal{F}$ is oriented.

If $A_0, \dots, A_k$ are successive ends of leaves, and assuming that A_0 is a right half-leaf, then A_1 is a left half-leaf and A_0 and A_1 are not separated from above.

Then, A_2 is a right half-leaf and A_1 and A_2 are not separated from below, and so on.

Thus, each non-trivial class of the asymptotic relation consists in alternately right and left ends of non-separated leaves, alternately from above and from below.

4.4. Points at infinity which are not limits of leaves: Center-like points. In this section, foliations are assumed to be non-singular.

Remark 4.15. Let $\mathcal{F}$ be a foliation of $\mathbb{R}^2$ and $o \in \mathbb{S}^1_{\mathcal{F}}$ a point so that

$$o = \bigcap_n (a_n, b_n), \quad n \in \mathbb{N},$$

where a_n, b_n are the limit points of the two ends of the same leaf L_n.

Then, a_n and b_n tend to o and o is not a limit point of an end of leaf of $\mathcal{F}$.

Proof. Consider Δ_n being the compact disc of $\mathbb{D}^2_{\mathcal{F}}$ whose boundary (as a disc) is $L_n \cup [a_n, b_n]$. Then, the Δ_n are totally ordered by the inclusion and $o \in \bigcap_n \Delta_n$. If a leaf L had an end on o, it should be contained in every Δ_n and hence contained in $\bigcap_n \Delta_n$. Thus, the two ends of L are distinct points in $\bigcap_n [a_n, b_n]$ contradicting the hypothesis. ■

We say that a point $o \in \mathbb{S}^1_{\mathcal{F}}$ satisfying the hypothesis of Remark 4.15 is a *center-like point*.

Here is a very simple example with this situation.

Example 4.16. The trivial horizontal foliation $\mathcal{H}$ admits two center-like points at infinity which are the limit points of the (vertical) y-axis (transverse to $\mathcal{H}$).

It is indeed easy to check that the following.

Remark 4.17. Given any foliation $\mathcal{F}$ of $\mathbb{R}^2$, $\mathbb{S}^1_{\mathcal{F}}$ carries at least 2 center-like points. To see that, just consider the (decreasing) intersection of the closure in $\mathbb{D}^2_{\mathcal{F}}$ of the half planes $\Delta^{\pm}_L$ for a maximal chain (given by Zorn lemma) for the inclusion.

But the situation may be much more complicated, as shown by the next example.

Example 4.18. Consider a simple closed curve $\gamma = \gamma^+ \cup \gamma^-$ of $\mathbb{R}^2$, where γ^+ and γ^- are the graphs of continuous functions $\varphi\colon [-1, 1] \to [0, 1]$ and $-\varphi$, respectively, where

- $\varphi(-1) = \varphi(1) = 0$,
- $\varphi(t) > 0$ for $t \in (-1, 1)$,
- the local maxima and minima of φ are dense in $[-1, 1]$ (some kind of Weierstrass function).

Let Δ be the open disc bounded by γ and endowed with the constant horizontal foliation $\mathcal{F}$.

Then, $\mathbb{S}^1_{\mathcal{F}} = \gamma$ and any local maximum point of γ^+ and any local minimum of γ^- are center-like points of $\mathbb{S}^1_{\mathcal{F}}$

The aim of this section is to show that the situation of Example 4.18 is in fact very common.

Lemma 4.19. *Let $\mathcal{F}$ be a foliation on $\mathbb{R}^2$. Assume that the union of leaves which are non-separated at their right side is dense in $\mathbb{R}^2$, and in the same way, that the union of leaves which are non-separated at their left side is dense in $\mathbb{R}^2$.*

Then, the set of center-like points on $\mathbb{S}^1_{\mathcal{F}}$ is a residual subset of $\mathbb{S}^1_{\mathcal{F}}$.

Proof. Fix a metric on $\mathbb{S}^1_{\mathcal{F}}$. Let $\mathcal{O}_n \subset \mathbb{S}^1_{\mathcal{F}}$ be the set of points belonging to an interval (a,b) of length less than $\frac{1}{n}$, where a, b are both ends of the same leaf of $\mathcal{F}$.

We will prove that $\mathcal{O}_n$ is a dense open subset of $\mathbb{S}^1_{\mathcal{F}}$. Then, $\bigcap_n O_n$ will be the announced residual subset.

The fact that $\mathcal{O}_n$ is open is by definition. We just need to prove the density of $\mathcal{O}_n$.

Recall that the ends of regular leaves are dense in $\mathbb{S}^1_{\mathcal{F}}$. Thus, we just need to prove that the ends of regular leaves are contained in the closure of $\mathcal{O}_n$.

Let L be a regular leaf and $\sigma\colon[-1,1] \to \mathbb{R}^2$ a positively oriented transverse segment with $\sigma_0 \in L$. We denote by L_t the leaf through σ_t, and we recall that, as L is regular, the right and left ends L_t^+, L_t^- of L_t tend to the right and left ends L^+ and L^-, respectively, as $t \to 0$.

Given any $r < s \in [-1,1]$, we denote by $U_{r,s}$, $U_{r,s}^{\text{right}}$, and $U_{r,s}^{\text{left}}$ the strip bounded by L_r and L_s, and the two closed half strips obtained by cutting $U_{r,s}$ along the segment $\sigma([r,s])$. Let $I_{r,s}^{\text{right}} \subset \mathbb{S}^1_{\mathcal{F}}$ and $I_{r,s}^{\text{left}} \subset \mathbb{S}^1_{\mathcal{F}}$ be the corresponding intervals on $\mathbb{S}^1_{\mathcal{F}}$. Notice that, as L is regular, these intervals have a length smaller than $\frac{1}{n}$ if r, s close to 0.

Our hypotheses imply that there are $t^{\text{right}}, t^{\text{left}} \in (r,s)$ so that $L_{t^{\text{right}}}$ is non-separated at the right, and $L_{t^{\text{left}}}$ is non-separated at the left.

This implies that both $U_{r,s}^{\text{right}}$ and $U_{r,s}^{\text{left}}$ contain entire leaves. Thus, $I_{r,s}^{\text{right}}$ and $I_{r,s}^{\text{left}}$ contain intervals whose both extremal points are ends of the same leaf. Taking r, s small enough, these intervals are contained in $\mathcal{O}_n$, showing that the points of $\mathbb{S}^1_{\mathcal{F}}$ corresponding to L^+ and L^- are in the closure of $\mathcal{O}_n$. This ends the proof. ∎

However, not every point o which is not the limit of an end of leaf is center-like.

Example 4.20. Let $\mathcal{F}_K$ be the foliation defined in Example 4.13 by restriction of the horizontal foliation on $\mathbb{R}^2 \setminus (K \times [0,+\infty))$, where K is a Cantor set $K \subset \mathbb{R}$. Consider a point $x \in K$ which is not the end point of a component of $\mathbb{R} \setminus K$. Then, the point $(x,0)$ corresponds to a point in $\mathbb{S}^1_{\mathcal{F}}$ which is not the limit of an end of a leaf and is not center-like.

Consider a point $o \in \mathbb{S}^1_{\mathcal{F}}$, and assume that it is not the limit point of any end of leaf. For any leaf L, we denote by $\Delta_L \subset \mathbb{D}^2_{\mathcal{F}}$ the compact disc containing o and whose frontier in $\mathbb{D}^2_{\mathcal{F}}$ is the segment $\bar{L}$ given by the closure of L. Then, $\Delta_L \cap \mathbb{S}^1_L$ is a segment I_L whose end points are the limit points of the ends of L. Note that the closed segments I_L are totally ordered by inclusion, and so are the disks Δ_L. Let us denote

$$I_o = \bigcap_L I_L \quad \text{and} \quad \Delta_o = \bigcap_L \Delta_L.$$

Then,

- if $o = I_o$, then o is a center-like point.

- Otherwise, $\partial\Delta_o \cap \overset{\circ}{\mathbb{D}}{}^2_{\mathcal{F}}$ consists in infinitely (countably) many leaves that are pairwise not separated and there is a subsequence of them whose limit is o.

5. The circle at infinity of a countable family of foliations

The aim of this section is the proof of Theorem 1.3, that is, to build the compactification associated to a countable family of foliations with saddles and prove its uniqueness.

Remark 5.1. Example 2.17 has already shown us that Theorem 1.3 is wrong for uncountable families.

The new difficulty in comparison to Theorem 4.1 is that there are no more separating families for the set of ends of all the foliations.

Example 5.2. Consider the restriction of the constant horizontal and vertical foliations to the strip $\{(x, y) \mid |x - y| < 1\}$, which is important for the study of Anosov flows. Then, every end of horizontal leaf has a unique successor or predecessor which is the end of a vertical leaf. Thus, no family can be separating.

For bypassing this difficulty, we will apply Theorem 2.1 instead of Theorem 1.1.

Proof of Theorem 1.3. The ends of regular leaves

$$\mathcal{R} = \bigcup_{i \in I} \mathcal{R}_i$$

of all the foliations $\mathcal{F}_i, i \in I$, are a family of disjoints ends of rays.

We have seen that for every foliation $\mathcal{F}_i$ the set of ends of regular leaves $\mathcal{R}_i$ admits a countable separating family, for instance, by considering regular leaves through a dense subset in $\mathbb{R}^2$.

Thus, Theorem 2.1 provides a compactification of $\mathbb{R}^2$ by $\mathbb{D}^2$ satisfying the announced properties for the regular leaves, that is, items (2) and (3).

For item (1), one needs to see that even the ends of non-regular leaves tend to points at infinity. This is given by Lemma 2.15.

For the uniqueness, a compactification satisfying the conclusion of Theorem 1.3 satisfies the fact that every non-trivial open interval $I \subset \mathbb{S}^1$ contains an uncountable subset of points which are limits of ends of leaves. As the family $\mathcal{F}_i$ is at most countable, one may choose this uncountable set as limits of ends of leaves of the same foliation $\mathcal{F}_i$. As the set of non-regular leaves of $\mathcal{F}_i$ is at most countable, one may choose this uncountable set as limits of ends of regular leaves of $\mathcal{F}_i$. Now, one may apply Theorem 2.1 which ensures the uniqueness of the compactification, ending the proof. ∎

5.1. Example: Countable families of polynomial vector fields.

Corollary 5.3. *Let $\mathcal{F} = \{\mathcal{F}_i\}_{i\in I}$, $I \subset \mathbb{N}$, be a countable family of foliations directed by polynomial vector fields on $\mathbb{R}^2$ whose singular points are all of saddle type. Then, the ends of leaves are either disjoint or coincide.*

Thus, according to Theorem 1.3, there is a unique compactification $\mathbb{D}^2_{\mathcal{F}} = \mathbb{R}^2 \cup \mathbb{S}^1_{\mathcal{F}}$ for which the ends of regular leaves of the same foliation tend to pairwise distinct points at the circle at infinity, and these ends of leaves are dense in $\mathbb{S}^1_{\mathcal{F}}$.

Proof. We just need to prove that the ends of leaves are either disjoint or equal for 2 such distinct foliations $\mathcal{F}$ and $\mathcal{G}$. Consider the tangency locus of $\mathcal{F}$ and $\mathcal{G}$, that is, an algebraic set in $\mathbb{R}^2$ which is either $\mathbb{R}^2$ (so that $\mathcal{F} = \mathcal{G}$ contradicting the assumption) or at most 1-dimensional. Thus, it consists in the union of a compact part and a finite family of disjoint rays $\delta_1, \dots, \delta_k$.

If it is compact, then every end of leaf of $\mathcal{F}$ is transverse to $\mathcal{G}$ and therefore cuts every leaf of $\mathcal{G}$ in at most 1 point: the ends are disjoints.

Otherwise, each ray δ_i either is tangent to both foliations and is therefore a common leaf (which is one of the announced possibilities) or is transverse to $\mathcal{F}$ and to $\mathcal{G}$ out of a finite set (because the tangencies on δ_i are an algebraic subset).

Thus, up to shrinking the non-tangent rays δ_i, we assume that they are transverse to both foliations. Therefore, the non-tangent rays δ_i cut every leaf of $\mathcal{F}$ in at most 1 point. This implies that every end of leaf of $\mathcal{F}$ which is not an end of leaf of $\mathcal{G}$ is transverse to $\mathcal{G}$ and thus is disjoint from any end of leaf of $\mathcal{G}$, concluding the proof. ∎

Remark 5.4. The compactification in Corollary 5.3 is in general distinct from the algebraic extension of the $\mathcal{F}_i$ on $\mathbb{RP}^2$: for instance, consider the trivial example of $\mathbb{R}^2$ endowed with the horizontal and vertical foliations. In this case, the compactification by the algebraic extension, all the leaves of the horizontal (resp., vertical) foliations tend to the same point at $\mathbb{RP}^1$ (which corresponds to 2 points for the circle at infinity).

5.2. Pseudo-Anosov flows on closed 3-manifolds. This section provides another point of view for results already shown in [6].

A pseudo-Anosov flow on a closed 3-manifold is a flow directed by a vector field X which is smooth out of a finite set $\mathcal{S}$ of periodic orbits. This flow leaves invariant 2 transverse foliations $\mathcal{F}^s$, $\mathcal{F}^u$ which are singular along the periodic orbits in $\mathcal{S}$. Along the singular set $\mathcal{S}$, the foliations $\mathcal{F}^s$, $\mathcal{F}^u$ are locally the product of a flow-orbit segment by a k-prongs singularity, $k > 2$. Finally, the natural action of the flow on its normal bundle N_X contracts uniformly, for positive times (resp., negative times), the vectors in N_X which are tangent to $\mathcal{F}^s$ (resp., to $\mathcal{F}^u$). Natural examples are

- the flows obtained as suspension of pseudo-Anosov homeomorphisms,
- the Anosov flows,
- the flow obtained from the example above by performing Dehn–Goodman–Fried surgeries along a finite set of periodic orbits.

Reference [7] shows that the orbit space of the lift $\widetilde{X}$ of X on the universal cover $\widetilde{M}$ of M is a plane $\mathbb{R}^2$, and the stable and unstable (singular) foliations of X induce a pair of foliations F^s, F^u with saddle singular points so that F^s and F^u are mutually transverse out of the singular points. Then, Theorem 1.3 applied to this pair of foliations gives back (by a different method) Fenley's compactification in [6, Theorem A]. Notice that Fenley emphasizes his compactification is not unique, whereas here the uniqueness is ensured by the requirement on the compactification in Theorem 1.3.

We will see in Section 8 that the compactification $\mathbb{D}^2_{F^s,F^u}$ associated to the pair of singular foliations F^s, F^u coincides with both compactifications $\mathbb{D}^2_{F^s}$ and $\mathbb{D}^2_{F^u}$ associated to each of these foliations, unless X is a (non-singular) suspension Anosov flow. The proof is detailed in the Anosov case; the pseudo-Anosov case will be obtained with a similar argument as a consequence of [6, Theorem 2.7]; see Remark 8.2.

5.3. Projections of $\mathbb{D}^2_{\mathcal{F}}$ on $\mathbb{D}^2_{\mathcal{F}_i}$ and center-like points on the circle at infinity. Let us start by presenting two very simple examples with opposite behaviors.

Example 5.5. Consider $\mathbb{R}^2$ endowed with the trivial horizontal and vertical foliations, $\mathcal{H}$ and $\mathcal{V}$, respectively. Then, the compactification $\mathbb{D}^2_{\mathcal{H},\mathcal{V}}$ is conjugated to the square $[-1,1]^2$ endowed with the trivial horizontal and vertical foliation. Every point $p \in \mathbb{S}^1_{\mathcal{H},\mathcal{V}}$, except the four vertices, is the limit of exactly 1 end of leaf, either horizontal (for p in the vertical sides) or vertical (for p in the horizontal sides).

The projection $\Pi_{\mathcal{H}}\colon \mathbb{D}^2_{\mathcal{H},\mathcal{V}} \to \mathbb{D}^2\mathcal{H}$ consists in collapsing the two horizontal sides, which are transformed into the center-like points of $\mathbb{S}^1_{\mathcal{H}}$.

The projection $\Pi_{\mathcal{H}}\colon \mathbb{D}^2_{\mathcal{H},\mathcal{V}} \to \mathbb{D}^2\mathcal{V}$ consists in collapsing the two vertical sides, which are transformed into the center-like points of $\mathbb{S}^1_{\mathcal{H}}$.

Example 5.6. Consider the strip $\{(x,y) \in \mathbb{R}^2 \mid |x-y| < 1\}$ endowed with the horizontal and vertical foliations $\mathcal{H}$ and $\mathcal{V}$, respectively. Then,

$$\mathbb{D}^2_{\mathcal{H},\mathcal{V}} = \mathbb{D}^2_{\mathcal{H}} = \mathbb{D}^2_{\mathcal{V}}$$

and consists in adding to two points $\pm\infty$ to the closed strip $\{(x,y) \in \mathbb{R}^2,\ |x-y| \leq 1\}$. Every point in the sides $|x-y| = 1$ are the limit of exactly 1 end of leaf of $\mathcal{H}$ and 1 end of leaf of $\mathcal{V}$, and the points $\pm\infty$ are center-like for both foliations.

These two examples show that pairs of very simple foliations may lead to different behaviors of the projection of the compactification associated to the pair on the compactification of each foliation.

Proposition 5.7 below shows that, for complicated foliations, the compactification of the pair of foliations in general coincides with the compactification of each foliation.

Proposition 5.7. *Let $\mathcal{F}$, $\mathcal{G}$ be two transverse foliations on $\mathbb{R}^2$. Assume that*

- *the union of leaves of $\mathcal{G}$ which are not separated at their right from another leaf is dense in $\mathbb{R}^2$,*
- *the union of leaves of $\mathcal{G}$ which are not separated at their left from another leaf is dense in $\mathbb{R}^2$.*

Then, the identity map on $\mathbb{R}^2$ extends as a homeomorphism from $\mathbb{D}^2_{\mathcal{F},\mathcal{G}} \to \mathbb{D}^2_{\mathcal{F}}$: in other words, $\mathbb{D}^2_{\mathcal{F},\mathcal{G}} = \mathbb{D}^2_{\mathcal{F}}$.

Proof. Assume that there is an open interval I of $\mathbb{S}^1_{\mathcal{F},\mathcal{G}}$ corresponding to no end of leaf of $\mathcal{F}$. Then, the ends of leaves of $\mathcal{G}$ are dense in I, and therefore, the projection of I on $\mathbb{S}^1_{\mathcal{G}}$ is injective.

Now, Lemma 4.19 implies that there are leaves L of $\mathcal{G}$ having both ends on I. Thus, up to changing positive to negative, every positive half-leaf of $\mathcal{F}$ through L has its end on I contradicting the definition of I.

So, the points of $\mathbb{S}^1_{\mathcal{F},\mathcal{G}}$ corresponding to ends of leaves of $\mathcal{F}$ are dense. Thus, $\mathbb{S}^1_{\mathcal{F},\mathcal{G}} = \mathbb{S}^1_{\mathcal{F}}$, concluding the proof. ∎

As a direct corollary of Proposition 5.7 and Lemma 4.19, one gets the following.

Corollary 5.8. *Let $\mathcal{F}$, $\mathcal{G}$ be two transverse foliations on $\mathbb{R}^2$ so that both $\mathcal{F}$ and $\mathcal{G}$ have density of leaves non-separated at the right and of leaves non-separated at the left.*

Then, generic points in $\mathbb{S}^1_{\mathcal{F},\mathcal{G}} = \mathbb{S}^1_{\mathcal{F}} = \mathbb{S}^1_{\mathcal{G}}$ are center-like for both foliations. (In other words, the set of points which are center-like for both foliations is a residual subset of $\mathbb{S}^1_{\mathcal{F},\mathcal{G}}$.)

5.4. Hyperbolic sectors. In the case of one foliation, we have seen that if several ends of leaves have the same limit points on the circle at infinity, then they are ordered as a segment of $\mathbb{Z}$ and two successive ends bound a hyperbolic sector. These hyperbolic sectors have a very precise model, which allows us to understand the position of a transverse foliation.

Lemma 5.9. *Let $\mathcal{F}$ and $\mathcal{G}$ be two transverse foliations on $\mathbb{R}^2$, and consider the projection $\pi_{\mathcal{F}}\colon \mathbb{D}^2_{\mathcal{F},\mathcal{G}} \to \mathbb{D}^2_{\mathcal{F}}$. Assume that $p \in \mathbb{D}^2_{\mathcal{F}}$ is the corner of a hyperbolic sector bounded by the ends A and B of leaves of $\mathcal{F}$.*

Then, there is a non-empty interval $I_{\mathcal{G}}$ of ends of leaves of $\mathcal{G}$ ending at p in $\mathbb{D}^2_{\mathcal{F}}$. Furthermore,

- *either $I_{\mathcal{G}}$ consists in a unique end of leaf C of $\mathcal{G}$ and A, B, C tend to tend the same point at infinity in $\mathbb{D}^2_{\mathcal{F},\mathcal{G}}$*
- *or $\pi_{\mathcal{F}}^{-1}(I_G)$ is a closed interval on the circle $\mathbb{S}^1_{\mathcal{F},\mathcal{G}}$ whose interior consists in regular ends of leaves of $\mathcal{G}$.*

Proof. Just use the model $[-1,1] \times [0,1]$, where p is the point $(0,1)$, $A = [-1,0) \times \{1\}$ and $B = (0,1] \times \{1\}$, and the horizontal segments $[-1,1] \times \{t\}$, $0 \le t < 1$, are $\mathcal{F}$-leaf segments. We can choose this model so that the vertical sides $\{-1\} \times [0,1]$ and $\{1\} \times [0,1]$ are leaves segments of $\mathcal{G}$. Consider the $\mathcal{G}$-leaves through $[-1,1] \times \{0\}$. The leaves reaching A and the leaves reaching B are two non-empty intervals, open in $[-1,1] \times \{0\}$ and disjoint. By connectedness of $[-1,1] \times \{0\}$, there are leaves, corresponding to a closed interval of $[-1,1]$ which reach neither A nor B: these leaves tend to $(0,1)$ in the model, that is, to $p \in \mathbb{S}^1_{\mathcal{F}}$.

Assume that this interval is not reduced to a single end of leaf of $\mathcal{G}$ and consider an end C in the interior of this interval and assume that C is, for instance, a right end. Consider the neighborhoods U_t^{right} of C defined in Section 3. Then, $\bigcap_t U_t^{\text{right}}$ consists of C and a (maybe empty) set of entire leaves of $\mathcal{G}$ contained in the hyperbolic sector. Assume that this set is not empty, and let D be such a leaf of $\mathcal{G}$. Every leaf of $\mathcal{F}$ cutting D has a half-leaf contained in the hyperbolic sector, contradicting the definition of hyperbolic sector. Thus, $C = \bigcap_t U_t^{\text{right}}$, meaning that C is a regular end of leaf of $\mathcal{G}$, ending the proof. ∎

As a straightforward consequence, one gets the following corollary.

Corollary 5.10. *Let $\mathcal{F} = \{\mathcal{F}_i\}_{i \in I}$, $I \subset \mathbb{N}$, be an at most countable family of pairwise transverse foliations on $\mathbb{R}^2$. Consider a point $p \in \mathbb{D}^2_{\mathcal{F}}$. Then,*

- *either at most 1 end of leaf of each $\mathcal{F}_i$ has p as its limit,*
- *or the set of ends tending to p is ordered as an interval of $\mathbb{Z}$ and, between any two successive ends of leaves of the same $\mathcal{F}_i$, there is exactly 1 end of a leaf of each $\mathcal{F}_j$, $j \neq i$.*

6. The circle at infinity for orientable laminations

6.1. The circle at infinity of a lamination. The way we proposed to compactify $\mathbb{R}^2$ can be generalized for any object providing a family of rays admitting a separating set.

For instance, what about laminations? The theory cannot be extended without hypotheses. An evident obstruction is if there are too few leaves to limit to a dense subset of a circle at infinity. But there are more subtle issues as shown by Example 6.1 below.

Example 6.1. There are closed laminations whose leaves are recurrent. For instance, consider a Plykin attractor on $\mathbb{R}^2$: it is a compact minimal lamination (by the unstable manifolds).

If you consider now a Plykin attractor on

$$\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\},$$

where ∞ belongs to the attractor, we get a closed lamination on $\mathbb{R}^2$, where every leaf is unbounded but recurrent.

Notice that the recurrent lamination in Example 6.1 is not orientable. We show that Poincaré–Bendixson argument applies to orientable laminations.

Lemma 6.2. *Let $\mathcal{L}$ be a closed orientable lamination of $\mathbb{R}^2$. Given any leaf L, either the closure $\overline{L}$ contains a compact leaf or L is a line.*

Proof. If $\overline{L} = L$, then L is either a compact leaf or is a line (i.e., is properly embedded in $\mathbb{R}^2$). Assume now that $\overline{L} \setminus L$ contains a point $x \in \mathbb{R}^2$. We fix an orientation of $\mathcal{L}$. Chose a segment $\sigma\colon [-1, 1]$ transverse to $\mathcal{L}$ so that $\sigma(0) = x$, and so that σ cuts positively every leaf. The hypothesis implies that L cuts σ in an infinite set. Consider 2 successive (for the order in the leaf L) intersection points z_0, z_1. Then, one gets a simple closed curve δ in $\mathbb{R}^2$ by concatenation of the segments $[z_0, z_1]_L$ and $[z_1, z_0]_\sigma$ joining z_0 to z_1, in L and in σ, respectively.

Consider the disc Δ bounded by δ. Every leaf cuts δ with the same orientation; that is, either every leaf enter in Δ or every leaf goes out of Δ. Up to reversing the orientation, one may assume that every leaf enters in Δ and in particular, L enters in Δ. In other words, there is a positive half-leaf L_+ included in Δ. This half-leaf cannot be recurrent (otherwise, it would cut again $[z_0, z_1]_\sigma$ and for that it needs to go out of Δ). Furthermore, we have the following claim.

Claim 6.3. No other leaf $L' \neq L$ can accumulate on L: $L \cap \overline{L'} = \emptyset$ if $L' \neq L$.

Proof. If L' accumulates on L, it cuts $[z_0, z_1]_\sigma$ in an infinite set. ■

Thus, the ω-limit set is $\omega(L) = \overline{L}_+ \setminus L_+$ and is not empty. Consider $y \in \overline{L}_+ \setminus L_+$. The leaf L_y is contained in Δ. Either L_y is compact and $\omega(L) = L_y$ and we are done, or $\overline{L}_y \setminus L_y \neq \emptyset$. In that case, repeating the argument of Claim 6.3 replacing L by L_y, one gets that L_y is not accumulated by any leaf, in particular, by L, contradicting the definition of L_y. ■

We are now ready to extend Theorem 4.1 to the case of orientable laminations.

Theorem 6.1. *Let $\mathcal{L}$ be a closed orientable lamination of $\mathbb{R}^2$ with no compact leaf, and assume that the set of leaves of $\mathcal{L}$ is uncountable. Then, there is a compactification $\mathbb{D}^2_{\mathcal{L}} \simeq$ of $\mathbb{R}^2$ by adding a circle at infinity $\mathbb{S}^1_{\mathcal{L}} = \partial\mathbb{D}^2_{\mathcal{F}}$ with the following properties:*

- *any half-leaf tends to a point at infinity,*
- *given a point $\theta \in \mathbb{S}^1_{\mathcal{L}}$, the set of ends of leaves tending to θ is at most countable,*
- *for any non-empty open subset I of $\mathbb{S}^1_{\mathcal{L}}$, the set of points in I corresponding to limits of ends of leaves is uncountable.*

Furthermore, this compactification of $\mathbb{R}^2$ by $\mathbb{D}^2$ with these three properties is unique, up to a homeomorphism of the disc $\mathbb{D}^2$.

Let me just give a sketch of the proof.

Proof. The lamination $\mathcal{L}$ is assumed to be oriented and without compact leaves so that every leaf is a line, according to Lemma 6.2.

According to Cantor–Bendixson theorem, see, for instance, [12], the lamination $\mathcal{L}$ can be written in a unique way as union $\mathcal{L} = \mathcal{L}_0 \cup \mathcal{L}_1$ of two disjoint laminations, where $\mathcal{L}_0$ is a closed lamination with no isolated leaves and $\mathcal{L}_1$ consists in a countable set of leaves.

A leaf $L \in \mathcal{L}_0$ is called *regular* if it is accumulated on both sides by leaves in $\mathcal{L}_0$ and is separated from any other leaf of $\mathcal{L}_0$. The same proof as for foliations shows that the set of leaves in $\mathcal{L}_0$ which are not regular are at most countable.

Finally, as for foliations, one considers the set $\mathcal{R}$ of germs of rays contained in regular leaves of $\mathcal{L}_0$. We consider a countable set $\mathcal{D}$ of regular leaves, whose union is dense in $\mathcal{L}_0$, and as for the case of foliations, one proves that the rays in $\mathcal{D}$ are separating for $\mathcal{R}$.

Then, we apply Theorem 1.1 and we get the announced canonical compactification. ■

When $\mathcal{L}$ is transversely a perfect compact set (that is, there is a transverse segment σ through every point $x \in \mathcal{L}$ so that $\sigma \cap \mathcal{L}$ is a compact set without isolated points), then the compactification given by Theorem 6.1 seems very natural: any homeomorphism h of $\mathbb{R}^2$ preserving $\mathcal{L}$ extends on $\mathbb{S}^1_{\mathcal{L}}$ as a homeomorphism H of $\mathbb{D}^2_{\mathcal{L}}$, and the restriction $H|_{\mathbb{S}^1_{\mathcal{L}}}$ is the identity map if and only if h preserves every leaf of $\mathcal{L}$. That is, it is more the case if $\mathcal{L}$ has isolated leaves.

For lamination with isolated leaves, Theorem 6.1 just ignores the countable part $\mathcal{L}_1$ of $\mathcal{L}$ (in the Cantor–Bendixson decomposition of $\mathcal{L}$). We will now propose a canonical compactification which takes into account this countable part.

We start by looking at two very different examples of countable oriented laminations.

Example 6.4. Consider the lamination $\mathcal{L} = \mathbb{R} \times \mathbb{Z}$. Then, $\mathcal{L}$ does not admit any compactification by a circle at infinity so that any homeomorphism h preserving $\mathcal{L}$ extends on the circle at infinity.

The reason is that any end of leaf is isolated for the cyclic order. Thus, the corresponding points on the circle at infinity of a compactification cannot be dense. Thus, some connected component Δ of the complement of the lamination is adjacent to a non-empty open interval J on the circle where no leaf is arriving. Any homeomorphism supported on this component Δ preserves the lamination, but some of them have no continuous extension on J.

Example 6.5. Consider a hyperbolic surface S of finite volume, and consider a set ℓ of essential disjoint simple closed geodesics on S. Then, the lift $\mathcal{L}$ of ℓ on the universal cover $\widetilde{S} = \mathring{\mathbb{D}}^2$ is a countable, discrete lamination by geodesics of the Poincaré disc so that the ends of leaves tend to points on the circle:

$$\mathbb{S}^1 = \partial\mathbb{D}^2,$$

and the set of such limit points is dense on $\mathbb{S}^1$ as the action of $\pi_1 S$ on $\mathbb{S}^1$ is minimal.

In this example, the lamination is transversely discrete, but the set of ends of leaves is a separating set for itself for the cyclic order.

In Example 6.5 above, what implies the existence of a separating set is the minimality of the action on the circle at infinity of a natural compactification.

In order to propose a canonical compactification for a closed, oriented lamination without compact leaves, we need to determine what part of a cyclically totally ordered set admits a separating subset, that is, what is done in the next proposition whose proof is left to the reader.

A totally cyclically ordered set E is *self-separating* if it is a separating subset for itself, and its cardinal is $\#E \geq 2$.

Proposition 6.6. *Let X be a set endowed with a total cyclic order. Consider the relation on X defined as follows: $x \approx y$ if one of the intervals $[x, y]$ or $[y, x]$ does not contain any self-separating subset E (with $\#E \geq 2$). Then,*

- *the relation $\approx$ is an equivalence relation,*
- *each equivalence class is an interval,*
- *the cyclic order on X induces a total cyclic order on the quotient $X/\approx$;*

  furthermore, $X/\approx$ is either a single point or is an infinite self-separating set.

Note that any two distinct points in a self-separating set belong to distinct classes so that $\#(X/\approx) = 1$ if and only if X does not contain any (non-trivial) self-separating subsets. Otherwise, $\#(X/\approx) = \infty$.

The canonical compactification is now given by Theorem 6.2 below.

Theorem 6.2. *Let $\mathcal{L}$ be a closed oriented lamination of the plane $\mathbb{R}^2$, with no compact leaf, and let $\mathcal{R}$ be the set of ends of leaves of $\mathcal{L}$. As the ends of leaves are disjoints rays, the set $\mathcal{R}$ is totally cyclically ordered. Assume that $\#(\mathcal{R}/\approx) > 1$.*

Then, there is a unique compactification $\mathbb{D}^2_{\mathcal{L}}$ of $\mathbb{R}^2$ by adding a circle at infinity $\mathbb{S}^1_{\mathcal{L}}$ so that

- *any end of leaf of $\mathcal{L}$ tends to a point in $\mathbb{S}^1_{\mathcal{L}}$,*
- *the set of points in $\mathbb{S}^1_{\mathcal{L}}$ which are the limit of an end of leaf is dense in $\mathbb{S}^1$,*
- *two ends of leaves tend to the same point in $\mathbb{S}^1_{\mathcal{L}}$ if and only if they belong to the same class in $\mathcal{R}/\approx$.*

Proof. Just apply Theorem 1.1 to a subset $E \subset \mathcal{R}$ containing exactly 1 representative in each class of $\approx$. One checks that the compactification obtained satisfies the announced properties and does not depend on the choice of E. ∎

Remark 6.7. Every class of $\approx$ in $\mathcal{R}$ is at most countable because the set of ends of regular leaves in the perfect part $\mathcal{L}_0$ is self-separating.

This compactification takes into account more leaves than the compactification given by Theorem 6.1, but it is still may have unexpected behaviors.

Example 6.8. Consider a non-compact hyperbolic surface S of finite volume and consider a closed lamination ℓ defined by two disjoint freely homotopic essential closed curves and a closed (but non-compact) non essential leaf whose both ends tend to the same puncture of S.

Then, the lift $\mathcal{L}$ of ℓ on the universal cover $\widetilde{S} = \mathring{\mathbb{D}}^2$ is a countable, discrete lamination of the Poincaré disc so that the ends of leaves each tend to points on the circle $\mathbb{S}^1 = \partial\mathbb{D}^2$; the set of such limit points is dense on $\mathbb{S}^1$ (again as the action of $\pi_1 S$ on $\mathbb{S}^1$ is minimal).

In this example, however, there are pairs of leaves which share the same limits of their both ends, and there are leaves whose both ends tend to the same point.

Given a closed oriented lamination $\mathcal{L}$ with no compact leaves and its Cantor–Bendixson decomposition $\mathcal{L} = \mathcal{L}_0 \cup \mathcal{L}_1$ ($\mathcal{L}_0$ is a closed lamination without isolated leaves and $\mathcal{L}_1$ is countable), Theorem 6.2 takes into account the part of the ends of leaves in $\mathcal{L}_1$ with separating subsets, in contrast with Theorem 6.1. From my personal taste, the main issue in Theorem 6.2 is that I did not find any natural criterion to calculate the equivalence classes of $\approx$ in $\mathcal{L}_1$. In fact, Lemma 6.9 below seems to present as paradoxal the fact that $\mathcal{L}_1$ may have separating subsets.

Lemma 6.9. *Let $D \subset \mathbb{R}$ be a countable compact subset, ordered by $\mathbb{R}$. Then, D does not contain any self-separating subsets $\mathcal{E} \subset D$.*

Proof. If $\mathcal{E}$ is a non-trivial self-separating subset, then there is an increasing bijection from $\mathcal{E} \setminus \{\min \mathcal{E}, \max \mathcal{E}\}$ to $\mathbb{Q} \cap (0, 1)$. This increasing bijection extends in a unique way in a (non-strictly) increasing map from $\mathbb{R} \to [0, 1]$. This map is continuous and the image of D is $[0, 1]$.

Thus, D is not countable. ∎

Lemma 6.9 tells us that the separating property of a closed countable lamination cannot be obtained locally (in foliated charts of the lamination). One deduces the following proposition.

Proposition 6.10. *Let $\mathcal{L}$ be a closed countable oriented lamination of $\mathring{\mathbb{D}}^2$ so that every end of leaf tends to a point on $\mathbb{S}^1$ and the set of such limit points are dense in $\mathbb{S}^1$.*

Then, given any non-empty open interval $I \subset \mathbb{S}^1$, there is $L \in \mathcal{L}$ whose both ends have their limits in I.

More precisely, any neighborhood of I in $\mathbb{D}^2$ contains an entire leaf of $\mathcal{L}$.

Proof. Consider two half leaves whose limit points are $x_0 \neq y_0 \in I$, $[x_0, y_0] \subset I$, and a simple segment $\sigma_0 \subset \mathbb{R}^2$ joining these two leaves, and so that the union of σ_0, the two half leaves, and $[x, y]$ is a closed path in $\mathbb{D}^2$ bounding a compact disc Δ_0. One may choose σ_0 to be the concatenation of a finite family of segments, successively transverse to $\mathcal{L}$ and lying along leaf segments. If no leaves are entirely contained in Δ_0, then every leaf having an end in I cuts σ_0. As the end points of leaves are infinitely many and the segment forming σ_0 are finitely many, one deduces that there are $x_0 < x < y < y_0$ and two half leaves ending at x and y and whose first intersection points with σ_0 are the endpoints of the same segment $\sigma \subset \sigma_0$ transverse to $\mathcal{L}$.

One considers the disc $\Delta \subset \mathbb{D}^2$ bounded by $[x, y] \subset I$, the two half leaves whose limits are points x and $y \in I$, and the segment σ transverse to $\mathcal{L}$ and joining these two leaves. If no leaves are contained in Δ, then every leaf having an end in I cuts σ.

On the other hand, any dense subset of an interval J of $\mathbb{R}$ is self-separating; thus, the set of points of $[x, y]$ which are limits of ends of leaves of $\mathcal{L}$ is self-separating. One deduces that the compact set $\sigma \cap \mathcal{L}$ contains a self-separating subset, but it is a countable compact set, and this contradicts Lemma 6.9. ∎

This proposition says that the separating property for a countable oriented lamination is obtained by leaves in small neighborhoods of the points at infinity.

6.2. Families of transverse laminations. Transversality does not imply in general the compactness of the intersection of two leaves of transverse laminations. But this compactness is our main hypothesis for the compactification associated to families of foliations. This leads to a new issue for defining the circle at infinity associated to a family of laminations.

However, if two lines $L_1, L_2 \subset \mathbb{R}$ intersect always with the same orientation, then $\#L_1 \cap L_2 \leq 1$. One deduces that Theorem 1.2 extends without difficulties to countable families of oriented closed laminations intersecting pairwise transversely and always with the same orientation.

Theorem 6.3. *Let $\mathcal{L} = \{\mathcal{L}_i\}$, $i \in I \subset \mathbb{N}$, be a family of closed orientable laminations of $\mathbb{R}^2$ with no compact leaves, and so that the set of leaves of $\mathcal{L}_i$ is uncountable. We assume that the laminations are pairwise transverse with constant orientation of the intersections. Then, there is a compactification $\mathbb{D}^2_{\mathcal{L}} \simeq$ of $\mathbb{R}^2$ by adding a circle at infinity $\mathbb{S}^1_{\mathcal{L}} = \partial\mathbb{D}^2_{\mathcal{L}}$ with the following properties:*

- *any half-leaf tends to a point at infinity,*
- *given a point $\theta \in \mathbb{S}^1_{\mathcal{L}}$, the set of ends of leaves tending to θ is at most countable,*
- *for any non-empty open subset I of $\mathbb{S}^1_{\mathcal{L}}$, the set of points in I corresponding to limits of ends of leaves is uncountable.*

Furthermore, this compactification of $\mathbb{R}^2$ by $\mathbb{D}^2$ with these three properties is unique, up to a homeomorphism of the disc $\mathbb{D}^2$.

7. Actions on a bifoliated plane

We have seen that any homeomorphism $h \in \mathrm{Homeo}(\mathbb{R}^2)$ preserving an at most countable family of transverse foliations $\mathcal{F}$ admits a unique extension as a homeomorphism on the compactification $\mathbb{D}^2_{\mathcal{F}}$.

Thus, if $H \hookrightarrow \mathrm{Homeo}(\mathbb{R}^2)$ is a group acting on $\mathbb{R}^2$ and preserving the (at most countable) family of transverse foliations $\mathcal{F}$, then this action extends in an action to $\mathbb{D}^2_{\mathcal{F}}$. By restriction to the circle at infinity, one gets an action of H on $\mathbb{S}^1_{\mathcal{F}}$.

If $H \hookrightarrow \mathrm{Homeo}(\mathbb{R}^2)$ is a group acting on $\mathbb{R}^2$ and preserving a family of foliations $\mathcal{F}$, we say that *the action is minimal on the leaves of $\mathcal{F}$* if $H(L)$ is dense in $\mathbb{R}^2$ for every leaf L of a foliation of the family $\mathcal{F}$.

7.1. Faithfulness.

Proposition 7.1. *Let $\mathcal{F}$ be a foliation and $h \in \mathrm{Homeo}(\mathbb{R}^2)$ a homeomorphism preserving $\mathcal{F}$. Then, the action of h on $\mathbb{S}^1_{\mathcal{F}}$ is the identity map if and only if $h(L) = L$ for any leaf L, and h preserves the orientation of the leaves.*

Proof. If h preserves every leaf and its orientation, then it preserves any limit of its ends. As these limits of ends are dense in $\mathbb{S}^1_{\mathcal{F}}$, one gets that the homeomorphism induced by h on $\mathbb{S}^1_{\mathcal{F}}$ is the identity map.

Conversely, if h induces the identity map on $\mathbb{S}^1_{\mathcal{F}}$, then for every leaf L the leaf $h(L)$ has the same limit of ends as L. According to Lemma 4.6, this implies $h(L) = L$ as announced. ∎

Corollary 7.2. *Let $\mathcal{F} = \{\mathcal{F}_i\}, i \in I \subset \mathbb{N}$, be a family of at least 2 transverse foliations. Let $h \in \mathrm{Homeo}(\mathbb{R}^2)$ be a homeomorphism preserving each foliation $\mathcal{F}_i$. Then, the action of h on $\mathbb{S}^1_{\mathcal{F}}$ is the identity map if and only if h itself is the identity map.*

Proof. If the induced homeomorphism induced by h on $\mathbb{S}^1_{\mathcal{F}}$ is the identity map, then the same happens to homeomorphism induced by h on every $\mathbb{S}^1_{\mathcal{F}_i}$ (because they are quotient of $\mathbb{S}^1_{\mathcal{F}}$). Thus, Proposition 7.1 implies that h preserves each leaf of each $\mathcal{F}_i$. As every point of $\mathbb{R}^2$ is the unique intersection point of the leaves through it, one deduces that every point of $\mathbb{R}^2$ is fixed by h and h is the identity map. ∎

7.2. Orientations and injectivity of the projections. Let $\mathcal{F}$ be a foliation of the plane $\mathbb{R}^2$, endowed with an orientation and a transverse orientation. Let $\mathcal{G} \subset \mathrm{Homeo}(\mathbb{R}^2)$ be a group of homeomorphisms preserving (globally) $\mathcal{F}$. Let $\mathcal{G}^+$ (resp., $\mathcal{G}_+$) be the index at most 2 subgroup consisting of the elements of $\mathcal{G}$ preserving the orientation (resp., the transverse orientation) of $\mathcal{F}$, and $\mathcal{G}^+_+ = \mathcal{G}^+ \cap \mathcal{G}_+$ the subgroup of elements preserving both orientations. Then, we have the following lemma.

Lemma 7.3. *If one of the groups $\mathcal{G}$, $\mathcal{G}^+$, $\mathcal{G}_+$, $\mathcal{G}^+_+$ acts minimally on the leaves of $\mathcal{F}$, then so does each of these 4 groups.*

We will indeed prove Lemma 7.4 for which Lemma 7.3 is a particular case.

Lemma 7.4. *Let $\mathcal{G}$ be a group acting minimally on the leaves of a foliation $\mathcal{F}$ of $\mathbb{R}^2$ and $\mathcal{H} \subset \mathcal{G}$ a subgroup of finite index. Then, $\mathcal{H}$ acts minimally on the leaves of $\mathcal{F}$.*

Proof. By assumption, the group $\mathcal{G}$ acts minimally on the leaves of $\mathcal{F}$, and consider a leaf L. As $\mathcal{H}$ is a finite-index subgroup, there are $g_1, \dots, g_n \in \mathcal{G}$ so that for any $g \in \mathcal{G}$ there is $i \in \{1, \dots, n\}$ with $g.\mathcal{H} = g_i\mathcal{H}$. Let us denote $\mathcal{H}_i = g_i \cdot \mathcal{H}$. In particular, $\mathcal{G} = \bigcup_i \mathcal{H}_i$, and then,

$$\mathbb{R}^2 = \bigcup_i \overline{\mathcal{H}_i(L)}.$$

Consider any open subset O of $\mathbb{R}^2$:

$$O = O \cap \bigcup_i \overline{\mathcal{H}_i(L)} = \bigcup_i (O \cap \overline{\mathcal{H}_i(L)}).$$

The open set O is a Baire space so that the union of finitely many closed sets with empty interior has empty interior: one deduces that at least one of the $O \cap \overline{\mathcal{H}_i(L)}$ has non-empty interior. One deduces that the union $\bigcup_i \overset{\circ}{\overline{\mathcal{H}_i(L)}}$ of the interiors of the $\overline{\mathcal{H}_i(L)}$ is dense in $\mathbb{R}^2$.

Notice that for every i and every g there is j so that $g(\mathcal{H}_i(L)) = \mathcal{H}_j(L)$.

Consider $\mathbb{R}^2 \setminus \bigcup_i \overset{\circ}{\overline{\mathcal{H}_i(L)}}$. It is a $\mathcal{G}$-invariant closed set, saturated for the foliation $\mathcal{F}$, and with empty interior. As every $\mathcal{G}$-orbit is dense, one deduces that this set is empty.

Thus,

$$\mathbb{R}^2 = \bigcup_i \overset{\circ}{\overline{\mathcal{H}_i(L)}}.$$

The open sets $\overset{\circ}{\overline{\mathcal{H}_i(L)}}$ are images of each other by homeomorphisms in $\mathcal{G}$, and in particular, they are all non-empty.

As $\mathbb{R}^2$ is connected, one deduces that the open sets $\overset{\circ}{\overline{\mathcal{H}_i(L)}}$ are not pairwise disjoint. Let $k \in \{1, \dots, n\}$ be the maximum number so that there are distinct $i_1, \dots, i_k$ with

$$\bigcap_1^k \overset{\circ}{\overline{\mathcal{H}_{i_j}(L)}} \neq \emptyset.$$

As the $\overset{\circ}{\overline{\mathcal{H}_i(L)}}$ are not pairwise disjoint, we know that $k \geq 2$. We will prove, arguing by contradiction.

Claim 7.5. $k = n$.

Proof. For that, we assume that $k < n$.

Then, we consider the union of all the intersections of k of these open sets. This union is an $\mathcal{F}$-saturated $\mathcal{G}$-invariant non-empty set and hence is dense. Its complement is an $\mathcal{F}$-saturated invariant closed set with empty interior and therefore is empty.

Thus, $\mathbb{R}^2$ is the union of these open sets. Now, again, the connexity of $\mathbb{R}^2$ implies that these open sets are not pairwise disjoint. This provides a non-empty intersection of 2 distinct of these sets, that is, a non-empty intersection of more than k of the $\overset{\circ}{\overline{\mathcal{H}_i(L)}}$, contradicting the choice of k. This shows $k = n$, proving the claim. ∎

Thus,

$$\bigcap_1^n \overset{\circ}{\overline{\mathcal{H}_i(L)}}$$

is a non-empty, $\mathcal{G}$-invariant open set saturated for the foliation $\mathcal{F}$, and thus, it is dense in $\mathbb{R}^2$.

We just proved that $\mathcal{H}(L)$ is dense in $\mathbb{R}^2$, concluding the proof. ∎

We will use the next straightforward corollary of Lemma 7.3.

Corollary 7.6. *Let $H \subset \mathrm{Homeo}(\mathbb{R}^2)$ be a group preserving a foliation $\mathcal{F}$ and acting minimally on the leaves. Assume that L is a leaf which is not separated at the right and from below. Then, the union of the leaves $h(L)$, $h \in H$, which are non-separated at the right and from below, is dense in $\mathbb{R}^2$ (the same holds, changing right by left and/or below by above).*

As a direct consequence of Proposition 5.7 and Corollary 7.6, we get the following proposition.

Proposition 7.7. *Let $\mathcal{F}$, $\mathcal{G}$ be two transverse foliations of $\mathbb{R}^2$ and $H \subset \mathrm{Homeo}(\mathbb{R}^2)$ preserving $\mathcal{F}$ and $\mathcal{G}$. Assume that the orbit of every leaf of $\mathcal{G}$ is dense in $\mathbb{R}^2$.*

If $\mathcal{G}$ has a non-separated leaf, then the projection of $\Pi_{\mathcal{F}}\colon \mathbb{D}^2_{\mathcal{F},\mathcal{G}} \to \mathbb{D}^2_{\mathcal{F}}$ is injective.

Proof. If $\mathcal{G}$ has a non-separated leaf L_1 at the right, it is non-separated from a leaf L_2 which is non-separated at the left. Now, Corollary 7.6 asserts that the leaves of $\mathcal{G}$ non-separated at the left as well as the leaves non-separated at the right are dense in $\mathbb{R}^2$. Now, Proposition 5.7 asserts that $\Pi_{\mathcal{F}}$ is a homeomorphism, concluding the proof. ∎

7.3. Minimality of the action on the circle at infinity. Theorem 7.1 is a slightly stronger version of Theorem 1.4.

Theorem 7.1. *Let $\mathcal{F}$ be a foliation on the plane $\mathbb{R}^2$ and $H \subset \mathrm{Homeo}(\mathbb{R}^2)$ preserving the foliation $\mathcal{F}$.*

(1) *If the action of H on $\mathbb{S}^1_{\mathcal{F}}$ is minimal, then the foliation $\mathcal{F}$ admits non-separated leaves from above and non-separated leaves from below.*
(2) *Conversely, if the foliation $\mathcal{F}$ admits non-separated leaves from above and non-separated leaves from below and if the orbit of every leaf is dense in $\mathbb{R}^2$, then the action of H on $\mathbb{S}^1_{\mathcal{F}}$ is minimal.*

We will see with Theorem 9.1 that the minimality of the action on the leaves is not a necessary condition for the minimality of the action on the circle at infinity.

Item (1) of Theorem 7.1 is a consequence of Proposition 7.8 below.

Proposition 7.8. *Let $\mathcal{F}$ be a foliation of $\mathbb{R}^2$, and assume that $\mathcal{F}$ has no non-separated leaves from below. (In other words, any two leaves L_1, L_2 are separated from below.) Given any leaf L, we denote by Δ^+_L the closure on $\mathbb{D}^2_{\mathcal{F}}$ of the upper half plane of $\mathbb{R}^2$ bounded by L.*

Then, $\bigcap_{L\in\mathcal{F}} \Delta^+_L$ is non-empty and consists in an unique point $O_{\mathcal{F}}$ on $\mathbb{S}^1_{\mathcal{F}}$. As a consequence, any $h \in \mathrm{Homeo}(\mathbb{D}^2_{\mathcal{F}})$ preserving $\mathcal{F}$ admits $O_{\mathcal{F}}$ as a fixed point:

$$h(O_{\mathcal{F}}) = O_{\mathcal{F}}.$$

Proof. We introduce a relation on the set (still denoted by $\mathcal{F}$) of the leaves of $\mathcal{F}$ as follows: $L_1 \preceq L_2$ if there is a positively oriented transverse segment σ starting at L_1 and ending at L_2. One easily checks that $\preceq$ is a partial order relation on $\mathcal{F}$.

Due to the connexity of $\mathbb{R}^2$, one gets the following claim.

Claim 7.9. Given any pair of leaves $L, \widetilde{L} \in \mathcal{F}$, there is $k \geq 0$ and $L_0, \dots, L_k \in \mathcal{F}$ so that

- for any $i \in \{0, \dots, k-1\}$ the leaves L_i and L_{i+1} are comparable for $\preceq$ (that is, $L_i \preceq L_{i+1}$ or $L_{i+1} \preceq L_i$),
- $L = L_0$ and $L' = L_k$.

Proof. There is a countable family of segments in $\mathbb{R}^2$ transverse to $\mathcal{F}$ so that every leaf L cuts at least one of these segments. The set of leaves cutting a given segment induces a connected open set of $\mathbb{R}^2$. Given any two points in $\mathbb{R}^2$, one considers a compact path joining these two points. By compacity, it is covered by a finite family of these open sets. One concludes easily. ∎

We denote by $\prec L, L' \succ \; \in \mathbb{N}$ the minimum value of such a number k. One easily checks that $\prec \cdot, \cdot \succ$ is a distance on the set of leaves $\mathcal{F}$.

Up to now, this could be done for any foliation $\mathcal{F}$. In this setting, our hypothesis that $\mathcal{F}$ does not admit leaves which are non-separate from below is translated as follows.

Claim 7.10. Assume that $L_0, L_1, L_2 \in \mathcal{F}$ are three leaves so that $L_0 \preceq L_1$ and $L_0 \preceq L_2$. Then, L_1 and L_2 are comparable for $\preceq$.

Proof. We assume that the leaves L_i are distinct; otherwise, there is nothing to do. Let $\sigma_i \colon [0,1] \to \mathbb{R}^2$, $i = 1, 2$, transverse to $\mathcal{F}$ and positively oriented so that $\sigma_i(0) \in L_0$ and $\sigma_i(1) \in L_i$.

Let $I = \{t \in [0,1] \mid L(\sigma_1(t)) \cap \sigma_2 \neq \emptyset\}$ and $J = \{t \in [0,1] \mid L(\sigma_1(t)) \cap \sigma_2 \neq \emptyset\}$. As $\mathbb{R}^2$ is simply connected, one shows that I and J are connected and each of them contains 0.

Let $t_1 = \sup I$ and $t_2 = \sup J$. For any $t \in [0, t_1)$, let $\tilde{t} \in J$ so that $L(\sigma_1(t)) = L(\sigma_2(\tilde{t}))$. In particular, $\tilde{t}$ tends to t_2 as t tends to t_1.

Thus, the leaves $L(\sigma_1(t_1))$ and $L(\sigma_2(t_2))$ are accumulated from below by the leaves $L(\sigma_1(t)) = L(\sigma_2(\tilde{t}))$; thus, are non-separated from below. By assumption on $\mathcal{F}$, this implies that they are equal:

$$L(\sigma_1(t_1)) = L(\sigma_2(t_2)).$$

If $t_1 < 1$ and $t_2 < 1$, then the leaf $L(\sigma_1(t))$ for $t > t_1$ close to t cuts σ_1 at a point $\sigma_2(\tilde{t})$ with $\tilde{t} > t_2$, close to t_2. This contradicts our choice of t_1 and t_2.

Thus, $t_1 = 1$ or (non-exclusive) $t_2 = 1$. In the first case, $L_1 = L(\sigma_1(t_1))$ cuts σ_2 and then $L_1 \preceq L_2$, and in the second case, L_2 cuts σ_1 and $L_2 \preceq L_1$. This ends the proof. ■

As a consequence of Claims 7.9 and 7.10, one deduces the following claim.

Claim 7.11. Given any two leaves $L, \widetilde{L}$, there is a leaf $\widehat{L}$ so that $L \preceq \widehat{L}$ and $\widetilde{L} \preceq \widehat{L}$. In particular, the distance $\prec \cdot, \cdot \succ$ is bounded by 2.

Proof. Consider a finite sequence of leaves $L = L_0, \dots, L_k = \widetilde{L}$, $k = \prec L, \widetilde{L} \succ$, and L_i comparable with L_{i+1}.

The minimality of k implies that L_{i-1} and L_{i+1} are not comparable (otherwise, one could delete L_i, getting a strictly smaller sequence).

Assume that there is $i \in \{1, \dots, k-1\}$ so that $L_{i-1} \succeq L_i$. If $L_i \succeq L_{i+1}$, then $L_{i-1} \succeq L_{i+1}$ which is forbidden by the observation above. Thus, $L_i \preceq L_{i+1}$ and Claim 7.10 implies again that L_{i-1} and L_{i+1} are comparable, which again is impossible. This proves that

$$\forall i \in \{1, \dots, k-1\}, L_{i-1} \preceq L_i.$$

In particular, $L_0 \prec L_1$. Furthermore, if $k > 2$, then $L_1 \prec L_2$ which is not possible as L_0 and L_2 are not comparable. So, $k \leq 2$: either $k = 1$, that is, L_0 and $L_k = L_1 = \widetilde{L}$ are comparable, or $k = 2$ and $L \prec L_1$ and $\widetilde{L} \prec L_1$. ■

As a consequence, one deduces the following claim.

Claim 7.12. There is an increasing sequence $L_i \prec L_{i+1}$, $i \in \mathbb{N}$, $L_i \in \mathcal{F}$, so that, given any leaf $L \in \mathcal{F}$, there is n with $L \prec L_n$.

Proof. One chooses a countable set of compact positively oriented segments

$$\sigma_i : [0, 1] \to \mathbb{R}^2$$

transverse to $\mathcal{F}$ so that any leaf cuts one of the σ_i (and thus is less than $L(\sigma_i(1))$ for $\preceq$). Then, one builds inductively the sequence L_i: L_{i+1} is obtained by applying Claim 7.11 to the leaves L_i and $L(\sigma_i(1))$. ■

Claim 7.13. The compact discs Δ_L^+ are decreasing with L for $\prec$: more precisely, if $L \prec \widetilde{L}$, then

$$\Delta_{\widetilde{L}}^+ \subset \Delta_L^+.$$

Proof. The hypothesis $L \prec \widetilde{L}$ implies that $\widetilde{L}$ is contained in Δ_L^+, and $\widetilde{L}$ is oriented in such a way that the half plane bounded by $\widetilde{L}$ and contained in Δ_L^+ is $\Delta_{\widetilde{L}}^+$. ■

Thus, Claims 7.12 and 7.13 imply

$$\bigcap_{L \in \mathcal{F}} \Delta_L^+ = \bigcap_{i \in \mathbb{N}} \Delta_{L_i}^+.$$

Now, $\bigcap_{L \in \mathcal{F}} \Delta_L^+ = \bigcap_{i \in \mathbb{N}} \Delta_{L_i}^+$ is a decreasing sequence of connected compact metric sets, saturated for $\mathcal{F}$, and therefore is a non-empty connected compact set saturated for $\mathcal{F}$. As it does not contain any leaf of $\mathcal{F}$, one deduces that $\bigcap_{L \in \mathcal{F}} \Delta_L^+ \cap \mathbb{R}^2 = \emptyset$; that is, $\bigcap_{L \in \mathcal{F}} \Delta_L^+$ is a compact interval U in $\mathbb{S}^1_{\mathcal{F}}$.

It remains to show that this interval $U = \bigcap_{L \in \mathcal{F}} \Delta_L^+$ is reduced to a point. Otherwise, there is a half-leaf L_+ whose limit belongs to the interior of U, and hence in the interior (for the topology of $\mathbb{D}^2_{\mathcal{F}}$) of the discs $\Delta_{L_i}^+$, for every i. This implies that $L_+ \cap \Delta_{L_i}^+ \neq \emptyset$ for every i. However, for i large enough, the leaf L_i is larger (for $\prec$) than the leaf L carrying the half-leaf L_+, and thus, $L \cap \Delta_{L_i}^+ = \emptyset$.

This contradiction ends the proof of Proposition 7.8. ■

Proof of item (1) *of Theorem* 7.1. If $\mathcal{F}$ does not admit non-separated leaves from below, then the point $O_{\mathcal{F}}$ in $\mathbb{S}^1_{\mathcal{F}}$ given by Proposition 7.8 is a global fixed point of H; thus, the action is not minimal. The same holds if $\mathcal{F}$ does not admit non-separated leaves from above. ■

Proof of item (2) *of Theorem* 7.1. We assume that H is a group acting minimally on the leaves of a foliation $\mathcal{F}$ having non-separated leaves, some of them from above and some of them from below. According to Lemma 7.3, up to considering a finite-index subgroup of H, acting minimally on the leaves of $\mathcal{F}$, one may assume that H preserves the orientation and transverse orientation of $\mathcal{F}$.

Recall that the ends of regular leaves are dense in $\mathbb{S}^1_{\mathcal{F}}$. Thus, it is enough to check that any neighborhood of any end of a regular leaf contains points in the orbit for H of any point of $\mathbb{S}^1_{\mathcal{F}}$. Consider a regular leaf L and $\sigma\colon [-1,1] \to \mathbb{R}^2$ a segment transverse to $\mathcal{F}$ with $\sigma(0) \in L$. We will show that the end of L^+ belongs to the closure of any H-orbit. (The same argument holds for the end of L^-.)

We denote by L_t the leaf through $\sigma(t)$. Consider the basis of neighborhood U_t^+ of the end L^+ given by the compact discs in $\mathbb{D}^2_{\mathcal{F}}$ given by the closure of the half plane bounded by L^+_{-t}, $\sigma([-t,t])$, and L^+_t.

Our hypothesis implies the following claim.

Claim 7.14. There is a dense subset of values of t so that L_t is not separated at the right. As a consequence, for every t, the topological disc U_t^+ contains entire leaves.

Proof. The first sentence is directly implied by the existence of leaves which are non-separated at the right, the fact that H preserves the orientations of the leaves and acts minimally on the leaves of $\mathcal{F}$.

The second sentence has been observed in Section 4. ∎

Any leaf L cuts $\mathbb{D}^2_{\mathcal{F}}$ in two discs, Δ^+_L and Δ^-_L (following the transverse orientation of $\mathcal{F}$) whose union $\Delta^+_L \cup \Delta^-_L$ is $\mathbb{D}^2_{\mathcal{F}}$.

Claim 7.15. Under the hypotheses, given any L, there are $g_1, g_2 \in H$ so that

$$g_1(\Delta^+_L) \subset \mathring{\Delta}^-_L \quad \text{and} \quad g_2(\Delta^-_L) \subset \mathring{\Delta}^+_L,$$

where $\mathring{\Delta}^-_L$ and $\mathring{\Delta}^+_L$ denote the interiors of Δ^-_L and Δ^+_L, respectively, for the topology of $\mathbb{D}^2_{\mathcal{F}}$.

As a consequence, both Δ^+_L and Δ^-_L contain points in any H-orbit of a point in $\mathbb{D}^2_{\mathcal{F}}$.

Proof. We prove the first inclusion; the other is obtained by reversing the transverse orientation of $\mathcal{F}$.

Consider L a leaf and $\sigma\colon [-1,1] \to \mathbb{R}^2$ a segment transverse to $\mathcal{F}$ (positively oriented for the transverse orientation of $\mathcal{F}$) so that $\sigma(0) \in L$. There is $-t \in [-1,0)$ so that the leaf L_{-t} is non-separated from below from a leaf L_2 because the leaves non-separated from below are dense in $\mathbb{R}^2$ due to the minimality of the action of H on the leaves and the fact that H preserves the transverse orientation of $\mathcal{F}$. Thus, $L_{-t} \subset \Delta^-_L$, $L_2 \subset \Delta^-_L$. Furthermore, $\Delta^-_{L_2}$ contains L_{-t} and thus contains L. One deduces that

$$\Delta^+_{L_2} \subset \Delta^-_L.$$

Now, there is $h \in H$ so that $h(L_2) = L_{-s}$ for some $-s \in (-1,0)$. One deduces that

$$\Delta^+_L \subset \mathring{\Delta}^+_{h(L_2)},$$

and thus,

$$h^{-1}\Delta^+_L \subset h^{-1}(\mathring{\Delta}^+_{h(L_2)}) = \mathring{\Delta}^+_{L_2} \subset \mathring{\Delta}^-_L.$$

This concludes the proof. ∎

We are ready to conclude the proof of Theorem 7.1: any neighborhood in $\mathbb{D}^2_{\mathcal{F}}$ of any point of $\mathbb{S}^1_{\mathcal{F}}$ contains an entire leaf L (Claim 7.14 above) and thus contains either Δ^+_L or Δ^-_L. According to Claim 7.15, this neighborhood contains points in any H-orbit of points in $\mathbb{D}^2_{\mathcal{F}}$. This shows the minimality of the action of H on $\mathbb{S}^1_{\mathcal{F}}$, concluding the proof. ∎

Theorem 7.2. *Let $\mathcal{F}, \mathcal{G}$ be two transverse foliations on the plane $\mathbb{R}^2$. Let $H \subset$* Homeo$(\mathbb{R}^2)$ *be a group preserving both foliations $\mathcal{F}$ and $\mathcal{G}$.*

(1) *If the action of H on $\mathbb{S}^1_{\mathcal{F},\mathcal{G}}$ is minimal, then both foliations $\mathcal{F}, \mathcal{G}$ have non-separated leaves from above and non-separated leaves from below.*

(2) *Conversely, if both foliations $\mathcal{F}$, $\mathcal{G}$ have non-separated leaves from above and non-separated leaves from below and if the orbit of every leaf of $\mathcal{F}$ and $\mathcal{G}$ is dense $\mathbb{R}^2$, then the action of H on $\mathbb{S}^1_{\mathcal{F},\mathcal{G}}$ is minimal.*

Proof. For item (1), if the action of H on $\mathbb{S}^1_{\mathcal{F},\mathcal{G}}$ is minimal, then both actions of H on $\mathbb{S}^1_{\mathcal{F}}$ and $\mathbb{S}^1_{\mathcal{G}}$ are minimal. Thus, item (1) follows from item (1) of Theorem 7.1.

Conversely, if the action on the leaves of $\mathcal{F}$ and $\mathcal{G}$ is assumed to be minimal and they have non-separated leaves, then Proposition 7.7 implies that both projections $\Pi_{\mathcal{F}}$ and $\Pi_{\mathcal{G}}$ are injective. That is, $\mathbb{S}^1_{\mathcal{F},\mathcal{G}} = \mathbb{S}^1_{\mathcal{F}} = \mathbb{S}^1_{\mathcal{G}}$. Now, the minimality of the action of H on this circle at infinity is given by item (2) of Theorem 7.1. ∎

8. Action of the fundamental group on the bifoliated plane of an Anosov flow

8.1. The bifoliated plane associated to an Anosov flow. Let X be an Anosov flow on a closed 3-manifold M. Then, Fenley and Barbot show that the lift of X on the universal cover of M is conjugated to $\mathbb{R}^3, \frac{\partial}{\partial x}$; in particular, the space of orbits of this lifted flow is a plane $\mathcal{P}_X \simeq \mathbb{R}^2$. Then, the center-stable and center-unstable foliations of X induce (by lifting to the universal cover and projecting on $\mathcal{P}_X$) a pair of transverse foliations $\mathcal{F}^s$, $\mathcal{F}^u$ on the plane $\mathcal{P}_X$. The triple $(\mathcal{P}_X, \mathcal{F}^s, \mathcal{F}^u)$ is called the *bifoliated plane* associated to X. Finally, the natural action of the fundamental group $\pi_1(M)$ on the universal cover of M projects on $\mathcal{P}_X$ in an action preserving both foliations $\mathcal{F}^s$ and $\mathcal{F}^u$.

Fenley and Barbot proved that if one of the foliations $\mathcal{F}^s$, $\mathcal{F}^u$ is trivial (that is, has no non-separated leaf and therefore is conjugate to an affine foliation by parallel straight lines), then the other is also trivial. In that case, one says that X is $\mathbb{R}$-*covered*. In that case, the bifoliated plane is conjugated to one of the two possible models:

- the plane $\mathbb{R}^2$ endowed with the trivial horizontal and vertical foliations; Solodov proved that this is equivalent to the fact that X is orbitally equivalent to the suspension flow of a linear automorphism of the torus $\mathbb{T}^2$;
- the restriction of the trivial horizontal and vertical foliation to the strip $|x - y| < 1$.

8.2. Injectivity of the projection of $\mathbb{D}^2_{\mathcal{F}^s,\mathcal{F}^u}$ on $\mathbb{D}^2_{\mathcal{F}^s}$ and $\mathbb{D}^2_{\mathcal{F}^u}$. The aim of this section is to prove Theorem 1.5 which is restated as Proposition 8.1 below and Theorem 8.1 (in the next section).

Proposition 8.1. *Let X be an Anosov flow on a 3-manifold. Then,*

- *either X is topologically equivalent to the suspension flow of a hyperbolic element of $GL(2, \mathbb{Z})$*

- *or both projections of the compactification $\mathbb{D}^2_{F^s,F^u}$ on $\mathbb{D}^2_{F^s}$ and $\mathbb{D}^2_{F^u}$ are homeomorphisms.*

Proof. Assume that the projection on $\mathbb{D}^2_{\mathcal{F}^u}$ is not injective. Thus, there is a non-trivial open interval I of $\mathbb{S}^1_{\mathcal{F}^s,\mathcal{F}^u}$ whose points are not limits of ends of leaves of $\mathcal{F}^u$. Thus, the limits of ends of leaves of $\mathcal{F}^s$ form a dense subset in I.

Consider two half leaves of $\mathcal{F}^s$ whose limits are points $x_0 \neq y_0$ in I, and assume that $[x_0, y_0] \subset I$. Consider a simple path σ_0, concatenation of segment of leaves of $\mathcal{F}^s$ and $\mathcal{F}^u$ joining these two half leaves so that the union of $[x_0, y_0]$, the two half leaves and σ_0, is a simple closed curve bounding a disc $\Delta_0 \subset \mathbb{D}^2_{\mathcal{F}^s,\mathcal{F}^u}$. Now, every $\mathcal{F}^s$-leaf ending in (x_0, y_0) has a half-leaf in Δ. If it is contained in Δ, then the $\mathcal{F}^u$-leaves crossing it have an end point in (x_0, y_0) contradicting the hypothesis. So, every $\mathcal{F}^s$-leaf having an end point in (x_0, y_0) crosses σ_0. As there are infinitely many $\mathcal{F}^s$-leaves with an end point in (x, y) and finitely many leaf segment forming σ_0, one deduces that there are $x_0 < x < y < y_0$ and two half $\mathcal{F}^s$-leaves L_0^+, L_1^+ ending at x and y, respectively, and a $\mathcal{F}^u$-leaf segment σ joining $L_0^+, L_1^+, \sigma(0) \in L_0^+, \sigma(1) \in L_1^+$ and crossing them with the same orientation. Consider the disc Δ bounded by $[x, y] \cup L_0^+ \cup L_1^+ \cup \sigma$. Let L_t^+ be the half $\mathcal{F}^s$-leaf entering in Δ through $\sigma(t)$. If the end L_t^+ is not a regular end, then there is an entire $\mathcal{F}^s$ leaf in Δ and then we have seen that this implies that the $\mathcal{F}^u$ leaves crossing it have an end point in (x, y), contradicting the hypotheses.

Thus, every half $\mathcal{F}^s$-leaf L_t^+ is a regular end. Furthermore, every $\mathcal{F}^u$-leaf crossing one of the $L_{t_0}^+$ crosses every L_t^+, $t \in [0, 1]$.

Then, the union of all these half leaves L_t^+ is what Fenley called a *product region* in [10]. Now, [10, Theorem 5.1] asserts that any Anosov flow admitting a product region is a suspension flow, concluding the proof. ■

Remark 8.2. This proof holds for pseudo-Anosov flow.

Let X be a pseudo-Anosov flow on a 3-manifold. Then,

- either X is topologically equivalent to the suspension flow of a hyperbolic element of $GL(2, \mathbb{Z})$,
- or both projections of the compactification $\mathbb{D}^2_{F^s,F^u}$ on $\mathbb{D}^2_{F^s}$ and $\mathbb{D}^2_{F^u}$ are homeomorphisms.

One just needs to replace [10, Theorem 5.1] by its version for pseudo-Anosov flows, [6, Theorem 2.7].

8.3. Minimality of the action on the circle at infinity. In order to prove Theorem 1.5, it remains to prove Theorem 8.1 below.

Theorem 8.1. *Let X be an Anosov flow on a closed 3-manifold M. Then, X is non-$\mathbb{R}$-covered if and only if the action of $\pi_1(M)$ on the circle $\mathbb{S}^1_{F^s,F^u}$ at infinity is minimal.*

Remark 8.3. If the manifold M is not orientable and if X is $\mathbb{R}$-covered, then [8] noticed that X is a suspension flow. Thus, on non-orientable manifolds M, Theorem 8.1 asserts the minimality of the action on the circle at infinity, except if M is a suspension manifold.

Remark 8.4. The bifoliated plane $(\mathcal{P}_X, \mathcal{F}^s, \mathcal{F}^u)$ remains unchanged if we consider a lift of X on a finite cover. Thus, it is enough to prove Theorem 8.1 in the case where M is oriented and the action of $\pi_1(M)$ preserves both orientation and transverse orientation of both foliations $\mathcal{F}^s$, $\mathcal{F}^u$.

Thus, up to now, we will assume that M is oriented and the action of $\pi_1(M)$ preserves both orientations and transverse orientations of both foliations $\mathcal{F}^s$, $\mathcal{F}^u$.

Remark 8.5. If X is $\mathbb{R}$-covered, then $\mathbb{S}^1_{\mathcal{F}^s}$ has exactly 2 center-like points, which are therefore preserved by the action of $\pi_1(M)$ on $\mathbb{S}^1_{\mathcal{F}^s}$: this action is not minimal, and thus, the action on $\mathbb{S}^1_{\mathcal{F}^s,\mathcal{F}^u}$ is not minimal.

Thus, we are left to prove Theorem 8.1 in the case where X is not $\mathbb{R}$-covered. We will start with the easier case, when X is assumed to be transitive. The non-transitive case will be done in the whole next section.

Proof of Theorem 8.1 when X is transitive. When X is non-$\mathbb{R}$-covered and transitive, then [9] proved that $\mathcal{F}^s$ and $\mathcal{F}^u$ admit non-separated leaves from above and non-separated leaves from below. Up to considering a finite cover of M, the action of $\pi_1(M)$ preserves the orientation and transverse orientation of $\mathcal{F}^s$. Moreover, the action is minimal on the set of leaves of $\mathcal{F}^s$. Thus, Theorem 7.1 asserts that the action of $\pi_1(M)$ on $\mathbb{S}^1_{\mathcal{F}^s}$ is minimal. As $\mathbb{S}^1_{\mathcal{F}^s} = \mathbb{S}^1_{\mathcal{F}^s,\mathcal{F}^u}$, this concludes the proof. ∎

9. Minimality of the action on the circle at infinity for non-transitive Anosov flows: Ending the proof of Theorem 1.5

For ending the proof of Theorem 1.5, we are left to prove the following theorem.

Theorem 9.1. *Let X be a non-transitive Anosov flow on a closed connected 3-manifold M. Then, the action of the fundamental group of M on the circle at infinity is minimal.*

This result is somewhat less intuitive, as the action of the fundamental group $\pi_1(M)$ on the leaves of M is not minimal, and even, if X has several attractors, may fail to admit a leaf whose orbit is dense.

The proof of the minimality of the action on the circle at infinity will require some background on Anosov flows, in particular, on non-transitive Anosov flows. In the whole section, X is a non-transitive Anosov flow on an orientable closed connected manifold M and the natural action of $\pi_1(M)$ on the bifoliated plane $(\mathcal{P}_X, \mathcal{F}^s, \mathcal{F}^u)$ preserves the orientations and transverse orientations of both foliations. Recall that we have seen that the compactification of both foliations coincides with the one of each foliation. We will denote by $\mathbb{D}^2_X$, $\mathbb{S}^1_X$ this compactification and the corresponding circle at infinity. We denote by $*$ this package of hypotheses and notations.

9.1. Background on non-transitive Anosov flows. Let X be a non-transitive Anosov flow. Thus, according to [1, 8], X is not $\mathbb{R}$-covered.

The flow X is a structurally stable flow so that Smale spectral decomposition theorem splits the non-wandering set of X in basic pieces ordered by *Smale order*: a basic piece is greater than another if its unstable manifolds cut the stable manifold of the other. For this order, the maximal basic pieces are the repellers and the minimal are the attractors. In [4], Brunella noticed that the basic pieces are separated by incompressible tori transverse to the flow.

Consider an attractor $\mathcal{A}$ of X. It is a compact set consisting of leaves of the unstable foliation of X; hence, it is a compact lamination by unstable leaves. Furthermore, the intersection of $\mathcal{A}$ with a transverse segment σ is a Cantor set. An unstable leaf W^u in $\mathcal{A}$ is called of *boundary type* if $W^u \cap \sigma$ belongs to the boundary of a connected component of $\sigma \setminus \mathcal{A}$.

A classical result from hyperbolic theory (see, for instance, [3]) asserts that the unstable leaves in $\mathcal{A}$ of boundary type are the unstable manifolds of a finite number of periodic orbits called *periodic orbits of boundary type*.

The same happens for repellers $\mathcal{R}$: they are compact laminations by stable leaves, transversally Cantor sets, and they admit finitely many leaves of boundary type, which are the stable manifolds of finitely many periodic orbits called of boundary type.

In this section, we will focus on attractors and repellers. Consider an attractor $\mathcal{A}$ of X, its lift $\widetilde{A}$ on the universal cover, and consider the projection of $\mathcal{A}$ on the bifoliated plane $\mathcal{P}_X$. This projection is a closed lamination by leaves of $\mathcal{F}^u$, and it cuts every transverse curve along a Cantor set. By a practical abuse of notation, we will still denote by $\mathcal{A}$ this lamination of $\mathcal{P}_X$; thus, $\mathcal{A}$ denotes at the same time a 2-dimensional lamination on M and a 1-dimensional lamination on $\mathcal{P}_X$.

The same happens for a repeller.

Let $\mathcal{A} \subset \mathcal{P}_X$ and $\mathcal{R} \subset \mathcal{P}_X$ be the unstable and stable laminations (respectively) corresponding to an attractor and a repeller of X. Then, the following hold.

- $\mathcal{A} \cap \mathcal{R} = \emptyset$. This seems obvious, but it will be a crucial property for us: given an unstable leaf L^u and a stable leaf L^s, this will be our unique criterion for knowing that they do not intersect.

- The periodic points contained in $\mathcal{A}$ (resp., $\mathcal{R}$) are dense in $\mathcal{A}$ (resp., $\mathcal{R}$).
- Each periodic orbit of X has a discrete $\pi_1(M)$-orbit in $\mathcal{P}_X$.
- The periodic orbits of boundary type are the $\pi_1(M)$-orbits of finitely many X-orbits and therefore are a discrete set in $\mathcal{P}_X$.
- Fenley [10] shows that the non-separated stable leaves of $\mathcal{F}^s$ (resp., $\mathcal{F}^u$) correspond to finitely many orbits of X, and hence to a discrete set of periodic points in $\mathcal{P}_X$.
- Thus, the periodic points p in $\mathcal{A}$ (resp., $\mathcal{R}$) which are not of boundary type and whose unstable (resp., stable) leaf is regular are dense in $\mathcal{A}$.
- If $\mathcal{A}_1, \dots, \mathcal{A}_k$ are the attractors of X, then the union of the stable leaves of $\mathcal{F}^s$ through the laminations $\mathcal{A}_1, \dots, \mathcal{A}_k$ of $\mathcal{P}_X$ are disjoint open subsets of $\mathcal{P}_X$ whose union is dense in $\mathcal{P}_X$. The same holds for the unstable leaves through the repellers.

As a straightforward consequence, one gets the following lemma.

Lemma 9.1. *There is a dense subset of $\mathcal{P}_X$ of points x whose stable leaf $L^s(x)$ contains a periodic point p in an attractor $\mathcal{A}$, not of boundary type, so that $L^u(p)$ is regular. A symmetric statement holds for repellers.*

9.2. Proof of Theorem 8.1. The two main steps of the proof of Theorem 8.1 are Propositions 9.2 and 9.4 below.

Proposition 9.2. *Let L^u be a leaf of $\mathcal{F}^u$ corresponding to an unstable leaf of X contained in a attractor of X. Let Δ_+ and Δ_- be the closures in $\mathbb{D}^2_X$ of the half planes in $\mathbb{R}^2$ bounded by L^s. Then, there are $g^+, g^- \in \pi_1(M)$ so that $g^-(\Delta^-) \subset \Delta^+$ and $g^+(\Delta^+) \subset \Delta^-$.*

The same statement holds for stable leaves in the repellers.

Corollary 9.3. *Let L^s and L^u be leaves of $\mathcal{F}^s$ and $\mathcal{F}^u$ in a repeller and in an attractor, respectively. Let $I \subset \mathbb{S}^1_X$ be a segment with non-empty interior and whose end points are the limits of both ends of the same leaf, L^s or L^u.*

Then, every orbit of the action of $\pi_1(M)$ contains points in I.

Proof. According to Proposition 9.2, there is $g \in \pi_1(M)$ so that $g(\mathbb{S}^1 \setminus I) \subset I$, ending the proof. ∎

Proposition 9.4. *Given any non-empty open interval $J \subset \mathbb{S}^1_X$, there is an L which is either a leaf of $\mathcal{F}^s$ in a repeller or a leaf of $\mathcal{F}^u$ in an attractor whose both ends have limits in J.*

Proof of Theorem 9.1 assuming Propositions 9.2 and 9.4. According to Proposition 9.4, every interval J with non-empty interior contains an interval I whose end points are the limit points of both ends of a stable or unstable leaf in a repeller or an attractor, respectively. Now, according to Corollary 9.3, the interval I contains a point in every $\pi_1(M)$ orbit in $\mathbb{S}^1_X$. Thus, any $\pi_1(M)$ orbit in $\mathbb{S}^1_X$ has points in any interval with non-empty interior: in other words, every $\pi_1(M)$ orbit is dense in $\mathbb{S}^1_X$, or in other words, the action of $\pi_1(M)$ on $\mathbb{S}^1_X$ is minimal, ending the proof. ∎

9.3. Proof of Proposition 9.2. Let L^u_0 be an unstable leaf in an attractor $\mathcal{A}_0$ and Δ^+_0 the closure of the upper half plane bounded by L^u_0. For proving Proposition 9.2, we want to prove that there is $f \in \pi_1(M)$ so that $f(\Delta^-_0) \subset \Delta^+_0$ (the other announced inclusion is identical).

Consider a point $p_0 \in L^u_0$ and L^s_0 the stable leaf through p_0.

Claim 9.5. There is an unstable leaf L^u_1 with the following properties:

- $L^u_1 \subset \Delta^+_0$,
- L^u_1 is contained in the basin of a repeller $\mathcal{R}_1$,
- L^u_1 contains a non-boundary periodic point $p_1 \in L^u_1$ of the repeller $\mathcal{R}_1$,
- L^u_1 cuts the stable leaf L^s_0 in a point $L^u_1 \cap L^s_0 = q_0$.

Proof. The union of unstable leaves in the basin of a repeller and carrying a non-boundary periodic point of this repeller is dense in $\mathbb{R}^2$. We can therefore choose such a leaf in Δ^+_0 and cutting L^s_0. ∎

Let L^s_1 be the stable leaf through p_1. It is a non-boundary stable leaf contained in the repeller $\mathcal{R}_1$. Note that L^s_1 is disjoint from the attractor $\mathcal{A}_0$. Thus,

- L^s_1 is disjoint from $L^u_0 \in \mathcal{A}_0$;
- the stable leaf L^s_1 is distinct, and therefore disjoint, from the stable leaf L^s_0.

In other words, the union $L^s_0 \cup L^u_0$ divides $\mathcal{P}_X$ into 4 quadrants and L^s_1 is contained in one of these quadrants. Let us denote by $C^{\pm,\pm}$ these 4 quadrants so that

$$\Delta^+_0 = C^{-,+} \cup C^{+,+} \quad \text{and} \quad L^s_1 \subset C^{+,+}.$$

Let us denote by $\Delta^+_1 = \Delta^+(L^s_1)$ the closure of the half plane bounded by L^s_1 and contained in Δ^+_0. Thus, Δ^+_1 is contained in the same quadrant $C^{+,+}$ as L^s_1. We denote by Δ^-_1 the closure of the other half plane bounded by L^s_1. Note that Δ^-_1 contains the 3 other quadrants; in particular, it contains Δ^-_0 and $C^{-,+}$.

As the leaf L^s_1 is (by assumption) not a boundary leaf of $\mathcal{R}_1$, it is accumulated on both sides by its $\pi_1(M)$-orbit. Thus, there is a leaf

$$L^s_2 = g(L^s_1)$$

in its orbit, cutting L_1^u at a point $x \in \Delta_1^-$ arbitrarily close to p_1 and hence $x \in C^{+,+}$. Notice that L_2^s is contained in the repeller $\mathcal{R}_1$ and thus is disjoint from $L_0^u \cup L_0^s$. Thus, it is contained in one quadrant. As it contained $x \in C^{+,+}$, one has

$$L_2^s \subset C^{+,+}.$$

Let $h \in \pi_1(M)$ be the generator of the stabilizer p_1 so that L_1^u is expanded by h. We consider the sequence of leaves $h^n(L_2^s)$ which cut $L_{1,-}^u$ at the point $h^n(x)$.

Claim 9.6. For n large enough, $h^n(L_2^s)$ is contained in the quadrant $C^{-,+}$.

Proof. Each leaf $h^n(L_2^s)$ intersects $L_1^u \subset \Delta_0^+$ and is disjoint from L_0^u (because $h^n(L_2^s)$ is contained in the repeller). Hence, $h^n(L_2^s)$ is contained in Δ_0^+ and is distinct and therefore disjoint from L_0^s. Thus, $h^n(L_2^s)$ is contained in one of the quadrants $C^{+,+}$ of $C^{-,+}$.

The point x_n tends to infinity in L_1^u, and so, it goes further than

$$q_0 = L_0^s \cap L_1^u.$$

Thus, for n large enough, $x_n \in C^{-,+}$. We proved that for n large enough $h^n(L_2^s) \subset C^{-,+}$, proving the claim. ∎

We conclude the proof of Proposition 9.2 by proving the following claim.

Claim 9.7. Consider n large enough so that $h^n(L_2^s) \subset C^{-,+}$.

Then, either $g(\Delta_1^-) \subset C^{+,+} \subset \Delta_0^+$ or $h^n g(\Delta_1^-) \subset C^{-,+} \subset \Delta_0^+$

As Δ_1^- contains Δ_0^-, the claim implies that either $g(\Delta_0^-) \subset \Delta_0^+$ or $h^n g(\Delta_0^-) \subset \Delta_0^+$, which concludes the proof of Proposition 9.2.

Proof of the claim. Assume that $g(\Delta_1^-)$ is not contained in $C^{+,+}$. As $g(\Delta_1^-)$ is one of the half-planes bounded by

$$g(L_1^s) = L_2^s \subset C^{+,+},$$

one gets that $g(\Delta_1^+)$ is the half plane bounded by L_2^s and contained in $C^{+,+}$. In particular, $g(\Delta_1^+)$ does not contain q_0. As p_1 and q_0 are on distinct sides of L_2^s, one deduces that $p_1 \in g(\Delta_1^+)$.

As p_1 is the fixed point of h, one deduces that

$$p_1 \in h^n g(\Delta_1^+).$$

Thus, $h^n g(\Delta_1^+)$ is the half plane bounded by $h^n(L_2^s)$ which is not contained in the quadrant $C^{-,+}$. Thus, $h^n g(\Delta_1^-)$ is the other half plane bounded by $h^n(L_2^s)$ and is contained in $C^{-,+}$, ending the proof. ∎

9.4. Proof of Proposition 9.4. We want to prove that any open interval I in the circle $\mathbb{S}^1_X$ contains the two ends of an unstable leaf in an attractor or the two ends of a stable leaf of a repeller.

Lemma 9.8. *Assuming $*$, there are dense subsets E^s_0, E^u_0 of $\mathbb{S}^1_X$ so that*

- *any $p \in E^s_0$ is the limit of an end of a regular leaf of $\mathcal{F}^s$ containing a periodic point x which belongs to an attractor $\mathcal{A}(p)$ and is not of boundary type,*
- *any $q \in E^u_0$ is the limit of an end of a regular leaf of $\mathcal{F}^u$ containing a periodic point y which belongs to a repeller $\mathcal{R}(q)$ and is not of boundary type.*

Proof. According to Lemma 9.1, the union of regular stable leaves containing periodic point of non-boundary type of an attractor is dense in $\mathcal{P}_X$. This family is therefore separating, according to Lemma 3.2. Thus, the limits of their ends are a dense subset of $\mathbb{S}^1_X$, as announced. ∎

Lemma 9.8 ensures the density of the endpoints of $\mathcal{F}^s$-leaves in the basins of attractors and of $\mathcal{F}^u$-leaves in the basins of repellers and carrying non-boundary-type periodic orbits. The next step is much more complicated: Lemma 9.9 ensures the density of the points which are endpoints of $\mathcal{F}^u$-leaves contained in the attractor itself (not in its basin) and of $\mathcal{F}^s$-leaves contained in the repeller and carrying non-boundary periodic orbits.

Lemma 9.9. *Assuming $*$, there are dense subsets $E^s, E^u \subset \mathbb{S}^1_X$ so that*

- *every $x \in E^s$ is the limit of the end of a regular leaf of $\mathcal{F}^s$ contained in a repeller $\mathcal{R}$, carrying a periodic point of non-boundary type,*
- *every $x \in E^u$ is the limit of the end of a regular leaf of $\mathcal{F}^u$ contained in an attractor $\mathcal{R}$, carrying a periodic point of non-boundary type.*

Proof. We just prove the density of E^u; the density of E^s is similar.

Consider a non-empty open interval $I \subset \mathbb{S}^1_X$. According to Lemma 9.8, there is a point $x \in I$ which is the limit of an end $L^s_+(p_0)$ of a regular leaf of $\mathcal{F}^s$ carrying a periodic point p_0 in a non-boundary-type unstable leaf $L^u(p_0)$ of an attractor $\mathcal{A}$.

The point p_0 is accumulated on both sides by periodic points in $\mathcal{A}$. We choose p_1 so that the limit y of $L^s_+(p_1)$ belongs to I (that is possible because $L^s_+(p_0)$ is regular) and $L^s_+(p_1)$ intersects $L^u(p_0)$ at a point q_1. Thus, let $J \subset I$ be the segment contained in I and whose end points are x and y. Notice that $y \neq x$; that is, J has non-empty interior, as $L^s(p_0)$ is a regular leaf.

Now, $L^u(p_0)$ is accumulated on both sides by regular unstable leaves contained in the attractor $\mathcal{A}$ and containing periodic point of non-boundary type. Let L^u_0 be such a leaf, with non-empty intersection with $L^s_+(p_0)$.

If L_0^u does not cut $L_+^s(p_1)$, then one end is contained in the half-strip bounded by $L_+^s(p_0)$, the segment of $[p_0, q_1]^u$ and $L_+^s(q_1)$. As a consequence, the limit of this end belongs to I so that $E^u \cap I \neq \emptyset$, and we are done.

Thus, we may assume now that L_0^u cuts $L_+^s(p_1)$.

Let h_0 and h_1 be the generators of the stabilizers of p_0 and p_1, respectively, so that h_0 expands $L_+^s(p_0)$ and h_1 expands $L_+^s(p_1)$.

We consider the images $\{h_0^n(L_0^u), h_1^n(L_0^u)\}_{n\in\mathbb{N}}$ of the leaf L_0^u by the positive iterates of h_0 and h_1. Each of these images is a regular unstable leaf in $\mathcal{A}$ and has a non-empty intersection with either $L_+^s(p_0)$ or $L_+^s(p_1)$. If one of these leaves does not cross both $L_+^s(p_0)$ and $L_+^s(p_1)$, then it has an end in the segment $J \subset I$, and we are done.

Assume now that every leaf in $\{h_0^n(L_0^u), h_1^n(L_0^u)\}_{n\in\mathbb{N}}$ crosses both $L_+^s(p_0)$ and $L_+^s(p_1)$. These images are leaves of $\mathcal{F}^u$, and therefore they are either disjoint or equal. For $L \in \{h_0^n(L_0^u), h_1^n(L_0^u)\}_{n\in\mathbb{N}}$, let $D(L) \subset \mathbb{D}^2_{\mathcal{F}}$ be the disc obtained as follows: one cuts along L the strip bounded by $L_+^s(p_0)$ and $L_+^s(p_1)$, getting two components; one considers the closure in $\mathbb{D}^2_{\mathcal{F}}$ of these components; now $D(L)$ is the one containing the segment $J \subset \mathbb{S}^1_{\mathcal{F}}$.

The disks $D(L)$ are naturally totally ordered by the inclusion, and we fix the indexation

$$\{h_0^n(L_0^u), h_1^n(L_0^u)\}_{n\in\mathbb{N}} = \{L_n^u\}_{n\in\mathbb{N}}$$

according to this order: for this indexation, $D(L_{n+1}^u) \subset D(L_n^u)$.

Consider $D = \bigcap_n (D(L_n^u))$. It is a compact subset of $\mathbb{D}^2_{\mathcal{F}}$ whose intersection with $\mathbb{S}^1_{\mathcal{F}}$ is the segment J.

Claim 9.10. $D \cap (L_+^s(p_0) \cup L_+^s(p_1)) = \emptyset$.

Proof. The leaves $h_0^n(L_0^u)$ have their intersection with $L^s(p_0)$ tending to x as $n \to \infty$: one deduces that $D \cap L^s(p_0) = \emptyset$. The leaves $h_1^n(L_1^u)$ have their intersection with $L^s(p_1)$ tending to y, and thus,

$$D \cap L^s(p_1) = \emptyset. \qquad \blacksquare$$

Claim 9.11. $D \setminus \mathbb{S}^1_X \neq \emptyset$.

Proof. There is a point z in the interior of J which is the limit of an end of leaf of $\mathcal{F}^u$. Thus, there is an half unstable leaf L_+^u contained in the strip bounded by $L_+^s(p_0)$ and $L_+^s(p_1)$, whose limit is z. Now, L_+^u is disjoint from all the L_n^u, and therefore,

$$L_+^u \subset D(L_n^u) \quad \text{for all } n.$$

This concludes the proof of the claim. ∎

Consider now a point $t \in D \setminus \mathbb{S}^1_{\mathcal{F}}$. The leaf $L^u(t)$ is disjoint from the leaves L^u_n for any n. Thus, it has an empty intersection with $(L^s_+(p_0) \cup L^s_+(p_1))$. As a consequence, one gets

$$L^u(t) \subset D.$$

In particular, $L^u(t)$ has both ends on J.

Suppose now that the point $t \in D \setminus \mathbb{S}^1_X$ has been chosen on the boundary of D. Thus, t is a limit of points in $L^u_n \subset \mathcal{A}$. As $\mathcal{A}$ is a closed subset of $\mathbb{R}^2 = \mathring{\mathbb{D}}^2_{\mathcal{F}}$, one deduces that $t \in \mathcal{A}$, and so, $L^u(t) \subset \mathcal{A}$.

One just found a leaf $L^u(t)$ contained in $\mathcal{A}$, having both ends in $J \subset I$. Let $D_t \subset D$ be the disc bounded by $L^u(t)$. We are not yet done, because L^u_t may fail to be a regular leaf.

Now, Proposition 9.2 implies that every unstable leaf, for instance, L^u_0, has an image by an element $k \in \pi_1(M)$ which is contained in D_t. Now, $k(L^u_0)$ is a regular unstable leaf in an attractor which has the limits of its both ends contained in $J \subset I$, that is, $E^u \cap I \neq \emptyset$, ending the proof. ■

We are now ready for ending the proof of Proposition 9.4, and therefore of Theorem 9.1 which ends the proof of Theorem 8.1 and Theorem 1.5.

Proof of Proposition 9.4. Let $I \subset \mathbb{S}^1_X$ be a non-empty open interval. According to Lemma 9.9, there is a regular unstable leaf L^u_0, contained in an attractor $\mathcal{A}$ and containing a periodic point of non-boundary type p_0, and having an end, say, $L^u_{0,+}$, whose limit is a point $x \in I$.

As L^u_0 is not a boundary leaf of $\mathcal{A}$, there are unstable leaves in $\mathcal{A}$ arbitrarily close to L^u_0 on both sides of L^u_0. As furthermore L^u_0 is a regular leaf, one can choose a leaf $L^u_1 \subset \mathcal{A}$ so that

- the limit of the end $L^u_{1,+}$ is a point $y \in I$ with $[x, y] \subset I$.
- there is a segment σ of a stable leaf having both end points a and b on L^u_0 and L^u_1, respectively.

We denote by D_σ the disc in $\mathbb{D}^2_X$ bounded by σ, $[x, y]$, $L^u_+(a) \subset L^u_0$ and $L^u_+(b) \subset L^u_1$.

Now, according to Lemma 9.8, there is a point $z \in [x, y]$ which is the limit of the end L^u_+ of a unstable leaf L^u which carries a periodic point q in a repeller $\mathcal{R}$, and q is not of boundary type. We denote by $h \in \pi_1(M)$ the generator of the stabilizer of q which is expanding along L^u.

The stable leaf $L^s(q)$ is contained in the repeller $\mathcal{R}$ and is accumulated on both sides by stable leaves in $\mathcal{R}$. We denote by L^s_0 a stable leaf in $\mathcal{R}$ crossing L^u_+ at a point x_0.

We consider $L^s_n = h^n(L^s_0)$. It is a stable leaf in $\mathcal{R}$ which cuts L^u_+ at the point $x_n = h^n(x_0)$.

Note that $x_n \to z$ as $n \to +\infty$. In particular, x_n belongs to the disc D_σ for n large.

As $\mathcal{R} \cap \mathcal{A} = \emptyset$, the leaves L^s_n are disjoint from L^u_0 and L^u_1. As two distinct stable leaves are disjoint, they are disjoint from σ (which is not disjoint from L^u_0 and L^u_1).

So, for large n, the leaf L^s_n is contained in D_σ and therefore has both its ends on $[x, y] \subset I$.

We have just exhibited a stable leaf in a repeller, whose both ends are in I. Thus, we ended the proof of Proposition 9.4. ■

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