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On some operators acting on line arrangements and their dynamics

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Abstract. We study some natural operators acting on configurations of points and lines in the plane and remark that many interesting configurations are fixed points for these operators. We review ancient and recent results on line or point arrangements through the realm of these operators. We study the first dynamical properties of the iteration of these operators on some line arrangements.

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1. Introduction

Line arrangements are an active field of research which is attractive from many viewpoints: topology (Zariski pairs), algebra (freeness), combinatorics, etc. For algebraic geometers, the results of Hirzebruch in [26] have been the starting point for the search of peculiar line arrangements in order to geometrically construct some ball quotient surfaces, which are surfaces with Chern numbers satisfying the equality in the Bogomolov–Miyaoka–Yau inequality $c_1^2 \leq 3c_2$. Arrangements of lines appear also to be important for the bounded negativity conjecture on algebraic surfaces [5, 6].

Motivated by the problem of constructing new interesting line arrangements, we formalize here the definitions of some operators acting on lines and points arrangements and we ask natural questions about their properties. For integers $n, m \geq 2$, one may, for example, define the line operator $\Lambda_{n,m}$ as the function which to a given line arrangement L_0 associates the union of the lines that contain at least m points among the points in L_0 that have multiplicity at least n. The idea of that definition comes naturally when considering the definition of (r_k, s_m)-configurations (see Section 3.2). These operators are probably known and used; however, to the knowledge of the author, they are not studied for themselves and this is one of our aims to (start to) do so and to convince the reader of their importance.

We will show that these operators appear quite naturally and ubiquitously when studying line arrangements with remarkable properties, such as reflexion line arrangements, simplicial arrangements, Zariski pairs or Sylvester–Gallai line arrangements.

For example, a Zariski pair is a pair of plane curves $\mathcal{A}, \mathcal{A}'$ which have the same combinatoric but have different topological properties, which properties are often difficult to detect. We obtain that the line operators are able to detect geometric differences between Zariski pairs of line arrangements $\mathcal{A}$ and $\mathcal{A}'$ when the moduli space of such $\mathcal{A}, \mathcal{A}'$ has dimension ≥ 1, but they do not see any differences when their moduli space is 0 dimensional.

Some natural questions on these operators Λ arise, for example, the question to understand what are their fixed points, i.e., if we can classify the arrangements $\mathcal{A}$ such that $\Lambda(\mathcal{A}) = \mathcal{A}$. As we explain in this paper, the answer is known for the $\Lambda_{2,2}$ operator, as a result of De Bruijn–Erdös. We give some examples of fixed points of operators Λ, but the question to classify them is quite open. It is also an interesting question to construct line arrangements $\mathcal{A}$ such that $\mathcal{A}$ and $\Lambda(\mathcal{A})$ have the same combinatorics but with $\Lambda(\mathcal{A}) \neq \mathcal{A}$, so that Λ is a (rational) self-map on the moduli space of these arrangements; we study such line arrangements on some cases. We also show that many results on line arrangements may be formulated as results on these operators Λ.

Another natural question is to understand line arrangements for which the action of an operator Λ diverges, i.e., for which the successive images $(\mathcal{A}_k)_{k\geq 0}$ by Λ of an arrangement $\mathcal{A}_0$ acquire more and more lines.

The other operators on line arrangements the author is aware of are related to the pentagram map defined by Schwartz in [35]. To a polygon of m lines, i.e., a m-tuple of lines $(\ell_1, \dots, \ell_m)$, the pentagram map associates the polygon $(\ell'_1, \dots, \ell'_m)$, where ℓ'_k is the line containing the points p_k, p_{k+2}, for $p_k = \ell_k \cap \ell_{k+1}$ (the indices being taken mod m). That creates a rational self-map on the space of polygons, which is of interest in classical projective geometry, algebraic combinatorics, moduli spaces, cluster algebras, and integrable systems, see; e.g., [7, 31].

The structure of the paper is as follows.

In Section 2, we define the line and point operators and give some speculations on their properties and the dynamics of their actions. The ring generated by these operators acts naturally on the free module of line (respectively, point) arrangements. We then remark that the Hirzebruch-type inequalities could be understood by using these operators and factorizing through the Chow group of points of the plane. We also review some classical theorems (Pappus, Pascal, and Desargues) and remark that they can be unified and formulated as statements on the operator $\Lambda_{\{2\},\{3\}}$.

In Section 3, we study some examples of action of operators on classical arrangements of lines (such as reflexion arrangements), on simplicial arrangements, on some free arrangements, and Zariski pairs. We also obtain an example of an arrangement of 81 lines with H-constant < -3, an interesting property for the bounded negativity conjecture. Although we do not claim for completeness, these two first sections may be also of interest for readers willing to have a short survey on line arrangements and related problems.

The next sections contain more original results.

In Section 4, we explore the divergence of some operators such as $\Lambda_{2,3}$, $\Lambda_{\{2\},\{3\}}$, $\Lambda_{3,2}$ and $\Lambda_{3,3}$ on some line arrangements.

Section 5 is devoted to the line operator $\Lambda_{\{2\},\{3\}}$. We discuss two families of arrangements of six lines, which we call the unassuming arrangements and the flashing arrangements (we study in details the former in [33]), for which the action of $\Lambda_{\{2\},\{3\}}$ is remarkable, recalling some classical theorems of geometry.

In Section 6, we construct flashing arrangements, i.e., line arrangements $\mathcal{A}$ such that $\Lambda(\mathcal{A}) \neq \mathcal{A}$ but $\Lambda(\Lambda(\mathcal{A})) = \mathcal{A}$ for operators $\Lambda \in \{\Lambda_{\{2\},\{k\}} \mid k = 3, 4, 5, 6\}$.

We conclude in Section 7 by recalling some notions used in this paper about matroids and by some remarks on realizable rank 3 matroids and the line operators Λ.

The computations involved use the Magma algebra software [11]; one can find our algorithms in the ancillary file of the arXiv version of this paper.

2. Definitions and occurrences of the operators, first questions

2.1. Notations and usual definitions. Let K be a field. We denote by $\mathbb{P}^2$ (respectively, $\check{\mathbb{P}}^2$) the set of points defined over K of the projective plane (respectively, of the dual projective plane).

For a set E, we denote by $\mathcal{P}(E)^{\mathrm{F}}$ the set of finite subsets of E. An element of $\mathcal{P}(\check{\mathbb{P}}^2)^{\mathrm{F}}$ is called a *line arrangement*, and an element of $\mathcal{P}(\mathbb{P}^2)^{\mathrm{F}}$ is called a *point configuration*. An arrangement of lines will be identified with the union of these lines in $\mathbb{P}^2$.

Let L_0 be an arrangement of lines. For $k \geq 2$, a point p is said *multiple* with multiplicity k for L_0, (or, for short, a k-*point* of L_0) if p is at the intersection of exactly k lines in L_0. We denote by $t_k(L_0)$ the number of k-points of L_0.

Let P_0 be a point configuration. A line which contains exactly k points in P_0 is said k-*rich*.

The cardinal of a set E is denoted by $|E|$.

When a line arrangement is known under the name of his inventor, e.g., Wiman, we refer to it as the "Wiman configuration" rather than the "Wiman arrangement", as is customary in the literature, and especially if it is part of a (r_k, s_m)-configuration as defined in Section 3.2.

2.2. The lines and points operators Λ and Ψ. For a subset $\mathfrak{n} \subset \mathbb{N} \setminus \{0,1\}$ of integers larger than or equal to 2, let us define the functions

$$\mathcal{L}_{\mathfrak{n}} : \mathcal{P}(\mathbb{P}^2)^{\mathrm{F}} \to \mathcal{P}(\check{\mathbb{P}}^2)^{\mathrm{F}} \quad \text{and} \quad \mathcal{P}_{\mathfrak{m}} : \mathcal{P}(\check{\mathbb{P}}^2)^{\mathrm{F}} \to \mathcal{P}(\mathbb{P}^2)^{\mathrm{F}}.$$

Hereby, we have the following.

- To a configuration P_0 of points in the plane, let $\mathcal{L}_{\mathfrak{n}}(P_0)$ be the set of lines defined by
$$\mathcal{L}_{\mathfrak{n}}(P_0) = \big\{\ell \mid |\ell \cap P_0| \in \mathfrak{n}\big\};$$
i.e., $\mathcal{L}_{\mathfrak{n}}(P_0)$ is the set of lines ℓ in the plane such that the number of points of P_0 contained in ℓ is in the set $\mathfrak{n}$.
- To an arrangement of lines L_0, let $\mathcal{P}_n(L_0)$ be the set of points
$$\mathcal{P}_{\mathfrak{n}}(L_0) = \big\{p \mid \exists\, k \in \mathfrak{n},\, p \text{ is a } k\text{-point of } L_0\big\}.$$

Then, for subsets $\mathfrak{n}, \mathfrak{m} \subset \mathbb{N} \setminus \{0,1\}$, let us define the $(\mathfrak{n},\mathfrak{m})$-*line operator* and the $(\mathfrak{n},\mathfrak{m})$-*point operator*

$$\Lambda_{\mathfrak{n},\mathfrak{m}} : \mathcal{P}(\check{\mathbb{P}}^2)^{\mathrm{F}} \to \mathcal{P}(\check{\mathbb{P}}^2)^{\mathrm{F}}, \quad \Psi_{\mathfrak{n},\mathfrak{m}} : \mathcal{P}(\mathbb{P}^2)^{\mathrm{F}} \to \mathcal{P}(\mathbb{P}^2)^{\mathrm{F}}$$

by

$$\Lambda_{\mathfrak{n},\mathfrak{m}}(L_0) = \mathcal{L}_{\mathfrak{m}}(\mathcal{P}_{\mathfrak{n}}(L_0)) \quad \text{and} \quad \Psi_{\mathfrak{n},\mathfrak{m}}(P_0) = \mathcal{P}_{\mathfrak{m}}(\mathcal{L}_{\mathfrak{n}}(P_0)).$$

If $\mathfrak{n} = \mathfrak{m}$, we may simply write $\Lambda_{\mathfrak{n}} = \Lambda_{\mathfrak{n},\mathfrak{n}}$ and $\Psi_{\mathfrak{n},\mathfrak{n}} = \Psi_{\mathfrak{n}}$. For $n \in \mathbb{N}, n \geq 2$, we will use the notation $\mathcal{L}_n, \mathcal{P}_n$ for

$$\mathcal{L}_n = \mathcal{L}_{\{k \geq n | k \in \mathbb{N}\}}, \quad \mathcal{P}_n = \mathcal{P}_{\{k \geq n | k \in \mathbb{N}\}},$$

and $\Lambda_n = \mathcal{L}_n \circ \mathcal{P}_n$, $\Psi_n = \mathcal{P}_n \circ \mathcal{L}_n$.

For integers $n \geq 2$, the operators $\mathcal{L}_{\{n\}}, \mathcal{P}_{\{n\}}$ are elementary, meaning that

$$\mathcal{L}_{\mathfrak{n}}(P_0) = \bigcup_{n \in \mathfrak{n}} \mathcal{L}_{\{n\}}(P_0), \quad \mathcal{P}_{\mathfrak{n}}(L_0) = \bigcup_{n \in \mathfrak{n}} \mathcal{P}_{\{n\}}(L_0),$$

for a point configuration P_0 or a line arrangement L_0; moreover, these unions are disjoint. We also have the relations

$$\Lambda_{\mathfrak{m},\mathfrak{n}}(L_0) = \bigcup_{(m,n) \in \mathfrak{m} \times \mathfrak{n}} \Lambda_{\{m\},\{n\}}(L_0). \tag{2.1}$$

Let Λ be a line operator and L_0 be a line arrangement. We define inductively a sequence $(L_n)_{n \in \mathbb{N}}$ of lines arrangements, called the Λ-sequence associated to L_0, by the relation

$$L_{n+1} = \Lambda(L_n).$$

Definition 1. We say that the Λ-sequence $(L_m)_m$ is Λ-*convergent* to a line arrangement L if $\exists M \in \mathbb{N}$ such that $\forall m \geq M$, $L_m = L$. We say that a sequence is *extinguishing* if it converges to $\emptyset$. If a sequence $(L_m)_m$ is extinguishing, its *length* is the smallest integer m such that $L_m = \emptyset$. A line arrangement L is Λ-*fixed* if $\Lambda(L) = L$. The operator Λ is said divergent for L if the number of lines of the associated Λ-sequence diverges to infinity.

The following questions are then natural.

Problem 2. Under which conditions on (Λ, L_0) is the Λ-sequence $(L_m)_m$ convergent? Or periodic? Or extinguishing?

For $n \in \mathbb{N}$, does there exist a Λ-sequence of length n? Do there exist sequences that are periodic with a non-trivial period?

For a given arrangement L_0, there always exists a line operator Λ such that $\Lambda(L_0) = \emptyset$. Does there always exist an operator Λ such that the Λ-sequence is convergent and non-extinguishing? Such that L_0 is Λ-fixed?

We give in Section 3 some examples which may help to refine these questions.

2.3. The dual operators. Fixing coordinates x_1, x_2, x_3 of $\mathbb{P}^2$, one can define the operator $\mathcal{D} : \mathcal{P}(\mathbb{P}^2)^{\mathrm{F}} \to \mathcal{P}(\check{\mathbb{P}}^2)^{\mathrm{F}}$ sending a point $p = (u_1 : u_2 : u_3)$ to the line (denoted by ${}^t p$) defined by $u_1x_1 + u_2x_2 + u_3x_3 = 0$ and the operator $\check{\mathcal{D}} : \mathcal{P}(\check{\mathbb{P}}^2)^{\mathrm{F}} \to \mathcal{P}(\mathbb{P}^2)^{\mathrm{F}}$ sending the line $\ell = \{u_1x_1 + u_2x_2 + u_3x_3 = 0\}$ to the point $[\ell] = (u_1 : u_2 : u_3)$.

The arrangement $\mathcal{D}(P_0)$ (respectively, $\check{\mathcal{D}}(L_0)$) is called the dual of P_0 (respectively, L_0). The operators $\mathcal{D}, \check{\mathcal{D}}$ are inverse of each other: $\check{\mathcal{D}}\mathcal{D} = \mathrm{Id}$, $\mathcal{D}\check{\mathcal{D}} = \mathrm{Id}$.

In practice, we will forget the accent and write $\mathcal{D}$ for both $\mathcal{D}$ and $\check{\mathcal{D}}$. Also, the line and point operators act naturally on both spaces $\mathcal{P}(\check{\mathbb{P}}^2)^{\mathrm{F}}$ and $\mathcal{P}(\mathbb{P}^2)^{\mathrm{F}}$, and we will take the same notations. By the properties of duality, one has

$$\mathcal{D} \circ \mathcal{P}_{\mathfrak{n}} = \mathcal{L}_{\mathfrak{n}} \circ \mathcal{D}, \quad \mathcal{L}_{\mathfrak{n}} \circ \mathcal{D} = \mathcal{D} \circ \mathcal{P}_{\mathfrak{n}}$$

so that

$$\mathcal{D} \circ \Lambda_{\mathfrak{m},\mathfrak{n}} \circ \mathcal{D} = \Psi_{\mathfrak{m},\mathfrak{n}}, \quad \mathcal{D} \circ \Psi_{\mathfrak{m},\mathfrak{n}} \circ \mathcal{D} = \Lambda_{\mathfrak{m},\mathfrak{n}}.$$

For a subset $\mathfrak{n}$ of integers ≥ 2, we denote by $\mathcal{D}_{\mathfrak{n}}$ the operator

$$\mathcal{D}_{\mathfrak{n}} = \mathcal{L}_{\mathfrak{n}} \circ \mathcal{D},$$

which to line arrangement $\mathcal{A}$ associates the line arrangement in the dual plane which is union of the lines containing exactly n points in $\mathcal{D}(\mathcal{A})$ for $n \in \mathfrak{n}$. It has the advantage of sending line arrangements to line arrangements and that for any two subsets $\mathfrak{n}, \mathfrak{m}$ of integers ≥ 2, the following relation holds:

$$\mathcal{D}_{\mathfrak{n}} \circ \mathcal{D}_{\mathfrak{m}} = \Lambda_{\mathfrak{m},\mathfrak{n}}.$$

2.4. The ring of operators on the free module of line arrangements. Let $\mathrm{Fr}_0(\check{\mathbb{P}}^2) = \mathbb{Z}^{(\mathcal{P}(\check{\mathbb{P}}^2)^{\mathrm{F}})}$ be the free module of line arrangements: an element of Fr_0 is a formal finite sum

$$\sum a_L[L],$$

where L runs over the arrangements of lines and a_L is an integer which is 0 unless for a finite number of arrangements. One can extend linearly the operators on $\mathrm{Fr}_0(\check{\mathbb{P}}^2)$, then adding the identity operator, one gets the (non-commutative) ring of operators $\mathbb{Z}[\Lambda]$.

Problem 3. Are there relations among the line operators in the ring $\mathbb{Z}[\Lambda]$?

The set of operators Λ is uncountable. One might also add to that ring the operators $\mathcal{D}_{\mathfrak{n}}$ and ask the same question.

Remark 4. A similar free module $\mathrm{Fr}_0(\mathbb{P}^2)$ exists for points configurations, with the natural ring of operators $\mathbb{Z}[\Psi]$.

2.5. Hirzebruch inequality and operators. Let L_0 be an arrangement of d lines in the complex plane. Let us recall that we denote by $t_k(L_0)$, the number of k-points of L_0. We recall that a line arrangement is called trivial if all its lines go through the same point, and quasi-trivial if it is the union of a trivial arrangement and a line containing only double points. Hirzebruch inequality is the following theorem.

Theorem 5 (Hirzebruch inequality for lines over $\mathbb{C}$, [26, Section 3.1]). *Assume that the line arrangement L_0 is neither trivial nor quasi-trivial; then,*

$$t_2(L_0) + t_3(L_0) \geq d + \sum_{k\geq 5}(k-4)t_k(L_0).$$

With our notations, one has

$$t_k(L_0) = |\mathcal{P}_{\{k\}}(L_0)|.$$

Looking at Theorem 5, it seems therefore natural to associate to a line arrangement L_0 the formal sum

$$\begin{aligned} S(L_0) &= \mathcal{P}_{\{2\}}(L_0) + \mathcal{P}_{\{3\}}(L_0) - \left(\sum_{k\geq 5}(k-4)\mathcal{P}_{\{k\}}(L_0)\right) - \check{\mathcal{D}}(L_0) \in \mathrm{Fr}_0(\mathbb{P}^2) \\ &= \mathcal{P}_{\{2\}}(L_0) + \mathcal{P}_{\{3\}}(L_0) - \left(\sum_{k\geq 5}\mathcal{P}_k(L_0)\right) - \check{\mathcal{D}}(L_0), \end{aligned}$$

and to consider the natural map

$$C : \mathrm{Fr}_0(\mathbb{P}^2) \to \mathrm{CH}_0(\mathbb{P}^2),$$

which to a point configuration $P \subset \mathbb{P}^2$ associates the element $\sum_{p \in P} p$ in the Chow group of $\mathbb{P}^2$, and extended linearly to any element of $\mathrm{Fr}_0(\mathbb{P}^2)$. To prove Hirzebruch inequality is then equivalent to prove that $C(S(L_0))$ has non-negative degree. Hirzebruch inequality is highly non-trivial, since it is false in positive characteristic; it uses the Bogomolov–Miyaoka–Yau inequality for surfaces over $\mathbb{C}$.

There are many other Hirzebruch-type inequalities for line arrangements over $\mathbb{C}$, (see, e.g., [32]).

Over $\mathbb{R}$, there is the Melchior inequality.

Theorem 6 (Melchior). *Let $\mathcal{A}$ be an arrangement of $d \geq 3$ real lines not in a pencil. Then,*

$$t_2(\mathcal{A}) \geq 3 + \sum_{r \geq 3} (r-3) t_r(\mathcal{A}).$$

It proves in particular that an arrangement of real lines not in a pencil has always some double points. The proof of Melchior inequality is combinatorial (see [32]).

2.6. Classical theorems (Pappus, Pascal, Desargues, etc.) In this section, we remark that many classical theorems of geometry can be formulated as statements about the operator $\Lambda_{\{2\},\{3\}}$. In Section 5.3, we will give some new results of classical flavor about that operator for some arrangements of six lines.

Let us recall some vocabulary.

Definition 7. An *hexagon* H is the data of a 6-tuple $(\ell_1, \dots, \ell_6)$ of distinct lines and a 6-tuple $P_6 = (p_1, \dots, p_6)$ of double points (called the vertices) on the arrangement H such that p_i is the intersection point of ℓ_i and ℓ_{i+1} (the indices being taken mod 6). The *three diagonals* of a hexagon are the three lines through the pairs of points p_i, p_{i+3}. The *opposite sides* of H are the three pairs of lines with no common points in the set $\{p_1, \dots, p_6\}$.

2.6.1. Pascal's hexagon theorem. Pascal's hexagon theorem states that if H is a hexagon such that its set of vertices P_6 is contained in an irreducible conic, then the three intersection points of the three pairs of opposite sides of H are contained in a line (see Figure 2.1). In other words and using our notation, the arrangement $\Lambda_{\{2\},\{3\}}(H)$ is one line when the vertices are generic on a conic. That line is called the Pascal line of the hexagon H.

Since one can form 60 hexagons with vertices in P_6, there are 60 Pascal lines (when P_6 is generic): that result is known as the hexagrammum mysticum theorem. Let $L_{15} = \mathcal{P}_{\{2\}}(P_6)$ be the union of the 15 lines containing two points in P_6. Using our notations, the hexagrammum mysticum theorem means that the line arrangement

$$\Lambda_{\{2\},\{3\}}(L_{15})$$

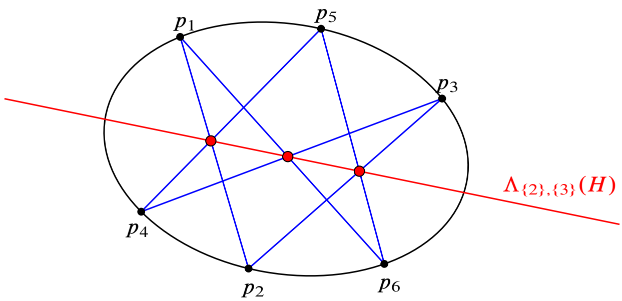


FIGURE 2.1
Pascal's hexagon theorem.

is the union of 60 lines. If P_6 is the set of the vertices of the regular hexagon, $\Lambda_{\{2\},\{3\}}(L_{15})$ has 67 lines; on random examples, we obtained 60 lines.

2.6.2. Pappus' and Steiner's theorems. Pappus' theorem can be derived from Pascal's hexagon theorem by degeneration of the conic into two lines ℓ_1, ℓ_2. Let p_1, p_2, p_3 (respectively, p_4, p_5, p_6) be three points on ℓ_1 (respectively, ℓ_2). Pappus theorem states that if $H = \{\ell_3, \dots, \ell_8\}$ is a hexagon formed by joining lines through $P_6 = \{p_1, \dots, p_6\}$, then there exists a line ℓ_9 (the Pappus line of H) such that $\Lambda_{\{2\},\{3\}}(H) = \{\ell_1, \ell_2, \ell_9\}$. When generic, the Pappus configuration $\{\ell_1, \dots, \ell_9\}$ is a 9_3-configuration: it has 9 triple points, each line contains three of these points. There are six hexagons H: the line arrangement $L_6 = \Lambda_{\{2\},\{3\}}(\mathcal{L}_{\{2\}}(P_6))$ is the union of the 6 Pappus lines.

Then, Steiner's theorem says that these six lines are union of two sets of three lines, each set of three lines meets in one point, and that the operation $\psi_{\{2\},\{3\}} \circ \mathcal{P}_{\{2\}}$ dual to $\Lambda_{\{2\},\{3\}} \circ \mathcal{L}_{\{2\}}$ applied to L_6:

$$P_6' = \psi_{\{2\},\{3\}}(\mathcal{P}_{\{2\}}(L_6)) = \psi_{\{2\},\{3\}} \circ \mathcal{P}_{\{2\}} \circ \Lambda_{\{2\},\{3\}} \circ \mathcal{L}_{\{2\}}(P_6)$$

is again a set $P_6' = \{p_1', \dots, p_6'\}$ of 6 points, with p_1', p_2', p_3' (respectively, p_4', p_5', p_6') on ℓ_1 (respectively, ℓ_2). Moreover, the points p_1', p_2', p_3' depend only on the intersection point of ℓ_1 and ℓ_2. The dynamics of the map $\{p_1, p_2, p_3\} \to \{p_1', p_2', p_3'\}$ has been studied in [27] (the author thanks Richard Schwartz for pointing out that reference). We remark that

$$\psi_{\{2\},\{3\}} \circ \mathcal{P}_{\{2\}} \circ \Lambda_{\{2\},\{3\}} \circ \mathcal{L}_{\{2\}} = \psi_{\{2\},\{3\}} \circ \psi_{\{3\},\{2\}} \circ \psi_{\{2\},\{2\}}.$$

Adopting the dual viewpoint, Steiner's theorem tells that the operator

$$\Lambda_{\{2\},\{3\}} \circ \Lambda_{\{3\},\{2\}} \circ \Lambda_{\{2\},\{2\}}$$

is a rational self map on the space of six lines concurrent in threes.

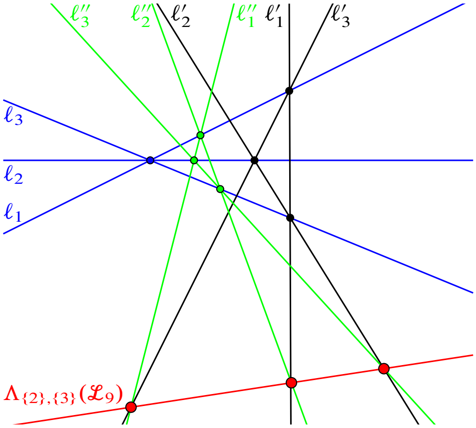


FIGURE 2.2
Desargues configuration.

2.6.3. Desargues' theorem. Let ℓ_1, ℓ_2, ℓ_3 be three lines meeting at the same point, let $\ell_1', \ell_2', \ell_3'$, (respectively, $\ell_1'', \ell_2'', \ell_3''$) be lines such that their three intersection points are on lines ℓ_1, ℓ_2, ℓ_3 (see Figure 2.2). Let $\mathcal{L}_9$ be the union of the nine lines $\ell_1, \dots, \ell_3''$. Desargues' theorem tells that $\Lambda_{\{2\},\{3\}}(\mathcal{L}_9)$ is one line.

The union $\mathcal{L}_{10}$ of $\mathcal{L}_9$ and the Desargues line $\Lambda_{\{2\},\{3\}}(\mathcal{L}_9)$ is a 10_3-configuration: $\mathcal{L}_{10}$ has 10 triple points and 10 lines such that each line contains 3 triple points. In fact, $\mathcal{L}_{10} = \Lambda_{2,3}(\mathcal{L}_9)$; moreover, $\Lambda_3(\mathcal{L}_{10}) = \mathcal{L}_{10}$.

2.6.4. Brianchon's theorem. Brianchon's theorem is the dual of Pascal's theorem, it states that when a hexagon H with vertices P_6 is circumscribed around a conic, then the three diagonals of the hexagon meet in a single point, in other words: $\Psi_{\{2\},\{3\}}(P_6)$ is a point.

The dual of Pascal's hexagrammum mysticum theorem may be formulated by saying that the point arrangement

$$\mathcal{P}_{\{3\}}(\Lambda_{\{2\},\{2\}}(H)) = \Psi_{\{2\},\{3\}}(\mathcal{P}_2(H))$$

is a set of (at least) 60 points.

2.7. Green and Tao results. In [19], Green and Tao obtained the following results.

Theorem 8 (Orchard problem; [19, Theorem 1.3] of Green–Tao). *Suppose that $\mathcal{P}$ is a finite set of n points in the real plane. Suppose that $n \geq n_0$ for some sufficiently large absolute constant n_0. Then, there are no more than $\lfloor n(n-3)/6 \rfloor + 1$ lines that are 3-rich; that is, they contain precisely 3 points of $\mathcal{P}$.*

For a line arrangement $\mathcal{C}$, let us denote by $|\mathcal{C}|$ the number of lines of $\mathcal{C}$. The above result can be rephrased as follows.

Theorem (Green–Tao). *Let $\mathcal{P}$ be a set of real points with $n = |\mathcal{P}| \geq n_0$ for some sufficiently large absolute constant n_0. Then, $|\mathcal{L}_{\{3\}}(\mathcal{P})| \leq \lfloor n(n=3)/6 \rfloor + 1$.*

Strongly related to that problem is the following result.

Theorem 9 ((Dirac–Motzkin conjecture; [19, Theorem 1.4] of Green–Tao)). *Suppose that $\mathcal{P}$ is a finite set of n points in the real plane, not all on one line. Suppose that $n \geq n_0$ for a sufficiently large absolute constant n_0. Then, $\mathcal{P}$ spans at least $n/2$ ordinary lines.*

In other words, we have the following theorem.

Theorem (Green–Tao). *There exists $n_0 \in \mathbb{N}$ such that for a set of points $\mathcal{P}$ not all on one line with $|\mathcal{P}| \geq n_0$, one has $|\mathcal{L}_{\{2\}}(\mathcal{P})| \geq |\mathcal{P}|/2$.*

There is an infinite number of point arrangements $\mathcal{P}$ such that $|\mathcal{L}_{\{2\}}(\mathcal{P})| = |\mathcal{P}|/2$, these arrangements are due to Böröczky. The two known point arrangements such that $|\mathcal{L}_{\{2\}}(\mathcal{P})| < |\mathcal{P}|/2$ are (i) the 7 points which are the vertices of a triangle, the mid-points and the center of that triangle, and (ii) an arrangement of 13 points due independently to Böröczky and McKee. As Green and Tao remark in their introduction of [19], the dual of point arrangements with few ordinary lines are line arrangements containing many triangles. In fact, the dual of the Böröczky points arrangements in (i) and (ii) of [19, Proposition 2.1] are the two known infinite families of simplicial line arrangements which are not quasi-trivial (see Section 3.3).

3. Action of operators on some examples of line arrangements

3.1. Some classical arrangements. In this section, we give some examples of line arrangements and how is the action of some operators on them.

Before that, let us recall the definition of H-constant which is used in order to study the bounded negativity conjecture. The H-constant of a line arrangement L_0 of d lines has been defined in [5] by the formula

$$H = \frac{d^2 - \sum_{m\geq 2} m^2 t_m}{\sum_{m\geq 2} t_m}.$$

It is proved in [5] (using inequalities of Hirzebruch) that over a field of characteristic zero, -4 is a lower bound for the H-constants of lines. Line arrangements with low H-constant have remarkable properties; for example, the line arrangement with lowest known H-constant ($\simeq -3.3582$) is the Wiman configuration of 45 lines constructed from some sporadic complex reflection group. Any known arrangements with H-constant < -3 were obtained by removing some lines from the Wiman arrangement.

3.1.1. Trivial, quasi-trivial, and finite plane arrangements.

Example 10 (The trivial arrangements). A line arrangement is *trivial* if this is a union of a finite number $n \geq 0$ of lines in the same pencil, i.e., of lines going through the same point. If $n \geq 2$, this is equivalent to $t_n = 1$. Then, for any line operator, $\Lambda(L_0) = \emptyset$.

Conversely, we have the following proposition.

Proposition 11. *Let L_0 be a line arrangement such that $\Lambda(L_0) = \emptyset$ for any line operator Λ. Then, L_0 is trivial.*

Proof. If L_0 is a line arrangement such that $\Lambda_2(L_0) = \emptyset$, then $\mathcal{P}_2$ is either empty or a point, therefore, L_0 is a trivial arrangement. ■

Example 12 (The quasi-trivial arrangements). A line arrangement of $n \geq 3$ lines is *quasi-trivial* if this is a union of a trivial arrangement and a line not going through the intersection point of the lines in the pencil. When $n \geq 4$, this is equivalent to the equality $t_{n-1} = 1$. The Λ_2-sequence associated to a quasi-trivial arrangement is constant (and non-extinguishing).

Example 13 (Finite plane arrangements). Let us suppose that the base field K is the closure of a finite field $\mathbb{F}_q$. Let $L_0 \in \check{\mathbb{P}}^2(\mathbb{F}_q)$ be a line arrangement. For a fixed line operator Λ, let us consider the Λ-sequence $(L_m)_m$ associated to L_0. For all $m \geq 0$, the intersection points of L_m are defined over $\mathbb{F}_q$, and therefore, the lines in L_{m+1} are in $\check{\mathbb{P}}^2(\mathbb{F}_q)$. Since these are finite sets, the Λ-sequence is always periodic. That simple example shows that proving that a sequence $(L_n)_{n\geq 0}$ associated to a line operator Λ and an arrangement L_0 acquire more and more lines is a non-trivial result over a field of characteristic 0.

The arrangement $L_0 = \check{\mathbb{P}}^2(\mathbb{F}_q)$ is called a finite plane arrangement. If $n, m \leq q+1$, the $\Lambda_{n,m}$-sequence associated to $\check{\mathbb{P}}^2(\mathbb{F}_q)$ is constant.

Quasi-trivial arrangements and finite plane arrangements are two examples of line arrangements L_0 such that $\Lambda_2(L_0) = L_0$. Conversely, we have the following.

Proposition 14. *Let $L_0 \neq \emptyset$ be an arrangement of lines. Suppose that $\Lambda_2(L_0) = L_0$. Then, L_0 is a quasi-trivial arrangement or a finite projective plane.*

Proposition 14 is an application of the following result.

Theorem 15 (De Bruijn–Erdös). *Let L_0 be an arrangement of $d \geq 3$ lines which is not a trivial arrangement and let t be the number of singular points. Then, $t \geq d$*

and $t = d$ if and only if L_0 is a quasi-trivial arrangement or a finite projective plane arrangement.

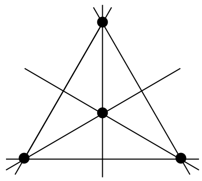
For a proof of Theorem 15, we refer to [16, Theorem 2.7].

Proof of Proposition 14. Let $P_0 = \{p_1, \dots, p_t\} = \mathcal{P}_2(L_0)$ be the set of singularities of the line arrangement $L_0 = \{\ell_1, \dots, \ell_d\}$. By the theorem of De Bruijn–Erdös, we have $t \geq d$.

Let us denote by $[\ell]$ the point in the dual plane corresponding to a line ℓ, and by ${}^t p$ the line in the dual plane defined by a point p in the plane. Since $\Lambda_2(L_0) = L_0$, for any two points $p \neq q \in P_0$, the line ℓ passing through p and q is contained in L_0. On the dual plane level, that implies that any two lines ${}^t p, {}^t q$ in the line arrangement $\check{P}_0 = \{{}^t p_1, \dots, {}^t p_t\}$ meet in a point $[\ell]$ such that $\ell \in L_0$. Therefore, the set $\{[\ell], | \ \ell \in L_0\}$ is the singularity set of $\check{P}_0$, and by the theorem of De Bruijn–Erdös, one get $d \geq t$. We obtain that $d = t$, and the result follows again from Theorem 15. ∎

3.1.2. Classical arrangements. A reference for most of the line arrangements presented here is [21, Chapter 6].

Example 16 (The complete quadrilateral). Let L_0 be the arrangement of six lines going through four points in general position.



Then, $P_0 = \mathcal{P}_2(L_0)$ is a set of 7 points and $\mathcal{L}_3(\mathcal{P}_2(L_0)) = L_0$, i.e., L_0 is fixed by $\Lambda_{2,3}$, unless in characteristic 2, for which $\Lambda_{3,2}(L_0) = \Lambda_3(L_0)$ is the Fano plane, i.e., the 7 lines in $\mathbb{P}^2(\mathbb{F}_2)$, giving the unique 7_3-configuration. In characteristic 0, the first terms of the associated Λ_2-sequence $(L_n)_n$ are such that

	$\lvert L_n \rvert$	H_{cst}	t_2	t_3	t_4	t_6
L_0	6	$-1,714$	3	4		
L_1	9	-2.077	6	4	3	
L_2	25	-2.464	60	24	3	10

where the column H_{cst} is an approximation of the H-constant; the line arrangement L_3 has 1471 lines. The number of lines in L_m grows to infinity, as shown in [12]. It would be interesting to know the limit of the H-constants of the configurations L_n.

Example 17 (Hesse and dual Hesse arrangements). The Hesse arrangement $\mathcal{H}$ is defined over fields containing a third root of unity $\omega \neq 1$. It is the union of 12 lines and its set P_0 of 21 singular points are such that

$$t_2 = 12, \quad t_4 = 9.$$

The nine 4-points and the 12 lines form a $(9_4, 12_3)$-configuration (see Section 3.2 for the definition of a (r_k, s_m)-configuration). One has $\mathcal{L}_3(P_0) = \mathcal{H}$, $\mathcal{P}_3(\mathcal{H}) = P_0$, thus, $\Lambda_3(\mathcal{H}) = \mathcal{H}$ and the associated Λ_3-sequence is constant. The Hesse arrangement is $\Lambda_{\{4\},\{3\}}$-constant and $\mathcal{L}_{\{2\}}(\mathcal{D}(\mathcal{H}))$ is projectively equivalent to $\mathcal{H}$.

The dual Hesse arrangement $\check{\mathcal{H}} = \mathcal{L}_{\{4\}}(\mathcal{D}(\mathcal{H}))$ is a $(12_3, 9_4)$-configuration of 9 lines with equations

$$\begin{aligned} &-x+z, -x+y, -y+z, -\omega x+z, -\omega x+y, \\ &-\omega^2 x+z, -\omega^2 x+y, -\omega^2 y+z, -\omega y+z, \end{aligned}$$

where $\omega^2+\omega+1=0$. The arrangement $\check{\mathcal{H}}$ has 12 triple points

$$\begin{aligned} &(1:0:0), (0:1:0), (0:0:1), (1:1:1), \\ &(1:1:\omega), (1:1:\omega^2), (1:\omega:1), (1:\omega^2:1), \\ &(\omega:1:1), (\omega^2:1:1), (\omega:\omega^2:1), (\omega^2:\omega:1), \end{aligned}$$

it is the unique known arrangement over $\mathbb{C}$ which has only triple points. One has $\Lambda_3(\check{\mathcal{H}}) = \check{\mathcal{H}}$, and $\Lambda_{3,4}(\check{\mathcal{H}}) = \check{\mathcal{H}}$. The line arrangement $\Lambda_{\{2\},\{4\}}(\mathcal{H})$ is projectively equivalent to $\check{\mathcal{H}}$.

The arrangement ML obtained by removing one line in $\check{\mathcal{H}}$ is a 8_3-arrangement known as the *MacLane* or *Möbius–Kantor* arrangement. The arrangement $\Lambda_{3,2}(\mathrm{ML})$ is isomorphic to the Hesse arrangement $\mathcal{H}$. The line arrangement $\Lambda_{3,2}(\check{\mathcal{H}})$ is an arrangement of 21 lines, it is the union of $\check{\mathcal{H}}$ and a line arrangement projectively equivalent to $\mathcal{H}$. Its singularities are $t_2 = 36, t_4 = 9, t_5 = 12$ and it is a free arrangement (see [28, Proposition 4.6]).

The (complex) line arrangements $\mathcal{H}$ and $\Lambda_{3,2}(\check{\mathcal{H}})$ are combinatorially simplicial, i.e., their singularities satisfy equation (3.1), cf. later in Section 3.3 the definition of simplicial line arrangement.

Example 18 (Ceva(n) arrangement). The Ceva(n) arrangement is given by the $3n$ lines

$$(x^n - y^n)(x^n - z^n)(y^n - z^n) = 0.$$

Ceva(2) is the complete quadrilateral, Ceva(3) is the dual Hesse arrangement. The arrangement Ceva(n) has n^2 triple points, three n-points and no other singularities;

the H-constant is $-\frac{3n^2}{n^2+3}$. Each line contains one n-point and n triple points, thus, for $n \geq 3$, the Λ_3-sequence is constant. In characteristic 0, this is the unique known infinite family of *Sylvester–Gallai* arrangements, i.e., line arrangements with no double points. These are also reflection arrangements, i.e., the lines are the fixed lines of some reflexion group (denoted by $G(n,n,3)$) acting on the plane.

The arrangement $\overline{\mathrm{Ceva}(n)}$ obtained as the union of Ceva(n) and the three lines $xyz = 0$ is a free arrangement. Each of the three lines adds up n double points, and the three n-points of Ceva(n) become $n+2$-points so that the singularities are

$$t_2 = 3n, \quad t_3 = n^2, \quad t_{n+2} = 3.$$

These line arrangements are combinatorially simplicial and one has $\Lambda_3(\overline{\mathrm{Ceva}(n)}) = \mathrm{Ceva}(n)$.

Example 19 (Polygonal arrangements $A_1(2m)$, extended Polygonal arrangements $A_1(4m+1)$, see [26]). For $m \geq 3$, the regular arrangement $A_1(2m)$ consists of $2m$ lines, of which m lines are determined by the edges of a regular m-gon in the Euclidean plane (contained in $\mathbb{P}^2$), while the other m are the lines of symmetry of that m-gon. For $m = 3$, we get the complete quadrilateral; for $m > 3$, one has

$$t_2 = m, \quad t_3 = \frac{m(m-1)}{2}, \quad t_m = 1, \quad t_r = 0 \quad \text{otherwise.}$$

Let $P_0 = \mathcal{P}_3(A_1(2m))$ be the set of triple or higher multiplicity points of the $A_1(2m)$ arrangement.

• Suppose that m is odd. Let us show that $\Lambda_3(A_1(2m)) = A_1(2m)$ when $m \geq 5$. Let us write $m = 2k+1$. Then, a line of symmetry contains k triple points (intersection by pairs of $2k$ lines of the faces) and one double point (intersection of the line with its opposite face). We get in that way $km = \frac{m(m-1)}{2}$ triple points and m double points. A face contains $m-1$ triple points, intersection of $m-1$ symmetry lines, and the $m-1$ other faces. We get $m(m-1)$ triple points, but each point is counted twice, so that the total number of triple points is $\frac{m(m-1)}{2}$. Therefore, as soon as m is odd and $m \geq 5$, the associated Λ_3-sequence is constant.

• Suppose that m is even, $m = 2k$. Let us show that when $k \geq 3$,

$$\Lambda_3(A_1(4k)) = A_1(4k+1), \quad \Lambda_3(A_1(4k+1)) = A_1(4k+1),$$

where $A_1(4k+1)$ is the union of $A_1(4k)$ and the line at infinity, its singularities are

$$t_2 = 3k, \quad t_3 = 2k(k-1), \quad t_4 = k, \quad t_{2k} = 1, \quad t_r = 0$$

otherwise. The arrangement $A_1(4k)$ has $2k$ faces, k lines of symmetries each containing two edges of the $2k$-gon, k lines of symmetries, each containing two double points, which are in the middle between two consecutive edges.

The number of double points on a face is 1 and there are $2k-1$ triple points: each face meets the other $2k-1$ faces and $2k-1$ lines of symmetries into $2m-1$ triple points, the remaining line of symmetry cuts the face at one double point.

A line of symmetry containing two edges cuts the $2k-1$ other lines of symmetry at the same point, a point of multiplicity $2k$, and cuts the $2k$ lines of the $2k$-gon at k triple points.

A line of symmetry containing two double points contains also the $2m$-point and $k-1$ triple points, intersection points of the $2k$ faces.

The $A_1(4k+1)$-configuration is obtained by adding the line at infinity, which contains then k 4-points, which were previously triple points on $A_1(4k)$ coming from two parallel faces and the parallel line of symmetry, and moreover, k double points, intersection of the k lines containing 2 edges of the $2k$-gon.

Thus, when $k-1 \geq 2$, we obtain that all lines in $A_1(4k)$ contain 3 or more m-points with $m \geq 3$.

For $k=2$, i.e., $A_1(9)$, the singularities are $t_2=6$, $t_3=4$, $t_4=3$ and we get that $\Lambda_3(A_1(9))$ is the complete quadrilateral $A_1(6)$.

Example 20 (Klein configuration). The Klein configuration $\mathcal{K}$ of 21 lines is a reflexion arrangement: it is the union of the hyperplanes fixed by the 21 involutions of the unique simple group of order 168 (acting faithfully on $\mathbb{P}^2$). The singularity set of $\mathcal{K}$ is such that

$$t_3=28, \quad t_4=21, \quad t_r=0 \quad \text{for } r \neq 3,4.$$

The arrangement $\mathcal{K}$ is defined over fields containing $\sqrt{-7}$; the H-constant is $H(\mathcal{K})=-3$. The 21 quadruple points and 21 lines form a (21_4)-configuration (see Section 3.2 for the definition). Each line contains 4 triple points and 4 quadruple points. The configuration is Λ_4-constant: $\Lambda_4(\mathcal{K})=\mathcal{K}$. It is also a fixed arrangement for $\Lambda_{3,4}$, but not for Λ_3 or for $\Lambda_{4,3}$.

The arrangement $\Lambda_3(\mathcal{K})$ has 133 lines, and singularities

$$t_2=2436, \quad t_3=588, \quad t_4=84, \quad t_5=168, \quad t_9=28, \quad t_{12}=21.$$

The 9-points and 12-points of $\Lambda_3(\mathcal{K})$ are the singular points of the Klein configuration.

The arrangement $\Lambda_{4,3}(\mathcal{K})$ has 49 lines, and singularities

$$t_2=252, \quad t_3=112, \quad t_8=21;$$

If it is fixed by $\Lambda_{4,3}$, one has

$$\Lambda_4(\Lambda_{4,3}(\mathcal{K}))=\mathcal{K};$$

moreover, $\Lambda_3(\Lambda_{4,3}(\mathcal{K}))$ contains 889 lines.

Example 21 (Grünbaum–Rigby configuration). See [21], see also [18]. The Grünbaum–Rigby configuration $\mathcal{C} = (P_0, L_0)$ of 91 points and 21 lines is such that

$$t_2 = 63, \quad t_3 = 7, \quad t_4 = 21,$$

with $H_{cst} = -30/13 \simeq -2.30$. The 21 4-points and the 21 lines form a (21_4)-configuration which, as an abstract configuration, is isomorphic to the (21_4)-configuration of the Klein arrangement (see Section 3.2 for the definition of a (r_k, s_m)-configuration). The 63 double points with the 21 lines form a $(63_2, 21_6)$-configuration, moreover, each line contains a unique triple point. One has $\Lambda_4(L_0) = L_0$: this is a fixed configuration for Λ_4.

Let $P_1 = \mathcal{P}_3(L_0)$ be the 28 points with multiplicity ≥ 3. The arrangement $\mathcal{L}_3(P_1) = \Lambda_3(L_0)$ has 50 lines, and the singularities are

$$t_2 = 259, \quad t_3 = 119, \quad t_7 = 29.$$

The points in P_1 are now points of multiplicity 7 in $\Lambda_3(L_0)$. The arrangement $\Lambda_3(L_0)$ is fixed by $\Lambda_{4,3}$.

The arrangement $\Lambda_4(\Lambda_3(L_0))$ has 29 lines, singularities

$$t_2 = 70, \quad t_3 = 21, \quad t_4 = 7, \quad t_5 = 21, \quad t_7 = 1,$$

and H-constant $-157/60$. It is a fixed arrangement for Λ_4.

The arrangement $\mathcal{L}_4(P_1)$ has 28 lines, the H-constant is $H = -294/113 \simeq -2.6$, and singularities

$$t_2 = 63, \quad t_3 = 28, \quad t_5 = 21, \quad t_7 = 1.$$

This arrangement is fixed for Λ_4; moreover, the 28 lines and 28 triple points form a 28_3-configuration.

Example 22 (The Wiman configuration). The Wiman configuration $\mathcal{W}$ is a reflexion arrangement, it is the union of 45 lines defined over a field containing a 15-th root of unity. Its singularities are

$$t_3 = 120, \quad t_4 = 45, \quad t_5 = 36.$$

The 45 lines and 45 4-points form a 45_4-configuration and the 45 lines and 36 5-points form a $(36_5, 45_4)$-configuration; with the 120 triple point that gives a $(120_3, 45_8)$-configuration. The configuration $\mathcal{W}$ is fixed by the $\Lambda_{5,4}$ operator. Its H-constant is $H(\mathcal{W}) = -225/67 \simeq -3.3582$, this is the lowest H-constant known for line arrangements.

The configuration $\Lambda_{4,5}(\mathcal{W})$ has 81 lines and singularities

$$t_2 = 180, \quad t_3 = 480, \quad t_5 = 36, \quad t_8 = 45,$$

and $H(\Lambda_{4,5}(\mathcal{W})) = -753/247 \simeq -3.0486 < -3$. It is fixed by the operator $\Lambda_{4,5}$.

3.2. The (r_k, s_m)-configurations.

3.2.1. Definition of (r_k, s_m)-configurations. For integers r, s, k, m, a (r_k, s_m)-configuration $(\mathcal{Q}, \mathcal{C})$ is an arrangement $\mathcal{C}$ of s lines and a set of r singular points $\mathcal{Q}$ of $\mathcal{C}$ such that each line of $\mathcal{C}$ contains exactly m points of $\mathcal{Q}$ and each point of $\mathcal{Q}$ is incident to exactly k lines of $\mathcal{C}$. One has $rk = sm$, moreover, if $r = s$, one simply speaks of a r_k-configuration. If the numbers r, s are unimportant for the problem considered, one speaks of a $[k, m]$-configuration (and of a k-configuration if $k = m$). References for points and lines configurations are, for example, [15, 20]. Note that in the definition of configuration, there can exist other m-rich lines and other k-points.

Lemma 23. *Let $\mathcal{C}$ be a $\Lambda_{\{k\},\{m\}}$-constant arrangement. Then, $(\mathcal{P}_{\{k\}}(\mathcal{C}), \mathcal{C})$ is a $[k, m]$-configuration.*

Let $(\mathcal{Q}, \mathcal{C})$ be a $[k, m]$-configuration. Then, each line of $\mathcal{C}$ is a line of $\Lambda_{\{k\},\{m\}}(\mathcal{C})$.

Proof. Suppose that $\mathcal{C}$ is a line arrangement such that

$$\Lambda_{\{k\},\{m\}}(\mathcal{C}) = \mathcal{C}.$$

That means each line of $\mathcal{C}$ contains m points in $\mathcal{P}_{\{k\}}(\mathcal{C})$, and therefore, $(\mathcal{P}_{\{k\}}(\mathcal{C}), \mathcal{C})$ is a $[k, m]$-configuration.

Suppose that $(\mathcal{Q}, \mathcal{C})$ is a $[k, m]$-configuration. If ℓ is a line of $\mathcal{C}$, then ℓ contains m exactly k-points, i.e., m points in $\mathcal{P}_{\{k\}}(\mathcal{C})$, thus, ℓ is line of $\Lambda_{\{k\},\{m\}}(\mathcal{C})$. ∎

It may happen that for a $[k, m]$-configuration $\mathcal{C}$, the line arrangement $\Lambda_{\{k\},\{m\}}(\mathcal{C})$ contains more lines, as shown in the following example.

Example 24. (A) Consider the arrangement $\mathcal{C}$ of 6 lines with equations $x, x + z, x - z, y, y + z, y - z$. The 9 points $(a : b : 1) \in \mathbb{P}^2$ with $a, b \in \{-1, 0, 1\}$ are the double points of $\mathcal{C}$, and there are two triple points at infinity. The 9 double points and the 6 lines form a $(9_2, 6_3)$-configuration. However, the arrangement $\Lambda_{\{2\},\{3\}}(\mathcal{C})$ contains $\mathcal{C}$ and the two lines with equation $x - y, x + y$.

(B) The Pappus 9_3-configuration P has also 9 double points which with the 9 lines forms a 9_2-configuration, moreover, $\Lambda_{\{2\},\{2\}}(\mathrm{P})$ has 36 lines.

(C) Consider a k-configuration $\mathcal{C}$; then, using appropriate projective transformations, one can take k projectively equivalent copies $\mathcal{C}_1, \dots, \mathcal{C}_k$ of $\mathcal{C}$ in $\mathbb{P}^2$ with one k-point p_i on each $\mathcal{C}_i$ in such a way that $p_1, \dots, p_k$ are contained on a line not in $\cup\mathcal{C}_i$; then, that line is in $\Lambda_{\{k\}}(\mathcal{C})$ but is not in $\mathcal{C} = \cup\mathcal{C}_i$.

Let $(\mathcal{Q}, \mathcal{C})$ and $(\mathcal{Q}', \mathcal{C}')$ be two $[k, m]$-configurations. Up to using a projective automorphism of $\mathbb{P}^2$, let us suppose that $\mathcal{Q} \cap \mathcal{Q}' = \emptyset$. Then, the union $(\mathcal{Q} \cup \mathcal{Q}', \mathcal{C} \cup \mathcal{C}')$ is a $[k, m]$-configuration. This leads to the following definition.

Definition 25. A $[k,m]$-configuration $(\mathcal{Q},\mathcal{C})$ is said *connected* if for any two points $p, p' \in \mathcal{Q}$, there exists a sequence of k-points $p_1 = p, p_2, \dots, p_j = p'$ such that p_i, p_{i+1} are two k-points on the same line of $\mathcal{C}$.

Problem 26. We do not know if there exist connected $[k,m]$-configurations with $k > 2$ such that $\Lambda_{\{k\},\{m\}}(\mathcal{C}) \neq \mathcal{C}$.

The dual arrangement of a $[k,m]$-configuration $(\mathcal{Q},\mathcal{C})$ is defined by $\check{\mathcal{C}} = \mathcal{D}(\mathcal{Q})$ and $(\mathcal{P}_m(\check{C}), \check{C})$ is an $[m,k]$-configuration. In particular, an n_k-configuration $(\mathcal{Q},\mathcal{C})$ has the feature that its dual is a n_k-configuration; it is called self dual if the two configurations are isomorphic, i.e., if there exists an isomorphism between planes that sends one arrangement into the other.

3.2.2. Examples of (r_k, s_m)-configurations. An arrangement $\mathcal{C}$ of $n \geq 3$ lines in general position is a $((\frac{n(n-1)}{2})_2, n_{n-1})$-configuration, and $\Lambda_{2,n-1}(\mathcal{C}) = \mathcal{C}$; one even has the relation $\Lambda_{\{2\},\{n-1\}}(\mathcal{C}) = \mathcal{C}$.

Another example of $\Lambda_{\{2\},\{m\}}$-constant arrangements with $m \geq 3$ is as follows: let $\mathcal{C}$ be the union of two pencils of m lines in general position. Each line of $\mathcal{C}$ contains exactly m double points, thus, $\mathcal{C}$ is $\Lambda_{\{2\},\{m\}}$-constant. More generally, if $\mathcal{C}$ is the union of $r > 1$ pencils of s lines in general position, then $\mathcal{C}$ is $\Lambda_{\{2\},\{(r-1)s\}}$-constant.

These arrangements have the property that each line contains a unique k-point with $k > 2$. But for example, it is possible to construct a $\Lambda_{\{2\},\{4\}}$-constant arrangement of 9 lines with 6 triple points and 18 double points such that each line contains 2 triple points and 4 double points.

We have seen that the Fano plane $\mathbb{P}^2(\mathbb{F}_2)$ is a 7_3-configuration. There exist n_3-configurations in characteristic $\neq 2, 3$ if and only if $n \geq 8$.

Consider the dual Hesse $(12_3, 9_4)$-configuration $\check{\mathcal{H}}$ (which exists over fields containing a third root of unity). Removing one line and the four points on that line gives the Möbius–Kantor 8_3-configuration MK. One has

$$\Lambda_3(\mathrm{MK}) = \mathrm{MK} \quad \text{and} \quad \Lambda_{2,3}(\mathrm{MK}) = \check{\mathcal{H}}.$$

The Pappus configuration (see Section 2.6.2) is a 9_3-configuration with singularities $t_2 = 9, t_3 = 9$; it is $\Lambda_{2,3}$ and Λ_3-constant. There are two other non equivalent 9_3-configurations, they are also Λ_3-constant. These configurations are defined over the rationals.

It is known that there exist n_4-configurations of real lines for $n = 18$ and $n \geq 20$ except possibly for $n \in \{23, 37, 43\}$. There exists no 19_4-configuration over the real field. There are only two real 18_4-configurations (over $\mathbb{R}$), the first was found by Bokowski and Schewe in [10] and is defined over $\mathbb{Q}(\sqrt{5})$, the second was found by Bokowski and Pilaud (see [8]) and is defined over $\mathbb{Q}(\sqrt[3]{108 + 12\sqrt{93}})$.

3.2.3. 2-arrangements. Let us define a $[k, m]$-arrangement as a $[k, m]$-configuration $(\mathcal{Q}, \mathcal{L})$ such that $t_k(\mathcal{L}) = \mathcal{Q}$: an n_2-arrangement is therefore an arrangement of n lines such that each line contains exactly two double points.

The k-configurations for $k \geq 3$ have been extensively studied, and it is easy to construct n_2-configurations by discarding some nodal points on n lines in general position. But if one is rather interested in 2-arrangements, much less seems to be known. Here is a list of the 2-arrangements we are aware of

(1) The triangle: three lines with $t_2 = 2$.

(2) The three 9_3-configurations and their nodes are also 9_2-arrangements.

(3) The Hesse arrangement is a 12_2-arrangement.

(4) The 15_2 icosahedral line arrangement.

If one imposes no triple points, one gets the following result, which the author owes to Piotr Pokora.

Theorem 27. *Let $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{C}}$ be an n_2-arrangement with $n \geq 3$ and assume that $t_3(\mathcal{L}) = 0$. Then, $\mathcal{L}$ is either the triangle (so $n = 3$) or the Hesse arrangement (so that $n = 12$).*

Proof. It is easy to see that if $n = 3$, then the only possible arrangement is the triangle with $t_2 = 3$. We can assume that $n \geq 4$. We can also assume, without loss of generality, that $t_n = 0$ and $t_{n-1} = 0$, since both trivial and quasi-trivial are not n_2-arrangements. Then, $\mathcal{L}$ satisfies the Hirzebruch inequality

$$t_2 + t_3 \geq n + \sum_{r \geq 5} (r - 4) t_r$$

(see Theorem 5). Since $t_2 = n$ and $t_3 = 0$, we get that for every $r \geq 5$, we have $t_r = 0$. Summing up, if $\mathcal{L}$ is an n_k-arrangement with $t_3 = 0$, then $t_2 = n$, $t_4 = \frac{n^2-3n}{12}$, and $t_r = 0$ for every $r \geq 5$. Moreover, our arrangement satisfies the equality in the above Hirzerbuch's inequality. Our goal is to check which line arrangements with only double and quadruple points satisfy Hirzebruch's inequality. It turns out, somehow surprisingly, that it follows from a result devoted to the existence of ball-quotient surfaces constructed as Hirzebruch–Kummer covers [4, Kapitel 3.1 G.], namely, the only arrangement satisfying all these conditions is the Hesse arrangement of 12 lines with 12 double and 9 quadruple points. ∎

3.3. Simplicial arrangements and Λ-operators. An arrangement of real lines is called *simplicial* if all the polygons cut out by the lines are triangles; this is equivalent to require the equality

$$3 + \sum_{r \geq 2} (r - 3) t_r = 0. \tag{3.1}$$

The quasi-trivial, the polygonal arrangements and extended polygonal arrangements (see Section 3.1.2) are the three known infinite families of simplicial arrangements. Few other sporadic examples of simplicial arrangements are known. The state of the art is given in [13] (together with the normals of the lines of simplicial line arrangements). The simplicial arrangements are labeled $A(n,k)$ or $A(n,k)_i$, where n is the number of lines and k, i are integers characterizing the arrangement.

One can check that among the 119 simplicial arrangements with less than 37 lines (which are listed in [13]), 100 are Λ-constant for some $\Lambda \in \{\Lambda_3, \Lambda_{3,4}, \Lambda_{3,5}, \Lambda_{3,6}, \Lambda_{2,3}, \Lambda_{2,4}, \Lambda_4\}$. For the other arrangements, one has the following relations:

$$\begin{aligned}
\Lambda_{2,4}(A(14,112)_3) &= A(15,128)_2, & \Lambda_{3,4}(A(24,316)) &= A(25,336)_3,\\
\Lambda_{3,4}(A(18,184)_2) &= A(19,200)_3, & \Lambda_{3,4}(A(24,320)) &= A(25,336)_6,\\
\Lambda_{3,4}(A(19,192)_1) &= A(18,180)_4, & \Lambda_{3,5}(A(25,320)) &= A(24,304),\\
\Lambda_{3,4}(A(20,220)_3) &= A(21,240)_4, & \Lambda_{3,5}(A(26,380)) &= A(31,480),\\
\Lambda_{3,4}(A(20,220)_4) &= A(21,240)_5, & \Lambda_{3,5}(A(29,400)) &= A(31,480),\\
\Lambda_{3,4}(A(23,304)) &= A(25,336)_6, & \Lambda_{3,5}(A(32,544)) &= A(33,576),\\
\Lambda_{3,6}(A(24,312)) &= A(25,336)_5, & \Lambda_{3,6}(A(36,684)) &= A(37,720)_2;
\end{aligned}$$

moreover, the arrangement $A(26,380)$ is $\Lambda_{\{3,5\},5}$-constant. For $\mathcal{A} = A(28,420)_4$, one has $\Lambda_{3,\{5\}}\Lambda_{3,5}(\mathcal{A}) = \mathcal{A}$, while $\Lambda_{3,5}(\mathcal{A})$ is $\Lambda_{3,5}$-constant equal to $A(29,448)_4$. Also, $A(28,420)_5$ and $A(28,420)_6$ are permuted under the operator $\Lambda_{\{3,4,5\},5}$. The image of both arrangements by $\Lambda_{3,5}$ is $A(31,480)$. The operator $\Lambda_{\{3,4,5\},5}$ permutes $A(27,400)$ and $A(29,440)$.

It is possible to obtain (many) other relations among these arrangements, such as $\Lambda_{4,5}(A(34,612)_1) = A(13,96)_3$.

3.4. Free arrangements. A line arrangement is free if the sheaf of vector fields tangent to this arrangement splits as a sum of two line bundles over $\mathbb{P}^2$ (see [30] for the definitions). Roughly speaking, Terao's conjecture says that the topology and the geometry of a free arrangement is determined by its combinatoric. For example, the Hesse arrangement is free. A free arrangement L_0 is such that the roots of its characteristic polynomial

$$T^2 + (1-d)T + 1 - d + \sum_{k\geq 2}(k-1)t_k(L_0)$$

are integers, where $d = |L_0|$.

One can construct free arrangements from a known free arrangement by the addition-deletion theorem of Terao (see, e.g., [1]). A natural question is whether one can construct all free arrangements starting from the arrangement ∅ and using the addition-deletion theorem. An arrangement (in $\mathbb{P}^2$) is said inductively free if it can be obtained by the addition theorem, recursively free if it can be obtained by addition and deletion of lines.

Cuntz–Hoge arrangement (see [14]) CH_{27} of 27 lines was the first found line arrangement which is free but non recursively free. It is defined over $\mathbb{Q}(\zeta)$, where ζ is a primitive fifth root of unity, this is, in fact, the reflection arrangement of the exceptional complex reflection group G_{27}. Its singularities are

$$t_2 = 15, \quad t_3 = 70, \quad t_7 = 6.$$

The line arrangement $\Lambda_{\{7\},\{2\}}(\mathrm{CH}_{27})$ is the icosahedral line arrangement of 15 lines with $t_2 = 15, t_3 = 10, t_5 = 6$, it is also known as the simplicial arrangement $\mathcal{A}(15, 120)$. The line arrangement $\Lambda_{\{7\},\{2\}}(\mathrm{CH}_{27})$ is contained in CH_{27}; the complementary arrangement L_{12} of 12 lines has only double points and $\Lambda_{\{2\},\{6\}}(L_{12}) = \mathcal{A}(15, 120)$.

The dual line arrangement $\mathcal{D}_{\{2\}}(\mathrm{CH}_{27})$ is also (projectively equivalent to) the icosahedral line arrangement $\mathcal{A}(15, 120)$.

In [2], Abe, Cuntz, Kawanoue, and Nozawa constructed two arrangements ACKN_{13} and ACKN_{15} that are free but not recursively free arrangements. In Section 4, we will discuss these arrangements which are interesting for understanding the dynamic of the operator $\Lambda_{\{3\},\{2\}}$. We found anew the line arrangements ACKN_{13} and ACKN_{15} in a completely different way when studying the operator $\Lambda_{\{2\},\{3\}}$ (see Subsections 5.1 and 5.3). The arrangement ACKN_{15} is also related to $\mathcal{A}(15, 120)$; see [33].

3.5. Zariski pair arrangements. There are several definitions of Zariski pairs (L, L'); roughly speaking, these are line arrangements that have the same combinatoric but such that the topology of the pairs $(\mathbb{P}^2, L)$, $(\mathbb{P}^2, L')$ is different.

• The Zariski pair $\mathrm{Ry}_{13}, \mathrm{Ry}'_{13}$ defined by Rybnikov in [34] are 13 line arrangements with $t_2 = 33, t_3 = 15$, they define the same matroid (meaning the 13 lines have the same configurations, see Section 7). However, one can check that the generic arrangements have different behavior under the map $\Lambda_{\{3\},\{2\}}$: the arrangement $\Lambda_{\{3\},\{2\}}(\mathrm{Ry}_{13})$ has 42 lines, but $\Lambda_{\{3\},\{2\}}(\mathrm{Ry}'_{13})$ has 45. Also, $\Lambda_3(\mathrm{Ry}_{13})$ has 11 lines, but $\Lambda_3(\mathrm{Ry}'_{13})$ possesses 10 lines.

We do not claim that the geometric differences we found between these line arrangements give another proof that one has Zariski pairs. However, we remark that, in [25, Proposition 4.6], the different topology of the pairs is detected thanks to an alignment of three specific points.

• The Zariski pair $\mathrm{ACCM}_{11}, \mathrm{ACCM}'_{11}$ defined by Artal, Carmona, Cogolludo, and Marco in [3] are arrangements of 11 lines. These line arrangements are defined over $\mathbb{Q}(\sqrt{5})$ and are conjugated under the Galois group. One of the lines has been added in order to trivialize the automorphism group of the combinatoric and $\Lambda_3(\mathrm{ACCM}_{11})$ removes that line: the line arrangement $\Lambda_3(\mathrm{ACCM}_{11})$ is the simplicial arrangement $\mathcal{A}(10,60)_3$, which is the union of the side of a pentagon and the lines through the center and the vertices of the pentagon.

• The Zariski pair $\mathrm{GB}_{12}, \mathrm{GB}'_{12}$ defined by Guerville–Ballé in [22] have 12 lines, defined over $\mathbb{Q}(\zeta_5)$, where ζ_5 is a primitive 5^{th}-root of unity. There is also a line which forces the triviality of the automorphism group of the combinatoric. Applying Λ_3 removes that line, and then, one gets projectively equivalent line arrangements with $t_2 = 13, t_3 = 6, t_4 = 4$.

• Various Zariski pairs of 13 lines over the rationals are defined by Guerville–Ballé and Viu in [25]. One of these pairs has the following non-bases (see Section 7):

$$(1,5,7),(1,8,10),(1,11,12),(2,5,6),(2,8,9),(2,11,13),(3,4,5),$$
$$(3,6,8),(3,7,11),(3,9,10),(3,12,13),(2,4,7,10,12),(1,4,6,9,13).$$

The moduli space of realizations has two one dimensional irreducible components $\mathcal{M}^{\pm}$. The normals of the 13 lines are the (projectivization of) the columns of the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & a^2 & b^2 & a^2 & a & a & b^2 & b & b \\ 0 & 0 & 1 & 1 & a^2 & a^2 & 1 & a & a & 1 & b & 1 & b \end{pmatrix},$$

where $ab = \pm 1$ for $\mathcal{M}^{\pm}$; moreover, the parameter a must be in some open set $U^{\pm}$ of $\mathbb{P}^1$ so that the line arrangement has no more singularities or is not degenerate. Let us denote by $\mathcal{A}^{\pm}(a)$ the 13 lines arrangement associated to the parameter $a \in U^{\pm}$. Then, the line arrangement $\Lambda_{\{3\},\{2\}}(\mathcal{A}^{+})$ has 18 lines, whereas $\Lambda_{\{3\},\{2\}}(\mathcal{A}^{-})$ has 30 lines: the different nature of $\mathcal{A}^{\pm}$ is detected by the $\Lambda_{\{3\},\{2\}}$ operator.

• The Zariski triple defined by Guerville–Ballé in [23] are arrangements of 12 lines, defined over $\mathbb{Q}(\zeta_7)$, where ζ_7 is a primitive 7th-root of unity. Their singularity set is $t_2 = 17, t_2 = 11, t_4 = t_5 = 1$. These line arrangements are rigid and are not projectively equivalent; however, they are conjugated under the Galois group. Therefore, they are indiscernible from the viewpoint of line operators since the action of the Galois group commutes with the action of the line operators. Similar arrangements are constructed in [24].

4. First divergent arrangements

4.1. Minimal Λ-constant arrangements. In Section 3, we obtained a classification of line arrangements L_0 such that $\Lambda_2(L_0) = L_0$. The triangle T, i.e., 3 lines in general position, is the non-trivial Λ_2-constant configuration which has the minimal number of lines.

The configuration L of 4 lines in general position is such that $\Lambda_{2,3}(L) = L$ and any other arrangement with 4 lines is sent to $\emptyset$ by $\Lambda_{2,3}$. The complete quadrilateral is also $\Lambda_{2,3}$-constant. One may ask the following question.

Problem 28. For a given operator Λ, find the minimal number $n > 0$ of lines such that there is an arrangement L with n lines and $\Lambda(L) = L$.

The complete quadrilateral $A_1(6)$ is $\Lambda_{3,2}$-constant, and so, are the simplicial arrangements $A(7, 32)$ and $A(9, 48)$. The Hesse configuration of 12 lines is $\Lambda_{3,2}$ and $\Lambda_{4,2}$-constant. Other $\Lambda_{4,2}$-constant arrangements are $A(13, 96)_3$, $A(15, 120)$; the first one is defined over $\mathbb{Q}$.

4.2. A condition for divergence. We have seen in Example 16 (see also Section 4.3 below), that four lines in general position over a field of characteristic 0 are the configuration with the minimal number of lines such that the associated Λ_2-sequence diverges, i.e., the number of lines of the associated Λ_2-sequence goes to infinity. More generally, for a fixed operator Λ, it would be interesting to known which configurations of lines are Λ-divergent, and with the minimal number of lines for that property. The following observation is elementary.

Lemma 29. *Let $\mathcal{A}$ be an arrangement of m lines. Suppose that $\Lambda_{n,k}(\mathcal{A})$ contains a line ℓ not on $\mathcal{A}$. Then, $m \geq nk$.*

Proof. The line goes though at least k points of multiplicity $\geq n$, since ℓ is not a line of $\mathcal{A}$ there are at least nk lines. ∎

4.3. Divergence of the Λ_2-operator. Let $\mathcal{C}_0$ be four real lines in general position and let $(\mathcal{C}_n)_{n\in\mathbb{N}}$ be the associated Λ_2-sequence.

Proposition 30. *The sequence $(\mathcal{C}_n)_{n\in\mathbb{N}}$ is divergent.*

That follows directly from Melchior inequality (see Theorem 6). Indeed, at each step, $\mathcal{C}_k$ is contained in $\mathcal{C}_{k+1}$ and no double point in $\mathcal{C}_k$ remains a double point in $\mathcal{C}_{k+1}$. The starting line arrangement being real, this is a sequence of real line arrangements, thus, $\mathcal{C}_k$ has some double points; therefore, the number of lines is strictly increasing with k.

We reproduce below another proof from [12], which is also interesting.

Proof. Since $\mathcal{C}_k \subset \mathcal{C}_{k+1}$, it is sufficient to prove that $\mathcal{C}_{k+1} \neq \mathcal{C}_k$ for all $k \geq 0$. Suppose that $\mathcal{C}_k = \mathcal{C}_{k+1}$ for some k, and let $\langle \mathcal{P}_2(\mathcal{C}_k)\rangle$ be the convex hull of the singular points of $\mathcal{C}_k$. Since $\mathcal{C}_0 \subset \mathcal{C}_k$, $\langle \mathcal{P}_2(\mathcal{C}_k)\rangle$ cannot be a line. If $\langle \mathcal{P}_2(\mathcal{C}_k)\rangle$ has 4 or more sides, then two nonadjacent sides (which are lines in $\mathcal{C}_k$) meet at some m-point of $\mathcal{C}_k$ outside $\langle \mathcal{P}_2(\mathcal{C}_k)\rangle$, which is impossible. Therefore, $\langle \mathcal{P}_2(\mathcal{C}_k)\rangle$ has 3 sides: let $a, b, c \in \mathcal{P}_2(\mathcal{C}_k)$

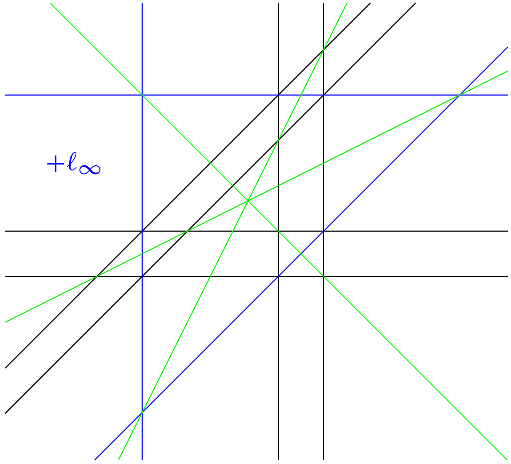


FIGURE 4.1
The line arrangement $\mathcal{C}_0$ and its images.

be the set of vertices of $\langle \mathcal{P}_2(\mathcal{C}_k)\rangle$. Suppose that there are points of $\mathcal{P}_2(\mathcal{C}_k)$ along at least two sides and not in $\{a,b,c\}$, say, $x \in ab,\ y \in ac$. Then, the line $xy \in \mathcal{C}_k$ cuts bc outside of $\langle \mathcal{P}_2(\mathcal{C}_k)\rangle$, which is again a contradiction. Suppose that there exists some point $x \in \mathcal{P}_2(\mathcal{C}_k)$ in the interior of $\langle \mathcal{P}_2(\mathcal{C}_k)\rangle$; then, the lines ax, bx, cx cut, respectively, bc, ac, ab at points of the sides not in $\{a,b,c\}$, which is a contradiction. Therefore, $\mathcal{C}_k$ is a quasi-trivial arrangement, this is impossible since $\mathcal{C}_0 \subset \mathcal{C}_k$, and one concludes that $\mathcal{C}_{k+1} \neq \mathcal{C}_k$. ■

4.4. About the divergence of the $\Lambda_{2,3}$-operator. By Lemma 29, a divergent $\Lambda_{2,3}$-sequence contains at least 6 lines. Figure 4.1 represents the successive images $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2$ by $\Lambda_{2,3}$ of the six line arrangement $\mathcal{C}_0$ which is the union of three pairs of parallel lines (in black).

The arrangement $\mathcal{C}_1 = \Lambda_{2,3}(\mathcal{C}_0)$ is the union of the black lines, the 3 blue lines and the line at infinity ℓ_∞ (which contains 3 double points of $\mathcal{C}_0$); the arrangement $\mathcal{C}_2 = \Lambda_{2,3}(\mathcal{C}_1)$ is the union of $\mathcal{C}_1$ and the three green lines. The following table gives the number of lines and singularities of $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$:

	$\lvert\mathcal{C}\rvert$	t_2	t_3	t_4	t_6	t_7
$\mathcal{C}_0$	6	15				
$\mathcal{C}_1$	10	9	6	3		
$\mathcal{C}_2$	13	12	16	3		
$\mathcal{C}_3$	28	87	31	15	3	3

Remark 31. The arrangement $\mathcal{C}_4$ has 946 lines. The arrangement $\mathcal{C}_2'$ obtained by removing the line at infinity of $\mathcal{C}_2$ has 12 lines and $t_2 = 9, t_3 = 19$. That line arrangement has been obtained by Zacharias in [36]. It is studied in [17], where it is proved that the 19 triple points of this configuration give another rational example of the non-containment of the third symbolic power into the second ordinary power of an ideal. The 12 real lines arrangements with 19 triple points (the upper bound) have been classified in [9].

4.5. About the divergence of the $\Lambda_{3,2}$-operator. Let us give two examples of line arrangements with interesting properties related to the $\Lambda_{3,2}$-operator.

- The sequence of the number of lines of the successive images by the operator $\Lambda_{3,2}$ of the dual Hesse configuration begins by $9, 21, 57, 7401$.
- The arrangement of 9 lines $\mathcal{C}_0$ obtained from the simplicial arrangement $A(10,60)_1$ by removing the line at infinity is such that

	$\lvert\mathcal{C}\rvert$	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}	t_{16}	t_{18}	t_{20}
$\mathcal{C}_0$	9	6	10												
$\mathcal{C}_1$	16	30	4	3	6										
$\mathcal{C}_2$	25	60	24	3		10									
$\mathcal{C}_3$	229	8472	1572	468	258	79	42	6	6	3	6	3	12	12	6

Here, $(\mathcal{C}_n)_n$ is the $\Lambda_{3,2}$-sequence associated to $\mathcal{C}_0$. We conjecture that the number of lines of $(\mathcal{C}_n)_n$ tends to infinity.

4.6. About the divergence of the Λ_3-operator. The equations of 15 lines $\ell_1, \dots, \ell_{15}$ of the simplicial arrangement $A(15,132)_1$ are given in [13, Table 7], the singularities of $A(15,132)_1$ are

$$t_2 = 9, \quad t_3 = 22, \quad t_5 = 3.$$

The arrangement $A(15,132)_1$ is defined over fields containing a root of $X^3 - 3X - 25$. The configuration $\mathcal{C}_0$ with the fewer lines we found and which seems divergent for the Λ_3-operator is the 14 line arrangement obtained by removing the line ℓ_2 (or ℓ_{10} or ℓ_{11}) in $A(15,132)_1$ (each line $\ell_2, \ell_{10}, \ell_{11}$ contains two 5-points). The singularities of $\mathcal{C}_0$ are

$$t_2 = 9,$$
$$t_3 = 20,$$
$$t_4 = 2,$$
$$t_5 = 1.$$

The following table gives the number of lines and singularities of the first terms of the associated Λ_3-sequence:

	$\|\mathcal{C}\|$	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	$\dots$
$\mathcal{C}_0$	14	9	20	2	1					
$\mathcal{C}_1$	18	27	12	10	3					
$\mathcal{C}_2$	24	63	12	6	12		1			
$\mathcal{C}_3$	42	235	46	6	8	8	12			
$\mathcal{C}_4$	305	16417	2847	849	314	101	56	30	16	$\dots$

The configuration $\mathcal{C}_0$ has another curious feature: the $\Lambda_{\{3,4\},3}$-sequence $(\widetilde{\mathcal{C}}_n)_{n\geq 0}$ associated to $\mathcal{C}_0$ converges to a sequence of 14 lines. The number of lines of the first 8 terms of the sequence is $14, 18, 24, 18, 23, 20, 26, 14$ so that one has $\widetilde{\mathcal{C}}_n \neq \widetilde{\mathcal{C}}_{n+1}$ for $n \leq 7$, moreover, one has $\widetilde{\mathcal{C}}_n = \widetilde{\mathcal{C}}_{n+1}$ for $n \geq 8$.

Let $\mathcal{C}'_0$ be the arrangement obtained from $A(15, 132)_1$ by removing the line ℓ_5. Forgetting the first term, the associated $\Lambda_{\{3,4\},3}$-sequence is periodic of period 2: for $n \geq 1$, one has $\mathcal{C}'_n = \mathcal{C}'_{n+2}$. The configuration $\mathcal{C}'_1$ has 13 lines and singularities $t_2 = 12$, $t_3 = 14$, $t_4 = 4$, the configuration $\mathcal{C}'_2$ has 17 lines and singularities

$$t_2 = 32, \quad t_3 = 4, \quad t_4 = 12, \quad t_5 = 3.$$

The arrangement $\mathcal{C}'_1$ is contained in $\mathcal{C}'_2$.

4.7. About the divergence of the $\Lambda_{\{2\},\{2\}}$-operator. Let $\mathcal{A}_0$ be the union of four lines in general position. Then, $\mathcal{A}_1 = \Lambda_{\{2\},\{2\}}(\mathcal{A}_0)$ is the union of three lines, and the associated sequence $(\mathcal{A}_k)_k$ is constant for $k \geq 1$. Any other configuration of ≤ 4 lines gives a sequence converging to the empty line arrangement.

If instead one takes the union $\mathcal{A}_0$ of 5 lines in general position, then $\mathcal{A}_1 = \Lambda_{\{2\},\{2\}}(\mathcal{A}_0)$ has 15 lines, and $\mathcal{A}_2$ has 2070 lines. The arrangements $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ are disjoints as sets of lines. We may wonder if that can be generalized: if for every $\mathcal{A}_k, \mathcal{A}_{k'}$ with $k \neq k'$, the line arrangements $\mathcal{A}_k, \mathcal{A}_{k'}$ have no common lines. We conjecture that the number of lines of $\mathcal{A}_k$ tends to ∞ with k.

5. On the $\Lambda_{\{2\},\{3\}}$-operator

5.1. A flashing configuration for $\Lambda_{\{2\},\{3\}}$. Define $\mathcal{S} = \{0, \pm 1, \frac{1}{2}, 2, \tau, \tau^2\}$, with $\tau^2 - \tau + 1 = 0$. For a fixed parameter $t \notin \mathcal{S}$, consider the configuration $\mathcal{F}_0 = \mathcal{F}_0(t)$

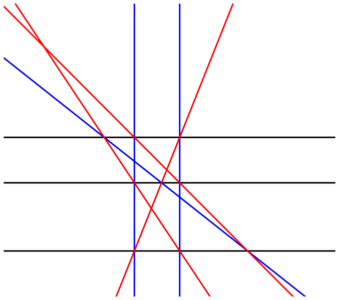


FIGURE 5.1
The 9 lines of $\mathcal{A}_1$, for the flashing arrangement of 6 lines for $\Lambda_{\{2\},\{3\}}$.

of 6 lines $\ell_1, \dots, \ell_6$, whose normals are the columns of the following matrix:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & t & 0 & 0 & t^2-t+1 \\ 0 & 1 & 1 & 0 & 1 & t \end{pmatrix}.$$

The singularities of $\mathcal{F}_0$ are $t_2 = 12, t_3 = 1$. Each line ℓ_1, ℓ_2, ℓ_3 contains 4 singular points: the triple point and 3 double points of $\mathcal{F}_0$; each of the three remaining lines contains 5 double points of $\mathcal{F}_0$. Then, $\mathcal{F}_1 = \Lambda_{\{2\},\{3\}}(\mathcal{F}_0)$ is an arrangement of 6 lines, union of the 3 lines ℓ_1, ℓ_2, ℓ_3 of $\mathcal{F}_0$ and three lines ℓ_7, ℓ_8, ℓ_9 with normals

$$(1:1:0), \quad (1:t:t), \quad (0:t:1).$$

It has the same configuration as $\mathcal{F}_0$, moreover, $\Lambda_{\{2\},\{3\}}(\mathcal{F}_1) = \mathcal{F}_0$, so that the sequence associated to $\mathcal{F}_0$ is periodic with period 2. The union of $\mathcal{F}_0$ and $\mathcal{F}_1$ is a 9 line arrangement $\mathcal{A}_1$ with 6 double points and 10 triple points. That line arrangement is a non-generic case of a Pappus configuration.

Definition 32. We say that an arrangement projectively equivalent to $\mathcal{F}_0(t)$ for some $t \notin \mathcal{S}$ is a flashing arrangement of six lines.

Figure 5.1 represents a Flashing arrangement; in black are the lines ℓ_1, ℓ_2, ℓ_3, in blue the lines ℓ_4, ℓ_5, ℓ_6 and in red the lines ℓ_7, ℓ_8, ℓ_9; the triple point is at infinity.

Using matroids (see Section 7), one can compute that the moduli space of line arrangements with such incidences is the above family $\mathcal{F}_0 = \mathcal{F}_0(t)$ for $t \notin \mathcal{S}$. The parameter $t \in \{0, \pm 1\}$ $(\subset \mathcal{S})$ gives a degenerate line arrangement. When $t = \tau \in \mathcal{S}$ with $\tau^2 - \tau + 1 = 0$, $\mathcal{F}_0(\tau)$ is an arrangement of six lines with two triple points such that the line arrangement $\Lambda_{\{2\},\{3\}}(\mathcal{F}_0)$ is Ceva(3) (a.k.a. the dual Hesse arrangement). When $t \in \{\frac{1}{2}, 2\}$, $\mathcal{F}_1 = \Lambda_{\{2\},\{3\}}(\mathcal{F}_0)$ has 7 lines with four double points and $\Lambda_{\{2\},\{3\}}(\mathcal{F}_1) = \emptyset$.

For the general case $t \notin \mathcal{S}$, the projective transformation of the plane

$$\gamma = \begin{pmatrix} -1 & 1 & 1-t \\ -t & t & 1-t \\ -t & 1 & 0 \end{pmatrix} \in \mathrm{PGL}_3$$

is such that $\gamma^2 = \mathrm{I_d}$ and $\mathcal{F}_1 = \gamma\mathcal{F}_0$; in particular, the action of $\Lambda_{\{2\},\{3\}}$ on the moduli space of flashing configurations is trivial.

We found the first example of a Flashing arrangement when searching a line arrangement $\mathcal{C}_0$ with the smallest number of lines such that the number of lines of the sequence defined by $\mathcal{C}_{k+1} = \Lambda_{\{2\},\{3\}}(\mathcal{C}_k)$ diverges to ∞.

Remark 33. In [29], Nazir and Yoshinaga define two line arrangements $\mathcal{A}^{\pm}$ of 9 lines with the same weak combinatoric (same singularities): $t_2 = 6, t_3 = 10$ as $\mathcal{A}_1 = \mathcal{F} \cup \Lambda_{\{2\},\{3\}}(\mathcal{F})$, for a flashing arrangement $\mathcal{F}$. However, they are rigid line arrangements. One can check moreover that such an arrangement $\mathcal{A}^{\pm}$ contains three flashing arrangements $\mathcal{F}$ of six lines. Each such an arrangement $\mathcal{F}$ is such that there exists a unique line in $\Lambda_{\{2\},\{3\}}(\mathcal{F})$ which is not on $\mathcal{A}^{\pm}$. The union of $\mathcal{A}^{\pm}$ and these three lines is the Ceva(4) arrangement.

5.2. A flashing configuration of 12 lines for $\Lambda_{\{3\},\{2\}}$. Let $\mathcal{F}_0$ be a flashing arrangement of six lines and define the dual of $\mathcal{F}_0$ by $\mathcal{A}_0 = \mathcal{D}_{\{2\}}(\mathcal{F}_0)$ (see notations in Section 5.4). This is a 12 lines arrangement, with $t_2 = 18, t_3 = 6, t_5 = 3$. The arrangement $\mathcal{A}_1 = \Lambda_{\{3\},\{2\}}(\mathcal{A}_0)$ has the same number and type of singularities; one has $\mathcal{A}_0 = \Lambda_{\{3\},\{2\}}(\mathcal{A}_1)$: the $\Lambda_{\{3\},\{2\}}$-sequence associated to $\mathcal{A}_0$ is periodic with period 2.

Definition 34. We call $\mathcal{A}_0$ a flashing configuration of 12 lines.

Arrangements $\mathcal{A}_0$ and $\mathcal{A}_1$ have nine common lines and the union of $\mathcal{A}_0$ and $\mathcal{A}_1$ is a 15 lines arrangement with singularities

$$t_2 = 36, \quad t_3 = 3, \quad t_5 = 6.$$

One can compute that the moduli space of line arrangements of 12 lines defining the same matroid as $\mathcal{A}_0$ is two dimensional. However, for a generic element $\mathcal{B}$ of that moduli space, $\Lambda_{\{3\},\{2\}}(\mathcal{B})$ has 15 lines with singularities $t_2 = 36, t_3 = 3, t_5 = 6$, therefore, $\Lambda_{\{3\},\{2\}}$ is not a self-map of that moduli space. The difference is also seen by looking at the 9 lines arrangement $\mathcal{D}(\mathcal{P}_3(\mathcal{B}))$ which has $t_2 = t_3 = 9$, whereas the arrangement $\mathcal{D}(\mathcal{P}_3(\mathcal{A}_0))$ is a 9 line arrangement with $t_2 = 6, t_3 = 10$.

The arrangement $\breve{\mathcal{F}}_0 = \mathcal{D}_2(\mathcal{F}_0)$ has 13 lines and singularities

$$t_2 = 21, \quad t_3 = t_4 = t_5 = 3.$$

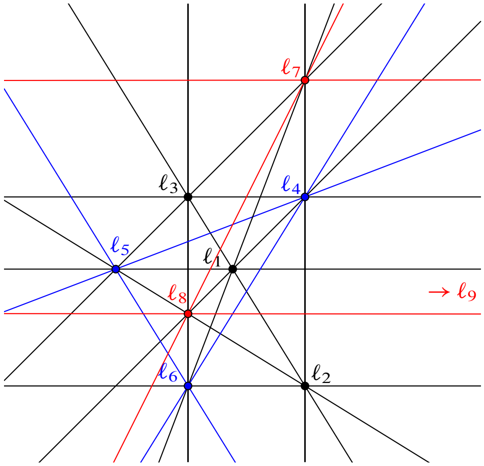


FIGURE 5.2
The 13 lines of $\breve{\mathcal{F}}_0 = \mathcal{L}_2(\mathcal{D}(\mathcal{C}_0))$.

Its moduli space $\mathcal{M}_{13}$ is one dimensional, and $\Lambda_{\{3\},2}(\mathcal{A}_1) = \breve{\mathcal{F}}_0$, so that we may recover the moduli space of flashing configuration of 12 lines from the moduli space $\mathcal{M}_{13}$. The difference between $\breve{\mathcal{F}}_0$ and $\mathcal{A}_1$ is the line going through points ℓ_1, ℓ_2, ℓ_3 in Figure 5.2. The black points correspond to the common lines ℓ_1, ℓ_2, ℓ_3 of $\mathcal{F}_0$ and $\breve{\mathcal{F}}_1 = \mathcal{D}_2(\mathcal{F}_1)$, the blue points correspond to ℓ_4, ℓ_5, ℓ_6 , the red points correspond to ℓ_7, ℓ_8, ℓ_9 (the last being at infinity), the three blue (respectively, red) lines are the lines of $\breve{\mathcal{F}}_0$ (respectively, $\breve{\mathcal{F}}_1$) which are not in the arrangement $\breve{\mathcal{F}}_1$ (respectively, $\breve{\mathcal{F}}_0$), the 10 black lines are common to both $\breve{\mathcal{F}}_0$ and $\breve{\mathcal{F}}_1$.

The arrangements of 13 lines $\breve{\mathcal{F}}_0$ has been studied in [2] as an example of a free but not recursively free line arrangement (see Section 3.4).

5.3. Unassuming arrangements. In [33], we study lines arrangements $\mathcal{C}_0$ of six lines such that $t_2(\mathcal{C}_0) = 15$, but the line arrangement $\mathcal{D}_2(\mathcal{C}_0)$ has singularities $t_2 = 27$, $t_3 = t_5 = 6$, moreover, the six points in the dual $\mathcal{D}(\mathcal{C}_0)$ are not contained in a conic. We call such arrangements "unassuming": which word describes something that is deceptively simple but has hidden qualities or advantages.

There is a one dimensional family of unassuming arrangements; the normals of the six lines are given by the columns of the following matrix:

$$M_t = \begin{pmatrix} 1 & 0 & 0 & 1 & \frac{1}{2}(1+t) & \frac{1}{2}(1-t) \\ 0 & 1 & 0 & 1 & \frac{1}{2}(1-t) & \frac{1}{2}(1+t) \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{5.1}$$

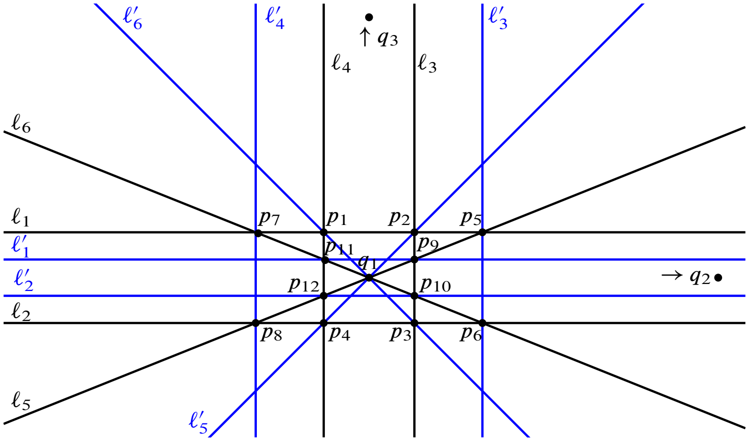


FIGURE 5.3
An unassuming arrangement and its image by $\Lambda_{\{2\},\{3\}}$.

for $t \in U = \mathbb{P}^1 \setminus \{0, \pm 1, \infty, \pm 2 \pm \sqrt{5}\}$. For fixed $t \in U$, let $(\mathcal{C}_k)_{k\geq 0}$ be the $\Lambda_{\{2\},\{3\}}$-sequence associated to $\mathcal{C}_0 = \mathcal{C}_0(t)$. We obtain the following theorem.

Theorem 35 ([33]). *The moduli space of unassuming arrangements is the union of U and a point. The image by the operator $\Lambda_{\{2\},\{3\}}$ of a generic unassuming arrangement is again an unassuming arrangement. For the general unassuming arrangement, the associated $\Lambda_{\{2\},\{3\}}$-sequence $(\mathcal{C}_k)_{k\geq 0}$ is such that $\mathcal{C}_m$ is not projectively equivalent to $\mathcal{C}_n$ for any $m \neq n$. For each n, there exist periodic unassuming arrangements with period n.*

See Figure 5.3 for a picture of such line arrangement (in black) and its image (in blue) by $\Lambda_{\{2\},\{3\}}$.

For an analytic proof, see [33]. From Figure 5.3, let us give a synthetic proof that there exists a one parameter family of real unassuming line arrangements which is preserved by $\Lambda_{\{2\},\{3\}}$, as follows.

Proof. Let (ℓ_1, ℓ_2) (respectively, (ℓ_3, ℓ_4)) be a pair of parallel lines such that ℓ_1 is orthogonal to ℓ_3. Let p_1, p_2, p_3, p_4 be the meeting points in $\mathbb{R}^2 \subset \mathbb{P}^2(\mathbb{R})$ of $\ell_1, \dots, \ell_4$ and let ℓ_5 be a generic line passing through the intersection point q_1 of the two diagonals of the 4-gon $p_1, \dots, p_4$ (there is a one parameter choice for such ℓ_5). Let ℓ_6 be the image of ℓ_5 by the orthogonal reflexion σ_1 with axis the line passing through q_1 and parallel to ℓ_1. The line ℓ_6 is also the image of ℓ_5 by the orthogonal reflexion σ_2 with axis the line passing through q_1 and parallel to ℓ_3. The lines $\ell_1, \dots, \ell_6$ are the black lines in Figure 5.3. The lines ℓ_1, ℓ_2 (respectively, ℓ_3, ℓ_4) meet at infinity at point

q_2 (respectively, q_3). We denote by $p_5, \dots, p_{12}$ the remaining double points as in Figure 5.3.

Since $p_{11} = \sigma_2(p_9)$, the line $\ell'_1 = \overline{p_9, p_{11}}$ is parallel to ℓ_1 and ℓ_2, and therefore, it contains q_2. Since $p_{12} = \sigma_2(p_{10})$, the line $\ell'_2 = \overline{p_{10}, p_{12}}$ is also parallel to ℓ_1 and ℓ_2, thus, it contains q_2.

Since $p_6 = \sigma_1(p_5)$, the line $\ell'_3 = \overline{p_5, p_6}$ is parallel to ℓ_3, ℓ_4 and contains q_3.

Since $p_8 = \sigma_1(p_7)$, the line $\ell'_4 = \overline{p_7, p_8}$ is parallel to ℓ_3, ℓ_4 and contains q_3.

The lines $\ell'_5 = \overline{p_2, p_4}$ and $\ell'_6 = \overline{p_1, p_3}$ contain q_1.

The lines $\ell'_1, \dots, \ell'_6$ containing exactly three double points of $\ell_1, \dots, \ell_6$ are the lines in blue in Figure 5.3.

Then, the situation for the blue of lines is the same as for the black lines, therefore, the configuration repeats itself. ■

By duality, one can rephrase the results in Theorem 35 on the action of $\Lambda_{\{2\},\{3\}}$ on unassuming arrangements as follows.

Theorem 36. *For a set $P_6 = \{p_1, \dots, p_6\}$ of six points, consider the following property.*

(P) *The union of the lines containing two points in P_6 possesses exactly six triple points $p'_1, \dots, p'_6$. The points of P_6 are not inscribed in a conic.*

Suppose that P_6 satisfies (P). Then, the set of triple points $P'_6 = \{p'_1, \dots, p'_6\} = \Psi_{\{2\},\{3\}}(P_6)$ satisfies (P), moreover, if the points in P_6 are real, there exists a unique set of six real points P_6^- satisfying (P) and such that

$$\Psi_{\{2\},\{3\}}(P_6^-) = P_6.$$

Figure 5.4 gives an example of such a set P_6 (points in black), and its image P'_6 (points in red) by $\Psi_{\{2\},\{3\}}$.

Remark 37. Let $\mathcal{C}_0$ be an unassuming arrangement, then the line arrangement $\mathcal{D}_2(\mathcal{C}_0)$ is the line arrangement we denoted by ACKN_{15} in Section 3.4, which is a free but non recursively free arrangement.

Remark 38. One may wonder if there exists a Cremona transformation τ sending the six lines of an unassuming arrangement $\mathcal{C}_0$ to $\mathcal{C}_1 = \Lambda_{\{2\},\{3\}}(\mathcal{C}_0)$. We checked that the images of $\mathcal{C}_0$ by the Cremona standard involutions based at three of the double points of $\mathcal{C}_0$ give either arrangements with fewer lines or unassuming arrangements projectively equivalent to $\mathcal{C}_0$. By the Noether–Castelnuovo theorem, the Cremona group is generated by standard involution and $PGL_3(\mathbb{C})$: the existence of such a τ seems therefore unlikely.

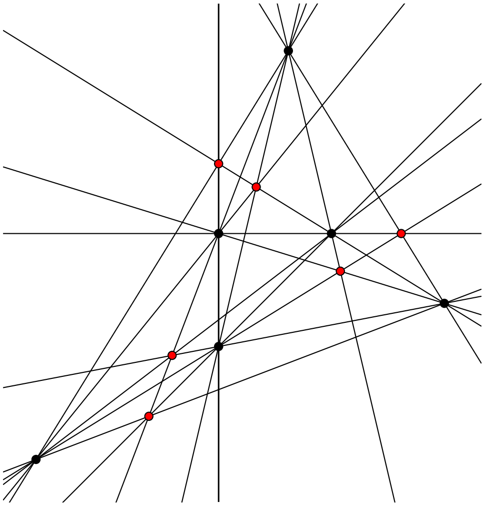


FIGURE 5.4
The points in P_6 and the six associated triple points.

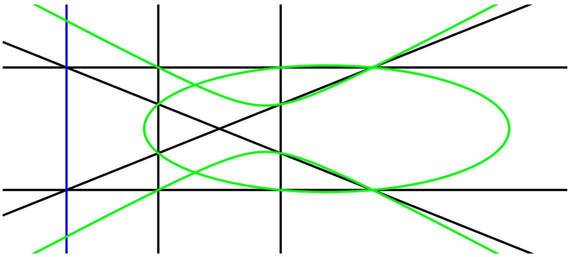


FIGURE 5.5
The two conics with the same Pascal line for $\mathcal{C}_0$.

Remark 39. One can compute that the set $\mathcal{S}$ of irreducible conics containing at least six of the 15 double points $\mathcal{P}_{\{2\}}(\mathcal{C}_0)$ of $\mathcal{C}_0$ is the union of 12 conics. Each conic C of $\mathcal{S}$ contains exactly 6 double points and the six lines of $\mathcal{C}_0$ form an hexagon for C. The Pascal line of that hexagon, i.e., the line containing the three double points of the three pairs of opposite sides, is a line of $\mathcal{C}_1 = \Lambda_{\{2\},\{3\}}(\mathcal{C}_0)$. Each line of $\mathcal{C}_1$ is the Pascal line of two conics of $\mathcal{S}$, see Figure 5.5.

5.4. About the divergence of $\Lambda_{\{2\},\{3\}}$. Let $\mathcal{C}_0$ be the arrangement of six lines with normals that are columns of matrix M_t in (5.1) for $t = \pm 2 \pm \sqrt{5} \in \mathbb{P}^1 \setminus U$. The following table gives the number of lines and singularities of the first terms of the associated $\Lambda_{\{2\},\{3\}}$-sequence:

	$\lvert\mathcal{C}\rvert$	t_2	t_3	t_4	t_5
$\mathcal{C}_0$	6	15			
$\mathcal{C}_1$	10	45			
$\mathcal{C}_2$	90	1710	120	210	45

We conjecture that the number of lines of $\mathcal{C}_k$ tends to infinity with k. The line arrangement $\mathcal{D}_{\{2\}}(\mathcal{C}_0)$ is the simplicial line arrangement $A(15, 120)$.

6. Flashing arrangements for $\Lambda_{\{2\},\{k\}}$, $k \in \{4, 5, 6, 7\}$

6.1. A generalization of the $\Lambda_{\{2\},\{3\}}$-flashing arrangements. For $n \geq 4$, let us construct line arrangements that have the same properties for $\Lambda_{\{2\},\{n\}}$ as the flashing arrangement for $\Lambda_{\{2\},\{3\}}$ in Section 5. We thus want to construct a line arrangement L of $3n$ lines $\ell_1, \dots, \ell_{3n}$ such that the line arrangement $L_2 = \{\ell_{n+1}, \dots, \ell_{2n}\}$ has a unique singular point ($t_n(L_2) = 1$), the lines arrangements $L_1 = \{\ell_1, \dots, \ell_n\}$ and $L_3 = \{\ell_{2n+1}, \dots, \ell_{3n}\}$ have only nodal singularities ($t_2(L_k) = \frac{1}{2}n(n-1)$ for $k = 1, 3$), the line arrangements $\mathcal{C}_0 = L_1 \cup L_2$ and $\mathcal{C}_1 = L_2 \cup L_3$ have nodes and one n-point ($t_2(\mathcal{C}_j) = \frac{1}{2}(3n^2 - n)$, $t_n(\mathcal{C}_j) = 1$ for $j = 1, 2$), moreover, the line arrangement $L = L_1 \cup L_2 \cup L_3$ has singularities

$$t_2 = n^2 - n, \quad t_3 = n^2, \quad t_n = 1.$$

We require, moreover, that each line ℓ_k contains n triple points of L. Then, up to permutation of the lines in each of the arrangements L_1, L_2, L_3 and up to relabeling the triple points, one may suppose that the incidence matrix of the $3n$ lines and n^2 triple points is

$$\begin{pmatrix} C_1 & I_n & I_n \\ C_2 & I_n & G_1 \\ \vdots & \vdots & \vdots \\ C_n & I_n & G_{n-1} \end{pmatrix},$$

where C_k is the $n \times n$ matrix having 1 in all the entries of k-th column and 0 in all other entries, I_n is the size n identity matrix, and $G_1, \dots, G_n$ are $n \times n$ matrices. Since the lines of that incidence matrix represent the triple points, each line of these matrices G_k must contain a unique 1, moreover, since the lines are distinct, the matrix G_1 cannot have a 1 on its diagonal. Similarly, once G_1 is fixed, there are new restrictions on G_2 etc, so that the matrix

$$I_n + \sum_{k=1}^{n-1} G_k$$

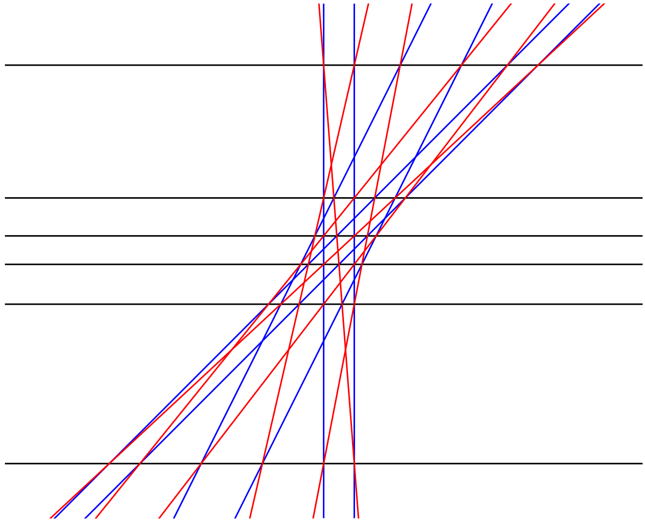


FIGURE 6.1
The 18 lines of $\mathcal{A}_1$ for the flash arrangement of $\Lambda_{\{2\},\{6\}}$.

is the matrix with 1 in each entry. We complete the above incidence matrix by adding as a line the size $1 \times 3n$ matrix $(0, \dots, 0, 1, \dots, 1, 0, \dots, 0)$, which represents the n-point.

Let G be the size $n \times n$ matrix such that $G(e_i) = e_{i+1}$, where the indices are taken modulo n and $(e_i)_i$ is the canonical basis. The matrix G has order n; for $G_1, \dots, G_{n-1}$, let us take $G_k = G^k$. We checked that for $n \in \{3, 4, 5, 6, 7\}$, the matroid associated to the above incidence matrix is representable in characteristic 0, in other words, there exist $3n$ lines in $\mathbb{P}^2(\mathbb{C})$ that have that configuration, see Figure 6.1 for the case $n = 6$.

In each case, the moduli space of such line arrangements is one dimensional. If $n \in \{3, 4, 6, 7\}$, that moduli space is an open subset of $\mathbb{P}^1$, in particular, it is irreducible; if $n = 5$, the moduli space is the union of two disjoint irreducible rational components defined over $\mathbb{Q}(\sqrt{5})$. For any $n \in \{3, 4, 5, 6, 7\}$, these line arrangements are made so that

$$\Lambda_{\{2\},\{n\}}(\mathcal{C}_j) = \mathcal{C}_k,$$

where $\{j, k\} = \{0, 1\}$, indeed: the n^2 double points which are the intersection points of the lines in L_1 and the lines in L_2 are also the n^2 double points which are the intersection of the lines in L_2 and the lines in L_3. The lines that are common to $\mathcal{C}_0$ and $\mathcal{C}_1$ are the lines of L_2. Each line in L_1 possess $2n - 1$ double points in $\mathcal{C}_0$, thus, they cannot be in $\mathcal{C}_1$, whereas $L_3 \subset \Lambda_{\{2\},\{n\}}(\mathcal{C}_0)$. In the next section, we describe with more details the case $k = 4$.

6.2. A flashing configuration of 8 lines for $\Lambda_{\{2\},\{4\}}$. For a parameter t, consider the following configuration of 12 lines $\ell_1, \dots, \ell_{12}$ whose normals are the columns of the following matrix:

$$\begin{pmatrix} 1 & 0 & t^2-\frac{1}{2}t & 2t^2 & 1 & 0 & t & 2 & 0 & 2t^2-t & t & 1 \\ 0 & 0 & t^2-t+\frac{1}{2} & 2t^2-2t+1 & 1 & 1 & t-1 & 2t-1 & 1 & 2t^2-3t+1 & t-\frac{1}{2} & 1 \\ 0 & 1 & t^2-\frac{1}{2}t & t^2 & 1 & t & 0 & t & 0 & t^2-t & t-\frac{1}{2} & t \end{pmatrix}$$

For $t \notin \mathcal{S} = \{\pm 1, 0, \frac{1}{2}, 2, \frac{1}{2}(1 \pm i)\}$, the image by $\Lambda_{\{2\},\{4\}}$ of the union $\mathcal{C}_0 = \mathcal{C}_0(t)$ of the first eight lines is the configuration $\mathcal{C}_1 = \{\ell_5, \dots, \ell_{12}\}$, moreover, $\Lambda_{\{2\},\{4\}}(\mathcal{C}_1) = \mathcal{C}_0$. The line arrangements $\mathcal{C}_0$ and $\mathcal{C}_1$ are projectively equivalent.

The 12 lines and 16 triple points of $\mathcal{A}_1$ form a $(16_3, 12_4)$-configuration. The arrangement $\mathcal{L}_{\{3\}}(\mathcal{D}(\mathcal{A}_1))$ has 16 lines with $t_2 = 48, t_4 = 12$ and form a $(12_4, 16_3)$-configuration.

Remark 40. The Reye configuration is a well-known $(12_4, 16_3)$-configuration (see, e.g., [15]), however, we checked that the matroids associated to the Reye configuration and to our $(12_4, 16_3)$-configuration are not isomorphic.

For $t = \frac{1}{2}(1 \pm i)$, one has that $\mathcal{C}_1 = \Lambda_{\{2\},\{4\}}(\mathcal{C}_0)$ is the 12 line arrangement $\{\ell_1, \dots, \ell_{12}\}$, and $\Lambda_{\{2\},\{4\}}(\mathcal{C}_1) = \emptyset$. Remarkably, for that value of t, the line arrangement $\mathcal{C}_1$ is the Ceva(4) line arrangement.

The dual arrangements $\check{\mathcal{C}}_k = \mathcal{D}_{\{2\}}(\check{\mathcal{C}}_k)$, $k = 1, 2$ have 22 lines; moreover,

$$\Lambda_{\{4\},\{2\}}(\check{\mathcal{C}}_0) = \check{\mathcal{C}}_1, \quad \Lambda_{\{4\},\{2\}}(\check{\mathcal{C}}_1) = \check{\mathcal{C}}_0;$$

their singularities are $t_2 = 99, t_4 = 8, t_7 = 4$. The arrangements $\check{\mathcal{C}}_0, \check{\mathcal{C}}_1$ are not free. The union $\check{\mathcal{C}}_0 \cup \check{\mathcal{C}}_1$ has 28 lines and singularities $t_2 = 180, t_4 = 5, t_7 = 8$.

7. Matroids and line operators

A matroid is a pair $(E, \mathcal{B})$, where E is a finite set and the elements of $\mathcal{B}$ are subsets of E (called *basis*), subject to the following properties:

- $\mathcal{B}$ is non-empty,
- if $A \in \mathcal{B}$ and $B \in \mathcal{B}$ are distinct basis and $a \in A \setminus B$, then there exists $b \in B \setminus A$ such that $(A \setminus \{a\}) \cup \{b\} \in \mathcal{B}$.

The basis have the same order n, called the *rank* of $(E, \mathcal{B})$. Order n subsets of E that are not basis are called *non-basis*. We identify E with $\{1, \dots, m\}$. A realization (over some field) of the matroid $(E, \mathcal{B})$ is the data of a size $n \times m$ matrix M, with columns $C_1, \dots, C_m$, such that an order n subset $\{i_1, \dots, i_n\}$ of E is a non-base if and only if the size n minor $|C_{i_1}, \dots, C_{i_n}|$ is zero.

Since we are mainly working on the projective plane, we are interested in rank 3 matroid. A realization of a matroid is then a labeled line arrangement $\mathcal{C} = \{L_1, \dots, L_m\}$ in $\mathbb{P}^2$ such that any set of three lines L_i, L_j, L_k of $\mathcal{C}$ meet at one point if and only if i, j, k is a non-basis of $(E, \mathcal{B})$.

Conversely, any labeled line arrangement $\mathcal{C} = \{L_1, \dots, L_m\}$ defines a rank 3 matroid $M(\mathcal{C}) = (\{1, \dots, m\}, \mathcal{B})$, where the non-base is any triple $\{i, j, k\}$ such that the intersection of L_i, L_j, L_k is non-empty.

Given a matroid $(E, \mathcal{B})$, one may compute the moduli space $\mathcal{M}_{\mathcal{B}}$ of realizations (see [33] for an example of such computation).

Examples in Section 3.5 show that the moduli space $\mathcal{M}_{\mathcal{B}}$ may have several connected components, and the action of some line operators Λ on some elements of these components may be different: the resulting line arrangements do not define the same matroids. These examples even show that inside the same irreducible components, two distinct realization may have images by Λ defining distinct matroids.

The example of unassuming arrangements shows that even for the trivial matroid (no non-bases) over $E = \{1, \dots, 6\}$, the line operators may have different behaviors.

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