Enseign. Math. (2) 72 (2026), 215–229
DOI 10.4171/LEM/1091

Equivariant discretizations of diffusions, random walks, and harmonic functions: Corrections and additions

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Abstract. The Martin boundary of a diffusion process on a manifold consists of positive harmonic functions. In a previous publication, we erroneously assumed that the same holds for discretizations of diffusions. In this article, we correct this mistake and extend some of our previous results.

Mathematics Subject Classification 2020: 53C99 (primary); 58J65, 60G50 (secondary).

Keywords: diffusion operator, random walk, Martin boundary, harmonic function, discretization.

1. Introduction

In this erratum, we discuss a mistake in the analysis of Martin boundaries, which occurs in [1, 3, 6]. The misunderstanding in these articles is that the Martin boundary of a μ-random walk on a countable set X consists of positive μ-harmonic functions. The fact that this does not hold for general random walks was pointed out by Anna Erschler to the second author of this erratum because it is of relevance in his paper [6]. We thank her for her input since it led the second author to notice the errors in [1, 3] and made us rethink the arguments in these latter articles.

We follow the notation and terminology of [3]. Let M be a connected manifold and L a diffusion operator on M. Recall that the diffusion on M associated to L is transient if and only if the *Green function* $G = G(x, y)$ of L is finite for one pair—or, equivalently, all pairs—of points $x \neq y$ in M. Fixing an origin $x_0 \in M$, we then obtain the *Martin kernel*

$$K = K(x, y) = G(x, y)/G(x_0, y),$$

defined and positive on the set of all triples of points $y \neq x, x_0$. Note that $K(\cdot, y)$ is L-harmonic on its domain for any $y \in M$. Say that a sequence (y_n) in M diverges if $d(x_0, y_n) \to \infty$. For any bounded open subset B of M and any diverging sequence (y_n), $K(x_0, y_n) = 1$ and $K(\cdot, y_n)$ is positive and L-harmonic on B for all n with

$x_0 \neq y_n$ and $d(B, y_n) > 0$. Hence, $K(\cdot, y_n)$ has subsequences such that $K(\cdot, y_n)$ converges pointwise or, equivalently, locally uniformly on M, by the ellipticity of L. The set of limit functions of diverging sequences so obtained make up the *Martin boundary* $\partial_L M$. Since L-harmonicity is a local property, $\partial_L M$ consists of positive L-harmonic functions, normalized to be equal to one at x_0. Convergence of diverging sequences induces topologies on $M \cup \partial_L M$ and $\partial_L M$, turning them into compact topological spaces.

Let X be a countable set and $\mu = (\mu_x)$ be a family of probability measures on X such that the associated random walk, also denoted by μ, is irreducible. Recall that μ is transient if and only if its Green's function $g = g(x, y)$ is finite. Fixing an origin $x_0 \in X$, we then obtain the *Martin kernel*

$$k = k(x, y) = g(x, y)/g(x_0, y).$$

Note that $g(\cdot, y)$ is positive, μ-superharmonic on X, and μ-harmonic on $X \setminus \{y\}$ so that $k(\cdot, y)$ is positive, μ-superharmonic on X, and μ-harmonic on $X \setminus \{y\}$. Say that a sequence (y_n) is divergent if it leaves any finite subset of X eventually. In complete analogy with the case of diffusions above, using divergent sequences (y_n) and the associated $k(\cdot, y_n)$, we obtain the *Martin boundary* $\partial_\mu X$. Again, convergence of sequences induces topologies on $X \cup \partial_\mu X$ and $\partial_\mu X$, turning them into compact topological spaces. The difference to the diffusion case is that being μ-harmonic is not a local property: for a function f on X to be μ-harmonic at $x \in X$ means that

$$f(x) = \sum \mu_x(y) f(y),$$

and therefore, it depends on the behavior of f on supp μ_x. Hence, it may happen that limits of sequences of Martin functions $k(\cdot, y_n)$ associated to diverging sequences (y_n) are μ-superharmonic but not μ-harmonic. However, if μ_x has finite support, then such limits are μ-harmonic at x [7, Lemma 7.18]. In fact, being not μ-harmonic is a phenomenon very much like the one encountered in Fatou's lemma in integration theory.

Remark 1.1. Our definition of the Martin boundary and compactification of X coincides with the one used in [7, Chapter 7]. There is another, smaller natural compactification so that one has to be careful when reading the literature; see the discussion in [7, page 188f].

Let now X be a discrete subset of M, where M is endowed with a diffusion operator L. Given M and X, we denote by $(F, V) \sim (F_x, V_x)_{x \in X}$ (regular Lyons–Sullivan) *LS-data* satisfying requirements (D1)–(D4) as in Section 2. Recall that LS-data induce a family $(\mu_y)_{y \in M}$ of probability measures on X, and the associated random walk μ

on X is the one where the error mentioned further up occurs. In fact, in [2], we obtain examples where not all functions in the Martin boundary $\partial_\mu X$ are μ-harmonic.

In the introduction of [3], we discuss coverings $\pi\colon M \to N$ of connected manifolds, where L is the pullback of a diffusion operator L_0 on N and where the LS-data (F, V) are chosen compatible with π. The result in the introduction, whose formulation needs a correction, is the second part of [3, Theorem D]. The corrected version involves the set of accumulation points $\partial_L X$ of X in $\partial_L M$ and the spaces $\mathcal{H}_F^+(M, L)$ of positive L-harmonic functions on M swept by F and $H^+(X, \mu)$ of positive μ-harmonic functions on X.

Theorem D (2). *Suppose that N is L_0-recurrent, that M is L-transient, and that X is endowed with the family μ of LS-measures associated to balanced LS-data. Then, restriction $R\colon \partial_L X \to \partial_\mu X$ of Martin kernels from M to X defines a Γ-equivariant, continuous, closed, and surjective map. Furthermore,*

$$R\colon \partial_L X \cap \mathcal{H}_F^+(M, L) \to \partial_\mu X \cap \mathcal{H}^+(X, \mu)$$

is a Γ-equivariant homeomorphism.

In [3, Theorem D (2)], it is asserted that $R\colon \partial_L X \to \partial_\mu X$ is a homeomorphism. However, the proof there rests on the aforementioned misunderstanding.

In the body of the text of [3], we discuss different cases, which extend or vary the setup in the introduction. This makes it necessary to correct the corresponding errors in each of these cases. In Section 2, we focus on the corrected version of [3, Theorem 3.30], a more general version of Theorem D (2). In Section 3, we discuss the changes required in the last part of [3, Subsection 3.4], which is concerned with what we call *uniform*. The definition of the latter needs a slight change, which also corrects the meaning of the term *appropriate* in [3, Theorem B]. Furthermore, we discuss the corresponding modifications required in [3, Sections 4 and 5]. Finally, in Section 4, we discuss random walks on countable sets and the changes required in [3, Section 6].

In the article [1] preceding our [3], the proofs of items (b) and (c) of [1, main theorem] and of [1, Theorem 2.8] also rest on the same misunderstanding regarding Martin boundaries. The above Theorem D (2) leads immediately to correct formulations of these. Since we need more geometric background, the part of [1] on negatively curved manifolds is discussed in the companion article [2].

2. The general case

This section includes the changes required in [3, Subsection 3.4], after the proof of [3, Theorem 3.29], in order to establish the corrected version of [3, Theorem 3.30].

We assume that $(F, V) = (F_x, V_x)_{x\in X}$ are (regular) LS-data for X. This means that the F_x are compact subsets of the relatively compact open subsets V_x of M and that there is a constant $C > 1$ such that

(D1) $x \in \mathring{F}_x$ and $F_x \subseteq V_x$ for all $x \in X$;

(D2) $F_x \cap V_y = \emptyset$ for all $x \neq y$ in X;

(D3) $F = \bigcup_{x\in X} F_x$ is closed and L-recurrent;

(D4) for all $x \in X$ and $y \in F_x$,

$$\frac{1}{C} < \frac{d\varepsilon(y, V_x)}{d\varepsilon(x, V_x)} < C,$$

where ε indicates exit measures of the L-diffusion. Our requirements (D1) and (D2) are more restrictive than the corresponding ones in [5] and conform to the ones in [1]. Given LS-data (F, V), we denote by $\mu = (\mu_y)_{y\in M}$ the corresponding family of LS-probability measures on X. We say that X is $*$-*recurrent* if it admits LS-data.

Throughout this section, we assume that M is L-transient and that the LS-data (F, V) are *balanced*; that is, there is a constant B such that the Green functions satisfy

(D5) $G_{V_x}(z, x) = B$ for all $x \in X$ and $z \in \partial F_x$.

This implies that we have $G(y, x) = BCg(y, x)$ for all $x \in X$ and $y \in M \setminus V_x$; see [1, Theorem 2.7] and [3, Theorem 3.29].

Lemma 2.1. (1) *For any $h \in \mathcal{H}^+(X, \mu)$, $Eh = Eh(y) = \mu_y(h)$ is the unique positive L-harmonic function on M extending h.*

(2) *The extension map $E\colon \mathcal{H}^+(X, \mu) \to \mathcal{H}^+_F(M, L)$ is continuous with respect to the compact-open topology.*

(3) *For any $\xi \in \partial_\mu X \cap \mathcal{H}^+(X, \mu)$, there is a unique $E\xi \in \partial_L M$ such that $Ek(\cdot, \xi) = K(\cdot, E\xi)$. Moreover, $E\xi \in \partial_L X \cap \mathcal{H}^+_F(M, L)$.*

Proof. (1) If $f \in \mathcal{H}^+(M, L)$ satisfies $Rf = h$, then $R(Eh - f) = 0$. Furthermore, by Theorem [3, Theorem 3.13], we have

$$Eh = ERf \leq f,$$

and hence, $f - Eh$ is a non-negative L-harmonic function vanishing on X. Therefore, $f - Eh = 0$ by the mean value property of L-harmonic functions.

(2) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}^+(X, \mu)$ converging to $f \in \mathcal{H}^+(X, \mu)$ pointwise. Consider the sequence of extensions $(Ef_n)_{n\in\mathbb{N}}$ in $\mathcal{H}^+_F(M, L)$ and choose $x_0 \in X$. Given any subsequence $(Ef_{n_k})_{k\in\mathbb{N}}$, since it is bounded at x_0, it follows from the gradient estimate that $(Ef_{n_k})_{k\in\mathbb{N}}$ is uniformly bounded and equicontinuous on any compact

domain of M containing x_0. We derive from the Arzela–Ascoli theorem that, after passing to a subsequence if necessary, we have that $Ef_{n_k} \to g \in C(M, \mathbb{R})$ locally uniformly. Using again that Ef_{n_k} is L-harmonic, we deduce that $g \in \mathcal{H}^+(M, L)$. Moreover, since $Ef_{n_k} \to g$ locally uniformly, we readily see that $f_{n_k} = REf_{n_k} \to Rg$ pointwise. This, together with the fact that $f_n \to f$ pointwise, gives that $f = Rg$, and hence, $g = Ef$. This means that any subsequence of $(Ef_n)_{n\in\mathbb{N}}$ has a subsequence converging locally uniformly to Ef, which yields that $Ef_n \to Ef$ in $H^+_F(M, L)$. Since the space $H^+(X, \mu)$ is metrizable, this establishes the continuity of the extension map.

(3) Let (x_n) be a sequence in X that converges to ξ in $X \cup \partial_\mu X$. Since $M \cup \partial_L M$ is compact, $K(\cdot, x_n)$ subconverges in M to some $K(\cdot, \eta)$. Since $k(\cdot, x_n) = RK(\cdot, x_n)$ on $X \setminus \{x_n\}$ for all n, we conclude that $k(\cdot, \xi) = RK(\cdot, \eta)$. Since $k(\cdot, \xi)$ is μ-harmonic and $K(\cdot, \eta)$ is L-harmonic, the first part of the proof implies the assertions. ∎

The following theorem is the corrected version of [1, Theorem 2.8] and [3, Theorem 3.30].

Theorem 2.2. *Restriction $R\colon \partial_L X \to \partial_\mu X$ of Martin kernels from M to X is continuous, closed, and surjective. Furthermore,*

$$R\colon \partial_L X \cap \mathcal{H}^+_F(M, L) \to \partial_\mu X \cap \mathcal{H}^+(X, \mu)$$

is a homeomorphism with inverse E.

In [1, Theorem 2.8] (in the case of Brownian motion) and in [3, Theorem 3.30], it is asserted, in somewhat different wording, that $R\colon \partial_L X \to \partial_\mu X$ is a homeomorphism. The proofs there rest on the erroneous assumption that points in $\partial_\mu X$ correspond to positive μ-harmonic functions on X.

Proof of Theorem 2.2. Let $\xi \in \partial_L X$. Then, there is a sequence (x_n) in X converging to ξ in $M \cup \partial_L M$. By definition, for any such sequence, the Martin kernels $K(\cdot, x_n)$ converge pointwise to the Martin kernel $K(\cdot, \xi)$ corresponding to ξ. Hence, their restrictions $k(\cdot, x_n)$ to X converge pointwise as functions on X. By definition, this means that the sequence of x_n converges to a point η in the Martin boundary $\partial_\mu X$ of X, where the associated Martin kernel $k(\cdot, \eta)$ is the restriction of $K(\cdot, \xi)$ to X. This means that $R\colon \partial_L X \to \partial_\mu X$ is well defined. Continuity of R is obvious, since limits correspond to pointwise convergence of Martin kernels. This yields that R is closed, $\partial_L X$ being compact.

Let (x_n) be a sequence in X which converges to a point ξ in $\partial_\mu X$. Now, $M \cup \partial_L M$ is compact. Hence, the sequence of x_n subconverges to a point $\eta \in \partial_L M$. It follows that the restriction of the associated Martin kernel $K(., \eta)$ to X equals $k(., \xi)$. Thus, R defines a surjection from $\partial_L X$ onto $\partial_\mu X$.

By Lemma 2.1 (3), $R\colon \partial_L X \cap \mathcal{H}_F^+(M,L) \to \partial_\mu X \cap \mathcal{H}^+(X,\mu)$ is bijective. Since R is continuous and closed, it is a homeomorphism. ∎

The spaces $\partial_L^{\min} M$ of minimal positive L-harmonic functions on M and $\partial_{L,F}^{\min} M$ of them swept by F are Borel subsets of $\partial_L M$. Recall the representation formula [3, (2.6)]; that is, for any $h \in \mathcal{H}^+(M,L)$, there exists a unique probability measure ν_h on $\partial_L^{\min} M$ such that

$$h(x) = h(x_0) \int_{\partial_L^{\min} M} K(x,\eta)\nu_h(d\eta), \tag{2.3}$$

where x_0 is the chosen origin for Martin kernels. Similarly, the space $\partial_\mu^{\min} X$ of minimal positive μ-harmonic functions on X is a Borel subset of $\partial_\mu X$, and there is a representation of positive μ-harmonic functions on X analogous to (2.3) (cf., for instance, [7, Chapter 7]).

Corollary 2.4. *For any $h \in \mathcal{H}^+(M,L)$ and ν_h according to (2.3),*

$$h \in \mathcal{H}_F^+(M,L) \text{ if and only if } \nu_h(\partial_{L,F}^{\min} M) = \nu_h(\partial_L^{\min} M).$$

Proof. We have $ERh \le h$ with equality if and only of $h \in \mathcal{H}_F^+(M,L)$. Hence, $ERh = h$ if and only if $ERK(\cdot,\xi) = K(\cdot,\xi)$ for ν_h-almost all ξ. ∎

Theorem 2.5. *We have $\partial_{L,F}^{\min} M \subseteq \partial_L X \cap \mathcal{H}_F^+(M,L)$, and restriction*

$$R\colon \partial_{L,F}^{\min} M \to \partial_\mu^{\min} X$$

is a homeomorphism with inverse E.

Proof. We start with the proof of the second assertion. To that end, let $\xi \in \partial_{L,F}^{\min} M$ and $h = K(\cdot,\xi)$. Suppose that $Rh \ge f$ for some $f \in \mathcal{H}^+(X,\mu)$. Then, $h = ERh \ge Ef$, and hence, Ef is a multiple of h. But then also $f = REf$ of Rh. This means that $R\xi \in \partial_\mu^{\min} X$. Injectivity of R follows readily from Lemma 2.1 (1).

To establish the surjectivity of R, let now $\xi \in \partial_\mu^{\min} X$ and $h = k(\cdot,\xi)$. We know from Theorem 2.2 that $E\xi \in \partial_L M \cap \mathcal{H}_F^+(M,L)$, and it suffices to show that Eh is minimal. Assume to the contrary that this is not the case. Then, there exist distinct $h_1, h_2 \in \mathcal{H}^+(M,L)$ and $0 < c < 1$ such that $Eh = ch_1 + (1-c)h_2$. Using that Eh is swept by F, it follows from [3, Theorem 3.13] that h_1 and h_2 are also swept by F. In particular, this yields that $Rh_1, Rh_2 \in \mathcal{H}^+(X,\mu)$ and $Rh_1 \ne Rh_2$, in view of Lemma 2.1 (1). Since $h = cRh_1 + (1-c)Rh_2$, we conclude that h is not minimal, which is a contradiction.

It remains to prove the first assertion. Again, let $\xi \in \partial_{L,F}^{\min} M$ and $h = K(\cdot,\xi)$. Then, $R\xi \in \partial_\mu^{\min} X$ by the first part of the proof. Then, $ER\xi \in \partial_L X \cap \mathcal{H}_F^+(M,L)$ by

Theorem 2.2. Since $h = ERh$ on X and h is swept by F, we have $\xi = ER\xi$, and the assertion follows. ∎

Corollary 2.6. *We have*

$$\partial^{\min}_{L,F} M = \partial_L X \cap \mathcal{H}^+_F(M, L) \ \textit{if and only if} \ \partial^{\min}_{\mu} X = \partial_\mu X \cap \mathcal{H}^+(X, \mu).$$

Proof. The left-hand sides of the equations are contained in the right-hand sides, in the first case by Theorem 2.5, in the second by [7, Theorem 7.53]. Now, the assertion follows from Theorems 2.2 and 2.5 since the homeomorphisms there are both given by R. ∎

Theorem 2.5 and Corollary 2.6 establish an analog of [3, Theorem 3.31] in the setting of this section.

3. The uniform case

This section addresses the changes required in [3, Subsection 3.4] after [3, Theorem 3.30], and in its applications in [3, Sections 4 and 5]. We redefine the notion of $*$-uniform so that the results of [3] for that case remain correct.

Let X be a $*$-recurrent subset of M and (F, V) balanced LS-data for X. Modifying the corresponding requirements in [3, Section 3.4], we say that (F, V) are *uniform* LS-data if there exist families $(U_x)_{x\in X}$ of Borel subsets and $(W_x)_{x\in X}$ of relatively compact open subsets of M satisfying the following:

(U1) $F_x \subseteq U_x \subseteq W_x$ and $F_x \cap W_y = \emptyset$ for any $x, y \in X$ with $x \neq y$,

(U2) $(U_x)_{x\in X}$ is a partition of M and $(W_x)_{x\in X}$ is locally finite,

(U3) for all $x \in X$, $y, z \in U_x$, and positive harmonic function h on W_x,

$$h(y) \geq ch(z),$$

(U4) for any $x \in X$ and $y \in U_x$, the balayage β_y of the Dirac measure δ_y onto the closed set $F_x \cup (M \setminus W_x)$, satisfies

$$\beta_y(F_x) \geq c.$$

Here, the constants $0 < c < 1$ in (U3) and (U4), if they exist, are chosen to coincide. If X admits uniform LS-data, then X is called $*$-*uniform*.

Theorem 3.1. *If M is L-transient and X is $*$-uniform (in the above modified sense), then*

$$\mathcal{H}^+(M, L) = \mathcal{H}^+_F(M, L).$$

Proof. Choose uniform LS-data (F_x, V_x) and families (U_x, W_x) as above, and let $h \in \mathcal{H}^+(M, L)$. Because h is bounded harmonic on the relatively compact sets W_x, we have

$$h(y) = \beta_y(h) = \beta_y(h|_{F_x}) + \beta_y(h|_{\partial W_x})$$

for all $x \in X$ and $y \in U_x$. By the above assumptions, we conclude that

$$\beta_y(h|_{F_x}) \geq c^2 h(y) \tag{3.2}$$

for any $y \in U_x$.

For any measure μ on U_x, let β_μ be the balayage of μ onto the closed set $F_x \cup (M \setminus W_x)$,

$$\beta_\mu = \int \beta_y \mu(dy).$$

More generally, if the support of μ is not contained in U_x, set

$$\beta_\mu = \sum_{x \in X} \beta_{\mu_x} = \sum_{x \in X} \int_{U_x} \beta_y \mu(dy),$$

where μ_x denotes the restriction of μ to U_x. Now, given any finite measure μ_n on M, set

$$\mu_{n+1} = \sum_x \beta_{\mu_{nx}}|_{\partial W_x} \quad \text{and} \quad \tau_{n+1} = \sum_x \beta_{\mu_{nx}}|_{F_x} = \beta_{\mu_n}|_F.$$

Recall here that the F_x and ∂W_y do not intersect, and note that the ∂W_x are not all disjoint since the U_x form a partition of M. However, the ∂W_x are a locally finite family of subsets of M.

For any finite Borel measure μ_n, we have

$$\mu_{n+1}(h) + \tau_{n+1}(h) = \mu_n(h).$$

Moreover, by (3.2),

$$\tau_{n+1}(h) \geq c^2 \mu_n(h).$$

Therefore, if $\mu_n(h) < \infty$, then

$$\mu_{n+1}(h) \leq (1 - c^2)\mu_n(h).$$

Now, to show the assertion, let $y \in M$. If $y \in F$, we are done. If $y \in M \setminus F$, let $\mu_0 = \delta_y$ and $\tau_0 = 0$. Then, $\mu_0(h) + \tau_0(h) = h(y) < \infty$. Recursively, by the above procedure, we obtain measures μ_n and τ_n such that

$$\mu_n(h) + \sum_{k \leq n} \tau_k(h) = h(y) \quad \text{and} \quad \mu_n(h) \leq (1 - c^2)^n h(y).$$

Now, $\sum_n \tau_n = \beta_F$, the balayage of δ_y onto F, and therefore, $\beta_F(h) = h(y)$. This is the assertion. ∎

Note that the main errors in [1, 3] rest on the equality of the spaces $\partial_\mu X$ and $\partial_\mu X \cap H^+(X,\mu)$. With the above modified definition, the equality holds at least in the $*$-uniform case.

Corollary 3.3. *If M is L-transient, X is $*$-uniform (in the above modified sense), and X is endowed with the family μ of LS-measures associated to uniform LS-data (F_x, V_x), then*

$$\partial_\mu X = \partial_\mu X \cap H^+(X,\mu).$$

Proof. Let $\xi \in \partial_\mu X$. By Theorem 2.2, there is an $\eta \in \partial_L X$ such that $k(\cdot,\xi) = RK(\cdot,\eta)$. Now, $K(\cdot,\eta) \in \mathcal{H}^+(M,L)$ is swept by F, by Theorem 3.1; hence, $k(\cdot,\xi) = RK(\cdot,\eta)$ is μ-harmonic. ∎

Remark 3.4. With the above change of the notions *uniform* and $*$-*uniform*, the corresponding results (and proofs) in [3] remain correct, in view of Corollary 3.3.

We end this section by showing that the applications in the subsequent sections of [3] remain virtually the same with the above modified definition of $*$-uniform. The case considered first in [3] is that of a covering $\pi\colon M \to N$ such that L is the lift of an operator L_0 on N.

(C) Let $z_0 \in N$, $F_0 \subseteq V_0$ be connected neighborhoods of z_0 with F_0 compact and V_0 open and relatively compact such that V_0 is evenly covered by π. For $x \in X = \pi^{-1}(z_0)$, let $F_x \subseteq V_x$ be the connected components of $F = \pi^{-1}(F_0)$ and $V = \pi^{-1}(V_0)$ containing x, respectively.

The first and second assertion of [3, Lemma 4.4] remain true, where the first asserts that all such tuples (F, V) constitute LS-data if the L_0-diffusion on N is recurrent. The final assertion of [3, Lemma 4.4] also remains true.

Lemma 3.5. *If N is compact, then X is $*$-uniform (in the above modified sense) with respect to some LS-data satisfying (C).*

Proof. The Riemannian metric on N associated to L_0 is complete since N is compact. Its pullback to M is complete and equals the Riemannian metric associated to L. Choose $R > 0$ less than half the injectivity radius of N at z_0. Then, the open ball $B(z_0, 2R)$ is evenly covered. Consider $F_0 \subseteq V_0 \subseteq B(z_0, R)$ such that the associated LS-data (F_x, V_x) for X as in [3, Lemma 4.4(2)] are balanced. For $x \in X$, consider the Dirichlet domain

$$D_x = \{z \in M : d(z,x) \le d(z,y) \text{ for all } y \in X\}$$

centered at x. Since $(D_x)_{x\in X}$ is a closed cover of M, by removing closed subsets from them, we obtain a partition $(U_x)_{x\in X}$ of M (not necessarily Γ-equivariant), consisting of Borel subsets. Finally, let $W_x = B(D_x, R)$ be the open R-tubular neighborhood of D_x with $x \in X$.

It is easily checked that $B(x, R) \cap W_y = \emptyset$ for all $x, y \in X$ with $x \neq y$, which implies that $F_x \subseteq B(x, R) \subseteq U_x$ and $F_x \cap W_y = \emptyset$ for all $x, y \in X$ with $x \neq y$. Keeping in mind that $\operatorname{diam} D_x \leq \operatorname{diam} M$ for any $x \in X$, it is easily checked that if $z \in M$ belongs to the intersection of n different W_x's, then there exist n pairwise non-homotopic loops based at z_0 of length at most $2 \operatorname{diam} M + 2R$. This yields that $(W_x)_{x \in X}$ is locally finite. Property (U3) follows from [4, Theorem 6], keeping in mind that the curvature of M is bounded and that D_x is star-shaped with respect to x and is contained in the closed ball $C(x, \operatorname{diam} M)$ for any $x \in X$.

It remains to show that (U4) is satisfied. To this end, let $r > 0$ such that $B(z_0, 2r) \subseteq F_0$. It is clear that there exists $c > 0$ such that

$$\varepsilon_y^{V_0 \setminus C(z_0, r)}(C(z_0, r)) \geq c$$

for any $y \in F_0 \setminus C(z_0, r)$. For $x \in X$, consider the positive harmonic function h_x on $W_x \setminus B(x, r)$ defined by

$$h_x(y) = \varepsilon_y^{W_x \setminus C(x, r)}(C(x, r)).$$

Then, $h_x(y) \geq \varepsilon_y^{V_x \setminus C(x,r)}(C(x, r)) \geq c$ for any $x \in X$ and $y \in F_x \setminus C(x, r)$. It is evident that for any point in $D_x \setminus F_x$ there exists a point on ∂F_x and a geodesic joining them, of length at most $\operatorname{diam} M$ and image contained in $D_x \setminus F_x^\circ$ (for instance, a segment of a geodesic emanating from x). Since h_x is harmonic in the r-tubular neighborhood of $D_x \setminus F_x^\circ$ for any $x \in X$ and the curvature of M is bounded, it follows from [4, Theorem 6] that there exists $C > 0$ such that

$$h_x(y) \geq C \min_{z \in \partial F_x} h_x(z) \geq Cc$$

for any $x \in X$ and $y \in D_x \setminus F_x$. From the definition of h_x, it is evident that for any $x \in X$ and $y \in U_x \setminus F_x \subseteq D_x \setminus F_x$, the balayage β_y of δ_y onto $F_x \cup (M \setminus W_x)$ satisfies $\beta_y(F_x) \geq h_x(y) \geq Cc$. ∎

Section 5 of [3] concerns properly discontinuous actions of groups Γ of automorphisms of (M, L). Let $x_0 \in M$, and let $\Gamma_0 \subseteq \Gamma$ be its stabilizer and $X = \Gamma x_0$ its Γ-orbit.

(A) Let $F_0 \subseteq V_0$ be Γ_0-invariant connected neighborhoods of x_0 such that F_0 is compact, V_0 is open and relatively compact, and $\gamma F_0 \cap V_0 = \emptyset$ for all $\gamma \in \Gamma$ not equal to e. For $x = \gamma x_0$, let $F_x = \gamma F_0$ and $V_x = \gamma V_0$.

All assertions of [3, Lemma 5.1] remain literally true, where the final one reads as follows.

Lemma 3.6. *If the action of* Γ *on* M *is uniform, then* X *is* $*$*-uniform (in the above modified sense) with respect to some LS-data satisfying* (A).

4. The case of random walks

This section addresses the changes required in [3, Section 6]. Since the discussion is concerned completely with random walks, with no diffusion involved, the setup is different and we need to address some other—albeit related—issues. Since we did not spell out the results there, we will spell them out here. As there, we decorate them with a double prime, since they are companions of the corresponding unprimed results in the diffusion case. For convenience, their order here differs from the original one.

Throughout this section, we work in the setting of [3, Section 6]. Let Y be a countable set, Γ a countable group acting on Y, and $(\nu_y)_{y\in Y}$ a Γ-invariant family of probability measures on Y such that the associated random walk ν on Y is irreducible.

Let $X \subseteq Y$ be a Γ-invariant subset. For a path $\omega \in Y^{\mathbb{N}_0}$, define the hitting time $R_X(\omega) = \inf\{k \geq 1 \mid \omega_k \in X\}$. We say that X is ν-recurrent if $P_y\{R_X(\omega) < \infty\} = 1$ for all $y \in Y$. Then, the family $\mu = (\mu_y)_{y\in Y}$ of hitting probabilities,

$$\mu_y(x) = P_y[\omega(R^X(\omega)) = x] = \sum_{i\geq 0} \sum_{y_1,\dots,y_i\in Y\setminus X} \nu_y(y_1)\nu_{y_1}(y_2)\cdots\nu_{y_i}(x), \tag{4.1}$$

where $y \in Y$ and $x \in X$, are probability measures on X and define a random walk with sample space $\Omega = Y \times X^{\mathbb{N}}$. By the definition of μ, we have

$$\mu_y(x) = \nu_y(x) + \sum_{z\in Y\setminus X} \nu_y(z)\mu_z(x) \tag{4.2}$$

for all $y \in Y$ and $x \in X$.

We denote by $\mathcal{H}^\infty(Y,\nu)$ and $\mathcal{H}^+(Y,\nu)$ the spaces of bounded, respectively, positive ν-harmonic functions on Y, and similarly for X and μ. We let $\mathcal{H}^+_X(Y,\nu)$ consist of functions $h \in \mathcal{H}^+(Y,\nu)$ swept by X, that is,

$$h(y) = \sum_{x\in X} \mu_y(x)h(x) \tag{4.3}$$

for any $y \in Y$. The extension map $E\colon \mathcal{H}^+(X,\mu) \to \mathcal{H}^+(Y,\nu)$ is defined by

$$(Eh)(y) = \sum_{x\in X} \mu_y(x)h(x) \tag{4.4}$$

for all $y \in Y$. It is checked as in Lemma 2.1 that Eh is the unique positive ν-harmonic extension of $h \in \mathcal{H}^+(X,\mu)$.

We denote by $\partial_\nu X$ the set of accumulation points of X in the Martin boundary $\partial_\nu Y$. Now, [3, Theorem C”] and its proof remain the same.

Theorem C". *In the above situation, we have the following.*

(1) *For any $h \in \mathcal{H}^+(Y, \nu)$, either h is μ-harmonic or strictly μ-superharmonic on X. More precisely, either $h(y) = \mu_y(h)$ for all $y \in Y$ or $h(y) > \mu_y(h)$ for all $y \in Y$.*

(2) *For any $h \in \mathcal{H}^\infty(Y, \nu)$, $Rh = h|_X$ belongs to $\mathcal{H}^\infty(X, \mu)$.*

(3) *$R\colon \mathcal{H}^\infty(Y, \nu) \to \mathcal{H}^\infty(X, \mu)$ is a Γ-equivariant isomorphism.*

(4) *Y is ν-transient if and only if X is μ-transient.*

By the ν-recurrence of X, the space of sequences in $Y^{\mathbb{N}_0}$ which have infinitely many of its members in X has full P_y-measure for all $y \in Y$, and the subsequences consisting of the corresponding starting members and members belonging to X will be called *X-subsequences*. Theorem D" in [3] takes now the following form.

Theorem D". *If Y is ν-transient, we have the following.*

(1) *Passage to X-subsequences induces a Γ-equivariant isomorphism between the Poisson boundaries of (Y, ν) and (X, μ).*

(2) *Restriction of Martin kernels to X yields a Γ-equivariant, continuous, closed, and surjective map $R\colon \partial_\nu X \to \partial_\mu X$. Furthermore,*

$$R\colon \partial_\nu X \cap \mathcal{H}^+_X(Y, \nu) \to \partial_\mu X \cap \mathcal{H}^+(X, \mu)$$

is a Γ-equivariant homeomorphism.

Here, the proof of the first assertion of Theorem D" remains the same as in [3]. The proof of the second assertion is analogous to the proof of Theorem 2.2 above.

Theorem A". *If $\Gamma\backslash Y$ is finite, then we have the following.*

(1) *For any $h \in \mathcal{H}^+(Y, \nu)$, the restriction $h|_X$ of h to X belongs to $\mathcal{H}^+(X, \mu)$. More precisely, $h(y) = \mu_y(h)$ for all $y \in Y$.*

(2) *The restriction map $\mathcal{H}^+(Y, \nu) \to \mathcal{H}^+(X, \mu)$ is a Γ-equivariant isomorphism of cones.*

Proof. It suffices to prove that $h(y) = \mu_y(h)$ for any $h \in \mathcal{H}^+(Y, \nu)$ and $y \in Y$. Let $h \in \mathcal{H}^+(Y, \nu)$. In view of (4.2), it is sufficient to show that $h(y) = \mu_y(h)$ for any $y \in Y \setminus X$. Choose a finite set A of representatives of the Γ-orbits in Y that do not intersect X. Since X is ν-recurrent and A is finite, there exist $c > 0$ and $N \in \mathbb{N}$ such that for any $y \in A$ there exists $n(y) \le N$, $y =: y_0, \dots, y_{n(y)-1} \in Y \setminus X$, and $\bar{y} := y_{n(y)} \in X$ such that

$$\nu_y(y_1)\nu_{y_1}(y_2)\cdots\nu_{y_{n(y)-1}}(\bar{y}) \ge c.$$

Since ν is irreducible and A is finite, there exists $c' > 0$ such that for any $y \in A$ there exists $k \in \mathbb{N}$ satisfying $\nu^k_{\bar{y}}(y) \geq c'$. Keeping in mind that ν-harmonic functions are also ν^k-harmonic, we derive that $\varphi(\bar{y}) \geq c'\varphi(y)$ for any $y \in A$ and any $\varphi \in \mathcal{H}^+(Y, \nu)$. Since ν is Γ-invariant, we readily see that $h_g = h \circ g$ is ν-harmonic for any $g \in \Gamma$, and therefore, $h(g\bar{y}) \geq c'h(gy)$ for any $y \in A$ and $g \in \Gamma$.

For any $y \in Y \setminus X$, using that h is ν-harmonic recursively, we obtain that

$$\begin{aligned} h(y) &= \sum_{y_1 \in Y} \nu_y(y_1)h(y_1) = \sum_{y_1 \in X} \nu_y(y_1)h(y_1) + \sum_{y_1 \in Y \setminus X} \nu_y(y_1)h(y_1) \\ &= \sum_{y_1 \in X} \nu_y(y_1)h(y_1) + \sum_{y_2 \in Y, y_1 \in Y \setminus X} \nu_y(y_1)\nu_{y_1}(y_2)h(y_2) = Q_n(y) + E_n(y) \end{aligned}$$

for any $n \in \mathbb{N}$, where

$$\begin{aligned} Q_n(y) &= \sum_{i=1}^{n} \sum_{y_i \in X, y_{i-1}, \ldots, y_1 \in Y \setminus X} \nu_y(y_1)\nu_{y_1}(y_2) \cdots \nu_{y_{i-1}}(y_i)h(y_i), \\ E_n(y) &= \sum_{y_n, \ldots, y_1 \in Y \setminus X} \nu_y(y_1)\nu_{y_1}(y_2) \cdots \nu_{y_{n-1}}(y_n)h(y_n). \end{aligned}$$

It is evident that $Q_n(y)$ is non-decreasing with respect to n, and thus, $E_n(y)$ is non-increasing with respect to n. From the definition of μ_y, it suffices to prove that $E_n(y) \to 0$ for any $y \in Y \setminus X$.

Bearing in mind that $y = gz$ for some $g \in \Gamma$ and $z \in A$, and that $N \geq n(z)$, we deduce that

$$Q_N(y) \geq \nu_{gz}(gz_1) \cdots \nu_{gz_{n(z)-1}}(g\bar{z})h(g\bar{z}) \geq ch(g\bar{z}) \geq cc'h(gz) = cc'h(y).$$

This, together with $h(y) = Q_N(y) + E_N(y)$, gives the estimate

$$E_N(y) \leq (1 - cc')h(y)$$

for any $y \in Y \setminus X$. For any $y \in Y \setminus X$ and $k \in \mathbb{N}$, we compute

$$\begin{aligned} E_{(k+1)N}(y) &= \sum_{y_{(k+1)N}, \ldots, y_1 \in Y \setminus X} \nu_y(y_1) \cdots \nu_{y_{(k+1)N-1}}(y_{(k+1)N})h(y_{(k+1)N}) \\ &= \sum_{y_{kN}, \ldots, y_1 \in Y \setminus X} \nu_y(y_1) \cdots \nu_{y_{kN-1}}(y_{kN})E_N(y_{kN}) \\ &\leq (1 - cc')E_{kN}(y), \end{aligned}$$

which yields that $E_{kN}(y) \leq (1 - cc')^k h(y)$ for any $k \in \mathbb{N}$. Since $(E_n(y))_{n \in \mathbb{N}}$ is non-increasing, we conclude that $E_n(y) \to 0$ for any $y \in Y \setminus X$, as we wished. ∎

Lemma 4.5. *If $\Gamma\backslash Y$ is finite and Y is ν-transient, then $\partial_\nu^{\min} Y \subseteq \partial_\nu X$.*

Proof. Let $\xi \in \partial_\nu^{\min} Y$. Then, $K(\cdot,\xi) \in H^+(Y,\nu)$ and Theorem A" yields that $RK(\cdot,\xi) \in \mathcal{H}^+(X,\mu)$. As in the proof of Theorem 2.5, minimality of $K(\cdot,\xi)$ implies the minimality of $RK(\cdot,\xi)$. This yields that $RK(\cdot,\xi) = k(\cdot,\eta)$ for some $\eta \in \partial_\mu^{\min} X$. We derive from Theorem D" that there exists $\xi' \in \partial_\nu X \cap H^+(Y,\nu)$ such that $RK(\cdot,\xi') = k(\cdot,\eta)$. Combining these, we deduce that $K(\cdot,\xi') = Ek(\cdot,\eta) = K(\cdot,\xi)$, which means that $\xi = \xi' \in \partial_\nu X$. ∎

The analog of [3, Theorem B] in the setting of random walks is the following.

Theorem B". *If $\Gamma\backslash Y$ is finite and Y is ν-transient, then the following hold.*

(1) *X is μ-transient and restriction of Martin kernels*

$$R\colon \partial_\nu^{\min} Y \to \partial_\mu^{\min} X$$

is a Γ-equivariant homeomorphism with inverse E.

(2) *$\partial_\nu X = \partial_\nu^{\min} Y$ if and only if $\partial_\mu X = \partial_\mu^{\min} X$, and then $\partial_\nu Y \cap H^+(Y,\nu) = \partial_\nu^{\min} Y$.*

Proof. Choose origin $x_0 \in X$ for Martin kernels associated to the ν-random walk on Y and the μ-random walk on X. We know from Theorems A" and D" that

$$R\colon \partial_\nu X \cap \mathcal{H}^+(Y,\nu) \to \partial_\mu X \cap \mathcal{H}^+(X,\mu)$$

is a Γ-equivariant homeomorphism with inverse E. As in the proof of Theorem 2.5, R maps minimal positive ν-harmonic functions of Y surjectively to minimal positive μ-harmonic functions on X. This means that R restricts to a homeomorphism

$$R\colon \partial_\nu^{\min} Y \cap \partial_\nu X \to \partial_\mu^{\min} X.$$

The first part of the proof is completed by Proposition 4.5.

We know from Theorem D" (2) that restriction $R\colon \partial_\nu X \to \partial_\mu X$ is surjective. In view of (1), it is easily checked that $R^{-1}(\partial_\mu^{\min} X) = \partial_\nu^{\min} Y$. This readily implies the equivalence of the first two statements of (2).

Assume now that $\partial_\mu X = \partial_\mu^{\min} X$. Let $\xi \in \partial_\nu Y \cap \mathcal{H}^+(Y,\nu)$ and consider a sequence $(y_n)_{n\in\mathbb{N}}$ in Y with $y_n \to \xi$ in $\mathcal{M}(Y,\nu)$. Fix a finite set A of representatives of the Γ-orbits in Y. After passing to a subsequence if necessary, we may suppose that there exist $z \in A$ and $(g_n)_{n\in\mathbb{N}} \subseteq \Gamma$ such that $y_n = g_n z$ for any $n \in \mathbb{N}$. Since the random walk on Y induced by ν is irreducible, we know that there exists $x_0 \in A \cap X$, $k_1, k_2 \in \mathbb{N}$ and $c > 0$ such that $\nu_y^{k_1}(x_0) \geq c$ and $\nu_{x_0}^{k_2}(y) \geq c$. Considering the divergent sequence $x_n = g_n x_0$ of X, we have that

$$\nu_{y_n}^{k_1}(x_n) \geq c \quad \text{and} \quad \nu_{x_n}^{k_2}(y_n) \geq c \quad \text{for any } n \in \mathbb{N},$$

which yields that the Martin kernels satisfy $K(\cdot, y_n) \le c^{-2} K(\cdot, x_n)$ pointwise, for sufficiently large $n \in \mathbb{N}$. It is evident that the same holds for their restrictions to X. After passing to a subsequence if necessary, we may assume that $x_n \to \eta \in \partial_\mu X$ in $\mathcal{M}(X, \mu)$. By assumption, $k(\cdot, \eta)$ is a minimal positive μ-harmonic function. Keeping in mind that $k(\cdot, y_n) \to RK(\cdot, \xi) \in \mathcal{H}^+(X, \mu)$, by virtue of Theorem A", it follows from the aforementioned estimate that $k(\cdot, \eta) = RK(\cdot, \xi)$. We conclude that $K(\cdot, \xi) = Ek(\cdot, \eta)$ which is a minimal positive ν-harmonic function, and thus, $\xi \in \partial_\nu^{\min} Y$. ∎

Acknowledgments. We are grateful to the Max Planck Institute for Mathematics in Bonn for its support and hospitality.

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(Reçu le 26 août 2024)

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