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Short note

A Picard little theorem for entire functions of matrices

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Abstract. An analog of Picard's little theorem for entire functions of matrices is proved.

1 Introduction

The Picard little theorem says that any non-constant entire function omits at most one complex value, i.e. its range is either the whole complex plane or the complex plane minus a single point. This theorem together with Picard's great theorem have a long history and many important generalizations that have been the impetus for powerful developments in complex analysis such as the Nevanlinna theory for value distribution of meromorphic functions (see e.g. [1]).

The purpose of this note is to prove an analog of Picard's little theorem for entire functions of complex matrices. More precisely, let $M_n(\mathbb{C})$ be the ring of all $n \times n$ matrices with complex entries. Then any entire function $f : \mathbb{C} \to \mathbb{C}$ with power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

defines a map $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by setting

$$f(A) = \sum_{k=0}^{\infty} a_k A^k$$

for any $A \in M_n(\mathbb{C})$. By analogy with Picard's little theorem, one may ask what can be said about the range of the map $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$. This question has been already considered in [4, Problem 210*], and in [3], a criterion for surjectivity of f has been proved for matrices with entries in an arbitrary algebraically closed field.

The main purpose of this note is to show that the range of f can be described almost completely by means of the *totally ramified values* of f, i.e. the complex numbers a such that all roots of the equation $f(z) = a$ have multiplicity at least 2.

For $a \in \mathbb{C}$, let E_a be the set of all matrices $A \in M_n(\mathbb{C})$ which have an eigenvalue a, and let $S_a \subset E_a$ be the set of matrices whose Jordan forms have at least one non-trivial Jordan block corresponding to a (i.e. the eigenspace of A corresponding to a has non-zero codimension). Our analog of Picard's little theorem for entire functions of matrices is the following.

Theorem 1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function.*

(i) *If f omits a complex value a, then*

$$f(M_n(\mathbb{C})) = M_n(\mathbb{C}) \setminus E_a.$$

(ii) *If f takes on all complex values and has no totally ramified values, then*

$$f(M_n(\mathbb{C})) = M_n(\mathbb{C}).$$

(iii) *If f has one totally ramified value a, then*

$$f(M_n(\mathbb{C})) = M_n(\mathbb{C}) \setminus S_a^f,$$

where $\emptyset \neq S_a^f \subset S_a$. Moreover, $S_a^f = S_a$ if and only if each root of the equation $f(z) = a$ has a multiplicity at least n. In particular, $S_a^f = S_a$ for $n = 2$.

(iv) *If f has two totally ramified values a and b, then*

$$f(M_n(\mathbb{C})) = M_n(\mathbb{C}) \setminus (S_a^f \cup S_b^f).$$

Moreover, S_a^f and S_b^f are non-empty proper subsets of S_a and S_b for $n \geq 3$, while $S_a^f = S_a$ and $S_b^f = S_b$ for $n = 2$.

To prove Theorem 1, we use some well-known algebraic facts about square complex matrices, Picard's little and great theorems as well as the well-known fact that any non-constant entire function has at most two totally ramified values.

The needed auxiliary facts for square complex matrices and totally ramified values of entire functions are given in Section 2, and then, in Section 3, we prove Theorem 1. Finally, in Section 4, we discuss some examples of entire functions satisfying the conditions of the four options listed in Theorem 1.

2 Auxiliary facts

We first recall some well-known algebraic facts about square complex matrices.

A Jordan block of size k is a $k \times k$ matrix of the form

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix},$$

where the missing entries are all zero. Every square complex matrix A is similar to a block diagonal matrix

$$J_A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_m \end{pmatrix},$$

where each block A_s is a Jordan block. The matrix J_A is unique up to the order of the Jordan blocks and is called the Jordan form of A. The entries on its main diagonal are equal to the eigenvalues of A.

Let $f:\mathbb{C}\to\mathbb{C}$ be an entire function.

Fact 1. *For any $X\in M_n(\mathbb{C})$ and any non-singular $P\in M_n(\mathbb{C})$, we have*

$$f(P^{-1}XP)=P^{-1}f(X)P.$$

Fact 2. *For any Jordan block $J_k(a)$, the following identity holds true:*

$$f(J_k(a))=\begin{pmatrix} f(a) & f'(a) & \dots & \dots & \frac{f^{(k-1)}(a)}{(k-1)!} \\ & f(a) & f'(a) & \dots & \frac{f^{(k-2)}(a)}{(k-2)!} \\ & & \ddots & \ddots & \vdots \\ & & & f(a) & f'(a) \\ & & & & f(a) \end{pmatrix},$$

where the missing entries are all zero.

Fact 3. *Let A be a block diagonal matrix with diagonal blocks $A_1,A_2,\dots,A_m$. Then $f(A)$ is a block diagonal matrix with diagonal blocks $f(A_1),f(A_2),\dots,f(A_m)$.*

Proof of Facts 1, 2, and 3. It is enough to prove each of these facts for the function $f(z)=z^m$, where m is a positive integer. This can be done easily by induction on m.

Given an upper triangular matrix $A\in M_n(\mathbb{C})$, denote by $d_0(A)$ its main diagonal and by $d_k(A)$, $1\le k\le n-1$, the k-th diagonal of A above $d_0(A)$. ∎

Fact 4. *Let $A\in M_n(\mathbb{C})$ be an upper diagonal matrix such that all entries on $d_0(A)$ are 0 and all entries on $d_1(A)$ are equal to a. Then, for any $1\le k\le n-1$, all entries on $d_1(A^k),\dots,d_{k-1}(A^k)$ are 0 and all entries on $d_k(A^k)$ are equal to a^k.*

Proof. Induction on k. ∎

Fact 5. *Let $A\in M_n(\mathbb{C})$ be an upper triangular matrix whose entries on $d_1(A)$ are all equal to a. Then the Jordan form of A has a single Jordan block $J_n(\lambda)$ if and only if $a\neq 0$ and all entries on $d_0(A)$ are equal to λ.*

Proof. Note first that all entries on $d_0(A)$ are equal to λ since the similar matrices have equal eigenvalues. Hence the Jordan form of A consists only of Jordan blocks of the form $J_s(\lambda)$. Denote by k the largest size of such a Jordan block. Then $k=n$ if and only if $(A-\lambda I)^{n-1}\neq 0$. On the other hand, we know from Fact 4 that the entry on $d_{n-1}((A-\lambda I)^{n-1})$ is equal to a^{n-1}, which proves Fact 5. ∎

We will need also the following analytic facts.

Fact 6 ([1, p. 45]). *Any entire function f has at most two totally ramified values. If there are two, say a and b, then each of the equations $f(z)=a$ and $f(z)=b$ has a root of multiplicity* 2.

Fact 7. *If a non-constant entire function omits a value, then it has no totally ramified values.*

Proof. See Example 3 for a more general result. ∎

3 Proof of Theorem 1

Given a matrix $A \in M_n(\mathbb{C})$, denote by J_A its Jordan form. Then it follows from Fact 1 that the equation $f(X) = A$ has a solution if and only if there is a matrix $X \in M_n(\mathbb{C})$ such that the matrices $f(J_X)$ and J_A are similar. This together with Facts 3, 2, 5, and 7 implies that the equation $f(X) = A$ has a solution if and only if, for any eigenvalue a of A, there exists z_a such that $f(z_a) = a$ and $f'(z_a) \neq 0$. This proves statements (i) and (ii).

Suppose now that f has one totally ramified value a. If a is not an eigenvalue of A, then the equation $f(X) = A$ has a solution since otherwise we will obtain another totally ramified value of $f(z)$, a contradiction. The reasoning above shows also that if a is an eigenvalue of A and $J_n(a) \in S_a$, then the equation $f(X) = J_n(a)$ has no solutions, i.e. $f(M_n(\mathbb{C})) = M_n(\mathbb{C}) \setminus S_a^f$, where $\emptyset \neq S_a^f \subseteq S_a$. Moreover, if the equation $f(z) = a$ has a root z_a with multiplicity less than n, then S_a^f is a proper subset of S_a. Indeed, it follows from Fact 2 that the Jordan form of $f(J_n(z_a))$ has at least one non-trivial Jordan block which shows that $f(J_n(z_a) \in S_a$. To prove (iii), it remains to show that if each root of the equation $f(z) = a$ has a multiplicity at least n, then $S_a^f = S_a$. Suppose that $f(X) = A$, where $A \in S_a$. Then the Jordan form of A has at least one non-trivial Jordan block $J_k(a)$. On the other hand, Fact 2 shows that each of these Jordan blocks is trivial, which is a contradiction.

Finally, suppose that f has two totally ramified values a and b. Then, by Fact 6, any of the equations $f(z) = a$ and $f(z) = b$ has a root of multiplicity 2 and (iv) follows by using the same reasoning as in the proof of (iii). ∎

Remark. Denote by $\mathrm{GL}_n(\mathbb{C})$ the group of non-singular $n \times n$ matrices with complex entries. Then any holomorphic function $f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ with Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$$

defines a map $f\colon \mathrm{GL}_n(\mathbb{C}) \to M_n(\mathbb{C})$ by setting

$$f(A) = \sum_{k=-\infty}^{\infty} a_k A^k.$$

Since $g(z) = f(e^z)$ is an entire function, it follows that f omits at most one value as well as that f has at most two ramified values. Note also that f has no ramified values if it omits a value. Hence one can repeat the proof of Theorem 1 line by line and see that its statement is true for holomorphic functions on $\mathbb{C} \setminus \{0\}$ as well. This follows also as a corollary of Theorem 1 since $f(\mathrm{GL}_n(\mathbb{C})) = g(M_n(\mathbb{C}))$.

4 Examples

In this section, we discuss some examples of entire functions satisfying the conditions of the four options listed in Theorem 1.

Example 1. (i) Any quadratic polynomial has one totally ramified value.

(ii) The polynomial $P(z) = z^k(z-1)$ $(k \geq 2)$ has no totally ramified values.

(iii) The general fact for polynomials is that they have at most one totally ramified value. This follows by the simple observation that if a is such a value for a polynomial P, then the zeros of $P - a$ contribute at least $\frac{\deg(P)}{2}$ to the zeros of P' counted with multiplicity.

Example 2. Consider the function

$$f(z) = \frac{a-b}{2}\sin(cz+d) + \frac{a+b}{2},$$

where $a \neq b$, $c \neq 0$, $d \in \mathbb{C}$. Then $f'(z) = \frac{(a-b)c}{2}\cos(cz+d) = 0 \Leftrightarrow \sin(cz+d) = \pm 1$ and we conclude that f has two totally ramified values a and b. Moreover, by [2, Theorem 2.1], these are all entire functions of order at most 1 (i.e. $f(z) = \mathrm{O}(e^{|z|^\gamma})$ for any $\gamma > 1$) with this property.

Example 3. As we know by Picard's great theorem, for any transcendental entire function f (i.e. not a polynomial), there is at most one $a \in \mathbb{C}$ such that the equation $f(z) = a$ has finitely many roots. In this case, $f = a + Pe^g$, where P is a monic polynomial and g is a non-constant entire function. Then it follows by Nevanlinna's theorem on deficient values [1, Theorem 2.4] that a is the only possible totally ramified value of f and this is so if and only if all zeros of P have multiplicity greater than 1. In particular, if a non-constant entire function f omits a value a, then $P = 1$ and f has no totally ramified values (Fact 7).

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