Elemente der Mathematik 81 (2026), 29–33
DOI 10.4171/EM/551

Short note Cauchy–Schwarz' inequality and beyond

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In memory of my son Horia

1 Introduction and the main result

The arithmetic-geometric mean (AGM) inequality and the Cauchy–Schwarz (CS) inequality are the most familiar inequalities in mathematics. The fact that (AGM) $\Rightarrow$ (CS) is well known (see [1, 2] or below). However, the reverse implication has been proved only recently [1] via calculus, and this raises the quest for an algebraic proof (see Corollary 1).

AGM Inequality. *If $x_1, \dots, x_m$ are positive numbers, then*

$$\Big(\sum_{j=1}^{m} x_j\Big)/m \geq \Big(\prod_{k=1}^{m} x_j\Big)^{1/m}. \tag{AGM}$$

CS Inequality. *For any positive numbers $a_1, \dots, a_m, b_1, \dots, b_m$, we have*

$$\Big(\sum_{j=1}^{m} a_j b_j\Big)^2 \leq \sum_{j=1}^{m} a_j^2 \sum_{j=1}^{m} b_j^2. \tag{CS}$$

A reformulation of (CS) which matches our purposes in this paper is the following. For any positive numbers $x_1, \dots, x_m, y_1, \dots, y_m$, we have

$$\sqrt{x_1 y_1} + \dots + \sqrt{x_m y_m} \leq \sqrt{x_1 + \dots + x_m}\sqrt{y_1 + \dots + y_m}.$$

This version has the following advantages. On the one hand, its proof is self-generated granting the superadditivity of the geometric mean in two dimensions, namely if

$$\mathrm{G}\colon [0,\infty)^2 \to [0,\infty)$$

is given by $\mathrm{G}(x,y) = \sqrt{xy}$ for $x, y \geq 0$, then for any $(x,y), (x',y') \in [0,\infty)^2$, we have

$$\mathrm{G}(x,y) + \mathrm{G}(x',y') \leq \mathrm{G}(x+x', y+y'),$$

which results by squaring both sides and straightforward computations via (AGM) for $m = 2$. Hence, we derive a proof of (AGM) $\Rightarrow$ (CS).

On the other hand, it can be readily extended to the following general Cauchy–Schwarz inequality, which we think is new in this form.

Theorem 1. *We have*

$$\sum_{j=1}^{m}\Big(\prod_{k=1}^{q} a_{kj}\Big)^{1/q} \le \Big(\prod_{k=1}^{q}\sum_{j=1}^{m} a_{kj}\Big)^{1/q} \qquad (\blacklozenge)$$

for any vectors $(a_{11},\dots,a_{1m}), (a_{21},\dots,a_{2m}),\dots,(a_{q1},\dots,a_{qm})$ *of* $(0,\infty)^m$.

Proof. We employ Cauchy's trick for the proof of (AGM). Hence, let $q_\star$ be a natural power of 2 such that $q_\star > q$. Repeatedly using the second version of (CS), we obtain that ($\blacklozenge$) holds true for any set of $q_\star$ vectors of $(0,\infty)^m$.

Now, let $(a_{11},\dots,a_{1m}), (a_{21},\dots,a_{2m}),\dots,(a_{q1},\dots,a_{qm})$ be q vectors in $(0,\infty)^m$ and add the set of $q_\star - q$ equal vectors $(a_{r1},\dots,a_{rm}) = (d_1,\dots,d_m)$ for $q+1 \le r \le q_\star$, where $d_j = (a_{1j}\cdots a_{qj})^{1/q}$ for $1 \le j \le m$. Since

$$\sum_{j=1}^{m} a_{rj} = \sum_{j=1}^{m} d_j \quad \text{for all } r \in \{q+1,\dots,q_\star\}$$

and for any integer j with $1 \le j \le m$, we have

$$\prod_{l=1}^{q_\star} a_{lj} = \Big(\prod_{k=1}^{q} a_{kj}\Big) d_j^{q_\star - q} = d_j^{q_\star}.$$

Divide both terms of inequality ($\blacklozenge$) for the above set of $q_\star$ vectors by

$$\Big(\sum_{j=1}^{m} d_j\Big)^{1-q/q_\star}$$

so that we get what we want. ■

Corollary 1. *The implication "(CS) ⇒ (AGM)" holds true.*

Proof. Let $x_1, x_2,\dots,x_m$ be positive numbers. Apply ($\blacklozenge$) for the m vectors obtained by cyclic permutation of the components of $(x_1, x_2,\dots,x_m)$. ■

Corollary 2. *Let* $\mathrm{P} \in \mathbb{R}[X]$ *be a polynomial with nonnegative coefficients. Then, for any* $x_1, x_2,\dots,x_m \in (0,\infty)$, *we have the inequality*

$$\mathrm{P}(\sqrt[m]{x_1\cdots x_m}) \le \sqrt[m]{\mathrm{P}(x_1)\cdots\mathrm{P}(x_m)}.$$

Proof. Since ($\blacklozenge$) remains true also for vectors whose components are nonnegative, take $\mathrm{P} = c_\nu X^\nu + c_{\nu-1}X^{\nu-1} + \cdots + c_0$ and apply ($\blacklozenge$) for the set of m vectors

$$\{(c_\nu x_j^\nu, c_{\nu-1}x_j^{\nu-1},\dots,c_0) : j = 1,\dots,m\} \subset \mathbb{R}^{\nu+1}.$$ ■

Remark. We recast ($\blacklozenge$) granting Hadamard–Schur's product $\mathbf{u} \star \mathbf{v}$ of two vectors

$$\mathbf{u} = (u_1,\dots,u_m) \quad \text{and} \quad \mathbf{v} = (v_1,\dots,v_m)$$

of $\mathbb{R}^m$, which is defined by

$$\mathbf{u} \star \mathbf{v} = (u_1 v_1, \ldots, u_m v_m),$$

and setting, for any $\lambda > 0$,

$$\|\mathbf{u}\|_\lambda = \left(\sum_{j=1}^{m} |u_j|^\lambda \right)^{1/\lambda},$$

as follows. For any vectors $\mathbf{u}_1, \ldots, \mathbf{u}_q$ of $\mathbb{R}^m$, we have

$$\|\mathbf{u}_1 \star \cdots \star \mathbf{u}_q\|_{1/q} \le \|\mathbf{u}_1\|_1 \cdots \|\mathbf{u}_q\|_1. \tag{$\natural$}$$

From this, we infer Hölder's inequality, namely the following result.

Theorem 2. *Let $k \in \mathbb{N}$ and let $p_1, \ldots, p_k, r$ be positive real numbers such that*

$$1/p_1 + \cdots + 1/p_k = 1/r.$$

Then, for any vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ of $\mathbb{R}^m$, we have

$$\|\mathbf{u}_1 \star \cdots \star \mathbf{u}_k\|_r \le \|\mathbf{u}_1\|_{p_1} \cdots \|\mathbf{u}_k\|_{p_k}. \tag{H}$$

Proof. By routine arguments, it is easily seen that we only need to show (H) when $r = 1$, and by continuity of the exponential function and the density of $\mathbb{Q}$ in $\mathbb{R}$, we may also assume that $p_1, \ldots, p_k$ are positive rationals, so that we may write $p_j = q/\nu_j$ for some positive integers q and ν_j such that $\nu_1 + \cdots + \nu_k = q$.

Now, by allowing repetitions in the sequence of vectors in ($\natural$), we deduce that, for arbitrary vectors $\mathbf{b}_1, \ldots, \mathbf{b}_k$ in $\mathbb{R}^m$, we have

$$\|\mathbf{b}_1^{\nu_1} \star \cdots \star \mathbf{b}_k^{\nu_k}\|_{1/q} \le \|\mathbf{b}_1\|_1^{\nu_1} \cdots \|\mathbf{b}_k\|_1^{\nu_k}, \tag{$\sharp$}$$

where, for a vector $\mathbf{w}$ of $\mathbb{R}^m$ and $\nu \in \mathbb{N}$, the vector $\mathbf{w}^\nu$ is Hadamard–Schur's product of $\mathbf{w}$ with itself taken ν times.

Then choose $\mathbf{b}_j \in \mathbb{R}^m$ such that $\mathbf{b}_j^{\nu_j} = |\mathbf{u}_j|^q$ for all $j = 1, \ldots, k$, where the components of the vector $|\mathbf{u}_j|$ are the moduli of the corresponding components of $\mathbf{u}_j$, and extract the q-th root of both terms in inequality ($\sharp$) thus obtained so that we get (H) for $r = 1$ and $p_j \in \mathbb{Q}$. The proof concludes. ∎

2 Superadditivity of geometric mean

In this section, we consider a few more or less known applications of (♦). One of these is the superadditivity of the "weighted geometric mean", a notion mostly employed in statistics and used to calculate the overall performance of an investment portfolio by assigning weights based on the proportion of each investment's value relative to the total portfolio value.

Here is the definition. Let $m \in \mathbb{N}$. A vector $\mathbf{w} = (w_1, \ldots, w_m)$ of $\mathbb{R}^m$ is called a "weight vector" if its components are positive and sum up to 1.

Now, given nonnegative real numbers $\xi_1, \dots, \xi_m$, the weighted geometric mean of $(\xi_1, \dots, \xi_m)$ with respect to the weight vector $\mathbf{w}$ is given by the formula

$$\mathrm{G}_{\mathbf{w}}(\xi_1, \dots, \xi_m) = {\xi_1}^{w_1} \cdots {\xi_m}^{w_m}.$$

In case $\xi_j > 0$ for all $j \in \{1, \dots, m\}$, $\mathrm{G}_{\mathbf{w}}(\xi_1, \dots, \xi_m)$ is the exponential of the weighted arithmetic mean of $(\log(\xi_1), \dots, \log(\xi_m))$ with respect to $\mathbf{w}$.

In particular, when $\mathbf{w} = (1/n, \dots, 1/n)$, we recover the ordinary geometric mean $(\xi_1, \dots, \xi_n)^{1/n}$ of $\xi_1, \dots, \xi_n$, which we denote by $\mathrm{G}(\xi_1, \dots, \xi_n)$.

Observe that either (♦) or Corollary 2 for $\mathrm{P} = 1 + X$ gives immediately the superadditivity of the geometric mean, a result known as Minkowski's inequality, namely, for any vectors $(x_1, \dots, x_m)$ and $(y_1, \dots, y_m)$ of $[0, \infty)^m$, we have

$$\mathrm{G}(x_1, \dots, x_m) + \mathrm{G}(y_1, \dots, y_m) \leq \mathrm{G}(x_1 + y_1, \dots, x_m + y_m). \tag{†}$$

In fact, more generally, the weighted geometric mean is superadditive.

Proposition 1. *Let* $\mathbf{w}$ *be a weight vector of* $\mathbb{R}^m$*. Then, for arbitrary vectors* $(x_1, \dots, x_m)$ *and* $(y_1, \dots, y_m)$ *of* $[0, \infty)^m$*, we have*

$$\mathrm{G}_{\mathbf{w}}(x_1, \dots, x_n) + \mathrm{G}_{\mathbf{w}}(y_1, \dots, y_m) \leq \mathrm{G}_{\mathbf{w}}(x_1 + y_1, \dots, x_m + y_m).$$

Proof. The inequality holds true when the weight vector $\mathbf{w}$ has rational components as follows immediately from (†) by allowing repetitions in the corresponding sequences of x_j's and y_j's. Then the conclusion results by the continuity of the exponentiation and the density of $\mathbb{Q}$ in $\mathbb{R}$. ∎

Proposition 2. *The geometric mean function* $\mathrm{G}\colon [0, \infty)^m \to \mathbb{R}$ *is concave.*

Proof. This is a particular case of a more general straightforward fact, namely if $\mathrm{K} \subseteq \mathbb{R}^m$ is a convex cone, then any superadditive and positively homogeneous function $\mathfrak{f}\colon \mathrm{K} \to \mathbb{R}$ of degree 1 is concave. ∎

Proposition 3. *The superadditivity of the geometric mean is equivalent to the arithmetic-geometric mean inequality.*

Proof. Granting [2, p. 34] or (†), we only need to check the implication "⇒". Suppose that $x_1, \dots, x_m$ are positive numbers, and let $\varphi\colon [0, \infty) \to \mathbb{R}$ be given for any $t \geq 0$ by setting

$$\varphi(t) = \mathrm{G}(1 + tx_1, \dots, 1 + tx_m) - t\mathrm{G}(x_1, \dots, x_m).$$

The superadditivity of the geometric mean implies that φ attains its minimum, which is 1, at $t = 0$; hence $\varphi'_+(0) \geq 0$. Since

$$\varphi'_+(0) = (x_1 + \cdots + x_m)/m - \mathrm{G}(x_1, \dots, x_m),$$

the conclusion follows. ∎

References

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