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A convergence result for the derivation of front propagation in nonlocal phase field models

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Abstract. We prove that the mean curvature of a smooth surface in $\mathbb{R}^n$, $n \geq 2$, arises as the limit of a sequence of functions that are intrinsically related to the difference between an n- and 1-dimensional fractional Laplacian of a phase transition. Depending on the order of the fractional Laplace operator, we recover the fractional mean curvature or the classical mean curvature of the surface. Moreover, we show that this is an essential ingredient for deriving the evolution of fronts in fractional reaction-diffusion equations such as those for atomic dislocations in crystals.

1. Introduction

In this paper, we prove that the fractional mean curvature and classical mean curvature of a smooth surface in $\mathbb{R}^n$, $n \geq 2$, arises as the ε-limit of a sequence of functions $\bar{a}_\varepsilon(x)$, $\varepsilon > 0$ (see (1.7)). These functions $\bar{a}_\varepsilon$ appear in the study of nonlocal reaction-diffusion equations from an interaction between an n-dimensional and a 1-dimensional fractional Laplacian of a phase transition ϕ, and thus play a key role in deriving the evolution of interfaces in nonlocal phase field models, see Section 1.1. The convergence result was first observed by Imbert and Souganidis in their 2009 unpublished preprint [14]. Since it is foundational for further study on fractional reaction-diffusion equations, we meticulously prove convergence to the mean curvature using a different approach than in [14].

Before presenting the main result, we describe the setting of the problem mathematically. Let Ω be a connected set in $\mathbb{R}^n$, $n \geq 2$, with smooth boundary $\Gamma := \partial\Omega$. Let $d = d(x)$ be the signed distance function associated to Ω given by

$$d(x) = \begin{cases} d(x,\Gamma) & \text{if } x \in \Omega, \\ -d(x,\Gamma) & \text{otherwise.} \end{cases} \tag{1.1}$$

For $\rho > 0$, define the neighborhood Q_ρ of Γ by

$$Q_\rho = \{x \in \mathbb{R}^n : |d(x)| < \rho\}.$$

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We assume there is some $\rho > 0$ such that

$$d \text{ is smooth in } Q_{2\rho}; \text{ in particular, } |\nabla d| = 1 \text{ in } Q_{2\rho}. \tag{1.2}$$

Remark 1.1. In what follows, one can replace d by any Lipschitz function $\tilde{d}$ such that $\tilde{d} \equiv d$ in Q_ρ and $\tilde{d}$ is smooth in $Q_{2\rho}$.

Let $0 < s < 1$ be fixed throughout the paper. We consider the fractional mean curvature of order $2s$ of the surface Γ as developed in [13]. Towards this end, define the singular measure

$$\nu(dz) = \frac{dz}{|z|^{n+2s}},$$

and set

$$\kappa[x, d] = \nu(\{z : d(x+z) > d(x), \nabla d(x) \cdot z < 0\}) - \nu(\{z : d(x+z) < d(x), \nabla d(x) \cdot z > 0\}).$$

By (1.2), $\kappa[x, d]$ is finite in Q_ρ precisely when $s \in (0, \frac{1}{2})$, see [13, Lemma 1], and it is the fractional mean curvature at $x \in \Gamma$. When $s \in [\frac{1}{2}, 1)$, we instead consider the classical mean curvature of the surface Γ; see, for instance, [11]. Let us simply recall that, since Γ is smooth, the mean curvature at $x \in \Gamma$ is well defined and is given by $-\Delta d(x)/(n-1)$. For $n \geq 1$, let $\mathcal{I}_n^s = -c_{n,s}(-\Delta)^s$ denote, up to a constant, the fractional Laplacian of order $2s$ in $\mathbb{R}^n$. Specifically, the operator $\mathcal{I}_n^s$ is a nonlocal integro-differential operator given by

$$\mathcal{I}_n^s u(x) = \text{P. V.} \int_{\mathbb{R}^n} (u(x+y) - u(x)) \frac{dy}{|y|^{n+2s}}, \quad x \in \mathbb{R}^n, \tag{1.3}$$

where P. V. indicates that the integral is taken in the principal value sense. Assume that W is a double-well potential satisfying

$$\begin{cases} W \in C^{2,\beta}(\mathbb{R}) & \text{for some } 0 < \beta < 1, \\ W(0) = W(1) = 0, & \\ W(u) > 0 & \text{for any } u \in (0,1), \\ W'(0) = W'(1) = 0, & \\ W''(0) = W''(1) > 0. & \end{cases} \tag{1.4}$$

The phase transition $\phi : \mathbb{R} \to (0,1)$ is then the unique solution to

$$\begin{cases} \mathcal{I}_1^s[\phi] = W'(\phi) & \text{in } \mathbb{R}, \\ \dot{\phi} > 0 & \text{in } \mathbb{R}, \\ \phi(-\infty) = 0, \quad \phi(+\infty) = 1, \quad \phi(0) = \frac{1}{2}, & \end{cases} \tag{1.5}$$

where $\mathcal{I}_1^s$ denotes the nonlocal operator in (1.3) with $n = 1$. See Section 2 for more on ϕ.

Define next the function $a_\varepsilon = a_\varepsilon(\xi; x)$ by

$$a_\varepsilon(\xi; x) = \int_{\mathbb{R}^n} \left(\phi\left(\xi + \frac{d(x+\varepsilon z) - d(x)}{\varepsilon} \right) - \phi(\xi + \nabla d(x) \cdot z) \right) \frac{dz}{|z|^{n+2s}}, \tag{1.6}$$

where $(\xi, x) \in \mathbb{R} \times \mathbb{R}^n$ and d is in (1.1). Roughly speaking, one may view a_ε as a nonlocal operator acting on $\phi = \phi(\xi)$.

Lemma 1.2. *For all $x \in Q_\rho$, it holds that*

$$a_\varepsilon\left(\frac{d(x)}{\varepsilon}; x\right) = \varepsilon^{2s} \mathcal{I}_n^s\left[\phi\left(\frac{d(\cdot)}{\varepsilon}\right)\right](x) - C_{n,s} \mathcal{I}_1^s \phi\left(\frac{d(x)}{\varepsilon}\right),$$

where $C_{n,s} > 0$ is given in (3.1).

The lemma is proven in Section 3. The corresponding function $\bar{a}_\varepsilon = \bar{a}_\varepsilon(x)$ is given by

$$\bar{a}_\varepsilon(x) = \frac{1}{\eta_\varepsilon} \int_{\mathbb{R}} a_\varepsilon(\xi; x) \dot{\phi}(\xi) d\xi, \tag{1.7}$$

where $x \in \mathbb{R}^n$ and

$$\eta_\varepsilon = \begin{cases} \varepsilon^{2s} & \text{if } s \in (0, \frac{1}{2}), \\ \varepsilon |\ln \varepsilon| & \text{if } s = \frac{1}{2}, \\ \varepsilon & \text{if } s \in (\frac{1}{2}, 1). \end{cases} \tag{1.8}$$

Our main result is the following.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a connected set with smooth boundary $\Gamma = \partial\Omega$. Let $d = d(x)$ be as in* (1.1) *and assume there is some $\rho > 0$ such that* (1.2) *holds. Then,*

$$\lim_{\varepsilon \to 0} \bar{a}_\varepsilon(x) = \begin{cases} \kappa[x, d] & \text{if } s \in (0, \frac{1}{2}), \\ \frac{1}{2} \frac{|S^{n-2}|}{n-1} \Delta d(x) & \text{if } s = \frac{1}{2}, \\ \frac{c_\star}{2} \frac{|S^{n-2}|}{n-1} \Delta d(x) & \text{if } s \in (\frac{1}{2}, 1), \end{cases}$$

uniformly in Q_ρ where, for $s \in (\frac{1}{2}, 1)$, the constant $c_\star > 0$ is given explicitly by

$$c_\star = \frac{n + 2s}{2} \left[\int_0^\infty \frac{q^{n-2}((n-1) - q^2)}{(1 + q^2)^{\frac{n+2s+2}{2}}} dq\right] \left[\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\phi(\xi + w) - \phi(\xi))^2}{|w|^{1+2s}} dw d\xi\right]. \tag{1.9}$$

Remark 1.4. For $n = 1$, $\bar{a}_\varepsilon(x) \equiv 0$. Indeed, the one-dimensional setting corresponds to $\Omega = (x_0, \infty)$ for some $x_0 \in \mathbb{R}$. Note that $\Gamma = \{x_0\}$, and the signed distance function is precisely $d(x) = x - x_0$. Therefore, $a_\varepsilon(\xi; x) \equiv 0$, and consequently, $\bar{a}_\varepsilon(x) \equiv 0$.

As addressed above, Theorem 1.3 was first observed by Imbert and Souganidis, see [14, Lemma 10] for $s \in (0, \frac{1}{2})$ and [14, Lemma 4] for $s \in [\frac{1}{2}, 1)$. While the general ideas of the proof are explained in [14], we found that certain aspects of their proof require further clarification and additional rigor. The critical case $s = \frac{1}{2}$ is particularly delicate and needs more attention. We give our own proof to complement their work and lay the groundwork for continued study on important physical models, such as those outlined below in Section 1.1 for dislocations in crystalline structures.

The proof of Theorem 1.3, especially for the critical case $s = \frac{1}{2}$, is somewhat technical, but we feel it is more direct than in [14]. Indeed, there are several advantages to our approach. First, we carefully decompose the domain of integration in (1.7) to expose the pieces that contribute to the mean curvature in the limit. Then, unlike in [14], we use the known asymptotic behavior of the phase transition ϕ and its derivatives at $\pm\infty$ (see Lemma 2.1) to carefully obtain precise error estimates. Kindly note that the asymptotic behavior (as well as existence and uniqueness) of the phase transition in [14, equation (13)] is assumed and differs from our setting, see (2.1).

We remark that the local counterpart of Theorem 1.3 corresponding to $s = 1$ is trivial. In light of Lemma 1.2, one formally computes

$$\varepsilon^2 \Delta\left[\phi\left(\frac{d(\cdot)}{\varepsilon}\right)\right](x) - \ddot{\phi}\left(\frac{d(x)}{\varepsilon}\right) = \varepsilon\dot{\phi}\left(\frac{d(x)}{\varepsilon}\right)\Delta d(x) \quad \text{for } x \in Q_\rho$$

and then defines $a_\varepsilon(\xi; x) := \varepsilon\dot{\phi}(\xi)\Delta d(x)$. Defining $\bar{a}_\varepsilon(x)$ accordingly with $\eta_\varepsilon = \varepsilon$, we trivially recover $\bar{a}_\varepsilon(x) \equiv \Delta d(x)$. We refer the reader to [2–4, 7] for related problems.

1.1. Application to nonlocal phase field models

We now show how Theorem 1.3 is crucial for mathematical analysis of nonlocal phase field problems. Let us briefly describe the evolutionary problem and present some formal computations.

Consider a connected set $\Omega_0 \subset \mathbb{R}^n$, $n \geq 2$, with smooth boundary $\Gamma_0 = \partial\Omega_0$. Let $d_0 = d_0(x)$ denote the signed distance function to Γ_0 (recall (1.1)) and assume there is some $\rho > 0$ such that (1.2) holds for d_0. Let $u^\varepsilon = u^\varepsilon(t, x)$ denote the solution to the fractional Allen–Cahn equation

$$\varepsilon\partial_t u^\varepsilon = \frac{1}{\eta_\varepsilon}\left(\varepsilon^{2s}\mathcal{I}_n^s[u^\varepsilon] - \widetilde{W}'(u^\varepsilon)\right) \quad \text{in } (0, \infty) \times \mathbb{R}^n \tag{1.10}$$

with η_ε defined in (1.8) and initial condition

$$u^\varepsilon(0, x) = \phi\left(\frac{d_0(x)}{\varepsilon}\right) \quad \text{on } \mathbb{R}^n, \tag{1.11}$$

where ϕ solves (1.5) and $\widetilde{W} := C_{n,s}W$ for an explicit constant $C_{n,s} > 0$ (given in (3.1)). Notice that if W satisfies (1.4), then so does $\widetilde{W}$.

When $n = 2$ and $s = \frac{1}{2}$, the PDE (1.10) is a rescaled version of the evolutionary Peierls–Nabarro model for atomic dislocations in crystalline structures, see [22, Section 1.2] and for the original model [17, 18, 23]. For the one-dimensional version of (1.10), we also refer to [8–10, 12, 20] and references therein. The set Γ_0 is understood as the initial dislocation curve in the crystal, and the parameter $\varepsilon > 0$ represents the scaling between the microscopic scale and the mesoscopic scale. We send $\varepsilon \to 0$ in (1.10) to describe the evolution $(\Gamma_t)_{t\geq 0}$ of the dislocation curve Γ_0 at the larger length scale. Indeed, as $\varepsilon \to 0$, the function u^ε approaches a piecewise function with plateaus corresponding to the global minima of W (i.e., 0 and 1) and whose jump set at time t is Γ_t. Here, we formally derive the evolution of $(\Gamma_t)_{t\geq 0}$ using Theorem 1.3.

Heuristically, let $(\Omega_t)_{t\geq 0}$ denote the evolution of the set Ω_0 according to (1.10), and assume that the boundaries $\Gamma_t = \partial\Omega_t$ are smooth. Let $d = d(t,x)$ denote the signed distance function to Ω_t and assume that (1.2) holds in

$$Q_{2\rho} = \big\{(t,x) \in [0,T]\times\mathbb{R}^n : |d(t,x)| < 2\rho\big\}$$

for some $T > 0$. The formal ansatz for deriving the evolution of the fronts $(\Gamma_t)_{t\in[0,T]}$ is

$$u^\varepsilon(t,x) \simeq \phi\left(\frac{d(t,x)}{\varepsilon}\right).$$

Plugging the ansatz into the equation, the left-hand side of (1.10) gives

$$\varepsilon\partial_t u^\varepsilon(t,x) = \dot{\phi}\left(\frac{d(t,x)}{\varepsilon}\right)\partial_t d(t,x).$$

Regarding the right-hand side, we find, for $(t,x)\in Q_\rho$,

$$\begin{aligned}\varepsilon^{2s}\mathcal{I}_n^s[u^\varepsilon(t,\cdot)](x) - \widetilde{W}'(u^\varepsilon(t,x)) &\simeq \varepsilon^{2s}\mathcal{I}_n^s\left[\phi\left(\frac{d(t,\cdot)}{\varepsilon}\right)\right](x) - C_{n,s}W'\left(\phi\left(\frac{d(t,x)}{\varepsilon}\right)\right)\\ &= \varepsilon^{2s}\mathcal{I}_n^s\left[\phi\left(\frac{d(t,\cdot)}{\varepsilon}\right)\right](x) - C_{n,s}\mathcal{I}_1^s[\phi]\left(\frac{d(t,x)}{\varepsilon}\right)\\ &= a_\varepsilon\left(\frac{d(t,x)}{\varepsilon};t,x\right),\end{aligned}$$

where the last line follows from Lemma 1.2. Note that $a_\varepsilon = a_\varepsilon(\xi;t,x)$ in (1.6) here depends on t though $d(t,x)$.

Freeze a point $(t,x)\in Q_\rho$. Let $\xi = d(t,x)/\varepsilon$ and assume separation of scales. That is, assume that ξ and (t,x) are unrelated. Then, multiplying the PDE in (1.10) by $\dot{\phi}(\xi)$ and integrating over $\xi\in\mathbb{R}$ gives

$$\int_{\mathbb{R}}(\varepsilon\partial_t u^\varepsilon)\dot{\phi}(\xi)\,d\xi = \frac{1}{\eta_\varepsilon}\int_{\mathbb{R}}\big(\varepsilon^{2s}\mathcal{I}_n^s[u^\varepsilon] - \widetilde{W}'(u^\varepsilon)\big)\dot{\phi}(\xi)\,d\xi.$$

From the above computations and (1.7), this yields

$$\partial_t d(t,x)\int_{\mathbb{R}}[\dot{\phi}(\xi)]^2\,d\xi \simeq \frac{1}{\eta_\varepsilon}\int_{\mathbb{R}} a_\varepsilon(\xi;t,x)\,d\xi = \bar{a}_\varepsilon(t,x).$$

Therefore, by Theorem 1.3, we conclude that, in Q_ρ,

$$\partial_t d(t,x) \simeq \left[\int_{\mathbb{R}}[\dot{\phi}(\xi)]^2\,d\xi\right]^{-1}\bar{a}_\varepsilon(t,x) \simeq \mu\begin{cases}\kappa[x,d(t,\cdot)] & \text{if } s\in(0,\frac{1}{2}),\\ \Delta d(t,x) & \text{if } s\in[\frac{1}{2},1)\end{cases}$$

for a constant $\mu > 0$ depending on n and s. This formally shows that the forming interphases as $\varepsilon \to 0$, $(\Gamma_t)_{t\in[0,T]}$, move according to either their fractional or classical mean curvature.

Of course, these are formal computations that need to be rigorously checked. We refer the reader to [22] for complete heuristics and a study of (1.10) in the case $s = \frac{1}{2}$. There, our initial configuration is a superposition of functions of the form (1.11) which corresponds to a finite collection of dislocations, and we show using Theorem 1.3 that they move independently and according to their mean curvature. The cases $s \in (0, \frac{1}{2})$ and $s \in (\frac{1}{2}, 1)$ will be treated separately in future work. Problem (1.10) was previously studied in [14] for any $s \in (0, 1)$, for the case of one dislocation curve (i.e., for the initial condition (1.11)), and under the additional assumption of the existence of certain correctors, see Assumption 3 there.

The stationary problem was studied in [1, 24] where they prove that the fractional Allen–Cahn energy, with the same scaling as in (1.8), Γ-converges to the $2s$-fractional perimeter functional when $s \in (0, \frac{1}{2})$ and classical perimeter functional when $s \in [\frac{1}{2}, 1)$. The local problem in which (1.10) is instead driven by the usual Laplacian Δ was studied by Modica–Mortola [15] for the stationary problem and Chen [7] for the evolutionary problem.

1.2. Organization of the paper

The rest of the paper is organized as follows. First, in Section 2, we review some properties of the phase transition ϕ. Preliminaries on a_ε and the proof of Lemma 1.2 are presented in Section 3. Section 4 contains preliminaries for the proof of Theorem 1.3 when $s \in [\frac{1}{2}, 1)$. We prove Theorem 1.3 for $s = \frac{1}{2}$, $s \in (\frac{1}{2}, 1)$, and $s \in (0, \frac{1}{2})$, respectively, in Sections 5, 6, and 7.

1.3. Notations

In the paper, we will denote by $C > 0$ any universal constant depending only on the dimension n, s, and W.

Denote by S^n the unit sphere in $\mathbb{R}^{n+1}$ and $\mathcal{H}^n$ the n-dimensional Hausdorff measure.

Given a function $u = u(x)$, defined on a set A, we write $u = O(\varepsilon)$ if there is $C > 0$ such that $|u(x)| \le C\varepsilon$ for all $x \in A$, and we write $u = o_\varepsilon(1)$ if $\lim_{\varepsilon\to 0} u(x) = 0$, uniformly in $x \in A$.

For a set A, we denote by $\mathbb{1}_A$ the characteristic function of the set A.

2. The phase transition ϕ

In this section, we present background and preliminary results on the phase transition ϕ. Let $H(\xi)$ denote the Heavyside function.

Lemma 2.1. *There is a unique solution $\phi \in C^{2,\beta}(\mathbb{R})$ of (1.5), with β as in (1.4). Moreover, there exists a constant $C > 0$ and $\kappa > 2s$ (only depending on s) such that*

$$\left|\phi(\xi) - H(\xi) + \frac{1}{2sW''(0)} \frac{\xi}{|\xi|^{1+2s}}\right| \le \frac{C}{|\xi|^{\kappa}} \quad \text{for } |\xi| \ge 1 \tag{2.1}$$

and

$$\frac{1}{C\,|\xi|^{2s+1}} \le \dot{\phi}(\xi) \le \frac{C}{|\xi|^{2s+1}}, \qquad |\ddot{\phi}(\xi)| \le \frac{C}{|\xi|^{2s+1}} \quad \text{for } |\xi| \ge 1. \tag{2.2}$$

If $s \in (\frac{1}{2}, 1)$, then

$$c_1 := \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(\phi(\xi+t) - \phi(\xi))^2}{|t|^{1+2s}} dt\, d\xi < \infty. \tag{2.3}$$

Proof. The existence of a unique solution of (1.5) is proven in [6] for $s = \frac{1}{2}$ and in [5, 19] for any $s \in (0,1)$ together with (2.3) when $s > \frac{1}{2}$. Estimate (2.1) is proven in [12] for $s = \frac{1}{2}$ and in [8] and [9], respectively, when $s \in (0, \frac{1}{2})$ and $s \in (\frac{1}{2}, 1)$. Finally, estimates (2.2) are shown in [5, 16, 19, 21]. ∎

We will use several times that, by (2.1), if $M > 1$,

$$\int_{-M}^{M} \dot{\phi}(\xi) d\xi = \phi(M) - \phi(-M) = 1 + O(M^{-2s}) \tag{2.4}$$

and

$$\int_{\{|\xi|>M\}} \dot{\phi}(\xi) d\xi = 1 - \int_{-M}^{M} \dot{\phi}(\xi) d\xi = O(M^{-2s}). \tag{2.5}$$

3. The function a_ε and fractional Laplacians of ϕ

Here, we present some preliminaries on a_ε and prove Lemma 1.2.

We will use the following lemma throughout the paper without reference. The proof is a standard computation in polar coordinates.

Lemma 3.1. *There exist $C_1, C_2 > 0$ such that for any $R > 0$,*

$$\int_{\{|z|<R\}} \frac{dz}{|z|^{n+2s-2}} = C_1 R^{2-2s} \quad \text{and} \quad \int_{\{|z|>R\}} \frac{dz}{|z|^{n+2s}} = \frac{C_2}{R^{2s}}.$$

Accordingly, by the regularity of ϕ and d, the integral in (1.6) is well defined for $x \in Q_\rho$:

$$\begin{aligned}
&\int_{\mathbb{R}} \left|\phi\left(\xi + \frac{d(x+\varepsilon z) - d(x)}{\varepsilon}\right) - \phi(\xi + \nabla d(x)\cdot z)\right| \frac{dz}{|z|^{n+2s}} \\
&\le C\left[\int_{\{|z|<1\}} \frac{|d(x+\varepsilon z) - d(x) - \nabla d(x)\cdot(\varepsilon z)|}{\varepsilon} \frac{dz}{|z|^{n+2s}} + \int_{\{|z|>1\}} \frac{dz}{|z|^{n+2s}}\right] \\
&\le C\left[\int_{\{|z|<1\}} \varepsilon |z|^2 \frac{dz}{|z|^{n+2s}} + \int_{\{|z|>1\}} \frac{dz}{|z|^{n+2s}}\right] \le C.
\end{aligned}$$

We will need the next result that allows us to view one-dimensional fractional Laplacians of functions defined over $\mathbb{R}$ equivalently as n-dimensional fractional Laplacians.

Lemma 3.2. *For a vector $e \in \mathbb{R}^n$ and a function $v \in C^{1,1}(\mathbb{R})$, let $v_e(x) = v(e \cdot x) : \mathbb{R}^n \to \mathbb{R}$. Then,*

$$\mathcal{I}_n^s[v_e](x) = |e|^{2s} C_{n,s} \mathcal{I}_1^s[v](e \cdot x),$$

where

$$C_{n,s} = \int_{\mathbb{R}^{n-1}} \frac{1}{(|y|^2 + 1)^{\frac{n+2s}{2}}} dy. \tag{3.1}$$

Consequently,

$$|e|^{2s} C_{n,s} \mathcal{I}_1^s[v](\xi) = \mathrm{P.V.} \int_{\mathbb{R}^n} (v(\xi + e \cdot z) - v(\xi)) \frac{dz}{|z|^{n+2s}}, \quad \xi \in \mathbb{R}.$$

Proof. The case $e = 0$ is trivial. Therefore, let us assume $e \neq 0$ and let $c := |e| > 0$. Begin by writing

$$\mathcal{I}_n^s[v_e](x) = \mathrm{P.V.} \int_{\mathbb{R}^n} (v(e \cdot x + e \cdot z) - v(e \cdot x)) \frac{dz}{|z|^{n+2s}}.$$

By rotation, it is enough to prove the result for $e = ce_1$. Observe for $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$ that

$$\begin{aligned}
\mathcal{I}_n^s[v_{ce_1}](x) &= \mathrm{P.V.} \int_{\mathbb{R}^n} (v(cx_1 + cz_1) - v(cx_1)) \frac{dz}{|z|^{n+2s}} \\
&= \mathrm{P.V.} \int_{\mathbb{R}} (v(cx_1 + cz_1) - v(cx_1)) \left(\int_{\mathbb{R}^{n-1}} \frac{1}{|(z_1, z')|^{n+2s}} dz' \right) dz_1.
\end{aligned}$$

Since

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}} \frac{1}{|(z_1, z')|^{n+2s}} dz' &= \int_{\mathbb{R}^{n-1}} \frac{1}{(|z'|^2 + z_1^2)^{\frac{n+2s}{2}}} dz' \\
&= \frac{1}{|z_1|^{n+2s}} \int_{\mathbb{R}^{n-1}} \frac{1}{(|y|^2 + 1)^{\frac{n+2s}{2}}} |z_1|^{n-1} dy = \frac{C_{n,s}}{|z_1|^{1+2s}},
\end{aligned}$$

we have

$$\begin{aligned}
\mathcal{I}_n^s[v_{ce_1}](x) &= C_{n,s} \mathrm{P.V.} \int_{\mathbb{R}} (v(cx_1 + cz_1) - v(cx_1)) \frac{dz_1}{|z_1|^{1+2s}} \\
&= c^{2s} C_{n,s} \mathrm{P.V.} \int_{\mathbb{R}} (v(cx_1 + z_1) - v(cx_1)) \frac{dz_1}{|z_1|^{1+2s}} \\
&= c^{2s} C_{n,s} \mathcal{I}_1^s[v](ce_1 \cdot x).
\end{aligned}$$

∎

We are now ready to prove Lemma 1.2.

Proof of Lemma 1.2. First, we write $a_\varepsilon = a_\varepsilon(d(x)/\varepsilon; x)$ as

$$\begin{aligned} a_\varepsilon = \text{P. V.} \int_{\mathbb{R}^n} \left(\phi\left(\frac{d(x+\varepsilon z)}{\varepsilon}\right) - \phi\left(\frac{d(x)}{\varepsilon}\right)\right) \frac{dz}{|z|^{n+2s}} \\ - \text{P. V.} \int_{\mathbb{R}^n} \left(\phi\left(\frac{d(x)}{\varepsilon} + \nabla d(x) \cdot z\right) - \phi\left(\frac{d(x)}{\varepsilon}\right)\right) \frac{dz}{|z|^{n+2s}}. \end{aligned} \tag{3.2}$$

Since $e = \nabla d(x)$ is well defined when $|d(x)| \le \rho$ and a unit vector, by applying Lemma 3.2 to the second integral in the right-hand side of (3.2) and a change of variables in the first integral, we obtain

$$\begin{aligned} a_\varepsilon &= \varepsilon^{2s}\, \text{P. V.} \int_{\mathbb{R}^n} \left(\phi\left(\frac{d(x+y)}{\varepsilon}\right) - \phi\left(\frac{d(x)}{\varepsilon}\right)\right) \frac{dy}{|y|^{n+1}} - C_{n,s} \mathcal{I}_1^s[\phi]\left(\frac{d(x)}{\varepsilon}\right) \\ &= \varepsilon^{2s} \mathcal{I}_n^s\left[\phi\left(\frac{d(\cdot)}{\varepsilon}\right)\right](x) - C_{n,s} \mathcal{I}_1^s[\phi]\left(\frac{d(x)}{\varepsilon}\right). \end{aligned}$$

■

4. Preliminaries for $s \in [\frac{1}{2}, 1)$

We give preliminaries for the proof Theorem 1.3 when $s \in [\frac{1}{2}, 1)$.

4.1. Mean curvature

Assume that d in (1.1) satisfies (1.2). Then, in Q_ρ, the eigenvalues of $D^2 d(x)$ are

$$\lambda_i(x) = \frac{-\kappa_i}{1 - \kappa_i d(x)}, \quad i = 1, \dots, n-1, \quad \lambda_n(x) = 0;$$

see, for example, [11, Lemma 14.17]. Moreover, since $|\nabla d| = 1$ in Q_ρ, we have the equation $D^2 d(x) \nabla d(x) = 0$ from which we see that $\nabla d(x)$ is an eigenvector, with norm 1, for $D^2 d(x)$ with associated eigenvalue $\lambda_n(x) = 0$.

Let us denote

$$A(y') := \frac{1}{2} \sum_{i=1}^{n-1} \lambda_i y_i^2, \quad y' = (y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1}.$$

Note, for $x \in Q_\rho$, that

$$\int_{S^{n-2}} A(\theta) d\theta = \frac{1}{2} \sum_{i=1}^{n-1} \lambda_i \int_{S^{n-2}} \theta_i^2 d\theta = \frac{1}{2} \sum_{i=1}^{n-1} \lambda_i \frac{|S^{n-2}|}{n-1} = \frac{1}{2} \frac{|S^{n-2}|}{n-1} \Delta d(x). \tag{4.1}$$

4.2. Change of variables in $\bar{a}_\varepsilon$

In light of (4.1), we write the following lemma for $\bar{a}_\varepsilon$ as an integral in terms of polar coordinates which exposes $A(\theta)$, $\theta \in S^{n-2}$. This will set the stage for proving Theorem 1.3 when $s \in [\frac{1}{2}, 1)$.

Lemma 4.1. *Let $s \in [\frac{1}{2}, 1)$ and $r \in (0, \frac{\rho}{2})$ be fixed. Then, for all $(t, x) \in Q_\rho$ and $0 < \varepsilon < r$,*

$$\bar{a}_\varepsilon(t, x) = O\left(\frac{\varepsilon^{2s}}{r^{2s}\eta_\varepsilon}\right) + \frac{\varepsilon^{2s}}{\eta_\varepsilon}\int_{\mathbb{R}} G(\xi)\dot{\phi}(\xi)d\xi,$$

where

$$G(\xi) = \int_0^r \frac{dp}{p^{2s+1}}\int_{S^{n-2}} d\theta \int_{-\frac{r}{p}}^{\frac{r}{p}} \left(\phi\left(\xi + \frac{p}{\varepsilon}(t + pb(\theta, t, r))\right) - \phi\left(\xi + \frac{tp}{\varepsilon}\right)\right)\frac{dt}{(t^2+1)^{\frac{n+2s}{2}}}$$

with

$$b(\theta, t, r) := A(\theta) + O(r(1 + t^2)). \tag{4.2}$$

Remark 4.2. Notice if $s \in (0, \frac{1}{2})$, then $O(\frac{\varepsilon^{2s}}{r^{2s}\eta_\varepsilon}) = O(\frac{1}{r^{2s}})$ is ineffectual for r small.

Proof. Let $T = (v_1, \dots, v_n)$ be an orthogonal matrix whose columns are a set of orthonormal eigenvectors $v_1, \dots, v_n$ for the eigenvalues $\lambda_1, \dots, \lambda_n$ with $v_n = \nabla d(x)$. We make the change of variables $\varepsilon z = Ty$ to write

$$a_\varepsilon(\xi; x) = \varepsilon^{2s}\int_{\mathbb{R}^n}\left(\phi\left(\xi + \frac{1}{\varepsilon}(d(x + Ty) - d(x))\right) - \phi\left(\xi + \frac{y_n}{\varepsilon}\right)\right)\frac{dy}{|y|^{n+2s}},$$

where $y = (y', y_n)$ with $y' \in \mathbb{R}^{n-1}$. Then, we split a_ε as follows, for $0 < \varepsilon < r$,

$$a_\varepsilon(\xi; x) = \int_{\{|y'|, |y_n| < r\}}(\cdots) + \int_{\{|y'| > r\}\cup\{|y_n| > r\}}(\cdots).$$

Since $0 < \phi < 1$, we have that

$$\begin{aligned}&\varepsilon^{2s}\left|\int_{\{|y'| > r\}\cup\{|y_n| > r\}}\left(\phi\left(\xi + \frac{1}{\varepsilon}(d(x + Ty) - d(x))\right) - \phi\left(\xi + \frac{y_n}{\varepsilon}\right)\right)\frac{dy}{|y|^{n+2s}}\right| \\ &\leq 2\varepsilon^{2s}\int_{\{|y| > r\}}\frac{dy}{|y|^{n+2s}} \leq \frac{C\varepsilon^{2s}}{r^{2s}}.\end{aligned}$$

Therefore,

$$a_\varepsilon(\xi; x) = O\left(\frac{\varepsilon^{2s}}{r^{2s}}\right) + \varepsilon^{2s}G(\xi), \tag{4.3}$$

where

$$G(\xi) = \int_{\{|y'|, |y_n| < r\}}\left(\phi\left(\xi + \frac{1}{\varepsilon}(d(x + Ty) - d(x))\right) - \phi\left(\xi + \frac{y_n}{\varepsilon}\right)\right)\frac{dy}{|y|^{n+2s}}.$$

Now, if $|y_n|, |y'| < r$, then $x + Ty \in Q_{2\rho}$ and by construction of T and the regularity of d, we have

$$\begin{aligned}d(x + Ty) - d(x) &= \nabla d(x)\cdot Ty + \frac{1}{2}D^2 d(x)Ty\cdot Ty + O(r|y|^2) \\ &= y_n + A(y') + O(r|y|^2).\end{aligned}$$

Consequently,

$$G(\xi) = \int_{\{|y'|,|y_n|<r\}} \left(\phi\left(\xi + \frac{1}{\varepsilon}(y_n + A(y') + O(r|y|^2)\right) - \phi\left(\xi + \frac{y_n}{\varepsilon}\right)\right)\frac{dy}{|y|^{n+2s}}.$$

Next, we make the further change of variable $t = y_n/|y'|$ to write

$$\begin{aligned} G(\xi) &= \int_{\{|y'|<r\}} \frac{dy'}{|y'|^{n+2s}} \int_{\{|y_n|<r\}} \frac{dy_n}{\left(\frac{y_n^2}{|y'|^2}+1\right)^{\frac{n+2s}{2}}} \\ &\quad \times \left(\phi\left(\xi + \frac{1}{\varepsilon}(y_n + A(y') + O(r|y|^2))\right) - \phi\left(\xi + \frac{y_n}{\varepsilon}\right)\right) \\ &= \int_{\{|y'|<r\}} \frac{dy'}{|y'|^{n+2s-1}} \int_{\{|t|<\frac{r}{|y'|}\}} \frac{dt}{(t^2+1)^{\frac{n+2s}{2}}} \\ &\quad \times \left(\phi\left(\xi + \frac{|y'|}{\varepsilon}\left(t + |y'|A\left(\frac{y'}{|y'|}\right) + |y'|O(r(1+t^2))\right)\right) - \phi\left(\xi + \frac{t|y'|}{\varepsilon}\right)\right). \end{aligned}$$

Finally, using polar coordinates $y' = p\theta$ with $p > 0$ and $\theta \in S^{n-2}$, we write

$$\begin{aligned} G(\xi) = \int_0^r \frac{dp}{p^{2s+1}} \int_{S^{n-2}} d\theta \int_{\{|t|<\frac{r}{p}\}} \frac{dt}{(t^2+1)^{\frac{n+2s}{2}}} \\ \times \left(\phi\left(\xi + \frac{p}{\varepsilon}(t + pA(\theta) + pO(r(1+t^2)))\right) - \phi\left(\xi + \frac{tp}{\varepsilon}\right)\right). \end{aligned}$$

With (4.3) and recalling that $\int_{\mathbb{R}} \dot\phi(\xi)d\xi = 1$, this completes the proof. ■

5. Proof of Theorem 1.3 for $s = \frac{1}{2}$

Throughout this section, assume that $s = \frac{1}{2}$.

Fix $r \in (0, \frac{\rho}{2})$ and let $0 < \varepsilon < r$. By Lemma 4.1, we have

$$\bar a_\varepsilon(x) = O\left(\frac{1}{r|\ln\varepsilon|}\right) + \frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} G(\xi)\dot\phi(\xi)d\xi, \tag{5.1}$$

where

$$G(\xi) = \int_0^r \frac{dp}{p^2}\int_{S^{n-2}} d\theta \int_{-\frac{r}{p}}^{\frac{r}{p}} \left(\phi\left(\xi + \frac{p}{\varepsilon}(t + pb(\theta,t,r))\right) - \phi\left(\xi + \frac{tp}{\varepsilon}\right)\right)\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}.$$

By the regularity of ϕ and recalling (4.2), we see that

$$\begin{aligned} &\left|\int_0^\varepsilon \frac{dp}{p^2}\int_{S^{n-2}} d\theta \int_{-1}^{1} \left(\phi\left(\xi + \frac{p}{\varepsilon}(t + pb(\theta,t,r))\right) - \phi\left(\xi + \frac{tp}{\varepsilon}\right)\right)\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}\right| \\ &\quad \le \frac{C}{\varepsilon}\int_0^\varepsilon dp \int_{S^{n-2}} d\theta \int_{-1}^{1} |A(\theta) + O(r(1+t^2))| \frac{dt}{(t^2+1)^{\frac{n+1}{2}}} \le \frac{C}{\varepsilon}\int_0^\varepsilon dp = C \end{aligned}$$

and also

$$\begin{aligned}&\left|\int_0^{\varepsilon}\frac{dp}{p^2}\int_{S^{n-2}}d\theta\int_{\{1<|t|<\frac{r}{p}\}}\Big(\phi\Big(\xi+\frac{p}{\varepsilon}(t+pb(\theta,r,t))\Big)-\phi\Big(\xi+\frac{tp}{\varepsilon}\Big)\Big)\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}\right|\\&\quad\le\frac{C}{\varepsilon}\int_0^{\varepsilon}dp\int_{S^{n-2}}d\theta\int_{\{1<|t|<\frac{r}{p}\}}(1+rt^2)\frac{dt}{|t|^{n+1}}\\&\quad\le\frac{C}{\varepsilon}\int_0^{\varepsilon}dp\Big[\int_{\{|t|>1\}}\frac{dt}{|t|^{n+1}}+r\int_{\{1<|t|<\frac{r}{p}\}}\frac{dt}{|t|}\Big]\\&\quad\le\frac{C}{\varepsilon}\int_0^{\varepsilon}(1+r\ln r-r\ln p)dp\le(1+r|\ln\varepsilon|).\end{aligned}$$

Together with (5.1), we have

$$\bar{a}_\varepsilon(x)=O\Big(\frac{1}{r|\ln\varepsilon|}\Big)+O(r)+\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}I(\xi)\dot{\phi}(\xi)d\xi, \tag{5.2}$$

where

$$I(\xi)=\int_{\varepsilon}^{r}\frac{dp}{p^2}\int_{S^{n-2}}d\theta\int_{-\frac{r}{p}}^{\frac{r}{p}}\Big(\phi\Big(\xi+\frac{p}{\varepsilon}(t+pb(\theta,t,r))\Big)-\phi\Big(\xi+\frac{tp}{\varepsilon}\Big)\Big)\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}.$$

Heuristics. We will prove that one of the main contributions comes from values of p between $\varepsilon^{\frac{1}{2}}$ and r and values of t such that, for $A(\theta)>0$,

$$-pA(\theta)<t<0, \tag{5.3}$$

and for $A(\theta)<0$,

$$0<t<-pA(\theta). \tag{5.4}$$

Indeed, by (2.1), if $A(\theta)>0$, for points t as in (5.3), the integrand function in $I(\xi)$ is close to 1, and thus,

$$\begin{aligned}\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}d\xi\dot{\phi}(\xi)\int_{\varepsilon^{\frac{1}{2}}}^{r}\frac{dp}{p^2}\int_{-pA(\theta)}^{0}(\cdots)dt&\simeq\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}d\xi\dot{\phi}(\xi)\int_{\varepsilon^{\frac{1}{2}}}^{r}\frac{dp}{p^2}\int_{-pA(\theta)}^{0}dt\\&=\frac{A(\theta)}{|\ln\varepsilon|}\int_{\mathbb{R}}d\xi\dot{\phi}(\xi)\int_{\varepsilon^{\frac{1}{2}}}^{r}\frac{dp}{p}\simeq\frac{A(\theta)}{2},\end{aligned}$$

see (5.12) below. Similarly, if $A(\theta)<0$, for points t as in (5.4) the integrand function is close to -1 and

$$\frac{1}{|\ln(\varepsilon)|}\int_{\mathbb{R}}d\xi\dot{\phi}(\xi)\int_{\varepsilon^{\frac{1}{2}}}^{r}\frac{dp}{p^2}\int_{0}^{-pA(\theta)}(\cdots)dt\simeq\frac{A(\theta)}{2}.$$

The other main contribution comes from values of p between ε and $\varepsilon^{\frac{1}{2}}$ and values of t between -1 and 1. Indeed, we will show that

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}d\xi\dot{\phi}(\xi)\int_{\varepsilon}^{\varepsilon^{\frac{1}{2}}}\frac{dp}{p^2}\int_{-1}^{1}(\cdots)dt\simeq\frac{A(\theta)}{|\ln\varepsilon|}\int_{\mathbb{R}}d\xi\dot{\phi}(\xi)\int_{\varepsilon}^{\varepsilon^{\frac{1}{2}}}\frac{dp}{p}=\frac{A(\theta)}{2},$$

see (5.20) and (5.21) below.

To formally prove the estimates above, we start by splitting $I(\xi)$ as

$$I(\xi) = \int_\varepsilon^r \frac{dp}{p^2} \int_{S^{n-2}} d\theta \int_{-1}^{1} (\cdots)dt + \int_\varepsilon^r \frac{dp}{p^2} \int_{S^{n-2}} d\theta \int_{\{1<|t|<\frac{r}{p}\}} (\cdots)dt \\ =: I_1(\xi) + I_2(\xi), \tag{5.5}$$

and then estimate $\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} \dot\phi(\xi) I_1(\xi)d\xi$ and $\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} \dot\phi(\xi) I_2(\xi)d\xi$ separately.

Step 1. Estimating $\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} \dot\phi(\xi) I_1(\xi)d\xi$. We will show that

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} \dot\phi(\xi) I_1(\xi)d\xi = \int_{S^{n-2}} A(\theta)d\theta + o_\varepsilon(1) + o_r(1), \tag{5.6}$$

where $o_\varepsilon(1)$ depends on the parameter r.

Note that if $|t| < 1$, then for some $C_0 > 0$,

$$|b(\theta, r, t) - A(\theta)| \le C_0 r. \tag{5.7}$$

Let r and δ be such that

$$C_0 r \le \delta < \frac{1}{2}. \tag{5.8}$$

We write

$$I_1(\xi) = \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta(\cdots) + \int_{S^{n-2}\cap\{A(\theta)<-3\delta\}} d\theta(\cdots) + \int_{S^{n-2}\cap\{|A(\theta)|\le 3\delta\}} d\theta(\cdots) \\ =: I_1^1(\xi) + I_1^2(\xi) + I_1^3(\xi). \tag{5.9}$$

Beginning with $I_1^1(\xi)$, we further split, for $R > 1$ to be chosen,

$$I_1^1(\xi) = \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac{1}{2}}}^{r} dp \int_{-1}^{1} (\cdots)dt \\ + \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} dp \int_{-1}^{1} (\cdots)dt \\ =: J_1(\xi) + J_2(\xi). \tag{5.10}$$

In what follows, we will use several times without reference that, recalling (5.7) and (5.8), if $A(\theta) > 3\delta$ and $|t| < 1$, then $b(\theta, t, r) > 0$ and by the monotonicity of ϕ,

$$\phi\left(\xi + \frac{p}{\varepsilon}(t + pb(\theta, t, r))\right) - \phi\left(\xi + \frac{tp}{\varepsilon}\right) > 0.$$

Step 1a. Estimating $\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} \dot\phi(\xi) J_1(\xi)d\xi$. We will show that

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} \dot\phi(\xi) J_1(\xi)d\xi = \frac{1}{2}\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta)d\theta + o_\varepsilon(1) + O(\delta) + O(r). \tag{5.11}$$

Begin by writing

$$\begin{aligned}
J_1(\xi) &= \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac12}}^{r} \frac{dp}{p^2} \int_{\delta p}^{1} (\cdots)dt \\
&\quad + \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac12}}^{r} \frac{dp}{p^2} \int_{-\delta p}^{\delta p} (\cdots)dt \\
&\quad + \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac12}}^{r} \frac{dp}{p^2} \int_{-p(A(\theta)-2\delta)}^{-\delta p} (\cdots)dt \\
&\quad + \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac12}}^{r} \frac{dp}{p^2} \int_{-p(A(\theta)+2\delta)}^{-p(A(\theta)-2\delta)} (\cdots)dt \\
&\quad + \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac12}}^{r} \frac{dp}{p^2} \int_{-1}^{-p(A(\theta)+2\delta)} (\cdots)dt \\
&=: J_1^1(\xi) + J_1^2(\xi) + J_1^3(\xi) + J_1^4(\xi) + J_1^5(\xi).
\end{aligned}$$

Notice that $p(A(\theta)+2\delta) < 1$ for $p \le r$ and r small enough.

The main contribution in $J_1(\xi)$ comes from $J_1^3(\xi)$. Indeed, we first show that

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} \dot\phi(\xi) J_1^3(\xi)d\xi = \frac12 \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta)d\theta + o_\varepsilon(1) + O(\delta) + O(r). \quad (5.12)$$

Notice that, for $R\delta > 2$ and δ as in (5.8), if

$$\theta \in S^{n-2}\cap\{A(\theta)>3\delta\}, \quad (R\varepsilon)^{\frac12} \le p \le r, \quad -p(A(\theta)-2\delta) \le t \le -p\delta, \quad |\xi| \le \frac{R\delta}{2},$$

then

$$\xi + \frac{tp}{\varepsilon} \le \xi - \frac{\delta p^2}{\varepsilon} \le \xi - R\delta \le -\frac{R\delta}{2}$$

and, recalling (5.7),

$$\xi + \frac{p}{\varepsilon}(t + pb(\theta,t,r)) \ge \xi + \frac{p}{\varepsilon}(t + pA(\theta) - pC_0 r) \ge \xi + \frac{\delta p^2}{\varepsilon} \ge \xi + R\delta \ge \frac{R\delta}{2}.$$

Consequently, by (2.1),

$$\begin{aligned}
&\phi\Big(\xi + \frac{p}{\varepsilon}(t + pb(\theta,t,r))\Big) - \phi\Big(\xi + \frac{tp}{\varepsilon}\Big) \\
&\quad = H\Big(\xi + \frac{p}{\varepsilon}(t + pb(\theta,t,r))\Big) - H\Big(\xi + \frac{tp}{\varepsilon}\Big) \\
&\qquad + O\bigg(\frac{1}{\xi + \frac{p}{\varepsilon}(t + pb(\theta,t,r))}\bigg) + O\bigg(\frac{1}{\xi + \frac{tp}{\varepsilon}}\bigg) = 1 + O\Big(\frac{1}{R\delta}\Big).
\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}} \dot{\phi}(\xi) J_1^3(\xi) d\xi$$

$$= \int_{-\frac{R\delta}{2}}^{\frac{R\delta}{2}} d\xi \dot{\phi}(\xi) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac{1}{2}}}^{r} \frac{dp}{p^2} \int_{-p(A(\theta)-2\delta)}^{-\delta p} \left(1+O\left(\frac{1}{R\delta}\right)\right) \frac{dt}{(t^2+1)^{\frac{n+1}{2}}}$$

$$+ \int_{\{|\xi|>\frac{R\delta}{2}\}} d\xi \dot{\phi}(\xi) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac{1}{2}}}^{r} \frac{dp}{p^2} \int_{-p(A(\theta)-2\delta)}^{-\delta p} (\cdots) dt. \tag{5.13}$$

The main contribution in $\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} \dot{\phi}(\xi) J_1^3(\xi) d\xi$ comes from the integral of 1 in (5.13). Indeed, since $|t| \le p(A(\theta)-2\delta) \le Cr$ implies

$$\frac{1}{(t^2+1)^{\frac{n+1}{2}}} = 1 + O(r),$$

we can write

$$\begin{aligned}
&\frac{1}{|\ln\varepsilon|} \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac{1}{2}}}^{r} \frac{dp}{p^2} \int_{-p(A(\theta)-2\delta)}^{-p\delta} \frac{dt}{(t^2+1)^{\frac{n+1}{2}}} \\
&\quad = (1+O(r)) \frac{1}{|\ln\varepsilon|} \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac{1}{2}}}^{r} \frac{1}{p^2} p(A(\theta)-3\delta) dp \\
&\quad = (1+O(r)) \frac{\ln r - \frac{1}{2}\ln R - \frac{1}{2}\ln\varepsilon}{|\ln\varepsilon|} \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} (A(\theta)-3\delta) d\theta \\
&\quad = \frac{1}{2}\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta) d\theta + o_\varepsilon(1) + O(\delta) + O(r).
\end{aligned}$$

With this and recalling (2.4), we infer that

$$\begin{aligned}
&\frac{1}{|\ln\varepsilon|} \int_{-\frac{R\delta}{2}}^{\frac{R\delta}{2}} d\xi \dot{\phi}(\xi) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac{1}{2}}}^{r} \frac{dp}{p^2} \int_{-p(A(\theta)-2\delta)}^{-p\delta} \frac{dt}{(t^2+1)^{\frac{n+1}{2}}} \\
&\quad = \frac{1}{2}\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta) d\theta + o_\varepsilon(1) + O(\delta) + O(r) + O\left(\frac{1}{R\delta}\right).
\end{aligned} \tag{5.14}$$

Next, we look at the error terms in (5.13). First, note that

$$\begin{aligned}
&\frac{1}{|\ln\varepsilon|} \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac{1}{2}}}^{r} \frac{dp}{p^2} \int_{-p(A(\theta)-2\delta)}^{-\delta p} \frac{dt}{(t^2+1)^{\frac{n+1}{2}}} \\
&\quad \le \frac{1}{|\ln\varepsilon|} \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac{1}{2}}}^{r} \frac{1}{p^2} p(A(\theta)-3\delta) dp \\
&\quad \le \frac{1}{|\ln\varepsilon|} \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac{1}{2}}}^{r} \frac{dp}{p} \le C \frac{\ln r - \frac{1}{2}\ln R - \frac{1}{2}\ln\varepsilon}{|\ln\varepsilon|} \le C.
\end{aligned}$$

With this, we estimate

$$\left|\frac{1}{|\ln\varepsilon|}\int_{-\frac{R\delta}{2}}^{\frac{R\delta}{2}} d\xi\dot{\phi}(\xi)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{(R\varepsilon)^{\frac{1}{2}}}^{r}\frac{dp}{p^2}\int_{-p(A(\theta)-2\delta)}^{-\delta p} O\left(\frac{1}{R\delta}\right)\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}\right|$$
$$\leq O\left(\frac{1}{R\delta}\right), \tag{5.15}$$

and similarly, using that

$$0<\phi<1$$

and using (2.5),

$$\begin{aligned}
0 &\leq \frac{1}{|\ln\varepsilon|}\int_{\{|\xi|>\frac{R\delta}{2}\}} d\xi\dot{\phi}(\xi)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \\
&\quad\times\int_{(R\varepsilon)^{\frac{1}{2}}}^{r}\frac{dp}{p^2}\int_{-p(A(\theta)-2\delta)}^{-\delta p}(\cdots)dt \\
&\leq \frac{2}{|\ln\varepsilon|}\int_{\{|\xi|>\frac{R\delta}{2}\}} d\xi\dot{\phi}(\xi)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \\
&\quad\times\int_{(R\varepsilon)^{\frac{1}{2}}}^{r}\frac{dp}{p^2}\int_{-p(A(\theta)-2\delta)}^{-\delta p}\frac{dt}{(t^2+1)^{\frac{n+1}{2}}} \\
&\leq O\left(\frac{1}{R\delta}\right).
\end{aligned}\tag{5.16}$$

Choosing

$$R=\delta^{-2} \tag{5.17}$$

from (5.13), (5.14), (5.15), and (5.16), estimate (5.12) follows.

We next show that

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot{\phi}(\xi)J_1^k(\xi)d\xi = O(\delta) \quad \text{for } k=1,2,4,5. \tag{5.18}$$

First, using that $0<\phi<1$, we get

$$\begin{aligned}
0 &\leq \int_{\mathbb{R}}\dot{\phi}(\xi)J_1^2(\xi)d\xi \leq 2\int_{\mathbb{R}} d\xi\dot{\phi}(\xi)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{(R\varepsilon)^{\frac{1}{2}}}^{r}\frac{dp}{p^2}\int_{-\delta p}^{\delta p} dt \\
&\leq 2\int_{\mathbb{R}} d\xi\dot{\phi}(\xi)\int_{(R\varepsilon)^{\frac{1}{2}}}^{r}\frac{dp}{p^2}2\delta p \\
&\leq C\delta|\ln\varepsilon|,
\end{aligned}$$

from which it follows that (5.18) holds for $k=2$. The estimate for $k=4$ is similar.

Regarding (5.18) for $k=5$, we use (5.7) and the monotonicity of ϕ to estimate

$$\begin{aligned}
0 &\le \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac12}}^{r} \frac{dp}{p^2} \int_{-1}^{-p(A(\theta)+2\delta)} (\cdots)dt \\
&\le \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac12}}^{r} \frac{dp}{p^2} \\
&\quad\times \int_{-1}^{-p(A(\theta)+2\delta)} \left\{\phi\left(\xi+\frac{tp}{\varepsilon}+\frac{p^2}{\varepsilon}(A(\theta)+C_0 r)\right)-\phi\left(\xi+\frac{tp}{\varepsilon}\right)\right\}dt \\
&= \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac12}}^{r} \frac{dp}{p^2} \int_{-1}^{-p(A(\theta)+2\delta)} dt \\
&\quad\times \int_0^1 \dot\phi\left(\xi+\frac{tp}{\varepsilon}+\tau\frac{p^2}{\varepsilon}(A(\theta)+C_0 r)\right)\frac{p^2(A(\theta)+C_0 r)}{\varepsilon}d\tau \\
&\le \frac{C}{\varepsilon}\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac12}}^{r} dp \int_0^1 d\tau \\
&\quad\times \int_{-1}^{-p(A(\theta)+2\delta)} \partial_t\left[\phi\left(\xi+\frac{tp}{\varepsilon}+\tau\frac{p^2}{\varepsilon}(A(\theta)+C_0 r)\right)\right]\frac{\varepsilon}{p}dt \\
&= C\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac12}}^{r} \frac{dp}{p} \int_0^1 \left\{\phi\left(\xi+\frac{p^2}{\varepsilon}[-A(\theta)-2\delta+\tau(A(\theta)+C_0 r)]\right)\right. \\
&\qquad\qquad \left. -\phi\left(\xi-\frac{p}{\varepsilon}+\tau\frac{p^2}{\varepsilon}(A(\theta)+C_0 r)\right)\right\}d\tau.
\end{aligned}$$

Now, if

$$A(\theta)\ge 0,\quad (R\varepsilon)^{\frac12}\le p\le r,\quad 0\le\tau\le 1,\quad |\xi|\le\frac{R\delta}{2},$$

and δ as in (5.8), then

$$\xi+\frac{p^2}{\varepsilon}[-A(\theta)-2\delta+\tau(A(\theta)+C_0 r)]\le \xi-\frac{\delta p^2}{\varepsilon}\le\frac{R\delta}{2}-R\delta=-\frac{R\delta}{2},$$

and for r and ε sufficiently small,

$$\xi-\frac{p}{\varepsilon}+\tau\frac{p^2}{\varepsilon}(A(\theta)+C_0 r)\le \xi-\frac{p}{2\varepsilon}\le-\frac{R\delta}{2}.$$

Recalling (5.17), by (2.1), we get

$$\phi\left(\xi+\frac{p^2}{\varepsilon}[-A(\theta)-2\delta+\tau(A(\theta)+C_0 r)]\right),\phi\left(\xi-\frac{p}{\varepsilon}+\tau\frac{p^2}{\varepsilon}(A(\theta)+C_0 r)\right)\le C\delta.$$

The computations above yield

$$0\le\int_{-\frac{R\delta}{2}}^{\frac{R\delta}{2}} d\xi\dot\phi(\xi)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{(R\varepsilon)^{\frac12}}^{r}\frac{dp}{p^2}\int_{-1}^{-p(A(\theta)+2\delta)}(\cdots)dt\le C\delta|\ln\varepsilon|.$$

On the other hand, estimating as above and by (2.5) and (5.17),

$$
\begin{aligned}
0 &\le \int_{\{|\xi|>\frac{R\delta}{2}\}} d\xi \dot{\phi}(\xi) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac{1}{2}}}^{r} \frac{dp}{p^2} \int_{-1}^{-p(A(\theta)+2\delta)} (\cdots) dt \\
&\le C \int_{\{|\xi|>\frac{R\delta}{2}\}} d\xi \dot{\phi}(\xi) \int_{S^{n-2}} d\theta \int_{(R\varepsilon)^{\frac{1}{2}}}^{r} \frac{dp}{p} \int_0^1 d\tau \le C\delta |\ln\varepsilon|.
\end{aligned}
$$

Together, we arrive at (5.18) for $k = 5$. Similar computations yield (5.18) for $k = 1$.

Combining (5.12) and (5.18), we obtain (5.11).

Step 1b. Estimating $\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} \dot{\phi}(\xi) J_2(\xi) d\xi$. We will show that

$$
\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} \dot{\phi}(\xi) J_2(\xi) d\xi = \frac{1}{2}\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta) d\theta + o_\varepsilon(1) + O(\delta) + O(r). \quad (5.19)
$$

We first write

$$
\begin{aligned}
J_2(\xi) &= \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p^2} \int_{\delta\frac{\varepsilon}{p}}^{1} (\cdots) dt \\
&\quad + \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p^2} \int_{-\delta\frac{\varepsilon}{p}}^{\delta\frac{\varepsilon}{p}} (\cdots) dt \\
&\quad + \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p^2} \int_{-1}^{-\delta\frac{\varepsilon}{p}} (\cdots) dt \\
&=: J_2^1(\xi) + J_2^2(\xi) + J_2^3(\xi).
\end{aligned}
$$

Here, the main contribution comes from J_2^1 and J_2^3. Indeed, we will show that

$$
\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} \dot{\phi}(\xi) J_2^1(\xi) d\xi = \frac{1}{4}\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta) d\theta + o_\varepsilon(1) + O(r) + O(\delta) \quad (5.20)
$$

and

$$
\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} \dot{\phi}(\xi) J_2^3(\xi) d\xi = \frac{1}{4}\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta) d\theta + o_\varepsilon(1) + O(r) + O(\delta). \quad (5.21)
$$

Beginning with J_2^3, we split, for $R_0 \ge K > 4$ to be chosen,

$$
\begin{aligned}
J_2^3(\xi) &= \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{\varepsilon}^{R_0\varepsilon} \frac{dp}{p^2} \int_{-1}^{-\delta\frac{\varepsilon}{p}} (\cdots) dt \\
&\quad + \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p^2} \int_{-1}^{-K\frac{\varepsilon}{p}} (\cdots) dt \\
&\quad + \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p^2} \int_{-K\frac{\varepsilon}{p}}^{-\delta\frac{\varepsilon}{p}} (\cdots) dt. \quad (5.22)
\end{aligned}
$$

Regarding the bounds of integration over t, note that $K\frac{\varepsilon}{p} \le 1$ if $K \le R_0$ and $p \ge R_0\varepsilon$. For the first integral on the right-hand side of (5.22), we estimate

$$\begin{aligned}
0 \le & \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{\varepsilon}^{R_0\varepsilon} \frac{dp}{p^2} \\
& \times \int_{-1}^{-\delta\frac{\varepsilon}{p}} \left\{\phi\left(\xi + \frac{p}{\varepsilon}(t + pb(\theta,t,r))\right) - \phi\left(\xi + \frac{tp}{\varepsilon}\right)\right\} \frac{dt}{(t^2+1)^{\frac{n+1}{2}}} \\
\le & \, C \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{\varepsilon}^{R_0\varepsilon} \frac{dp}{p^2} \int_{-1}^{-\delta\frac{\varepsilon}{p}} \frac{p^2}{\varepsilon} b(\theta,t,r)dt \le \frac{C}{\varepsilon}\int_{\varepsilon}^{R_0\varepsilon} dp \le CR_0.
\end{aligned}$$

It follows that

$$0 \le \frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} d\xi\dot{\phi}(\xi) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{\varepsilon}^{R_0\varepsilon} \frac{dp}{p^2}\int_{-1}^{-\delta\frac{\varepsilon}{p}} (\cdots)dt \le \frac{CR_0}{|\ln\varepsilon|}. \tag{5.23}$$

For the second integral in (5.22), by (5.7) and the monotonicity of ϕ, we have

$$\begin{aligned}
0 \le & \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p^2} \\
& \times \int_{-1}^{-K\frac{\varepsilon}{p}} \left\{\phi\left(\xi + \frac{p}{\varepsilon}(t + pb(\theta,t,r))\right) - \phi\left(\xi + \frac{tp}{\varepsilon}\right)\right\} \frac{dt}{(t^2+1)^{\frac{n+1}{2}}} \\
\le & \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p^2} \\
& \times \int_{-1}^{-K\frac{\varepsilon}{p}} \left\{\phi\left(\xi + \frac{tp}{\varepsilon} + \frac{p^2}{\varepsilon}(A(\theta) + C_0 r)\right) - \phi\left(\xi + \frac{tp}{\varepsilon}\right)\right\} \frac{dt}{(t^2+1)^{\frac{n+1}{2}}} \\
\le & \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p^2} \int_{-1}^{-K\frac{\varepsilon}{p}} dt \\
& \times \int_0^1 \dot{\phi}\left(\xi + \frac{tp}{\varepsilon} + \tau\frac{p^2}{\varepsilon}(A(\theta) + C_0 r)\right)\frac{p^2}{\varepsilon}(A(\theta) + C_0 r)d\tau \\
\le & \frac{C}{\varepsilon}\int_{S^{n-2}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} dp \int_0^1 d\tau \int_{-1}^{-K\frac{\varepsilon}{p}} \partial_t\left[\phi\left(\xi + \frac{pt}{\varepsilon} + \tau\frac{p^2}{\varepsilon}(A(\theta) + C_0 r)\right)\right]\frac{\varepsilon}{p}dt \\
= & \, C\int_{S^{n-2}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p} \\
& \times \int_0^1 \left\{\phi\left(\xi - K + \tau\frac{p^2}{\varepsilon}(A(\theta) + C_0 r)\right) - \phi\left(\xi - \frac{p}{\varepsilon} + \tau\frac{p^2}{\varepsilon}(A(\theta) + C_0 r)\right)\right\} d\tau.
\end{aligned}$$

If

$$R_0\varepsilon \le p \le (R\varepsilon)^{\frac{1}{2}}, \quad 0 \le \tau \le 1, \quad |\xi| \le \frac{K}{2},$$

then we have that, for some $\widetilde{C} \geq 1$,

$$\xi - K + \tau \frac{p^2}{\varepsilon}(A(\theta) + C_0 r) \leq -\frac{K}{2} + \widetilde{C} R,$$
$$\xi - \frac{p}{\varepsilon} + \tau \frac{p^2}{\varepsilon}(A(\theta) + C_0 r) \leq \frac{K}{2} - R_0 + \widetilde{C} R.$$

Let R_0 and K be such that

$$4 \leq 4\widetilde{C} R = K \leq R_0. \tag{5.24}$$

Then,

$$\xi - K + \tau \frac{p^2}{\varepsilon}(A(\theta) + C_0 r), \xi - \frac{p}{\varepsilon} + \tau \frac{p^2}{\varepsilon}(A(\theta) + C_0 r) \leq -\frac{K}{4},$$

and by (2.1),

$$\phi\left(\xi - K + \tau \frac{p^2}{\varepsilon}(A(\theta) + C_0 r)\right), \phi\left(\xi - \frac{p}{\varepsilon} + \tau \frac{p^2}{\varepsilon}(A(\theta) + C_0 r)\right) \leq \frac{C}{K}.$$

This implies that, for $|\xi| \leq K/2$,

$$\begin{aligned}\int_{S^{n-2}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p} \int_0^1 \Bigg\{\phi\left(\xi - K \frac{p^2}{\varepsilon}(A(\theta) + C_0 r)\right) \\ - \phi\left(\xi - \frac{p}{\varepsilon} + \frac{p^2}{\varepsilon}(A(\theta) + C_0 r)\right)\Bigg\} d\tau \\ \leq \frac{C}{K} \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p} \leq \frac{C|\ln \varepsilon|}{K}.\end{aligned}$$

The computations above yield

$$0 \leq \int_{-\frac{K}{2}}^{\frac{K}{2}} d\xi \dot{\phi}(\xi) \int_{S^{n-2} \cap \{A(\theta) > 3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p^2} \int_{-1}^{-K\frac{\varepsilon}{p}} (\cdots) dt \leq \frac{C|\ln \varepsilon|}{K}.$$

On the other hand, estimating as above but using that $0 < \phi < 1$ and (2.5), we obtain

$$\begin{aligned}0 \leq \int_{\{|\xi| > \frac{K}{2}\}} d\xi \dot{\phi}(\xi) \int_{S^{n-2} \cap \{A(\theta) > 3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p^2} \int_{-1}^{-K\frac{\varepsilon}{p}} (\cdots) dt \\ \leq C \int_{\{|\xi| > \frac{K}{2}\}} d\xi \dot{\phi}(\xi) \int_{S^{n-2}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p} \int_0^1 d\tau \leq \frac{C|\ln \varepsilon|}{K}.\end{aligned}$$

We conclude that

$$0 \leq \frac{1}{|\ln \varepsilon|} \int_{\mathbb{R}} d\xi \dot{\phi}(\xi) \int_{S^{n-2} \cap \{A(\theta) > 3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p^2} \int_{-1}^{-K\frac{p}{\varepsilon}} (\cdots) dt \leq \frac{C}{K}. \tag{5.25}$$

We next estimate the third term on the right-hand side of (5.22). We first notice that, if $|t| \le K\frac{\varepsilon}{p} \le 1$ and $p \ge R_0\varepsilon$, then

$$\frac{1}{(t^2+1)^{\frac{n+1}{2}}} = 1 + O\left(\frac{K}{R_0}\right).$$

We set

$$R_0 = K^2; \tag{5.26}$$

then,

$$\begin{aligned}
&\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p^2} \\
&\quad\times \int_{-K\frac{\varepsilon}{p}}^{-\delta\frac{\varepsilon}{p}} \left\{\phi\left(\xi + \frac{p}{\varepsilon}(t + pb(\theta,t,r))\right) - \phi\left(\xi + \frac{tp}{\varepsilon}\right)\right\} \frac{dt}{(t^2+1)^{\frac{n+1}{2}}} \\
&= \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p^2} \int_{-K\frac{\varepsilon}{p}}^{-\delta\frac{\varepsilon}{p}} (\cdots)(1 + O(K^{-1}))dt.
\end{aligned}$$

Using again (5.7) and the monotonicity of ϕ, we get

$$\begin{aligned}
&\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p^2} \int_{-K\frac{\varepsilon}{p}}^{-\delta\frac{\varepsilon}{p}} \left\{\phi\left(\xi + \frac{p}{\varepsilon}(t + pb(\theta,t,r))\right) - \phi\left(\xi + \frac{tp}{\varepsilon}\right)\right\} dt \\
&\le \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p^2} \\
&\quad\times \int_{-K\frac{\varepsilon}{p}}^{-\delta\frac{\varepsilon}{p}} \left\{\phi\left(\xi + \frac{tp}{\varepsilon} + \frac{p^2}{\varepsilon}(A(\theta) + C_0 r)\right) - \phi\left(\xi + \frac{tp}{\varepsilon}\right)\right\} dt \\
&= \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p^2} \int_{-K\frac{\varepsilon}{p}}^{-\delta\frac{\varepsilon}{p}} dt \\
&\quad\times \int_0^1 \dot\phi\left(\xi + \frac{tp}{\varepsilon} + \tau\frac{p^2}{\varepsilon}(A(\theta) + C_0 r)\right) \frac{p^2}{\varepsilon}(A(\theta) + C_0 r) d\tau \\
&= \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p} \int_0^1 d\tau \\
&\quad\times \int_{-K\frac{\varepsilon}{p}}^{-\delta\frac{\varepsilon}{p}} \partial_t\left[\phi\left(\xi + \frac{tp}{\varepsilon} + \tau\frac{p^2}{\varepsilon}(A(\theta) + C_0 r)\right)\right](A(\theta) + C_0 r) dt \\
&= \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p} \int_0^1 d\tau \\
&\quad\times \int_{-K\frac{\varepsilon}{p}}^{-\delta\frac{\varepsilon}{p}} \partial_t\left[\phi\left(\xi + \frac{tp}{\varepsilon} + \tau\frac{p^2}{\varepsilon}(A(\theta) + C_0 r)\right)\right] A(\theta) dt + O(r|\ln\varepsilon|).
\end{aligned}$$

Similarly,

$$\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p^2} \int_{-K\frac{\varepsilon}{p}}^{-\delta\frac{\varepsilon}{p}} \left\{\phi\Big(\xi + \frac{p}{\varepsilon}(t + pb(\theta,t,r))\Big) - \phi\Big(\xi + \frac{tp}{\varepsilon}\Big)\right\} dt$$

$$\geq \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p} \int_0^1 d\tau$$

$$\times \int_{-K\frac{\varepsilon}{p}}^{-\delta\frac{\varepsilon}{p}} \partial_t\left[\phi\Big(\xi + \frac{tp}{\varepsilon} + \tau\frac{p^2}{\varepsilon}(A(\theta) - C_0 r)\Big)\right] A(\theta)dt + O(r|\ln\varepsilon|).$$

For the main terms in the integrals above, we write

$$\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta A(\theta) \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p} \int_0^1 d\tau$$

$$\times \int_{-K\frac{\varepsilon}{p}}^{-\delta\frac{\varepsilon}{p}} \partial_t\left[\phi\Big(\xi + \frac{tp}{\varepsilon} + \tau\frac{p^2}{\varepsilon}(A(\theta) \pm C_0 r)\Big)\right] dt$$

$$= \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta A(\theta) \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p}$$

$$\times \int_0^1 \left\{\phi\Big(\xi - \delta + \tau\frac{p^2}{\varepsilon}(A(\theta) \pm C_0 r)\Big) - \phi\Big(\xi - K + \tau\frac{p^2}{\varepsilon}(A(\theta) \pm C_0 r)\Big)\right\} d\tau$$

$$= \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta A(\theta) \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \left\{\phi(\xi - \delta) - \phi(\xi - K) + O\Big(\frac{p^2}{\varepsilon}\Big)\right\} \frac{dp}{p}.$$

As above, we find that

$$\int_{\mathbb{R}} d\xi \dot{\phi}(\xi) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta A(\theta) \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \phi(\xi - K)\frac{dp}{p} \leq \frac{C|\ln\varepsilon|}{K}.$$

Moreover, regarding the error term,

$$\int_{\mathbb{R}} d\xi \dot{\phi}(\xi) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta A(\theta) \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p}\frac{p^2}{\varepsilon} \leq CR.$$

The main contribution comes from the following integral:

$$\int_{\mathbb{R}} d\xi \dot{\phi}(\xi) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta A(\theta) \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \phi(\xi - \delta)\frac{dp}{p}$$

$$= \int_{\mathbb{R}} d\xi \dot{\phi}(\xi) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta A(\theta) \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \phi(\xi)\frac{dp}{p} + O(\delta|\ln\varepsilon|)$$

$$= \int_{\mathbb{R}} \frac{1}{2}\frac{d}{d\xi}(\phi^2(\xi))d\xi \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta A(\theta) \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p} + O(\delta|\ln\varepsilon|)$$
$$= \frac{1}{2}\left(\frac{1}{2}|\ln\varepsilon| + \frac{1}{2}\ln R - \ln R_0\right)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta)d\theta + O(\delta|\ln\varepsilon|),$$

where we used that $\phi(\infty) = 1$ and $\phi(-\infty) = 0$. Putting it all together, we get

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} d\xi\dot{\phi}(\xi)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p^2}\int_{-K\frac{\varepsilon}{p}}^{-\delta\frac{\varepsilon}{p}}(\cdots)dt$$
$$= \frac{1}{4}\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta)d\theta + o_\varepsilon(1) + O(r) + O(K^{-1}) + O(\delta). \tag{5.27}$$

Recalling (5.17), (5.24), and (5.26), from (5.23), (5.25), and (5.27), we get (5.21).

We now check that the estimate for J_2^1 in (5.20) holds. For R_0, K as in (5.24) and (5.26), we write

$$J_2^1(\xi) = \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{\varepsilon}^{R_0\varepsilon} \frac{dp}{p^2}\int_{\delta\frac{\varepsilon}{p}}^{1}(\cdots)dt$$
$$+ \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p^2}\int_{\delta\frac{\varepsilon}{p}}^{K\frac{\varepsilon}{p}}(\cdots)dt$$
$$+ \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p^2}\int_{K\frac{\varepsilon}{p}}^{1}(\cdots)dt. \tag{5.28}$$

Similar computations as for the estimates (5.23) and (5.25) yield

$$0 \le \frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} d\xi\dot{\phi}(\xi)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{\varepsilon}^{R_0\varepsilon} \frac{dp}{p^2}\int_{\delta\frac{\varepsilon}{p}}^{1}(\cdots)dt \le \frac{CR_0}{|\ln\varepsilon|} \tag{5.29}$$

and

$$0 \le \frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} d\xi\dot{\phi}(\xi)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p^2}\int_{K\frac{p}{\varepsilon}}^{1}(\cdots)dt \le \frac{C}{K}. \tag{5.30}$$

The second term on the right-hand side of (5.28) is similar to (5.27). Indeed, as above,

$$\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p^2}\int_{\delta\frac{\varepsilon}{p}}^{K\frac{\varepsilon}{p}}(\cdots)dt$$
$$= \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta A(\theta)\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \{\phi(\xi+K) - \phi(\xi+\delta)\}\frac{dp}{p}$$
$$+ O(R) + O(K^{-1}|\ln\varepsilon|) + O(r|\ln\varepsilon|). \tag{5.31}$$

By (2.1), for $|\xi| \le \frac{K}{2}$,

$$\phi(\xi + K) = 1 + O\left(\frac{1}{K}\right).$$

Therefore,

$$\begin{aligned}
&\int_{-\frac{K}{2}}^{\frac{K}{2}} d\xi \dot{\phi}(\xi) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p^2} \int_{\delta\frac{\varepsilon}{p}}^{K\frac{\varepsilon}{p}} (\cdots)dt \\
&= \int_{-\frac{K}{2}}^{\frac{K}{2}} d\xi \dot{\phi}(\xi) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta A(\theta) \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \{1 - \phi(\xi) + O(K^{-1}) + O(\delta)\} \frac{dp}{p} \\
&\quad + O(R) + O(K^{-1}|\ln\varepsilon|)) + O(r|\ln\varepsilon|) \\
&= \int_{-\frac{K}{2}}^{\frac{K}{2}} d\xi \{\dot{\phi}(\xi) - \frac{1}{2}\frac{d}{d\xi}(\phi(\xi))^2\} \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta A(\theta) \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p} \\
&\quad + O(K^{-1}|\ln\varepsilon|) + O(\delta|\ln\varepsilon|) + O(R) + O(r|\ln\varepsilon|) \\
&= \left[\phi\left(\frac{K}{2}\right) - \phi\left(-\frac{K}{2}\right) - \frac{1}{2}\left(\phi^2\left(\frac{K}{2}\right) - \phi^2\left(-\frac{K}{2}\right)\right)\right] \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta A(\theta) \\
&\quad \times \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p} + O(K^{-1}|\ln\varepsilon|) + O(\delta|\ln\varepsilon|) + O(R) + O(r|\ln\varepsilon|).
\end{aligned}$$

Since, again by (2.1),

$$\phi\left(\frac{K}{2}\right) - \phi\left(-\frac{K}{2}\right) - \frac{1}{2}\left(\phi^2\left(\frac{K}{2}\right) - \phi^2\left(-\frac{K}{2}\right)\right) = \frac{1}{2} + O(K^{-1}),$$

we get

$$\begin{aligned}
&\int_{-\frac{K}{2}}^{\frac{K}{2}} d\xi \dot{\phi}(\xi) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p^2} \int_{\delta\frac{\varepsilon}{p}}^{K\frac{\varepsilon}{p}} (\cdots)dt \\
&= \left(\frac{1}{2} + O(K^{-1})\right) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta A(\theta) \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p} \\
&\quad + O(K^{-1}|\ln\varepsilon|) + O(\delta|\ln\varepsilon|) + O(R) + O(r|\ln\varepsilon|) \\
&= \frac{1}{2} \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta A(\theta) \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{dp}{p} \\
&\quad + O(K^{-1}|\ln\varepsilon|) + O(\delta|\ln\varepsilon|) + O(R) + O(r|\ln\varepsilon|) \\
&= \frac{1}{2}\left(\frac{1}{2}|\ln\varepsilon| + \frac{1}{2}\ln R - \ln R_0\right) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta) d\theta \\
&\quad + O(K^{-1}|\ln\varepsilon|) + O(\delta|\ln\varepsilon|) + O(R) + O(r|\ln\varepsilon|).
\end{aligned}$$

On the other hand, by (2.5) and (5.31), we get

$$\int_{\{|\xi|>\frac{K}{2}\}} d\xi\dot\phi(\xi)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{dp}{p^2}\int_{\delta\frac{\varepsilon}{p}}^{K\frac{\varepsilon}{p}}(\cdots)dt$$
$$= \int_{\{|\xi|>\frac{K}{2}\}} d\xi\dot\phi(\xi)\Big[\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta A(\theta)\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\{\phi(\xi+K)-\phi(\xi+\delta)\}\frac{dp}{p}$$
$$+O(R)+O(K^{-1}|\ln\varepsilon|)+O(r|\ln\varepsilon|)\Big]$$
$$\le C|\ln\varepsilon|\int_{\{|\xi|>\frac{K}{2}\}}\dot\phi(\xi)d\xi = O(K^{-1}|\ln\varepsilon|).$$

The previous two estimates give

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} d\xi\dot\phi(\xi)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{dp}{p^2}\int_{\delta\frac{\varepsilon}{p}}^{K\frac{\varepsilon}{p}}(\cdots)dt$$
$$=\frac14\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta)d\theta+o_\varepsilon(1)+O(r)+O(K^{-1})+O(\delta). \tag{5.32}$$

From (5.28), (5.29), (5.30), (5.32), and recalling (5.17), (5.24), and (5.26), we get (5.20).

Lastly, we will show that

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot\phi(\xi)J_2^2(\xi)d\xi = O(\delta). \tag{5.33}$$

Indeed,

$$0\le J_2^2(\xi)=\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{dp}{p^2}$$
$$\times\int_{-\delta\frac{\varepsilon}{p}}^{\delta\frac{\varepsilon}{p}}\left\{\phi\left(\xi+\frac{p}{\varepsilon}(t+pb(\theta,t,r))\right)-\phi\left(\xi+\frac{tp}{\varepsilon}\right)\right\}\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}$$
$$\le C\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{dp}{p^2}\int_{-\delta\frac{\varepsilon}{p}}^{\delta\frac{\varepsilon}{p}}\frac{p^2}{\varepsilon}b(\theta,r,t)dt$$
$$\le\frac{C}{\varepsilon}\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{\varepsilon}^{(R\varepsilon)^{\frac12}}\delta\frac{\varepsilon}{p}dp\le C\delta|\ln\varepsilon|,$$

from which we obtain (5.33).

Therefore, combining (5.20), (5.21), and (5.33), we have (5.19).

Completion of Step 1. Recall (5.9) and (5.10). With (5.11) and (5.19), we can finally write

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot\phi(\xi)I_1^1(\xi)d\xi=\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta)d\theta+o_\varepsilon(1)+O(\delta)+O(r). \tag{5.34}$$

In the same way, we obtain

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot\phi(\xi)I_1^2(\xi)d\xi \\ =\int_{S^{n-2}\cap\{A(\theta)<-3\delta\}}A(\theta)d\theta+o_\varepsilon(1)+O(\delta)+O(r). \tag{5.35}$$

Finally, let us show that

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot\phi(\xi)I_1^3(\xi)d\xi=o_\varepsilon(1)+o_\delta(1)+O(r). \tag{5.36}$$

If one of the eigenvalues λ_i is different than zero, then $\mathcal{H}^{n-2}(\{\theta\in S^{n-2}|A(\theta)=0\})=0$. In particular, $\mathcal{H}^{n-2}(\{\theta\in S^{n-2}||A(\theta)|<3\delta\})=o_\delta(1)$. Therefore, integrating in t as before,

$$\begin{aligned}\left|\int_{\mathbb{R}}\dot\phi(\xi)I_1^3(\xi)d\xi\right|&\le C\int_{\mathbb{R}}d\xi\dot\phi(\xi)\int_{S^{n-2}\cap\{|A(\theta)|<3\delta\}}d\theta\int_\varepsilon^r\frac{dp}{p}\\&\le C|\ln\varepsilon|\int_{S^{n-2}\cap\{|A(\theta)|<3\delta\}}d\theta=|\ln\varepsilon|o_\delta(1),\end{aligned}$$

which implies (5.36).

If instead, $\lambda_i=0$ for all $i=1,\dots,n-1$, then $A(\theta)\equiv0$ and $S^{n-2}\cap\{|A(\theta)|<3\delta\}=S^{n-2}$. In this case, we write, for δ and R as in (5.8) and (5.17),

$$\begin{aligned}I_1^3(\xi)&=\int_{S^{n-2}}d\theta\int_{(R\varepsilon)^{\frac12}}^r\frac{dp}{p^2}\int_{-1}^{-2\delta p}(\cdots)dt+\int_{S^{n-2}}d\theta\int_{(R\varepsilon)^{\frac12}}^r\frac{dp}{p^2}\int_{-2\delta p}^{2\delta p}(\cdots)dt\\&+\int_{S^{n-2}}d\theta\int_{(R\varepsilon)^{\frac12}}^r\frac{dp}{p^2}\int_{2\delta p}^{1}(\cdots)dt+\int_{S^{n-2}}d\theta\int_\varepsilon^{(R\varepsilon)^{\frac12}}\frac{dp}{p^2}\int_{-1}^{-\delta\frac{\varepsilon}{p}}(\cdots)dt\\&+\int_{S^{n-2}}d\theta\int_\varepsilon^{(R\varepsilon)^{\frac12}}\frac{dp}{p^2}\int_{-\delta\frac{\varepsilon}{p}}^{\delta\frac{\varepsilon}{p}}(\cdots)dt+\int_{S^{n-2}}d\theta\int_\varepsilon^{(R\varepsilon)^{\frac12}}\frac{dp}{p^2}\int_{\delta\frac{\varepsilon}{p}}^{1}(\cdots)dt.\end{aligned}$$

As for the estimates of $\int_{\mathbb{R}}\dot\phi(\xi)(J_1^1(\xi)+J_1^2(\xi)+J_1^5(\xi))d\xi$ (recall (5.18)),

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}d\xi\dot\phi(\xi)\int_{S^{n-2}}d\theta\int_{(R\varepsilon)^{\frac12}}^r\frac{dp}{p^2}\int_{-1}^{-2\delta p}(\cdots)dt=O(\delta),$$

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}d\xi\dot\phi(\xi)\int_{S^{n-2}}d\theta\int_{(R\varepsilon)^{\frac12}}^r\frac{dp}{p^2}\int_{-2\delta p}^{2\delta p}(\cdots)dt=O(\delta)$$

and

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}d\xi\dot\phi(\xi)\int_{S^{n-2}}d\theta\int_{(R\varepsilon)^{\frac12}}^r\frac{dp}{p^2}\int_{2\delta p}^{1}(\cdots)dt=O(\delta).$$

As for the estimates of $\int_{\mathbb{R}}\dot\phi(\xi)(J_2^1(\xi)+J_2^2(\xi)+J_2^3(\xi))d\xi$ (recall (5.20), (5.21) and (5.33)),

$$\begin{aligned}&\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}d\xi\dot\phi(\xi)\int_{S^{n-2}}d\theta\int_{\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{dp}{p^2}\int_{-1}^{-\delta\frac{\varepsilon}{p}}(\cdots)dt\\&\quad=\frac14\int_{S^{n-2}}A(\theta)d\theta+o_\varepsilon(1)+O(r)+O(\delta)=o_\varepsilon(1)+O(r)+O(\delta),\\&\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}d\xi\dot\phi(\xi)\int_{S^{n-2}}d\theta\int_{\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{dp}{p^2}\int_{-\delta\frac{\varepsilon}{p}}^{\delta\frac{\varepsilon}{p}}(\cdots)dt=O(\delta)\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}d\xi\dot\phi(\xi)\int_{S^{n-2}}d\theta\int_{\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{dp}{p^2}\int_{\delta\frac{\varepsilon}{p}}^{1}(\cdots)dt\\&\quad=\frac14\int_{S^{n-2}}A(\theta)d\theta+o_\varepsilon(1)+O(r)+O(\delta)=o_\varepsilon(1)+O(r)+O(\delta).\end{aligned}$$

Estimate (5.36) then follows.

From (5.34), (5.35) and (5.36), and recalling (5.8), we choose $\delta=C_0r$ to finally get (5.6).

Step 2. Estimating $\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot\phi(\xi)I_2(\xi)d\xi$. We will show that

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot\phi(\xi)I_2(\xi)d\xi=o_r(1).\tag{5.37}$$

Recalling (4.2), we see that for $|t|>1$, there is $C_1>0$ such that

$$A(\theta)-C_1rt^2\le b(\theta,t,r)\le A(\theta)+C_1rt^2.$$

Then, for C_1 as above and $R>2$, to be determined, by the monotonicity of ϕ,

$$\begin{aligned}I_2(\xi)\le&\int_\varepsilon^r\frac{dp}{p^2}\int_{S^{n-2}}d\theta\\&\times\int_{\{|t|>1\}}\left\{\phi\left(\xi+\frac{tp}{\varepsilon}+\frac{p^2}{\varepsilon}(A(\theta)+C_1rt^2)\right)-\phi\left(\xi+\frac{tp}{\varepsilon}\right)\right\}\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}\\=&\int_\varepsilon^r\frac{dp}{p^2}\int_{S^{n-2}}d\theta\int_{\{|t|>\frac{1}{2\sqrt{rRC_1p}}\}}(\cdots)dt\\&+\int_\varepsilon^r\frac{dp}{p^2}\int_{S^{n-2}}d\theta\int_{\{1<|t|<\frac{1}{2\sqrt{rRC_1p}}\}}(\cdots)dt=:I_2^1(\xi)+I_2^2(\xi).\end{aligned}$$

We first estimate

$$I_2^1(\xi) \le 2\int_\varepsilon^r \frac{dp}{p^2}\int_{S^{n-2}} d\theta \int_{\{|t|>\frac{1}{2\sqrt{rRC_1 p}}\}} \frac{dt}{|t|^{n+1}}$$
$$\le C\int_\varepsilon^r (rR)^{\frac{n}{2}} p^{\frac{n}{2}-2}dp \le C(rR)^{\frac{n}{2}}\int_\varepsilon^r \frac{dp}{p} \le C(rR)^{\frac{n}{2}}|\ln\varepsilon|,$$

so that

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} \dot\phi(\xi) I_2^1(\xi)d\xi \le C(rR)^{\frac{n}{2}}. \tag{5.38}$$

Next, let us estimate $\int_{\mathbb{R}} \dot\phi(\xi) I_2^2(\xi)d\xi$. If

$$|\xi| \le \frac{|t|p}{\varepsilon} - \frac{p}{R\varepsilon}, \quad 0\le p\le r<1, \quad 1\le |t| \le \frac{1}{2\sqrt{rRC_1 p}}, \quad R>2, \quad 0\le\tau\le 1,$$

and R, r are such that

$$r|A(\theta)| \le \frac{1}{4R}, \tag{5.39}$$

then

$$\left|\xi + \frac{tp}{\varepsilon} + \tau\frac{p^2}{\varepsilon}(A(\theta)+C_1 rt^2)\right| \ge \frac{|t|p}{\varepsilon} - |\xi| - \frac{p}{4R\varepsilon} - \frac{p}{4R\varepsilon} \ge \frac{p}{2R\varepsilon}.$$

Therefore, by (2.2), for some $\tau\in(0,1)$,

$$\begin{aligned}
&\phi\left(\xi + \frac{tp}{\varepsilon} + \frac{p^2}{\varepsilon}(A(\theta)+C_1 rt^2)\right) - \phi\left(\xi + \frac{tp}{\varepsilon}\right)\\
&= \dot\phi\left(\xi + \frac{tp}{\varepsilon} + \tau\frac{p^2}{\varepsilon}(A(\theta)+C_1 rt^2)\right)\frac{p^2}{\varepsilon}(A(\theta)+C_1 rt^2)\\
&\le \frac{C}{|\xi + \frac{tp}{\varepsilon} + \tau\frac{p^2}{\varepsilon}(A(\theta)+C_1 rt^2)|^2}\frac{p^2}{\varepsilon}(1+rt^2) \le C(1+rt^2)R^2\varepsilon,
\end{aligned}$$

from which we find that

$$\begin{aligned}
&\int_\varepsilon^r \frac{dp}{p^2}\int_{S^{n-2}} d\theta \int_{\{1<|t|<\frac{1}{2\sqrt{rRC_1 p}}\}} \frac{dt}{(t^2+1)^{\frac{n+1}{2}}}\\
&\quad\times \int_{-\frac{|t|p}{\varepsilon}+\frac{p}{R\varepsilon}}^{\frac{|t|p}{\varepsilon}-\frac{p}{R\varepsilon}} \dot\phi(\xi)\left\{\phi\left(\xi+\frac{tp}{\varepsilon}+\frac{p^2}{\varepsilon}(A(\theta)+C_1 rt^2)\right) - \phi\left(\xi+\frac{tp}{\varepsilon}\right)\right\}d\xi\\
&\le CR^2\varepsilon\int_{\mathbb{R}} d\xi\dot\phi(\xi)\int_{S^{n-2}} d\theta\int_\varepsilon^r \frac{dp}{p^2}\int_{\{1<|t|<\frac{1}{2\sqrt{rRC_1 p}}\}} \frac{1+rt^2}{|t|^{n+1}}dt\\
&\le CR^2\varepsilon\int_\varepsilon^r \frac{dp}{p^2}\int_{\{1<|t|<\frac{1}{2\sqrt{rRC_1\varepsilon}}\}} \left(\frac{1}{|t|^{n+1}} + \frac{r}{|t|}\right)dt \le CR^2(1+r|\ln\varepsilon|).
\end{aligned}$$

Consequently,

$$\frac{1}{|\ln\varepsilon|}\int_{\varepsilon}^{r}\frac{dp}{p^2}\int_{S^{n-2}}d\theta\int_{\{1<|t|<\frac{1}{2\sqrt{rRC_1p}}\}}\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}\int_{\{|\xi|<\frac{|t|p}{\varepsilon}-\frac{p}{R\varepsilon}\}}\dot{\phi}(\xi)$$
$$\times\left\{\phi\left(\xi+\frac{tp}{\varepsilon}+\frac{p^2}{\varepsilon}(A(\theta)+C_1rt^2)\right)-\phi\left(\xi+\frac{tp}{\varepsilon}\right)\right\}d\xi$$
$$\le CR^2\left(\frac{1}{|\ln\varepsilon|}+r\right). \tag{5.40}$$

Next, again by (2.2), for $|t|>1$, $R>2$ and for some $\tau_1,\tau_2\in(-1,1)$,

$$\int_{\{\frac{|t|p}{\varepsilon}-\frac{p}{R\varepsilon}<|\xi|<\frac{|t|p}{\varepsilon}+\frac{p}{R\varepsilon}\}}\dot{\phi}(\xi)d\xi$$
$$=\left(\phi\left(\frac{|t|p}{\varepsilon}+\frac{p}{R\varepsilon}\right)-\phi\left(\frac{|t|p}{\varepsilon}-\frac{p}{R\varepsilon}\right)\right)$$
$$+\left(\phi\left(-\frac{|t|p}{\varepsilon}+\frac{p}{R\varepsilon}\right)-\phi\left(-\frac{|t|p}{\varepsilon}-\frac{p}{R\varepsilon}\right)\right)$$
$$=\dot{\phi}\left(\frac{|t|p}{\varepsilon}+\tau_1\frac{p}{R\varepsilon}\right)\frac{2p}{R\varepsilon}+\dot{\phi}\left(-\frac{|t|p}{\varepsilon}+\tau_2\frac{p}{R\varepsilon}\right)\frac{2p}{R\varepsilon}\le C\left(\frac{\varepsilon}{tp}\right)^2\frac{p}{R\varepsilon}=C\frac{\varepsilon}{Rpt^2}.$$

Therefore,

$$\int_{\varepsilon}^{r}\frac{dp}{p^2}\int_{S^{n-2}}d\theta\int_{\{1<|t|<\frac{1}{2\sqrt{rRC_1p}}\}}\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}\int_{\{\frac{|t|p}{\varepsilon}-\frac{p}{R\varepsilon}<|\xi|<\frac{|t|p}{\varepsilon}+\frac{p}{R\varepsilon}\}}\dot{\phi}(\xi)$$
$$\times\left\{\phi\left(\xi+\frac{tp}{\varepsilon}+\frac{p^2}{\varepsilon}(A(\theta)+C_1rt^2)\right)-\phi\left(\xi+\frac{tp}{\varepsilon}\right)\right\}d\xi$$
$$\le C\int_{S^{n-2}}d\theta\int_{|t|>1}dt\frac{1+rt^2}{|t|^{n+1}}\int_{\varepsilon}^{r}\frac{dp}{p^2}\frac{p^2}{\varepsilon}\int_{\{\frac{|t|p}{\varepsilon}-\frac{p}{R\varepsilon}<|\xi|<\frac{|t|p}{\varepsilon}+\frac{p}{R\varepsilon}\}}\dot{\phi}(\xi)d\xi$$
$$\le\frac{C}{R}\int_{\varepsilon}^{r}\frac{dp}{p}\int_{|t|>1}\frac{dt}{|t|^{n+1}}\le\frac{C|\ln\varepsilon|}{R}.$$

Consequently,

$$\frac{1}{|\ln\varepsilon|}\int_{\varepsilon}^{r}\frac{dp}{p^2}\int_{S^{n-2}}d\theta\int_{\{1<|t|<\frac{1}{2\sqrt{rRC_1p}}\}}\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}\int_{\{\frac{|t|p}{\varepsilon}-\frac{p}{R\varepsilon}<|\xi|<\frac{|t|p}{\varepsilon}+\frac{p}{R\varepsilon}\}}\dot{\phi}(\xi)$$
$$\times\left\{\phi\left(\xi+\frac{tp}{\varepsilon}+\frac{p^2}{\varepsilon}(A(\theta)+C_1rt^2)\right)-\phi\left(\xi+\frac{tp}{\varepsilon}\right)\right\}d\xi\le\frac{C}{R}. \tag{5.41}$$

Finally, if

$$|\xi|\ge\frac{|t|p}{\varepsilon}+\frac{p}{R\varepsilon},\quad 0\le p\le r,\quad 1\le|t|\le\frac{1}{2\sqrt{rRC_1p}},\quad R>2,\quad 0\le\tau\le1,$$

and (5.39) holds true, then

$$\left|\xi+\frac{tp}{\varepsilon}+\tau\frac{p^2}{\varepsilon}(A(\theta)+C_1rt^2)\right|\geq|\xi|-\frac{|t|p}{\varepsilon}-\frac{p^2}{\varepsilon}|A(\theta)|-\frac{p^2}{\varepsilon}C_1rt^2\geq\frac{p}{2R\varepsilon},$$

and as before, by (2.2),

$$\phi\left(\xi+\frac{tp}{\varepsilon}+\frac{p^2}{\varepsilon}(A(\theta)+C_1rt^2)\right)-\phi\left(\xi+\frac{tp}{\varepsilon}\right)\leq C(1+rt^2)R^2\varepsilon.$$

Therefore,

$$\begin{aligned}&\int_\varepsilon^r\frac{dp}{p^2}\int_{S^{n-2}}d\theta\int_{\{1<|t|<\frac{1}{2\sqrt{rRC_1p}}\}}\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}\int_{\{|\xi|>\frac{|t|p}{\varepsilon}+\frac{p}{R\varepsilon}\}}\dot\phi(\xi)\\&\times\left\{\phi\left(\xi+\frac{tp}{\varepsilon}+\frac{p^2}{\varepsilon}(A(\theta)+C_1rt^2)\right)-\phi\left(\xi+\frac{tp}{\varepsilon}\right)\right\}d\xi\\&\leq CR^2\varepsilon\int_\varepsilon^r\frac{dp}{p^2}\int_{\{1<|t|<\frac{1}{2\sqrt{rRC_1\varepsilon}}\}}\frac{1+rt^2}{|t|^{n+1}}dt\leq CR^2(1+r|\ln\varepsilon|),\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{|\ln\varepsilon|}\int_\varepsilon^r\frac{dp}{p^2}\int_{S^{n-2}}d\theta\int_{\{1<|t|<\frac{1}{2\sqrt{rRC_1p}}\}}\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}\int_{\{|\xi|>\frac{|t|p}{\varepsilon}+\frac{p}{R\varepsilon}\}}\dot\phi(\xi)\\&\times\left\{\phi\left(\xi+\frac{tp}{\varepsilon}+\frac{p^2}{\varepsilon}(A(\theta)+C_1rt^2)\right)-\phi\left(\xi+\frac{tp}{\varepsilon}\right)\right\}d\xi\leq CR^2\Big(\frac{1}{|\ln\varepsilon|}+r\Big).\end{aligned}\tag{5.42}$$

From (5.40), (5.41), and (5.42), choosing $R=r^{-\frac13}$, we have that (5.39) holds true for r small enough, and

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot\phi(\xi)I_2^2(\xi)d\xi\leq o_r(1).$$

Together with (5.38), this gives

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot\phi(\xi)I_2(\xi)d\xi\leq o_r(1).$$

The lower bound for $\int_{\mathbb{R}}\dot\phi(\xi)I_2(\xi)d\xi$ is obtained in a similar way. Estimate (5.37) follows.

Conclusion. Recalling (5.5), we combine (5.6) and (5.37) to finally obtain

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot\phi(\xi)I(\xi)d\xi=\int_{S^{n-2}}A(\theta)d\theta+o_\varepsilon(1)+o_r(1),\tag{5.43}$$

where $o_\varepsilon(1)$ depends on the parameter r. From (5.2) and (5.43), we first send $\varepsilon\to0$ and then $r\to0$ to arrive at

$$\lim_{\varepsilon\to0}\bar a_\varepsilon(x)=\int_{S^{n-2}}A(\theta)d\theta,$$

uniformly in Q_ρ. Recalling (4.1), this gives the desired result. ∎

6. Proof of Theorem 1.3 for $s \in (\frac{1}{2}, 1)$

Throughout this section, we assume that $s \in (\frac{1}{2}, 1)$.

Fix $r \in (0, \frac{\varrho}{2})$ and take $0 < \varepsilon < r$. By Lemma 4.1, we have

$$\bar{a}_\varepsilon(x) = O\left(\frac{\varepsilon^{2s-1}}{r^{2s}}\right) + \varepsilon^{2s-1}\int_{\mathbb{R}} G(\xi)\dot{\phi}(\xi)d\xi, \tag{6.1}$$

where

$$G(\xi) = \int_0^r \frac{dp}{p^{2s+1}}\int_{S^{n-2}} d\theta \int_{-\frac{r}{p}}^{\frac{r}{p}} \frac{dt}{(t^2+1)^{\frac{n+2s}{2}}} \times \left\{\phi\left(\xi + \frac{p}{\varepsilon}(t + pb(\theta,t,r))\right) - \phi\left(\xi + \frac{tp}{\varepsilon}\right)\right\}.$$

We start by splitting $G(\xi)$, for $R > 0$ to be chosen, as

$$\begin{aligned} G(\xi) &= \int_0^{R\varepsilon} \frac{dp}{p^{2s+1}}\int_{S^{n-2}} d\theta \int_{-\frac{r}{p}}^{\frac{r}{p}} (\cdots)dt + \int_{R\varepsilon}^{r} \frac{dp}{p^{2s+1}}\int_{S^{n-2}} d\theta \int_{-\frac{r}{p}}^{\frac{r}{p}} (\cdots)dt \\ &=: I_1(\xi) + I_2(\xi), \end{aligned} \tag{6.2}$$

and estimate $\varepsilon^{2s-1}\int_{\mathbb{R}} \dot{\phi}(\xi)I_1(\xi)d\xi$ and $\varepsilon^{2s-1}\int_{\mathbb{R}} \dot{\phi}(\xi)I_2(\xi)d\xi$ separately.

Step 1. Estimating $\varepsilon^{2s-1}\int_{\mathbb{R}} \dot{\phi}(\xi)I_1(\xi)d\xi$. We will show that

$$\varepsilon^{2s-1}\int_{\mathbb{R}} \dot{\phi}(\xi)I_1(\xi)d\xi = c_1c_2\int_{S^{n-2}} A(\theta)d\theta + o_\varepsilon(1) + o_r(1), \tag{6.3}$$

where $c_1 > 0$ is defined in (2.3) and $c_2 > 0$ depends only on n and s and is to be determined. Recalling (4.2), we write

$$\begin{aligned} I_1(\xi) &= \int_0^{R\varepsilon} \frac{dp}{p^{2s+1}}\int_{S^{n-2}} d\theta \int_{-\frac{r}{p}}^{\frac{r}{p}} \left\{\phi\left(\xi + \frac{p}{\varepsilon}(t + pb(\theta,t,r))\right) - \phi\left(\xi + \frac{tp}{\varepsilon}\right)\right\} \\ &\quad \times \frac{dt}{(t^2+1)^{\frac{n+2s}{2}}} \\ &= \int_0^{R\varepsilon} \frac{dp}{p^{2s+1}}\int_{S^{n-2}} d\theta \int_{-\frac{r}{p}}^{\frac{r}{p}} \frac{dt}{(t^2+1)^{\frac{n+2s}{2}}} \\ &\quad \times \int_0^1 \dot{\phi}\left(\xi + \frac{p}{\varepsilon}(t + \tau pb(\theta,t,r))\right)\frac{p^2}{\varepsilon}(A(\theta) + O(r(1+t^2)))d\tau \\ &= \frac{1}{\varepsilon}\int_0^{R\varepsilon} \frac{dp}{p^{2s-1}}\int_{S^{n-2}} d\theta A(\theta) \int_{-\frac{r}{p}}^{\frac{r}{p}} \frac{dt}{(t^2+1)^{\frac{n+2s}{2}}} \\ &\quad \times \int_0^1 \dot{\phi}\left(\xi + \frac{p}{\varepsilon}(t + \tau pb(\theta,t,r))\right)d\tau \\ &\quad + \frac{O(r)}{\varepsilon}\int_0^{R\varepsilon} \frac{dp}{p^{2s-1}}\int_{S^{n-2}} d\theta \int_{-\frac{r}{p}}^{\frac{r}{p}} \frac{dt}{(t^2+1)^{\frac{n+2s-2}{2}}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\varepsilon}\int_0^{R\varepsilon}\frac{dp}{p^{2s-1}}\int_{S^{n-2}} d\theta A(\theta)\int_{-\frac{r}{p}}^{\frac{r}{p}}\frac{dt}{(t^2+1)^{\frac{n+2s}{2}}}\\
&\quad\times\int_0^1\Big(\dot\phi\Big(\xi+\frac{pt}{\varepsilon}\Big)+O\Big(\tau\frac{p^2}{\varepsilon}b(\theta,t,r)\Big)\Big)d\tau\\
&\quad+\frac{O(r)}{\varepsilon}\int_0^{R\varepsilon}\frac{dp}{p^{2s-1}}\int_{S^{n-2}}d\theta\int_{-\frac{r}{p}}^{\frac{r}{p}}\frac{dt}{(t^2+1)^{\frac{n+2s-2}{2}}} = I_1^1(\xi)+E_1(\xi)+E_2(\xi),
\end{aligned}
$$

where we take

$$
\begin{aligned}
I_1^1(\xi) &= \frac{1}{\varepsilon}\int_0^{R\varepsilon}\frac{dp}{p^{2s-1}}\int_{S^{n-2}}d\theta A(\theta)\int_{-\frac{r}{p}}^{\frac{r}{p}}\dot\phi\Big(\xi+\frac{pt}{\varepsilon}\Big)\frac{dt}{(t^2+1)^{\frac{n+2s}{2}}},\\
E_1(\xi) &= \frac{1}{\varepsilon^2}\int_0^{R\varepsilon}dp p^{3-2s}\int_{-\frac{r}{p}}^{\frac{r}{p}}O(1+r(1+t^2))\frac{dt}{(t^2+1)^{\frac{n+2s}{2}}} \qquad (6.4)
\end{aligned}
$$

and

$$
E_2(\xi) = \frac{O(r)}{\varepsilon}\int_0^{R\varepsilon}\frac{dp}{p^{2s-1}}\int_{-\frac{r}{p}}^{\frac{r}{p}}\frac{dt}{(t^2+1)^{\frac{n+2s-2}{2}}}. \qquad (6.5)
$$

Integrating by parts in t, we have

$$
\begin{aligned}
&I_1^1(\xi)\\
&= \frac{1}{2\varepsilon}\int_0^{R\varepsilon}\frac{dp}{p^{2s-1}}\int_{S^{n-2}}d\theta A(\theta)\int_{-\frac{r}{p}}^{\frac{r}{p}}\Big(\dot\phi\Big(\xi+\frac{pt}{\varepsilon}\Big)+\dot\phi\Big(\xi-\frac{pt}{\varepsilon}\Big)\Big)\frac{dt}{(t^2+1)^{\frac{n+2s}{2}}}\\
&= \frac{1}{2}\int_0^{R\varepsilon}\frac{dp}{p^{2s}}\int_{S^{n-2}}d\theta A(\theta)\int_{-\frac{r}{p}}^{\frac{r}{p}}\partial_t\Big[\phi\Big(\xi+\frac{pt}{\varepsilon}\Big)-\phi\Big(\xi-\frac{pt}{\varepsilon}\Big)\Big]\frac{dt}{(t^2+1)^{\frac{n+2s}{2}}}\\
&= \frac{1}{2}\int_0^{R\varepsilon}\frac{dp}{p^{2s}}\int_{S^{n-2}}d\theta A(\theta)\Big[\Big(\phi\Big(\xi+\frac{pt}{\varepsilon}\Big)-\phi\Big(\xi-\frac{pt}{\varepsilon}\Big)\Big)\frac{1}{(t^2+1)^{\frac{n+2s}{2}}}\Big|_{t=-\frac{r}{p}}^{t=\frac{r}{p}}\\
&\qquad\qquad+(n+2s)\int_{-\frac{r}{p}}^{\frac{r}{p}}\Big(\phi\Big(\xi+\frac{pt}{\varepsilon}\Big)-\phi\Big(\xi-\frac{pt}{\varepsilon}\Big)\Big)\\
&\qquad\qquad\times\frac{t}{(t^2+1)^{\frac{n+2s+2}{2}}}dt\Big] = I_1^2(\xi)+E_3(\xi),
\end{aligned}
$$

where

$$
\begin{aligned}
I_1^2(\xi) &= \frac{n+2s}{2}\int_0^{R\varepsilon}\frac{dp}{p^{2s}}\int_{S^{n-2}}d\theta A(\theta)\\
&\quad\times\int_{-\frac{r}{p}}^{\frac{r}{p}}\Big(\phi\Big(\xi+\frac{pt}{\varepsilon}\Big)-\phi\Big(\xi-\frac{pt}{\varepsilon}\Big)\Big)\frac{t}{(t^2+1)^{\frac{n+2s+2}{2}}}dt
\end{aligned}
$$

and

$$
E_3(\xi) = \frac{1}{2}\int_0^{R\varepsilon}\frac{dp}{p^{2s}}\int_{S^{n-2}}d\theta A(\theta)\Big(\phi\Big(\xi+\frac{pt}{\varepsilon}\Big)-\phi\Big(\xi-\frac{pt}{\varepsilon}\Big)\Big)\frac{1}{(t^2+1)^{\frac{n+2s}{2}}}\Big|_{t=-\frac{r}{p}}^{t=\frac{r}{p}}. \qquad (6.6)
$$

Note that the integrals above are well defined as

$$|I_1^2(\xi)|, |E_3(\xi)| \le \frac{C}{\varepsilon}\int_0^{R\varepsilon} p^{1-2s}dp \le CR^{2-2s}\varepsilon^{1-2s}.$$

Therefore, integrating by parts with respect to ξ, we get

$$\begin{aligned}\int_{\mathbb{R}} I_1^2(\xi)\dot\phi(\xi)d\xi &= \frac{n+2s}{2}\int_0^{R\varepsilon}\frac{dp}{p^{2s}}\int_{S^{n-2}} d\theta A(\theta)\int_{-\frac{r}{p}}^{\frac{r}{p}} dt\frac{t}{(t^2+1)^{\frac{n+2s+2}{2}}}\\ &\quad\times\int_{\mathbb{R}}\dot\phi(\xi)\left(\phi\left(\xi+\frac{pt}{\varepsilon}\right)-\phi\left(\xi-\frac{pt}{\varepsilon}\right)\right)d\xi\\ &= -\frac{n+2s}{2}\int_0^{R\varepsilon}\frac{dp}{p^{2s}}\int_{S^{n-2}} d\theta A(\theta)\int_{-\frac{r}{p}}^{\frac{r}{p}} dt\frac{t}{(t^2+1)^{\frac{n+2s+2}{2}}}\\ &\quad\times\int_{\mathbb{R}}\phi(\xi)\left(\dot\phi\left(\xi+\frac{pt}{\varepsilon}\right)-\dot\phi\left(\xi-\frac{pt}{\varepsilon}\right)\right)d\xi.\end{aligned}$$

Integrating by parts again in t, we find

$$\begin{aligned}\int_{\mathbb{R}} I_1^2(\xi)\dot\phi(\xi)d\xi &= -\varepsilon\frac{n+2s}{2}\int_0^{R\varepsilon}\frac{dp}{p^{1+2s}}\int_{S^{n-2}} d\theta A(\theta)\int_{\mathbb{R}} d\xi\phi(\xi)\\ &\quad\times\int_{-\frac{r}{p}}^{\frac{r}{p}}\partial_t\left[\phi\left(\xi+\frac{pt}{\varepsilon}\right)+\phi\left(\xi-\frac{pt}{\varepsilon}\right)-2\phi(\xi)\right]\frac{t}{(t^2+1)^{\frac{n+2s+2}{2}}}dt\\ &= -\varepsilon\frac{n+2s}{2}\int_0^{R\varepsilon}\frac{dp}{p^{1+2s}}\int_{S^{n-2}} d\theta A(\theta)\int_{\mathbb{R}} d\xi\phi(\xi)\\ &\quad\times\left[\left(\phi\left(\xi+\frac{pt}{\varepsilon}\right)+\phi\left(\xi-\frac{pt}{\varepsilon}\right)-2\phi(\xi)\right)\frac{t}{(t^2+1)^{\frac{n+2s+2}{2}}}\Bigg|_{t=-\frac{r}{p}}^{t=\frac{r}{p}}\right.\\ &\quad-\int_{-\frac{r}{p}}^{\frac{r}{p}}\left(\phi\left(\xi+\frac{pt}{\varepsilon}\right)+\phi\left(\xi-\frac{pt}{\varepsilon}\right)-2\phi(\xi)\right)\\ &\quad\left.\times\frac{d}{dt}\left[\frac{t}{(t^2+1)^{\frac{n+2s+2}{2}}}\right]dt\right] = F+E_4,\end{aligned}$$

where

$$\begin{aligned}F &= \varepsilon\frac{n+2s}{2}\int_0^{R\varepsilon}\frac{dp}{p^{1+2s}}\int_{S^{n-2}} d\theta A(\theta)\int_{\mathbb{R}} d\xi\phi(\xi)\\ &\quad\times\int_{-\frac{r}{p}}^{\frac{r}{p}} dt\left(\phi\left(\xi+\frac{pt}{\varepsilon}\right)+\phi\left(\xi-\frac{pt}{\varepsilon}\right)-2\phi(\xi)\right)\frac{d}{dt}\left[\frac{t}{(t^2+1)^{\frac{n+2s+2}{2}}}\right]\end{aligned}$$

and

$$\begin{aligned}E_4 &= -\varepsilon\frac{n+2s}{2}\int_0^{R\varepsilon}\frac{dp}{p^{1+2s}}\int_{S^{n-2}} d\theta A(\theta)\int_{\mathbb{R}} d\xi\phi(\xi)\\ &\quad\times\left(\phi\left(\xi+\frac{pt}{\varepsilon}\right)+\phi\left(\xi-\frac{pt}{\varepsilon}\right)-2\phi(\xi)\right)\frac{t}{(t^2+1)^{\frac{n+2s+2}{2}}}\Bigg|_{t=-\frac{r}{p}}^{t=\frac{r}{p}}.\end{aligned} \tag{6.7}$$

In F, we will make the change of variable $w = pt/\varepsilon$ in t and then $q = p/(\varepsilon|w|)$ in p. In this regard, it is helpful to first write

$$\begin{aligned}\frac{d}{dt}\left[\frac{t}{(t^2+1)^{\frac{n+2s+2}{2}}}\right] &= \frac{1}{(t^2+1)^{\frac{n+2s+2}{2}}} - (n+2s+2)\frac{t^2}{(t^2+1)^{\frac{n+2s+4}{2}}}\\ &= \frac{1}{(\frac{\varepsilon^2}{p^2}w^2+1)^{\frac{n+2s+2}{2}}} - (n+2s+2)\frac{\frac{\varepsilon^2}{p^2}w^2}{(\frac{\varepsilon^2}{p^2}w^2+1)^{\frac{n+2s+4}{2}}}\\ &= \frac{1}{(q^{-2}+1)^{\frac{n+2s+2}{2}}} - (n+2s+2)\frac{q^{-2}}{(q^{-2}+1)^{\frac{n+2s+4}{2}}}\\ &= q^{n+2s+2}\left[\frac{1}{(1+q^2)^{\frac{n+2s+2}{2}}} - \frac{n+2s+2}{(1+q^2)^{\frac{n+2s+4}{2}}}\right].\end{aligned}$$

Therefore, making the aforementioned change of variables,

$$\begin{aligned}F &= \varepsilon\frac{n+2s}{2}\int_0^{R\varepsilon}\frac{dp}{p^{1+2s}}\int_{S^{n-2}} d\theta A(\theta)\int_{\mathbb{R}} d\xi\phi(\xi)\int_{-\frac{r}{\varepsilon}}^{\frac{r}{\varepsilon}}\frac{\varepsilon}{p}dw\\ &\quad\times(\phi(\xi+w)+\phi(\xi+w)-2\phi(\xi))\left[\frac{1}{(\frac{\varepsilon^2}{p^2}w^2+1)^{\frac{n+2s+2}{2}}} - \frac{(n+2s+2)\frac{\varepsilon^2}{p^2}w^2}{(\frac{\varepsilon^2}{p^2}w^2+1)^{\frac{n+2s+4}{2}}}\right]\\ &= \varepsilon^2\frac{n+2s}{2}\int_{S^{n-2}} d\theta A(\theta)\int_{\mathbb{R}} d\xi\phi(\xi)\int_{-\frac{r}{\varepsilon}}^{\frac{r}{\varepsilon}} dw\int_0^{\frac{R}{|w|}}\frac{dq}{(\varepsilon|w|)^{1+2s}q^{2+2s}}\\ &\quad\times(\phi(\xi+w)+\phi(\xi-w)-2\phi(\xi))q^{n+2s+2}\left[\frac{1}{(1+q^2)^{\frac{n+2s+2}{2}}} - \frac{n+2s+2}{(1+q^2)^{\frac{n+2s+4}{2}}}\right]\\ &= \varepsilon^{1-2s}\frac{n+2s}{2}\int_{S^{n-2}} d\theta A(\theta)\int_{\mathbb{R}} d\xi\phi(\xi)\int_{-\frac{r}{\varepsilon}}^{\frac{r}{\varepsilon}} dw\frac{\phi(\xi+w)+\phi(\xi-w)-2\phi(\xi)}{|w|^{1+2s}}\\ &\quad\times\int_0^{\frac{R}{|w|}} q^n\left[\frac{1}{(1+q^2)^{\frac{n+2s+2}{2}}} - \frac{n+2s+2}{(1+q^2)^{\frac{n+2s+4}{2}}}\right]dq.\end{aligned}$$

Note that, by estimate (2.2) for $\ddot{\phi}$ and Taylor's theorem, for $|w| < |\xi|/2$,

$$|\phi(\xi+w)+\phi(\xi-w)-2\phi(\xi)| \le C\frac{w^2}{1+|\xi|^{2s+1}}.$$

Thus,

$$\begin{aligned}&\int_{\mathbb{R}} d\xi\phi(\xi)\int_{\mathbb{R}}\frac{|\phi(\xi+w)+\phi(\xi-w)-2\phi(\xi)|}{|w|^{1+2s}}dw\\ &\le C\int_{\mathbb{R}} d\xi\phi(\xi)\left[\int_{\{|w|<\frac{\max\{|\xi|,1\}}{2}\}}\frac{|w|^{1-2s}}{1+|\xi|^{2s+1}}dw + \int_{\{|w|>\frac{\max\{|\xi|,1\}}{2}\}}\frac{1}{|w|^{1+2s}}dw\right]\\ &\le C\int_{\mathbb{R}}\left(\frac{1}{1+|\xi|^{4s-1}}+\frac{1}{1+|\xi|^{2s}}\right)d\xi \le C.\end{aligned}$$

Moreover, for any $M > 1$,

$$
\begin{aligned}
&\int_{\mathbb{R}} d\xi\phi(\xi)\int_{\{|w|>M\}}\frac{|\phi(\xi+w)+\phi(\xi-w)-2\phi(\xi)|}{|w|^{1+2s}}dw\\
&=\int_{-M}^{M}d\xi(\cdots)+\int_{\{|\xi|>M\}}d\xi(\cdots)\\
&\le C\int_{-M}^{M}d\xi\int_{\{|w|>M\}}\frac{dw}{|w|^{1+2s}}+C\int_{\{|\xi|>M\}}\left(\frac{1}{1+|\xi|^{4s-1}}+\frac{1}{1+|\xi|^{2s}}\right)d\xi\\
&\le \frac{C}{M^{2s-1}}+\frac{C}{M^{2(2s-1)}}\\
&\le \frac{C}{M^{2s-1}}.
\end{aligned}
$$

Therefore, for c_1 defined in (2.3), we have

$$
\begin{aligned}
&-\frac{1}{2}\int_{\mathbb{R}} d\xi\phi(\xi)\int_{-\frac{r}{\varepsilon}}^{\frac{r}{\varepsilon}}\frac{\phi(\xi+w)+\phi(\xi-w)-2\phi(\xi)}{|w|^{1+2s}}dw\\
&\quad=-\frac{1}{2}\int_{\mathbb{R}} d\xi\phi(\xi)\int_{-R^{\frac{1}{2}}}^{R^{\frac{1}{2}}}\frac{\phi(\xi+w)+\phi(\xi-w)-2\phi(\xi)}{|w|^{1+2s}}dw\\
&\qquad+O(R^{-(s-\frac{1}{2})})+O\Big(\Big(\frac{\varepsilon}{r}\Big)^{2s-1}\Big)\\
&\quad=\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(\phi(\xi+w)-\phi(\xi))^2}{|w|^{1+2s}}dwd\xi+O(R^{-(s-\frac{1}{2})})+O\Big(\Big(\frac{\varepsilon}{r}\Big)^{2s-1}\Big)\\
&\quad=c_1+O(R^{-(s-\frac{1}{2})})+O\Big(\Big(\frac{\varepsilon}{r}\Big)^{2s-1}\Big).
\end{aligned}
$$

Next, define c_2 by

$$
c_2 := (n+2s)\int_0^{\infty}\left[\frac{n+2s+2}{(1+q^2)^{\frac{n+2s+4}{2}}}-\frac{1}{(1+q^2)^{\frac{n+2s+2}{2}}}\right]q^n dq. \tag{6.8}
$$

For $|w| < R^{\frac{1}{2}}$, we have

$$
\begin{aligned}
&\int_0^{\frac{R}{|w|}}\left[\frac{n+2s+2}{(1+q^2)^{\frac{n+2s+4}{2}}}-\frac{1}{(1+q^2)^{\frac{n+2s+2}{2}}}\right]q^n dq\\
&=\int_0^{\infty}\left[\frac{n+2s+2}{(1+q^2)^{\frac{n+2s+4}{2}}}-\frac{1}{(1+q^2)^{\frac{n+2s+2}{2}}}\right]q^n dq+O\big(R^{-\frac{1+2s}{2}}\big)\\
&=c_2+O\big(R^{-\frac{1+2s}{2}}\big).
\end{aligned}
$$

The previous estimates imply that

$$
\begin{aligned}
F &= -\varepsilon^{1-2s}\frac{n+2s}{2}\int_{S^{n-2}} d\theta A(\theta)\int_{\mathbb{R}} d\xi\phi(\xi)\\
&\quad\times\int_{-R^{\frac12}}^{R^{\frac12}} dw\frac{\phi(\xi+w)+\phi(\xi-w)-2\phi(\xi)}{|w|^{1+2s}}\\
&\quad\times\int_0^{\frac{R}{|w|}} q^n\left[\frac{n+2s+2}{(1+q^2)^{\frac{n+2s+4}{2}}}-\frac{1}{(1+q^2)^{\frac{n+2s+2}{2}}}\right]dq+\varepsilon^{1-2s}(o_\varepsilon(1)+o_{R^{-1}}(1))\\
&= -\varepsilon^{1-2s}\frac{n+2s}{2}\int_{S^{n-2}} d\theta A(\theta)\int_{\mathbb{R}} d\xi\phi(\xi)\\
&\quad\times\int_{-R^{\frac12}}^{R^{\frac12}} dw\frac{\phi(\xi+w)+\phi(\xi-w)-2\phi(\xi)}{|w|^{1+2s}}\\
&\quad\times\int_0^{\infty} q^n\left[\frac{n+2s+2}{(1+q^2)^{\frac{n+2s+4}{2}}}-\frac{1}{(1+q^2)^{\frac{n+2s+2}{2}}}\right]dq+\varepsilon^{1-2s}(o_\varepsilon(1)+o_{R^{-1}}(1))\\
&= \varepsilon^{1-2s}\left(c_1c_2\int_{S^{n-2}} A(\theta)d\theta+o_\varepsilon(1)+o_{R^{-1}}(1)\right).
\end{aligned}
$$

We conclude that

$$
\varepsilon^{2s-1}F = c_1c_2\int_{S^{n-2}} A(\theta)d\theta+o_\varepsilon(1)+o_{R^{-1}}(1). \tag{6.9}
$$

It remains to check the errors terms.

Step 1a. Estimating $\varepsilon^{2s-1}\int_{\mathbb{R}}\dot\phi(\xi)E_1(\xi)d\xi$ and $\varepsilon^{2s-1}\int_{\mathbb{R}}\dot\phi(\xi)E_2(\xi)d\xi$. We will show that

$$
\varepsilon^{2s-1}\int_{\mathbb{R}}\dot\phi(\xi)E_1(\xi)d\xi = o_\varepsilon(1) \tag{6.10}
$$

and

$$
\varepsilon^{2s-1}\int_{\mathbb{R}}\dot\phi(\xi)E_2(\xi)d\xi = o_r(1). \tag{6.11}
$$

Observe that

$$
\int_0^{R\varepsilon}\frac{dp}{p^{2s-1}}\int_{\frac{r}{p}}^{\frac{r}{p}}\frac{dt}{(t^2+1)^{\frac{n+2s-2}{2}}} \le \int_0^{R\varepsilon}\frac{dp}{p^{2s-1}}\int_{\mathbb{R}}\frac{dt}{(t^2+1)^{\frac{n+2s-2}{2}}} = CR^{2-2s}\varepsilon^{2-2s}.
$$

First, recalling (6.4), we estimate

$$
|E_1(\xi)| \le \frac{C}{\varepsilon^2}\int_0^{R\varepsilon} R^2\varepsilon^2\frac{dp}{p^{2s-1}}\int_{-\frac{r}{\varepsilon}}^{\frac{r}{\varepsilon}}\frac{dt}{(t^2+1)^{\frac{n+2s-2}{2}}} \le CR^{4-4s}\varepsilon^{2-2s},
$$

so that (6.10) holds. Next, recalling (6.5), we have

$$
\varepsilon^{2s-1}E_2(\xi) = O(rR^{2-2s}).
$$

Choosing

$$R = r^{-1}, \tag{6.12}$$

we arrive at (6.11).

Step 1b. Estimating $\varepsilon^{2s-1}\int_{\mathbb{R}} E_3(\xi)\dot{\phi}(\xi)d\xi$ and E_4. We will show that

$$\varepsilon^{2s-1}\int_{\mathbb{R}} E_3(\xi)\dot{\phi}(\xi)d\xi = o_\varepsilon(1) \tag{6.13}$$

and

$$\varepsilon^{2s-1}E_4 = o_\varepsilon(1). \tag{6.14}$$

First, recalling (6.6), we write

$$\begin{aligned}
|E_3(\xi)| &\le C\int_0^{R\varepsilon}\left[\phi\left(\xi+\frac{r}{\varepsilon}\right)-\phi\left(\xi-\frac{r}{\varepsilon}\right)\right]\frac{1}{(\frac{r^2}{p^2}+1)^{\frac{n+2s}{2}}}\frac{dp}{p^{2s}}\\
&\le C\int_0^{R\varepsilon}\frac{p^{n+2s}}{r^{n+2s}}\frac{dp}{p^{2s}}\\
&= \frac{C}{r^{n+2s}}\int_0^{R\varepsilon}p^n\,dp\\
&= \frac{C}{r^{n+2s}}(R\varepsilon)^{n+1} = o_\varepsilon(1),
\end{aligned}$$

which gives (6.13).

Next, by (2.1), for $|\xi| > 2r/\varepsilon$, since $H(\xi \pm \frac{r}{\varepsilon}) = H(\xi)$,

$$\left|\phi\left(\xi\pm\frac{r}{\varepsilon}\right)-\phi(\xi)\right| \le \frac{C}{|\xi|^{2s}}.$$

Therefore, recalling (6.7), we have

$$\begin{aligned}
|E_4| &= \Bigg|\varepsilon(n+2s)\int_0^{R\varepsilon}\frac{dp}{p^{1+2s}}\int_{S^{n-2}}d\theta A(\theta)\int_{\mathbb{R}}d\xi\dot{\phi}(\xi)\\
&\quad\times\left[\left(\phi\left(\xi+\frac{r}{\varepsilon}\right)-\phi(\xi)\right)+\left(\phi\left(\xi-\frac{r}{\varepsilon}\right)-\phi(\xi)\right)\right]\frac{\frac{r}{p}}{(\frac{r^2}{p^2}+1)^{\frac{n+2s+2}{2}}}\Bigg|\\
&\le C\varepsilon\int_0^{R\varepsilon}\frac{dp}{p^{1+2s}}\frac{p^{n+2s+1}}{r^{n+2s+1}}\left[\int_{-\frac{2r}{\varepsilon}}^{\frac{2r}{\varepsilon}}d\xi+\int_{\{|\xi|>\frac{2r}{\varepsilon}\}}\frac{d\xi}{|\xi|^{2s}}\right]\\
&\le \frac{C\varepsilon}{r^{n+2s+1}}\int_0^{R\varepsilon}p^n\frac{r}{\varepsilon}dp \le \frac{C\varepsilon^{n+1}R^{n+1}}{r^{n+2s}} = o_\varepsilon(1),
\end{aligned}$$

which gives (6.14).

Completion of Step 1. Recall from above that

$$\varepsilon^{2s-1}\int_{\mathbb{R}}\dot{\phi}(\xi)I(\xi)d\xi = \varepsilon^{2s-1}\left[F+E_4+\int_{\mathbb{R}}\dot{\phi}(\xi)[E_1(\xi)+E_2(\xi)+E_3(\xi)]d\xi\right].$$

Combining this with (6.9), (6.10), (6.11), (6.12), (6.13), and (6.14) gives (6.3).

Step 2. Estimating $\varepsilon^{2s-1}\int_{\mathbb{R}}\dot\phi(\xi)I_2(\xi)d\xi$. We will show that

$$\varepsilon^{2s-1}\int_{\mathbb{R}}\dot\phi(\xi)I_2(\xi)d\xi = o_r(1). \tag{6.15}$$

By the monotonicity of ϕ and recalling (4.2), we have

$$\begin{aligned}
I_2(\xi) &\le \int_{R\varepsilon}^{r}\frac{dp}{p^{2s+1}}\int_{S^{n-2}}d\theta \\
&\quad\times\int_{-\frac{r}{p}}^{\frac{r}{p}}\left\{\phi\left(\xi+\frac{tp}{\varepsilon}+\frac{Cp^2}{\varepsilon}(1+t^2)\right)-\phi\left(\xi+\frac{tp}{\varepsilon}\right)\right\}\frac{dt}{(t^2+1)^{\frac{n+2s}{2}}}\\
&=\int_{R\varepsilon}^{r}\frac{dp}{p^{2s+1}}\int_{S^{n-2}}d\theta\int_{-\frac{r}{p}}^{\frac{r}{p}}\frac{dt}{(t^2+1)^{\frac{n+2s}{2}}}\\
&\quad\times\int_0^1\dot\phi\left(\xi+\frac{tp}{\varepsilon}+\tau\frac{Cp^2}{\varepsilon}(1+t^2)\right)\frac{Cp^2}{\varepsilon}(1+t^2)d\tau\\
&=\frac{C}{\varepsilon}\int_{R\varepsilon}^{r}\frac{dp}{p^{2s-1}}\int_{S^{n-2}}d\theta\int_{-\frac{r}{p}}^{\frac{r}{p}}\frac{dt}{(t^2+1)^{\frac{n+2s-2}{2}}}\\
&\quad\times\int_0^1\dot\phi\left(\xi+\frac{tp}{\varepsilon}+\tau\frac{Cp^2}{\varepsilon}(1+t^2)\right)d\tau\\
&=\frac{C}{\varepsilon}\int_{R\varepsilon}^{r}\frac{dp}{p^{2s-1}}\int_{S^{n-2}}d\theta\int_0^1 d\tau\\
&\quad\times\int_{-\frac{r}{p}}^{\frac{r}{p}}\partial_t\left[\phi\left(\xi+\frac{tp}{\varepsilon}+\tau\frac{Cp^2}{\varepsilon}(1+t^2)\right)\right]\frac{\varepsilon}{p+2\tau Cp^2t}\frac{dt}{(t^2+1)^{\frac{n+2s-2}{2}}}.
\end{aligned}$$

Using that $p+2\tau Cp^2t\ge p-2\tau Crp\ge p/2>0$ for $|t|<r/p$ and r small enough, we integrate by parts to get

$$\begin{aligned}
I_2(\xi) &\le C\int_{R\varepsilon}^{r}\frac{dp}{p^{2s}}\int_{S^{n-2}}d\theta\int_0^1 d\tau\\
&\quad\times\int_{-\frac{r}{p}}^{\frac{r}{p}}\partial_t\left[\phi\left(\xi+\frac{tp}{\varepsilon}+\tau\frac{Cp^2}{\varepsilon}(1+t^2)\right)\right]\frac{dt}{(t^2+1)^{\frac{n+2s-2}{2}}}\\
&=C\int_{R\varepsilon}^{r}\frac{dp}{p^{2s}}\int_{S^{n-2}}d\theta\\
&\quad\times\int_0^1 d\tau\Bigg[\phi\left(\xi+\frac{tp}{\varepsilon}+\tau\frac{Cp^2}{\varepsilon}(1+t^2)\right)\frac{1}{(t^2+1)^{\frac{n+2s-2}{2}}}\Bigg|_{t=-\frac{r}{p}}^{t=\frac{r}{p}}\\
&\qquad\qquad+(n+2s-2)\int_{-\frac{r}{p}}^{\frac{r}{p}}\phi\left(\xi+\frac{tp}{\varepsilon}+\tau\frac{Cp^2}{\varepsilon}(1+t^2)\right)\frac{t}{(t^2+1)^{\frac{n+2s}{2}}}dt\Bigg]\\
&\le C\int_{R\varepsilon}^{r}\frac{dp}{p^{2s}}\le C(R\varepsilon)^{-(2s-1)}.
\end{aligned}$$

Similarly, one can prove that

$$I_2(\xi) \geq -C(R\varepsilon)^{-(2s-1)}.$$

We conclude that

$$\varepsilon^{2s-1} I_2(\xi) = O(R^{-(2s-1)}).$$

Recalling (6.12), the estimate (6.15) follows.

Conclusion. Recalling (6.2), we combine (6.3) and (6.15) to finally obtain

$$\varepsilon^{2s-1} \int_{\mathbb{R}} \dot{\phi}(\xi) G(\xi) d\xi = c_1 c_2 \int_{S^{n-2}} A(\theta) d\theta + o_\varepsilon(1) + o_r(1), \tag{6.16}$$

where $o_\varepsilon(1)$ depends on the parameter r. From (6.1) and (6.16), we first send $\varepsilon \to 0$ and then $r \to 0$ to arrive at

$$\lim_{\varepsilon \to 0} \bar{a}_\varepsilon(x) = c_1 c_2 \int_{S^{n-2}} A(\theta) d\theta,$$

uniformly in Q_ρ. Recalling (4.1), this gives the desired result with

$$c_\star = c_1 c_2.$$

Lastly, we rewrite c_2 in (6.8) to show that $c_\star$ can be written as (1.9). Since

$$-\frac{1}{q}\frac{d}{dq}\left[\frac{1}{(1+q^2)^{\frac{n+2s+2}{2}}}\right] = \frac{n+2s+2}{(1+q^2)^{\frac{n+2s+4}{2}}} \quad \text{for } q > 0,$$

we can write

$$c_2 = (n+2s) \int_0^\infty q^n \left[-\frac{1}{q}\frac{d}{dq}\left[\frac{1}{(1+q^2)^{\frac{n+2s+2}{2}}}\right] - \frac{1}{(1+q^2)^{\frac{n+2s+2}{2}}}\right] dq.$$

Integrating by parts, we have

$$-\int_0^\infty q^{n-1} \frac{d}{dq}\left[\frac{1}{(1+q^2)^{\frac{n+2s+2}{2}}}\right] dq = \int_0^\infty (n-1) q^{n-2} \frac{1}{(1+q^2)^{\frac{n+2s+2}{2}}} dq,$$

so that

$$\begin{aligned} c_2 &= (n+2s) \int_0^\infty \left[(n-1) q^{n-2} \frac{1}{(1+q^2)^{\frac{n+2s+2}{2}}} - \frac{q^n}{(1+q^2)^{\frac{n+2s+2}{2}}}\right] dq \\ &= (n+2s) \int_0^\infty \frac{q^{n-2}((n-1)-q^2)}{(1+q^2)^{\frac{n+2s+2}{2}}} dq. \end{aligned}$$

Recalling (2.3), this gives (1.9) for $c_\star = c_1 c_2$. ∎

7. Proof of Theorem 1.3 for $s \in (0, \frac{1}{2})$

Throughout this section, we assume that $s \in (0, \frac{1}{2})$.

In what follows we denote, as usual, $y = (y', y_n)$ with $y' \in \mathbb{R}^{n-1}$. Moreover, we will make the change of variable $z = Ty$ where T is an orthonormal matrix such that

$$\nabla d(x) \cdot (Ty) = y_n. \tag{7.1}$$

We start with some preliminary results.

Lemma 7.1. *Let $s \in (0, \frac{1}{2})$. Then, there exists $C > 0$ such that for all $R, \tau > 0$,*

$$\int_{\{|y_n|<\tau, |y_n|<R|y'|^2\}} \frac{dy}{|y|^{n+2s}} \leq CR^{\frac{1+2s}{2}} \tau^{\frac{1-2s}{2}}. \tag{7.2}$$

Proof. Making a change of variables in y', we compute

$$\begin{aligned}
\int_{\{|y_n|<\tau, |y_n|<R|y'|^2\}} \frac{dy}{|y|^{n+2s}} &= \int_{\{|y_n|<\tau\}} \frac{dy_n}{|y_n|^{n+2s}} \int_{\{|y'|>|y_n|^{\frac{1}{2}} R^{-\frac{1}{2}}\}} \frac{dy'}{(\frac{|y'|^2}{|y_n|^2}+1)^{\frac{n+2s}{2}}} \\
&= \int_{\{|y_n|<\tau\}} \frac{dy_n}{|y_n|^{1+2s}} \int_{\{|z'|>|y_n|^{-\frac{1}{2}} R^{-\frac{1}{2}}\}} \frac{dz'}{(|z'|^2+1)^{\frac{n+2s}{2}}} \\
&\leq \int_{\{|y_n|<\tau\}} \frac{dy_n}{|y_n|^{1+2s}} \int_{\{|z'|>|y_n|^{-\frac{1}{2}} R^{-\frac{1}{2}}\}} \frac{dz'}{|z'|^{n+2s}} \\
&= CR^{\frac{1+2s}{2}} \int_{\{|y_n|<\tau\}} \frac{dy_n}{|y_n|^{\frac{1+2s}{2}}} = CR^{\frac{1+2s}{2}} \tau^{\frac{1-2s}{2}}.
\end{aligned}$$

∎

Lemma 7.2. *Let $s \in (0, \frac{1}{2})$. There exist $\tau_0, R > 0$ such that for all $\tau \leq \tau_0$, $0 \leq \sigma < \tau/2$ and $x \in Q_\rho$,*

$$\int_{\{d(x+z)>d(x)-\sigma,\ -\tau<\nabla d(x)\cdot z<-2\sigma\}} \frac{dz}{|z|^{n+2s}} \leq \int_{\{|y_n|<\tau, |y_n|<R|y'|^2\}} \frac{dy}{|y|^{n+2s}} \tag{7.3}$$

and

$$\int_{\{d(x+z)<d(x)+\sigma,\ 2\sigma<\nabla d(x)\cdot z<\tau\}} \frac{dz}{|z|^{n+2s}} \leq \int_{\{|y_n|<\tau, |y_n|<R|y'|^2\}} \frac{dy}{|y|^{n+2s}}. \tag{7.4}$$

Proof. Since $d \in C^2(Q_{2\rho})$ and Lipschitz continuous in $\mathbb{R}^n$, there exists $C > 0$ such that, for all $x \in Q_\rho$ and $z \in \mathbb{R}^n$,

$$|d(x+z) - d(x) - \nabla d(x) \cdot z| \leq C|z|^2$$

so that

$$\begin{aligned}
&\{z : d(x+z) > d(x) - \sigma,\ -\tau < \nabla d(x) \cdot z < -2\sigma\} \\
&= \{z : d(x+z) - d(x) - \nabla d(x) \cdot z > -\sigma - \nabla d(x) \cdot z \\
&\qquad > -\frac{\nabla d(x) \cdot z}{2} > \sigma, \nabla d(x) \cdot z > -\tau\} \\
&\subseteq \{z : |\nabla d(x) \cdot z| < \tau, |\nabla d(x) \cdot z| < C|z|^2\}.
\end{aligned}$$

Then, performing the change of variables $z = Ty$ with T as in (7.1), we get

$$\begin{aligned}\int_{\{d(x+z)>d(x)-\sigma,\ -\tau<\nabla d(x)\cdot z<-2\sigma\}} \frac{dz}{|z|^{n+2s}} &\le \int_{\{|\nabla d(x)\cdot z|<\tau, |\nabla d(x)\cdot z|<C|z|^2\}} \frac{dz}{|z|^{n+2s}} \\ &= \int_{\{|y_n|<\tau, |y_n|<C(|y'|^2+|y_n|^2)\}} \frac{dy}{|y|^{n+2s}} \\ &\le \int_{\{|y_n|<\tau, |y_n|<C(|y'|^2+\tau|y_n|)\}} \frac{dy}{|y|^{n+2s}}.\end{aligned}$$

Let $\tau_0 > 0$ be such that $1 - C\tau_0 = \frac{1}{2}$, then for all $\tau \le \tau_0$,

$$\begin{aligned}\int_{\{|y_n|<\tau, |y_n|<C(|y'|^2+\tau|y_n|)\}} \frac{dy}{|y|^{n+2s}} &\le \int_{\{|y_n|<\tau, |y_n|<C(|y'|^2+\tau_0|y_n|)\}} \frac{dy}{|y|^{n+2s}} \\ &= \int_{\{|y_n|<\tau, |y_n|<R|y'|^2\}} \frac{dy}{|y|^{n+2s}},\end{aligned}$$

where $R = 2C$. This proves (7.3). Estimate (7.4) follows with a similar argument. ■

The following well-known result, see [13, Lemma 1], is a consequence of Lemmas 7.1 and 7.2.

Proposition 7.3. *Let $s \in (0, \frac{1}{2})$ and $x \in Q_\rho$. Then, the quantities*

$$\begin{aligned}\kappa^+[x, d] &= \nu(\{z : d(x+z) > d(x),\ \nabla d(x)\cdot z < 0\}), \\ \kappa^-[x, d] &= \nu(\{z : d(x+z) < d(x),\ \nabla d(x)\cdot z > 0\})\end{aligned}$$

are finite.

Let us proceed with the proof of Theorem 1.3. We begin by writing

$$\begin{aligned}\frac{a_\varepsilon(\xi; x)}{\varepsilon^{2s}} &= \int_{\mathbb{R}^n} \left(\phi\left(\xi + \frac{d(x+z)-d(x)}{\varepsilon}\right) - \phi\left(\xi + \frac{\nabla d(x)\cdot z}{\varepsilon}\right)\right) \frac{dz}{|z|^{n+2s}} \\ &= \int_{\{d(x+z)>d(x),\ \nabla d(x)\cdot z<0\}} (\cdots) + \int_{\{d(x+z)<d(x),\ \nabla d(x)\cdot z>0\}} (\cdots) \\ &\quad + \int_{\{d(x+z)>d(x),\ \nabla d(x)\cdot z>0\}} (\cdots) + \int_{\{d(x+z)<d(x),\ \nabla d(x)\cdot z<0\}} (\cdots) \\ &=: I_1(\xi) + I_2(\xi) + I_3(\xi) + I_4(\xi).\end{aligned} \tag{7.5}$$

Step 1: Estimating $\int_{\mathbb{R}} \dot\phi(\xi) I_1(\xi) d\xi$ and $\int_{\mathbb{R}} \dot\phi(\xi) I_2(\xi) d\xi$. We will show that

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}} \dot\phi(\xi) I_1(\xi) d\xi = \kappa^+[x, d] \tag{7.6}$$

and

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}} \dot\phi(\xi) I_2(\xi) d\xi = -\kappa^-[x, d]. \tag{7.7}$$

For $\delta > 0$, we write

$$\begin{aligned} I_1(\xi) &= \int_{\{d(x+z)-d(x)>\delta,\ \nabla d(x)\cdot z<-\delta\}} (\cdots) + \int_{\{d(x+z)-d(x)>0,\ -\delta<\nabla d(x)\cdot z<0\}} (\cdots) \\ &\quad + \int_{\{0<d(x+z)-d(x)<\delta,\ \nabla d(x)\cdot z<-\delta\}} (\cdots) \\ &=: J_1(\xi) + J_2(\xi) + J_3(\xi). \end{aligned} \tag{7.8}$$

We first estimate $\int_{\mathbb{R}} \dot\phi(\xi) J_1(\xi) d\xi$. Then, by (2.1), for $|\xi| < \delta/(2\varepsilon)$ and $z \in \mathbb{R}^n$ such that $d(x+z) - d(x) > \delta$ and $\nabla d(x) \cdot z < -\delta$, we have

$$\begin{aligned} &\phi\left(\xi + \frac{d(x+z)-d(x)}{\varepsilon}\right) - \phi\left(\xi + \frac{\nabla d(x)\cdot z}{\varepsilon}\right) \\ &= H\left(\xi + \frac{d(x+z)-d(x)}{\varepsilon}\right) - H\left(\xi + \frac{\nabla d(x)\cdot z}{\varepsilon}\right) \\ &\quad + O\left(\left|\xi + \frac{d(x+z)-d(x)}{\varepsilon}\right|^{-2s}\right) + O\left(\left|\xi + \frac{\nabla d(x)\cdot z}{\varepsilon}\right|^{-2s}\right) = 1 + O\left(\frac{\varepsilon^{2s}}{\delta^{2s}}\right). \end{aligned}$$

Consequently, by (2.4),

$$\int_{-\frac{\delta}{2\varepsilon}}^{\frac{\delta}{2\varepsilon}} \dot\phi(\xi) J_1(\xi) d\xi = \int_{\{d(x+z)-d(x)>\delta,\ \nabla d(x)\cdot z<-\delta\}} \frac{dz}{|z|^{n+2s}} + O\left(\frac{\varepsilon^{2s}}{\delta^{2s}}\right).$$

Moreover, by (2.5) and Proposition 7.3,

$$\begin{aligned} \left| \int_{\{|\xi|>\frac{\delta}{2\varepsilon}\}} \dot\phi(\xi) J_1(\xi) d\xi \right| &\le 2 \int_{\{|\xi|>\frac{\delta}{2\varepsilon}\}} d\xi \dot\phi(\xi) \int_{\{d(x+z)-d(x)>0,\ \nabla d(x)\cdot z<0\}} \frac{dz}{|z|^{n+2s}} \\ &= O\left(\frac{\varepsilon^{2s}}{\delta^{2s}}\right). \end{aligned}$$

From the last two estimates, we infer that

$$\int_{\mathbb{R}} \dot\phi(\xi) J_1(\xi) d\xi = \int_{\{d(x+z)-d(x)>0,\ \nabla d(x)\cdot z<0\}} \frac{dz}{|z|^{n+2s}} + O\left(\frac{\varepsilon^{2s}}{\delta^{2s}}\right) + o_\delta(1). \tag{7.9}$$

Next, by (7.3) with $\sigma = 0$ and $\tau = \delta$, for δ small enough, and by (7.2),

$$\int_{\mathbb{R}} \dot\phi(\xi) J_2(\xi) d\xi = O\big(\delta^{\frac{1-2s}{2}}\big). \tag{7.10}$$

Finally, by Proposition 7.3,

$$\begin{aligned} |J_3(\xi)| &\le 2 \int_{\{d(x+z)-d(x)>0,\ \nabla d(x)\cdot z<0\}} \mathbb{1}_{\{0<d(x+z)-d(x)<\delta\}}(z) \frac{dz}{|z|^{n+2s}} \\ &\le 2 \int_{\{d(x+z)-d(x)>0,\ \nabla d(x)\cdot z<0\}} \frac{dz}{|z|^{n+2s}} \le C. \end{aligned}$$

Since for $x \in Q_\rho$, the set $\{z : d(z) = d(x)\}$ is a smooth surface, we have that

$$\mathbb{1}_{\{0<d(x+z)-d(x)<\delta\}}(z) \to 0 \quad \text{a.e. as } \delta \to 0. \tag{7.11}$$

Therefore, by the Dominated Convergence Theorem, $J_3(\xi) = o_\delta(1)$ and

$$\int_{\mathbb{R}} \dot\phi(\xi) J_3(\xi) d\xi = o_\delta(1). \tag{7.12}$$

From (7.8), (7.9), (7.10), and (7.12), letting first $\varepsilon \to 0$ and then $\delta \to 0$, (7.6) follows. The limit in (7.7) can be proven with a similar argument.

Step 2: Estimating $\int_{\mathbb{R}} \dot\phi(\xi) I_3(\xi) d\xi$ and $\int_{\mathbb{R}} \dot\phi(\xi) I_4(\xi) d\xi$. We will show that

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}} \dot\phi(\xi) I_3(\xi) d\xi = 0 \tag{7.13}$$

and

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}} \dot\phi(\xi) I_4(\xi) d\xi = 0. \tag{7.14}$$

For $\delta > 0$, we write

$$\begin{aligned} I_3(\xi) = & \int_{\{z:d(x+z)-d(x)>\delta,\ \nabla d(x)\cdot z>2\delta\}} (\cdots) + \int_{\{d(x+z)-d(x)>0,\ 0<\nabla d(x)\cdot z<2\delta\}} (\cdots) \\ & + \int_{\{0<d(x+z)-d(x)<\delta,\ \nabla d(x)\cdot z>2\delta\}} (\cdots) \\ =: & J_1(\xi) + J_2(\xi) + J_3(\xi). \end{aligned} \tag{7.15}$$

We first estimate $\int_{\mathbb{R}} \dot\phi(\xi) J_1(\xi) d\xi$. Then, by (2.1), for $|\xi| < \delta/(2\varepsilon)$ and $z \in \mathbb{R}^n$ such that $d(x+z) - d(x) > \delta$ and $\nabla d(x) \cdot z > 2\delta$,

$$\begin{aligned} & \phi\left(\xi + \frac{d(x+z)-d(x)}{\varepsilon}\right) - \phi\left(\xi + \frac{\nabla d(x)\cdot z}{\varepsilon}\right) \\ & = H\left(\xi + \frac{d(x+z)-d(x)}{\varepsilon}\right) - H\left(\xi + \frac{\nabla d(x)\cdot z}{\varepsilon}\right) \\ & \quad + O\left(\left|\xi + \frac{d(x+z)-d(x)}{\varepsilon}\right|^{-2s}\right) + O\left(\left|\xi + \frac{\nabla d(x)\cdot z}{\varepsilon}\right|^{-2s}\right) = O\left(\frac{\varepsilon^{2s}}{\delta^{2s}}\right). \end{aligned}$$

Therefore, for $|\xi| < \delta/(2\varepsilon)$ and performing the change of variables $z = Ty$ with T as in (7.1),

$$|J_1(\xi)| \le O\left(\frac{\varepsilon^{2s}}{\delta^{2s}}\right) \int_{\{\nabla d(x)\cdot z>2\delta\}} \frac{dz}{|z|^{n+2s}} \le O\left(\frac{\varepsilon^{2s}}{\delta^{2s}}\right) \int_{\{|y|>2\delta\}} \frac{dy}{|y|^{n+2s}} = O\left(\frac{\varepsilon^{2s}}{\delta^{4s}}\right),$$

which, together with (2.4), implies

$$\int_{-\frac{\delta}{2\varepsilon}}^{\frac{\delta}{2\varepsilon}} \dot\phi(\xi) J_1(\xi) d\xi = O\left(\frac{\varepsilon^{2s}}{\delta^{4s}}\right).$$

Moreover, for all $\xi \in \mathbb{R}^n$,

$$|J_1(\xi)| \leq \int_{\{|y|>2\delta\}} \frac{2}{|y|^{n+2s}} dy = O\left(\frac{1}{\delta^{2s}}\right)$$

so that, by (2.5),

$$\int_{\{|\xi|>\frac{\delta}{2\varepsilon}\}} \dot{\phi}(\xi) J_1(\xi) d\xi = O\left(\frac{\varepsilon^{2s}}{\delta^{4s}}\right).$$

From the last two estimates, we infer that

$$\int_{\mathbb{R}} \dot{\phi}(\xi) J_1(\xi) d\xi = O\left(\frac{\varepsilon^{2s}}{\delta^{4s}}\right). \tag{7.16}$$

Next, let us estimate $\int_{\mathbb{R}} \dot{\phi}(\xi) J_2(\xi) d\xi$. By the monotonicity of ϕ, making the change of variables $z = Ty$ with T as in (7.1), and then taking $p = |y'|$, $t = y_n/p$, we estimate

$$\begin{aligned}
J_2(\xi) &\leq \int_{\{0<\nabla d(x)\cdot z<2\delta\}} \left(\phi\left(\xi + \frac{\nabla d(x)\cdot z + C|z|^2}{\varepsilon}\right) - \phi\left(\xi + \frac{\nabla d(x)\cdot z}{\varepsilon}\right)\right) \frac{dz}{|z|^{n+2s}} \\
&= \int_{\{0<y_n<2\delta\}} \left(\phi\left(\xi + \frac{y_n + C(|y'|^2 + y_n^2)}{\varepsilon}\right) - \phi\left(\xi + \frac{y_n}{\varepsilon}\right)\right) \frac{dy}{|y|^{n+2s}} \\
&= \mathcal{H}^{n-2}(S^{n-2}) \int_0^\infty \frac{dp}{p^{1+2s}} \int_0^{\frac{2\delta}{p}} \left(\phi\left(\xi + \frac{tp + Cp^2(1+t^2)}{\varepsilon}\right)\right. \\
&\qquad\qquad \left. - \phi\left(\xi + \frac{tp}{\varepsilon}\right)\right) \frac{dt}{(1+t^2)^{\frac{n+2s}{2}}} \\
&= \int_0^r \frac{dp}{p^{1+2s}} (\cdots) + \int_r^\infty \frac{dp}{p^{1+2s}} (\cdots) =: J_2^1(\xi) + J_2^2(\xi),
\end{aligned}$$

with $r \geq \delta$ to be determined. For the first term above, we have

$$\begin{aligned}
J_2^1(\xi) &= \frac{C}{\varepsilon} \int_0^r dp\, p^{1-2s} \int_0^{\frac{2\delta}{p}} \frac{dt}{(1+t^2)^{\frac{n+2s-2}{2}}} \int_0^1 \dot{\phi}\left(\xi + \frac{tp + \tau C p^2(1+t^2)}{\varepsilon}\right) d\tau \\
&= \frac{C}{\varepsilon} \int_0^r dp\, p^{1-2s} \int_0^{\frac{2\delta}{p}} \frac{dt}{(1+t^2)^{\frac{n+2s-2}{2}}} \\
&\quad \times \int_0^1 \partial_t \left[\phi\left(\xi + \frac{tp + \tau C p^2(1+t^2)}{\varepsilon}\right)\right] \frac{\varepsilon}{p(1+2t\tau Cp)} d\tau \\
&\leq C \int_0^r \frac{dp}{p^{2s}} \int_0^1 d\tau \int_0^{\frac{2\delta}{p}} \partial_t \left[\phi\left(\xi + \frac{tp + \tau C p^2(1+t^2)}{\varepsilon}\right)\right] dt,
\end{aligned}$$

where we used that $p(1+2t\tau Cp) \geq p/2$ if $0 \leq tp \leq 2\delta$ and δ is small enough. Integrating with respect to t, we obtain

$$\begin{aligned} J_2^1(\xi) &\leq C \int_0^r \frac{dp}{p^{2s}} \int_0^1 \left[\phi\left(\xi + \frac{2\delta + \tau C(p^2 + 4\delta^2)}{\varepsilon}\right) - \phi\left(\xi + \frac{\tau C p^2}{\varepsilon}\right)\right] d\tau \\ &\leq C \int_0^r \frac{dp}{p^{2s}} = C r^{1-2s}. \end{aligned}$$

We also estimate

$$J_2^2(\xi) \leq C \int_r^\infty \frac{dp}{p^{1+2s}} \int_0^{\frac{2\delta}{p}} dt = \frac{C\delta}{r^{1+2s}}.$$

Choosing $r = r(\delta)$ such that $r = o_\delta(1)$ and $\delta/r^{1+2s} = o_\delta(1)$, we obtain

$$J_2(\xi) \leq o_\delta(1).$$

The lower bound can be proven with a similar argument. We conclude that

$$\int_{\mathbb{R}} \dot{\phi}(\xi) J_2(\xi) d\xi = o_\delta(1). \tag{7.17}$$

Finally, for τ_0 as in Lemma 7.2 and $\delta < \tau_0/2$, from (7.4) and (7.2),

$$\begin{aligned} |J_3(\xi)| &\leq 2 \int_{\{0<d(x+z)-d(x)<\delta,\ \nabla d(x)\cdot z > 2\delta\}} \frac{dz}{|z|^{n+2s}} \\ &= 2 \int_{\{d(x+z)-d(x)<\delta,\ \nabla d(x)\cdot z > 2\delta\}} \mathbb{1}_{\{0<d(x+z)-d(x)<\delta\}}(z) \frac{dz}{|z|^{n+2s}} \\ &\leq 2 \int_{\{d(x+z)-d(x)<\delta,\ 2\delta < \nabla d(x)\cdot z < \tau_0\}} \frac{dz}{|z|^{n+2s}} + 2 \int_{\{\nabla d(x)\cdot z > \tau_0\}} \frac{dz}{|z|^{n+2s}} \leq C. \end{aligned}$$

Recalling (7.11), we see that by the Dominated Convergence Theorem, $J_3(\xi) = o_\delta(1)$ and

$$\int_{\mathbb{R}} \dot{\phi}(\xi) J_3(\xi) d\xi = o_\delta(1). \tag{7.18}$$

From (7.15), (7.16), (7.17) and (7.18), letting first $\varepsilon \to 0$ and then $\delta \to 0$, (7.13) follows. The limit in (7.7) can be proven with a similar argument.

Conclusion. Recalling (7.5), we combine (7.6), (7.7), (7.13), and (7.14) to complete the proof. ■

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