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Parabolic obstacle problems with a drift term: Existence, uniqueness and asymptotic behavior

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Abstract. In this paper we are concerned with the obstacle problem related to an operator with a drift-type lower order term that in the linear case represents the one related to the Fokker–Plank equation, whose (normalized) solution describes the evolution of the probability density for a stochastic process. The main novelty is the presence in the coefficient of the lower order term of a singularity in the spatial variable and minimal-in-time integrability assumption. We prove the well-posedness of a global solution to the obstacle problem and we describe the asymptotic behavior of such a solution. In particular, in the autonomous case, we prove that the global solution of our obstacle problem converges to the solution of the corresponding elliptic obstacle.

1. Introduction

Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$, $N > 2$, and $T > 0$. In the present paper we study the parabolic obstacle problem on the cylinder $\Omega_T := \Omega \times (0, T)$ related to the operator

$$\mathcal{H}(u) := u_t - \operatorname{div}[A(x, t, \nabla u) + B(x, t, u)]. \tag{1.1}$$

Here the vector fields

$$A = A(x, t, \xi) : \Omega_T \times \mathbb{R}^N \to \mathbb{R}^N,$$

and

$$B = B(x, t, s) : \Omega_T \times \mathbb{R} \to \mathbb{R}^N$$

are both Carathéodory functions, that is, they are measurable with respect to $(x, t) \in \Omega_T$ for all $\xi \in \mathbb{R}^N$ and for all $s \in \mathbb{R}$ respectively and continuous with respect to $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$ respectively for almost every $(x, t) \in \Omega_T$. For A we require strong monotonicity and Lipschitz continuity with respect to the ξ-variable—namely, we assume that there exist α, β positive constants such that

$$[A(x, t, \xi) - A(x, t, \eta)] \cdot (\xi - \eta) \geqslant \alpha |\xi - \eta|^2, \tag{1.2}$$

$$|A(x, t, \xi) - A(x, t, \eta)| \leqslant \beta |\xi - \eta|, \tag{1.3}$$

$$A(x, t, 0) = 0, \tag{1.4}$$

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for almost every $(x,t) \in \Omega_T$ and for all $\xi, \eta \in \mathbb{R}^N$. Finally, we assume that

$$|B(x,t,s_1) - B(x,t,s_2)| \leqslant b(x,t)|s_1 - s_2|, \tag{1.5}$$

$$B(x,t,0) = 0, \tag{1.6}$$

for almost every $(x,t) \in \Omega_T$ and for any $s_1, s_2 \in \mathbb{R}$. We require that $b \in L^2(\Omega_T)$ and that b can be split as

$$b = b_1 + b_2 + b_3, \tag{1.7}$$

where

$$b_1 \in L^\infty(\Omega_T), \tag{1.8}$$

while b_2 satisfies the following assumption:

$$b_2(\cdot,t) \in L^{N,\infty}(\Omega) \quad \text{for a.e. } t \in (0,T), \tag{1.9}$$

$$(0,T) \ni t \mapsto \|b_2(\cdot,t)\|_{L^{N,\infty}(\Omega)} \quad \text{belongs to } L^\infty(0,T), \tag{1.10}$$

and finally,

$$b_3 \in L^2(0,T,L^\infty(\Omega)). \tag{1.11}$$

Here $L^{N,\infty}(\Omega)$ denotes the Marcinkiewicz space. An example of function b that we have in mind is

$$b(x,t) := (e^{-t} + h(t))\frac{g(x)}{\Lambda} + \Big(e^{-t}\frac{\Lambda}{|x|}\Big), \quad x \neq 0, \ \ \Lambda > 0, \tag{1.12}$$

defined in the unit ball of $\mathbb{R}^N$ centered in the origin, with $g(x) \in L^\infty(\Omega)$ and $h(t) \in L^2(0,T)$.

The main difficulty when studying the operator appearing in (1.1) consists in the lack of coercivity of such an operator and the presence of a singular term in the drift coefficient $b(x,t)$. As well as establishing existence and a uniqueness result, we provide estimates in time for the related solution to obstacle problem. In the linear case, the operator $\mathcal{H}(u)$ in (1.1) represents the one related to Fokker–Plank equation, whose (normalized) solution describes the evolution of the probability density for a stochastic process related to the Itô stochastic differential equation. A similar operator also occurs in the diffusion model for semiconductor devices (see [13]) or the electrochemistry model (see [12]). A wide literature occurs on this topic and related applications. For an almost complete classical overview, see [31] and references therein. See also [17, 18, 21, 32]. More recently, the case of a nonsmooth growth drift term has been studied in [6, 7, 15, 16, 19, 30]. About the theory of elliptic obstacle problems as well as of elliptic variational inequalities, we refer to the classical overviews of Bensoussan and Lions (see [4]) and of Kinderlerer and Stampacchia (see [20]). Concerning the parabolic case, first existence results related to problems with time-independent obstacles have been treated by Lions and Stampacchia (see [25]) in the linear case and by Brezis (see [10]) for the more general parabolic problems. The case of regular-in-time obstacle functions has been considered by Brezis in [11]. More generally, obstacle functions that are less regular in time are studied in [1]. In the present paper, we treat time-continuous obstacles in $L^2(0,T,W^{1,2}(\Omega))$ as in [8].

To formulate our problem, we consider for simplicity the particular case of the *zero* obstacle function. Throughout the paper you will find the general case. Set

$$\mathcal{W}(0,T) := \left\{ v \in L^2(0,T,W_0^{1,2}(\Omega)) : v_t \in L^2(0,T,W^{-1,2}(\Omega)) \right\}.$$

We define its convex set

$$\mathcal{K}_0(\Omega_T) := \left\{ v \in \mathcal{W}(0,T) : \ v \geqslant 0 \ \text{ a.e. in } \Omega_T \right\}.$$

Definition 1.1. For

$$u_0 \in L^2(\Omega) \quad \text{such that } u_0 \geqslant 0 \text{ a.e. in } \Omega, \tag{1.13}$$

and

$$f \in L^2(\Omega_T), \tag{1.14}$$

we say that a function $u \in \mathcal{K}_0(\Omega_T)$ is a solution to the obstacle problem related to (1.1) with initial datum u_0 and forcing term f if $u(\cdot,0) = u_0$ and it satisfies the following variational inequality:

$$\begin{aligned} &\int_0^T \langle u_t, v-u \rangle \, \mathrm{d}t + \int_{\Omega_T} A(x,t,\nabla u) \cdot \nabla(v-u) \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + \int_{\Omega_T} B(x,t,u) \cdot \nabla(v-u) \, \mathrm{d}x \, \mathrm{d}t \\ &\geqslant \int_{\Omega_T} f(v-u) \, \mathrm{d}x \, \mathrm{d}t \quad \text{for all } v \in \mathcal{K}_0(\Omega_T). \end{aligned} \tag{1.15}$$

Our first result is the following a priori estimate, which states the continuity of a solution with respect to the initial datum and the forcing term:

Theorem 1.1. *Assume* (1.2)–(1.11), (1.13), *and* (1.14). *Let* $\tilde{f} \in L^2(\Omega_T)$ *and* $w_0 \in L^2(\Omega)$. *Let* u *and* w *be solutions to the obstacle problem for* (1.1) *with data* u_0, f *and* w_0, $\tilde{f}$ *respectively. Then*

$$\int_\Omega |u(t) - w(t)| \, \mathrm{d}x \leqslant \int_\Omega |u_0 - w_0| \, \mathrm{d}x + \int_{\Omega_t} |f - \tilde{f}| \, \mathrm{d}x \, \mathrm{d}s, \tag{1.16}$$

for every $t \in (0,T)$.

As a consequence, a solution to the obstacle problem (in the sense of Definition 1.1) is unique, despite the lack of coercivity and the singularity in the drift term (see [26]). However, assumptions (1.8)–(1.11) do not guarantee in general the existence of a solution; see [5, 14, 16]. A bound on the singular part of the function b is then required.

Let b satisfy (1.8)–(1.11) and let

$$\mathcal{D}_b := \operatorname*{ess\,sup}_{t\in(0,T)} \|b_2(\cdot,t)\|_{L^{N,\infty}(\Omega)}. \tag{1.17}$$

From now on we assume that

$$\mathcal{D}_b < \frac{\alpha}{S_{N,2}}, \tag{1.18}$$

where $S_{N,2}$ is the Sobolev constant appearing in (2.5) below. Then we have the following:

Theorem 1.2. *Assume* (1.2)–(1.11), (1.13), (1.14), *and* (1.18). *Then the obstacle problem for* (1.1) *admits a unique non-negative solution. Moreover, the following Lewy–Stampacchia inequality holds:*

$$0 \leqslant u_t - \operatorname{div}[A(\cdot,\cdot,\nabla u) + B(\cdot,\cdot,u)] - f \leqslant \max\{-f, 0\}. \tag{1.19}$$

As a consequence (see Remark 4.6 below), if f is non-negative, u is the unique non-negative solution of the Cauchy–Dirichlet problem with non-negative initial datum u_0 related to operator (1.1)—namely,

$$\mathcal{H}(u) = f.$$

Let us remark that condition (1.18) does not imply in general a bound on the norm of $b(\cdot,t)$ in $L^{N,\infty}(\Omega)$. Indeed, if $b(x,t)$ is the function in (1.12), then condition (1.18) reduces to

$$\Lambda < \frac{\alpha}{2}(N-2).$$

On the other hand, an easy calculation shows that for almost every $t \in (0,T)$,

$$\|b(t)\|_{L^{N,\infty}(\Omega)} \geqslant \frac{C(N)}{\Lambda}. \tag{1.20}$$

Anyway, we point out that if

$$b_2 \in C([0,T], L^N(\Omega)), \tag{1.21}$$

condition (1.18) is always satisfied by a suitable decomposition of b (see Section 4 and [14, 27] for details) and so we have the next result.

Corollary 1.3. *Assume* (1.2)–(1.8), (1.11), (1.13), (1.14), *and* (1.21). *Then, the conclusions of Theorem* 1.2 *hold true.*

The global well-posedness of the obstacle problem related to (1.1) is addressed, whenever the structure data are defined in $\Omega_\infty := \Omega \times (0,\infty)$. Roughly speaking, a global solution is a function $u \in C_{\mathrm{loc}}([0,\infty), L^2(\Omega)) \cap L^2_{\mathrm{loc}}(0,\infty, W_0^{1,2}(\Omega))$ that satisfies (1.15) for all $T > 0$. Once we have established the existence and the uniqueness of a continuous-in-time global solution, we examine its asymptotic behavior.

Despite the presence of a lower order term in (1.1) and the obstacle function, we provide a description of the time behavior for the solution. More generally, assuming global-in-time summability of the structure data and the forcing term, under a bound similar to (1.18), we establish exponential decay for the spatial L^2-norm of the difference of two solutions, with constants continuously depending on the data (see Theorem 5.2). As consequence, we obtain that if u and w are global solutions to the obstacle problem related to (1.1) with initial data u_0 and w_0 and forcing terms f and $\widetilde{f}$, respectively, then

$$\lim_{t\to\infty} \|u(t) - w(t)\|_{L^2(\Omega)} = 0.$$

Hence, global solutions to the obstacle problem related to (1.1) have the same asymptotic profile, no matter what the initial values and forcing terms are.

Finally, in the autonomous case, we prove that the global solution of the obstacle problem related to (1.1) converges to the solution of the corresponding elliptic obstacle problem (see Theorem 6.1). In particular, in the case of the zero obstacle function we have the following result:

Theorem 1.4. *Let Ω be the unit ball in $\mathbb{R}^N$ and*

$$B(x) = \Big(\frac{g(x)}{\Lambda} + \frac{\Lambda}{|x|}\Big)\frac{x}{|x|}, \quad x \neq 0,$$

with $g \in L^\infty(\Omega)$ and $0 < \Lambda < \frac{N-2}{2}$. Then there exists a unique global solution $u \in C([0,\infty), L^2(\Omega))$ to the obstacle problem related to

$$\mathcal{H}(u) = u_t - \operatorname{div}(\nabla u + B(x)u),$$

with initial datum $u_0 \in L^2(\Omega)$ and forcing term $f \in L^2(\Omega)$. Moreover,

$$\lim_{t\to\infty} \|u(t) - \widetilde{u}\|_{L^2(\Omega)} = 0,$$

with $\widetilde{u} \in W_0^{1,2}(\Omega)$ being the unique solution to the elliptic obstacle problem related to

$$\widetilde{\mathcal{H}}(u) = -\operatorname{div}(\nabla\widetilde{u} + B(x)\widetilde{u}),$$

with forcing term f.

As far as we know, the results above are new also in the case of operator without lower order term and when the drift coefficient b is as in Corollary 1.3. When dealing with inequalities, new and greater difficulties arise compared with the case of equations [7, 16, 28]. Then new strategies occur.

A fundamental tool in proving our results is the validity of the variational inequality on any subcylinder $\Omega \times (0,t)$ for all $t > 0$; see Proposition 3.2. New test functions arise and a regularizing-in-time procedure is necessary. The paper is organized as follows: in Section 2, we introduce some preliminary tools. After this, in Section 3, we describe the obstacle problem for a more general class of obstacle functions. In this section we also prove the aforementioned localization property for the variational inequality. In Section 4 the proof of existence and uniqueness of a solution to the obstacle problem is given, and we establish a Lewy–Stampacchia-type inequality for the solution. In Section 5 we introduce the notion of global solution and we give a complete description of its asymptotic behavior. Finally, in Section 6 we address the autonomous case.

2. Preliminary results

2.1. Notation

We will denote by C (or by similar symbols such as $C_1, C_2, \ldots$) a generic positive constant, which may vary from line to line. To highlight the dependence of a constant C with respect to a set of parameters, we adopt the notation $C(\cdot, \ldots, \cdot)$.

Among remarkable constants appearing in the paper, we mention the constant $\mathcal{C}_2 = \mathcal{C}_2(\Omega, N) > 0$ appearing in the Poincaré inequality

$$\mathcal{C}_2(\Omega, N)\|\varphi\|^2_{L^2(\Omega)} \leqslant \|\nabla\varphi\|^2_{L^2(\Omega)} \quad \text{for all } \varphi \in W_0^{1,2}(\Omega). \tag{2.1}$$

For all $k > 0$, we let $T_k : \mathbb{R} \to \mathbb{R}$ be the truncation at height k—namely,

$$T_k(y) := \min\bigl\{k, \max\{-k, y\}\bigr\} \quad \text{for } y \in \mathbb{R}.$$

We also introduce the function $\Theta_k : \mathbb{R} \to \mathbb{R}$ defined as

$$\Theta_k(y) := \int_0^y T_k(\rho)\,\mathrm{d}\rho \quad \text{for } y \in \mathbb{R}. \tag{2.2}$$

Observe that

$$k|y| - \frac{k^2}{2} \leqslant \Theta_k(y) \leqslant k|y| \quad \text{for } y \in \mathbb{R}. \tag{2.3}$$

2.2. Gronwall's lemma

We shall use the following version of Gronwall's lemma:

Lemma 2.1. *Let $t_0 < T$. Assume that $y, z, \gamma \in C([t_0, T])$ with $z \geqslant 0$ in $[t_0, T]$. If*

$$y(t) \leqslant z(t) + \int_{t_0}^t \gamma(s)y(s)\,\mathrm{d}s \quad \textit{for all } t \in [t_0, T],$$

then

$$y(t) \leqslant z(t) + \int_{t_0}^t z(s)\gamma(s)\exp\Bigl(\int_{t_0}^t \gamma(r)\,\mathrm{d}r\Bigr)\,\mathrm{d}s \quad \textit{for all } t \in [t_0, T].$$

Moreover, if z is nondecreasing, then

$$y(t) \leqslant z(t)\exp\Bigl(\int_{t_0}^t \gamma(s)\,\mathrm{d}s\Bigr) \quad \textit{for all } t \in [t_0, T].$$

2.3. Function spaces

Let Ω be a bounded domain in $\mathbb{R}^N$. Given $1 < p < \infty$ and $1 \leqslant q < \infty$, the Lorentz space $L^{p,q}(\Omega)$ consists of all real measurable functions f defined on Ω for which the quantity

$$\|f\|_{p,q}^q := p \int_0^\infty [\lambda_f(k)]^{\frac{q}{p}} k^{q-1}\,\mathrm{d}k$$

is finite. Here and in what follows, λ_f denotes the distribution function of f, defined as

$$\lambda_f(k) := \left|\left\{x \in \Omega : |f(x)| > k\right\}\right| \quad \text{for all } k > 0.$$

For $p = q$, the Lorentz space $L^{p,p}(\Omega)$ reduces to the Lebesgue space $L^p(\Omega)$. For $q = \infty$, the space $L^{p,\infty}(\Omega)$ consists of all measurable functions f defined on Ω such that

$$\|f\|_{p,\infty}^p := \operatorname*{ess\,sup}_{k>0} k^p \lambda_f(k) < \infty,$$

and it coincides with the Marcinkiewicz space, denoted by $L^{p,\infty}(\Omega)$. The quantity $\|\cdot\|_{p,q}$ is not a norm, since the triangle inequality generally fails. Note that $\|\cdot\|_{p,q}$ is equivalent to a norm and $L^{p,q}(\Omega)$ becomes a Banach space when endowed with it (see [3, 29]).

For Lorentz spaces, the following inclusions hold:

$$L^r(\Omega) \subset L^{p,q}(\Omega) \subset L^{p,r}(\Omega) \subset L^{p,\infty}(\Omega) \subset L^q(\Omega),$$

whenever $1 \leqslant q < p < r \leqslant \infty$. Moreover, for $1 < p < \infty$, $1 \leqslant q \leqslant \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, (where we use the notation $\frac{1}{\infty} = 0$) , if $f \in L^{p,q}(\Omega)$, $g \in L^{p',q'}(\Omega)$, we have the Hölder-type inequality

$$\|fg\|_1 \leqslant \|f\|_{p,q}\,\|g\|_{p',q'}. \tag{2.4}$$

Remark 2.2. We recall that $L^\infty(\Omega)$ is not dense in $L^{N,\infty}(\Omega)$. Then it makes sense to define

$$\mathcal{D}_f := \inf_{g \in L^\infty(\Omega)} \|f - g\|_{L^{N,\infty}(\Omega)}.$$

The condition $\mathcal{D}_f < \gamma$ for some $\gamma > 0$ is equivalent to saying that there exists $f_1 \in L^\infty(\Omega)$ such that

$$\|f - f_1\|_{L^{N,\infty}(\Omega)} < \gamma.$$

Then, if $f = f_1 + f_2$ with $f_2 \in L^{N,\infty}(\Omega)$, a bound on $\|f_2\|_{L^{N,\infty}(\Omega)}$ measures how far f is from $L^\infty(\Omega)$. This motivates the notation used in (1.17).

The Sobolev embedding theorem in Lorentz spaces [2, 29] reads as follows:

Theorem 2.3. *Let us assume that $1 < p < N$, $1 \leqslant q \leqslant p$. Then every function $u \in W_0^{1,1}(\Omega)$ satisfying $|\nabla u| \in L^{p,q}(\Omega)$ actually belongs to $L^{p^*,q}(\Omega)$, where $p^* := \frac{Np}{N-p}$ is the Sobolev conjugate exponent of p and*

$$\|u\|_{p^*,q} \leqslant S_{N,p}\,\|\nabla u\|_{p,q}, \tag{2.5}$$

where $S_{N,p}$ is the Sobolev constant given by $S_{N,p} = \omega_N^{-1/N}\frac{p}{N-p}$.

2.4. Continuity with respect to time and compactness

We recall two well-known results (see [33, Chapter III, page 106]). The first one involves the set

$$\mathcal{W}(0,T) := \left\{v \in L^2(0,T,W_0^{1,2}(\Omega)) : v_t \in L^2(0,T,W^{-1,2}(\Omega))\right\},$$

equipped with the norm

$$\|u\|_{\mathcal{W}(0,T)} := \|u\|_{L^2(0,T,W^{1,2}(\Omega))} + \|u_t\|_{L^2(0,T,W^{-1,2}(\Omega))}.$$

Lemma 2.4. *If $u \in \mathcal{W}(0,T)$, then $u \in C([0,T],L^2(\Omega))$ and*

$$\|u\|_{C([0,T],L^2(\Omega))} \leqslant C\|u\|_{W(0,T)},$$

for some constant $C > 0$ independent of u. Moreover, the function $t \in [0,T] \mapsto \|u(\cdot,t)\|^2_{L^2(\Omega)}$ is absolutely continuous and

$$\frac{1}{2}\left(\|u(\cdot,t)\|^2_{L^2(\Omega)}\right)_t = \langle u_t(\cdot,t), u(\cdot,t)\rangle \quad \textit{for a.e. } t \in [0,T].$$

Due to the classical compactness result of Aubin–Lions, we have the following:

Lemma 2.5. *The set $\mathcal{W}(0,T)$ is compactly embedded into $L^2(\Omega_T)$.*

2.5. Mollification in time

A peculiar tool in the theory of parabolic equations is provided by mollification with respect to the time variable introduced by Landes in [22]. Given $1 \leqslant r \leqslant \infty$ and a function $v \in L^r(0,T,X)$ where X is any separable Banach space, this kind of regularization is denoted by $[\![v]\!]_h$ for $h > 0$ and it formally solves the ODE

$$\partial_t[\![v]\!]_h = -\frac{1}{h}([\![v]\!]_h - v),$$

with initial datum $[\![v]\!]_h(0) = v_0 \in X$. The construction of such regularization goes as follows: the mollification in time $[\![v]\!]_h$ of a function $v \in L^r(0,T,X)$ with $1 \leqslant r \leqslant \infty$ is defined as

$$[\![v]\!]_h(t) := e^{-\frac{t}{h}}v_0 + \frac{1}{h}\int_0^t e^{\frac{s-t}{h}}v(s)\,\mathrm{d}s, \tag{2.6}$$

for $h \in (0,T]$ and $t \in [0,T]$. The next lemma collects the basic properties of $[\![\cdot]\!]_h$ if X is either a Lebesgue space or Sobolev space (see, e.g., [9, Lemma 3.1]).

Lemma 2.6. *Assume that $1 \leqslant p < \infty$.*

(1) *If $v \in L^p(\Omega_T)$ and $v_0 \in L^p(\Omega)$, then*

$$\|[\![v]\!]_h\|_{L^p(\Omega_T)} \leqslant \|v\|_{L^p(\Omega_T)} + C(p)h^{\frac{1}{p}}\|v_0\|_{L^p(\Omega)},$$
$$[\![v]\!]_h \to v \quad \textit{strongly in } L^p(\Omega_T) \textit{ as } h \to 0,$$
$$\partial_t[\![v]\!]_h = \frac{1}{h}(v - [\![v]\!]_h).$$

(2) *If* $\nabla v \in L^p(\Omega_T)$ *and* $\nabla v_0 \in L^p(\Omega)$*, then* $\nabla[\![v]\!]_h = [\![\nabla v]\!]_h$ *and*

$$\|\nabla[\![v]\!]_h\|_{L^p(\Omega_T)} \leqslant \|\nabla v\|_{L^p(\Omega_T)} + C(p)h^{\frac{1}{p}}\|\nabla v_0\|_{L^p(\Omega)},$$
$$[\![\nabla v]\!]_h \to \nabla v \quad \textit{strongly in } L^p(\Omega_T) \textit{ as } h \to 0.$$

(3) *If* $v \in L^\infty(0, T, L^p(\Omega))$ *and* $v_0 \in L^p(\Omega)$*, then* $[\![v]\!]_h \in C([0, T], L^p(\Omega))$ *and*

$$\|[\![v]\!]_h\|_{L^\infty(0,T,L^p(\Omega))} \leqslant \|v\|_{L^\infty(0,T,L^p(\Omega))} + \|v_0\|_{L^p(\Omega)}.$$

Moreover,

$$[\![v]\!]_h \to v \quad \textit{strongly in } L^p(\Omega_T) \textit{ as } h \to 0,$$
$$[\![v]\!]_h \to v \quad \textit{strongly in } C([0, T], L^p(\Omega)) \textit{ as } h \to 0,$$

and $[\![v]\!]_h(0) = v_0$.

(4) *If* $v_k \to v$ *strongly in* $L^p(\Omega_T)$*, then*

$$[\![v_k]\!]_h \to [\![v]\!]_h \quad \textit{and} \quad \partial_t[\![v_k]\!]_h \to \partial_t[\![v]\!]_h \quad \textit{strongly in } L^p(\Omega_T).$$

(5) *If* $\nabla v_k \to \nabla v$ *strongly in* $L^p(\Omega_T)$*, then*

$$\nabla[\![v_k]\!]_h \to \nabla[\![v]\!]_h \quad \textit{strongly in } L^p(\Omega_T).$$

(6) *If* $v_k \rightharpoonup v$ *weakly in* $L^p(\Omega_T)$ *(or* $\nabla v_k \rightharpoonup \nabla v$*) weakly in* $L^p(\Omega_T)$*, then* $[\![v_k]\!]_h \rightharpoonup [\![v]\!]_h$ *weakly in* $L^p(\Omega_T)$ *(or* $\nabla[\![v_k]\!]_h \rightharpoonup \nabla[\![v]\!]_h$ *weakly in* $L^p(\Omega_T)$*).*

3. Statement of the problem

The aim of this section is to introduce the obstacle problem for a general class of obstacle functions. More precisely, let ψ be a measurable function defined in Ω_T satisfying the following regularity conditions:

$$\psi \in C([0, T], L^2(\Omega)) \cap L^2(0, T, W^{1,2}(\Omega)), \tag{3.1}$$
$$\psi \leqslant 0 \quad \text{a.e. in } \partial\Omega \times (0, T), \tag{3.2}$$
$$\psi_t \in L^2(\Omega_T), \tag{3.3}$$
$$\psi(\cdot, 0) \in W_0^{1,2}(\Omega). \tag{3.4}$$

Let us consider the convex subset $\mathcal{K}_\psi(\Omega_T)$ of $\mathcal{W}(0, T)$ defined as

$$\mathcal{K}_\psi(\Omega_T) := \{v \in \mathcal{W}(0, T) : \ v \geqslant \psi \ \text{ a.e. in } \Omega_T\}.$$

We provide the following immediate generalization of Definition 1.1:

Definition 3.1. For
$$u_0 \in L^2(\Omega) \tag{3.5}$$
satisfying the compatibility condition
$$u_0 \geqslant \psi(\cdot, 0) \quad \text{a.e. in } \Omega, \tag{3.6}$$
and
$$f \in L^2(\Omega_T), \tag{3.7}$$
we say that a function $u \in \mathcal{K}_\psi(\Omega_T)$ is a solution to the obstacle problem related to (1.1) with initial datum u_0 and forcing term f if $u(\cdot, 0) = u_0$ and the variational inequality
$$\int_0^T \langle u_t, v - u\rangle \, \mathrm{d}t + \int_{\Omega_T} A(x,t,\nabla u)\cdot \nabla(v-u)\,\mathrm{d}x\,\mathrm{d}t \\ + \int_{\Omega_T} B(x,t,u)\cdot\nabla(v-u)\,\mathrm{d}x\,\mathrm{d}t \geqslant \int_{\Omega_T} f(v-u)\,\mathrm{d}x\,\mathrm{d}t, \tag{3.8}$$
holds for any $v \in \mathcal{K}_\psi(\Omega_T)$.

As already noticed in the introduction, a feature that deserves to be highlighted for our problem is a localization property, that is, any solution of the obstacle problem settled in Ω_T solves the same obstacle problem in any subcylinder of the form Ω_τ with $\tau \in (0, T)$. Before we give the proof of the localization property, we recall an approximation result for functions v belonging to
$$L^\infty(0, T, L^2(\Omega)) \cap L^2(0, T, W_0^{1,2}(\Omega)),$$
which satisfy $v \geq \psi$ almost everywhere in Ω_T via mollification in time when the obstacle satisfies (3.1)–(3.4). It can be found in [8, Lemma 3.1] and we state it below.

Lemma 3.1. *Let (3.1)–(3.4) hold and let $v \in L^\infty(0, T, L^2(\Omega)) \cap L^2(0, T, W_0^{1,2}(\Omega))$ be such that $v \geq \psi$ almost everywhere in Ω_T. For $h \in (0, T]$ set $w_h := \max\{\psi, [\![v]\!]_h\}$ where $[\![v]\!]_h$ is defined as in (2.6) with an initial datum $v_0 \in W_0^{1,2}(\Omega)$ such that $v_0 \geqslant \psi(\cdot, 0)$ almost everywhere in Ω. Then $w_h \in \mathcal{K}_\psi(\Omega_T)$ and $w_h \to v$ strongly in $L^2(0, T, W^{1,2}(\Omega))$ as $h \to 0$ and $w_h(\cdot, 0) = v_0$. Moreover, if $0 \leqslant t_1 < t_2 \leqslant T$, then*
$$\limsup_{h\to 0^+} \int_{\Omega\times(t_1,t_2)} (w_h)_t(w_h - v)\,\mathrm{d}x\,\mathrm{d}t \leqslant 0.$$

We are in a position to state and prove the aforementioned localization property.

Proposition 3.2. *Assume (1.2)–(1.11) and let the obstacle function ψ satisfy (3.1)–(3.4) and (3.6) with $u_0 \in L^2(\Omega)$. If $u \in \mathcal{K}_\psi(\Omega_T)$ satisfies $u(0) = u_0$ and inequality (3.8) for every $v \in \mathcal{K}_\psi(\Omega_T)$ with f satisfying (3.7), then for every $\tau \in (0, T)$, we also have*
$$\int_0^\tau \langle u_t, v - u\rangle \, \mathrm{d}t + \int_{\Omega_\tau} A(x,t,\nabla u)\cdot \nabla(v-u)\,\mathrm{d}x\,\mathrm{d}t \\ + \int_{\Omega_\tau} B(x,t,u)\cdot\nabla(v-u)\,\mathrm{d}x\,\mathrm{d}t \geqslant \int_{\Omega_\tau} f(v-u)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.9}$$

Proof. For $\theta \in (0,\tau)$, let $\xi_\theta \in C^0([0,T])$ be defined as follows:

$$\xi_\theta(t) := \chi_{[0,\tau-\theta]} + \frac{1}{\theta}(\tau - t)\chi_{(\tau-\theta,\tau]} = \begin{cases} 1 & \text{if } t \in [0,\tau-\theta], \\ \frac{1}{\theta}(\tau-t) & \text{if } t \in (\tau-\theta,\tau], \\ 0 & \text{if } t \in (\tau, T]. \end{cases}$$

Let us consider $v \in \mathcal{K}_\psi(\Omega_T)$ such that $v_t \in L^2(\Omega_T)$ and, for $h \in (0,T]$, let $w_h := \max\{[\![u]\!]_h, \psi\}$ where $[\![u]\!]_h$ is defined assuming that it satisfies the initial condition $[\![u]\!]_h(\cdot,0) = \psi(\cdot,0)$. Clearly $w_h \in \mathcal{K}_\psi(\Omega_T)$ by Lemma 3.1. Therefore, the function

$$\widetilde{v}_{\theta,h} := \xi_\theta v + (1-\xi_\theta) w_h,$$

is admissible for inequality (3.8). Thus, we have

$$\begin{aligned} &\int_{\Omega_T} (\widetilde{v}_{\theta,h})_t (\widetilde{v}_{\theta,h} - u)\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} [A(x,t,\nabla u) + B(x,t,u)]\cdot\nabla(\widetilde{v}_{\theta,h} - u)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \frac{1}{2}\|\widetilde{v}_{\theta,h}(0) - u_0\|^2_{L^2(\Omega)} - \frac{1}{2}\|\widetilde{v}_{\theta,h}(T) - u(T)\|^2_{L^2(\Omega)} \\ &\quad \geqslant \int_0^T f(\widetilde{v}_\theta - u)\,\mathrm{d}t, \end{aligned} \tag{3.10}$$

where we used the fact that

$$\begin{aligned} &\int_0^T \langle u_t, \widetilde{v}_{\theta,h} - u\rangle\,\mathrm{d}t \\ &\quad = \int_{\Omega_T} (\widetilde{v}_{\theta,h})_t (\widetilde{v}_\theta - u)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\|\widetilde{v}_\theta(0) - u_0\|^2_{L^2(\Omega)} - \frac{1}{2}\|\widetilde{v}_\theta(T) - u(T)\|^2_{L^2(\Omega)}. \end{aligned}$$

We decompose the second term in the left-hand side of (3.10) as follows:

$$\begin{aligned} &\int_{\Omega_T} [A(x,t,\nabla u) + B(x,t,u)]\cdot\nabla(\widetilde{v}_\theta - u)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_{\Omega\times(0,\tau-\theta)} [A(x,t,\nabla u) + B(x,t,u)]\cdot\nabla(v - u)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_{\Omega\times[\tau-\theta,\tau)} [A(x,t,\nabla u) + B(x,t,u)]\cdot\nabla(\widetilde{v}_{\theta,h} - u)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_{\Omega\times(\tau,T)} [A(x,t,\nabla u) + B(x,t,u)]\cdot\nabla(w_h - u)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.11}$$

Since $0 \leqslant \xi_\theta(t) \leqslant 1$ for every $t \in (0,T)$, we deduce that $|\nabla(\widetilde{v}_{\theta,h} - u)| \leqslant |\nabla v| + |\nabla w_h| + |\nabla u|$ and so the second term in the right-hand side of (3.11) vanishes as $\theta \to 0^+$ by assumptions (1.3)–(1.11) and applying the dominated convergence theorem. Hence, we have

$$\lim_{\theta\to 0^+}\int_{\Omega_T}[A(x,t,\nabla u)+B(x,t,u)]\cdot\nabla(\widetilde{v}_\theta-u)\,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_{\Omega\times(0,\tau)}[A(x,t,\nabla u)+B(x,t,u)]\cdot\nabla(v-u)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{\Omega\times(\tau,T)}[A(x,t,\nabla u)+B(x,t,u)]\cdot\nabla(w_h-u)\,\mathrm{d}x\,\mathrm{d}t.$$

In a similar way, we obtain

$$\lim_{\theta\to 0^+}\int_{\Omega_T} f(\widetilde{v}_\theta-u)\,\mathrm{d}x\,\mathrm{d}t=\int_{\Omega\times(0,\tau)} f(v-u)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{\Omega\times(\tau,T)} f(w_h-u)\,\mathrm{d}x\,\mathrm{d}t.$$

We deal with the first integral in (3.10) and we decompose it as follows:

$$\int_{\Omega_T}(\widetilde{v}_{\theta,h})_t(\widetilde{v}_{\theta,h}-u)\,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_{\Omega\times(0,\tau-\theta)} v_t(v-u)\,\mathrm{d}x\,\mathrm{d}t+\int_{\Omega\times(\tau-\theta,\tau)}(\widetilde{v}_{\theta,h})_t(\widetilde{v}_{\theta,h}-u)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{\Omega\times(\tau,T)}(w_h)_t(w_h-u)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.12}$$

Then, we observe that

$$\lim_{\theta\to 0^+}\int_{\Omega\times(0,\tau-\theta)} v_t(v-u)\,\mathrm{d}x\,\mathrm{d}t=\int_{\Omega\times(0,\tau)} v_t(v-u)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.13}$$

Now we focus on the second integral in the right-hand side of (3.12). By the definition of $\widetilde{v}_{\theta,h}$ we see that

$$\int_{\Omega\times(\tau-\theta,\tau)}(\widetilde{v}_{\theta,h})_t(\widetilde{v}_{\theta,h}-u)\,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_{\Omega\times(\tau-\theta,\tau)}\xi_\theta'\xi_\theta|v-w_h|^2\,\mathrm{d}x\,\mathrm{d}t+\int_{\Omega\times(\tau-\theta,\tau)}\xi_\theta'(v-w_h)(w_h-u)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{\Omega\times(\tau-\theta,\tau)}\big(\xi_\theta v_t+(1-\xi_\theta)(w_h)_t\big)\big(\xi_\theta(v-w_h)+w_h-u\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$\equiv I_1(\theta)+I_2(\theta)+I_3(\theta). \tag{3.14}$$

We are interested in estimating the limits of $I_1(\theta)$, $I_2(\theta)$, and $I_3(\theta)$ as $\theta\to 0^+$; hence, we do not highlight the dependence on h of such integrals. First, we have

$$\lim_{\theta\to 0^+} I_1(\theta)=-\frac{1}{2}\|(v-w_h)(\tau)\|^2_{L^2(\Omega)}. \tag{3.15}$$

Then, we also obtain

$$\lim_{\theta\to 0^+} I_2(\theta) \leqslant \|(v-w_h)(w_h-u)(\tau)\|_{L^1(\Omega)}. \tag{3.16}$$

Moreover, by the dominated convergence theorem, we deduce

$$\lim_{\theta\to 0^+} I_3(\theta) = 0. \tag{3.17}$$

If we gather (3.15), (3.16), and (3.17) and if we pass to the limit as $\theta \to 0^+$ in (3.14), we obtain

$$\begin{aligned}\limsup_{\theta\to 0^+} &\int_{\Omega\times(\tau-\theta,\tau)} (\widetilde{v}_{\theta,h})_t(\widetilde{v}_{\theta,h}-u)\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant -\frac{1}{2}\|(v-w_h)(\tau)\|^2_{L^2(\Omega)} + \|(v-w_h)(w_h-u)(\tau)\|_{L^1(\Omega)}. \end{aligned}\tag{3.18}$$

Subsequently, if we gather (3.13) and (3.18), and if we pass to the limit as $\theta \to 0^+$ in (3.12), we have

$$\begin{aligned}\limsup_{\theta\to 0^+} &\int_{\Omega_T} (\widetilde{v}_{\theta,h})_t(\widetilde{v}_{\theta}-u)\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \int_{\Omega_\tau} v_t(v-u)\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\|(v-w_h)(\tau)\|^2_{L^2(\Omega)} + \|(v-w_h)(w_h-u)(\tau)\|_{L^1(\Omega)} \\ &\quad + \int_{\Omega\times(\tau,T)} (w_h)_t(w_h-u)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Now, if we take into account the previous inequalities we pass to the limit as $\theta \to 0^+$ in (3.10), we obtain

$$\begin{aligned}&\int_{\Omega_\tau} v_t(v-u)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\|v(0)-u_0\|^2_{L^2(\Omega)} - \frac{1}{2}\|(v-w_h)(\tau)\|^2_{L^2(\Omega)} \\ &\qquad + \int_{\Omega_\tau} [A(x,t,\nabla u) + B(x,t,u)]\cdot\nabla(v-u)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_{\Omega\times(\tau,T)} [A(x,t,\nabla u) + B(x,t,u)]\cdot\nabla(w_h-u)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \|(v-w_h)(w_h-u)(\tau)\|_{L^1(\Omega)} - \frac{1}{2}\|w_h(T)-u(T)\|^2_{L^2(\Omega)} \\ &\qquad + \int_{\Omega\times(\tau,T)} (w_h)_t(w_h-u)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad \geqslant \int_{\Omega_\tau} f(v-u)\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega\times(\tau,T)} f(w_h-u)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Then we pass to the limit as $h \to 0^+$, and since by Lemma 2.6 we have

$$\limsup_{h\to 0^+} \int_{\Omega\times(\tau,T)} (w_h)_t(w_h-u)\,\mathrm{d}x\,\mathrm{d}t \leqslant 0,$$

we conclude that

$$\int_{\Omega_\tau} v_t(v-u)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\|v(0)-u_0\|^2_{L^2(\Omega)} - \frac{1}{2}\|(v-u)(\tau)\|^2_{L^2(\Omega)} \\ + \int_{\Omega_\tau} [A(x,t,\nabla u) + B(x,t,u)]\cdot\nabla(v-u)\,\mathrm{d}x\,\mathrm{d}t \geqslant \int_{\Omega_\tau} f(v-u)\,\mathrm{d}x\,\mathrm{d}t. \quad (3.19)$$

Once observed that the first three terms in the left-hand side of (3.19) equals

$$\int_0^\tau \langle u_t, v-u\rangle\,\mathrm{d}x\,\mathrm{d}t,$$

we get the inequality

$$\int_0^\tau \langle u_t, v-u\rangle\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_\tau} [A(x,t,\nabla u) + B(x,t,u)]\cdot\nabla(v-u)\,\mathrm{d}x\,\mathrm{d}t \\ \geqslant \int_{\Omega_\tau} f(v-u)\,\mathrm{d}x\,\mathrm{d}t,$$

for every test function $v \in \mathcal{K}_\psi(\Omega_T)$ with the additional property $v_t \in L^2(\Omega_T)$. This last requirement can be removed by an approximation argument (see Remark 3.3 below). ■

Remark 3.3. Inequality (3.9) holds for admissible test function w only satisfying

$$w \in L^2(0,T,W_0^{1,2}(\Omega)) \quad \text{such that } w \geqslant \psi \text{ a.e. in } \Omega_T, \quad (3.20)$$

without requiring the existence of the time derivative of w. Indeed, if w satisfies (3.20), we can define the function $w_h := \max\{[\![w]\!]_h, \psi\}$ where $[\![w]\!]_h$ is chosen such that $[\![w]\!]_h(\cdot,0) = \psi(\cdot,0)$ and by Lemma 3.1 we have $[\![w]\!]_h \in \mathcal{K}_\psi(\Omega_T)$. Therefore, we can use w_h as a test function for (3.9) and we have

$$\int_0^\tau \langle u_t, w_h-u\rangle\;\mathrm{d}t + \int_{\Omega_\tau} A(x,t,\nabla u)\cdot\nabla(w_h-u)\,\mathrm{d}x\,\mathrm{d}t \\ + \int_{\Omega_\tau} B(x,t,u)\cdot\nabla(w_h-u)\;\mathrm{d}x\,\mathrm{d}t \geqslant \int_{\Omega_\tau} f(w_h-u)\,\mathrm{d}x\,\mathrm{d}t. \quad (3.21)$$

Again by Lemma 3.1, we have $w_h \to w$ strongly in $L^2(0,T,W_0^{1,2}(\Omega))$ as $h \to 0^+$, we pass to the limit in (3.21) and we obtain that (3.9) holds for all w satisfying (3.20).

4. A priori estimates, existence, and uniqueness

We start the present section by stating and proving the following more general version of Theorem 1.1:

Theorem 4.1. *Assume* (1.2)–(1.11). *Let the obstacle function* ψ *satisfy* (3.1)–(3.4) *and assume that* (3.5)–(3.7) *hold and that* $w_0 \geq \psi(\cdot, 0)$ *almost everywhere in* Ω, *with* $w_0 \in L^2(\Omega)$. *Let* u *and* w *be solutions to the obstacle problem for* (1.1) *with data* u_0, f *and* w_0, $\tilde{f}$, *respectively, where* f *and* $\tilde{f}$ *belong to* $L^2(\Omega_T)$. *Then*

$$\int_\Omega |u(t) - w(t)| \,\mathrm{d}x \leqslant \int_\Omega |u_0 - w_0| \,\mathrm{d}x + \int_{\Omega_t} \left| f - \tilde{f} \right| \mathrm{d}x\,\mathrm{d}s \tag{4.1}$$

for every $t \in (0, T)$.

Proof. We fix $\varepsilon > 0$. By Proposition 3.2 we know that inequalities (3.8) and (5.10) hold true in Ω_t, for every $t \in (0, T)$. Testing such inequalities in Ω_t by (respectively) $u - T_\varepsilon(u - w)$ and $w - T_\varepsilon(w - u)$ and summing the resulting inequalities, we obtain

$$\begin{aligned}\int_\Omega \Theta_\varepsilon(u(t) - w(t)) \,\mathrm{d}x &+ \int_{\Omega_t} [A(x, s, \nabla u) - A(x, s, \nabla w)] \nabla T_\varepsilon(u - w) \,\mathrm{d}x\,\mathrm{d}s \\ &+ \int_{\Omega_t} [B(x, s, u) - B(x, s, w)] \nabla T_\varepsilon(u - w) \,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant \int_\Omega \Theta_\varepsilon(u_0 - w_0) \,\mathrm{d}x + \int_{\Omega_t} (f - \tilde{f}) T_\varepsilon(u - w) \,\mathrm{d}x\,\mathrm{d}s,\end{aligned} \tag{4.2}$$

where $\Theta_\varepsilon(z)$ is as in (2.2). By (4.2) and (2.3), we have

$$\begin{aligned}\int_\Omega \Theta_\varepsilon(u(t) - w(t)) \,\mathrm{d}x &+ \int_{\Omega_t} [A(x, s, \nabla u) - A(x, s, \nabla w)] \nabla T_\varepsilon(u - w) \,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant \varepsilon \int_\Omega |u_0 - w_0| \,\mathrm{d}x + \varepsilon \int_{\Omega_t} |f - \tilde{f}| \,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \varepsilon \int_{\Omega_t} b(x, s) |\nabla T_\varepsilon(u - w)| \,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

We divide by ε and we obtain

$$\begin{aligned}\frac{1}{\varepsilon} \int_\Omega \Theta_\varepsilon(u(t) - w(t)) \,\mathrm{d}x &+ \frac{\alpha}{\varepsilon} \int_{\Omega_t} |\nabla T_\varepsilon(u - w)|^2 \,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant \int_\Omega |u_0 - w_0| \,\mathrm{d}x + \int_{\Omega_t} \left| f - \tilde{f} \right| \mathrm{d}x\,\mathrm{d}s + \int_{\Omega_t} b(x, s) |\nabla T_\varepsilon(u - w)| \,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

By Young's inequality we have

$$b |\nabla T_\varepsilon(u - w)| \leqslant \frac{\varepsilon}{2\alpha} b^2 + \frac{\alpha}{2\varepsilon} |T_\varepsilon(u - w)|^2.$$

By the previous inequalities, we deduce

$$\begin{aligned}\frac{1}{\varepsilon} \int_\Omega \Theta_\varepsilon(u(t) - w(t)) \,\mathrm{d}x &+ \frac{\alpha}{2\varepsilon} \int_{\Omega_t} |\nabla T_\varepsilon(u - w)|^2 \,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant \int_\Omega |u_0 - w_0| \,\mathrm{d}x + \int_{\Omega_t} \left| f - \tilde{f} \right| \mathrm{d}x\,\mathrm{d}s + \frac{\varepsilon}{2\alpha} \int_{\Omega_t} |b(x, s)|^2 \,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{4.3}$$

Once observed that by (2.3) it follows that

$$\lim_{\varepsilon\to 0^+} \frac{\Theta_\varepsilon(z)}{\varepsilon} = |z|,$$

and dropping the second non-negative term in the left-hand side of (4.3), we obtain assertion (4.1) as $\varepsilon \to 0$. ∎

Theorem 4.1 gives the following uniqueness result:

Corollary 4.2. *Under the assumptions of Theorem 4.1, the obstacle problem admits a unique solution in the sense of Definition 3.1.*

Another immediate consequence of Theorem 4.1 is the following result:

Corollary 4.3. *Under the assumptions of Theorem 4.1, for every $\lambda > 0$ we have*

$$\begin{aligned}\operatorname*{ess\,sup}_{0<\tau<t} \lambda \big|\{x \in \Omega : |u(x,\tau) - w(x,\tau)| > \lambda\}\big| \\ \le \int_\Omega |u_0 - w_0|\,\mathrm{d}x + \int_{\Omega_t} |f - \tilde{f}|\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

As far as uniqueness is achieved, now we prove existence of a solution to the obstacle problem.

Theorem 4.4. *Assume (1.2)–(1.11) and (1.18). Let $u_0 \in L^2(\Omega)$, $f \in L^2(\Omega_T)$, and $\psi : \Omega \to \mathbb{R}$ satisfy (3.1)–(3.4), (3.6), and*

$$\psi_t - \operatorname{div}\big(A(\cdot,\cdot,\nabla\psi) + B(\cdot,\cdot,\psi)\big) \in L^2(\Omega_T). \tag{4.4}$$

Then, the obstacle problem for (1.1) with initial value u_0 and forcing term f admits a unique solution $u \in \mathcal{K}_\psi(\Omega_T)$. Moreover, the following Lewy–Stampacchia inequality holds:

$$\begin{aligned}0 &\leqslant u_t - \operatorname{div}[A(\cdot,\cdot,\nabla u) + B(\cdot,\cdot,u)] - f \\ &\leqslant \big(\psi_t - \operatorname{div}[A(\cdot,\cdot,\nabla\psi) + B(\cdot,\cdot,\psi)] - f\big)^+.\end{aligned} \tag{4.5}$$

In order to get existence for the obstacle problem, we implement a penalization technique. More precisely, for every fixed $\lambda > 0$, we consider the sequence $(u_{\lambda,n})_{n\in\mathbb{N}}$ of solutions to the problems

$$\begin{cases} (u_{\lambda,n})_t - \operatorname{div}\Big(A(x,t,\nabla u_{\lambda,n}) + \dfrac{B(x,t,T_n(u_{\lambda,n}))}{1+\frac{1}{n}b(x,t)}\Big) & \\ \qquad = f + \Psi^+ Z_\lambda(\psi - u_{\lambda,n}) & \text{in } \Omega_T, \\ u_{\lambda,n} = 0 & \text{on } \partial\Omega \times (0,T), \\ u_{\lambda,n}(\cdot,0) = u_0 & \text{in } \Omega \end{cases} \tag{4.6}$$

where

$$\Psi := \psi_t - \operatorname{div}(A(\cdot,\cdot,\nabla\psi) + B(\cdot,\cdot,\psi)) - f \tag{4.7}$$

and $Z_\lambda \in W^{1,\infty}(\mathbb{R})$ is defined as

$$Z_\lambda(s) = 0 \quad \text{if } s \leqslant -\lambda, \tag{4.8}$$

$$Z_\lambda(s) = 1 + \frac{s}{\lambda} \quad \text{if } -\lambda < s \leqslant 0, \tag{4.9}$$

$$Z_\lambda(s) = 1 \quad \text{if } s > 0. \tag{4.10}$$

As a direct consequence of the definition of Z_λ, we have

$$|Z_\lambda(s)| \leq 1 \quad \text{for every } \ s \in \mathbb{R}. \tag{4.11}$$

Because of (3.7) and (4.4), the function Ψ^+ belongs to $L^2(\Omega_T)$. Therefore, the existence of a solution $u_{\lambda,n} \in C([0,T], L^2(\Omega)) \cap L^2(0,T,W_0^{1,2}(\Omega))$ follows by the classical theory of monotone operators (see [24]). For the sequence $(u_{\lambda,n})_{n\in\mathbb{N}}$, we have a priori estimates and compactness in suitable spaces, as we state in the following result:

Lemma 4.5. *For all $\lambda > 0$ and for all $n \in \mathbb{N}$, there exists a positive constant C independent of λ and n such that*

$$\|u_{\lambda,n}\|_{L^\infty(0,T,L^2(\Omega))} + \|\nabla u_{\lambda,n}\|_{L^2(\Omega_T)} + \|(u_{\lambda,n})_t\|^2_{L^2(0,T,W^{-1,2}(\Omega))} \leqslant C. \tag{4.12}$$

In addition, there exists $u_\lambda \in C([0,T], L^2(\Omega)) \cap L^2(0,T,W_0^{1,2}(\Omega))$ with $(u_\lambda)_t \in L^2(0,T,W^{-1,2}(\Omega))$ such that $u_\lambda(\cdot,0) = u_0$ and up to a subsequence (denoted again $u_{\lambda,n}$) as $n \to \infty$, we have

$$u_{\lambda,n} \to u_\lambda \quad \textit{strongly in } L^2(\Omega_T), \tag{4.13}$$

$$u_{\lambda,n} \to u_\lambda \quad \textit{a.e. in } \Omega_T, \tag{4.14}$$

$$u_{\lambda,n} \overset{*}{\rightharpoonup} u_\lambda \quad \textit{weakly}^* \textit{ in } L^\infty(0,T;L^2(\Omega)), \tag{4.15}$$

$$u_{\lambda,n} \rightharpoonup u_\lambda \quad \textit{weakly in } L^2\big(0,T,W_0^{1,2}(\Omega)\big), \tag{4.16}$$

$$(u_{\lambda,n})_t \rightharpoonup (u_\lambda)_t \quad \textit{weakly in } L^2(0,T,W^{-1,2}(\Omega)), \tag{4.17}$$

$$\nabla u_{\lambda,n} \to \nabla u_\lambda \quad \textit{a.e. in } \Omega_T. \tag{4.18}$$

Proof. We test the equation in problem (4.6) by $u_{\lambda,n}$ and we obtain (thanks to the structure assumptions and to (4.11)) the following estimate:

$$\begin{aligned} &\frac{1}{2}\int_\Omega |u_{\lambda,n}|^2\,dx + \alpha\int_{\Omega_t} |\nabla u_{\lambda,n}|^2\,dx\,ds \\ &\quad\leqslant \frac{1}{2}\int_\Omega |u_0|^2\,dx + \int_{\Omega_t} b(x,t)|u_{\lambda,n}||\nabla u_{\lambda,n}|\,dx\,ds \\ &\qquad + \int_{\Omega_t} |f||u_{\lambda,n}|\,dx\,ds + \int_{\Omega_t} |\Psi^+||u_{\lambda,n}|\,dx\,ds, \end{aligned} \tag{4.19}$$

for every $t \in (0,T)$ arbitrarily fixed. By (1.7), the second integral in the right-hand side of (4.19) can be decomposed as follows:

$$\int_{\Omega_t} b(x,s)|u_{\lambda,n}||\nabla u_{\lambda,n}|\,\mathrm{d}x\,\mathrm{d}s = \sum_{i=1}^{3}\int_{\Omega_t} b_i(x,s)|u_{\lambda,n}||\nabla u_{\lambda,n}|\,\mathrm{d}x\,\mathrm{d}s. \tag{4.20}$$

By the Hölder inequality (2.4) and the Sobolev inequality (2.5), we have

$$\begin{aligned}\int_{\Omega_t} & b_2(x,s)|u_{\lambda,n}||\nabla u_{\lambda,n}|\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant \int_0^t \|b_2(s)\|_{L^{N,\infty}(\Omega)}\|u_{\lambda,n}(s)\|_{L^{2^*,2}(\Omega)}\|\nabla u_{\lambda,n}(s)\|_{L^2(\Omega)}\,\mathrm{d}s\\ &\leqslant S_{N,2}\mathcal{D}_b\|\nabla u_{\lambda,n}\|^2_{L^2(\Omega_t)}.\end{aligned} \tag{4.21}$$

Hence, we reabsorb the second term in the right hand-side of (4.19) by the left hand-side, since by assumption (1.18) we have that

$$\theta := \alpha - S_{N,2}\mathcal{D}_b > 0. \tag{4.22}$$

We obtain

$$\begin{aligned}\frac{1}{2}\|u_{\lambda,n}(t)\|^2_{L^2(\Omega)} + \theta\|\nabla u_{\lambda,n}\|^2_{L^2(\Omega_t)} &\le \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \int_{\Omega_t} b_1|u_{\lambda,n}||\nabla u_{\lambda,n}|\,\mathrm{d}x\,\mathrm{d}s\\ &+ \int_{\Omega_t} b_3|u_{\lambda,n}||\nabla u_{\lambda,n}|\,\mathrm{d}x\,\mathrm{d}s + \int_{\Omega_t}|f||u_{\lambda,n}|\,\mathrm{d}x\,\mathrm{d}s\\ &+ \int_{\Omega_t}|\Psi^+||u_{\lambda,n}|\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{4.23}$$

By means of Young's inequality, we have

$$\begin{aligned}\int_{\Omega_t} & b_1|u_{\lambda,n}||\nabla u_{\lambda,n}|\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant \frac{2}{\theta}\|b_1\|^2_{L^\infty(\Omega_T)}\int_0^t \|u_{\lambda,n}(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + \frac{\theta}{8}\|\nabla u_{\lambda,n}(s)\|^2_{L^2(\Omega_t)},\end{aligned} \tag{4.24}$$

$$\begin{aligned}\int_{\Omega_t} & b_3|u_{\lambda,n}||\nabla u_{\lambda,n}|\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant \frac{2}{\theta}\int_0^t \|b_3(s)\|^2_{L^\infty(\Omega)}\|u_{\lambda,n}(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + \frac{\theta}{8}\|\nabla u_{\lambda,n}\|^2_{L^2(\Omega_t)}.\end{aligned} \tag{4.25}$$

By means of Young's inequality and Poincaré's inequality, we have

$$\int_{\Omega_t}|f||u_{\lambda,n}|\,\mathrm{d}x\,\mathrm{d}s \leqslant c(N,\Omega,\theta)\int_0^t \|f(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + \frac{\theta}{8}\|\nabla u_{\lambda,n}\|^2_{L^2(\Omega_t)}, \tag{4.26}$$

$$\int_{\Omega_t}|\Psi^+||u_{\lambda,n}|\,\mathrm{d}x\,\mathrm{d}s \leqslant c(N,\Omega,\theta)\int_0^t \|\Psi^+(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + \frac{\theta}{8}\|\nabla u_{\lambda,n}\|^2_{L^2(\Omega_t)}. \tag{4.27}$$

By (4.23)–(4.27), we deduce

$$
\begin{aligned}
&\|u_{\lambda,n}(t)\|^2_{L^2(\Omega)} + \theta\|\nabla u_{\lambda,n}\|^2_{L^2(\Omega_t)} \\
&\quad\leqslant \frac{4}{\theta}\|b_1\|^2_{L^\infty(\Omega_s)}\int_0^t \|u_{\lambda,n}(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + \frac{4}{\theta}\int_0^t \|b_3(s)\|^2_{L^\infty(\Omega)}\|u_{\lambda,n}(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s \\
&\qquad + c(N,\Omega,\theta)\int_0^t \big(\|f(s)\|^2_{L^2(\Omega)} + \|\Psi^+(s)\|^2_{L^2(\Omega)}\big)\,\mathrm{d}s + \|u_0\|^2_{L^2(\Omega)}. \qquad (4.28)
\end{aligned}
$$

We apply Lemma 2.1 by choosing

$$
y(t) = \|u_{\lambda,n}(t)\|^2_{L^2(\Omega)}, \qquad (4.29)
$$

$$
z(t) = c(N,\Omega,\theta)\int_0^t \big(\|f(s)\|^2_{L^2(\Omega)} + \|\Psi^+(s)\|^2_{L^2(\Omega)}\big)\,\mathrm{d}s + \|u_0\|^2_{L^2(\Omega)}, \qquad (4.30)
$$

$$
\gamma(s) = \frac{4}{\theta}\big(\|b_1\|^2_{L^\infty(\Omega_T)} + \|b_3(s)\|^2_{L^\infty(\Omega)}\big), \qquad (4.31)
$$

and we conclude that

$$
\|u_{\lambda,n}\|_{L^\infty(0,T,L^2(\Omega))} \leqslant C_1, \qquad (4.32)
$$

where C_1 is a positive constant independent of λ and n. Combining (4.28) with (4.32), we also obtain

$$
\|\nabla u_{\lambda,n}\|_{L^2(\Omega_\tau)} \leqslant C_2, \qquad (4.33)
$$

where C_2 is a positive constant independent of λ and n. With the aid of the equation in (4.6) and taking into account estimates (4.32) and (4.33), we obtain

$$
\|(u_{\lambda,n})_t\|_{L^2(0,T,W^{-1,2}(\Omega))} \leqslant C_3,
$$

where C_3 is a positive constant independent of λ and n. Therefore, thanks also to the compactness lemma of Aubin–Lions, we obtain the existence of a limit function u_λ for which (4.13)–(4.17) hold. It is then also clear that $u_\lambda(\cdot,0) = u_0$, since $u_{\lambda,n}(\cdot,0) = u_0$ for all $n \in \mathbb{N}$.

It remains to prove that (4.18) holds. To this aim, we test problem (4.6) by $T_k(u_{\lambda,n} - u_\lambda)$, and recalling the definition of Θ_k in (2.2), we have

$$
\begin{aligned}
&\int_\Omega \Theta_k(u_{\lambda,n}(T) - u_\lambda(T))\,\mathrm{d}x \\
&\qquad + \int_{\Omega_T} [A(x,t,\nabla u_{\lambda,n}) - A(x,t,\nabla u_\lambda)]\cdot\nabla T_k(u_{\lambda,n} - u_\lambda)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leqslant -\int_{\Omega_T} A(x,t,\nabla u_\lambda)\cdot\nabla T_k(u_{\lambda,n} - u_\lambda)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_{\Omega_T} (f + \Psi^+ Z_\lambda(\psi - u_{\lambda,n}))T_k(u_{\lambda,n} - u_\lambda)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \int_{\Omega_T} \frac{B(x,t,T_n(u_{\lambda,n}))}{1+\frac{1}{n}b(x,t)}\cdot\nabla T_k(u_{\lambda,n} - u_\lambda)\,\mathrm{d}x\,\mathrm{d}t. \qquad (4.34)
\end{aligned}
$$

Observe that

$$\lim_{n\to+\infty}\int_{\Omega_T} A(x,t,\nabla u_\lambda)\cdot\nabla T_k(u_{\lambda,n}-u_\lambda)\,\mathrm{d}x\,\mathrm{d}t = 0,$$

because of (4.16). Moreover, we have that

$$\lim_{n\to+\infty}\int_{\Omega_T} (f+\Psi^+ Z_\lambda(\psi-u_{\lambda,n}))T_k(u_{\lambda,n}-u_\lambda)\,\mathrm{d}x\,\mathrm{d}t = 0,$$

because of (4.14) together with the dominated convergence theorem. Now we show that

$$\lim_{n\to+\infty}\int_{\Omega_T} \chi_{\{|u_{\lambda,n}-u_\lambda|\leqslant k\}}\frac{B(x,t,T_n(u_{\lambda,n}))}{1+\frac{1}{n}b(x,t)}\cdot\nabla(u_{\lambda,n}-u_\lambda)\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{4.35}$$

First observe that

$$\chi_{\{|u_{\lambda,n}-u_\lambda|\leqslant k\}}\frac{B(x,t,T_n(u_{\lambda,n}))}{1+\frac{1}{n}b(x,t)} \to B(x,t,u_{\lambda,n}) \quad \text{a.e. in } \Omega_T,$$

because of (4.14). On the other hand, we have

$$\chi_{\{|u_{\lambda,n}-u_\lambda|\leqslant k\}}\frac{|B(x,t,T_n(u_{\lambda,n}))|}{1+\frac{1}{n}b(x,t)} \leqslant b(x,t)(|u_\lambda|+k)\in L^2(\Omega_T),$$

because of (1.5) and (1.6). Hence, it follows that

$$\chi_{\{|u_{\lambda,n}-u_\lambda|\leqslant k\}}\frac{B(x,t,T_n(u_{\lambda,n}))}{1+\frac{1}{n}b(x,t)} \to B(x,t,u_\lambda) \quad \text{strongly in } L^2(\Omega_T),$$

and this allows to conclude that (4.35) holds true. Previous facts allow us to pass to the limit in (4.34) to get

$$\lim_{n\to+\infty}\int_{\Omega_T} [A(x,t,\nabla u_{\lambda,n})-A(x,t,\nabla u_\lambda)]\cdot\nabla T_k(u_{\lambda,n}-u_\lambda)\,\mathrm{d}x\,\mathrm{d}t = 0,$$

which leads to (4.18), arguing as in the proof of [23, Lemma 3.3]. ■

Now we are in a position to prove Theorem 4.4.

Proof of Theorem 4.4. It is sufficient to prove the existence of a solution to the obstacle problem, since the uniqueness is given by Corollary 4.2. For reader's convenience, we split the proof in several steps.

Step 1: convergence scheme for a penalization problem.

With convergences (4.13)–(4.16) and (4.18) proved above, we can pass to the limit as $n\to\infty$ in (4.6) and we obtain that the function u_λ obtained in Lemma 4.5 solves the following problem:

$$\begin{cases} (u_\lambda)_t - \operatorname{div}(A(x,t,\nabla u_\lambda)+B(x,t,u_\lambda)) & \\ \qquad = f+\Psi^+ Z_\lambda(\psi-u_\lambda) & \text{in } \Omega_T,\\ u_\lambda = 0 & \text{on } \partial\Omega\times(0,T),\\ u_\lambda(\cdot,0)=u_0 & \text{in } \Omega. \end{cases} \tag{4.36}$$

Moreover, by (4.12), we also have

$$\|u_\lambda\|_{L^\infty(0,T,L^2(\Omega))} + \|\nabla u_\lambda\|_{L^2(\Omega_T)} + \|(u_\lambda)_t\|^2_{L^2(0,T,W^{-1,2}(\Omega))} \leqslant C, \tag{4.37}$$

where C is independent of λ. Proceeding as in the proof of Lemma 4.5, we deduce the existence of a function $u \in C([0,T], L^2(\Omega)) \cap L^2(0,T,W_0^{1,2}(\Omega))$ such that $u(\cdot,0) = u_0$ and (up to a subsequence) as $\lambda \to 0^+$ we have

$$\begin{aligned} u_\lambda &\to u && \text{strongly in } L^2(\Omega_T), \\ u_\lambda &\to u && \text{a.e. in } \Omega_T, \\ u_\lambda &\overset{*}{\rightharpoonup} u && \text{weakly* in } L^\infty(0,T;L^2(\Omega)), \\ u_\lambda &\rightharpoonup u && \text{weakly in } L^2(0,T,W_0^{1,2}(\Omega)), \\ (u_\lambda)_t &\rightharpoonup u_t && \text{weakly in } L^2(0,T,W^{-1,2}(\Omega)). \end{aligned} \tag{4.38}$$

Step 2: we prove that $u \geqslant \psi$ a.e. in Ω_T.

We choose $(\psi - u_\lambda)^+$ as test function in the first equation in problem (4.36) and we obtain

$$\begin{aligned} &\int_0^t \langle (u_\lambda)_t, (\psi - u_\lambda)^+ \rangle \,\mathrm{d}s + \int_{\Omega_t} \big(A(x,s,\nabla u_\lambda) + B(x,s,u_\lambda)\big)\nabla(\psi - u_\lambda)^+ \,\mathrm{d}x\,\mathrm{d}s \\ &\quad = \int_{\Omega_t} f(\psi - u_\lambda)^+ \,\mathrm{d}x\,\mathrm{d}s + \int_{\Omega_t} \Psi^+ Z_\lambda(\psi - u_\lambda)(\psi - u_\lambda)^+ \,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{4.39}$$

We estimate the first integral in the left-hand side of (4.39). Thanks to compatibility condition (3.6), we deduce

$$\begin{aligned} &\int_0^t \langle (u_\lambda)_t, (\psi - u_\lambda)^+ \rangle \,\mathrm{d}s \\ &\quad = \int_0^t \langle \psi_t, (\psi - u_\lambda)^+ \rangle \,\mathrm{d}s - \int_0^t \langle (\psi - u_\lambda)_t, (\psi - u_\lambda)^+ \rangle \,\mathrm{d}s \\ &\quad = \int_0^t \langle \psi_t, (\psi - u_\lambda)^+ \rangle \,\mathrm{d}s - \frac{1}{2}\|(\psi(t) - u_\lambda(t))^+\|^2_{L^2(\Omega)}. \end{aligned}$$

Hence, it follows that (recalling the definition of Ψ in (4.7))

$$\begin{aligned} &\frac{1}{2}\|(\psi(t) - u_\lambda(t))^+\|^2_{L^2(\Omega)} + \int_{\Omega_t} \big(A(x,s,\nabla\psi) - A(x,s,\nabla u_\lambda)\big)\nabla(\psi - u_\lambda)^+ \,\mathrm{d}x\,\mathrm{d}s \\ &\qquad + \int_{\Omega_t} \big(B(x,s,u_\lambda) - B(x,s,\psi)\big)\nabla(\psi - u_\lambda)^+ \,\mathrm{d}x\,\mathrm{d}s \\ &\quad \leqslant -\int_{\Omega_t} \Psi^-(\psi - u_\lambda)^+ \,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{4.40}$$

The right-hand side of (4.40) is nonpositive, and so by the structure assumptions, we have

$$\begin{aligned} &\frac{1}{2}\|(\psi(t) - u_\lambda(t))^+\|^2_{L^2(\Omega)} + \alpha\int_{\Omega_t} \left|\nabla(\psi - u_\lambda)^+\right|^2 \,\mathrm{d}x\,\mathrm{d}s \\ &\quad \leqslant \int_{\Omega_t} b(x,s)(\psi - u_\lambda)^+ |\nabla(\psi - u_\lambda)^+| \,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{4.41}$$

The right-hand side of (4.41) can be decomposed as follows:

$$\begin{aligned}\int_{\Omega_t} b(x,s)(\psi-u_\lambda)^+|\nabla(\psi-u_\lambda)^+|\,\mathrm{d}x\,\mathrm{d}s \\ = \int_{\Omega_t} b_1(x,s)(\psi-u_\lambda)^+|\nabla(\psi-u_\lambda)^+|\,\mathrm{d}x\,\mathrm{d}s \\ + \int_{\Omega_t} b_2(x,s)(\psi-u_\lambda)^+|\nabla(\psi-u_\lambda)^+|\,\mathrm{d}x\,\mathrm{d}s \\ + \int_{\Omega_t} b_3(x,s)(\psi-u_\lambda)^+|\nabla(\psi-u_\lambda)^+|\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{4.42}$$

With computation similar to the one leading to (4.21), we observe that

$$\begin{aligned}&\int_{\Omega_t} b_2(x,s)(\psi-u_\lambda)^+|\nabla(\psi-u_\lambda)^+|\,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant \int_0^t \|b_2(s)\|_{L^{N,\infty}(\Omega)}\|(\psi-u_\lambda)^+(s)\|_{L^{2^*,2}(\Omega)}\|\nabla(\psi-u_\lambda)^+(s)\|_{L^2(\Omega)}\,\mathrm{d}s \\ &\leqslant S_{N,2}\mathcal{D}_b\|\nabla(\psi-u_\lambda)^+\|^2_{L^2(\Omega_t)}.\end{aligned} \tag{4.43}$$

Hence, by (4.41)–(4.43) and thanks to assumption (1.18), we obtain

$$\begin{aligned}&\frac{1}{2}\|(\psi(t)-u_\lambda(t))^+\|^2_{L^2(\Omega)} + \theta\int_{\Omega_t}|\nabla(\psi-u_\lambda)^+|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant \int_{\Omega_t} b_1(x,s)(\psi(s)-u_\lambda(s))^+|\nabla(\psi(s)-u_\lambda(s))^+|\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \int_{\Omega_t} b_3(x,s)(\psi(s)-u_\lambda(s))^+|\nabla(\psi(s)-u_\lambda(s))^+|\,\mathrm{d}x\,\mathrm{d}s,\end{aligned} \tag{4.44}$$

where θ is as in (4.22). By means of Young's inequality we have

$$\begin{aligned}&\int_{\Omega_t} b_1(x,s)(\psi(s)-u_\lambda(s))^+|\nabla(\psi(s)-u_\lambda(s))^+|\,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant \frac{1}{2\theta}\|b_1\|^2_{L^\infty(\Omega_T)}\int_0^t \|(\psi(s)-u_\lambda(s))^+\|^2_{L^2(\Omega)}\,\mathrm{d}s \\ &\quad + \frac{\theta}{2}\|\nabla(\psi-u_\lambda)^+\|^2_{L^2(\Omega_t)},\end{aligned} \tag{4.45}$$

and also

$$\begin{aligned}&\int_{\Omega_t} b_3(x,s)(\psi(s)-u_\lambda(s))^+|\nabla(\psi(s)-u_\lambda(s))^+|\,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant \frac{1}{2\theta}\int_0^t \|b_3(s)\|^2_{L^\infty(\Omega)}\|(\psi(s)-u_\lambda(s))^+\|^2_{L^2(\Omega)}\,\mathrm{d}s \\ &\quad + \frac{\theta}{2}\|\nabla(\psi-u_\lambda)^+\|^2_{L^2(\Omega_t)}.\end{aligned} \tag{4.46}$$

Gathering (4.44)–(4.46), we obtain

$$\|(\psi(t)-u_\lambda(t))^+\|^2_{L^2(\Omega)} \leqslant \frac{1}{\theta}\int_0^t \big(\|b_1\|^2_{L^\infty(\Omega_T)}+\|b_3(s)\|^2_{L^\infty(\Omega)}\big)\|(\psi(s)-u_\lambda(s))^+\|^2_{L^2(\Omega)}\,\mathrm{d}s.$$

We apply Lemma 2.1 by choosing $y(t):=\|(\psi(t)-u_\lambda(t))^+\|^2_{L^2(\Omega)}$, $z(t)\equiv 0$, and $\gamma(t):=\frac{1}{\theta}(\|b_1\|^2_{L^\infty(\Omega_T)}+\|b_3(t)\|^2_{L^\infty(\Omega)})$, and we conclude that $\|(\psi(t)-u_\lambda(t))^+\|^2_{L^2(\Omega)}=0$ for all $t\in(0,T)$, that is, $u_\lambda \geqslant \psi$ almost everywhere in Ω_T. Finally, it follows that

$$u \geqslant \psi \quad \text{a.e. in } \Omega_T \tag{4.47}$$

due to the convergence in (4.38).

Step 3: we prove that (3.8) *holds (i.e., u solves the obstacle problem).*

We test problem (4.36) by $T_\varepsilon(u_\lambda-u)$, where $\varepsilon>0$ is arbitrarily fixed. We obtain

$$\begin{aligned}\int_\Omega \Theta_\varepsilon(u_\lambda-u)(x,T)\,\mathrm{d}t &+ \int_{\Omega_T}[A(x,t,\nabla u_\lambda)-A(x,t,\nabla u)]\cdot\nabla T_\varepsilon(u_\lambda-u)\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant -\int_{\Omega_T} A(x,t,\nabla u)\cdot\nabla T_\varepsilon(u_\lambda-u)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_{\Omega_T}(f+\Psi^+Z_\lambda(\psi-u_\lambda))T_\varepsilon(u_\lambda-u)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad-\int_{\Omega_T} B(x,t,u_\lambda)\cdot\nabla T_\varepsilon(u_\lambda-u)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{4.48}$$

where Θ_ε is as in (2.2). We explicitly remark that the left-hand side of (4.48) is non-negative, while the right-hand side vanishes in the limit as $\lambda\to 0^+$. Hence, it follows that

$$\lim_{\lambda\to 0^+}\int_\Omega \Theta_\varepsilon(u_\lambda-u)(x,T)\,\mathrm{d}x=0, \tag{4.49}$$

$$\lim_{\lambda\to 0^+}\int_{\Omega_T}[A(x,t,\nabla u_\lambda)-A(x,t,\nabla u)]\cdot\nabla T_\varepsilon(u_\lambda-u)\,\mathrm{d}x\,\mathrm{d}t=0. \tag{4.50}$$

Once again, (4.50) implies

$$\nabla u_\lambda \to \nabla u \quad \text{a.e. in } \Omega_T. \tag{4.51}$$

Moreover, because of (2.3), one has

$$|u_\lambda(x,T)-u(x,T)| \leqslant \frac{1}{\varepsilon}\Theta_\varepsilon(u_\lambda-u)(x,T)+\frac{\varepsilon}{2}. \tag{4.52}$$

Observe that (4.49) and (4.52) imply

Passing to the limit as $\varepsilon\to 0^+$, we obtain

$$\lim_{\lambda\to 0^+}\int_\Omega |u_\lambda(x,T)-u(x,T)|\,\mathrm{d}x=0. \tag{4.53}$$

Let $v \in \mathcal{K}_\psi(\Omega_T)$ and let η_λ be a cut-off function with respect to the space variable such that $\eta_\lambda \in C_0^\infty(\Omega)$, $0 \leqslant \eta_\lambda \leqslant 1$ in Ω, $\eta_\lambda = 1$ in $\Omega^\lambda := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \lambda\}$, and $\lambda|\nabla\eta_\lambda| \leqslant C$. We define $v_\lambda := v + \lambda\eta_\lambda$. We use $T_k(v_\lambda - u_\lambda)$ as a test function in the first equation of (4.36) and we obtain

$$\int_0^T \langle (u_\lambda)_t, T_k(v_\lambda - u_\lambda)\rangle \,\mathrm{d}s + \int_{\Omega_T} (A(x,s,\nabla u_\lambda) + B(x,s,u_\lambda))\nabla T_k(v_\lambda - u_\lambda)\,\mathrm{d}x\,\mathrm{d}s$$
$$= \int_{\Omega_T} \Psi^+ Z_\lambda(\psi - u_\lambda)T_k(v_\lambda - u_\lambda)\,\mathrm{d}x\,\mathrm{d}s + \int_{\Omega_T} f\,T_k(v_\lambda - u_\lambda)\,\mathrm{d}x\,\mathrm{d}s. \tag{4.54}$$

Integrating by parts, the first term in the left-hand side of (4.39) satisfies the identity

$$\int_0^T \langle (u_\lambda)_t, T_k(v_\lambda - u_\lambda)\rangle \,\mathrm{d}s = \int_0^T \langle (v_\lambda)_t, T_k(v_\lambda - u_\lambda)\rangle \,\mathrm{d}s$$
$$+ \int_\Omega \Theta_k(v_\lambda(x,0) - u_0)\,\mathrm{d}x - \int_\Omega \Theta_k(v_\lambda - u_\lambda)(x,T)\,\mathrm{d}x. \tag{4.55}$$

As Θ_k is Lipschitz continuous, we have

$$\left|\int_\Omega \Theta_k(v_\lambda - u_\lambda)(x,T)\,\mathrm{d}x - \int_\Omega \Theta_k(v-u)(x,T)\,\mathrm{d}x\right|$$
$$\leqslant \int_\Omega |\Theta_k(v_\lambda - u_\lambda)(x,T) - \Theta_k(v-u)(x,T)|\,\mathrm{d}x$$
$$\leqslant \int_\Omega |v_\lambda - v|(x,T)\,\mathrm{d}x + \int_\Omega |u_\lambda - u|(x,T)\,\mathrm{d}x. \tag{4.56}$$

Because of (4.53), we have

$$\lim_{\lambda\to 0^+} \int_\Omega \Theta_k(v_\lambda - u_\lambda)(x,T)\,\mathrm{d}x = \int_\Omega \Theta_k(v-u)(x,T)\,\mathrm{d}x.$$

Observing that $(v_\lambda)_t = v_t$, we can pass to the limit in (4.55) and we obtain

$$\lim_{\lambda\to 0^+} \int_0^T \langle (u_\lambda)_t, T_k(v_\lambda - u_\lambda)\rangle \,\mathrm{d}s$$
$$= \int_0^T \langle v_t, T_k(v-u)\rangle \,\mathrm{d}s + \int_\Omega \Theta_k(v(x,0) - u_0)\,\mathrm{d}x - \int_\Omega \Theta_k(v-u)(x,T)\,\mathrm{d}x$$
$$= \int_0^T \langle u_t, T_k(v-u)\rangle \,\mathrm{d}s. \tag{4.57}$$

Now we estimate the integral

$$\int_{\Omega_T} \Psi^+ Z_\lambda(\psi - u_\lambda)T_k(v_\lambda - u_\lambda)\,\mathrm{d}x\,\mathrm{d}s.$$

If either $v_\lambda \leqslant u_\lambda$ or if $v_\lambda > u_\lambda$, the monotonicity of Z_λ implies

$$\Psi^+ Z_\lambda(\psi - u_\lambda)T_k(v_\lambda - u_\lambda) \geqslant \Psi^+ Z_\lambda(\psi - v_\lambda)T_k(v_\lambda - u_\lambda).$$

Recalling that $v \geqslant \psi$ almost everywhere in Ω_T, we have $\psi - v_\lambda = \psi - v - \lambda\eta_\lambda \leqslant -\lambda\eta_\lambda = -\lambda$ in Ω^λ. Combining these properties, we arrive at

$$\begin{aligned}
&\int_{\Omega_T} \Psi^+ Z_\lambda(\psi - u_\lambda) T_k(v_\lambda - u_\lambda)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad \geqslant \int_{\Omega_T} \Psi^+ Z_\lambda(\psi - v_\lambda) T_k(v_\lambda - u_\lambda)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad = \int_{\Omega_T \setminus \Omega_T^\lambda} \Psi^+ Z_\lambda(\psi - v_\lambda) T_k(v_\lambda - u_\lambda)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad \geqslant - \int_{\Omega_T \setminus \Omega_T^\lambda} |\Psi^+||v_\lambda - u_\lambda|\,\mathrm{d}x\,\mathrm{d}s \\
&\quad \geqslant -\Big(\int_{\Omega_T \setminus \Omega_T^\lambda} |\Psi^+|^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{1/2}\Big(\int_{\Omega_T} |v_\lambda - u_\lambda|^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{1/2}.
\end{aligned} \tag{4.58}$$

We observe that the term

$$\Big(\int_{\Omega_T} |v_\lambda - u_\lambda|^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{1/2}$$

is uniformly bounded with respect to λ, while

$$\lim_{\lambda \to 0^+} \int_{\Omega_T \setminus \Omega_T^\lambda} |\Psi^+|^2\,\mathrm{d}x\,\mathrm{d}s = 0,$$

since $|\Omega_T \setminus \Omega_T^\lambda| \to 0$ as $\lambda \to 0^+$. We deduce that

$$\lim_{\lambda \to 0^+} \int_{\Omega_T} \Psi^+ Z_\lambda(\psi - u_\lambda) T_k(v_\lambda - u_\lambda)\,\mathrm{d}x\,\mathrm{d}s \geqslant 0.$$

The remaining terms in (4.54) pass to the limit because of (4.51). Hence, we obtain

$$\begin{aligned}
&\int_0^T \langle u_t, T_k(v - u)\rangle\,\mathrm{d}s + \int_{\Omega_T} (A(x,s,\nabla u) + B(x,s,u))\nabla T_k(v - u)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad \geqslant \int_{\Omega_T} f\,T_k(v - u)\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

Now the variational inequality follows passing to the limit as $k \to \infty$.

Step 4: we prove that the Lewy–Stampacchia inequality holds.

Since $0 \leqslant Z_\lambda \leqslant 1$, we have

$$0 \leqslant (u_\lambda)_t - \operatorname{div}(A(x,t,\nabla u_\lambda) + B(x,t,u_\lambda)) - f \leqslant \Psi^+$$

in the sense of distribution. Therefore, the Lewy–Stampacchia inequality follows by passing to the limit as $\lambda \to 0^+$ in the latter relation. ∎

Remark 4.6. As a byproduct of the existence and uniqueness result together with the validity of the Lewy–Stampacchia inequality stated in Theorem 4.4, we obtain that for all non-negative initial values $u_0 \in L^2(\Omega)$ and for all non-negative source terms $f \in L^2(\Omega_T)$, the problem

$$\begin{cases} u_t - \operatorname{div}(A(x,t,\nabla u) + B(x,t,u)) = f & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = 0 & \text{in } \Omega \end{cases}$$

admits a unique non-negative solution $u \in C([0,T], L^2(\Omega)) \cap L^2(0,T,W_0^{1,2}(\Omega))$.

We conclude this section by proving Corollary 1.3 stated in the introduction.

Proof of Corollary 1.3. To prove the claimed result, it suffices to observe that $b_2 = b_2 - T_M(b_2) + T_M(b_2)$ for $M > 0$ and

$$\operatorname*{ess\,sup}_{0<t<T} \|(b_2 - T_M(b_2))(t)\|_{L^{N,\infty}(\Omega)} \leqslant \max_{0\leqslant t\leqslant T} \|(b_2 - T_M(b_2))(t)\|_{L^N(\Omega)}. \tag{4.59}$$

Then, one can choose $M_0 > 0$ such that the right-hand side of (4.59) with $M = M_0$ is strictly less that $\frac{\alpha}{S_{N,2}}$. The result then follows by Theorem 4.4, leaving unchanged the choice of b_3 and replacing b_1 and b_2 by $b_1 + T_{M_0}(b_2) \in L^\infty(\Omega_T)$ and $b_2 - T_{M_0}(b_2)$, respectively. ∎

5. Global solutions

In this section we denote $\Omega_\infty := \Omega \times (0,\infty)$ and we assume that

$$A = A(x,t,\xi) : \Omega_\infty \times \mathbb{R}^N \to \mathbb{R}^N$$

and

$$B = B(x,t,s) : \Omega_\infty \times \mathbb{R} \to \mathbb{R}^N$$

are both Carathéodory functions. We require that assumptions (1.2)–(1.4) on A are satisfied for almost every $(x,t) \in \Omega_\infty$ and for all $\xi, \eta \in \mathbb{R}^N$. Moreover, $b \in L^2_{\text{loc}}(0,\infty,L^2(\Omega))$ is such that (1.7) holds where b_1, b_2, and b_3 satisfy (1.8)–(1.11) for all $T > 0$. Then, let assumptions (1.5)–(1.6) on B hold for almost every $(x,t) \in \Omega_\infty$ and for all $s_1, s_2 \in \mathbb{R}$. For the obstacle function, we assume that $\psi : \Omega_\infty \to \mathbb{R}$ satisfies (3.1)–(3.3) for all $T > 0$. Finally, we assume compatibility condition (3.6).

Here, for a global solution to the obstacle problem, we mean the following:

Definition 5.1. Let $u_0 \in L^2(\Omega)$ and $f \in L^2_{\text{loc}}(0,\infty,L^2(\Omega))$. Under the structure assumptions above, by a global solution to the obstacle problem related to (1.1) with initial data u_0 and forcing term f, we mean a function $u \in C_{\text{loc}}([0,\infty),L^2(\Omega)) \cap L^2_{\text{loc}}([0,\infty),W_0^{1,2}(\Omega))$ with $u(\cdot,0) = u_0$ such that for all $T > 0$, one has $u \geqslant \psi$ almost everywhere in Ω_T and inequality (3.8) holds true for every $v \in \mathcal{K}_\psi(\Omega_T)$.

We establish existence and uniqueness of a global solution.

Theorem 5.1. *Under the assumptions above on A, B, ψ, u_0, and f, assume also that*

$$\mathcal{D}_{b,T} := \operatorname*{ess\,sup}_{0<t<T} \|b_2(\cdot,t)\|_{L^{N,\infty}(\Omega)} < \frac{\alpha}{S_{N,2}} \tag{5.1}$$

and

$$\operatorname{div}[A(\cdot,\cdot,\nabla\psi) + B(\cdot,\cdot,\psi)] \in L^2(\Omega_T), \tag{5.2}$$

hold for all $T > 0$, where $S_{N,2}$ in (5.1) is the Sobolev constant appearing in (2.5). Then there exists a unique global solution to the obstacle problem related to (1.1) with initial data u_0 and forcing term f.

Proof. Let $T_0 > 0$ be arbitrarily fixed. For every $m \in \mathbb{N}$, let us denote $u^{(m)} \in \mathcal{K}_\psi(\Omega_{mT_0})$ as the solution to the obstacle problem in the parabolic cylinder Ω_{mT_0}, whose existence and uniqueness is guaranteed by Theorem 4.1. By Proposition 3.2, the function $u^{(m+1)}$ solves the obstacle problem also in the parabolic cylinder Ω_{mT_0}. Due to the uniqueness of a solution of the obstacle problem recalled above, we get $u^{(m+1)} = u^{(m)}$ in Ω_{mT_0}. Therefore, the function $u := u^{(m)}$ in Ω_{mT_0} for every $m \in \mathbb{N}$ is well defined and by construction is the unique global solution to the obstacle problem in the sense of Definition 5.1. ■

A quantitative description of the asymptotic behavior of a global solution is provided by next result. As a byproduct, we get that global solutions to the obstacle problem related to (1.1) have the same asymptotic profile, no matter what the initial values and forcing terms are.

Theorem 5.2. *Let the structure assumptions of Theorem 5.1 be in charge. Let u and w be global solutions to the obstacle problem related to (1.1) in the sense of Definition 5.1 with initial data u_0 and w_0 and forcing terms f and $\tilde{f}$, respectively, where $f - \tilde{f} \in L^1(\Omega_\infty) \cap L^2(\Omega_\infty)$. Assume there exists $w_0 \in L^2(\Omega)$ satisfying $w_0(x) \geq \psi(x,0)$ almost everywhere in Ω. Let $b \in L^2(\Omega_\infty)$ satisfy (1.7) with*

$$b_1 \in L^2(\Omega_\infty) \cap L^\infty(\Omega_\infty), \tag{5.3}$$

$$b_3 \in L^2(0,\infty, L^\infty(\Omega)), \tag{5.4}$$

and b_2 satisfies

$$b_2(\cdot,t) \in L^{N,\infty}(\Omega) \quad \text{for a.e. } t \in (0,\infty),$$
$$(0,\infty) \ni t \mapsto \|b_2(\cdot,t)\|_{L^{N,\infty}(\Omega)} \quad \text{belongs to } L^\infty(0,\infty).$$

Let

$$\mathcal{D}_{b,\infty} := \operatorname*{ess\,sup}_{0<t<\infty} \|b_2(\cdot,t)\|_{L^{N,\infty}(\Omega)} < \frac{\alpha}{S_{N,2}}, \tag{5.5}$$

where $S_{N,2}$ is the Sobolev constant appearing in (2.5). Then we get

$$\|u(t) - w(t)\|^2_{L^2(\Omega)} \leqslant \Lambda_0 e^{-Mt} \quad \text{for all } t > 0 \tag{5.6}$$

where

$$M := \frac{1}{2}\mathcal{C}_2(\Omega, N)(\alpha - \mathcal{D}_{b,\infty} S_{N,2}) \tag{5.7}$$

with $\mathcal{C}_2(\Omega, N)$ *the Poincaré constant defined in* (2.1) *and*

$$\begin{aligned}\Lambda_0 := C \exp(c\|b_3\|^2_{L^2(0,\infty,L^\infty(\Omega))})\big[\|u_0 - w_0\|^2_{L^2(\Omega)} + \|f - \tilde{f}\|^2_{L^2(\Omega_\infty)} \\ + \|b_1\|^{2N}_{L^\infty(\Omega_\infty)}\|b_1\|^2_{L^2(\Omega_\infty)}\big(\|u_0 - w_0\|^2_{L^2(\Omega)} + \|f - \tilde{f}\|^2_{L^1(\Omega_\infty)}\big)\big],\end{aligned} \tag{5.8}$$

for some positive constants $c = c(\alpha, N, \mathcal{D}_{b,\infty}, \Omega)$ *and* $C = C(\alpha, N, \mathcal{D}_{b,\infty}, \Omega)$*. As a consequence,*

$$\lim_{t\to\infty} \|u(t) - w(t)\|_{L^2(\Omega)} = 0.$$

Proof. Since by assumption u and w are global solutions, for all $t > 0$, we have $u, w \in \mathcal{K}_\psi(\Omega_t)$ with $u(\cdot, 0) = u_0$ and $w(\cdot, 0) = w_0$ and for all $v \in \mathcal{K}_\psi(\Omega_t)$, the following variational inequalities hold true:

$$\begin{aligned}\int_0^t \langle u_t, v - u\rangle \, ds + \int_{\Omega_t} A(x, s, \nabla u) \cdot \nabla(v - u)\, dx\, ds \\ + \int_{\Omega_t} B(x, s, u) \cdot \nabla(v - u)\, dx\, ds \geqslant \int_{\Omega_s} f(v - u)\, dx\, ds,\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}\int_0^t \langle w_t, v - w\rangle \, ds + \int_{\Omega_t} A(x, s, \nabla w) \cdot \nabla(v - w)\, dx\, ds \\ + \int_{\Omega_t} B(x, s, w) \cdot \nabla(v - w)\, dx\, ds \geqslant \int_{\Omega_t} \tilde{f}(v - w)\, dx\, ds.\end{aligned} \tag{5.10}$$

In both (5.9) and (5.10), we use $\frac{u+w}{2}$ as a test function and we sum the resulting estimates to get the following:

$$\begin{aligned}\frac{1}{2}\int_\Omega (u - w)^2(\cdot, t)\, dx + \int_{\Omega_t} [A(x, s, \nabla u) - A(x, s, \nabla w)] \cdot \nabla(u - w)\, dx\, ds \\ \leqslant \frac{1}{2}\int_\Omega (u_0 - w_0)^2\, dx + \int_{\Omega_t} (f - \tilde{f})(u - w)\, dx\, ds \\ + \int_{\Omega_t} [B(x, s, w) - B(x, s, u)] \cdot \nabla(u - w)\, dx\, ds.\end{aligned}$$

By using assumptions (1.2) and (1.5)–(1.7), we have

$$\begin{aligned}\frac{1}{2}\int_\Omega (u - w)^2(\cdot, t)\, dx + \alpha \int_{\Omega_t} |\nabla(u - w)|^2\, dx\, ds \\ \leqslant \frac{1}{2}\int_\Omega (u_0 - w_0)^2\, dx + \int_{\Omega_t} |f - \tilde{f}||u - w|\, dx\, ds \\ + \int_{\Omega_t} b_1|u - w||\nabla(u - w)|\, dx\, ds + \int_{\Omega_t} b_2|u - w||\nabla(u - w)|\, dx\, ds \\ + \int_{\Omega_t} b_3|u - w||\nabla(u - w)|\, dx\, ds.\end{aligned} \tag{5.11}$$

The fourth term in the right-hand side of (5.11) can be estimated with similar computations, which leads to (4.21); more precisely, it holds that

$$\int_{\Omega_t} b_2|u-w||\nabla u-\nabla w|\,\mathrm{d}x\,\mathrm{d}s \leqslant S_{N,2}\mathcal{D}_{b,\infty}\int_{\Omega_t}|\nabla(u-w)|^2\,\mathrm{d}x\,\mathrm{d}s.$$

Hence, thanks to assumption (5.5) we get

$$\theta := \alpha - S_{N,2}\mathcal{D}_{b,\infty} > 0,$$

and we deduce

$$\frac{1}{2}\int_{\Omega}(u-w)^2(\cdot,t)\,\mathrm{d}x + \theta\int_{\Omega_t}|\nabla(u-w)|^2\,\mathrm{d}x\,\mathrm{d}s \leqslant \frac{1}{2}\int_{\Omega}(u_0-w_0)^2\,\mathrm{d}x$$
$$+\int_{\Omega_t}|f-\widetilde{f}||u-w|\,\mathrm{d}x\,\mathrm{d}s + \int_{\Omega_t}(b_3+b_1)|u-w||\nabla(u-w)|\,\mathrm{d}x\,\mathrm{d}s. \quad (5.12)$$

The second term in the right-hand side of (5.11) can be estimated by means of the Young and Poincaré inequalities, and we obtain

$$\int_{\Omega_t}|f-\widetilde{f}||u-w|\,\mathrm{d}x\,\mathrm{d}s$$
$$\leqslant c(N,\Omega,\theta)\int_{\Omega_t}|f-\widetilde{f}|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{\theta}{6}\int_{\Omega_t}|\nabla u-\nabla w|^2\,\mathrm{d}x\,\mathrm{d}s.$$

On the other hand, we have

$$\int_{\Omega_t} b_3|u-w||\nabla(u-w)|\,\mathrm{d}x\,\mathrm{d}s$$
$$\leqslant \frac{3}{2\theta}\int_{\Omega_t} b_3^2|u-w|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{\theta}{6}\int_{\Omega_t}|\nabla(u-w)|^2\,\mathrm{d}x\,\mathrm{d}s.$$

Now, we estimate the term

$$\int_{\Omega_t} b_1|u-w||\nabla(u-w)|\,\mathrm{d}x\,\mathrm{d}s.$$

First notice that by Young inequality we obtain

$$\int_{\Omega_t} b_1|u-w||\nabla(u-w)|\,\mathrm{d}x\,\mathrm{d}s$$
$$\leqslant \frac{3}{2\theta}\int_{\Omega_t} b_1^2|u-w|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{\theta}{6}\int_{\Omega_t}|\nabla(u-w)|^2\,\mathrm{d}x\,\mathrm{d}s. \quad (5.13)$$

Furthermore, we have

$$\int_{\Omega_t} b_1^2|u-w|^2\,\mathrm{d}x\,\mathrm{d}s = \int_0^t\int_{\Omega\cap\{|u(s)-w(s)|\leqslant\lambda\}} b_1^2|u-w|^2\,\mathrm{d}x\,\mathrm{d}s$$
$$+\int_0^t\int_{\Omega\cap\{|u(s)-w(s)|>\lambda\}} b_1^2|u-w|^2\,\mathrm{d}x\,\mathrm{d}s$$
$$\leqslant \lambda^2\int_{\Omega_t} b_1^2\,\mathrm{d}x\,\mathrm{d}s + \|b_1\|^2_{L^\infty(\Omega_T)}\int_0^t\int_{\Omega\cap\{|u(s)-w(s)|>\lambda\}}|u-w|^2\,\mathrm{d}x\,\mathrm{d}s. \quad (5.14)$$

On the other hand, by Hölder's inequality, we deduce

$$\int_0^t \int_{\Omega\cap\{|u(s)-w(s)|>\lambda\}} |u-w|^2 \,\mathrm{d}x\,\mathrm{d}s \leqslant \operatorname*{ess\,sup}_{s\in(0,\infty)} \big|\{|u(s)-w(s)|>\lambda\}\big|^{2/N} \int_0^t \Big(\int_\Omega |u-w|^{2^*}\Big)^{2/2^*} \mathrm{d}x\,\mathrm{d}s. \tag{5.15}$$

Because of Corollary 4.3, we have

$$\operatorname*{ess\,sup}_{0<s<\infty} \lambda\big|\{x\in\Omega : |u(x,s)-w(x,s)|>\lambda\}\big| \leqslant \int_\Omega |u_0-w_0|\,\mathrm{d}x + \int_0^\infty \int_\Omega |f-\tilde{f}|\,\mathrm{d}x\,\mathrm{d}s. \tag{5.16}$$

Estimates (5.15) and (5.16) together with the Sobolev inequality imply the following inequality:

$$\int_0^t \int_{\Omega\cap\{|u(s)-w(s)|>\lambda\}} |u-w|^2 \,\mathrm{d}x\,\mathrm{d}s \leqslant C_0(N)\Big(\frac{\int_\Omega |u_0-w_0|\,\mathrm{d}x + \int_0^\infty \int_\Omega |f-\tilde{f}|\,\mathrm{d}x\,\mathrm{d}s}{\lambda}\Big)^{2/N} \cdot \int_{\Omega_t} |\nabla u - \nabla w|^2 \,\mathrm{d}x\,\mathrm{d}s, \tag{5.17}$$

where $C_0 = C_0(N) = S_{N,2}^2$ is a positive constant depending only on N. We gather (5.13)–(5.17) and we obtain

$$\begin{aligned}\int_{\Omega_t} b_1|u-w||\nabla(u-w)|\,\mathrm{d}x\,\mathrm{d}s &\leqslant \frac{3\lambda^2}{2\theta}\int_{\Omega_t} b_1^2\,\mathrm{d}x\,\mathrm{d}s \\ &+ \frac{3}{2\theta}C_0(N)\|b_1\|^2_{L^\infty(\Omega_T)}\Big(\frac{\int_\Omega |u_0-w_0|\,\mathrm{d}x + \int_0^\infty \int_\Omega |f-\tilde{f}|\,\mathrm{d}x\,\mathrm{d}s}{\lambda}\Big)^{2/N} \\ &\times \int_{\Omega_t} |\nabla u-\nabla w|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{\theta}{6}\int_{\Omega_t}|\nabla(u-w)|^2\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{5.18}$$

Using all previous estimates in (5.12), we obtain

$$\begin{aligned}&\frac{1}{2}\int_\Omega (u-w)^2(\cdot,t)\,\mathrm{d}x + \frac{\theta}{2}\int_{\Omega_t}|\nabla(u-w)|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant \frac{1}{2}\int_\Omega (u_0-w_0)^2\,\mathrm{d}x + C_1(N,\Omega,\theta)\int_{\Omega_t}|f-\tilde{f}|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{3\lambda^2}{2\theta}\int_{\Omega_t} b_1^2\,\mathrm{d}x\,\mathrm{d}s \\ &+ \frac{3}{2\theta}C_0(N)\|b_1\|^2_{L^\infty(\Omega_\infty)}\Big(\frac{\int_\Omega |u_0-w_0|\,\mathrm{d}x + \int_0^\infty \int_\Omega |f-\tilde{f}|\,\mathrm{d}x\,\mathrm{d}s}{\lambda}\Big)^{2/N} \\ &\times \int_{\Omega_t}|\nabla u-\nabla w|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{3}{2\theta}\int_{\Omega_t} b_3^2|u-w|^2\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{5.19}$$

We choose λ such that

$$\frac{3}{2\theta}\Big(C_0(N)^{N/2}\|b_1\|^N_{L^\infty(\Omega_\infty)}\frac{\int_\Omega |u_0-w_0|\,\mathrm{d}x+\int_0^\infty\int_\Omega |f-\widetilde{f}|\,\mathrm{d}x\,\mathrm{d}s}{\lambda}\Big)^{2/N}=\frac{\theta}{4},$$

namely,

$$\lambda = C_1(N,\theta)C_0(N)^{N/2}\|b_1\|^N_{L^\infty(\Omega_\infty)}\Big(\int_\Omega |u_0-w_0|\,\mathrm{d}x+\int_0^\infty\int_\Omega |f-\widetilde{f}|\,\mathrm{d}x\,\mathrm{d}s\Big). \tag{5.20}$$

So, in particular, we get

$$\lambda = C(N,\theta)\|b_1\|^N_{L^\infty(\Omega_\infty)}\Big(\int_\Omega |u_0-w_0|\,\mathrm{d}x+\int_0^\infty\int_\Omega |f-\widetilde{f}|\,\mathrm{d}x\,\mathrm{d}s\Big). \tag{5.21}$$

Using also Poincaré's inequality, from (5.22) and thanks to the choice of λ, we deduce

$$\begin{aligned}
&\|(u-w)(t)\|^2_{L^2(\Omega)}+M\int_0^t\|(u-w)(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s\\
&\leqslant \|u_0-w_0\|^2_{L^2(\Omega)}+c_1\|f-\widetilde{f}\|^2_{L^2(\Omega_\infty)}\\
&\qquad + c_2\|b_1\|^{2N}_{L^\infty(\Omega_\infty)}\|b_1\|^2_{L^2(\Omega_\infty)}\big(\|u_0-w_0\|^2_{L^2(\Omega)}+\|f-\widetilde{f}\|^2_{L^1(\Omega_\infty)}\big)\\
&\qquad + c_3\int_0^t\|b_3(s)\|^2_{L^\infty(\Omega)}\|(u-w)(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s,
\end{aligned} \tag{5.22}$$

where the constant M is as in (5.7), while the constants c_1, c_2, c_3 depend on α, N, $\mathcal{D}_{b,\infty}$, and Ω. We merely rewrite (5.22) as follows:

$$\begin{aligned}
&\|(u-w)(t)\|^2_{L^2(\Omega)}\\
&\qquad\leqslant \|u_0-w_0\|^2_{L^2(\Omega)}+c_1\|f-\widetilde{f}\|^2_{L^2(\Omega_\infty)}\\
&\qquad\quad + c_2\|b_1\|^{2N}_{L^\infty(\Omega_\infty)}\|b_1\|^2_{L^2(\Omega_\infty)}\big(\|u_0-w_0\|^2_{L^2(\Omega)}+\|f-\widetilde{f}\|^2_{L^1(\Omega_\infty)}\big)\\
&\qquad\quad + \int_0^t\big(c_3\|b_3(s)\|^2_{L^\infty(\Omega)}-M\big)\|(u-w)(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s.
\end{aligned}$$

We set

$$\begin{aligned}
z_0 :=\ &\|u_0-w_0\|^2_{L^2(\Omega)}+c_1\|f-\widetilde{f}\|^2_{L^2(\Omega_\infty)}\\
&+ c_2\|b_1\|^{2N}_{L^\infty(\Omega_\infty)}\|b_1\|^2_{L^2(\Omega_\infty)}\big(\|u_0-w_0\|^2_{L^2(\Omega)}+\|f-\widetilde{f}\|^2_{L^1(\Omega_\infty)}\big).
\end{aligned} \tag{5.23}$$

We apply Gronwall's inequality of Lemma 2.1, with

$$\begin{aligned}
z(t)&\equiv z_0,\\
\gamma(s)&=c_3\|b_3(s)\|^2_{L^\infty(\Omega)}-M,\\
y(t)&=\|(u-w)(t)\|^2_{L^2(\Omega)},
\end{aligned}$$

and we obtain

$$\|u(t)-w(t)\|^2_{L^2(\Omega)} \leqslant z_0 \exp\Big(\int_0^t \big(c_3\|b_3(s)\|^2_{L^\infty(\Omega)} - M\big)\,\mathrm{d}s\Big)$$
$$\leqslant z_0 \exp\Big(c_3 \int_0^\infty \|b_3(s)\|^2_{L^\infty(\Omega)}\,\mathrm{d}s\Big) e^{-Mt}.$$

We deduce then the existence of positive constants c and C as in the statement such that (5.6) holds choosing Λ_0 as in (5.8). ■

6. The autonomous case

Let us consider here the autonomous case, that is, when functions A, B together with the forcing term f and the obstacle functions ψ are all independent of the time variable t. Let us state our assumptions in this particular case.

The vector fields

$$A = A(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$$

and

$$B = B(x,s) : \Omega \times \mathbb{R} \to \mathbb{R}^N$$

are both Carathéodory functions. We assume that there exist positive constants α, β such that

$$[A(x,\xi) - A(x,\eta)] \cdot (\xi - \eta) \geqslant \alpha|\xi-\eta|^2, \tag{6.1}$$
$$|A(x,\xi) - A(x,\eta)| \leqslant \beta|\xi - \eta|, \tag{6.2}$$
$$A(x,0) = 0, \tag{6.3}$$

for almost every $x \in \Omega$ and for all $\xi, \eta \in \mathbb{R}^N$. We assume that

$$|B(x,s_1) - B(x,s_2)| \leqslant b(x)|s_1 - s_2|, \tag{6.4}$$
$$B(x,0) = 0, \tag{6.5}$$

for almost every $x \in \Omega$ and for any $s_1, s_2 \in \mathbb{R}$, where $b : \Omega \to \mathbb{R}$ is split as

$$b = b_1 + b_2, \tag{6.6}$$

with

$$b_1 \in L^\infty(\Omega) \tag{6.7}$$

and

$$b_2 \in L^{N,\infty}(\Omega). \tag{6.8}$$

Let $\psi : \Omega \to \mathbb{R}$ be a measurable function satisfying

$$\psi \in W^{1,2}(\Omega), \tag{6.9}$$
$$\psi \leqslant 0 \quad \text{a.e. on } \partial\Omega. \tag{6.10}$$

Finally, assume

$$f \in L^2(\Omega). \tag{6.11}$$

As proved in [14], under the further assumption

$$\mathcal{D}_b := \|b_2\|_{L^{N,\infty}(\Omega)} < \frac{\alpha}{S_{N,2}}, \tag{6.12}$$

where $S_{N,2}$ is the Sobolev constant appearing in (2.5), there exists $\tilde{u} \in \mathcal{K}_\psi(\Omega)$ such that

$$\int_\Omega A(x, \nabla\tilde{u}) \cdot \nabla(\tilde{v} - \tilde{u})\,\mathrm{d}x + \int_\Omega B(x, \tilde{u}) \cdot \nabla(\tilde{v} - \tilde{u})\,\mathrm{d}x \geqslant \int_\Omega f(\tilde{v} - \tilde{u})\,\mathrm{d}x \quad \text{for all } \tilde{v} \in \mathcal{K}_\psi(\Omega) \tag{6.13}$$

where

$$\mathcal{K}_\psi(\Omega) := \{\varphi \in W_0^{1,2}(\Omega) : \varphi \geqslant \psi \quad \text{a.e. in } \Omega\}.$$

In other words, $\tilde{u}$ solves an elliptic obstacle problem with forcing term f.

Our next result describes the asymptotic behavior of the global solutions to the autonomous parabolic obstacle problem. First of all we prove that the solution $\tilde{u}$ of the elliptic obstacle problem with forcing term f is unique and then we show that all the global solutions to the parabolic obstacle problem, whatever their initial datum u_0 is, tend to $\tilde{u}$ as $t \to +\infty$.

Theorem 6.1. *Assume that $A = A(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ and $B = B(x,s) : \Omega \times \mathbb{R} \to \mathbb{R}^N$ are Carathéodory functions and that (6.1)–(6.12) hold. Let u be the global solution to the parabolic obstacle problem related to (1.1) with initial data $u_0 \in L^2(\Omega)$ satisfying $u_0 \geqslant \psi$ almost everywhere in Ω and forcing term $f \in L^2(\Omega)$. Then, there exists a unique solution $\tilde{u} \in \mathcal{K}_\psi(\Omega)$ to the elliptic variational inequality in (6.13). Moreover, we have*

$$\|u(t) - \tilde{u}\|^2_{L^2(\Omega)} \leqslant \|u_0 - \tilde{u}\|^2_{L^2(\Omega)} \big(1 + c\|b_1\|^{2N}_{L^\infty(\Omega)} \|b_1\|^2_{L^2(\Omega)} t\big) e^{-Mt}, \tag{6.14}$$

where

$$M := \frac{1}{2}\mathcal{C}_2(\Omega, N)(\alpha - \mathcal{D}_b S_{N,2}) \tag{6.15}$$

with $\mathcal{C}_2(\Omega, N)$ being the Poincaré constant in (2.1) and $c = c(\alpha, N, \mathcal{D}_b, \Omega)$. As a consequence, we have

$$\lim_{t\to\infty} \|u(t) - \tilde{u}\|_{L^2(\Omega)} = 0. \tag{6.16}$$

Proof. Let $\tilde{u} \in \mathcal{K}_\psi(\Omega)$ be a solution of the elliptic obstacle problem. We point out that a priori we do not know if such a solution is unique, but if we prove that the function $u - \tilde{u}$ satisfies estimate (6.14), then as an immediate consequence we can conclude that $\tilde{u}$ is the unique solution of the elliptic obstacle problem belonging to $\mathcal{K}_\psi(\Omega)$. As a matter of fact, by (6.14) it follows that (6.16) holds true and thus, if there exists another solution $\tilde{u}_1$ of the elliptic obstacle problem belonging to $\mathcal{K}_\psi(\Omega)$, then as $t \to +\infty$ the function $u(t)$

converges (strongly in $L^2(\Omega)$) also to $\widetilde{u}_1$. Consequently, by the uniqueness of the limit in $L^2(\Omega)$, $\widetilde{u}$ coincides with $\widetilde{u}_1$, that is, the solution of the elliptic obstacle problem is unique.

Hence, to conclude the proof, we prove now that estimate (6.14) holds true. To this aim, let $T > 0$ be arbitrarily fixed and let $v \in \mathcal{K}_\psi(\Omega_T)$. As ψ is time independent, we have $v(\cdot, s) \in \mathcal{K}_\psi(\Omega)$ for all $s \in (0, T)$. Therefore, we may test (6.13) by $v(\cdot, s)$ and then integrate the resulting inequality with respect to $s \in (0, T)$ to get

$$\int_0^T \langle \widetilde{u}_t, v - \widetilde{u}\rangle \, \mathrm{d}s + \int_{\Omega_T} A(x, \nabla\widetilde{u}) \cdot \nabla(v - \widetilde{u}) \, \mathrm{d}x \, \mathrm{d}s + \int_{\Omega_T} B(x, \widetilde{u}) \cdot \nabla(v - \widetilde{u}) \, \mathrm{d}x \, \mathrm{d}s \geqslant \int_{\Omega_T} f(v - \widetilde{u}) \, \mathrm{d}x \, \mathrm{d}s,$$

where we used $\widetilde{u}_t \equiv 0$. Since T is arbitrarily fixed, we obtain that $\widetilde{u}(x, t) \equiv \widetilde{u}(x)$ is a global solution to the parabolic obstacle problem related to (1.1) with initial value $\widetilde{u}(x)$ and forcing term f.

By the definition of global solution for a given $t > 0$, we know that

$$u \in \mathcal{K}_\psi(\Omega_t), \tag{6.17}$$

with $u(\cdot, 0) = u_0$ and for all $v \in \mathcal{K}_\psi(\Omega_t)$ the variational inequality

$$\int_0^t \langle u_t, v - u\rangle \, \mathrm{d}s + \int_{\Omega_t} A(x, \nabla u) \cdot \nabla(v - u) \, \mathrm{d}x \, \mathrm{d}s + \int_{\Omega_t} B(x, u) \cdot \nabla(v - u) \, \mathrm{d}x \, \mathrm{d}s \geqslant \int_{\Omega_t} f(v - u) \, \mathrm{d}x \, \mathrm{d}s \tag{6.18}$$

holds. Since u and $\widetilde{u}$ are global solutions to the parabolic obstacle problem related to (1.1) with initial values u_0 and $\widetilde{u}$ respectively and for the same forcing term f, Theorem 4.1 gives us

$$\int_\Omega |u(t) - \widetilde{u}| \, \mathrm{d}x \leqslant \int_\Omega |u_0 - \widetilde{u}| \, \mathrm{d}x. \tag{6.19}$$

As ψ is time independent, we have $\widetilde{u} \in \mathcal{K}_\psi(\Omega_t)$ and also $u(\cdot, s) \in \mathcal{K}_\psi(\Omega)$ for all $s \in (0, t)$. Hence, one can use use $\frac{u+\widetilde{u}}{2}$ as a test function for (6.18) to obtain

$$\int_0^t \langle u_t, \widetilde{u} - u\rangle \, \mathrm{d}s + \int_{\Omega_t} A(x, \nabla u) \cdot \nabla(\widetilde{u} - u) \, \mathrm{d}x \, \mathrm{d}s + \int_{\Omega_t} B(x, u) \cdot \nabla(\widetilde{u} - u) \, \mathrm{d}x \, \mathrm{d}s \geqslant \int_{\Omega_t} f(\widetilde{u} - u) \, \mathrm{d}x \, \mathrm{d}s. \tag{6.20}$$

Similarly, one can use $\frac{u(\cdot,s)+\widetilde{u}}{2}$ as a test function in (6.13) and then integrate the resulting inequality with respect to $s \in (0, t)$ to obtain

$$\int_{\Omega_t} A(x, \nabla\widetilde{u}) \cdot \nabla(u - \widetilde{u}) \, \mathrm{d}x \, \mathrm{d}s + \int_{\Omega_t} B(x, \widetilde{u}) \cdot \nabla(\widetilde{u} - u) \, \mathrm{d}x \, \mathrm{d}s \geqslant \int_{\Omega_t} f(u - \widetilde{u}) \, \mathrm{d}x \, \mathrm{d}s. \tag{6.21}$$

If we add inequalities (6.20) and (6.21), we obtain

$$\frac{1}{2}\int_{\Omega}|u(t)-\widetilde{u}|^2\,\mathrm{d}x+\int_{\Omega_t}[A(x,\nabla u)-A(x,\nabla\widetilde{u})]\cdot\nabla(u-\widetilde{u})\,\mathrm{d}x\,\mathrm{d}s$$
$$\leqslant\frac{1}{2}\int_{\Omega}|u_0-\widetilde{u}|^2\,\mathrm{d}x+\int_{\Omega_t}[B(x,\widetilde{u})-B(x,u)]\cdot\nabla(u-\widetilde{u})\,\mathrm{d}x\,\mathrm{d}s.$$

By using assumptions (6.1)–(6.8), we have

$$\frac{1}{2}\int_{\Omega}|u(t)-\widetilde{u}|^2\,\mathrm{d}x+\alpha\int_{\Omega_t}|\nabla(u-\widetilde{u})|^2\,\mathrm{d}x\,\mathrm{d}s$$
$$\leqslant\frac{1}{2}\int_{\Omega}|u_0-\widetilde{u}|^2\,\mathrm{d}x+\int_{\Omega_t}b_1(x)|u-\widetilde{u}||\nabla(u-\widetilde{u})|\,\mathrm{d}x\,\mathrm{d}s$$
$$+\int_{\Omega_t}b_2(x)|u-\widetilde{u}||\nabla(u-\widetilde{u})|\,\mathrm{d}x\,\mathrm{d}s. \tag{6.22}$$

The last term in the right-hand side of (6.22) can be estimated with similar computations, which leads to (4.21), because of assumption (6.12). So, by setting $\theta:=\alpha-\mathcal{D}_b S_{N,2}$, it holds that

$$\frac{1}{2}\int_{\Omega}|u(t)-\widetilde{u}|^2\,\mathrm{d}x+\theta\int_{\Omega_t}|\nabla(u-\widetilde{u})|^2\,\mathrm{d}x\,\mathrm{d}s$$
$$\leqslant\frac{1}{2}\int_{\Omega}|u_0-\widetilde{u}|^2\,\mathrm{d}x+\int_{\Omega_t}b_1(x)|u-\widetilde{u}||\nabla(u-\widetilde{u})|\,\mathrm{d}x\,\mathrm{d}s.$$

By Young's inequality, we have

$$\int_{\Omega_t}b_1(x)|u-\widetilde{u}||\nabla(u-\widetilde{u})|\,\mathrm{d}x\,\mathrm{d}s\leqslant\frac{1}{2\theta}\int_{\Omega_t}b_1^2(x)|u-\widetilde{u}|^2\,\mathrm{d}x\,\mathrm{d}s+\frac{\theta}{2}\int_{\Omega_t}|\nabla(u-\widetilde{u})|^2\,\mathrm{d}x\,\mathrm{d}s.$$

By the previous estimates, it follows that

$$\int_{\Omega}|u(t)-\widetilde{u}|^2\,\mathrm{d}x+\theta\int_{\Omega_t}|\nabla(u-\widetilde{u})|^2\,\mathrm{d}x\,\mathrm{d}s$$
$$\leqslant\int_{\Omega}|u_0-\widetilde{u}|^2\,\mathrm{d}x+\frac{1}{\theta}\int_{\Omega_t}b_1^2(x)|u-\widetilde{u}|^2\,\mathrm{d}x\,\mathrm{d}s. \tag{6.23}$$

For $\lambda>0$ arbitrarily fixed, we have

$$\int_{\Omega_t}b_1^2(x)|u-\widetilde{u}|^2\,\mathrm{d}x\,\mathrm{d}s=\int_0^t\int_{\Omega\cap\{|u(s)-\widetilde{u}(s)|\leqslant\lambda\}}b_1^2(x)|u-\widetilde{u}|^2\,\mathrm{d}x\,\mathrm{d}s$$
$$+\int_0^t\int_{\Omega\cap\{|u(s)-\widetilde{u}(s)|>\lambda\}}b_1^2(x)|u-w|^2\,\mathrm{d}x\,\mathrm{d}s$$
$$\equiv I_1+I_2.$$

First, observe that

$$I_1\leqslant\lambda^2\|b_1\|^2_{L^2(\Omega)}t. \tag{6.24}$$

Using Sobolev's inequality and (6.19), and proceeding as done to prove (5.17) above, we obtain

$$I_2 \leqslant C_0(N)\|b_1\|^2_{L^\infty(\Omega)}\Big(\frac{\int_\Omega |u_0-\tilde{u}|\,dx}{\lambda}\Big)^{2/N}\int_{\Omega_t}|\nabla(u-\tilde{u})|^2\,dx\,ds. \tag{6.25}$$

Then we choose λ such that

$$\frac{1}{\theta}C_0(N)\|b_1\|^2_{L^\infty(\Omega)}\Big(\frac{\int_\Omega |u_0-\tilde{u}|\,dx}{\lambda}\Big)^{2/N} = \frac{\theta}{2},$$

namely,

$$\lambda = \Big(\frac{2}{\theta^2}C_0(N)\Big)^{N/2}\|b_1\|^N_{L^\infty(\Omega)}\|u_0-\tilde{u}\|_{L^1(\Omega)}. \tag{6.26}$$

By (6.23) and thanks the previous choice of λ, we deduce

$$\begin{aligned}\int_\Omega |u(t)-\tilde{u}|^2\,dx + \frac{\theta}{2}\int_{\Omega_t}|\nabla(u-\tilde{u})|^2\,dx\,ds \\ \leqslant \int_\Omega |u_0-\tilde{u}|^2\,dx + \frac{1}{\theta}\lambda^2\|b_1\|^2_{L^2(\Omega)}t,\end{aligned}$$

and by Poincaré's inequality, we obtain that the following estimate holds:

$$\|u(t)-\tilde{u}\|^2_{L^2(\Omega)} \leqslant \|u_0-\tilde{u}\|^2_{L^2(\Omega)} + \frac{\lambda^2}{\theta}\|b_1\|^2_{L^2(\Omega)}t - M\int_0^t \|u(s)-\tilde{u}\|^2_{L^2(\Omega)}\,ds,$$

with M as in (6.15). We apply Gronwall's inequality of Lemma 2.1, with

$$\begin{aligned}z(t) &= \|u_0-\tilde{u}\|^2_{L^2(\Omega)} + \frac{\lambda^2}{\theta}\|b_1\|^2_{L^2(\Omega)}t,\\ \gamma(s) &\equiv -M,\\ y(t) &= \|u(t)-\tilde{u}\|^2_{L^2(\Omega)},\end{aligned}$$

where we used the fact that z is nondecreasing. Therefore, we have

$$\|u(t)-\tilde{u}\|^2_{L^2(\Omega)} \leqslant z(t)e^{-Mt}. \tag{6.27}$$

Taking into account that λ is provided by (6.26), by Hölder's inequality, we obtain

$$z(t) \leqslant \|u_0-\tilde{u}\|^2_{L^2(\Omega)}\big(1 + c(N,\theta,\Omega)\|b_1\|^{2N}_{L^\infty(\Omega)}\|b_1\|^2_{L^2(\Omega)}t\big). \tag{6.28}$$

Finally, (6.14) follows by (6.27) and (6.28). ∎

Remark 6.2. We let

$$g \in C([0,T],L^2(\Omega)) \cap L^2(0,T,W^{1,2}(\Omega)), \tag{6.29}$$

$$g_t \in L^2(\Omega_T), \tag{6.30}$$

satisfying the compatibility condition

$$\psi \leqslant g \quad \text{on } \partial\Omega \times (0,T). \tag{6.31}$$

Moreover, we will assume

$$\psi(\cdot,0) \in g(\cdot,0) + W_0^{1,2}(\Omega). \tag{6.32}$$

Our results still holds true if g is considered as a boundary datum for the obstacle problem, that is, if the convex set $\mathcal{K}_\psi(\Omega_T)$ is replaced by

$$\begin{aligned}\mathcal{K}_{\psi,g}(\Omega_T) := \big\{ v \in C^0([0,T], L^2(\Omega)) : \ v_t \in L^2(0,T,W^{-1,2}(\Omega)),\\ v \in g + L^2(0,T,W_0^{1,2}(\Omega)), \text{ and } v \geqslant \psi \text{ a.e. in } \Omega_T \big\}.\end{aligned}$$

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