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Porosity of the free boundary in a class of higher-dimensional elliptic problems

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Abstract. We investigate a class of n-dimensional ($n \geq 2$) free boundary elliptic problems, which includes the dam problem, the aluminum problem, and the lubrication problem. We establish that the free boundary in this class is a porous set, which implies its Hausdorff dimension being less than n, which in turn leads to its Lebesgue measure being zero. As a corollary, we obtain the uniqueness of the so-called reservoirs-connected solution of the dam problem in higher dimension. Our proof of the porosity relies on the Lipschitz continuity of the solution and an appropriately constructed barrier function.

1. Introduction

We are interested in the study of some properties of the following problem:

$$\begin{cases} \text{Find } (u,\chi) \in H^1(\Omega) \times L^\infty(\Omega) \text{ such that:} \\ \text{(i)} \quad u = 0 \quad \text{on } \Gamma \\ \text{(ii)} \quad u \geq 0, \quad 0 \leq \chi \leq 1, \quad u(1-\chi) = 0 \quad \text{a.e. in } \Omega \\ \text{(iii)} \quad \displaystyle\int_\Omega \big(a(x)\nabla u + \chi h(x) e_n\big) \cdot \nabla \xi \, dx \leq 0, \\ \qquad\qquad \forall \xi \in H^1(\Omega), \quad \xi = 0 \quad \text{on } \partial\Omega \setminus \Gamma, \quad \xi \geq 0 \quad \text{on } \Gamma \end{cases} \tag{P}$$

where $e_n = (0,\dots,0,1)$, $\Omega = \{(x', x_n) \in \mathbb{R}^n \ / \ x' \in B'_\rho,\ \gamma(x') < x_n < \gamma(x') + l\}$, $B'_\rho = \{x' \in \mathbb{R}^{n-1} \ / \ |x'| < \rho\}$, $x' = (x_1, \dots, x_{n-1})$, ρ and l are two positive numbers, $\gamma : B'_\rho \to \mathbb{R}$ is a $C^{1,1}$ function, and $\Gamma = \{(x', \gamma(x') + l) : x' \in B'_\rho\}$.

Here, $a(x) = (a_{ij}(x))$ is an $n \times n$ matrix and $h(x)$ denotes a scalar function such that the following conditions hold for positive constants λ, Λ, $\overline{h}$, $\underline{h}$, and $p > n$:

$$\sum_{i,j} |a_{ij}(x)| \leq \Lambda \qquad \text{for a.e. } x \in \Omega, \tag{1.1}$$

$$a(x)\xi \cdot \xi \geq \lambda |\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^n \text{ and for a.e. } x \in \Omega, \tag{1.2}$$

$$\underline{h} \leq h(x) \leq \overline{h} \qquad \text{for a.e. } x \in \Omega, \tag{1.3}$$

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$$h_{x_n} \in L^p(\Omega \cup \Gamma), \tag{1.4}$$

$$h_{x_n}(x) \geq 0 \qquad \text{for a.e. } x \in \Omega. \tag{1.5}$$

This formulation encompasses various free boundary problems, notably the dam problem [2, 3, 6, 8, 15, 19], where Ω denotes a porous medium, $a(x)$ represents its permeability matrix, and $h(x) = a_{nn}(x)$. Additionally, the weak formulation extends to include the lubrication problem [4], with $a(x) = h^3(x)I_2$, where I_2 denotes the 2×2 identity matrix, and $h(x)$ is a scalar function associated with the Reynolds equation. A third model addressed herein is the aluminum electrolysis problem [5, 13], in which $a(x)$ is a variable diagonal matrix.

Our primary objective in this study is to investigate the free boundary $(\partial\{u > 0\}) \cap \Omega$, denoted in this paper by FB, which separates the sets $\{u > 0\}$ and $\{u = 0\}$. FB delineates the interface between the fluid-containing region and the fluid-free region in scenarios such as dam and lubrication problems, as well as in the context of aluminum electrolysis, where it separates regions with liquid and solid aluminum. The regularity of the set FB has been explored by numerous researchers in various contexts. For instance, in [1], H. W. Alt established that it is a Lipschitz continuous curve $x_n = \Phi(x')$ under the condition that $a(x)$ is the identity matrix in dimension $n \geq 2$, a result that can readily be extended to a constant matrix. In the two-dimensional case, J. Carrillo and M. Chipot [6] established that FB is the graph of a continuous curve $x_2 = \Phi(x_1)$ in the homogeneous case. Furthermore, A. Lyaghfouri [15] generalized this result from [6] to the nonhomogeneous case under the condition that $a_{12}(x) = 0$ and $h(x) = a_{22}(x)$ is nondecreasing with respect to x_2.

When a and h are both nonconstant and unrelated to each other, proving the continuity of the free boundary becomes more challenging. This complexity was initially addressed in the two-dimensional context by M. Chipot [11], and later explored by S. Challal and the author [9] under varied assumptions. Moreover, under general assumptions, the authors demonstrated in [10], that FB is represented locally by the graphs of a family of continuous functions. However, all these proofs primarily focused on two dimensions and were not easily extendable to higher dimensions. Given that under assumption (1.5) FB is the graph of a lower semi-continuous function $\phi(x')$, a pivotal aspect of demonstrating the continuity was to establish the upper semi-continuity of ϕ, which in dimension two rested on the existence, by the maximum principle, of two points in arbitrarily close proximity to a free boundary point, such that u vanishes above these points. This facilitated the construction of a barrier function for comparison with u on a rectangle positioned above the free boundary point with the objective to show that u vanishes in that rectangle, hence implying the upper semi-continuity of ϕ. However, this approach becomes clearly complicated in dimensions $n \geq 3$, as it relies on comparing the solution u to a suitably constructed barrier function along the entire lateral surface of a cylinder centered at a free boundary point. Such a comparison requires a locally uniform asymptotic behavior of u near the free boundary—a property that is believed to hold but is challenging to rigorously establish. Unlike in the two-dimensional case, the maximum principle in higher dimensions does not guarantee that u vanishes on the lateral boundary of a sequence of arbitrarily small cylinders containing an interior free boundary point. In two dimensions, this was possible due to the combined effect of the maximum principle, the properties of the function χ, and the simple structure of the lateral boundary—consisting merely of two vertical line segments as cylindrical domains, in this case, reduced to rectangles, thus simplifying the analysis considerably.

It is worth mentioning that in the special case of the homogeneous dam problem in dimensions $n \geq 3$, Alt [1] successfully applied the Baiocchi transform to convert the original problem into an obstacle-like problem, which has the same free boundary and managed to prove that it is a graph of a smooth function. Nevertheless, this powerful approach does not extend to the more general nonhomogeneous setting, unless the permeability matrix $a(x)$ is independent of the vertical coordinate x_n as shown in [12, Theorem 5.2]. The primary obstacle is technical in nature: the Baiocchi transform of the solution satisfies a considerably more complex equation, making the analysis of the related free boundary more complicated to investigate.

In this paper, our aim is to investigate the regularity of the set FB within the broader framework of n-dimensional case ($n \geq 2$). Our main contribution lies in establishing the porosity of FB, a result that notably entails its Hausdorff dimension being less than n, consequently resulting in its Lebesgue measure being zero, leading to $\chi = \chi_{\{u>0\}}$.

We begin with two remarks.

Remark 1.1. Consider the change of variables $y = \Phi(x) = (\Phi_i(x))$ and $x = \Psi(y) = (\Psi_i(y))$, where

$$\Phi_i(x) = \begin{cases} x_i, & \text{if } i = 1, \ldots, n-1, \\ x_n - \gamma(x'), & \text{if } i = n, \end{cases}$$

and

$$\Psi_i(y) = \begin{cases} y_i, & \text{if } i = 1, \ldots, n-1, \\ y_n + \gamma(y'), & \text{if } i = n. \end{cases}$$

Then $\Psi : U = B'_\rho \times (0, l) \to \Omega$ and $\Phi : \Omega \to U$ are $C^{1,1}$ diffeomorphisms with $\Psi = \Phi^{-1}$ and $J_\Phi(x) = J_\Psi(y) = 1$ for all $x \in \Omega$ and all $y \in U$. Moreover, setting $b(y) = D\Phi(\Psi(y))a(\Psi(y))^t D\Phi(\Psi(y))$, $v(y) = uo\Psi(y)$, $\theta(y) = \chi o\Psi(y)$, and $k(y) = ho\Psi(y)$, it is not difficult to verify that (v, θ) is a solution of the following problem:

$$(Q) \begin{cases} \text{Find } (v, \theta) \in H^1(U) \times L^\infty(U) \text{ such that :} \\ \text{(i)} \quad v = 0 \quad \text{on } S = B'_\rho \times \{l\}, \\ \text{(ii)} \quad v \geq 0, \quad 0 \leq \theta \leq 1, \quad v(1-\theta) = 0 \quad \text{a.e. in } U, \\ \text{(iii)} \quad \displaystyle\int_U \big(b(y)\nabla v + \theta k(y)e_n\big) \cdot \nabla\zeta dy \leq 0, \\ \qquad\qquad \forall \zeta \in H^1(U), \qquad \zeta = 0 \quad \text{on } \partial U \setminus S, \qquad \zeta \geq 0 \quad \text{on } S. \end{cases}$$

Therefore, without sacrificing generality, we may consider Ω to be a cylinder $B'_\rho \times (0, l)$ and Γ the flat surface $B'_\rho \times l$. This assumption will persist throughout the paper.

Remark 1.2. (i) Henceforth, we denote by (u, χ) a solution to problem (P).

(ii) From (P) (iii), we derive the equation $\operatorname{div}(a(x)\nabla u) = -(h\chi)_{x_n}$ satisfied in distributional sense. Since $\chi \in L^\infty(\Omega)$ and due to (1.1)–(1.3), it follows (see [14, Theorem 8.24, page 202] that u is locally Höder continuous in Ω.

(iii) Since $u = 0$ on Γ and Γ is $C^{1,1}$, we also have (see [14, Theorem 8.29, page 205]) that u is locally Höder continuous in $\Omega \cup \Gamma$.

Prior to presenting our principal contribution, it is essential to recall a definition (see, e.g., [21]).

Definition 1.3. We define a set $E \subset \mathbb{R}^n$ as porous with a porosity parameter $\delta > 0$ if there exists a positive number r_0 such that for every point $x \in E$ and every radius $r \in (0, r_0)$, there exists a point $y \in \mathbb{R}^n$ such that the open ball $B_{\delta r}(y)$ is entirely contained within the complement of E in the ball $B_r(x)$, that is, $B_{\delta r}(y) \subset B_r(x) \setminus E$.

Remark 1.4. A porous set with porosity δ possesses a Hausdorff dimension that does not surpass $n - c\delta^n$ (see [18,20]), where $c = c(n) > 0$ represents a constant dependent solely on n. Consequently, a porous set has Lebesgue measure zero.

We shall additionally presume that

$$a \in C^{0,\alpha}_{\text{loc}}(\Omega \cup \Gamma) \quad \text{for some } \alpha \in (0, 1). \tag{1.6}$$

Under the aforementioned assumptions, our main result is presented below.

Theorem 1.5. *The free boundary* (FB) *is locally a porous set.*

Corollary 1.6.

$$\chi = \chi_{\{u>0\}} \quad a.e.\ in\ \Omega.$$

The next two remarks are about the dam problem and the so-called reservoirs-connected solution.

Remark 1.7. Let $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ be a bounded open domain with Lipschitz boundary, representing a porous medium. We assume that the boundary $\partial\Omega$ is partitioned into three disjoint components:

- S_1, the impervious portion,
- S_2, the part in contact with the air, and
- S_3, which comprises the union of all bottom reservoirs supplying the dam.

Let φ be a non-negative Lipschitz continuous function defined on $S_2 \cup S_3$, with $\varphi = 0$ on S_2 and $\varphi > 0$ on S_3. As noted earlier in the introduction, under the vertical flow condition $a(x)e_n = a_{nn}(x)e_n$ and when $h(x) = a_{nn}(x)$, the general problem (P) reduces to the classical dam problem (see [2, 6]):

$$\begin{cases} \text{Find } (u,\chi) \in H^1(\Omega) \times L^\infty(\Omega) \text{ such that:} \\ \text{(i)} \quad u = \varphi \quad \text{on} \quad S_2 \cup S_3, \\ \text{(ii)} \quad u \geq 0, \quad 0 \leq \chi \leq 1, \quad u(1-\chi) = 0 \quad \text{a.e. in } \Omega, \\ \text{(iii)} \quad \displaystyle\int_\Omega a(x)\big(\nabla u + \chi e_n\big) \cdot \nabla \xi dx \leq 0, \\ \qquad\qquad \forall \xi \in H^1(\Omega), \qquad \xi = 0 \quad \text{on } S_3, \qquad \xi \geq 0 \quad \text{on } S_2. \end{cases}$$

In [15], the author established the uniqueness of the reservoirs-connected solution, assuming that the free boundary set FB has zero Lebesgue measure, for dimensions $n \geq 2$.

Remark 1.8. The concept of a reservoirs-connected solution was initially introduced in [6] under the name S_3-connected solution. This terminology was subsequently relabeled as reservoirs-connected solution in [7].

A solution (u,χ) to the dam problem is said to be reservoirs-connected if every connected component C of the saturated region $\{u > 0\}$ has a nonempty intersection with the bottom boundary of a reservoir. In other words, each saturated component remains in direct contact with at least one of the reservoirs bases.

Assuming the permeability matrix $a(x)$ satisfying conditions (1.1)–(1.2) and (1.6), and that $a(x)e_n = a_{nn}(x)e_n$ with $a_{nn}(x)$ satisfying (1.4)–(1.5), and further assuming that S_2 is given by the graph of a $C^{1,1}$ function, we obtain the following uniqueness result:

Corollary 1.9. *Under the above assumptions, the reservoirs-connected solution of the dam problem is unique.*

We start by recalling a crucial regularity result: the Lipschitz continuity of u. Under assumptions (1.1)–(1.4) and (1.6), we have [9, 16, 17]:

Proposition 1.10.

$$u \in C^{0,1}_{\text{loc}}(\Omega \cup \Gamma).$$

Remark 1.11. The Lipschitz continuity of u stands as a pivotal element in establishing our main result. It is noteworthy that this level of regularity is optimal, as it is directly tied to the discontinuity exhibited by ∇u across the set FB. Additionally, it is vital to highlight that the Lipschitz continuity of u has played a key role in proving the regularity results concerning the free boundary in both [9] and [10].

We note that numerous properties of solutions to problem (P), established in the two-dimensional case in [11], readily extend to higher dimension. In the ensuing discussion, we revisit some of these properties essential for our subsequent analysis. The first property is the monotonicity of χ with respect to x_n.

Proposition 1.12.

$$\chi_{x_n} \leq 0 \quad \textit{in } \mathcal{D}'(\Omega).$$

Proposition 1.12, the continuity of u, and the maximum principle imply that if u is positive at a given point in Ω, then it remains positive throughout a cylinder below that point. Consequently, if u vanishes at a specific point within Ω, it also vanishes above that point.

Proposition 1.13. *Let $x_0 = (x_0', x_{0n}) \in \Omega$.*

(1) *If $u(x_0) > 0$, then there exists $\varepsilon > 0$ such that*

$$u(x', x_n) > 0, \quad \forall (x', x_n) \in B_\varepsilon(x_0) \cup (B_\varepsilon'(x_0') \times (0, x_{0n})).$$

(2) *If $u(x_0) = 0$, then $u(x_0', x_n) = 0$ for all $x_n \geq x_{0n}$.*

Proposition 1.13 prompts the definition of the subsequent function for all $x' \in B_\rho'$:

$$\phi(x') = \begin{cases} 0 & \text{if} \quad u(x', x_n) = 0 \ \ \forall x_n \in (0, l), \\ \sup\left\{x_n \in (0, l) \mid u(x', x_n) > 0\right\} & \text{otherwise.} \end{cases}$$

The function ϕ is well defined and satisfies thanks to Proposition 1.13:

Proposition 1.14. *ϕ is lower semi-continuous on B_ρ' and*

$$\left\{u > 0\right\} = \left\{x_n < \phi(x')\right\}.$$

The last property states that if u is zero in a particular open ball within Ω, then χ is zero almost everywhere within that ball and above it.

Proposition 1.15. *Let $x_0 = (x_0', x_{0n}) \in \Omega$ and $r > 0$ such that $B_r(x_0) \subset\subset \Omega$. If $u = 0$ in $B_r(x_0)$, then we have*

$$\chi = 0 \quad \text{a.e. in } B_r(x_0) \cup (B_r'(x_0') \times (x_{0n}, l)).$$

The subsequent sections of this paper are structured as follows: in Section 2, we develop a supersolution for problem (P), referred to as a "barrier function." This tool will play a pivotal role in proving Theorem 1.5. Section 3 presents the detailed proofs of Theorem 1.5 and Corollary 1.6.

2. A barrier function

Let $x_0 = (x_0', x_{0n}) \in \Omega$, $\overline{x}_0 = (x_0', l)$, and $r_0 > 0$ such that $\overline{B}_{2r_0}(x_0) \subset \Omega$, which implies that $2r_0 < \min(x_{0n}, l - x_{0n})$, and $K_0 = \overline{B}'_{2r_0}(x_0') \times [x_{0n} - r_0, l] \subset \Omega \cup \Gamma$.

From Proposition 1.10, there exists a constant $C_1 > 0$ depending on $n, p, \lambda, \Lambda, h, a, r_0$, and l such that

$$|\nabla u(x)| \leq C_1 \quad \text{a.e. } x \in K_0. \tag{2.1}$$

We consider a function $\eta \in C^2(\mathbb{R})$ such that $-1 \leq \eta(t) \leq 1$ in $\mathbb{R}$ and

$$\eta(t) = \begin{cases} -1 & \text{if } t \leq \dfrac{1}{4}, \\ 1 & \text{if } t \geq 1. \end{cases}$$

Let $\delta \in (0, 1)$ be a positive real number and let

$$\theta(t) = \eta\Big(\frac{3(t-1)}{4\delta(2-\delta)} + 1\Big).$$

Then $\theta \in C^2(\mathbb{R})$ is such that $-1 \leq \theta(t) \leq 1$ in $\mathbb{R}$ and

$$\theta(t) = \begin{cases} -1 & \text{if } t \leq (1-\delta)^2, \\ 1 & \text{if } t \geq 1. \end{cases}$$

Setting $g(y') = \theta(|y'|^2)$ for $y' \in \mathbb{R}^{n-1}$, we have $g \in C^2(\mathbb{R}^{n-1})$ and

$$g(y') = \begin{cases} -1 & \text{if } |y'| \leq 1-\delta, \\ 1 & \text{if } |y'| \geq 1. \end{cases}$$

Let $r \in (0, r_0)$. Then the function $f_r(x') = l + rg(\frac{x'-x'_0}{r})$ satisfies $f_r \in C^2(\mathbb{R}^{n-1})$, $l - r \leq f_r(x') \leq l + r$ in $\mathbb{R}^{n-1}$, and

$$f_r(x') = \begin{cases} l-r & \text{if } |x'-x'_0| \leq (1-\delta)r, \\ l+r & \text{if } |x'-x'_0| \geq r. \end{cases}$$

We infer from (1.3)–(1.4) that for almost every $x' \in B'_\rho$, the function $h(x', .)$ has a representative that is continuous up to $x_n = l$ and in particular possesses a trace $h(x', l)$ for almost every $x' \in B'_\rho$. Therefore, we can define the extension of h to $\widetilde{\Omega} = B'_\rho \times (0, 2l)$ as follows:

$$\widehat{h}(x', x_n) = \begin{cases} h(x', x_n) & \text{if } (x', x_n) \in \Omega, \\ h(x', l) - h(x', 2l - x_n) & \text{if } (x', x_n) \in \widetilde{\Omega} \setminus \Omega. \end{cases}$$

We also define the extension of the function a to $\widetilde{\Omega}$ by:

$$\widetilde{a}(x', x_n) = \begin{cases} a(x', x_n) & \text{if } (x', x_n) \in \Omega, \\ a(x', 2l - x_n) & \text{if } (x', x_n) \in \widetilde{\Omega} \setminus \Omega. \end{cases}$$

We observe that $\widetilde{a}$ and $\widehat{h}$ satisfy (1.1)–(1.5) and (1.6) in $\widetilde{\Omega}$ with the same constants.

Finally, let $\varepsilon \in (0, 1)$ be a fixed real number. We consider the unique solution v of the following elliptic Dirichlet boundary value problem:

$$\begin{cases} \operatorname{div}(\widetilde{a}(x)\nabla v + \widehat{h}e_n) = 0 & \text{in } D_r, \\ v = \dfrac{\varepsilon}{3}(f_r(x') - x_n) & \text{on } \partial D_r, \end{cases} \tag{2.2}$$

where $D_r = \{(x', x_n) \in \mathbb{R}^n \ : \ |x' - x'_0| < 2r \text{ and } l - 2r < x_n < f_r(x')\}$. We note that

$$0 \le v \le \frac{\varepsilon}{3}(l + r - (l - 2r)) = \varepsilon r \quad \text{on } \partial D_r. \tag{2.3}$$

First we establish an estimate of v.

Proposition 2.1. *There exists a positive constant C_2 dependent solely on n, p, λ, δ, and $|h_{x_n}|_{L^p(\Omega)}$ such that*

$$0 < v \le C_2 \varepsilon^{1-\frac{n}{p}} \quad \text{in } D_r. \tag{2.4}$$

Proof. Since $\operatorname{div}(\widetilde{a}(x)\nabla v) = -\widehat{h}_{x_n} \le 0$ in D_r and $v \ge 0$ on ∂D_r, we obtain by the weak maximum principle (see [14, Theorem 8.1, page 179]) that $v \ge 0$ in D_r.

Now, because of the boundary condition, the strong maximum principle (see [14, Theorem 8.19, page 198]) leads to $v > 0$ in D_r.

To prove the second inequality, we introduce the set

$$\Delta = \big\{(y', y_n) : |y'| < 2 \text{ and } -2 < y_n < g(y')\big\}$$

and the following functions defined in Δ:

$$w(y) = \frac{1}{r}v(\overline{x}_0 + ry), \quad b(y) = \widetilde{a}(\overline{x}_0 + ry), \quad k(y) = \widehat{h}(\overline{x}_0 + ry), \quad y = \frac{1}{r}(x - \overline{x}_0).$$

Then it is straightforward to verify using (2.2) that w satisfies

$$\begin{cases} \operatorname{div}(b(y)\nabla w) = -k_{y_n} = -r\widehat{h}_{x_n}(\overline{x}_0 + ry) \text{in } \Delta, \\ \qquad\qquad\qquad w = \dfrac{\varepsilon}{3}(g(y') - y_n) \text{on } \partial\Delta. \end{cases} \tag{2.5}$$

Clearly b and k satisfy (1.1)–(1.6) with the same constants. By applying [14, Theorem 8.16, page 191], we get

$$\sup_{\Delta} w \le \sup_{\partial\Delta} w + \frac{C}{\lambda}|k_{y_n}|_{L^{q/2}(\Delta)},$$

where $q = 2p > n$ and C is a positive constant depending only on n and p. Moreover,

$$\begin{aligned} |k_{y_n}|_{L^{q/2}(\Delta)} &= |r\widehat{h}_{x_n}(\overline{x}_0 + ry)|_{L^{q/2}(\Delta)} \\ &= \Big(\int_{\Delta} r^p \widehat{h}^p_{x_n}(\overline{x}_0 + ry)dy\Big)^{1/p} = \Big(\int_{D_r} \frac{r^p}{r^n}\widehat{h}^p_{x_n}(x)dx\Big)^{1/p} \\ &= r^{1-\frac{n}{p}}|\widehat{h}_{x_n}|_{p,D_r} \le r^{1-\frac{n}{p}}|\widehat{h}_{x_n}|_{p,\widetilde{\Omega}} = 2^{\frac{1}{p}} r^{1-\frac{n}{p}}|h_{x_n}|_{p,\Omega}. \end{aligned}$$

Hence, we get by using (2.3) for $0 < r < \varepsilon < 1$ and $C_2 = 1 + \frac{2^{\frac{1}{p}}C}{\lambda}|h_{x_n}|_{L^p(\Omega)}$,

$$\begin{aligned} \sup_{D_r} v = \sup_{\Delta} w &\le \varepsilon + \frac{2^{\frac{1}{p}}C}{\lambda}\varepsilon^{1-\frac{n}{p}}|h_{x_n}|_{L^p(\Omega)} \\ &= \Big(\varepsilon^{\frac{n}{p}} + \frac{2^{\frac{1}{p}}C}{\lambda}|h_{x_n}|_{L^p(\Omega)}\Big)\varepsilon^{1-\frac{n}{p}} \le C_2\varepsilon^{1-\frac{n}{p}}. \end{aligned}$$

∎

The following regularity and estimate results on $G_r = \{x_n = f_r(x')\} \cap \{|x' - x'_0| \leq r\}$ will be needed to show that v is a supersolution to problem (P) in $\Omega_r = B'_r(x'_0) \times (l - 2r, l)$.

Proposition 2.2. *We have that $v \in C^{1,\alpha}(D_r \cup G_r)$ and there exists a positive constant C_3 independent of ε and r such that for all $r \in (0, r_1)$, $r_1 = \min(r_0, \varepsilon)$,*

$$|\nabla v(x)| \leq C_3 \varepsilon^{1-\frac{n}{p}}, \quad \forall x \in G_r. \tag{2.6}$$

Proof. Let $T = \{y_n = g(y')\} \cap \{|y'| \leq 1\}$ and $\Delta' = \Delta \cap \{|y'| \leq 3/2\}$. Since b and k satisfy (1.1)–(1.6) with the same constants, T is a $C^{1,\alpha}$ boundary portion of $\partial\Delta$, $w = 0$ on T, we deduce from (2.5) and the subsequent remark to [14, Corollary 8.36, page 212] that $w \in C^{1,\alpha}(\Delta \cup T)$ and satisfies the following estimate:

$$|w|_{1,\alpha,\Delta'} \leq C(|w|_{0,\Delta} + |k_{y_n}|_{p,\Delta}) \tag{2.7}$$

where $C = C(n, \lambda, M, d', T)$, $d' = \operatorname{dist}(\Delta', \partial\Delta \setminus T)$, M is an upper bound of

$$\max_{i,j}(|b_{ij}|_{0,\alpha,\Delta'}) = r^\alpha \max_{i,j}(|\widetilde{a}_{ij}|_{0,\alpha,D'_r}) \leq \max_{i,j}(|\widetilde{a}_{ij}|_{0,\alpha,\widetilde{K}_0}), \text{ and since } 0 < r < \varepsilon < 1,$$
$$\begin{aligned} D'_r &= D_r \cap \{l - 3r/2 \leq x_n \leq l + r/2\} \subset\subset \widetilde{K}_0 \\ &= \overline{B}'_{2r_0}(x'_0) \times [x_{0n} - r_0, 2l - x_{0n} + r_0]. \end{aligned}$$

Since $\max_{i,j}(|\widetilde{a}_{ij}|_{0,\alpha,\widetilde{K}_0}) = \max_{i,j}(|a_{ij}|_{0,\alpha,K_0})$, it follows that C can be chosen to depend solely on n, λ, δ, and $\max_{i,j}(|a_{ij}|_{0,\alpha,K_0})$.

Moreover, from the proof of Proposition 2.1, $|k_{y_n}|_{p,\Delta} \leq 2^{\frac{1}{p}} r^{1-\frac{n}{p}} |h_{x_n}|_{p,\Omega}$. Hence, we infer from (2.4) and (2.7), that we have for all $0 < r < r_1$,

$$|w|_{1,\alpha,\Delta'} \leq C\big(C_2 \varepsilon^{1-\frac{n}{p}} + 2^{\frac{1}{p}} \varepsilon^{1-\frac{n}{p}} |h_{x_n}|_{p,\Omega}\big) \leq C\big(C_2 + 2^{\frac{1}{p}} |h_{x_n}|_{p,\Omega}\big) \varepsilon^{1-\frac{n}{p}}.$$

We obtain for another constant $C = C(n, \lambda, \delta, |h_{x_n}|_{p,\Omega}, \max_{i,j}(|a_{ij}|_{0,\alpha,K_0}))$, denoted by C_3, $|\nabla w|_{0,\Delta'} \leq C_3 \varepsilon^{1-\frac{n}{p}}$. This leads, in particular, to

$$|\nabla w(y', g(y'))| \leq C_3 \varepsilon^{1-\frac{n}{p}} \qquad \forall y' \in \overline{B}'_1$$

Therefore, we obtain

$$|\nabla v(x', f_r(x'))| = \left|\nabla w\Big(\frac{x' - x'_0}{r}, g\Big(\frac{x' - x'_0}{r}\Big)\Big)\right| \leq C_3\, \varepsilon^{1-\frac{n}{p}}, \quad \forall x' \in G_r. \qquad \blacksquare$$

Let now θ_0 be a number such that $\theta_0 \geq 2 \max_{-2 \leq t \leq 2} |\theta'(t)|$ and

$$\varepsilon_0 = \Big(\frac{h}{C_3 \Lambda \sqrt{1 + \theta_0^2}}\Big)^{\frac{p}{p-n}}.$$

From now on, we will denote by v the function defined in (2.1) for $\varepsilon = \varepsilon_0$ and extended by 0 to $\{x_n > f_r(x')\}$. The next lemma asserts that v is a supersolution to problem (P) in Ω_r.

Lemma 2.3. *We have for all* $0 < r < r_1$ *that*

$$\int_{\Omega_r} (a(x)\nabla v + \chi_{\{v>0\}} h(x)e_n).\nabla\zeta dx \geq 0,$$
$$\forall \zeta \in H^1(\Omega_r), \qquad \zeta \geq 0 \quad in\ \Omega_r, \quad \zeta = 0 \quad on\ \partial\Omega_r \setminus \Gamma. \tag{2.8}$$

Proof. Set $C_r = \Omega_r \cap \{x_n < f_r(x')\} = \Omega_r \cap \{v > 0\}$, and let $\nu = (\nu_1, \ldots, \nu_n)$ be the outward unit normal vector to $\partial C_r \cap (\{x_n = l\} \cup \Omega_r)$. First, we have for $|x' - x_0'| \leq r$,

$$|\nabla_{x'} f_r(x')| = \left|\theta'\Big(\frac{|x'-x_0'|^2}{r^2}\Big)\right| . \frac{2|x'-x_0'|}{r} \leq \theta_0,$$

which leads to

$$\nu_n = \frac{1}{\sqrt{1 + |\nabla_{x'} f_r(x')|^2}} \geq \frac{1}{\sqrt{1+\theta_0^2}} \quad \text{on } \partial C_r \cap \Omega_r = \big\{x_n = f_r(x')\big\} \cap \Omega_r. \tag{2.9}$$

Next, since

$$\nu_n = 1 \geq \frac{1}{\sqrt{1+\theta_0^2}}$$

on $\{x_n = l\} \cap \partial C_r$, we deduce from (1.1), (1.3), (2.6), and (2.9) that for $0 < r < r_1$,

$$a(x)\nabla v.\nu + h(x)\nu_n \geq -C_3\Lambda.\varepsilon_0^{1-\frac{n}{p}} + \frac{h}{\sqrt{1+\theta_0^2}}$$
$$= 0 \quad \text{on } \partial C_r \cap (\{x_n = l\} \cup \Omega_r). \tag{2.10}$$

We conclude by taking into account (2.2) and (2.10), for $\zeta \in H^1(\Omega_r)$ with $\zeta \geq 0$ in Ω_r and $\zeta = 0$ on $\partial\Omega_r \setminus \Gamma$,

$$\int_{\Omega_r} \big(a(x)\nabla v + \chi_{\{v>0\}} h(x)e_n\big).\nabla\zeta dx = \int_{C_r} \big(a(x)\nabla v + h(x)e_n\big).\nabla\zeta dx$$
$$= \int_{\partial C_r \cap (\{x_n = l\} \cup \Omega_r)} \big(a(x)\nabla v.\nu + h(x)\nu_n\big)\zeta d\sigma \geq 0 \qquad \blacksquare$$

Remark 2.4. Functions similar to the one constructed previously are often termed as barrier functions.

The proof of Theorem 1.5 is based on comparing the function u to the barrier function v. We initiate this process by establishing two lemmas.

Lemma 2.5. *Assume that* $u \leq v$ *on* $\Omega \cap \partial\Omega_r$, *and let* $\Omega_r^s = \Omega_r \cap \{0 \leq u - v < s\} \cap \{v > 0\}$ $(s > 0)$. *Then we have*

$$\int_{\Omega_r^s} a(x)\nabla(u-v)^+.\nabla(u-v)^+ dx = o(s) \ \ as \ \ s \to 0 \quad uniformly\ in\ r \in (0, r_1]. \tag{2.11}$$

Proof. For $s, \eta > 0$, we consider the functions

$$H_s(t) = \min\Big(\frac{t^+}{s}, 1\Big), \quad d_\eta(x) = H_\eta((x_n - f_r(x'))^+).$$

Given that $u \le v$ on $\Omega \cap \partial\Omega_r$, and since $u = 0$ and $v \ge 0$ on $\partial\Omega_r \cap \Gamma$ and $H_s(t) = 0$ for $t \le 0$, we have $H_s(u - v) = 0$ on $\partial\Omega_r$. Moreover, since $d_\eta = 0$ below $\{x_n = f_r(x')\}$, we have $d_\eta = 0$ on $\Omega \cap \partial\Omega_r$. Therefore, $\zeta = H_s(u - v) + d_\eta(1 - H_s(u)) \in H^1(\Omega_r)$ is a non-negative function that vanishes on $\Omega \cap \partial\Omega_r$, and hence, we have

$$\int_{\Omega_r} \big(a(x)\nabla u + \chi h(x)e_n\big).\nabla\zeta dx \le 0,$$

which leads to

$$\begin{aligned}&\int_{\Omega_r} \big(a(x)\nabla u + \chi h(x)e_n\big).\nabla(H_s(u - v))dx\\ &\qquad \le -\int_{\Omega_r} \big(a(x)\nabla u + \chi h(x)e_n\big).\nabla(d_\eta(1 - H_s(u)))dx. \end{aligned} \tag{2.12}$$

Since $H_s(u - v) = 0$ on $\partial\Omega_r$, we get by (2.8) that

$$-\int_{\Omega_r} \big(a(x)\nabla v + \chi_{\{v>0\}}h(x)e_n\big).\nabla(H_s(u - v))dx \le 0. \tag{2.13}$$

Adding (2.12) and (2.13), we get

$$\begin{aligned}&\int_{\Omega_r} \big(a(x)\nabla(u - v) + (\chi - \chi_{\{v>0\}})h(x)e_n\big).\nabla(H_s(u - v))dx\\ &\qquad \le -\int_{\Omega_r} \big(a(x)\nabla u + \chi h(x)e_n\big).\nabla(d_\eta(1 - H_s(u)))dx,\end{aligned}$$

which can be written as

$$\begin{aligned}\int_{\Omega_r} a(x)\nabla(u - v).\nabla(H_s(u - v))dx &\le \int_{\Omega_r} h(x)(\chi_{\{v>0\}} - \chi)e_n.\nabla(H_s(u - v))dx\\ &\quad - \int_{\Omega_r} \Big(a(x)\nabla u + \chi h(x)e_n\Big).\nabla(d_\eta(1 - H_s(u)))dx\end{aligned}$$

or

$$\begin{aligned}&\int_{\Omega_r\cap\{v>0\}} a(x)\nabla(u - v).\nabla(H_s(u - v))dx\\ &\qquad \le \int_{\Omega_r} h(x)(\chi_{\{v>0\}} - \chi)e_n.\nabla(H_s(u - v))dx\\ &\qquad\quad - \int_{\Omega_r\cap\{v=0\}} a(x)\nabla u.\nabla(H_s(u))dx\\ &\qquad\quad - \int_{\Omega_r} \big(a(x)\nabla u + \chi h(x)e_n\big).\nabla(d_\eta(1 - H_s(u)))dx. \end{aligned} \tag{2.14}$$

Since $H_s(u-v)=0$ in $\{u\leq v\}$, and $\chi=1$ almost everywhere in $\{u>0\}\supset\{u>v\}$, we have

$$\int_{\Omega_r}(\chi_{\{v>0\}}-\chi)h(x)e_n.\nabla(H_s(u-v))dx=-\int_{\Omega_r\cap\{v=0\}}\chi h(x)e_n.\nabla(H_s(u))dx$$
$$=\int_{\Omega_r\cap\{v=0\}}\chi h(x)e_n.\nabla(1-H_s(u))dx \tag{2.15}$$

Given that $d_\eta=0$ in $\{v>0\}$, we have

$$\int_{\Omega_r}\big(a(x)\nabla u+\chi h(x)e_n\big).\nabla(d_\eta(1-H_s(u)))dx$$
$$=\int_{\Omega_r\cap\{v=0\}}\big(a(x)\nabla u+\chi h(x)e_n\big).\nabla(d_\eta(1-H_s(u)))dx. \tag{2.16}$$

Taking into account (2.14)–(2.16) and the fact that $\nabla(1-H_s(u))=-\nabla(H_s(u))$, we obtain

$$\begin{aligned}
&\int_{\Omega_r\cap\{v>0\}}H_s'(u-v)a(x)\nabla(u-v).\nabla(u-v)dx\\
&\quad\leq\int_{\Omega_r\cap\{v=0\}}a(x)\nabla u.\nabla(1-H_s(u))dx\\
&\qquad+\int_{\Omega_r\cap\{v=0\}}\chi h(x)e_n.\nabla(1-H_s(u))dx\\
&\qquad-\int_{\Omega_r\cap\{v=0\}}\big(a(x)\nabla u+\chi h(x)e_n\big).\nabla(d_\eta(1-H_s(u)))dx\\
&\quad=\int_{\Omega_r\cap\{v=0\}}\big(a(x)\nabla u+\chi h(x)e_n\big).\nabla((1-d_\eta)(1-H_s(u)))dx\\
&\quad=-\int_{\Omega_r\cap\{v=0\}}(1-d_\eta)\big(a(x)\nabla u+\chi h(x)e_n\big).\nabla(H_s(u))dx\\
&\qquad-\int_{\Omega_r\cap\{v=0\}}(1-H_s(u))\big(a(x)\nabla u+\chi h(x)e_n\big).\nabla d_\eta dx\\
&\quad=I_1^{s,\eta}+I_2^{s,\eta}.
\end{aligned}\tag{2.17}$$

Since $\lim_{\eta\to 0}d_\eta(x)=1$ for almost every $x\in\Omega_r\cap\{v=0\}$, we have by the Lebesgue theorem,

$$\lim_{\eta\to 0}I_1^{s,\eta}=0. \tag{2.18}$$

Now, observe that we have $\nabla d_\eta = H'_\eta((x_n - f_r(x'))^+)(-\nabla_{x'} f_r(x'), 1)$ in $\Omega_r \cap \{v = 0\}$. In particular, we see that d_η is an increasing function in x_n, leading to

$$\begin{aligned}
I_2^{s,\eta} &= -\int_{\Omega_r \cap \{u=v=0\}} \chi h(x) e_n.\nabla d_\eta dx \\
&\quad - \int_{\Omega_r \cap \{u>v=0\}} (1 - H_s(u))\big(a(x)\nabla u + h(x)e_n\big).\nabla d_\eta dx \\
&= -\int_{\Omega_r \cap \{u=v=0\}} H'_\eta((x_n - f_r(x'))^+)\chi h(x) dx \\
&\quad - \int_{\Omega_r \cap \{u>v=0\}} (1 - H_s(u))\big(a(x)\nabla u + h(x)e_n\big).\nabla d_\eta dx \\
&\leq -\int_{\Omega_r \cap \{u>v=0\}} (1 - H_s(u))\big(a(x)\nabla u + h(x)e_n\big).\nabla d_\eta dx. \qquad (2.19)
\end{aligned}$$

Using (1.3) and (2.1), we get from (2.19), for $C'_1 = \Lambda C_1 + \overline{h}$,

$$\begin{aligned}
|I_2^{s,\eta}| &\leq \frac{C'_1}{\eta} \int_{\{u>0\} \cap \{f_r(x') < x_n < f_r(x') + \eta\}} (1 - H_s(u)) dx \\
&\leq \frac{C'_1}{\eta} \int_{\{f_r(x') < x_n < f_r(x') + \eta\} \cap \{f_r(x') < \phi(x')\}} (1 - H_s(u)) dx \\
&= C'_1 \int_{\{f_r < \phi\}} \Big(\frac{1}{\eta} \int_{f_r(x')}^{f_r(x')+\eta} (1 - H_s(u)) dx_n \Big) dx'. \qquad (2.20)
\end{aligned}$$

For each $x' \in \{f_r < \phi\}$, we have by the continuity of $1 - H_s(u)$,

$$\lim_{\eta \to 0} \frac{1}{\eta} \int_{f_r(x')}^{f_r(x')+\eta} (1 - H_s(u)) dx_n = 1 - H_s(u(x', f_r(x'))). \qquad (2.21)$$

Using (2.20)–(2.21) and the Lebesgue theorem, we obtain by letting $\eta \to 0$,

$$\limsup_{\eta \to 0} |I_2^{s,\eta}| \leq C'_1 \int_{\{f_r < \phi\}} (1 - H_s(u(x', f_r(x')))) dx'. \qquad (2.22)$$

Hence, combining (2.17)–(2.18) and (2.22), we arrive at

$$\begin{aligned}
&\frac{1}{s} \int_{\Omega_r \cap \{0 < u - v < s\} \cap \{v > 0\}} a(x)\nabla(u - v)^+.\nabla(u - v)^+ dx \\
&\quad \leq C'_1 \int_{\{f_r < \phi\}} (1 - H_s(u(x', f_r(x')))) dx'.
\end{aligned}$$

Since $\{x_n < \phi(x')\} = \{u > 0\}$, we have $u(x', f_r(x')) > 0$ whenever $f_r(x') < \phi(x')$, leading to $\lim_{s\to 0}(1 - H_s(u(x', f_r(x')))) = 0$. The lemma follows by using the Lebesgue theorem in the above inequality. ■

Lemma 2.6. *If* $u \leq \varepsilon_0 r/3 = C_0 r$ *on* $\partial\Omega_r \cap \Omega$, *then* $u \equiv 0$ *in* $B'_r(x'_0) \times (l - r, l)$.

Proof. Assume that $u \le C_0 r$ on $\partial\Omega_r \cap \Omega$. First, we claim that $u \le v$ on $\partial\Omega_r \cap \Omega$. This is due to the fact that $v \ge C_0 r$ on $\partial\Omega_r \cap \Omega$. Indeed, we observe that $\partial\Omega_r \cap \Omega = (\partial B'_r(x'_0) \times [l-2r, l]) \cup (\bar{B}'_r(x'_0) \times \{x_n = l-2r\})$. So, we distinguish two cases:

Case 1: Assume that $x \in \partial B'_r(x'_0) \times [l-2r, l]$.

Then $v(x) = \frac{\varepsilon_0}{3}(f_r(x') - x_n) = \frac{\varepsilon_0}{3}(l + r - x_n) \ge \frac{\varepsilon_0}{3}(l + r - l) = \frac{\varepsilon_0}{3} r = C_0 r$.

Case 2: Assume that $x \in \bar{B}'_r(x'_0) \times \{x_n = l - 2r\}$.

Then $v(x) = \frac{\varepsilon_0}{3}(f_r(x') - x_n) \ge \frac{\varepsilon_0}{3}(l - r - (l-2r)) = \frac{\varepsilon_0}{3} r = C_0 r$.

Next, let $\zeta \in \mathcal{D}(\Omega \cap \{x_n < f_r^*(x')\})$, where $f_r^*(x') = \min(l, f_r(x')$. Then we have by the Lebesgue theorem,

$$\begin{aligned}\int_{\Omega_r} a(x)\nabla(u-v)^+ \cdot \nabla\zeta dx &= \lim_{s\to 0} \int_{\Omega_r} H_s(u-v)a(x)\nabla(u-v)^+ \cdot \nabla\zeta dx \\ &= \lim_{s\to 0} I_s. \qquad (2.23)\end{aligned}$$

Since $H_s(u-v) = 0$ whenever $u \le v$, we have

$$\nabla(u-v)^+ \cdot \nabla(H_s(u-v)\zeta) = \nabla(u-v) \cdot \nabla(H_s(u-v)\zeta) \quad \text{a.e. in } \Omega_r.$$

Therefore, we can write

$$\begin{aligned}I_s = &\int_{\Omega_r} a(x)\nabla(u-v) \cdot \nabla(H_s(u-v)\zeta) dx \\ &- \int_{\Omega_r} \zeta a(x)\nabla(u-v) \cdot \nabla(H_s(u-v)) dx\end{aligned}$$

or

$$\begin{aligned}I_s &= \int_{\Omega_r} a(x)\nabla(u-v) \cdot \nabla(H_s(u-v)\zeta) dx \\ &\quad - \frac{1}{s}\int_{\Omega_r \cap \{0<u-v<s\} \cap \{v>0\}} \zeta a(x)\nabla(u-v) \cdot \nabla(u-v) dx \\ &= I_s^1 - I_s^2. \qquad (2.24)\end{aligned}$$

First, we observe by using (1.2), that

$$|I_s^2| \le \max_{\bar{\Omega}_r} |\zeta| \cdot \frac{1}{s}\int_{\Omega_r \cap \{0<u-v<s\} \cap \{v>0\}} a(x)\nabla(u-v) \cdot \nabla(u-v) dx,$$

which leads by Lemma 2.5 to

$$\lim_{s\to 0} I_s^2 = 0. \qquad (2.25)$$

Next, we claim that $I_s^1 = 0$. Indeed,

$$I_s^1 = \int_{\Omega_r} a(x)\nabla u \cdot \nabla(H_s(u-v)\zeta) dx - \int_{\Omega_r} a(x)\nabla v \cdot \nabla(H_s(u-v)\zeta) dx. \qquad (2.26)$$

Since $u \le v$ on $\Omega \cap \partial\Omega_r$, we have $H_s(u-v) = 0$ on $\Omega \cap \partial\Omega_r$. Moreover, $\zeta \in \mathcal{D}(\Omega \cap \{x_n < f_r^*(x')\})$ and $\Omega_r \cap \{v > 0\} = \Omega_r \cap \{x_n < f_r^*(x')\}$ imply that $H_s(u - v)\zeta \in H_0^1(\Omega_r \cap \{x_n < f_r^*(x')\})$. Therefore, we obtain from (2.2),

$$\begin{aligned}
\int_{\Omega_r} a(x)\nabla v \cdot \nabla(H_s(u-v)\zeta)dx &= \int_{\Omega_r \cap \{x_n < f_r^*(x')\}} a(x)\nabla v \cdot \nabla(H_s(u-v)\zeta)dx \\
&= -\int_{\Omega_r \cap \{x_n < f_r^*(x')\}} h(x)(H_s(u-v)\zeta)_{x_n}dx \\
&= -\int_{\Omega_r} h(x)(H_s(u-v)\zeta)_{x_n}dx. \qquad (2.27)
\end{aligned}$$

Now, since $\pm\chi(\Omega_r)H_s(u-v)\zeta$ are test functions for (P), $\chi = 1$ almost everywhere in $\Omega_r \cap \{u > 0\}$, and $H_s(u-v) = 0$ whenever $u = 0$, we have

$$\begin{aligned}
\int_{\Omega_r} a(x)\nabla u.\nabla(H_s(u-v)\zeta)dx &= -\int_{\Omega_r} \chi h(x)(H_s(u-v)\zeta)_{x_n}dx \\
&= -\int_{\Omega_r} h(x)(H_s(u-v)\zeta)_{x_n}dx. \qquad (2.28)
\end{aligned}$$

It follows from (2.26)–(2.28) that $I_s^1 = 0$. Hence, we get from (2.23)–(2.25) that

$$\int_{\Omega_r} a(x)\nabla(u-v)^+ \cdot \nabla\zeta dx = 0, \quad \forall\zeta \in \mathcal{D}(\Omega \cap \{x_n < f_r^*(x')\}).$$

Given that $(u-v)^+ = 0$ on $\partial\Omega_r$, the extension $w = \chi(\Omega_r)(u-v)^+$ by 0 of $(u-v)^+$ to $\Omega \setminus \Omega_r$ belongs to $H^1(\Omega)$, and we obtain

$$\int_{\Omega \cap \{x_n < f_r^*(x')\}} a(x)\nabla w.\nabla\zeta dx = 0, \quad \forall\zeta \in \mathcal{D}(\Omega \cap \{x_n < f_r^*(x')\}).$$

Taking into account (1.1)–(1.2) and the fact that $w \ge 0$ in Ω and $w = 0$ in $\Omega \setminus \Omega_r$, we get by the strong maximum principle that $w = 0$ in $\Omega \cap \{x_n < f_r^*(x')\}$, which leads to $(u - v)^+ = 0$ in $\Omega_r \cap \{x_n < f_r^*(x')\}$, or equivalently, $u \le v$ in $\Omega_r \cap \{x_n < f_r^*(x')\}$. We recall that $v = 0$ on $\Omega_r \cap \{x_n = f_r^*(x')\}$. Therefore, we get $u = 0$ on $\Omega_r \cap \{x_n = f_r^*(x')\}$, and consequently, we obtain by using Proposition 1.13 (ii), that $u = 0$ in $\Omega_r \cap \{x_n \ge f_r^*(x')\}$. Since $f_r^*(x') = l - r$ when $|x' - x_0'| < (1-\delta)r$, we conclude that $u = 0$ in the cylinder $B'_{(1-\delta)r}(x_0') \times (l - r, l)$.

Finally, by observing that δ can be chosen as small as we wish, we conclude that $u = 0$ within $B'_r(x_0') \times (l - r, l)$. ∎

Lemma 2.7. *Let $x_0 = (x_0', x_{n0}) \in \Omega$ and $r, l' > 0$ be such that $\Omega' = B'_r(x_0') \times (0, l') \subset \Omega$. If $u(x', l') = 0$ for all $x' \in B'_r(x_0')$, then the restriction of (u, χ) to Ω' is a solution to problem (P) in Ω', that is, we have*

$$\int_{\Omega'} \big(a(x)\nabla u + \chi h(x)e_n\big).\nabla\zeta dx \le 0,$$
$$\forall\zeta \in H^1(\Omega'), \qquad \zeta = 0 \quad \text{on } \partial\Omega' \setminus \{x_n = l'\}, \qquad \zeta \ge 0 \quad \text{on } \{x_n = l'\}.$$

Proof. Let ζ be as in the lemma and let $s > 0$ be small enough ($l - 2s > l'$ and $s < r$), and consider the two functions $\alpha_s(x) = H_s(r - |x' - x_0'|)$ and $\beta_s(x) = 1 - H_s(x_n - l + 2s)$, where H_s is as in the proof of Lemma 2.5. Note that without loss of generality, we can assume that ζ is continuous in $\bar{\Omega}'$ with compact support $K \subset\subset B'_{r-s}(x_0') \times (0, l']$ for s small enough. We observe that $\alpha_s = 0$ on $\partial B'_r(x_0')$, $\beta_s = 0$ in $\{x_n \geq l - s\}$, and $\zeta = 0$ on $\{x_n = 0\}$. Therefore, $\alpha_s\beta_s\zeta$ vanishes on $(\partial B'_r(x_0') \times [0, l]) \cup (\bar{B}'_r(x_0') \times \{x_n = 0\})$. Extending $\alpha_s\beta_s\zeta$ by zero outside $B'_r(x_0') \times (0, l)$, this function becomes a test function for problem (P) and we obtain

$$\int_\Omega \big(a(x)\nabla u + \chi h(x)e_n\big) \cdot \nabla(\alpha_s\beta_s\zeta)dx \leq 0,$$

which, since we have by Proposition 1.13 (ii) and Proposition 1.15, $u = \chi = 0$ in $B'_r(x_0') \times [l', l]$, and since $\beta_s = 1$ in $\{x_n \leq l - 2s\} \supset \{x_n < l'\} \supset \Omega'$, can be written as

$$\int_{\Omega'} \big(a(x)\nabla u + \chi h(x)e_n\big) \cdot \nabla(\alpha_s\zeta)dx \leq 0.$$

Since α_s does not depend on x_n, and since by Proposition 1.10 $|\nabla u| \leq C$ in K for some positive constant C, we obtain

$$\begin{aligned}\int_{\Omega'} \alpha_s\big(a(x)\nabla u + \chi h(x)e_n\big) \cdot \nabla\zeta dx \ &\leq -\int_{\Omega'} \zeta\big(a(x)\nabla u + \chi h(x)e_n\big) \cdot \nabla\alpha_s dx \\ = -\int_{\Omega'} \zeta a(x)\nabla u \cdot \nabla\alpha_s dx &= \frac{1}{s}\int_{(B'_r(x_0')\setminus B'_{r-s}(x_0'))\times(0,l')} \zeta a(x)\nabla_{x'} u \cdot \frac{x' - x_0'}{|x' - x_0'|}dx \\ \leq \frac{\Lambda C}{s}\int_{(B'_r(x_0')\setminus B'_{r-s}(x_0'))\times(0,l')} |\zeta|dx = 0,\end{aligned}$$

since $K \subset B'_{r-s}(x_0') \times (0, l']$. ■

3. Proof of the main results

Proof of Theorem 1.5. Let $x_0 = (x_0', x_{0n}) \in FB$, $r \in (0, r_1)$, and let $k = [\frac{l - x_{0n}}{r}]$ be the greatest integer less than or equal to $\frac{l - x_{0n}}{r}$. This is equivalent to

$$x_{0n} \leq l - kr < x_{0n} + r. \tag{3.1}$$

We distinguish two cases:

Case 1: There exists $j \in \{1, \ldots, k\}$ such that u is not identically zero within the cylinder $\Omega_r^j = B'_r(x_0') \times (l - jr, l)$. Let $j_0 = \min\{j \in \{1, \ldots, k\} : u$ is not identically zero in $\Omega_r^j\}$. Then, either $j_0 = 1$ or $j_0 \geq 2$.

(i) $j_0 = 1$:

In this case, u is not identically zero within the cylinder $\Omega_r^1 = B'_r(x_0') \times (l - r, l)$. Consequently, from Lemma 2.6, we infer that $\sup_{\bar{B}'_r(x_0')\times[l-r,l]} u > C_0 r$. Thus, there exists

$y_1 = (y'_1, y_{1n}) \in \overline{B}'_r(x'_0) \times [l - r, l]$ such that $u(y_1) > C_0 r$. Using (2.1), we further conclude that for each x in $B_{\delta_1 r}(y_1)$, where $\delta_1 = \min(1, \frac{C_0}{2C_1})$,

$$u(x) \geq u(y_1) - C_1|x - y_1| \geq C_0 r - C_1\delta_1 r \geq \Big(C_0 - C_1\frac{C_0}{2C_1}\Big)r = \frac{C_0}{2}r.$$

We deduce that $B_{\delta_1 r}(y_1)$ is contained within the set $\{u > 0\}$, which implies by Proposition 1.13 (i) that $u > 0$ below $B_{\delta_1 r}(y_1)$. Picking the point $x_1 = (y'_1, x_{0n})$, it follows that $B_{\delta_1 r}(x_1)$ also lies within $\{u > 0\}$. Furthermore, since $\delta_1 \leq 1$ and $x_{1n} = x_{0n}$, we have $|x_1 - x_0| = |x'_1 - x'_0| \leq r$, and it is easy to verify that $B_{\delta_1 r}(x_1) \subset B_{2r}(x_0)$. Thus, we have established that

$$B_{\delta_1 r}(x_1) \subset B_{2r}(x_0) \setminus FB, \quad \forall r \in (0, r_1). \tag{3.2}$$

<u>(ii) $j_0 \geq 2$:</u>

In this case, u is identically zero within the cylinder $\Omega_r^{j_0-1} = B'_r(x'_0) \times (l - (j_0 - 1)r, l)$. Using Lemma 2.7, we know that the restriction of (u, χ) to $U_r^{j_0} = B'_r(x'_0) \times (0, l - (j_0 - 1)r)$ is a solution to problem (P) in $U_r^{j_0}$.

Given that u is not identically zero within $\Omega_r^{j_0}$, it is not identically zero within $B'_r(x'_0) \times (l - j_0 r, l - (j_0 - 1)r)$. Therefore, we can argue as in (i) to show that (3.2) holds with the same number δ_1.

<u>Case 2:</u> For any $j \in \{1, \dots, k\}$, u is identically zero within Ω_r^j.

In particular, u is identically zero within $\Omega_r^k = B'_r(x'_0) \times (l - kr, l)$.

Let $x_1 = (x'_0, l - kr + r/2)$ and $\delta_2 = \frac{1}{2}$. For each x in $B_{\delta_2 r}(x_1)$, we have $B_{\delta_2 r}(x_1) \subset B_{2r}(x_0)$. Indeed, using (3.1),

$$\begin{aligned}|x - x_0| &\leq |x - x_1| + |x_1 - x_0| < \frac{r}{2} + \Big|l - kr + \frac{r}{2} - x_{0n}\Big| \\ &= \frac{r}{2} + l - kr + \frac{r}{2} - x_{0n} = l - kr + r - x_{0n} < r + r = 2r.\end{aligned}$$

Moreover, we claim that $B_{\delta_2 r}(x_1)$ is contained within the set $\{u = 0\}$. For that, it is enough to verify that $B_{\delta_2 r}(x_1) \subset \Omega_r^k = B'_r(x'_0) \times (l - kr, l)$.

It is obvious that $B_{\delta_2 r}(x_1) \subset B'_r(x'_0) \times (0, l)$ because $x'_1 = x'_0$ and $\delta_2 r < r$. Furthermore, let $x = (x', x_n) \in B_{\delta_2 r}(x_1)$. It is enough to verify that $x_n \geq l - kr$. Indeed, since $x_n - x_{1n} = x_n - (l - kr + r/2) > -\delta_2 r = -r/2$, we obtain

$$x_n > l - kr + \frac{r}{2} - \frac{r}{2} = l - kr.$$

Thus, we have established that

$$B_{\delta_2 r}(x_1) \subset B_{2r}(x_0) \setminus FB, \quad \forall r \in (0, r_1). \tag{3.3}$$

To conclude, let $\delta_0 = \frac{1}{2}\min(\delta_1, \delta_2)$, and use (3.2) and (3.3) for $r/2$ with $r \in (0, r_1)$. We get, since $r/2 \in (0, r_1)$,

If Case 1 holds, then: $\quad B_{\delta_0 r}(x_1) \subset B_{\delta_1 r/2}(x_1) \subset B_{2(r/2)}(x_0) \setminus FB = B_r(x_0) \setminus FB.$

If Case 2 holds, then: $\quad B_{\delta_0 r}(x_1) \subset B_{\delta_2 r/2}(x_1) \subset B_{2(r/2)}(x_0) \setminus FB = B_r(x_0) \setminus FB.$

This completes the proof, demonstrating that the free boundary is locally a porous set with porosity δ_0. ■

Proof of Corollary 1.6. First we have, by (P) (ii),

$$\chi = 1 \quad \text{a.e. in } \{u > 0\}. \tag{3.4}$$

Next, let x be a point in $\mathrm{Int}(\{u = 0\})$, the interior of the set $\{u = 0\}$. There exists a ball $B_r(x)$ of center x and radius r such that $u = 0$ in $B_r(x)$. By Proposition 1.15, we have $\chi = 0$ almost everywhere in $B_r(x)$. Therefore, we obtain

$$\chi = 0 \quad \text{a.e. in } \mathrm{Int}(\{u = 0\}). \tag{3.5}$$

Since $FB = \partial\{u > 0\} \cap \Omega$ is of Lebesgue measure zero due to its porosity, we conclude from (3.4)–(3.5) that $\chi = \chi_{\{u>0\}}$ almost everywhere in Ω. ■

Proof of Corollary 1.9. In [15], the author has established that the reservoirs-connected solution of the dam problem is unique provided that the set FB has Lebesgue measure zero for $n \geq 2$. Therefore, this uniqueness directly stems from Theorem 1.5 and Remark 1.4. ■

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