

Nonlocal degenerate parabolic-hyperbolic equations on bounded domains

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Abstract. We study well-posedness of degenerate mixed-type parabolic-hyperbolic equations $\partial_t u + \operatorname{div}(f(u)) = \mathcal{L}[b(u)]$ on bounded domains with general Dirichlet boundary/exterior conditions. The nonlocal diffusion operator \mathcal{L} can be any symmetric Lévy operator (e.g. fractional Laplacians) and b is nondecreasing and allowed to have degenerate regions ($b' = 0$). We propose an entropy solution formulation for the problem and show uniqueness of bounded entropy solutions under general assumptions. Existence of solutions is shown in a separate paper. The uniqueness proof is based on the Kružkov doubling of variables technique and incorporates several a priori results derived from our entropy formulation: an L^∞ -bound, an energy estimate, strong initial trace, weak boundary traces, and a *nonlocal* boundary condition. Our work can be seen as both extending nonlocal theories from the whole space to domains and local theories on domains to the nonlocal case. Unlike local theories, our formulation does not assume energy estimates. They are now a consequence of the formulation, but as opposed to previous nonlocal theories, they play an essential role in our arguments.

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1. Introduction

In this paper we study well-posedness on bounded domains $\Omega \subset \mathbb{R}^d$ for mixed-type hyperbolic-parabolic equations with nonlinear and nonlocal diffusion:

$$\begin{cases} \partial_t u + \operatorname{div}(f(u)) = \mathcal{L}[b(u)] & \text{in } \mathcal{Q} := (0, T) \times \Omega, \\ u = u^c & \text{in } \mathcal{Q}^c := (0, T) \times \Omega^c, \\ u(0, \cdot) = u_0 & \text{on } \Omega, \end{cases} \quad (1.1)$$

where $T > 0$, the initial/boundary (exterior) data u_0, u^c are bounded, the “fluxes” f, b are (at least) locally Lipschitz and b is nondecreasing (possibly degenerate),¹ “div” is the x -divergence, and the nonlocal diffusion operator \mathcal{L} is defined for $\phi \in C_c^\infty(\mathbb{R}^d)$ as a singular integral:

$$\begin{aligned} \mathcal{L}[\phi](x) &:= \text{P.V.} \int_{|z|>0} (\phi(x+z) - \phi(x)) \, d\mu(z) \\ &:= \lim_{\epsilon \rightarrow 0} \int_{|z|>\epsilon} (\phi(x+z) - \phi(x)) \, d\mu(z), \end{aligned} \quad (1.2)$$

where the (Lévy) measure $\mu \geq 0$ is symmetric and $\int_{|z|>0} |z|^2 \wedge 1 \, d\mu(z) < \infty$.² This class of anomalous diffusion operators coincides with the generators of the symmetric pure-jump Lévy processes [15, 20, 99, 100], including α -stable, tempered, relativistic, and compound Poisson processes. The corresponding generators include the fractional Laplacians [15, 99]

$$\mathcal{L} = -(-\Delta)^{\frac{\alpha}{2}}, \quad \alpha \in (0, 2), \quad \text{where } d\mu_\alpha(z) := C_{d,\alpha} |z|^{-(d+\alpha)} \, dz,$$

anisotropic operators $-\sum_{i=1}^N (-\partial_{x_i}^2)^{\frac{\alpha_i}{2}}$, $\alpha_i \in (0, 2)$, relativistic Schrödinger operators $m^\alpha I - (m^2 I - \Delta)^{\frac{\alpha}{2}}$, degenerate and 0-order operators, and numerical discretizations [50, 51] of these. Since the operators \mathcal{L} are naturally defined on the whole space, they require Dirichlet data on Ω^c for (1.1) to be well defined. In addition, inflow conditions are needed in hyperbolic regions. We will discuss this in more detail below.

Equation (1.1) is a possibly degenerate nonlinear convection-diffusion equation. Convection and diffusion phenomena are ubiquitous in sciences, economics, and engineering, their history dating back centuries. The study of nonlocal models is a strong trend across disciplines, motivated by applications [45, 86, 112], but also by deep connections between different fields and theoretical breakthroughs within mathematics [25, 34, 35, 81, 107]. Of the many applications of equations like (1.1), we mention reservoir simulation [58], sedimentation processes [33], and traffic flow [111] in the local case (e.g. $\mathcal{L} = \Delta$); detonation

¹Both f and b may be strongly degenerate in the sense that f' or b' is 0 on some interval.

²Hence $-\mathcal{L}$ is a nonnegative, self-adjoint (on L^2), pseudo-differential operator of order $\alpha \in [0, 2]$.

in gases [43], radiation hydrodynamics [93, 95], and semiconductor growth [112] in the nonlocal case; and porous media flow [48, 88, 106] and mathematical finance [45, 91] in both cases.

The solution structure of equation (1.1) is very rich. Special cases include scalar conservation laws ($b = 0$) [46, 63, 84], fractional conservation laws ($b(u) = u$) [1, 22, 54], nonlinear fractional diffusion equations ($f = 0$), degenerate porous-medium-type equations [47–49, 108], strongly degenerate Stefan-type problems [52], and problems of mixed hyperbolic-parabolic type [5, 41, 56]. Solutions may have fundamentally different behaviour in different regions: the problem is hyperbolic with possible discontinuous shock solutions where the diffusion degenerates ($b'(u) = 0$). In diffusive regions ($b'(u) > 0$), the problem is “parabolic” and the behaviour of solutions is determined by the order α of \mathcal{L} : smooth solutions in the subcritical/diffusion-dominated case $\alpha \in (1, 2)$ [54], possibly discontinuous solutions in the supercritical/convection-dominated case $\alpha \in (0, 1)$ [1, 2]. The critical case $\alpha = 1$ is delicate, but can sometimes give regular solutions [44, 73]. In subcritical diffusive regions, boundary conditions are expected to be satisfied pointwise, while in other regions, there may be sharp boundary layers and loss of boundary conditions (at “outflow”).

To accommodate nonsmooth solutions, a weak/distributional solutions concept is needed, and then solutions are nonunique unless additional (entropy) conditions are imposed [2, 46]. Kruřkov [77] developed an entropy solution theory and well-posedness for the Cauchy problem for scalar conservation laws (1.1) with $b = 0$. After a long time and attempts by different authors, Carrillo [36] came up with a highly nontrivial extension to local (degenerate) convection-diffusion equations (1.1) with $\mathcal{L} = \Delta$. Inspired by ideas from nonlocal viscosity solution theories (see e.g. [17, 67]), Alibaud [1] introduced an entropy solution theory and well-posedness for the Cauchy problem for (1.1) with linear diffusion $b(u) = u$ and $\mathcal{L} = -(-\Delta)^{\frac{\alpha}{2}}$. Cifani and Jakobsen [41] then extended these ideas to cover the Cauchy problem for (1.1) in full generality, including nonlinear possibly degenerate diffusions and arbitrary pure-jump Lévy diffusion operators. In nonlocal entropy solution theories, a key point is to split the nonlocal operator into a singular and a nonsingular part, where the singular part vanishes after taking limits, and the nonsingular part leads to beneficial cancellations in the Kruřkov doubling of variables argument. This idea has no local analogue, and the theory is quite different from local theories [36, 40, 70]. We refer to [3–5, 21, 42, 56, 65, 72] and references therein for more results on Cauchy problems and recent extensions.

Hyperbolic and mixed type boundary value problems. In hyperbolic problems, boundary conditions can be imposed only at inflow, where characteristics go into the domain. When $b = 0$ and (1.1) is a nonlinear scalar conservation law,

$$\begin{cases} \partial_t u + \operatorname{div}(f(u)) = 0 & \text{in } Q, \\ u = u^c & \text{on } \Gamma := (0, T) \times \partial\Omega, \\ u(0, \cdot) = u_0 & \text{on } \Omega, \end{cases} \quad (1.3)$$

characteristics depend on u itself and the correct way to impose boundary conditions only at inflow is

$$\operatorname{sgn}(u - u^c)(f(u) - f(k)) \cdot \hat{n} \geq 0 \quad \forall k : u \wedge u^c \leq k \leq u \vee u^c \quad \text{on } (0, T) \times \partial\Omega, \quad (1.4)$$

where \hat{n} is the outward unit normal. This strong formulation of Bardos, Leroux, and Nedelec [16] (the BLN condition) requires BV regularity and strong boundary traces for u . Otto [89] (see also [84]) introduced an L^∞ -theory using weak boundary traces and a weak entropy formulation on the boundary, and then this theory was simplified by the semi-Kruřkov boundary entropy formulation of Vovelle [109] – see also [59]. In [39] divergence-measure fields were analyzed to obtain weak traces and applied to (1.3). The divergence structure of the equation makes entropy solutions regular at the boundaries, and L^∞ -solutions of (1.3) are shown to have strong traces in [80, 90, 105]. The weak trace approach of Otto is in a sense bypassed, but the case of general fluxes $f = f(t, x, u)$ is not well understood. In the BV setting, an extensive overview of well-posedness for (1.3) is given in [96]; see also [46, 101]. Measure-valued solutions are considered in [102, 103], renormalized solutions in [9, 38, 92], kinetic solutions in [66], saturated solutions in [83], and Robin and Neumann conditions in [12].

When both convection and nonlinear local diffusion ($\mathcal{L} = \Delta$) are present in (1.1), the problem is of mixed type and will be hyperbolic in regions where the diffusion degenerates. A correct formulation taking into account inflow conditions is needed, and the first result was given in [36] for homogeneous Dirichlet conditions. The case of $u^c \neq 0$ is more delicate in the sense that the question of traces becomes more visible. Well-posedness is proved by Mascia, Porretta, and Terracina [85] using a careful parabolic extension of the Otto formulation. Michel and Vovelle [87] recast the problem as a single integral inequality using semi-Kruřkov entropies, and also proved existence and convergence of finite volume schemes. Further improvements can be found in [104]. Strong traces exist in some cases [61, 79], strong entropy solutions can be found in [11, 97], and kinetic solutions in [74, 75]. We also mention work on related boundary conditions [13, 32], on anisotropic diffusion [76, 82, 110], and on doubly and triply nonlinear degenerate parabolic equations [7, 8, 10, 14, 18, 19, 23, 24, 37, 53].

Nonlocal boundary value problems. Our nonlocal operators \mathcal{L} are naturally defined on the whole space. For problem (1.1) to be well defined, we need to restrict \mathcal{L} to the bounded domain Ω . Likely the most classical way of doing this is to impose Dirichlet data on the complement Ω^c as in (1.1). In probability theory this corresponds to stopping or killing the underlying (particle) process the first time it exits Ω [55]. This approach is by far the most popular one in the PDE setting, with a large literature including important contributions by Caffarelli and Silvestre – see e.g. [31, 34, 35, 94, 107]. As opposed to the local case, there are many other ways to restrict \mathcal{L} to a bounded domain and impose Dirichlet conditions. Equivalent definitions of nonlocal operators in \mathbb{R}^d [78] via PDEs, probability, calculus of variations, spectral theory etc., may differ on domains and lead to different interpretations of Dirichlet conditions. Prominent examples are approaches

based on censoring (regional fractional Laplacians) or spectral theory (spectral fractional Laplacians) [25, 62] where data is only required on the boundary $\partial\Omega$. But the resulting operators \mathcal{L} are no longer translation invariant, and their integral representations depend explicitly on Ω . See e.g. [62] for a comparison of the three approaches discussed above and [26, 27] for a discussion in a nonlinear PDE setting.

Problem (1.1) is a nonlocal problem of mixed type, and any formulation of it needs to take into account inflow conditions in hyperbolic regions. Up to now the literature on such problems is very limited. For purely parabolic equations ($f \equiv 0$), boundary value problems have been studied in e.g. [26–29]. Here there are no hyperbolic regions and inflow conditions are not needed. When the diffusion is linear and nondegenerate ($b = \text{Id}$), \mathcal{L} is the spectral fractional Laplacian, and $u^c \equiv 0$, there are results in [30, 68, 69]. Here the boundary condition is satisfied in a pointwise sense when $\alpha \in (1, 2)$. Recently, Huaroto and Neves [64] showed existence of L^∞ -entropy solutions of a problem similar to (1.1), but with local boundary conditions and $\mathcal{L} = -(-\Delta)_\Omega^{\frac{\alpha}{2}}$, the regional fractional Laplacian of order $\alpha \in (1, 2)$.³ This setting is well adapted to analysis in Sobolev spaces $H^s(\Omega)$ with strong boundary traces, and the proofs use the (local) vanishing viscosity method, a priori estimates on smooth solutions, and compactness arguments to pass to the limit. From a nice L^1 -contraction type of estimate, they also obtain uniqueness of the vanishing viscosity limits. However as explained in [64, page 3], there is currently no general uniqueness result for entropy solutions of this model.⁴

Main contributions of the paper.

- (1) An appropriate entropy solution formulation for problem (1.1)
- (2) A uniqueness result for entropy solutions of (1.1)

The most difficult part of well-posedness for (1.1) is uniqueness. Existence of entropy solutions of (1.1) holds under mild additional assumptions (see Theorem 2.8 below), but due to the length of the current paper and additional technicalities and difficulties, the existence theory is given in a separate paper [6].

Entropy solutions of (1.1) are defined to be L^∞ -functions satisfying pointwise exterior conditions and a family of weak semi-Kruřkov entropy inequalities incorporating both the equation in the interior and inflow conditions in hyperbolic parts of the boundary. The definition includes splitting the nonlocal operator into singular and nonsingular parts as in [1, 41] and a version of the entropy formulation on the boundary given in [85, 87].

³For functions $\phi \in C^2(\mathbb{R}^d)$ supported in Ω and $x \in \Omega$, from the definitions we immediately have that

$$-(-\Delta)_\Omega^{\frac{\alpha}{2}}\phi(x) = -(-\Delta)^{\frac{\alpha}{2}}\phi(x) + \kappa(x)\phi(x) \quad \text{where } \kappa(x) = C_{d,\alpha} \int_{\Omega^c} |z|^{-(d+\alpha)} dz \sim d_{\partial\Omega}(x)^{-\alpha}$$

is singular on the boundary. This operator does not belong to the class of operators considered in this paper.

⁴In no nontrivial case does the theory of [64] coincide with ours, even for existence. The proofs are also quite different from ours, e.g. we do not use smooth approximations, but rather the entropy formulation directly.

See Definition 2.2 for the precise statement. The entropy inequalities include initial and boundary terms that impose strong initial traces (Lemma 2.5), hyperbolic weak traces on the boundary (Proposition 4.8), and a boundary condition (Proposition 4.9) that coincides with the BLN condition (1.4) whenever strong traces exist [109].

The most important contribution of the paper is the uniqueness result for our class of entropy solutions (Theorem 5.1). As in the local case,⁵ it takes the form of a partial L^1 -contraction estimate, $\|u(t, \cdot) - v(t, \cdot)\|_{L^1} \leq \|u_0 - v_0\|_{L^1}$, for solutions u and v coinciding on Ω^c . This result seems to be the first uniqueness result for degenerate parabolic-hyperbolic equations that are nonlocal. The proof is based on a highly non-trivial adaptation of the Kružkov doubling of variables technique and combines nonlocal elements from [1,41] with arguments to handle boundary conditions from [87]. Working in a domain requires a lot of extra ingredients and is technically more involved than working on the whole space. To handle nonlocal problems on domains, we develop new and useful identities and bounds for Lévy operators in Section 3, we prove nonlocal versions of weak trace, boundary conditions, energy estimates, a purely new result on boundary integrability in Section 4, and finally, we change the procedure of the uniqueness argument itself in Section 5. In the local case [87], the doubling of variables arguments roughly speaking use carefully chosen boundary layer sequences and passage to the limit (i) to undo the doubling and then (ii) to see the boundary. To construct the inner boundary layer sequences, rather nice and explicit solutions of simple Poisson problems are then used. Such arguments are difficult to generalize to our family of nonlocal operators which may not be uniformly elliptic and have integral representations that are more difficult to handle. We overcome this problem by taking cruder “outer” boundary layer sequences and reverse the order of the limits (i) and (ii) mentioned above. The price for this approach is more regularity requirements on b and \bar{u}^c in problem (1.1); see Remark 4.2 for more details.

One key part of uniqueness proofs on domains is making sense of and controlling fluxes at the boundary. To give some insight into our new nonlocal formulation, consider e.g. the in-flux of $|u - \bar{u}|$ at the boundary. In the local case ($\mathcal{L} = \Delta$) it is formally given as the measure

$$(\operatorname{sgn}(u - \bar{u}^c)(f(u) - f(\bar{u}^c)) - \nabla|b(u) - b(\bar{u}^c)|) \cdot \nabla \chi_\Omega \quad \text{on } \Gamma = (0, T) \times \partial\Omega,$$

where χ_Ω is the characteristic function on Ω . The existence and control of such a (weak) trace is part of the solution concept in [85], but not in [87] where it instead follows from the entropy inequalities and the assumption of finite energy. Here we work with a nonlocal weak trace given by

$$\operatorname{sgn}(u - \bar{u}^c)(f(u) - f(\bar{u}^c)) \cdot \nabla \chi_\Omega - \mathcal{B}(|b(u) - b(\bar{u}^c)|, \chi_\Omega) \quad \text{in } M := (0, T) \times \mathbb{R}^d,$$

where \mathcal{B} is the bilinear operator whose integral is the bilinear form associated to \mathcal{L} . The latter measure has a singular part on Γ and an absolutely continuous part in $M \setminus \Gamma$ which

⁵Problem (1.1) with $\mathcal{L} = \Delta$; see e.g. [36, 85, 87] for uniqueness results.

is due to the nonlocal \mathcal{B} -term. Like in [87], we also prove the necessary properties of this trace (Proposition 4.8) from the entropy formulation using a (nonlocal) energy estimate. Contrary to the local case, this energy estimate is not a part of the formulation, but follows as a consequence of it (see Proposition 4.1).⁶

Outline of paper. Section 2 is reserved for assumptions, concept of solution, and main results, and in Section 3 we provide preliminary results on nonlocal operators. We collect important properties of entropy solutions, like energy estimates, boundary conditions, and weak traces, in Section 4. These properties are then used to prove uniqueness in Section 5. The [appendices](#) contain auxiliary results and calculations.

2. Assumptions, concept of solution, and main results

In this section we state the notation and assumptions, present the (entropy) solution concept we will use, and give the main results – including uniqueness, a priori estimates, and boundary trace/condition results. We also state the existence result from [6].

Notation. Let $\phi \vee^+ \psi := \max\{\phi, \psi\} = \phi \vee \psi$, $\phi \vee^- \psi := \min\{\phi, \psi\} = \phi \wedge \psi$, and $\text{sgn}^\pm(a) = \pm 1$ if $\pm a > 0$ and zero otherwise. In addition to the open and bounded set $\Omega \subset \mathbb{R}^d$, we will be working with the sets

$$\begin{aligned} Q &= (0, T) \times \Omega, & \Gamma &= (0, T) \times \partial\Omega, & \bar{\Omega} &= \Omega \cup \partial\Omega, \\ Q^c &= (0, T) \times \Omega^c, & \bar{\Gamma} &= [0, T] \times \partial\Omega, & M &= (0, T) \times \mathbb{R}^d. \end{aligned}$$

We will write $|\Omega|$ to mean the d -dimensional Lebesgue measure of Ω , and $|Q| := T|\Omega|$. The $(d - 1)$ -dimensional Hausdorff measure on $\partial\Omega$ is denoted by $d\sigma(x)$. In the double variable scenario $(t, x), (s, y) \in \mathbb{R} \times \mathbb{R}^d$, we will for brevity often omit the differentials $dt \, dx \, ds \, dy$ from integrals featuring all four variables. Instead, we specify the relevant variables for each domain by writing

$$\Omega_x, \quad \Omega_y, \quad Q_x, \quad Q_y, \quad \Gamma_x, \quad \Gamma_y, \quad M_x, \quad M_y,$$

where x indicates that the domain of integration is over (t, x) , while y indicates integration over (s, y) .

The Lévy operator \mathcal{L} is as defined in (1.2), and we define its corresponding bilinear operator by

$$\mathcal{B}[\phi, \psi](x) := \lim_{\epsilon \rightarrow 0} \frac{1}{2} \int_{|z| > \epsilon} (\phi(x+z) - \phi(x))(\psi(x+z) - \psi(x)) \, d\mu(z), \quad (2.1)$$

⁶Energy estimates play no role in nonlocal problems on the whole space [41], but are still necessary for local ones [71].

where the limit will typically be taken in $L^1(\mathbb{R}^d)$ or $L^1_{\text{loc}}(\mathbb{R}^d)$; Proposition 3.8 lists conditions on ϕ, ψ ensuring that this limit exists. We denote truncated operators by

$$\mathcal{L}^{\geq r}, \mathcal{L}^{< r}, \mathcal{L}^{r' > \dots \geq r}, \mathcal{B}^{\geq r}, \mathcal{B}^{< r}, \mathcal{B}^{r' > \dots \geq r}, \quad (2.2)$$

where the domains of integration are restricted to $\{|z| \geq r\}$, $\{|z| < r\}$, and $\{r' > |z| \geq r\}$; observe that $\mathcal{L}^{\geq r}, \mathcal{L}^{< r}, \mathcal{L}^{r' > \dots \geq r}$ are themselves Lévy operators. When the functions ϕ, ψ depend on two space variables $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$, we will write $\mathcal{L}_x, \mathcal{L}_y, \mathcal{B}_x, \mathcal{B}_y$ to specify in which of the variables the operators act. And acting in both variables is the frequently occurring Lévy operator

$$\begin{aligned} \mathcal{L}_{x+y}[\phi](x, y) &:= \lim_{\epsilon \rightarrow 0} \int_{|z| \geq \epsilon} (\phi(x+z, y+z) - \phi(x, y)) \, d\mu(z), \\ \mathcal{B}_{x+y}[\phi, \psi](x, y) &:= \frac{1}{2} \lim_{\epsilon \rightarrow 0} \int_{|z| \geq \epsilon} (\phi(x+z, y+z) - \phi(x, y)) \\ &\quad \times (\psi(x+z, y+z) - \psi(x, y)) \, d\mu(z), \end{aligned} \quad (2.3)$$

and the (less frequent) mixed bilinear operators

$$\begin{aligned} \mathcal{B}_{x,y}[\phi, \psi](x, y) &:= \frac{1}{2} \lim_{\epsilon \rightarrow 0} \int_{|z| \geq \epsilon} (\phi(x+z, y) - \phi(x, y)) \\ &\quad \times (\psi(x, y+z) - \psi(x, y)) \, d\mu(z), \\ \mathcal{B}_{x,x+y}[\phi, \psi](x, y) &:= \frac{1}{2} \lim_{\epsilon \rightarrow 0} \int_{|z| \geq \epsilon} (\phi(x+z, y) - \phi(x, y)) \\ &\quad \times (\psi(x+z, y+z) - \psi(x, y)) \, d\mu(z), \end{aligned} \quad (2.4)$$

with $\mathcal{B}_{y,x}$ and $\mathcal{B}_{y,x+y}$ defined similarly. For all these operators, we denote their truncated versions by notation analogous to that in (2.2).

We identify $L^2(\Omega)$ as a subspace of $L^2(\mathbb{R}^d)$ through zero extensions and define $H_0^{\mathcal{L}}(\Omega) \subset L^2(\Omega)$ to be the Hilbert space with norm

$$\|\phi\|_{H_0^{\mathcal{L}}(\Omega)}^2 := \|\phi\|_{L^2(\Omega)}^2 + \int_{\mathbb{R}^d} \mathcal{B}[\phi, \phi] \, dx.$$

The Lipschitz constant of a function g on a set K is denoted $L_{g,K}$ or simply L_g if the set is clear from the context.

Assumptions. In this paper we will use the following assumptions:

- (A $_{\Omega}$) $\Omega \subset \mathbb{R}^d$ is open, bounded with C^2 -boundary $\partial\Omega$, and outward-pointing normal \hat{n}
- (A $_f$) $f = (f_1, f_2, \dots, f_d) \in W_{\text{loc}}^{1,\infty}(\mathbb{R}; \mathbb{R}^d)$,
- (A $_b$) $b \in W_{\text{loc}}^{1,\infty}(\mathbb{R}; \mathbb{R})$ is nondecreasing, and the weak derivative $b' \in \text{TV}_{\text{loc}}(\mathbb{R})$,
- (A $_{u^c}$) $u^c \in (C^2 \cap L^\infty)(Q^c)$ and has an extension $\bar{u}^c \in (C^2 \cap L^\infty)([0, T] \times \mathbb{R}^d)$,

(A $_{u_0}$) $u_0 \in L^\infty(\Omega)$,

(A $_\mu$) μ is a symmetric Radon measure on $\mathbb{R}^d \setminus \{0\}$, $\mu \geq 0$, and $\int_{\mathbb{R}^d \setminus \{0\}} (|z|^2 \wedge 1) d\mu(z) < \infty$.

We extend μ to \mathbb{R}^d by setting $\mu(\{0\}) = 0$: this is for notational brevity. On \mathbb{R}^d , μ is a locally unbounded Borel measure, but still σ -finite, and so the coming product measures will be unambiguous.

Remark 2.1. (a) In (A $_f$) and (A $_b$), we can assume without loss of generality that $f(0) = 0$ and $b(0) = 0$ (add constants to f and b), and that f and b are globally Lipschitz and b' has bounded total variation (solutions are uniformly bounded by Lemma 2.4).

(b) In (A $_b$) the condition $b' \in \text{TV}_{\text{loc}}(\mathbb{R})$ implies $b \in W_{\text{loc}}^{1,\infty}(\mathbb{R})$, the standard assumption for the Cauchy problem. Our stronger assumption still allows for classical power type and strongly degenerate nonlinearities $b(r) = r^m$ for $m > 1$ (porous medium) and $b(r) = \max\{r - L, 0\}$ (one-phase Stefan).

(c) Under our assumptions, we have $\mathcal{L}[b(\bar{u}^c)] \in L^1(Q)$ (Corollary 3.7) and $\partial_t \bar{u}^c + \text{div}_x(f(\bar{u}^c)) - \mathcal{L}[b(\bar{u}^c)] \in L^1(Q)$, which will be important in many calculations and results concerning $u - \bar{u}^c$.

Entropy solutions. For the definition of entropy solutions we introduce the semi-Kruřkov entropy–entropy flux pairs

$$(u - k)^\pm, \quad F^\pm(u, k) := \text{sgn}^\pm(u - k)(f(u) - f(k)) \quad \text{for } u, k \in \mathbb{R},$$

where $(\cdot)^+ := \max\{\cdot, 0\}$ and $(\cdot)^- := (-\cdot)^+$, and the usual splitting of the nonlocal operator,

$$\mathcal{L}[\phi](x) = \mathcal{L}^{<r}[\phi](x) + \mathcal{L}^{\geq r}[\phi](x) \quad \text{for } \phi \in C_c^\infty(\mathbb{R}^d), r > 0, x \in \mathbb{R}^d,$$

where $\mathcal{L}^{<r}$ and $\mathcal{L}^{\geq r}$ are defined in (2.2). To motivate the definition (see Appendix A for a rigorous computation), assume u is a smooth solution of (1.1), $0 \leq \varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$, and $k \in \mathbb{R}$. Replace $(u, f(u), b(u))$ by $(u - k, f(u) - f(k), b(u) - b(k))$ in (1.1), multiply the resulting PDE by $\text{sgn}^\pm(u - k)\varphi$, and integrate over Q . After several integrations by parts and chain rules, we get the equality

$$\begin{aligned} & - \int_Q ((u - k)^\pm \partial_t \varphi + F^\pm(u, k) \cdot \nabla \varphi) dx dt \\ & \quad - \int_M \mathcal{L}^{\geq r}[b(u) - b(k)] \text{sgn}^\pm(u - k) \varphi \chi_\Omega dx dt \\ & \quad - \int_M (b(u) - b(k))^\pm \mathcal{L}^{<r}[\varphi] dx dt \\ & = \int_\Omega (u_0 - k)^\pm \varphi(0, \cdot) dx + L_f \int_\Gamma (\bar{u}^c - k)^\pm \varphi d\sigma(x) dt \\ & \quad - \int_\Gamma \varphi d\mu_c^\pm - \int_M \varphi d\mu_d^{\pm, r}, \end{aligned}$$

where the μ -measures are absolutely continuous and given by

$$d\mu_c^\pm := (F^\pm(u, k) \cdot \hat{n} + L_f(u - k)^\pm) d\sigma(x) dt$$

and

$$d\mu_d^{\pm, r} := (\mathcal{L}^{<r}[b(u) - b(k)] \operatorname{sgn}^\pm(u - k) \chi_\Omega - \mathcal{L}^{<r}[(b(u) - b(k))^\pm]) d\sigma(x) dt.$$

Had both measures been nonnegative, we could have omitted their contributions in the above equality; this would have yielded the entropy inequality from the definition below. While the “classical” measure μ_c^\pm is indeed nonnegative, this is not necessarily the case for the new measure $\mu_d^{\pm, r}$. However, the negative part of $\mu_d^{\pm, r}$ is supported in $E := \operatorname{supp}(b(u^c) - b(k))^\pm \setminus Q$, and so $\mu_d^{\pm, r}$ will at least be nonnegative for test functions $0 \leq \varphi \in C_c^\infty(M \setminus E)$; conveniently, such test functions suffice for our well-posedness argument.

Our definition is then the following one.

Definition 2.2 (Entropy solution). A function $u \in L^\infty(M)$ is an *entropy solution* of (1.1) if

- (a) (Entropy inequalities in \bar{Q}) For all $r > 0$, and all $k \in \mathbb{R}$ and $0 \leq \varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$ satisfying

$$(b(u^c) - b(k))^\pm \varphi = 0 \quad \text{in } Q^c, \tag{2.5}$$

the following inequality holds:

$$\begin{aligned} & - \int_Q ((u - k)^\pm \partial_t \varphi + F^\pm(u, k) \cdot \nabla \varphi) dx dt \\ & \quad - \int_Q \mathcal{L}^{\geq r}[b(u)] \operatorname{sgn}^\pm(u - k) \varphi dx dt - \int_M (b(u) - b(k))^\pm \mathcal{L}^{<r}[\varphi] dx dt \\ & \leq \int_\Omega (u_0 - k)^\pm \varphi(0, \cdot) dx + L_f \int_\Gamma (\bar{u}^c - k)^\pm \varphi d\sigma(x) dt. \end{aligned} \tag{2.6}$$

- (b) (Data in Q^c) $u = u^c$ a.e. in Q^c .

This definition is an extension of both [41, Definition 2.1] (see also [1]) for nonlocal problems in the whole space and [87, Definition 2.1] (see also [85]) for local problems on bounded domains. In the hyperbolic case ($b' \equiv 0$), it is equivalent [109] to the original definition of Otto [89]. By (A_Ω) – (A_μ) , all the integrals in (2.6) are well defined.

Remark 2.3. (a) The condition $(b(u^c) - b(k))^+ \varphi = 0$ is weaker than $|b(u^c) - b(k)|\varphi = 0$. When $b(u^c) < b(k)$, the first condition holds for any φ while the second then implies that $\varphi = 0$. This gives a hint to why standard Kružkov entropy–entropy flux pairs are too restrictive in this setting; see [109] for counter-examples to uniqueness in the hyperbolic case when $b = 0$.

- (b) The term $\mathcal{L}^{\geq r}[b(u)]$ is a convolution of a bounded Lebesgue-measurable function $b(u)$ and a finite Borel measure μ . It is well defined in the sense that $\mathcal{L}^{\geq r}$ maps

all Borel representatives of $b(u)$ to the same element in $L^\infty(M)$; see [60, Proposition 8.49] for details.

A priori results and uniqueness. A first consequence of our definition is the L^∞ a priori estimate for solutions.

Lemma 2.4 (L^∞ -bound). *Assume (A_Ω) – (A_μ) and u is an entropy solution of (1.1). Then*

$$\min \left\{ \operatorname{ess\,inf}_\Omega u_0, \operatorname{ess\,inf}_{Q^c} u^c \right\} \leq u(t, x) \leq \max \left\{ \operatorname{ess\,sup}_\Omega u_0, \operatorname{ess\,sup}_{Q^c} u^c \right\} \quad \text{for a.e. } (t, x) \in Q.$$

The weakly posed initial conditions u_0 are in fact assumed in the strong L^1 -sense.

Lemma 2.5 (Time continuity at $t = 0$). *Assume (A_Ω) – (A_μ) and u is an entropy solution of (1.1). Then*

$$\operatorname{ess\,lim}_{t \rightarrow 0^+} \|u(t, \cdot) - u_0\|_{L^1(\Omega)} = 0.$$

The rather standard proofs of these two results can be found in Appendix B. We continue by listing additional a priori estimates that are needed for the uniqueness proof. The proofs are given in Section 4. Recall that \mathcal{B} is the bilinear operator associated with \mathcal{L} defined in (2.1), and define

$$F(u, k) := (F^+ + F^-)(u, k) = \operatorname{sgn}(u - k)(f(u) - f(k)).$$

Proposition 4.1 (Energy estimate). *Assume (A_Ω) – (A_μ) and u is an entropy solution of (1.1). Then*

$$\begin{aligned} & \int_M \mathcal{B}[b(u) - b(\bar{u}^c), b(u) - b(\bar{u}^c)] \, dx \, dt \\ & \leq \int_\Omega H(u_0, \bar{u}^c(0)) \, dx - \int_Q [(u - \bar{u}^c) \bar{u}_t^c + F(u, \bar{u}^c) \cdot \nabla \bar{u}^c] b'(\bar{u}^c) \, dx \, dt \\ & \quad + \int_Q \mathcal{L}[b(\bar{u}^c)](b(u) - b(\bar{u}^c)) \, dx \, dt, \end{aligned} \tag{4.1}$$

where $H(u, k) := \int_k^u (b(\xi) - b(k)) \, d\xi \geq 0$. Moreover, the right-hand side of (4.1) is finite.

Remark 2.6. This global energy estimate for $b(u) - b(\bar{u}^c)$ implies a local energy estimate for $b(u)$: by the inequality $x^2 \leq 2(x - y)^2 + 2y^2$ we have

$$\begin{aligned} \int_M \mathcal{B}[b(u), b(u)] \varphi \, dx \, dt & \leq 2 \int_M \mathcal{B}[b(u) - b(\bar{u}^c), b(u) - b(\bar{u}^c)] \varphi \, dx \, dt \\ & \quad + 2 \int_M \mathcal{B}[b(\bar{u}^c), b(\bar{u}^c)] \varphi \, dx \, dt \end{aligned}$$

for any $0 \leq \varphi \in C_c^\infty(M)$, where $\mathcal{B}[b(\bar{u}^c), b(\bar{u}^c)]$ is locally bounded since $b(\bar{u}^c)$ is bounded and locally Lipschitz. If $\int_M \mathcal{B}[b(\bar{u}^c), b(\bar{u}^c)] \, dx \, dt < \infty$, we can send $\varphi \rightarrow 1$ to get a global energy estimate for $b(u)$.

To write down our results regarding boundary integrability, weak trace, and boundary condition, we need different sequences of test functions to approximate the (characteristic function of the) domain. We say $(\zeta_\delta)_{\delta>0} \subset C_c^\infty(\mathbb{R}^d)$ is a *boundary layer sequence* if, for some Borel function $\zeta: \partial\Omega \rightarrow [0, 1]$,

$$0 \leq \zeta_\delta(x) \leq 1, \quad x \in \mathbb{R}^d, \quad \forall \delta > 0, \quad \lim_{\delta \rightarrow 0} \zeta_\delta(x) = \begin{cases} 1, & x \in \Omega, \\ \zeta(x), & x \in \partial\Omega, \\ 0, & x \in \mathbb{R}^d \setminus \Omega. \end{cases} \quad (2.7)$$

Boundary layer sequences $(\bar{\zeta}_\delta)_{\delta>0}$ and $(\zeta_\delta)_{\delta>0}$ are called *outer* and *inner boundary layer sequences* if

$$\bar{\zeta}_\delta = 1 \quad \text{in } \bar{\Omega} \quad \text{and} \quad \lim_{\delta \rightarrow 0} \bar{\zeta}_\delta(x) = \chi_{\bar{\Omega}}(x), \quad x \in \mathbb{R}^d, \quad (2.8)$$

$$\text{supp } \zeta_\delta \subset \Omega \quad \text{and} \quad \lim_{\delta \rightarrow 0} \zeta_\delta(x) = \chi_\Omega(x), \quad x \in \mathbb{R}^d. \quad (2.9)$$

Typically, we will require these sequences to satisfy $\limsup_{\delta \rightarrow 0} \|\nabla \zeta_\delta\|_{L^1(\mathbb{R}^d)} < \infty$. All boundary layer sequences satisfy the weak limit $-\nabla \zeta_\delta \rightarrow \hat{n} \delta_{\partial\Omega}$: for any $\psi \in C^1(\mathbb{R}^d)$,

$$\lim_{\delta \rightarrow 0} \int_{\mathbb{R}^d} \psi(-\nabla \zeta_\delta) \, dx = \lim_{\delta \rightarrow 0} \int_{\mathbb{R}^d} \text{div}(\psi) \zeta_\delta \, dx = \int_{\Omega} \text{div}(\psi) \, dx = \int_{\partial\Omega} \psi \cdot \hat{n} \, d\sigma(x).$$

We note that outer and inner boundary layer sequences are easily constructed by setting

$$\bar{\zeta}_\delta := \rho_\delta * \chi_{\Omega_\delta}, \quad \zeta_\delta := \rho_\delta * \chi_{\Omega_{-\delta}},$$

where $\rho_\delta(x) = \frac{1}{\delta^d} \rho(\frac{x}{\delta})$ is a standard mollifier with support in $\{|x| < \delta\}$, $\Omega_\delta = \{x: \text{dist}(x, \Omega) < \delta\}$, and $\Omega_{-\delta} = \{x: \text{dist}(x, \Omega^c) > \delta\}$. By (A_Ω) , $\partial\Omega$ and $\partial\Omega_{\pm\delta}$ are C^2 with finite Hausdorff \mathcal{H}^{d-1} -measure when δ is small. Moreover,

$$\lim_{\delta \rightarrow 0} \|\nabla \zeta_\delta\|_{L^1(\mathbb{R}^d)} = \lim_{\delta \rightarrow 0} \|\nabla \bar{\zeta}_\delta\|_{L^1(\mathbb{R}^d)} = |\partial\Omega|_{\mathcal{H}^{d-1}} < \infty.$$

The next result restricts how much $b(u)$ can differ from $b(\bar{u}^c)$ close to the boundary Γ . It is a purely nonlocal result and prevents blow-up of certain integrals.

Proposition 4.6 (Boundary integrability). *Assume (A_Ω) – (A_μ) and u is an entropy solution of (1.1). Then*

$$\int_0^T \int_{\substack{x \in \Omega \\ x+z \in \Omega^c}} |b(u(t, x)) - b(\bar{u}^c(t, x))| \, d\mu(z) \, dx \, dt \leq C, \quad (4.23)$$

where $C = C(\Omega, T, f, b, \mu, \bar{u}^c, u_0)$ is finite and given by (4.11).

The next two results have local analogues and are essential for the uniqueness proof.

Proposition 4.8 (Weak trace). *Assume (A_Ω) – (A_μ) and u is an entropy solution of (1.1). Let ν be the finite Borel measure from (4.22). Then, for any boundary layer sequence $(\zeta_\delta)_{\delta>0}$, $r > 0$, and $\varphi \in C_b^\infty([0, T] \times \mathbb{R}^d)$, we have*

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \left(\int_Q F(u, \bar{u}^c) \cdot (\nabla \zeta_\delta) \varphi \, dx \, dt - \int_M \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \zeta_\delta] \varphi \, dx \, dt \right) \\ & = \int_{\bar{\Gamma}} (1 - \zeta) \varphi \, d\nu - \int_M \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \chi_\Omega] \varphi \, dx \, dt, \end{aligned} \quad (4.26)$$

where $\bar{\Gamma} = [0, T] \times \partial\Omega$, $F(u, \bar{u}^c) = \text{sgn}(u - \bar{u}^c)(f(u) - f(\bar{u}^c))$, and ζ is the limit of ζ_δ on $\partial\Omega$. The right-hand side of (4.26) is nonpositive, and the expression inside the lim on the left-hand side is bounded by $\tilde{C} \|\varphi\|_{C^1(M)}$ for a $\tilde{C} > 0$ independent of δ , φ .

Remark 2.7. The $\mathcal{B}^{<r}$ term on the right-hand side is finite by boundary integrability Proposition 4.6.

The $\mathcal{B}^{<r}$ terms are new and represent the contribution from the nonlocal diffusion, while the hyperbolic part (the F and ν terms) coincides with the weak trace of Otto's⁷ for scalar conservation laws [84, 109]. The “flux density” $F(u, \bar{u}^c) \cdot \nabla \chi_\Omega - \mathcal{B}[|b(u) - b(\bar{u}^c)|, \chi_\Omega]$ measures the inflow of $|u - \bar{u}^c|$ into Q . The convective part is a singular measure concentrated on Γ , while the nonlocal diffusive part is absolutely continuous and supported on M . The term $-\mathcal{B}[|b(u) - b(\bar{u}^c)|, \chi_\Omega](t, x)$ (see (2.1)) represents the inflow density from $(t, x) \in Q^c$ into Q or minus the outflow density from $(t, x) \in Q$ to Q^c .

The weak trace becomes important in the uniqueness proof during the Kruřkov doubling of variables technique, where we want to replace “difficult” terms by “easy” terms using the boundary condition below. Define the Kruřkov boundary entropy–entropy flux pairs as

$$\begin{aligned} \mathcal{F} &= \mathcal{F}(u, \bar{u}^c, k) := F(u, \bar{u}^c) + F(u, k) - F(\bar{u}^c, k), \\ \Sigma &= \Sigma(u, \bar{u}^c, k) := |b(u) - b(\bar{u}^c)| + |b(u) - b(k)| - |b(\bar{u}^c) - b(k)|. \end{aligned}$$

Proposition 4.9 (Boundary condition). *Assume (A_Ω) – (A_μ) and u is an entropy solution of (1.1). Then, for any inner boundary layer sequence $(\zeta_\delta)_{\delta>0}$, $k \in \mathbb{R}$, $r > 0$, and $0 \leq \varphi \in C_b^\infty([0, T] \times \mathbb{R}^d)$,*

$$\lim_{\delta \rightarrow 0} \left(\int_Q (\mathcal{F} \cdot \nabla \zeta_\delta) \varphi \, dx \, dt - \int_M \mathcal{B}^{<r} [\Sigma, \zeta_\delta] \varphi \, dx \, dt \right) \leq 0. \quad (4.33)$$

Moreover, the expression inside the lim is bounded by $\tilde{C} \|\varphi\|_{C^1(M)}$ for a $\tilde{C} > 0$ independent of δ , k , φ .

⁷It is an equivalent restatement in terms of semi-Kruřkov entropy–entropy fluxes due to Vovelle [109] of Otto's result (see [84, Definition 7.2 and Lemma 7.34]).

At this point we could formulate an equivalent definition of entropy solutions in the spirit of Mascia, Porretta, and Terracina [85, Definition 1.2], by combining Lemma 2.5, Propositions 4.6, 4.8, 4.9, and the classical Kružkov entropy inequalities in Q . However, the way that Definition 2.2 is formulated now makes it easier to use in the existence proof where we take limits of approximate solutions (see [6]).

We now state the uniqueness result, the main contribution of this paper.

Theorem 5.1 (Uniqueness). *Assume (A_Ω) – (A_μ) . Let u, v be entropy solutions of (1.1) in the sense of Definition 2.2 with initial data u_0, v_0 and exterior data $u^c = v^c$. Then, for a.e. $t \in (0, T)$, we have*

$$\int_{\Omega} |u(t, x) - v(t, x)| \, dx \leq \int_{\Omega} |u_0(x) - v_0(x)| \, dx.$$

In particular, if $u_0 = v_0$ also, then $u = v$ a.e. in $(0, T) \times \mathbb{R}^d$.

Finally, we state the existence result from [6]. Here we need slightly stronger assumptions on \mathcal{L}/μ :

(A'_μ) μ is such that the symbol $m(\xi) := \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot z)) \, d\mu(z) \rightarrow \infty$ as $|\xi| \rightarrow \infty$. The symbol m is the multiplier of $-\mathcal{L}$, and the condition is sufficient (and necessary) for certain strong L^2 -compactness to follow from \mathcal{L} -energy estimates. This condition requires a nonfinite μ and is implied if the absolutely continuous (with respect to the Lebesgue measure) part of μ is nonfinite; see [6] for details. In particular, it is satisfied for fractional Laplacians where the symbol is $m_\alpha(\xi) = |\xi|^\alpha$.

Theorem 2.8 (Existence). *Assume (A_Ω) – (A_μ) and either (A'_μ) or $\mu(\mathbb{R}^d) < \infty$. Then there exists an entropy solution u of (1.1).*

The proof in [6] uses a series of approximations of the nonlocal operators, entropy process solutions, nonlinear weak- \star compactness techniques, and strong L^2 -compactness inherited from the energy estimate and novel arguments to control time translations. We also show that (A'_μ) is sufficient and necessary for \mathcal{L} -energy estimates to induce strong spatial compactness in L^2 .

Remark 2.9. (a) We extend the uniqueness result Theorem 5.1 to entropy-process solutions in [6].

(b) We have to work in L^∞ , and not in $C([0, T]; L^1) \cap L^\infty$, since the L^1 -contraction in Theorem 5.1 is not strong enough to control space translations of solutions in terms of space translations of the data. Our existence proof is therefore based on weak compactness arguments that do not preserve the $C([0, T]; L^1(\Omega))$ -regularity of approximate solutions.

3. Preliminaries on the Lévy operator

This section is devoted to proving various identities and control estimates for the operators \mathcal{L} and \mathcal{B} . We will say here that the expressions $\mathcal{L}[\phi]$ and $\mathcal{B}[\psi, \varphi]$ are well defined in $L^1(\mathbb{R}^d)$ or $L^1_{\text{loc}}(\mathbb{R}^d)$ when the limits $\lim_{r \searrow 0} \mathcal{L}^{\geq r}[\phi]$ and $\lim_{r \searrow 0} \mathcal{B}^{\geq r}[\psi, \varphi]$ exist in $L^1(\mathbb{R}^d)$ or $L^1_{\text{loc}}(\mathbb{R}^d)$ respectively.

3.1. Identities for \mathcal{L} and \mathcal{B}

We begin by establishing a connection between \mathcal{L} and \mathcal{B} and follow up with a product rule for \mathcal{B} . These results resemble well-known identities for Δ and ∇ , and the proofs boil down to truncations of the operators \mathcal{L} and \mathcal{B} , interchanging integrals, and change of variables. To ensure that Fubini’s theorem can be applied, some “absolute integrability” of the relevant components is required. For this purpose, an assumption of compact support is included in the propositions (though various other assumptions would suffice) as we will frequently encounter compactly supported functions.

Proposition 3.1 (Integration by parts formulas). *Assume (A_μ) , $\phi, \psi \in L^\infty(\mathbb{R}^d)$, and that ϕ or ψ has compact support. If both $\mathcal{L}[\phi]$ and $\mathcal{B}[\phi, \psi]$ are well defined in $L^1(\mathbb{R}^d)$, then*

$$\int_{\mathbb{R}^d} \mathcal{L}[\phi]\psi \, dx = - \int_{\mathbb{R}^d} \mathcal{B}[\phi, \psi] \, dx. \tag{3.1}$$

If both $\mathcal{L}[\phi]$ and $\mathcal{L}[\psi]$ are well defined in $L^1(\mathbb{R}^d)$, then

$$\int_{\mathbb{R}^d} \mathcal{L}[\phi]\psi \, dx = \int_{\mathbb{R}^d} \phi \mathcal{L}[\psi] \, dx. \tag{3.2}$$

Proof. We prove only (3.1) as the proof of (3.2) is similar. Fix $r > 0$, and note that $(z, x) \mapsto (\phi(x + z) - \phi(x))\psi(x)$ is absolutely integrable on the set $\{(z, x) \in \mathbb{R}^d \times \mathbb{R}^d : |z| \geq r\}$ with respect to $d\mu(z) \, dx$ (see the proof of Proposition 3.8 for similar expressions treated in detail). We may then use Fubini’s theorem to compute

$$\begin{aligned} \int_{\mathbb{R}^d} \mathcal{L}^{\geq r}[\phi]\psi \, dx &= \int_{\mathbb{R}^d} \int_{|z| \geq r} (\phi(x + z) - \phi(x))\psi(x) \, d\mu(z) \, dx \\ &= \int_{|z| \geq r} \int_{\mathbb{R}^d} (\phi(x) - \phi(x - z))\psi(x - z) \, dx \, d\mu(z) \\ &= - \int_{\mathbb{R}^d} \int_{|z| \geq r} (\phi(x + z) - \phi(x))\psi(x + z) \, d\mu(z) \, dx, \end{aligned}$$

where we made the substitutions $x \mapsto x - z$ and then $z \mapsto -z$, exploiting the symmetry of μ . Thus

$$\begin{aligned} \int_{\mathbb{R}^d} \mathcal{L}^{\geq r}[\phi]\psi \, dx &= \frac{1}{2} \int_{\mathbb{R}^d} \int_{|z| \geq r} (\phi(x + z) - \phi(x))(\psi(x) - \psi(x + z)) \, d\mu(z) \, dx \\ &= - \int_{\mathbb{R}^d} \mathcal{B}^{\geq r}[\phi, \psi] \, dx. \end{aligned}$$

The result follows by letting $r \rightarrow 0$. ■

Remark 3.2. If ψ in the previous proposition has compact support, the result holds true even when $\mathcal{L}[\phi]$ is merely well defined in the L^1_{loc} -sense; the proof is identical.

The next formula is a nonlocal analogue of the product rule $\nabla\phi \cdot \nabla(\psi\varphi) = (\nabla\phi \cdot \nabla\psi)\varphi + (\nabla\phi \cdot \nabla\varphi)\psi$.

Proposition 3.3 (Product rule). *Assume (A_μ) , $\phi, \psi, \varphi \in L^\infty(\mathbb{R}^d)$, and either ϕ, ψ , or φ , has compact support. Then, if $\mathcal{B}[\phi, \psi\varphi]$, $\mathcal{B}[\phi, \psi]$, and $\mathcal{B}[\phi, \varphi]$ are well defined in $L^1(\mathbb{R}^d)$, we have*

$$\int_{\mathbb{R}^d} \mathcal{B}[\phi, \psi\varphi] \, dx = \int_{\mathbb{R}^d} \mathcal{B}[\phi, \psi]\varphi \, dx + \int_{\mathbb{R}^d} \mathcal{B}[\phi, \varphi]\psi \, dx.$$

Proof. Fix $r > 0$. Similar to the previous proof, the compact support of one of ϕ, ψ, φ will guarantee that the following integrands are absolutely integrable with respect to $d\mu(z) \, dx$ on the given domain. By Fubini’s theorem we compute

$$\begin{aligned} & \int_{\mathbb{R}^d} (\mathcal{B}^{\geq r}[\phi, \psi\varphi] - \mathcal{B}^{\geq r}[\phi, \psi]\varphi) \, dx \\ &= \frac{1}{2} \int_{\mathbb{R}^d} \int_{|z| \geq r} (\phi(x+z) - \phi(x))(\varphi(x+z) - \varphi(x))\psi(x+z) \, d\mu(z) \, dx \\ &= \frac{1}{2} \int_{|z| \geq r} \int_{\mathbb{R}^d} (\phi(x) - \phi(x-z))(\varphi(x) - \varphi(x-z))\psi(x) \, dx \, d\mu(z) \\ &= \frac{1}{2} \int_{\mathbb{R}^d} \int_{|z| \geq r} (\phi(x+z) - \phi(x))(\varphi(x+z) - \varphi(x))\psi(x) \, d\mu(z) \, dx \\ &= \int_{\mathbb{R}^d} \mathcal{B}^{\geq r}[\phi, \varphi]\psi \, dx, \end{aligned}$$

where we made the substitution $x \mapsto x - z$ followed by $z \mapsto -z$ and exploited the symmetry of $d\mu(z)$. The result follows by letting $r \rightarrow 0$. ■

3.2. Bounds for \mathcal{L} and \mathcal{B}

We here provide bounds and inequalities that will be used to make sense of the expressions $\mathcal{L}[\phi]$ and $\mathcal{B}[\psi, \varphi]$ in nontrivial situations. We begin with a standard estimate for smooth functions.

Lemma 3.4. *Let $p \in [1, \infty]$, $\phi \in C_c^2(\mathbb{R}^d)$, and let $D^2\phi(x)$ denote the Hessian matrix of ϕ at x . Then*

$$\|\mathcal{L}[\phi]\|_{L^p(\mathbb{R}^d)} \leq \left[2\|\phi\|_{L^p(\mathbb{R}^d)} + \frac{1}{2}\|D^2\phi\|_{L^p(\mathbb{R}^d)} \right] \int_{\mathbb{R}^d} (|z|^2 \wedge 1) \, d\mu(z),$$

where $\|D^2\phi\|_{L^p(\mathbb{R}^d)}$ denotes the L^p -norm of $x \mapsto |D^2\phi(x)|$, with $|D^2\phi|$ being the spectral radius of $D^2\phi$.

Up to the choice of constants, this result is proved e.g. in [57]; see Lemma 2.1 and Remark 2.2 there.

When $\phi \in L^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$, the previous lemma guarantees that $\mathcal{L}[\phi]$ makes sense as a distribution if we canonically define $\langle \mathcal{L}[\phi], \varphi \rangle := \langle \phi, \mathcal{L}[\varphi] \rangle$ for all $\varphi \in C_c^2(\mathbb{R})$. Moreover, it follows that $\mathcal{L}^{\geq r}[\phi] \rightarrow \mathcal{L}[\phi]$ in the sense of distributions as $r \rightarrow 0$. However, a distributional notion of $\mathcal{L}[\phi]$ will often be too weak for our purpose, and so the next proposition is useful.

Proposition 3.5 (Lévy operator on compositions). *Assume $h: \mathbb{R} \rightarrow \mathbb{R}$ admits a weak derivative of bounded variation and $\phi \in L^\infty(\mathbb{R}^d)$ has compact support. If $\mathcal{L}[\phi]$ is well defined in $L^1(\mathbb{R}^d)$, then so is $\mathcal{L}[h(\phi)]$, and moreover we have*

$$\|\mathcal{L}[h(\phi)]\|_{L^1(\mathbb{R}^d)} \leq [\|h'\|_{L^\infty(\mathbb{R})} + |h'|_{\text{TV}(\mathbb{R})}] \|\mathcal{L}[\phi]\|_{L^1(\mathbb{R}^d)}. \quad (3.3)$$

Remark 3.6. Note that $|h'|_{\text{TV}(\mathbb{R})} < \infty$ implies $\|h'\|_{L^\infty(\mathbb{R})} < \infty$ so that h is actually Lipschitz.

Proof of Proposition 3.5. As h' is of bounded variation, associate it with its left-continuous representation (so that the following pointwise estimates are unambiguous). We consider first the case when the Lévy measure is finite $\mu(\mathbb{R}^d) < \infty$, so that \mathcal{L} is a zero-order operator; in particular we have $\mathcal{L}[\psi] = \mu * \psi - \mu(\mathbb{R}^d)\psi$ for any $\psi \in L^\infty(\mathbb{R}^d)$. Due to its bounded variation, h' admits an essentially unique representation $h' = \beta_+ - \beta_-$ for two nondecreasing functions β_+, β_- satisfying

$$|h'|_{\text{TV}(\mathbb{R})} = |\beta_+|_{\text{TV}(\mathbb{R})} + |\beta_-|_{\text{TV}(\mathbb{R})} = \beta_+(\infty) + \beta_-(\infty) - \beta_+(-\infty) - \beta_-(-\infty).$$

Introducing then the monotone function $\beta(v) := \beta_+(v) + \beta_-(v) - \frac{1}{2}|h'|_{\text{TV}(\mathbb{R})}$, which has been shifted to satisfy $2\|\beta\|_{L^\infty(\mathbb{R})} = |h'|_{\text{TV}(\mathbb{R})}$, we observe that we have the inequality

$$|h'(v+w) - h'(v)| \leq \text{sgn}(w)(\beta(v+w) - \beta(v)), \quad (3.4)$$

for any $v, w \in \mathbb{R}$.

In the next computation, let $\delta := \phi(x+z) - \phi(x)$ for notational simplicity. Subtracting $h'(\phi)\mathcal{L}[\phi]$ from $\mathcal{L}[h(\phi)]$ we find for a.e. $x \in \mathbb{R}^d$,

$$\begin{aligned} |\mathcal{L}[h(\phi)] - h'(\phi)\mathcal{L}[\phi]|(x) &= \left| \int_{\mathbb{R}^d} h(\phi(x) + \delta) - h(\phi(x)) - h'(\phi(x))\delta \, d\mu(z) \right| \\ &\leq \int_{\mathbb{R}^d} \int_0^1 |(h'(\phi(x) + s\delta) - h'(\phi(x)))\delta| \, ds \, d\mu(z) \\ &\leq \int_{\mathbb{R}^d} \int_0^1 (\beta(\phi(x) + s\delta) - \beta(\phi(x)))\delta \, ds \, d\mu(z), \end{aligned} \quad (3.5)$$

where we used (3.4). By monotonicity of β , we further have $(\beta(\phi + s\delta) - \beta(\phi))\delta \leq (\beta(\phi + \delta) - \beta(\phi))\delta$ whenever $s \in [0, 1]$. And so by (3.5) we infer

$$\begin{aligned} \|\mathcal{L}[h(\phi)] - h'(\phi)\mathcal{L}[\phi]\|_{L^1(\mathbb{R}^d)} &\leq \int_{\mathbb{R}^d} 2\mathcal{B}[\beta(\phi), \phi] \, dx \\ &\leq 2\|\beta\|_{L^\infty(\mathbb{R}^d)} \|\mathcal{L}[\phi]\|_{L^1(\mathbb{R}^d)}, \end{aligned} \quad (3.6)$$

where we shifted \mathcal{B} over to \mathcal{L} using Proposition 3.1. Thus, (3.3) follows for zero-order \mathcal{L} when combining the triangle inequality with (3.6) and the bound $2\|\beta\|_{L^\infty(\mathbb{R}^d)} = |h'|_{\text{TV}(\mathbb{R})}$.

For general \mathcal{L} , we first note that $\mathcal{L}[h(\phi)]$ is necessarily well defined in $L^1(\mathbb{R}^d)$: Let $r' > r > 0$ and note that

$$\begin{aligned} \limsup_{r', r \rightarrow 0} \|\mathcal{L}^{\geq r'}[h(\phi)] - \mathcal{L}^{\geq r}[h(\phi)]\|_{L^1(\mathbb{R}^d)} &= \limsup_{r', r \rightarrow 0} \|\mathcal{L}^{r' > \dots \geq r}[h(\phi)]\|_{L^1(\mathbb{R}^d)} \\ &\lesssim \limsup_{r', r \rightarrow 0} \|\mathcal{L}^{r' > \dots \geq r}[\phi]\|_{L^1(\mathbb{R}^d)} = 0, \end{aligned}$$

where we used (3.3) on the zero-order Lévy operator $\mathcal{L}^{r' > \dots \geq r}$ and the fact that $\mathcal{L}[\phi]$ is well defined in $L^1(\mathbb{R}^d)$. The inequality (3.3) then follows for $\mathcal{L}[h(\phi)]$ by using the corresponding one for $\mathcal{L}^{\geq r}[h(\phi)]$ and letting $r \rightarrow 0$. ■

We combine the previous two results to compute a local L^1 -bound for $\mathcal{L}[h(\phi)]$ with h as above and with ϕ bounded and locally C^2 .

Corollary 3.7. *Let h be as in Proposition 3.5, and let $\phi \in C_b^2(\mathbb{R}^d)$. Then for any compact $U \subset \mathbb{R}^d$ and any smooth cut-off function $\psi \in C_c^\infty(\mathbb{R}^d)$ satisfying $\psi(x) = 1$ whenever $\text{dist}(x, U) \leq 1$ we have*

$$\begin{aligned} \|\mathcal{L}[h(\phi)]\|_{L^1(U)} &\leq [\|h'\|_{L^\infty(\mathbb{R})} + |h'|_{\text{TV}(\mathbb{R})}] \\ &\quad \times \left[4\|\phi\|_{L^\infty(\mathbb{R}^d)} \|\psi\|_{L^1(\mathbb{R}^d)} + \frac{1}{2} \|D^2(\phi\psi)\|_{L^1(\mathbb{R}^d)} \right] \\ &\quad \times \int_{\mathbb{R}^d} (|z|^2 \wedge 1) \, d\mu(z), \end{aligned} \tag{3.7}$$

where $\|D^2(\phi\psi)\|_{L^1(\mathbb{R}^d)}$ is as in Lemma 3.4. Thus $\mathcal{L}[h(\phi)] \in L_{\text{loc}}^1(\mathbb{R}^d)$, and in particular, we have

$$\lim_{r \rightarrow 0} \mathcal{L}^{\geq r}[h(\phi)] = \mathcal{L}[h(\phi)] \quad \text{in } L_{\text{loc}}^1(\mathbb{R}^d).$$

Proof. To prove (3.7), we decompose $\mathcal{L} = \mathcal{L}^{\geq 1} + \mathcal{L}^{< 1}$. As $\mathcal{L}^{< 1}[h(\phi)] = \mathcal{L}^{< 1}[h(\phi\psi)]$ on U , we find

$$\begin{aligned} \|\mathcal{L}^{< 1}[h(\phi)]\|_{L^1(U)} &\leq \|\mathcal{L}^{< 1}[h(\phi\psi)]\|_{L^1(\mathbb{R}^d)} \\ &\leq [\|h'\|_{L^\infty(\mathbb{R})} + |h'|_{\text{TV}(\mathbb{R})}] \\ &\quad \times \left[2\|\phi\psi\|_{L^1(\mathbb{R}^d)} + \frac{1}{2} \|D^2(\phi\psi)\|_{L^1(\mathbb{R}^d)} \right] \int_{\mathbb{R}^d} (|z|^2 \wedge 1) \, d\mu(z), \end{aligned}$$

by Proposition 3.3 and Lemma 3.4. And using that $\mathcal{L}^{\geq 1}[h(\phi)] = \psi \mathcal{L}^{\geq 1}[h(\phi)]$ on U , we compute

$$\begin{aligned} \|\mathcal{L}^{\geq 1}[h(\phi)]\|_{L^1(U)} &\leq \|\psi \mathcal{L}^{\geq 1}[h(\phi)]\|_{L^1(\mathbb{R}^d)} \\ &\leq 2\|\psi\|_{L^1(\mathbb{R}^d)} \|h'\|_{L^\infty(\mathbb{R})} \|\phi\|_{L^\infty(\mathbb{R}^d)} \mu(\{|z| \geq 1\}). \end{aligned}$$

Putting these estimates together (plus a few coarse estimates) gives (3.7). The fact that $\lim_{r \rightarrow 0} \mathcal{L}^{\geq r}[h(\phi)] = \mathcal{L}[h(\phi)]$ in $L^1_{\text{loc}}(\mathbb{R})$ follows if we can prove L^1 -convergence on U as the latter set was arbitrary. This fact follows immediately from (3.7) with $\mathcal{L}^{< r}$ as the respective Lévy operator; we get

$$\|\mathcal{L}[h(\phi)] - \mathcal{L}^{\geq r}[h(\phi)]\|_{L^1(U)} = \|\mathcal{L}^{< r}[h(\phi)]\|_{L^1(U)} \lesssim \int_{|z| < r} (|z|^2 \wedge 1) \, d\mu(z),$$

which tends to zero as $r \rightarrow 0$. ■

We end the section by listing a few sufficient conditions on $\phi, \psi \in L^\infty(\mathbb{R}^d)$ for $\mathcal{B}[\phi, \psi]$ to be well defined. Here we will say that “the integrand of $\mathcal{B}[\phi, \psi]$ is absolutely integrable with respect to $d\mu(z) \, dx$ ” means

$$\frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} |\phi(x+z) - \phi(x)| |\psi(x+z) - \psi(x)| \, d\mu(z) \, dx < \infty. \quad (3.8)$$

This condition ameliorates the expression $\mathcal{B}[\phi, \psi]$ and in particular ensures it to be well defined in $L^1(\mathbb{R}^d)$; even the pointwise definition (2.1) is then necessarily meaningful almost everywhere.

Proposition 3.8 (List of \mathcal{B} -compatible pairs). *Let $\phi, \psi \in L^\infty(\mathbb{R}^d)$. Then the integrand of $\mathcal{B}[\phi, \psi]$ is absolutely integrable with respect to $d\mu(z) \, dx$ if either of the following conditions is satisfied:*

- (i) $\phi, \psi \in H^{\mathcal{L}}(\mathbb{R}^d)$. The integral in (3.8) is then bounded by

$$\left(\int_{\mathbb{R}^d} \mathcal{B}[\phi, \phi] \, dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} \mathcal{B}[\psi, \psi] \, dx \right)^{\frac{1}{2}}.$$

- (ii) $\phi \in H^{\mathcal{L}}(\mathbb{R}^d)$, ψ is Lipschitz continuous, and ϕ or ψ is supported in a compact set $E \subset \mathbb{R}^d$. The integral in (3.8) is then bounded by

$$\sqrt{2|E|} \left(\int_{\mathbb{R}^d} (|z|^2 \wedge 1) \, d\mu(z) \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} \mathcal{B}[\phi, \phi] \, dx \right)^{\frac{1}{2}} (L_\psi \vee 2\|\psi\|_{L^\infty}).$$

- (iii) ϕ is of bounded variation, ψ is Lipschitz continuous, and ϕ or ψ is supported in a compact set $E \subset \mathbb{R}^d$. The integral in (3.8) is then bounded by

$$\left(\int_{\mathbb{R}^d} (|z|^2 \wedge 1) \, d\mu(z) \right) (|\phi|_{\text{TV}} \vee 2|E| \|\phi\|_{L^\infty}) (L_\psi \vee 2\|\psi\|_{L^\infty})$$

- (iv) The Lévy measure μ is finite, and ϕ or ψ is supported in a compact set $E \subset \mathbb{R}^d$. The integral in (3.8) is then bounded by

$$4|E| \mu(\mathbb{R}^d) \|\phi\|_{L^\infty} \|\psi\|_{L^\infty}.$$

Here, $|E|$ denotes the d -dimensional Lebesgue measure of E .

Proof. (i) This follows from the definition of $H^{\mathcal{L}}(\mathbb{R}^d)$ and the Cauchy–Schwarz inequality.

(ii) Here we have

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} |\phi(x+z) - \phi(x)| |\psi(x+z) - \psi(x)| d\mu(z) dx \\ & \leq \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} |\phi(x+z) - \phi(x)| |\psi(x+z) - \psi(x)| (\chi_E(x) + \chi_E(x+z)) d\mu(z) dx, \end{aligned}$$

where χ_E is the indicator function on E . By splitting the integral on the right-hand side into two, one integral featuring $\chi_E(x)$ and the other $\chi_E(x+z)$, one may perform the substitution of variable $x \mapsto x-z$ followed by $z \mapsto -z$ to see that the two parts coincide. Thus, the integral is bounded by

$$\begin{aligned} & \int_{\mathbb{R}^d \times \mathbb{R}^d} |\phi(x+z) - \phi(x)| |\psi(x+z) - \psi(x)| \chi_E(x) d\mu(z) dx \\ & \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} [|\phi(x+z) - \phi(x)|] [(|z| \wedge 1) \chi_E(x)] d\mu(z) dx (L_\psi \vee 2 \|\psi\|_{L^\infty}). \end{aligned}$$

The result then follows from Hölder's inequality applied to the two square brackets.

(iii) We again introduce $\chi_E(x)$ into the integral and get the desired bound from the calculation

$$\begin{aligned} & \int_E |\phi(x+z) - \phi(x)| |\psi(x+z) - \psi(x)| dx \\ & \leq \int_E |\phi(x+z) - \phi(x)| dx (L_\psi |z| \wedge 2 \|\psi\|_{L^\infty}) \\ & \leq (|\phi|_{\text{TV}} |z| \wedge 2 |E| \|\phi\|_{L^\infty}) (L |z| \wedge 2 \|\psi\|_{L^\infty(\mathbb{R}^d)}) \\ & \leq (|z|^2 \wedge 1) (|\phi|_{\text{TV}} \vee 2 |E| \|\phi\|_{L^\infty}) (L_\psi \vee 2 \|\psi\|_{L^\infty}). \end{aligned}$$

(iv) As before, we only need to control the integral of $|\phi(x+z) - \phi(x)| |\psi(x+z) - \psi(x)| \chi_E(x) \leq 4 \|\phi\|_{L^\infty} \|\psi\|_{L^\infty} \chi_E(x)$. Integrating the latter quantity gives the claimed bound. ■

We will frequently encounter the expression $\mathcal{B}[\chi_\Omega, \varphi]$, where χ_Ω is the characteristic function on Ω and where φ is bounded and Lipschitz continuous. Because $\partial\Omega$ has finite Hausdorff \mathcal{H}^{d-1} -measure (due to its regularity), we get that $|\chi_\Omega|_{\text{TV}(\mathbb{R}^d)} = |\partial\Omega|_{\mathcal{H}^{d-1}} < \infty$.⁸ By the previous proposition we conclude that $\mathcal{B}[\chi_\Omega, \varphi]$ is well defined in $L^1(\mathbb{R}^d)$.

We conclude the section with a similar, but more specific, proposition for the double variable scenario $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$. The corresponding operators are defined in Section 2, specifically (2.3) and (2.4).

⁸This identity follows from the definition of $|\cdot|_{\text{TV}(\mathbb{R}^d)}$ and the Gauss divergence theorem.

Proposition 3.9. *Let $\phi, \psi \in L^\infty(\mathbb{R}^d)$ and $\varphi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ depend on x, y , and (x, y) respectively, and $\mathcal{B}_x[\phi, \phi]$ and $\mathcal{B}_y[\psi, \psi]$ be well defined in $L^1_{\text{loc}}(\mathbb{R}^d)$. Then the integrands of*

$$\begin{aligned} &\mathcal{B}_{x+y}[|\phi - \psi|, \varphi], \quad \mathcal{B}_{x,x+y}[|\phi - \psi|, \varphi], \quad \mathcal{B}_{y,x+y}[|\phi - \psi|, \varphi], \\ &\mathcal{B}_{x,y}[|\phi - \psi|, \varphi], \quad \text{and} \quad \mathcal{B}_{y,x}[|\phi - \psi|, \varphi], \end{aligned}$$

are all absolutely integrable with respect to $d\mu(z) dx dy$ in the analogous sense of (3.8), and the resulting integral is bounded by

$$\begin{aligned} &(2|E|)^{\frac{3}{2}} \left[\left(\int_E \mathcal{B}_x[\phi, \phi] dx \right)^{\frac{1}{2}} + \left(\int_E \mathcal{B}_y[\psi, \psi] dy \right)^{\frac{1}{2}} \right] \\ &\quad \times \left(\int_{\mathbb{R}^d} (|z|^2 \wedge 1) d\mu(z) \right)^{\frac{1}{2}} (L_\varphi \vee \|\varphi\|_{L^\infty}), \end{aligned}$$

where $E \subset \mathbb{R}^d$ is any compact set such that $\text{supp}(\varphi) \subseteq E \times E$, $|E|$ denotes its d -dimensional Lebesgue measure, and L_φ is a Lipschitz constant for φ .

Proof. We only prove the \mathcal{B}_{x+y} -case, as the others can be dealt with similarly. Arguing as in part (ii) of the proof of Proposition 3.8, we can restrict (x, y) to the compact set $E \times E$, and we compute

$$\begin{aligned} &\int_{E \times E} \int_{\mathbb{R}^d} \left| |\phi(x+z) - \psi(y+z)| - |\phi(x) - \psi(y)| \right| \\ &\quad \times |\varphi(x+z, y+z) - \varphi(x, y)| d\mu(z) dx dy \\ &\leq \int_{E \times E} \int_{\mathbb{R}^d} (|\phi(x+z) - \phi(x)| + |\psi(y+z) - \psi(y)|) \\ &\quad \times ((\sqrt{2}L_\varphi|z|) \wedge (2\|\varphi\|_{L^\infty})) d\mu(z) dx dy. \end{aligned}$$

The desired bound then follows from an appropriate application of Hölder’s inequality. ■

4. Properties of entropy solutions

In this section we establish some results and a priori estimates for entropy solutions of (1.1).

4.1. Finite energy

We will repeatedly need that entropy solutions have finite energy.

Proposition 4.1 (Energy estimate). *Assume (A_Ω) – (A_μ) and u is an entropy solution of (1.1). Then*

$$\begin{aligned}
 & \int_M \mathcal{B}[b(u) - b(\bar{u}^c), b(u) - b(\bar{u}^c)] \, dx \, dt \\
 & \leq \int_\Omega H(u_0, \bar{u}^c(0)) \, dx \\
 & \quad - \int_Q [(u - \bar{u}^c)\bar{u}_t^c + F(u, \bar{u}^c) \cdot \nabla \bar{u}^c] b'(\bar{u}^c) \, dx \, dt \\
 & \quad + \int_Q \mathcal{L}[b(\bar{u}^c)](b(u) - b(\bar{u}^c)) \, dx \, dt, \tag{4.1}
 \end{aligned}$$

where $F(u, \bar{u}^c) = \text{sgn}(u - \bar{u}^c)(f(u) - f(\bar{u}^c))$ and $H(u, k) := \int_k^u (b(\xi) - b(k)) \, d\xi \geq 0$. Moreover, the right-hand side of (4.1) is finite.

Proof. The family of entropy inequalities (2.6) are those corresponding to the semi-Kružkov entropy pairs

$$u \mapsto ((u - k)^\pm, F^\pm(u, k)).$$

We will now construct a second family of entropy inequalities corresponding to the entropy pairs $u \mapsto (H^\pm(u, k), G^\pm(u, k))$ defined by

$$\begin{aligned}
 H^+(u, k) &= \int_k^\infty (u - \xi)^+ b'(\xi) \, d\xi, & G^+(u, k) &= \int_k^\infty F^+(u, \xi) b'(\xi) \, d\xi, \\
 H^-(u, k) &= \int_{-\infty}^k (u - \xi)^- b'(\xi) \, d\xi, & G^-(u, k) &= \int_{-\infty}^k F^-(u, \xi) b'(\xi) \, d\xi.
 \end{aligned} \tag{4.2}$$

We proceed by performing the calculations for the (+) and (−) cases simultaneously, though we stress that one should consider them separate calculations.

Let $k \in \mathbb{R}$ and $0 \leq \varphi \in C_c^\infty(M)$ be such that

$$(b(u^c) - b(k))^\pm \varphi = 0, \tag{4.3}$$

for all $(t, x) \in Q^c$. Observe that (4.3) is still valid when replacing k with ξ provided $\pm \xi \geq \pm k$. In particular, it follows that, for all $\pm \xi \geq \pm k$, we have (2.6) with ξ replacing k , yielding

$$\begin{aligned}
 & - \int_Q ((u - \xi)^\pm \varphi_t + F^\pm(u, \xi) \cdot \nabla \varphi) \, dx \, dt \\
 & \quad - \int_Q \mathcal{L}^{\geq r}[b(u)] \text{sgn}^\pm(u - \xi) \varphi \, dx \, dt \\
 & \quad - \int_M (b(u) - b(\xi))^\pm \mathcal{L}^{< r}[\varphi] \, dx \, dt \\
 & \leq L_f \int_\Gamma (\bar{u}^c - \xi)^\pm \varphi \, d\sigma(x) \, dt, \tag{4.4}
 \end{aligned}$$

where there is no “initial term” on the right-hand side as $\varphi(0, x) = 0$. Multiplying (4.4) by the nonnegative quantity $b'(\xi)$ and integrating over $\xi \in (k, \infty)$ in the (+) case and $\xi \in (-\infty, k)$ in the (–) case we get

$$\begin{aligned}
 & - \int_Q (H^\pm(u, k)\varphi_t + G^\pm(u, k) \cdot \nabla\varphi) \, dx \, dt \\
 & \mp \int_M \mathcal{L}^{\geq r}[b(u)](b(u) - b(k))^\pm \varphi \, dx \, dt \\
 & - \frac{1}{2} \int_M ((b(u) - b(k))^\pm)^2 \mathcal{L}^{< r}[\varphi] \, dx \, dt \\
 & \leq L_f \int_\Gamma H^\pm(\bar{u}^c, k)\varphi \, d\sigma(x) \, dt, \tag{4.5}
 \end{aligned}$$

where H^\pm and G^\pm are as in (4.2). We have for later convenience extended the domain of integration for the second integral in (4.5) from Q to M ; this is possible as $(b(u) - b(k))^\pm \varphi$ is zero a.e. in Q^c . To summarize, (4.5) is valid for all $k \in \mathbb{R}$ and $0 \leq \varphi \in C_c^\infty(M)$ satisfying (4.3).

Next we wish to set $k = \bar{u}^c(t, x)$ in (4.5), and while this cannot be done directly, we will accomplish this by a “doubling of variables” argument. For parameters $(s, y) \in M$ and $\varepsilon \in (0, 1)$, we introduce the pair $(\kappa^\pm, \psi) \in \mathbb{R} \times C_c^\infty(M)$ defined as

$$\begin{aligned}
 \kappa^\pm &= \kappa_{s,y,\varepsilon}^\pm = \bar{u}^c(s, y) \pm \varepsilon L_{\bar{u}^c}, \\
 \psi &= \psi_{s,y,\varepsilon}(t, x) = \tilde{\varphi}(t, x) \rho_\varepsilon(t - s, x - y),
 \end{aligned}$$

where $0 \leq \tilde{\varphi} \in C_c^\infty(M)$ is arbitrary, ρ_ε is a standard mollifier with support in the centred ball of radius ε in $\mathbb{R} \times \mathbb{R}^d$ and $L_{\bar{u}^c}$ is a local Lipschitz constant for \bar{u}^c valid on $\{(t, x) \in M: \text{dist}((t, x), \text{supp } \tilde{\varphi}) < 1\}$. The support of ρ_ε implies that ψ is zero when $|(t - s, x - y)| \geq \varepsilon$, while $|(t - s, x - y)| < \varepsilon$ implies that $\pm \kappa^\pm \geq \pm \bar{u}^c(t, x)$. In particular, we have

$$(b(\bar{u}^c(t, x)) - b(\kappa_{s,y,\varepsilon}^\pm))^\pm \psi_{s,y,\varepsilon}(t, x) = 0,$$

for all t, s, x, y, ε , and we may thus use the pair (κ^\pm, ψ) as the constant and test function in (4.5). Doing so, and additionally writing φ instead of $\tilde{\varphi}$ for notional simplicity, we get after integrating over $(s, y) \in M$,

$$\begin{aligned}
 & - \int_{Q_x \times M_y} [H^\pm(u, \kappa^\pm)\varphi_t + G^\pm(u, \kappa^\pm) \cdot \nabla_x \varphi] \rho_\varepsilon \\
 & \mp \int_{M_x \times M_y} \mathcal{L}_x^{\geq r}[b(u)](b(u) - b(\kappa^\pm))^\pm \varphi \rho_\varepsilon \\
 & - \frac{1}{2} \int_{M_x \times M_y} ((b(u) - b(\kappa^\pm))^\pm)^2 \mathcal{L}_x^{< r}[\varphi \rho_\varepsilon] \\
 & \leq - \int_{Q_x \times M_y} [H^\pm(u, \kappa^\pm)\partial_s \rho_\varepsilon + G^\pm(u, \kappa^\pm) \cdot \nabla_y \rho_\varepsilon] \varphi. \tag{4.6}
 \end{aligned}$$

For brevity, we have here suppressed the differentials $dt dx ds dy$ from the integrals and added sub-indexes to the operators and the domains of integration in line with the notation from Section 2. We also used the identities $\partial_t \rho_\varepsilon = -\partial_s \rho_\varepsilon$ and $\nabla_x \rho_\varepsilon = -\nabla_y \rho_\varepsilon$ and dropped the term on the right-hand side of (4.5) since

$$H^\pm(\bar{u}^c(t, x), \kappa^\pm) \varphi \rho_\varepsilon = 0,$$

for all t, x, s, y, ε .

Next we integrate by parts in (s, y) on the right-hand side of (4.6), take ε so small that the boundary terms (in time) vanish, and then add the integral of the quantity $\pm \mathcal{L}_y^{\geq r} [b(\kappa^\pm)] (b(u) - b(\kappa^\pm))^\pm \varphi \rho_\varepsilon$ to each side:

$$\begin{aligned} & - \int_{Q_x \times M_y} [H^\pm(u, \kappa^\pm) \varphi_t + G^\pm(u, \kappa^\pm) \cdot \nabla_x \varphi] \rho_\varepsilon \\ & \mp \int_{M_x \times M_y} \mathcal{L}_{x+y}^{\geq r} [b(u) - b(\kappa^\pm)] (b(u) - b(\kappa^\pm))^\pm \varphi \rho_\varepsilon \\ & - \frac{1}{2} \int_{M_x \times M_y} ((b(u) - b(\kappa^\pm))^\pm)^2 \mathcal{L}_x^{< r} [\varphi \rho_\varepsilon] \\ & \leq \mp \int_{Q_x \times M_y} [(u - \kappa^\pm)^\pm \partial_s \kappa^\pm + F^\pm(u, \kappa^\pm) \cdot \nabla_y \kappa^\pm] b'(\kappa^\pm) \varphi \rho_\varepsilon \\ & \pm \int_{Q_x \times M_y} \mathcal{L}_y^{\geq r} [b(\kappa^\pm)] (b(u) - b(\kappa^\pm))^\pm \varphi \rho_\varepsilon. \end{aligned} \quad (4.7)$$

With \mathcal{L}_{x+y} as defined in (2.3), we here used that $\mathcal{L}_{x+y}^{\geq r} [w_1 - w_2] = \mathcal{L}_x^{\geq r} [w_1] - \mathcal{L}_y^{\geq r} [w_2]$ when w_1 and w_2 are functions in x and y respectively, and we used the formulas

$$\begin{aligned} \partial_s H^\pm(u, \kappa^\pm) &= \mp (u - \kappa^\pm)^\pm b'(\kappa^\pm) \partial_s \kappa^\pm, \\ \nabla_y G^\pm(u, \kappa^\pm) &= \mp F^\pm(u, \kappa^\pm) b'(\kappa^\pm) \nabla_y \kappa^\pm, \end{aligned}$$

which follow from (4.2). Fixing $r' > r$ we split up the operator $\mathcal{L}_{x+y}^{\geq r} = \mathcal{L}_{x+y}^{\geq r'} + \mathcal{L}_{x+y}^{r' > \dots \geq r}$. Looking at the first inequality of Corollary C.2 for the choice $\eta(\phi) = \frac{1}{2}(\phi^\pm)^2$, we get the pointwise inequality

$$\pm \mathcal{L}_{x+y}^{r' > \dots \geq r} [b(u) - b(\kappa^\pm)] (b(u) - b(\kappa^\pm))^\pm \leq \frac{1}{2} \mathcal{L}_{x+y}^{r' > \dots \geq r} [((b(u) - b(\kappa^\pm))^\pm)^2].$$

We thus conclude

$$\begin{aligned} & \mp \int_{M_x \times M_y} \mathcal{L}_{x+y}^{\geq r} [b(u) - b(\kappa^\pm)] (b(u) - b(\kappa^\pm))^\pm \varphi \rho_\varepsilon \\ & \geq \mp \int_{M_x \times M_y} \mathcal{L}_{x+y}^{\geq r'} [b(u) - b(\kappa^\pm)] (b(u) - b(\kappa^\pm))^\pm \varphi \rho_\varepsilon \\ & - \frac{1}{2} \int_{M_x \times M_y} ((b(u) - b(\kappa^\pm))^\pm)^2 \mathcal{L}_x^{r' > \dots \geq r} [\varphi] \rho_\varepsilon, \end{aligned}$$

where we used the self-adjointness of $\mathcal{L}_{x+y}^{r'>\dots\geq r}$ (Proposition 3.1) to move it over to $\varphi\rho_\varepsilon$ on which the operator satisfies $\mathcal{L}_{x+y}^{r'>\dots\geq r}[\varphi\rho_\varepsilon] = \mathcal{L}_x^{r'>\dots\geq r}[\varphi]\rho_\varepsilon$. Exploiting this in (4.7), letting $r \rightarrow 0$, and then renaming r' as r we get

$$\begin{aligned}
 & - \int_{Q_x \times M_y} [H^\pm(u, \kappa^\pm)\varphi_t + G^\pm(u, \kappa^\pm) \cdot \nabla_x \varphi]\rho_\varepsilon \\
 & \mp \int_{M_x \times M_y} \mathcal{L}_{x+y}^{\geq r}b(u) - b(\kappa^\pm)^\pm \varphi\rho_\varepsilon \\
 & - \frac{1}{2} \int_{M_x \times M_y} ((b(u) - b(\kappa^\pm))^\pm)^2 \mathcal{L}_x^{< r}[\varphi]\rho_\varepsilon \\
 & \leq \mp \int_{Q_x \times M_y} [(u - \kappa^\pm)^\pm \partial_s \kappa^\pm + F^\pm(u, \kappa^\pm) \cdot \nabla_y \kappa^\pm] b'(\kappa^\pm) \varphi\rho_\varepsilon \\
 & \pm \int_{Q_x \times M_y} \mathcal{L}_y^{\geq r}[b(\kappa^\pm)](b(u) - b(\kappa^\pm))^\pm \varphi\rho_\varepsilon,
 \end{aligned}$$

which only differs from (4.7) in that ρ_ε is outside of the operator $\mathcal{L}_x^{< r}$.

We may now let $\varepsilon \rightarrow 0$ and by the regularity of \bar{u}^c , b , H^\pm , and G^\pm we get, through standard estimates,

$$\begin{aligned}
 & - \int_Q [H^\pm(u, \bar{u}^c)\varphi_t + G^\pm(u, \bar{u}^c) \cdot \nabla \varphi] \\
 & \mp \int_M \mathcal{L}^{\geq r}b(u) - b(\bar{u}^c)^\pm \varphi \\
 & - \frac{1}{2} \int_M ((b(u) - b(\bar{u}^c))^\pm)^2 \mathcal{L}^{< r}[\varphi] \\
 & \leq \mp \int_Q [(u - \bar{u}^c)^\pm \bar{u}_t^c + F^\pm(u, \bar{u}^c) \cdot \nabla \bar{u}^c] b'(\bar{u}^c) \varphi \\
 & \pm \int_Q \mathcal{L}^{\geq r}[b(\bar{u}^c)](b(u) - b(\bar{u}^c))^\pm \varphi, \tag{4.8}
 \end{aligned}$$

where \bar{u}^c is here a function in (t, x) , now the only variables of integration.

Set $\varphi(t, x) = \theta(t)\phi(x)$, where $0 \leq \theta \in C_c^\infty((0, T))$ and $0 \leq \phi \in C_c^\infty(\mathbb{R}^d)$, and where $\phi(x) = 1$ whenever $\text{dist}(x, \Omega) \leq r$ so that both $\nabla \phi$ and $\mathcal{L}^{< r}[\phi]$ are zero in Q . Then (4.8) collapses to

$$\begin{aligned}
 & - \int_Q H^\pm(u, \bar{u}^c)\theta' \\
 & \mp \int_M \mathcal{L}^{\geq r}b(u) - b(\bar{u}^c)^\pm \theta \\
 & \leq \mp \int_Q [(u - \bar{u}^c)^\pm \bar{u}_t^c + F^\pm(u, \bar{u}^c) \cdot \nabla \bar{u}^c] b'(\bar{u}^c) \theta \\
 & \pm \int_Q \mathcal{L}^{\geq r}[b(\bar{u}^c)](b(u) - b(\bar{u}^c))^\pm \theta,
 \end{aligned}$$

where we used that $b(u) - b(\bar{u}^c) = 0$ in $M \setminus Q$. Add the (+) case to the (-) case, and we further get

$$\begin{aligned}
 & - \int_Q H(u, \bar{u}^c) \theta' \\
 & \quad + \int_M \mathcal{B}^{\geq r} [b(u) - b(\bar{u}^c), b(u) - b(\bar{u}^c)] \theta \\
 & \leq - \int_Q [(u - \bar{u}^c) \bar{u}_t^c + F(u, \bar{u}^c) \cdot \nabla \bar{u}^c] b'(\bar{u}^c) \theta \\
 & \quad + \int_Q \mathcal{L}^{\geq r} [b(\bar{u}^c)] (b(u) - b(\bar{u}^c)) \theta, \tag{4.9}
 \end{aligned}$$

where $H = H^+ + H^-$, $F = F^+ + F^-$, and we shifted $-\mathcal{L}^{\geq r}$ over to $\mathcal{B}^{\geq r}$ using Proposition 3.1.

Finally, for $\epsilon > 0$ let $\theta = \theta_\epsilon$ be such that $\theta_\epsilon(t) = 1$ for $t \in [\epsilon, T - \epsilon]$ and $\|\theta'_\epsilon\|_{L^1(0, T)} = 2$. Letting $r, \epsilon \rightarrow 0$ we may for the first term in (4.9) use the continuity of u at $t = 0$ (Lemma 2.5) and the regularity of H to conclude that

$$\liminf_{r, \epsilon \rightarrow 0} \left(- \int_Q H(u, \bar{u}^c) \theta'_\epsilon \right) \geq - \int_\Omega H(u_0, \bar{u}^c(0)),$$

where $\bar{u}^c(0) = \bar{u}^c(0, \cdot)$. For the second term, we apply Fatou's lemma to conclude

$$\liminf_{r, \epsilon \rightarrow 0} \int_M \mathcal{B}^{\geq r} [b(u) - b(\bar{u}^c), b(u) - b(\bar{u}^c)] \theta_\epsilon \geq \int_M \mathcal{B} [b(u) - b(\bar{u}^c), b(u) - b(\bar{u}^c)].$$

By Corollary 3.7, the last term in (4.9) converges to its canonical limit, and we conclude as $r, \epsilon \rightarrow 0$,

$$\begin{aligned}
 & - \int_\Omega H(u_0, \bar{u}^c(0)) \\
 & \quad + \int_M \mathcal{B} [b(u) - b(\bar{u}^c), b(u) - b(\bar{u}^c)] \\
 & \leq - \int_Q [(u - \bar{u}^c) \bar{u}_t^c + F(u, \bar{u}^c) \cdot \nabla \bar{u}^c] b'(\bar{u}^c) \\
 & \quad + \int_Q \mathcal{L} [b(\bar{u}^c)] (b(u) - b(\bar{u}^c)).
 \end{aligned}$$

The proposition is proved. ■

4.2. Regularity results near the boundary

In Lemma 4.3 below we present a family of inequalities that follow from Definition 2.2. From this family, we will derive appropriate boundary regularity of an entropy solution, which is vital for the uniqueness argument. The proof is similar to that of Proposition 4.1,

but here we perform coarser computations resulting in a less explicit inequality (4.10) compared to (4.1). This is due to low-regularity terms that are difficult to handle when “going to the diagonal” in the doubling of variables argument. Hence, these terms are replaced by upper bounds; while precise limits can be computed, we have instead chosen simpler and coarser calculations.

Remark 4.2. The coming lemma is analogous to [87, Lemma 4.2], but the two proofs differ in that the main steps are interchanged. In [87] the (local analogue of the) below inequality is proved first for test functions compactly supported inside the domain, done through a doubling of variables argument. Afterwards, the result is extended up to the boundary using a boundary layer sequence satisfying both $\inf_{\delta>0} \inf_{x \in \Omega} \Delta \zeta_\delta(x) \geq 0$ and $\sup_{\delta>0} \|\nabla \zeta_\delta\|_{L^1(\Omega)} < \infty$. Both properties of ζ_δ are crucial for this extension argument, and it is not clear whether an analogous boundary layer sequence always exists in our case (where \mathcal{L} replaces Δ). Thus, we instead prove the below inequality *directly* for test function support up to the boundary. This approach bypasses the need for a special boundary layer sequence, but a new difficulty arises: the admissibility condition (2.5) outside the domain hinders a straightforward doubling of variables argument, and the data \bar{u}^c must be “lifted” as done in the proof of Proposition 4.1. This lifting leads to expressions like $\mathcal{L}[b(\bar{u}^c \pm \varepsilon)]$, which we control by imposing regularity on \bar{u}^c and b separately rather than just on their composition $b(\bar{u}^c)$ as done in [87].

Lemma 4.3. *Assume (A_Ω) – (A_μ) and u is an entropy solution of (1.1). Then for $0 \leq \varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$ and $k \in \mathbb{R}$,*

$$\begin{aligned} & - \int_Q [(u - (\bar{u}^c \vee^\pm k))^\pm \varphi_t + F^\pm(u, \bar{u}^c \vee^\pm k) \cdot \nabla \varphi] \, dx \, dt \\ & \quad + \int_M \mathcal{B}[(b(u) - b(\bar{u}^c \vee^\pm k))^\pm, \varphi] \, dx \, dt \\ & \leq \int_Q [(1 + dL_f)L_{\bar{u}^c} + |\mathcal{L}[b((\bar{u}^c \vee^\pm k))]|] \varphi \, dx \, dt \\ & \quad + \int_\Omega [\|u_0\|_{L^\infty(\Omega)} + \|\bar{u}^c\|_{L^\infty(M)}] \varphi(0, \cdot) \, dx, \end{aligned} \tag{4.10}$$

where $F(u, v) := \text{sgn}(u - v)(f(u) - f(v))$. Moreover, the right-hand side is bounded by $C \|\varphi\|_{L^\infty}$ where C is the finite constant given by

$$\begin{aligned} C := & |Q|(1 + dL_f)L_{\bar{u}^c} + |\Omega|(\|u_0\|_{L^\infty(\Omega)} + \|\bar{u}^c\|_{L^\infty(M)}) \\ & + \left(\sup_k \|\mathcal{L}[b(\bar{u}^c \wedge k)]\|_{L^1(Q)} \right) \vee \left(\sup_k \|\mathcal{L}[b(\bar{u}^c \vee k)]\|_{L^1(Q)} \right). \end{aligned} \tag{4.11}$$

Remark 4.4. The nonlocal terms in (4.10) and (4.11) are well posed, but this is not obvious. The terms involving \mathcal{L} will be discussed in the proof below, so we now discuss why $g_k^\pm(t, x) := (b(u) - b(\bar{u}^c \vee^\pm k))^\pm$ has compact support and finite energy (making the \mathcal{B} -pairing well defined by Proposition 3.8 (ii)). Compact support follows since

$0 \leq g_k^\pm \leq |b(u) - b(\bar{u}^c)|$, where the right-hand side is supported on Q . As for the energy, we can bound its nonsingular part using the previous bound and Proposition 3.8 (iv):

$$\int_M \mathcal{B}^{\geq 1}[g_k^\pm, g_k^\pm] dx dt \leq 4|Q|\mu(\{|z| \geq r\})\|b(u) - b(\bar{u}^c)\|_{L^\infty(M)}.$$

For the singular part we can, since g_k^\pm is supported in Q , write $\int_M \mathcal{B}^{< 1}[g_k^\pm, g_k^\pm] dx dt = \int_U \mathcal{B}^{< 1}[g_k^\pm, g_k^\pm] dx dt$ for $U = \{(t, x) \in M: \text{dist}(x, \Omega) < 1\}$. Next, $b(\bar{u}^c \vee^\pm k) = b(\bar{u}^c) \vee^\pm b(k)$ by monotonicity of b , and so

$$g_k^\pm = ((b(u) - b(\bar{u}^c)) \mp (b(k) - b(\bar{u}^c)))^\pm.$$

And since $\mathcal{B}[\eta(\phi), \eta(\phi)] \leq \mathcal{B}[\phi, \phi]$ for $|\eta'| < 1$, and $\mathcal{B}[\phi + \psi, \phi + \psi] \leq 2\mathcal{B}[\phi, \phi] + 2\mathcal{B}[\psi, \psi]$, we get from the previous identity that

$$\begin{aligned} \int_M \mathcal{B}^{< 1}[g_k^\pm, g_k^\pm] dx dt &\leq 2 \int_U \mathcal{B}^{< 1}[b(u) - b(\bar{u}^c), b(u) - b(\bar{u}^c)] dx dt \\ &\quad + 2 \int_U \mathcal{B}^{< 1}[b(\bar{u}^c), b(\bar{u}^c)] dx dt, \end{aligned}$$

where the right-hand side is finite by Proposition 4.1 and since $b(\bar{u}^c)$ is bounded and locally Lipschitz.

Proof of Lemma 4.3. We start from the definition of entropy solutions and the entropy inequality (2.6). Seeking to replace k by $\bar{u}^c \vee^\pm k$, we double the variables; the first part of this proof is similar to that of Proposition 4.1. In particular, we also here carry out the proofs of the (+) and (-) cases simultaneously.

For parameters $(s, y) \in M$, $k \in \mathbb{R}$, and $\varepsilon \in (0, 1)$, we introduce the pair $(\kappa^\pm, \psi) \in \mathbb{R} \times C_c^\infty(M)$ defined as

$$\begin{aligned} \kappa^\pm &= \kappa_{s,y,\varepsilon,k}^\pm = (\bar{u}^c(s, y) \pm \varepsilon L_{\bar{u}^c}) \vee^\pm k, \\ \psi &= \psi_{s,y,\varepsilon}(t, x) = \varphi(t, x) \rho_\varepsilon(t - s, x - y), \end{aligned} \tag{4.12}$$

where $0 \leq \varphi \in C_c^\infty(M)$ is arbitrary, ρ_ε is a standard mollifier with support in the centred ball of radius ε in $\mathbb{R} \times \mathbb{R}^d$, and $L_{\bar{u}^c}$ is a Lipschitz constant for \bar{u}^c valid on $\{(t, x) \in M: \text{dist}((t, x), \text{supp } \bar{\varphi}) < 1\}$. The support of ρ_ε implies that ψ is zero when $|(t - s, x - y)| \geq \varepsilon$, while $|(t - s, x - y)| < \varepsilon$ implies that $\pm \kappa^\pm \geq \pm \bar{u}^c(t, x)$. In particular, we have

$$(b(\bar{u}^c(t, x)) - b(\kappa^\pm))^\pm \psi(t, x) = 0, \tag{4.13}$$

for all t, s, x, y, ε . Thus, the pair (κ^\pm, ψ) qualifies for use in the entropy inequality (2.6) as the constant and test function (which are denoted k, φ in said inequality, and should not to be confused with the same symbols used here). Inserting this pair in (2.6), we get after

integrating over $(s, y) \in M$,

$$\begin{aligned}
 & - \int_{Q_x \times M_y} [(u - \kappa^\pm)^\pm \varphi_t + F^\pm(u, \kappa^\pm) \cdot \nabla_x \varphi] \rho_\varepsilon \\
 & \quad - \int_{Q_x \times M_y} \mathcal{L}_x^{\geq r} [b(u)] \operatorname{sgn}(u - \kappa^\pm)^\pm \varphi \rho_\varepsilon \\
 & \quad - \int_{M_x \times M_y} (b(u) - b(\kappa^\pm))^\pm \mathcal{L}_x^{< r} [\varphi \rho_\varepsilon] \\
 & \leq - \int_{Q_x \times M_y} [(u - \kappa^\pm)^\pm \partial_s \rho_\varepsilon + F^\pm(u, \kappa^\pm) \cdot \nabla_y \rho_\varepsilon] \varphi,
 \end{aligned}$$

where we use the sub-index notation from Section 2. We removed both the “initial term” and the “boundary term” from the right-hand-side of (2.6) as here they are both zero, the former because $\varphi = 0$ for $t = 0$, and the latter because of a similar argument that gave (4.13). We also used the two identities $\partial_t \rho_\varepsilon = -\partial_s \rho_\varepsilon$ and $\nabla_x \rho_\varepsilon = -\nabla_y \rho_\varepsilon$.

Integrating by parts in (s, y) on the right-hand side of (4.6), taking ε so small that the boundary term vanishes, and adding the integral of $\mathcal{L}_y^{\geq r} [b(\kappa^\pm)](b(u) - b(\kappa^\pm))^\pm \varphi \rho_\varepsilon$ over $Q_x \times M_y$ to each side, we further get

$$\begin{aligned}
 & - \int_{Q_x \times M_y} [(u - \kappa^\pm)^\pm \varphi_t + F^\pm(u, \kappa^\pm) \cdot \nabla_x \varphi] \rho_\varepsilon \\
 & \quad - \int_{Q_x \times M_y} \mathcal{L}_{x+y}^{\geq r} [b(u) - b(\kappa^\pm)] \operatorname{sgn}(u - \kappa^\pm)^\pm \varphi \rho_\varepsilon \\
 & \quad - \int_{M_x \times M_y} (b(u) - b(\kappa^\pm))^\pm \mathcal{L}_x^{< r} [\varphi \rho_\varepsilon] \\
 & \leq \int_{Q_x \times M_y} [\partial_s((u - \kappa^\pm)^\pm) + \operatorname{div}_y(F^\pm(u, \kappa^\pm))] \varphi \rho_\varepsilon \\
 & \quad + \int_{Q_x \times M_y} \mathcal{L}_y^{\geq r} [b(\kappa^\pm)] \operatorname{sgn}(u - \kappa^\pm)^\pm \varphi \rho_\varepsilon. \tag{4.14}
 \end{aligned}$$

With \mathcal{L}_{x+y} as defined in (2.3), we here used that $\mathcal{L}_{x+y}^{\geq r} [w_1 - w_2] = \mathcal{L}_x^{\geq r} [w_1] - \mathcal{L}_y^{\geq r} [w_2]$ when w_1 and w_2 are functions in x and y respectively, and we also integrated by parts in (s, y) on the right-hand side. By Lipschitz continuity, it is clear that the weak derivatives on the right-hand side of (4.14) satisfy

$$|\partial_s((u - \kappa^\pm)^\pm)| + |\operatorname{div}_y(F^\pm(u, \kappa^\pm))| \leq L_{\bar{u}^c} + dL_f L_{\bar{u}^c},$$

since $L_{\bar{u}^c}$ is a Lipschitz constant for κ on Q_y .

Observe next that the domain of integration for the second integral on the left-hand side of (4.14) may be extended from $Q_x \times M_y$ to $M_x \times M_y$ as $\operatorname{sgn}^\pm(u - \kappa)^\pm \rho_\varepsilon = 0$ on

$Q_x^c \times M_y$ (by the same argument used for (4.13)). Doing so, we may further infer

$$\begin{aligned} \int_{M_x \times M_y} \mathcal{L}_{x+y}^{\geq r} [b(u) - b(\kappa^\pm)] \operatorname{sgn}(u - \kappa^\pm)^\pm \varphi \rho_\varepsilon &\leq \int_{M_x \times M_y} \mathcal{L}_{x+y}^{\geq r} [(b(u) - b(\kappa^\pm))^\pm] \varphi \rho_\varepsilon \\ &= \int_{M_x \times M_y} (b(u) - b(\kappa^\pm))^\pm \mathcal{L}_x^{\geq r} [\varphi] \rho_\varepsilon, \end{aligned}$$

where we used the second inequality of Corollary C.2, the self-adjointness of $\mathcal{L}_{x+y}^{\geq r}$ (Proposition 3.1), and the identity $\mathcal{L}_{x+y}^{\geq r} [\varphi \rho_\varepsilon] = \rho_\varepsilon \mathcal{L}_x^{\geq r} [\varphi]$. Exploiting these bounds in (4.14), we find that

$$\begin{aligned} & - \int_{Q_x \times M_y} [(u - \kappa^\pm)^\pm \varphi_t + F^\pm(u, \kappa^\pm) \cdot \nabla_x \varphi] \rho_\varepsilon \\ & \quad - \int_{M_x \times M_y} (b(u) - b(\kappa^\pm))^\pm \mathcal{L}_x^{\geq r} [\varphi] \rho_\varepsilon \\ & \quad - \int_{Q_x \times M_y} (b(u) - b(\kappa^\pm))^\pm \mathcal{L}_x^{< r} [\varphi \rho_\varepsilon] \\ & \leq \int_{Q_x \times M_y} [(1 + dL_f)L_{\bar{u}^c} + |\mathcal{L}_y^{\geq r}[b(\kappa^\pm)]|] \varphi \rho_\varepsilon. \end{aligned} \quad (4.15)$$

Next we let $r \rightarrow 0$. The limits on the left-hand side of (4.15) are straightforward by Lemma 3.4 and so we turn our attention to the right-hand side. It is easily verified that the function $h_{k,\varepsilon}^\pm(\cdot) := b((\cdot \pm \varepsilon L_{\bar{u}^c}) \vee^\pm k)$ satisfies

$$\|(h_{k,\varepsilon}^\pm)'\|_{L^\infty(\mathbb{R})} \leq \|b'\|_{L^\infty(\mathbb{R})} \quad \text{and} \quad |(h_{k,\varepsilon}^\pm)'|_{\text{TV}(\mathbb{R})} \leq |b'|_{\text{TV}(\mathbb{R})} + \|b'\|_{L^\infty(\mathbb{R})} \quad (4.16)$$

(the global regularity of b is due to Remark 2.1), and that the definition (4.12) of κ^\pm implies that

$$h_{k,\varepsilon}^\pm(\bar{u}^c) = b(\kappa^\pm). \quad (4.17)$$

By Corollary 3.7 we then get $\lim_{r \rightarrow 0} \mathcal{L}_y^{\geq r} [b(\kappa^\pm)] = \mathcal{L}_y [b(\kappa^\pm)]$ in $L^1_{\text{loc}}(M_y)$. Using this in (4.15) yields

$$\begin{aligned} & - \int_{Q_x \times M_y} [(u - \kappa^\pm)^\pm \varphi_t + F^\pm(u, \kappa^\pm) \cdot \nabla_x \varphi] \rho_\varepsilon \\ & \quad - \int_{M_x \times M_y} (b(u) - b(\kappa^\pm))^\pm \mathcal{L}_x [\varphi] \rho_\varepsilon \\ & \leq \int_{Q_x \times M_y} [(1 + dL_f)L_{\bar{u}^c} + |\mathcal{L}_y [b(\kappa^\pm)]|] \varphi \rho_\varepsilon. \end{aligned} \quad (4.18)$$

Finally, we wish to send $\varepsilon \rightarrow 0$. The left-hand side can be dealt with using classical arguments (cf. e.g. [41]) and so we again focus on the right-hand side. We will first establish the limit

$$\lim_{r \rightarrow 0} \sup_{\varepsilon \in (0,1)} \int_{Q_x \times M_y} |\mathcal{L}_y^{< r} [b(\kappa^\pm)]| \varphi \rho_\varepsilon = 0. \quad (4.19)$$

Letting U_y denote the set of points in M_y such that $\text{dist}(y, \Omega) \leq 1$, we see that the support of $(s, y) \mapsto \rho_\varepsilon(t - s, x - y)$ lies in U_y for all $(t, x) \in Q_x$ and $\varepsilon \in (0, 1)$. Thus, for $\varepsilon \in (0, 1)$,

$$\int_{Q_x \times M_y} |\mathcal{L}_y^{<r}[b(\kappa^\pm)]| \varphi \rho_\varepsilon \leq \|\varphi\|_{L^\infty(M)} \int_{U_y} |\mathcal{L}_y^{<r}[h_{k,\varepsilon}^\pm(\bar{u}^c)]|,$$

where we use the notation from (4.17). By the explicit bound (3.7) of Corollary 3.7, the uniform bounds (4.16), and assumption (A_{u^c}) , there is a finite constant $C' > 0$ only depending on U_y and \bar{u}^c such that

$$\|\mathcal{L}_y^{<r}[h_{k,\varepsilon}^\pm(\bar{u}^c)]\|_{L^1(U_y)} \leq C'(\|b'\|_{L^\infty(\mathbb{R})} + |b'|_{\text{TV}(\mathbb{R})}) \int_{|z|<r} (|z|^2 \wedge 1) d\mu(z).$$

This term converges to 0 as $r \rightarrow 0$ by the dominated convergence theorem and (A_μ) .

Thus we attain (4.19), and exploiting it, we may compute

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{Q_x \times M_y} |\mathcal{L}_y[b(\kappa^\pm)]| \varphi \rho_\varepsilon &= \lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_{Q_x \times M_y} |\mathcal{L}_y^{\geq r}[b(\kappa^\pm)]| \varphi \rho_\varepsilon \\ &= \int_Q |\mathcal{L}[b(\bar{u}^c \vee^\pm k)]| \varphi \, dx \, dt, \end{aligned}$$

where \bar{u}^c on the rightmost side is a function in (t, x) . Here we approximated \mathcal{L}_y by $\mathcal{L}_y^{\geq r}$ uniformly in ε by (4.19), used that $\mathcal{L}_y^{\geq r}$ is a zero-order operator to pass to the limit in ε , and then concluded by the limit definition of \mathcal{L} . Sending $\varepsilon \rightarrow 0$ in (4.18), we then find that

$$\begin{aligned} & - \int_Q [(u - (\bar{u}^c \vee^\pm k))^\pm \varphi_t + F^\pm(u, (\bar{u}^c \vee^\pm k)) \cdot \nabla \varphi] \, dx \, dt \\ & \quad - \int_M (b(u) - b((\bar{u}^c \vee^\pm k)))^\pm \mathcal{L}[\varphi] \, dx \, dt \\ & \leq \int_Q [(1 + dL_f)L_{\bar{u}^c} + |\mathcal{L}[b((\bar{u}^c \vee^\pm k))]|] \varphi \, dx \, dt, \end{aligned} \quad (4.20)$$

where \bar{u}^c is now a function in (t, x) . Finally, we want to extend the support of φ to include $t = 0$ and $t = T$ as we have so far assumed it to have compact support in $M = (0, T) \times \mathbb{R}^d$. In (4.20) we make the substitution $\varphi \mapsto \theta_\varepsilon \varphi$, where $(\theta_\varepsilon)_{\varepsilon > 0} \subset C_c^\infty((0, T))$ is a family of nonnegative functions satisfying $\lim_{\varepsilon \rightarrow 0} \theta_\varepsilon(t) = 1$ for $t \in (0, T)$ and $\|\theta'_\varepsilon\|_{L^1((0, T))} = 2$ for all ε , while the new φ is like the previous one. Exploiting the time continuity of u at $t = 0$, we get the inequality

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_Q (u - (\bar{u}^c \vee^\pm k))^\pm \varphi \theta'_\varepsilon \, dx \, dt &\leq \int_\Omega (u_0 - (\bar{u}^c(0, \cdot) \vee^\pm k))^\pm \varphi(0, \cdot) \, dx \\ &\leq \int_\Omega (\|u_0\|_{L^\infty(\Omega)} + \|\bar{u}^c\|_{L^\infty(M)}) \varphi(0, \cdot) \, dx. \end{aligned}$$

Thus, replacing φ by $\theta_\varepsilon \varphi$ in (4.20) and letting $\varepsilon \rightarrow 0$ we get (4.10) after shifting the operator \mathcal{L} to \mathcal{B} ; this shift is justified by Propositions 3.1 and 3.8 and Remark 4.4.

Finally, that the nonlocal terms in (4.11) are finite (and well defined) follows again by Corollary 3.7 applied to $h_{k,0}^\pm(\bar{u}^c)$, where $h_{k,0}^\pm$ is as in (4.16). ■

Corollary 4.5. *Under the assumptions of Lemma 4.3, there exists a finite signed Borel measure ν_k^\pm on $[0, T] \times \mathbb{R}^d$, which integrates to zero⁹ and is nonpositive on $[0, T] \times (\mathbb{R}^d \setminus \Omega)$, and such that*

$$\begin{aligned} \int_{[0,T] \times \mathbb{R}^d} \varphi \, d\nu_k^\pm &= - \int_{\mathcal{Q}} [(u - (\bar{u}^c \vee^\pm k))^\pm \varphi_t + F^\pm(u, \bar{u}^c \vee^\pm k) \cdot \nabla \varphi] \, dx \, dt \\ &\quad + \int_M \mathcal{B}[(b(u) - b(\bar{u}^c \vee^\pm k))^\pm, \varphi] \, dx \, dt, \end{aligned} \quad (4.21)$$

for any $\varphi \in C_b^\infty([0, T] \times \mathbb{R}^d)$. Moreover, the total variation norm of ν_k^\pm admits the bound $\|\nu_k^\pm\| \leq 2C$, where C is as in (4.11).

Proof. For brevity, let $T_k^\pm: C_b^\infty([0, T] \times \mathbb{R}^d) \rightarrow \mathbb{R}$ denote the operator such that $T_k^\pm(\varphi)$ coincides with the right-hand side of (4.21) or, equivalently, the left-hand side of (4.10). Pick $0 \leq \psi \in C_c^\infty(\mathbb{R}^d)$ such that $\psi = 1$ for $|x| \leq 1$ and set $\psi_n(x) = \psi(x/n)$. Note that $\lim_{n \rightarrow \infty} T_k^\pm(\psi_n) = 0$. For an arbitrary $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$ satisfying $|\varphi| \leq 1$, we can exploit the linearity of T_k^\pm to compute

$$T_k^\pm(\varphi) = \lim_{n \rightarrow \infty} T_k^\pm(\varphi + \psi_n) \leq 2C.$$

The inequality follows from applying (4.10) (possible as $\varphi + \psi_n$ is nonnegative for a sufficiently large n) followed by using $|\varphi + \psi_n| \leq 2$. The linearity of T_k^\pm then implies $|T_k^\pm(\varphi)| \leq 2C \|\varphi\|_{L^\infty}$ for any $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$ and so by the Riesz representation theorem (see e.g. [98]) there exists a signed Borel measure ν_k^\pm , of total variation norm $\|\nu_k^\pm\| \leq 2C$, such that $T_k^\pm(\varphi) = \int_{[0,T] \times \mathbb{R}^d} \varphi \, d\nu_k^\pm$ for all $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$. That this relation can be extended to $\varphi \in C_b^\infty([0, T] \times \mathbb{R}^d)$ follows by inserting $\varphi \psi_n$, with ψ_n as before, and letting $n \rightarrow \infty$; each side converges to the canonical limit.

As it is clear that ν_k^\pm integrates to zero, it remains to prove that it is nonpositive on $[0, T] \times (\mathbb{R}^d \setminus \Omega)$. This can be seen as follows: Let ζ_δ denote a boundary layer sequence and observe that, for any $0 \leq \varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$, we have

$$\int_{[0,T] \times (\mathbb{R}^d \setminus \Omega)} \varphi \, d\nu_k^\pm = \lim_{\delta \rightarrow 0} \int_{[0,T] \times \mathbb{R}^d} (1 - \zeta_\delta) \varphi \, d\nu_k^\pm = \lim_{\delta \rightarrow 0} T_k^\pm((1 - \zeta_\delta) \varphi) \leq 0,$$

where we used (4.10). ■

We now demonstrate three forms of boundary regularity which are consequences of the general identity (4.21). They will be referred to as *boundary integrability*, *weak trace*, and *boundary condition*. The two latter regularity properties have local analogues; in [87] they are referred to as “weak normal trace” and “boundary condition” respectively. The former however, boundary integrability, has no local analogue in either [87] or [85], and is in this sense a truly nonlocal feature of the equation.

⁹ $\int_{[0,T] \times \mathbb{R}^d} d\nu_k^\pm = \nu_k^\pm([0, T] \times \mathbb{R}^d) = 0.$

4.2.1. Boundary integrability. Consider (4.21) when $k = -\|\bar{u}^c\|_{L^\infty(M)}$ in the (+) case and $k = \|\bar{u}^c\|_{L^\infty(M)}$ in the (-) case; if we add the two resulting identities and let ν denote the sum of the corresponding measures, we get

$$\begin{aligned} \int_{[0,T] \times \mathbb{R}^d} \varphi \, d\nu &= - \int_Q [|u - \bar{u}^c| \varphi_t + F(u, \bar{u}^c) \cdot \nabla \varphi] \, dx \, dt \\ &\quad + \int_M \mathcal{B}[|b(u) - b(\bar{u}^c)|, \varphi] \, dx \, dt, \end{aligned} \quad (4.22)$$

for all $\varphi \in C_b^\infty([0, T] \times \mathbb{R}^d)$. Here $F := F^+ + F^-$ so that

$$F(u, \bar{u}^c) = \operatorname{sgn}(u - \bar{u}^c)(f(u) - f(\bar{u}^c)),$$

and ν is a finite signed Borel measure which integrates to zero and whose total variation norm satisfies $\|\nu\| \leq 4C$ with C as in (4.11). This identity puts a restriction on how much $b(u)$ can differ from $b(\bar{u}^c)$ close to Γ , as shown in the following proposition.

Proposition 4.6 (Boundary integrability). *Assume (A_Ω) – (A_μ) and u is an entropy solution of (1.1). Then*

$$\int_0^T \int_{\substack{x \in \Omega \\ x+z \in \Omega^c}} |b(u(t, x)) - b(\bar{u}^c(t, x))| \, d\mu(z) \, dx \, dt \leq C, \quad (4.23)$$

where C is finite and given by (4.11).

Proof. Let $(\bar{\zeta}_\delta)_{\delta>0}$ be an outer boundary layer sequence (cf. (2.8)). Replacing φ by $\bar{\zeta}_\delta$ in (4.22), we get

$$\int_M \mathcal{B}[|b(u) - b(\bar{u}^c)|, \bar{\zeta}_\delta] \, dx \, dt = \int_{[0,T] \times \mathbb{R}^d} \bar{\zeta}_\delta \, d\nu \leq 2C, \quad (4.24)$$

where the inequality follows as the positive part of ν is bounded by $2C$. For brevity, we now set $g(t, x) := |b(u(t, x)) - b(\bar{u}^c(t, x))|$ for $(t, x) \in M$. Writing out the \mathcal{B} term, we find

$$\begin{aligned} &\int_M \mathcal{B}[g, \bar{\zeta}_\delta] \, dx \, dt \\ &= \int_M \int_{\mathbb{R}^d} (g(t, x+z) - g(t, x))(\bar{\zeta}_\delta(x+z) - \bar{\zeta}_\delta(x)) \, d\mu(z) \, dx \, dt. \end{aligned} \quad (4.25)$$

As g has finite energy (it is dominated by that of $b(u) - b(\bar{u}^c)$) and $\bar{\zeta}_\delta \in C_c^\infty(\mathbb{R}^d)$, the integrand on the right-hand side is absolutely integrable, and so we need not worry about the order of integration. Exploiting that $\bar{\zeta}_\delta = 1$ in $\bar{\Omega}$ and that $g(t, x) = 0$ for $x \in \Omega^c$, the integral in (4.25) reduces to

$$\begin{aligned} &\int_0^T \int_{\substack{x \in \Omega \\ x+z \in \Omega^c}} g(t, x)(1 - \bar{\zeta}_\delta(x+z)) \, d\mu(z) \, dx \, dt \\ &\quad + \int_0^T \int_{\substack{x+z \in \Omega \\ x \in \Omega^c}} g(t, x+z)(1 - \bar{\zeta}_\delta(x)) \, d\mu(z) \, dx \, dt \end{aligned}$$

$$\begin{aligned}
 &= 2 \int_0^T \int_{\substack{x \in \Omega \\ x+z \in \Omega^c}} g(t, x) (1 - \bar{\zeta}_\delta(x+z)) \, d\mu(z) \, dx \, dt \\
 &\rightarrow 2 \int_0^T \int_{\substack{x \in \Omega \\ x+z \in \Omega^c}} g(t, x) (1 - \chi_{\bar{\Omega}}(x+z)) \, d\mu(z) \, dx \, dt,
 \end{aligned}$$

as $\delta \rightarrow 0$: the equality holds as, in the second integral, we can perform the change of variables $x \mapsto x - z$ followed by $z \mapsto -z$, and the limit is valid by monotonicity and Fubini's theorem. Inserting this limit on the left-hand side of (4.24), we get (4.23) save for the weight $1 - \chi_{\bar{\Omega}}(x+z)$ in the integrand. On the domain of integration, $x \in \Omega$ and $x+z \in \Omega^c$, this weight coincides with $1 - \chi_{\partial\Omega}(x+z)$. By (A $_\Omega$), we infer that $\chi_{\partial\Omega}(x) = 0$ for a.e. $x \in \mathbb{R}^d$, and so $x \mapsto \int_{\mathbb{R}^d} \chi_{\partial\Omega}(x+z) \, d\mu(z)$ is zero for a.e. $x \in \mathbb{R}^d$; see Remark 2.3 (b) (while μ is not finite here, we can use finite approximations $\mu_n(E) := \mu(E \setminus \{0 < |z| < n^{-1}\})$ to get the same conclusion by monotonicity). That is, the weight $(1 - \chi_{\bar{\Omega}}(x+z))$ can safely be removed from the integral without changing its value. ■

A corollary of the previous proposition is the following \mathcal{B} -pairing property of $b(u) - b(\bar{u}^c)$.

Corollary 4.7. *Under the assumptions of Proposition 4.6, the integrand of $\mathcal{B}[b(u) - b(\bar{u}^c), \varphi\chi_\Omega]$ is absolutely integrable with respect to $d\mu(z) \, dx \, dt$ for all $\varphi \in C^\infty([0, T] \times \mathbb{R}^d)$.*

Proof. Similarly to the previous proof, we set $g(t, x) := b(u(t, x)) - b(\bar{u}^c(t, x))$ for brevity, and compute

$$\begin{aligned}
 &\int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^d} |g(t, x+z) - g(t, x)| |(\varphi\chi_\Omega)(t, x+z) - (\varphi\chi_\Omega)(t, x)| \, d\mu(z) \, dx \, dt \\
 &= \int_0^T \int_{\substack{x \in \Omega \\ x+z \in \Omega}} |g(t, x+z) - g(t, x)| |\varphi(t, x+z) - \varphi(t, x)| \, d\mu(z) \, dx \, dt \\
 &\quad + \int_0^T \int_{\substack{x \in \Omega \\ x+z \in \Omega^c}} |g(t, x)| |\varphi(t, x)| \, d\mu(z) \, dx \, dt \\
 &\quad + \int_0^T \int_{\substack{x \in \Omega^c \\ x+z \in \Omega}} |g(t, x+z)| |\varphi(t, x+z)| \, d\mu(z) \, dx \, dt,
 \end{aligned}$$

where we used that both g and $\varphi\chi_\Omega$ are supported in \mathcal{Q} . The first integral on the right-hand side is finite as g is of finite energy (Proposition 4.1) and φ is smooth, while the two latter integrals (which by a change of variables can be seen to coincide) are finite due to Proposition 4.6. ■

4.2.2. Weak trace. We next establish a weak trace identity resulting from (4.22).

Proposition 4.8 (Weak trace). *Assume (A_Ω) – (A_μ) and u is an entropy solution of (1.1). Let ν be the finite Borel measure from (4.22). Then, for any boundary layer sequence $(\zeta_\delta)_{\delta>0}$, $r > 0$, and $\varphi \in C_b^\infty([0, T] \times \mathbb{R}^d)$, we have*

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \left(\int_Q F(u, \bar{u}^c) \cdot (\nabla \zeta_\delta) \varphi \, dx \, dt - \int_M \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \zeta_\delta] \varphi \, dx \, dt \right) \\ &= \int_{\bar{\Gamma}} (1 - \zeta) \varphi \, d\nu - \int_M \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \chi_\Omega] \varphi \, dx \, dt, \end{aligned} \quad (4.26)$$

where $\bar{\Gamma} = [0, T] \times \partial\Omega$, $F(u, \bar{u}^c) = \text{sgn}(u - \bar{u}^c)(f(u) - f(\bar{u}^c))$, and ζ is the limit of ζ_δ on $\partial\Omega$. The right-hand side of (4.26) side is nonpositive, and the expression inside the lim on the left-hand side is bounded by $\tilde{C} \|\varphi\|_{C^1(M)}$ for a $\tilde{C} > 0$ independent of δ, φ .

Proof of Proposition 4.8. Choosing $(1 - \zeta_\delta)\varphi$ as the test function in (4.22), we get after a little rewriting,

$$\begin{aligned} & \int_Q F(u, \bar{u}^c) \cdot (\nabla \zeta_\delta) \varphi \, dx \, dt - \int_M \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \zeta_\delta] \varphi \, dx \, dt \\ &= \int_{[0, T] \times \mathbb{R}^d} (1 - \zeta_\delta) \varphi \, d\nu + \int_Q [|u - \bar{u}^c| \varphi_t + F(u, \bar{u}^c) \cdot \nabla \varphi] (1 - \zeta_\delta) \, dx \, dt \\ &\quad - \int_M \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \varphi] (1 - \zeta_\delta) \, dx \, dt \\ &\quad - \int_M \mathcal{B}^{\geq r} [|b(u) - b(\bar{u}^c)|, (1 - \zeta_\delta) \varphi] \, dx \, dt, \end{aligned} \quad (4.27)$$

where we decomposed $\mathcal{B} = \mathcal{B}^{\geq r} + \mathcal{B}^{<r}$ for some fixed $r > 0$ and used the product rule for $\mathcal{B}^{<r}$ (Proposition 3.3). Note that the very last part of the proposition follows from (4.27), the uniform bound $|\zeta_\delta| \leq 1$, the total variation bound $\|\nu\| \leq 4C$ with C as in (4.11), the finite energy, and the compact support of $|b(u) - b(\bar{u}^c)|$; the integrals involving $\mathcal{B}^{<r}$ and $\mathcal{B}^{\geq r}$ on the right-hand side are bounded by (ii) and (iv) from Proposition 3.8.

Letting $\delta \rightarrow 0$ in (4.27) we obtain

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \left(\int_Q F(u, \bar{u}^c) \cdot (\nabla \zeta_\delta) \varphi \, dx \, dt - \int_M \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \zeta_\delta] \varphi \, dx \, dt \right) \\ &= \int_{\bar{\Gamma}} (1 - \zeta) \varphi \, d\nu + \int_{[0, T] \times (\mathbb{R}^d \setminus \bar{\Omega})} \varphi \, d\nu - \int_M \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \varphi] (1 - \chi_\Omega) \, dx \, dt \\ &\quad - \int_M \mathcal{B}^{\geq r} [|b(u) - b(\bar{u}^c)|, (1 - \chi_\Omega) \varphi] \, dx \, dt. \end{aligned} \quad (4.28)$$

It remains to show that the right-hand side of (4.28) coincides with that of (4.26). Consider the special case of an outer boundary layer sequence $(\tilde{\zeta}_\delta)_{\delta>0}$ (cf. (2.8)). Using that $\tilde{\zeta}_\delta = 1$

on $\bar{\Omega}$, (4.28) reads

$$\begin{aligned}
 & - \lim_{\delta \rightarrow 0} \int_M \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \bar{\xi}_\delta] \varphi \, dx \, dt \\
 & = \int_{[0, T] \times (\mathbb{R}^d \setminus \bar{\Omega})} \varphi \, dv - \int_M \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \varphi] (1 - \chi_\Omega) \, dx \, dt \\
 & \quad - \int_M \mathcal{B}^{\geq r} [|b(u) - b(\bar{u}^c)|, (1 - \chi_\Omega) \varphi] \, dx \, dt.
 \end{aligned} \tag{4.29}$$

But in the case of an outer boundary layer sequence, the limit on the left-hand side of (4.29) may be computed explicitly, giving

$$\begin{aligned}
 & - \lim_{\delta \rightarrow 0} \int_M \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \bar{\xi}_\delta] \varphi \, dx \, dt \\
 & = - \int_M \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \chi_\Omega] \varphi \, dx \, dt.
 \end{aligned} \tag{4.30}$$

For brevity we skip this computation as it is almost identical to the proof of Proposition 4.6. In conclusion, the three last terms on the right-hand side of (4.28) may be replaced by the right-hand side of (4.30) and so we attain (4.26). Finally, that the right-hand side of (4.26) is nonpositive is a consequence of Corollary 4.5 and that $\mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \chi_\Omega] \geq 0$ (as can be seen by writing it out). ■

4.2.3. Boundary condition. Recall that $F = F^+ + F^-$, and define the quantities

$$\begin{aligned}
 \mathcal{F} & = \mathcal{F}(u, \bar{u}^c, k) := F(u, \bar{u}^c) + F(u, k) - F(\bar{u}^c, k), \\
 \Sigma & = \Sigma(u, \bar{u}^c, k) := |b(u) - b(\bar{u}^c)| + |b(u) - b(k)| - |b(\bar{u}^c) - b(k)|.
 \end{aligned} \tag{4.31}$$

As was observed in [87], these quantities may alternatively be written as

$$\begin{aligned}
 \mathcal{F}/2 & = F^+(u, \bar{u}^c \vee^+ k) + F^-(u, \bar{u}^c \vee^- k), \\
 \Sigma/2 & = (b(u) - b(\bar{u}^c \vee^+ k))^+ + (b(u) - b(\bar{u}^c \vee^- k))^- ,
 \end{aligned} \tag{4.32}$$

which is verified by considering the three cases $u < \bar{u}^c \vee^- k$, $u > \bar{u}^c \vee^+ k$, and $u \in [\bar{u}^c \vee^- k, \bar{u}^c \vee^+ k]$. In particular, we infer from (4.32) that $|\mathcal{F}| \leq 2L_f |u - \bar{u}^c|$ and $0 \leq \Sigma \leq 2|b(u) - b(\bar{u}^c)|$, and that Σ has finite energy by Remark 4.4.

Proposition 4.9 (Boundary condition). *Assume (A_Ω) – (A_μ) and u is an entropy solution of (1.1). Then, for any inner boundary layer sequence $(\xi_\delta)_{\delta > 0}$, $k \in \mathbb{R}$, $r > 0$, and $0 \leq \varphi \in C_b^\infty([0, T] \times \mathbb{R}^d)$,*

$$\lim_{\delta \rightarrow 0} \left(\int_Q (\mathcal{F} \cdot \nabla \xi_\delta) \varphi \, dx \, dt - \int_M \mathcal{B}^{<r} [\Sigma, \xi_\delta] \varphi \, dx \, dt \right) \leq 0. \tag{4.33}$$

Moreover, the expression inside the lim is bounded by $\tilde{C} \|\varphi\|_{C^1(M)}$ for a $\tilde{C} > 0$ independent of δ , k , φ .

Proof. For a fixed k , the (+) and (−) cases of (4.21) result in two identities, each featuring their own measure ν_k^+ and ν_k^- . Setting $\nu_k := \nu_k^+ + \nu_k^-$, and adding these two identities together, we get for any $0 \leq \varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$,

$$\begin{aligned} \int_{[0, T] \times \mathbb{R}^d} \varphi \, d\nu_k &= - \int_Q [(u - (\bar{u}^c \vee^+ k))^+ + (u - (\bar{u}^c \vee^- k))^-] \varphi_t \, dx \, dt \\ &\quad - \frac{1}{2} \int_Q \mathcal{F} \cdot \nabla \varphi \, dx \, dt + \frac{1}{2} \int_M \mathcal{B}[\Sigma, \varphi] \, dx \, dt, \end{aligned} \quad (4.34)$$

where we used the formula (4.32). Next we substitute $\varphi \mapsto (\bar{\zeta}_\delta - \zeta_\delta)\varphi$ in (4.34), where the new φ is as before, while ζ_δ and $\bar{\zeta}_\delta$ represent an inner and outer boundary layer sequence respectively (cf. (2.9) and (2.8)). By the product rules for ∇ and \mathcal{B} we find

$$\begin{aligned} - \int_Q \mathcal{F} \cdot \nabla((\bar{\zeta}_\delta - \zeta_\delta)\varphi) \, dx \, dt &= \int_Q (\mathcal{F} \cdot \nabla \zeta_\delta)\varphi \, dx \, dt - \int_Q (\mathcal{F} \cdot \nabla \varphi)(1 - \zeta_\delta) \, dx \, dt, \\ \int_M \mathcal{B}[\Sigma, (\bar{\zeta}_\delta - \zeta_\delta)\varphi] \, dx \, dt &= \int_M \mathcal{B}[\Sigma, \bar{\zeta}_\delta]\varphi \, dx \, dt - \int_M \mathcal{B}[\Sigma, \zeta_\delta]\varphi \, dx \, dt \\ &\quad + \int_M \mathcal{B}[\Sigma, \varphi](\bar{\zeta}_\delta - \zeta_\delta) \, dx \, dt, \end{aligned}$$

where, for the first identity, we used that $\bar{\zeta}_\delta = 1$ in $\bar{\Omega}$. If we first substitute $\varphi \mapsto (\bar{\zeta}_\delta - \zeta_\delta)\varphi$ in (4.34) and then use the previous two identities, we get after some rewriting,

$$\begin{aligned} &\int_Q (\mathcal{F} \cdot \nabla \zeta_\delta)\varphi \, dx \, dt - \int_M \mathcal{B}^{<r}[\Sigma, \zeta_\delta]\varphi \, dx \, dt \\ &= 2 \int_{[0, T] \times \mathbb{R}^d} (\bar{\zeta}_\delta - \zeta_\delta)\varphi \, d\nu_k + \int_M \mathcal{B}^{\geq r}[\Sigma, \zeta_\delta]\varphi \, dx \, dt - \int_M \mathcal{B}[\Sigma, \bar{\zeta}_\delta]\varphi \, dx \, dt \\ &\quad + 2 \int_Q [(u - (\bar{u}^c \vee^+ k))^+ + (u - (\bar{u}^c \vee^- k))^-](\bar{\zeta}_\delta - \zeta_\delta)\varphi_t \, dx \, dt \\ &\quad + \int_Q (\mathcal{F} \cdot \nabla \varphi)(1 - \zeta_\delta) \, dx \, dt - \int_M \mathcal{B}[\Sigma, \varphi](\bar{\zeta}_\delta - \zeta_\delta) \, dx \, dt, \end{aligned} \quad (4.35)$$

where we decomposed $\mathcal{B}[\Sigma, \zeta_\delta] = \mathcal{B}^{\geq r}[\Sigma, \zeta_\delta] + \mathcal{B}^{<r}[\Sigma, \zeta_\delta]$ for some fixed $r > 0$. We pause here to note that the last part of the proposition follows from (4.35). Indeed, one has the uniform bounds $|\zeta_\delta|, |\bar{\zeta}_\delta| \leq 1$, the total variation bound $\|\nu_k\| \leq 4C$ with C as in (4.11), the bounds $|\mathcal{F}| \leq 2L_f|u - \bar{u}^c|$ and $0 \leq \Sigma \leq 2|b(u) - b(\bar{u}^c)|$, and finite energy of Σ (as explained below (4.32)). In particular, the two integrals involving $\mathcal{B}^{\geq r}[\Sigma, \zeta_\delta]$ and $\mathcal{B}[\Sigma, \varphi]$ can be bounded by (ii) and (iv) from Proposition 3.8, while the integral involving $\mathcal{B}[\Sigma, \bar{\zeta}_\delta]$ is bounded by Corollary 4.7; this latter claim is not so obvious and so we explain it in detail: Since $\Sigma(t, x) = 0$ for $x \in \Omega^c$ and $\bar{\zeta}_\delta = 1$ for $x \in \Omega$, we get that $2\mathcal{B}[\Sigma, \bar{\zeta}_\delta]$

may be written

$$\begin{aligned} & \int_{\mathbb{R}^d} (\Sigma(t, x+z) - \Sigma(t, x)) (\bar{\zeta}_\delta(x+z) - \bar{\zeta}_\delta(x)) d\mu(z) \\ &= \begin{cases} \int_{x+z \in \Omega^c} \Sigma(t, x) (1 - \bar{\zeta}_\delta(x+z)) d\mu(z) & \text{if } x \in \Omega, \\ \int_{x+z \in \Omega} \Sigma(t, x+z) (1 - \bar{\zeta}_\delta(x)) d\mu(z) & \text{if } x \in \Omega^c. \end{cases} \end{aligned}$$

Using $0 \leq \Sigma \leq 2|b(u) - b(\bar{u}^c)|$ and $0 \leq (1 - \bar{\zeta}_\delta) \leq 1$ in this last expression, we obtain the bound

$$0 \leq \mathcal{B}[\Sigma, \bar{\zeta}_\delta] \leq 2\mathcal{B}[|b(u) - b(\bar{u}^c)|, \chi_\Omega], \quad (4.36)$$

for a.e. $(t, x) \in M$. By Corollary 4.7, the right-hand side of (4.36) is integrable over M .

Next we let $\delta \rightarrow 0$ in (4.35): We have the pointwise limit $(\bar{\zeta}_\delta - \zeta_\delta) \rightarrow \chi_{\partial\Omega}$ and the two $L^1(M)$ -limits $\mathcal{B}^{\geq r}[\Sigma, \zeta_\delta] \rightarrow \mathcal{B}^{\geq r}[\Sigma, \chi_\Omega]$ and $\mathcal{B}[\Sigma, \bar{\zeta}_\delta] \rightarrow \mathcal{B}[\Sigma, \chi_\Omega]$, where the latter is a consequence of dominated convergence (we have (4.36)) and the fact that χ_Ω and $\chi_{\bar{\Omega}}$ differ on a Lebesgue null set (see the discussion at the end of the proof of Proposition 4.6). Thus, when $\delta \rightarrow 0$, we obtain from (4.35),

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \left(\int_Q (\mathcal{F} \cdot \nabla \zeta_\delta) \varphi \, dx \, dt - \int_M \mathcal{B}^{< r}[\Sigma, \zeta_\delta] \varphi \, dx \, dt \right) \\ &= 2 \int_{\bar{\Gamma}} \varphi \, dv_k - \int_M \mathcal{B}^{< r}[\Sigma, \chi_\Omega] \varphi \, dx \, dt, \end{aligned}$$

where $\bar{\Gamma} := [0, T] \times \partial\Omega$. The proposition is thus proved, because v_k is nonpositive on $\bar{\Gamma}$ (by Corollary 4.5) and because $\mathcal{B}^{< r}[\Sigma, \chi_\Omega] \geq 0$ (which is proved similarly to (4.36)). ■

5. Uniqueness of entropy solutions

This section is devoted to proving the uniqueness of entropy solutions of (1.1). The proof deploys the classical Kruřkov “doubling of variables” device as developed for nonlocal problems in [1, 41], and follows a similar path to the one developed in the work of Otto [89], Michel and Vovelle [87], and Mascia, Porretta, and Terracina [85]. Its length is due to the many nontrivial limits that need to be computed. To get a rough understanding of the coming proof, we give a formal overview of its steps.

For two entropy solutions u, v with the same exterior data $\bar{u}^c = \bar{v}^c$, the proof starts off classically by doubling the variables, $u = u(t, x)$ and $v = v(s, y)$, followed by combining the two entropy inequalities for a nonnegative test function in the variables (t, x, s, y) which is compactly supported in $Q \times Q$. The test function is chosen as a product of boundary layer sequences (for compact support) and a standard mollifier in the variables $t - s$ and $x - y$.

Just as in the classical hyperbolic case [77] the mollifier commutes with the symmetric operators $(\partial_t + \partial_s)$ and $(\nabla_x + \nabla_y)$, which is necessary for avoiding blow-up when we later let the mollifier approximate $\delta(t - s, x - y)$. In the degenerate parabolic-hyperbolic case, the Δ -term poses some challenge at this step since $\Delta_x + \Delta_y$ does *not* commute with the mollifier. This difficulty was overcome by Carrillo who in [36] showed that the necessary “cross-terms” could be safely introduced into the entropy inequality, replacing this operator by the more appropriate $(\nabla_x + \nabla_y) \cdot (\nabla_x + \nabla_y)$. In our nonlocal case, a similar difficulty arises for \mathcal{L} , but only for its singular part (which has been subjected to a convex inequality [1, 41]). Of course, this singular part vanishes as $r \rightarrow 0$, which is how the proof proceeds. The remaining “good” part of \mathcal{L} is then written in the appropriate (x, y) -symmetric form using Proposition D.1 and partially shifted onto the test function to avoid blow-up as $r \rightarrow 0$.

We then let the support of the test function expand up to the boundary. Here we introduce a new $r > 0$ to once again divide \mathcal{L} into a singular and a nonsingular part. As the boundary layers converge to the characteristic function on Ω , we avoid potential blow-up of the convection term and the singular part of \mathcal{L} by resorting to the boundary condition of Proposition 4.9. Essentially, this condition lets us exchange cumbersome differences of u and v with more manageable ones, such as a difference of u and \bar{u}^c which admits a weak trace by Proposition 4.8.

Finally, we “go to the diagonal” by letting the mollifier tend to $\delta(t - s, x - y)$ in an appropriate sense. Here, we again deploy the weak trace result of Proposition 4.8 to compute some of the limits and, conveniently, these new trace terms cancel out the ones from before, which could not otherwise be dropped due to having a “bad” sign. After the completion of this process, the desired conclusion is reached modulo some debris from the singular part of \mathcal{L} . But these undesirable terms are small by Proposition 4.6 (boundary integrability) and vanish when, once more, we let $r \rightarrow 0$, completing the proof.

Theorem 5.1 (Uniqueness). *Assume (A_Ω) – (A_μ) . Let u, v be entropy solutions of (1.1) in the sense of Definition 2.2 with initial data u_0, v_0 and exterior data $u^c = v^c$. Then for a.e. $t \in (0, T)$ we have*

$$\int_{\Omega} |u(t, x) - v(t, x)| \, dx \leq \int_{\Omega} |u_0(x) - v_0(x)| \, dx.$$

In particular, if also $u_0 = v_0$, then $u = v$ a.e. in $(0, T) \times \mathbb{R}^d$.

Remark 5.2. We can pick the *same* C^2 -extension for both u^c and v^c , i.e. $\bar{u}^c = \bar{v}^c$. This is unproblematic, as all the a priori calculations of Section 4 can be carried out for any smooth extension.

Proof of Theorem 5.1. (1) *Doubling the variables.* We begin with the standard approach of doubling the variables. Consider u and v functions in (t, x) and (s, y) respectively, and let $0 \leq \varphi \in C_c^\infty(Q_x \times Q_y)$, where the sub-index notation is as introduced in Section 2. Considering $(s, y) \in M_y$ fixed, in the entropy inequalities (2.6) for u we may let the

constant be replaced with $v(s, y)$ and the test function replaced with $(t, x) \mapsto \varphi(t, x, s, y)$. Combining the (+) and (-) cases, we obtain

$$\begin{aligned} & - \int_{Q_x} |u - v| \partial_t \varphi + F(u, v) \cdot \nabla_x \varphi \\ & - \int_{M_x} \mathcal{L}_x^{\geq r} [b(u)] \operatorname{sgn}(u - v) \varphi + |b(u) - b(v)| \mathcal{L}_x^{< r} [\varphi] \leq 0, \end{aligned} \quad (5.1)$$

where we for brevity suppress the differentials $dt \, dx$ and have added sub-indexes to clarify in which variables the operators act. As (5.1) holds for all $(s, y) \in M_y$, it can be integrated over this set, so that we obtain

$$\begin{aligned} & - \int_{Q_x \times Q_y} |u - v| \partial_t \varphi + F(u, v) \cdot \nabla_x \varphi \\ & - \int_{M_x \times M_y} \mathcal{L}_x^{\geq r} [b(u)] \operatorname{sgn}(u - v) \varphi + |b(u) - b(v)| \mathcal{L}_x^{< r} [\varphi] \leq 0, \end{aligned} \quad (5.2)$$

where we for the first integral used that φ is supported in $Q_x \times Q_y$. Swapping the roles of u and v (and the role of (t, x) and (s, y)) we get the analogous inequality

$$\begin{aligned} & - \int_{Q_x \times Q_y} |v - u| \partial_s \varphi + F(v, u) \cdot \nabla_y \varphi \\ & - \int_{M_x \times M_y} \mathcal{L}_y^{\geq r} [b(v)] \operatorname{sgn}(v - u) \varphi + |b(v) - b(u)| \mathcal{L}_y^{< r} [\varphi] \leq 0. \end{aligned} \quad (5.3)$$

With \mathcal{L}_{x+y} as defined in (2.3), we have $\mathcal{L}_{x+y}^{\geq r} [b(u) - b(v)] = \mathcal{L}_x^{\geq r} [b(u)] - \mathcal{L}_y^{\geq r} [b(v)]$, and so by adding (5.2) to (5.3) we obtain

$$\begin{aligned} & - \int_{Q_x \times Q_y} |u - v| (\partial_t + \partial_s) \varphi + F(u, v) \cdot (\nabla_x + \nabla_y) \varphi \\ & - \int_{M_x \times M_y} \mathcal{L}_{x+y}^{\geq r} [b(u) - b(v)] \operatorname{sgn}(u - v) \varphi + |b(u) - b(v)| (\mathcal{L}_x^{< r} + \mathcal{L}_y^{< r}) [\varphi] \leq 0, \end{aligned} \quad (5.4)$$

where we used that $F(u, v) = F(v, u)$ and that $\operatorname{sgn}(u - v) = -\operatorname{sgn}(v - u)$. Exploiting the second inequality of Corollary C.2 with the self-adjointness of $\mathcal{L}_{x+y}^{\geq r}$ (Proposition 3.1) we see that

$$\begin{aligned} \int_{M_x \times M_y} \mathcal{L}_{x+y}^{\geq r} [b(u) - b(v)] \operatorname{sgn}(u - v) \varphi & \leq \int_{M_x \times M_y} \mathcal{L}_{x+y}^{\geq r} [|b(u) - b(v)|] \varphi \\ & = \int_{M_x \times M_y} |b(u) - b(v)| \mathcal{L}_{x+y}^{\geq r} [\varphi]. \end{aligned}$$

Using this in (5.4) and further letting $r \rightarrow 0$, we conclude

$$\begin{aligned} & - \int_{Q_x \times Q_y} (|u - v| (\partial_t + \partial_s) \varphi + F(u, v) \cdot (\nabla_x + \nabla_y) \varphi) \\ & - \int_{M_x \times M_y} (|b(u) - b(v)| \mathcal{L}_{x+y} [\varphi]) \leq 0. \end{aligned} \quad (5.5)$$

(2) *Going to the boundary.* With a slight abuse of notation, we let $r > 0$ represent a new arbitrary constant so that we may (again) split up the operator $\mathcal{L}_{x+y} = \mathcal{L}_{x+y}^{\geq r} + \mathcal{L}_{x+y}^{< r}$. By Proposition 3.1 we may shift $\mathcal{L}_{x+y}^{< r}$ to $\mathcal{B}_{x+y}^{< r}$ so to get

$$- \int_{M_x \times M_y} |b(u) - b(v)| \mathcal{L}_{x+y}^{< r} [\varphi] = \int_{M_x \times M_y} \mathcal{B}_{x+y}^{< r} [|b(u) - b(v)|, \varphi].$$

The integral on the right-hand side is well defined because of Proposition 3.9 and the local energy of $b(u)$ and $b(v)$ (see Remark 2.6). Using this relation in (5.5) and rearranging, we get

$$\begin{aligned} & - \int_{Q_x \times Q_y} |u - v| (\partial_t + \partial_s) \varphi - \int_{M_x \times M_y} |b(u) - b(v)| \mathcal{L}_{x+y}^{\geq r} [\varphi] \\ & \leq \int_{Q_x \times Q_y} F(u, v) \cdot (\nabla_x + \nabla_y) \varphi - \int_{M_x \times M_y} \mathcal{B}_{x+y}^{< r} [|b(u) - b(v)|, \varphi]. \end{aligned} \quad (5.6)$$

Next we set

$$\varphi(t, x, s, y) = \zeta_\delta(x) \zeta_\delta(y) \rho(t - s, x - y) \alpha(t),$$

where $(\zeta_\delta)_{\delta > 0}$ is an inner boundary layer sequence satisfying $\sup_{\delta > 0} \|\nabla \zeta_\delta\|_{L^1(\Omega)} < \infty$ (cf. (2.7)), and where both $\rho \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$ and $\alpha \in C_c^\infty((0, T))$ are nonnegative. We assume that the support of ρ is contained in a sufficiently small ball around zero so that φ is indeed compactly supported in $Q_x \times Q_y$. For brevity, we will write $\zeta_\delta^x, \zeta_\delta^y, \rho, \alpha$ to mean the expressions $\zeta_\delta(x), \zeta_\delta(y), \rho(t - s, x - y), \alpha(t)$ respectively. Inserting for φ in (5.6) we get

$$\begin{aligned} & - \int_{Q_x \times Q_y} |u - v| \zeta_\delta^x \zeta_\delta^y \rho \alpha' - \int_{M_x \times M_y} |b(u) - b(v)| \mathcal{L}_{x+y}^{\geq r} [\zeta_\delta^x \zeta_\delta^y] \rho \alpha \\ & \leq \int_{Q_x \times Q_y} F(u, v) \cdot (\nabla_x \zeta_\delta^x) \zeta_\delta^y \rho \alpha - \int_{M_x \times M_y} \mathcal{B}_{x,x+y}^{< r} [|b(u) - b(v)|, \zeta_\delta^x \zeta_\delta^y] \rho \alpha \\ & \quad + \int_{Q_x \times Q_y} F(u, v) \cdot (\nabla_y \zeta_\delta^y) \zeta_\delta^x \rho \alpha - \int_{M_x \times M_y} \mathcal{B}_{y,x+y}^{< r} [|b(u) - b(v)|, \zeta_\delta^x \zeta_\delta^y] \rho \alpha, \end{aligned} \quad (5.7)$$

where we used the cross-terms formula for $\mathcal{B}_{x+y}^{< r}$ from Proposition D.1 (the resulting integrals are, again, well defined by Proposition 3.9), and moved ρ outside the nonlocal operators since it is invariant under translations $(x, y) \mapsto (x + z, y + z)$.

We now wish to let $\delta \rightarrow 0$; observe that the left-hand side of (5.7) has a well-defined limit as $\mathcal{L}_{x+y}^{\geq r}$ is a zero-order operator. Thus, letting $\delta \rightarrow 0$ we obtain

$$- \int_{Q_x \times Q_y} |u - v| \rho \alpha' - \int_{M_x \times M_y} |b(u) - b(v)| \mathcal{L}_{x+y}^{\geq r} [\chi_\Omega^x \chi_\Omega^y] \rho \alpha \leq \lim_{\delta \rightarrow 0} (I_u + I_v), \quad (5.8)$$

where χ_Ω^x and χ_Ω^y denote the characteristic function χ_Ω as a function in x and y respectively and where I_u and I_v denote the first and second lines on the right-hand side of (5.7). That $I_u + I_v$ admits a limit as $\delta \rightarrow 0$ will be proved by breaking it up into eight appropriate terms.

We begin by exploiting the product rule given by Lemma D.2 to see that I_u may be rewritten

$$\begin{aligned} I_u &= \int_{\mathcal{Q}_x \times \mathcal{Q}_y} F(u, v) \cdot (\nabla_x \zeta_\delta^x) \zeta_\delta^y \rho \alpha - \int_{M_x \times M_y} \mathcal{B}_x^{<r} [|b(u) - b(v)|, \zeta_\delta^x] \zeta_\delta^y \rho \alpha \\ &\quad - \int_{M_x \times M_y} |b(u) - b(v)| (\mathcal{B}_y^{<r} [\zeta_\delta^y, \rho] \zeta_\delta^x - \mathcal{B}_{x,y}^{<r} [\zeta_\delta^x, \zeta_\delta^y] \rho) \alpha, \end{aligned}$$

and by adding and subtracting terms we further obtain

$$\begin{aligned} I_u &= \int_{\mathcal{Q}_x \times \mathcal{Q}_y} (\mathcal{F}_u \cdot \nabla_x \zeta_\delta^x) \zeta_\delta^y \rho \alpha - \int_{M_x \times M_y} \mathcal{B}_x^{<r} [\Sigma_u, \zeta_\delta^x] \zeta_\delta^y \rho \alpha \\ &\quad - \int_{M_x \times M_y} |b(u) - b(v)| (\mathcal{B}_y^{<r} [\zeta_\delta^y, \rho] \zeta_\delta^x - \mathcal{B}_{x,y}^{<r} [\zeta_\delta^x, \zeta_\delta^y] \rho) \alpha \\ &\quad - \int_{\mathcal{Q}_x \times \mathcal{Q}_y} (F(u, \bar{u}^c) - F(v, \bar{u}^c)) \cdot (\nabla_x \zeta_\delta^x) \zeta_\delta^y \rho \alpha \\ &\quad + \int_{M_x \times M_y} \mathcal{B}_x^{<r} [|b(u) - b(\bar{u}^c)| - |b(v) - b(\bar{u}^c)|, \zeta_\delta^x] \zeta_\delta^y \rho \alpha, \end{aligned} \quad (5.9)$$

where $\mathcal{F}_u = \mathcal{F}(u(t, x), v(s, y), \bar{u}^c(t, x))$ and $\Sigma_u = \Sigma(u(t, x), v(s, y), \bar{u}^c(t, x))$, and with Σ, \mathcal{F} as in (4.31). Analogously, we may write I_v as

$$\begin{aligned} I_v &= \int_{\mathcal{Q}_x \times \mathcal{Q}_y} (\mathcal{F}_v \cdot \nabla_y \zeta_\delta^y) \zeta_\delta^x \rho \alpha - \int_{M_x \times M_y} \mathcal{B}_y^{<r} [\Sigma_v, \zeta_\delta^y] \zeta_\delta^x \rho \alpha \\ &\quad - \int_{M_x \times M_y} |b(u) - b(v)| (\mathcal{B}_x^{<r} [\zeta_\delta^x, \rho] \zeta_\delta^y - \mathcal{B}_{x,y}^{<r} [\zeta_\delta^x, \zeta_\delta^y] \rho) \alpha \\ &\quad - \int_{\mathcal{Q}_x \times \mathcal{Q}_y} (F(v, \bar{v}^c) - F(u, \bar{v}^c)) \cdot (\nabla_y \zeta_\delta^y) \zeta_\delta^x \rho \alpha \\ &\quad + \int_{M_x \times M_y} \mathcal{B}_y^{<r} [|b(v) - b(\bar{v}^c)| - |b(u) - b(\bar{v}^c)|, \zeta_\delta^y] \zeta_\delta^x \rho \alpha, \end{aligned} \quad (5.10)$$

where $\mathcal{F}_v = \mathcal{F}(v(s, y), u(t, x), \bar{v}^c(s, y))$ and $\Sigma_v = \Sigma(v(s, y), u(t, x), \bar{v}^c(s, y))$. Note that, although \bar{u}^c and \bar{v}^c coincide as functions, they depend on the variables (t, x) and (s, y) respectively. Finally, by grouping terms from (5.9) and (5.10) we can further write

$$I_u + I_v = J_u^0 + J_v^0 + J_u^1 + J_v^1 + J_u^2 + J_v^2 + J_u^3 + J_v^3, \quad (5.11)$$

where we have introduced

$$\begin{aligned} J_u^0 &:= \int_{\mathcal{Q}_x \times \mathcal{Q}_y} (\mathcal{F}_u \cdot \nabla_x \zeta_\delta^x) \zeta_\delta^y \rho \alpha - \int_{M_x \times M_y} \mathcal{B}_x^{<r} [\Sigma_u, \zeta_\delta^x] \zeta_\delta^y \rho \alpha, \\ J_v^0 &:= \int_{\mathcal{Q}_x \times \mathcal{Q}_y} (\mathcal{F}_v \cdot \nabla_y \zeta_\delta^y) \zeta_\delta^x \rho \alpha - \int_{M_x \times M_y} \mathcal{B}_y^{<r} [\Sigma_v, \zeta_\delta^y] \zeta_\delta^x \rho \alpha, \\ J_u^1 &:= - \int_{\mathcal{Q}_x \times \mathcal{Q}_y} (F(u, \bar{u}^c) \cdot \nabla_x \zeta_\delta^x) \zeta_\delta^y \rho \alpha + \int_{M_x \times M_y} \mathcal{B}_x^{<r} [|b(u) - b(\bar{u}^c)|, \zeta_\delta^x] \zeta_\delta^y \rho \alpha, \end{aligned}$$

$$\begin{aligned}
 J_v^1 &:= - \int_{Q_x \times Q_y} (F(v, \bar{v}^c) \cdot \nabla_y \zeta_\delta^y) \zeta_\delta^x \rho \alpha + \int_{M_x \times M_y} \mathcal{B}_y^{<r} [|b(v) - b(\bar{v}^c)|, \zeta_\delta^y] \zeta_\delta^x \rho \alpha, \\
 J_u^2 &:= \int_{Q_x \times Q_y} (F(u, \bar{v}^c) \cdot \nabla_y \zeta_\delta^y) \zeta_\delta^x \rho \alpha - \int_{M_x \times M_y} |b(u) - b(v)| \mathcal{B}_y^{<r} [\zeta_\delta^y, \rho] \zeta_\delta^x \alpha, \\
 J_v^2 &:= \int_{Q_x \times Q_y} (F(v, \bar{u}^c) \cdot \nabla_x \zeta_\delta^x) \zeta_\delta^y \rho \alpha - \int_{M_x \times M_y} |b(u) - b(v)| \mathcal{B}_x^{<r} [\zeta_\delta^x, \rho] \zeta_\delta^y \alpha, \\
 J_u^3 &:= \int_{M_x \times M_y} (|b(u) - b(v)| \mathcal{B}_{x,y}^{<r} [\zeta_\delta^x, \zeta_\delta^y] - \mathcal{B}_y^{<r} [|b(u) - b(\bar{v}^c)|, \zeta_\delta^y] \zeta_\delta^x) \rho \alpha, \\
 J_v^3 &:= \int_{M_x \times M_y} (|b(u) - b(v)| \mathcal{B}_{x,y}^{<r} [\zeta_\delta^x, \zeta_\delta^y] - \mathcal{B}_x^{<r} [|b(v) - b(\bar{u}^c)|, \zeta_\delta^x] \zeta_\delta^y) \rho \alpha.
 \end{aligned}$$

The limit as $\delta \rightarrow 0$ of each of these terms will now be examined.

(3) *The limit of J_u^0 and J_v^0 .* We will not compute the exact value of these limits (though they can be derived from the proof of Proposition 4.9). We only demonstrate that they are nonpositive, which suffices for our purpose. Starting with J_u^0 , we observe that it can be written

$$J_u^0 = \int_{Q_y} \left[\int_{Q_x} (\mathcal{F}_u \cdot \nabla_x \zeta_\delta^x) \rho \alpha - \int_{M_x} \mathcal{B}_x^{<r} [\Sigma_u, \zeta_\delta^x] \rho \alpha \right] \zeta_\delta^y.$$

By the boundary condition for u (Proposition 4.9), the expression inside the lim is uniformly bounded on Q_y as $\delta \rightarrow 0$ and it admits a nonnegative limit. Applying dominated convergence, we conclude that

$$\lim_{\delta \rightarrow 0} J_u^0 \leq 0.$$

An analogous calculation gives the corresponding result $\lim_{\delta \rightarrow 0} J_v^0 \leq 0$.

(4) *The limit of J_u^1 and J_v^1 .* Starting with J_u^1 , we again do a little rewriting to see that

$$J_u^1 = \int_{Q_y} \left[\int_{M_x} (-F(u, \bar{u}^c) \cdot \nabla_x \zeta_\delta^x + \mathcal{B}_x^{<r} [|b(u) - b(\bar{u}^c)|, \zeta_\delta^x]) \rho \alpha \right] \zeta_\delta^y.$$

By the weak trace result (Proposition 4.8), the expression inside the lim is uniformly bounded on Q_y as $\delta \rightarrow 0$ and it admits a limit given by said proposition. Thus, dominated convergence gives us

$$\lim_{\delta \rightarrow 0} J_u^1 = L_u^1 := - \int_{\Gamma_x \times Q_y} \rho \alpha \, dv_u + \int_{M_x \times Q_y} \mathcal{B}_x^{<r} [|b(u) - b(\bar{u}^c)|, \chi_\Omega^x] \rho \alpha,$$

where ν_u is a finite Borel measure on $\bar{\Gamma}_x$, but we integrate over Γ_x as α is compactly supported in $(0, T)$. As before, we write χ_Ω^x to stress the x -dependence of the characteristic function.

Analogously, we also obtain

$$\lim_{\delta \rightarrow 0} J_v^1 = L_v^1 := - \int_{Q_x \times \Gamma_y} \rho \alpha \, dv_v + \int_{Q_x \times M_y} \mathcal{B}_y^{<r} [|b(v) - b(\bar{v}^c)|, \chi_\Omega^y] \rho \alpha.$$

(5) *The limit of J_u^2 and J_v^2 .* As before, we start with J_u^2 and rewrite it as

$$\begin{aligned} J_u^2 &= \int_{Q_x} \left(\int_{Q_y} (F(u, \bar{v}^c) \cdot \nabla_y \zeta_\delta^y) \rho \alpha \right) \zeta_\delta^x \\ &\quad - \int_{|z| < r} \left(\int_{M_x \times M_y} |b(u) - b(v)| (\zeta_\delta^y(y+z) - \zeta_\delta^y(y)) \right. \\ &\quad \quad \left. \times (\rho(x-y-z) - \rho(x-y)) \alpha \zeta_\delta^x \right) d\mu(z), \end{aligned}$$

where for the second integral we used Fubini's theorem and wrote out only the arguments featuring translations in z . For the first integral, we exploit the regularity of \bar{v}^c , ρ , α , and f to get

$$\lim_{\delta \rightarrow 0} \int_{Q_x} \left(\int_{Q_y} (F(u, \bar{v}^c) \cdot \nabla_y \zeta_\delta^y) \rho \alpha \right) \zeta_\delta^x = - \int_{Q_x \times \Gamma_y} (F(u, \bar{v}^c) \cdot \hat{n}) \rho \alpha,$$

where \hat{n} is an outward-pointing normal vector on $\partial\Omega$. Here we used dominated convergence; this is justified as the inner integral is uniformly bounded over Q_x and $\delta > 0$ (we recall that $\sup_{\delta > 0} \|\nabla \zeta_\delta\|_{L^1(\Omega)} < \infty$). For the second part of J_u^2 we again wish to deploy dominated convergence; the inner integral has a canonical limit for fixed z , and for all $\delta > 0$ it is dominated by

$$\begin{aligned} \|b(u) - b(v)\|_{L^\infty(M_x \times M_y)} \sup_{\delta > 0} (\|\nabla \zeta_\delta^y\|_{L^1(M_y)} |z| \wedge 2 \|\zeta_\delta^y\|_{L^1(M_y)}) \\ \times (L_\rho |z| \wedge 2 \|\rho\|_{L^\infty(M_x \times M_y)}) \|\alpha \zeta_\delta^x\|_{L^1(M_x)}, \end{aligned}$$

where L_ρ is a Lipschitz constant for ρ on $M_x \times M_y$. As this bound is less than $C(|z|^2 \wedge 1)$, for an appropriate C , we infer from dominated convergence that the second part of J_u^2 admits the limit

$$\begin{aligned} - \int_{|z| < r} \left(\int_{M_x \times M_y} |b(u) - b(v)| (\chi_\Omega^y(y+z) - \chi_\Omega^y(y)) \right. \\ \quad \left. \times (\rho(x-y-z) - \rho(x-y)) \alpha \chi_\Omega^x \right) d\mu(z), \end{aligned}$$

as $\delta \rightarrow 0$. Fubini's theorem may again be applied to move the integral in z inside, and so we conclude

$$\lim_{\delta \rightarrow 0} J_u^2 = L_u^2 := - \int_{Q_x \times \Gamma_y} (F(u, \bar{v}^c) \cdot \hat{n}) \rho \alpha - \int_{Q_x \times M_y} |b(u) - b(v)| \mathcal{B}_y^{<r} [\chi_\Omega^y, \rho] \alpha,$$

and by an analogous argument we also obtain

$$\lim_{\delta \rightarrow 0} J_v^2 = L_v^2 := - \int_{\Gamma_x \times Q_y} (F(v, \bar{u}^c) \cdot \hat{n}) \rho \alpha - \int_{M_x \times Q_y} |b(u) - b(v)| \mathcal{B}_x^{<r} [\chi_\Omega^x, \rho] \alpha.$$

(6) *The limit of J_u^3 and J_v^3 .* We begin by studying J_u^3 . By Fubini's theorem, we rewrite J_u^3 as

$$J_u^3 = \int_{|z|<r} \left(\int_{M_x \times M_y} \left[|b(u) - b(v)| (\zeta_\delta^x(x+z) - \zeta_\delta^x(x)) (\zeta_\delta^y(y+z) - \zeta_\delta^y(y)) \right. \right. \\ \left. \left. - (|b(u) - b(\bar{v}^c(y+z))| - |b(u) - b(\bar{v}^c(y))|) \right. \right. \\ \left. \left. \times (\zeta_\delta^y(y+z) - \zeta_\delta^y(y)) \zeta_\delta^x \right] \rho \alpha \right) d\mu(z),$$

where only the arguments featuring translations in z have been written out. For fixed z it is obvious that the inner integral converges to its canonical limit as $\delta \rightarrow 0$. Moreover, the inner integral admits the following bound:

$$|z|^2 \left(\|b(u) - b(v)\|_{L^\infty(M_x \times M_y)} \|\nabla \zeta_\delta^x\|_{L^1(\mathbb{R}_x^d)} \|\nabla \zeta_\delta^y\|_{L^1(\mathbb{R}_y^d)} \right. \\ \left. + L_{b(\bar{v}^c)} \|\nabla \zeta_\delta^y\|_{L^1(\mathbb{R}_y^d)} \|\zeta_\delta^x\|_{L^1(\mathbb{R}_x^d)} \right) \|\rho \alpha\|_{L^\infty(M_x \times M_y)} \leq C|z|^2,$$

where $L_{b(\bar{v}^c)}$ is a local Lipschitz constant of $b(\bar{v}^c)$ and where C is some large constant independent of $\delta > 0$ as $\sup_\delta \|\nabla \zeta_\delta\|_{L^1(\mathbb{R}^d)} < \infty$ and $\sup_\delta \|\zeta_\delta\|_{L^1(\mathbb{R}^d)} \leq |\Omega|$. As this last bound is integrable on $|z| < r$ with respect to $d\mu$, it follows by dominated convergence that

$$\lim_{\delta \rightarrow 0} J_u^3 = L_u^3 := \int_{M_x \times M_y} \left(|b(u) - b(v)| \mathcal{B}_{x,y}^{<r}[\chi_\Omega^x, \chi_\Omega^y] - \mathcal{B}_y^{<r}[|b(u) - b(\bar{v}^c)|, \chi_\Omega^y] \chi_\Omega^x \right) \rho \alpha.$$

And by an analogous argument, we also obtain

$$\lim_{\delta \rightarrow 0} J_v^3 = L_v^3 := \int_{M_x \times M_y} \left(|b(u) - b(v)| \mathcal{B}_{x,y}^{<r}[\chi_\Omega^x, \chi_\Omega^y] - \mathcal{B}_x^{<r}[|b(v) - b(\bar{u}^c)|, \chi_\Omega^x] \chi_\Omega^y \right) \rho \alpha.$$

(7) *Going to the diagonal.* In summary, we have so far established the inequality

$$- \int_{Q_x \times Q_y} |u - v| \rho \alpha' - \int_{M_x \times M_y} |b(u) - b(v)| \mathcal{L}_{x+y}^{\geq r}[\chi_\Omega^x, \chi_\Omega^y] \rho \alpha \\ \leq L_u^1 + L_v^1 + L_u^2 + L_v^2 + L_u^3 + L_v^3, \quad (5.12)$$

which follows from (5.8) and (5.11), together with the above-established limits. Observe that we have chosen to omit L_u^0 and L_v^0 from the right-hand side as these terms are non-positive. We now wish to insert a more explicit expression for ρ . For two small fixed parameters $\epsilon, \varepsilon > 0$ we set

$$\rho(t-s, x-y) = \theta_\epsilon(t-s) \tilde{\rho}_\varepsilon(x-y),$$

where θ_ϵ and ρ_ε are standard mollifiers in \mathbb{R} and \mathbb{R}^d respectively. More precisely, we have $\theta_\epsilon = \epsilon^{-1} \theta(\cdot/\epsilon)$ and $\tilde{\rho}_\varepsilon = \varepsilon^{-d} \tilde{\rho}(\cdot/\varepsilon)$, where θ and $\tilde{\rho}$ are positive, smooth, and symmetric functions supported in the unit ball of \mathbb{R} and \mathbb{R}^d that additionally satisfy

$\int_{\mathbb{R}} \theta \, dt = \int_{\mathbb{R}^d} \tilde{\rho} \, dx = 1$. For notational simplicity, we will write ρ instead of $\tilde{\rho}$, and the abbreviations θ_ϵ and ρ_ϵ to mean the expressions $\theta_\epsilon(t - s)$ and $\rho_\epsilon(x - y)$. If, on the left-hand side of (5.12), we replace the (old) ρ with the new expression $\theta_\epsilon \rho_\epsilon$ we get

$$\begin{aligned} & - \int_{Q_x \times Q_y} |u(t, x) - v(s, y)| \theta_\epsilon \rho_\epsilon \alpha' \\ & - \int_{M_x \times M_y} |b(u(t, x)) - b(v(s, y))| \mathcal{L}_{x+y}^{\geq r} [\chi_\Omega^x \chi_\Omega^y] \theta_\epsilon \rho_\epsilon \alpha, \end{aligned} \quad (5.13)$$

where the arguments of u and v have been included to clarify the next computation. We wish to *go to the diagonal*, i.e. to let $\epsilon, \epsilon \rightarrow 0$. This task is fairly straightforward for the above expression. Indeed, subtracting

$$\begin{aligned} & - \int_{Q_x \times Q_y} |u(t, x) - v(t, x)| \theta_\epsilon \rho_\epsilon \alpha' \\ & - \int_{M_x \times M_y} |b(u(t, x)) - b(v(t, x))| \mathcal{L}_x^{\geq r} [\chi_\Omega^x] \theta_\epsilon \rho_\epsilon \alpha, \end{aligned} \quad (5.14)$$

from (5.13) and making the substitution of variables $s = t + \tau$ and $y = x + \xi$, a (mostly) standard calculation shows that the result is bounded by

$$\begin{aligned} & \sup_{\substack{|\tau| < \epsilon \\ |\xi| < \epsilon}} \|v(\cdot + \tau, \cdot + \xi) - v\|_{L^1(Q_y)} \|\alpha'\|_{L^\infty((0, T))} \\ & + \sup_{\substack{|\tau| < \epsilon \\ |\xi| < \epsilon}} \|b(v)(\cdot + \tau, \cdot + \xi) - b(v)\|_{L^1(Q_y)} 2C_r \|\chi_\Omega^x\|_{L^\infty(M_x)} \|\alpha\|_{L^\infty(M_x)} \\ & + \|b(u) - b(v)\|_{L^\infty(M_x \times M_y)} 2C_r \sup_{|\xi| < \epsilon} \|\chi_\Omega^x(\cdot)(\chi_\Omega^x(\cdot + \xi) - 1)\|_{L^1(M_x)} \|\alpha\|_{L^\infty((0, T))}, \end{aligned}$$

where $C_r = \mu(\{|z| \geq r\})$. By translation regularity of L^1_{loc} -functions, this last bound tends to zero as $\epsilon, \epsilon \rightarrow 0$. In particular, the limit of (5.13) as $\epsilon, \epsilon \rightarrow 0$ coincides with that of (5.14) which is given by

$$\begin{aligned} & - \int_Q |u(t, x) - v(t, x)| \alpha'(t) \, dx \, dt \\ & - \int_M |b(u(t, x)) - b(v(t, x))| \mathcal{L}^{\geq r} [\chi_\Omega](x) \alpha(t) \, dx \, dt. \end{aligned} \quad (5.15)$$

Observe next that the second integral in (5.15) is nonpositive. This follows as $|b(u(t, x)) - b(v(t, x))|$ is nonnegative and supported in Q (since $u^c = v^c$ a.e. in Q^c), while $\mathcal{L}^{\geq r} [\chi_\Omega]$ is nonpositive in Q due to χ_Ω taking its maximum in Q . We conclude then from (5.12) and the above analysis that

$$\begin{aligned} & - \int_Q |u(t, x) - v(t, x)| \alpha'(t) \, dx \, dt \\ & \leq \lim_{r \rightarrow 0} \limsup_{\epsilon \rightarrow 0} \lim_{\epsilon \rightarrow 0} [L_u^1 + L_v^1 + L_u^2 + L_v^2 + L_u^3 + L_v^3], \end{aligned} \quad (5.16)$$

where the choice of taking \limsup in ε is for the sake of brevity: While the intermediate ε -limit does in fact exist for all the coming terms, several of these limits further vanish as $r \rightarrow 0$. Thus, it will often be sufficient (and much simpler) to instead obtain appropriate bounds on $\limsup_{\varepsilon \rightarrow 0}$ for these terms.

The remainder of the proof is to show that the right-hand side of (5.16) is nonpositive. For clarity, we restate the definitions of the right-hand-side terms, with the old ρ replaced by $\theta_\varepsilon \rho_\varepsilon$. They are

$$\begin{aligned} L_u^1 &= - \int_{\Gamma_x \times Q_y} \theta_\varepsilon \rho_\varepsilon \alpha \, dv_u + \int_{M_x \times Q_y} \mathcal{B}_x^{<r} [|b(u) - b(\bar{u}^c)|, \chi_\Omega^x] \theta_\varepsilon \rho_\varepsilon \alpha, \\ L_v^1 &= - \int_{Q_x \times \Gamma_y} \theta_\varepsilon \rho_\varepsilon \alpha \, dv_v + \int_{Q_x \times M_y} \mathcal{B}_y^{<r} [|b(v) - b(\bar{v}^c)|, \chi_\Omega^y] \theta_\varepsilon \rho_\varepsilon \alpha, \\ L_u^2 &= - \int_{Q_x \times \Gamma_y} (F(u, \bar{v}^c) \cdot \hat{n}) \theta_\varepsilon \rho_\varepsilon \alpha - \int_{Q_x \times M_y} |b(u) - b(v)| \mathcal{B}_y^{<r} [\chi_\Omega^y, \rho_\varepsilon] \theta_\varepsilon \alpha, \\ L_v^2 &= - \int_{\Gamma_x \times Q_y} (F(v, \bar{u}^c) \cdot \hat{n}) \theta_\varepsilon \rho_\varepsilon \alpha - \int_{M_x \times Q_y} |b(u) - b(v)| \mathcal{B}_x^{<r} [\chi_\Omega^x, \rho_\varepsilon] \theta_\varepsilon \alpha, \\ L_u^3 &= \int_{M_x \times M_y} (|b(u) - b(v)| \mathcal{B}_{x,y}^{<r} [\chi_\Omega^x, \chi_\Omega^y] - \mathcal{B}_y^{<r} [|b(u) - b(\bar{v}^c)|, \chi_\Omega^y] \chi_\Omega^x) \theta_\varepsilon \rho_\varepsilon \alpha, \\ L_v^3 &= \int_{M_x \times M_y} (|b(u) - b(v)| \mathcal{B}_{x,y}^{<r} [\chi_\Omega^x, \chi_\Omega^y] - \mathcal{B}_x^{<r} [|b(v) - b(\bar{u}^c)|, \chi_\Omega^x] \chi_\Omega^y) \theta_\varepsilon \rho_\varepsilon \alpha. \end{aligned}$$

Because the right-hand side of (5.16) is bounded by the sum of the termwise limits, that is,

$$\lim_{r \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \lim_{\epsilon \rightarrow 0} [L_u^1 + L_v^1 + L_u^2 + L_v^2 + L_u^3 + L_v^3] \leq \sum_{\substack{k=1,2,3 \\ w=u,v}} \lim_{r \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \lim_{\epsilon \rightarrow 0} L_w^k, \quad (5.17)$$

we proceed by studying the limit of each L -term, as was similarly done for the J -terms.

(8) *The limits of L_u^1 and L_v^1 .* Starting with L_u^1 , we introduce

$$\varrho_\varepsilon(x) := \int_{\Omega} \rho_\varepsilon(x - y) \, dy. \quad (5.18)$$

Note that ϱ_ε is smooth, has compact support, and is bounded between 0 and 1 on \mathbb{R}^d . Moreover, it satisfies

$$\lim_{\varepsilon \rightarrow 0} \varrho_\varepsilon(x) = \begin{cases} 1, & x \in \Omega, \\ \frac{1}{2}, & x \in \partial\Omega, \\ 0, & x \in \Omega^c \setminus \partial\Omega, \end{cases} \quad (5.19)$$

by the regularity of the boundary. With this notation, we let $\varepsilon \rightarrow 0$ and find that L_u^1 can be written

$$\lim_{\varepsilon \rightarrow 0} L_u^1 = - \int_{\Gamma_x} \varrho_\varepsilon \alpha \, dv_u + \int_{M_x} \mathcal{B}_x^{<r} [|b(u) - b(\bar{u}^c)|, \chi_\Omega^x] \varrho_\varepsilon \alpha, \quad (5.20)$$

where we used that α has compact support in $(0, T)$ and that $\lim_{\epsilon \rightarrow 0} \int_0^T \theta_\epsilon(t-s) ds = 1$ for all $t \in (0, T)$. By Corollary 4.7, the expression $\mathcal{B}_x^{<r} [|b(u) - b(\bar{u}^c)|, \chi_\Omega^x]$ is well defined in $L^1(M_x)$ and must consequently vanish (in the L^1 sense) as $r \rightarrow 0$. We conclude from (5.19), (5.20), and dominated convergence that

$$\lim_{r \rightarrow 0} \limsup_{\epsilon \rightarrow 0} \lim_{\epsilon \rightarrow 0} L_u^1 = -\frac{1}{2} \int_\Gamma \alpha \, d\nu_u.$$

For L_v^1 the analysis is analogous, except for the small difference that α goes from being a function in t to one in s as $\epsilon \rightarrow 0$. This leads to the analogous conclusion

$$\lim_{r \rightarrow 0} \limsup_{\epsilon \rightarrow 0} \lim_{\epsilon \rightarrow 0} L_v^1 = -\frac{1}{2} \int_\Gamma \alpha \, d\nu_v.$$

(9) *The limit of L_u^2 and L_v^2 .* As we will see, these two limits are the opposite of (and thus cancel out) the previous two limits. However, they are far more laborious to compute.

Starting with L_u^2 , we first note that the limit $\epsilon \rightarrow 0$ is standard, giving

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} L_u^2 &= - \int_0^T \int_{\Omega_x \times \partial\Omega_y} (F(u, \bar{v}^c) \cdot \hat{n}) \alpha \rho_\epsilon \, dx \, d\sigma(y) \, dt \\ &\quad - \int_0^T \int_{\Omega_x \times \mathbb{R}_y^d} |b(u) - b(v)| \mathcal{B}_y^{<r} [\chi_\Omega^y, \rho_\epsilon] \alpha \, dx \, dy \, dt, \end{aligned} \tag{5.21}$$

where v and \bar{v}^c on the right-hand side of (5.21) are functions in (t, y) . We wish to replace $F(u(t, x), \bar{v}^c(t, y))$ with $F(u(t, x), \bar{u}^c(t, x))$ in the first integral on the right-hand side of (5.21). To do so, we recall that \bar{v}^c is the same function as \bar{u}^c (but in different variables) so we may compute

$$\begin{aligned} &\left| \int_0^T \int_{\Omega_x \times \partial\Omega_y} (F(u, \bar{v}^c) - F(u, \bar{u}^c)) \cdot \hat{n} \alpha \rho_\epsilon \, dx \, d\sigma(y) \, dt \right| \\ &\leq L_f \int_0^T \int_{\Omega_x \times \partial\Omega_y} |\bar{v}^c(t, y) - \bar{u}^c(t, x)| \alpha(t) \rho_\epsilon(x-y) \, dx \, d\sigma(y) \, dt \\ &\leq L_f L_{\bar{u}^c} T \|\alpha\|_{L^\infty((0,T))} \int_{\Omega_x \times \partial\Omega_y} |x-y| \rho_\epsilon(x-y) \, dx \, d\sigma(y), \end{aligned} \tag{5.22}$$

where L_f and $L_{\bar{u}^c}$ are local Lipschitz constants of f and \bar{u}^c . Using that $\int_{\mathbb{R}^d} |x-y| \rho_\epsilon(x-y) \, dx \leq \epsilon$ we note that the rightmost side of (5.22) is of size $\lesssim \epsilon$. Moreover, by the divergence theorem we have the identity

$$- \int_{\partial\Omega_y} \hat{n} \rho_\epsilon \, d\sigma(y) = \int_{\Omega_y} -\nabla_y \rho_\epsilon \, dy = \int_{\Omega_y} \nabla_x \rho_\epsilon \, dy = \nabla_x \varrho_\epsilon,$$

where ϱ_ϵ is as in (5.18). Summarizing, the first part of (5.21) can be written

$$- \int_0^T \int_{\Omega_x \times \partial\Omega_y} (F(u, \bar{v}^c) \cdot \hat{n}) \alpha \rho_\epsilon \, dx \, d\sigma(y) \, dt = \int_{Q_x} F(u, \bar{u}^c) \cdot \nabla_x \varrho_\epsilon \alpha \, dx \, dt + O(\epsilon). \tag{5.23}$$

Next we will find an analogous representation for the second integral in (5.21). For brevity, we will in the coming analysis ignore α and the integral in t and suppress the t -dependence of the remaining expressions. We begin by observing that

$$\begin{aligned} & \left| \int_{\Omega_x \times \mathbb{R}_y^d} (|b(u) - b(v)| - |b(u) - b(\bar{u}^c)|) \mathcal{B}_y^{<r}[\chi_\Omega^y, \rho_\varepsilon] \, dx \, dy \right| \\ & \leq \int_{\Omega_x \times \mathbb{R}_y^d} |(b(v) - b(\bar{v}^c)) \mathcal{B}_y^{<r}[\chi_\Omega^y, \rho_\varepsilon]| \, dx \, dy \\ & \quad + \int_{\Omega_x \times \mathbb{R}_y^d} |(b(\bar{u}^c) - b(\bar{v}^c)) \mathcal{B}_y^{<r}[\chi_\Omega^y, \rho_\varepsilon]| \, dx \, dy. \end{aligned} \quad (5.24)$$

We show that both expressions on the right-hand side of (5.24) vanish uniformly in ε as $r \rightarrow 0$. For the first one, we exploit that $|b(v) - b(\bar{v}^c)|$ is zero for $y \in \Omega^c$ and obtain

$$\begin{aligned} & \int_{\Omega_x \times \mathbb{R}_y^d} |(b(v) - b(\bar{v}^c)) \mathcal{B}_y^{<r}[\chi_\Omega^y, \rho_\varepsilon]| \, dx \, dy \\ & \leq \int_{\substack{x, y \in \Omega \\ y+z \in \Omega^c \\ |z| < r}} \frac{1}{2} |b(v(y)) - b(\bar{v}^c(y))| |\rho_\varepsilon(x - y - z) - \rho_\varepsilon(x - y)| \, dx \, dy \, d\mu(z) \\ & \leq \int_{\substack{y \in \Omega \\ y+z \in \Omega^c \\ |z| < r}} |b(v(y)) - b(\bar{v}^c(y))| \, dy \, d\mu(z) \rightarrow 0, \quad r \rightarrow 0, \end{aligned}$$

where the limit holds by the boundary integrability of Proposition 4.6. The second expression on the right-hand side of (5.24) is more tricky. First, since \bar{u}^c and \bar{v}^c are the same function and differ only in their arguments (they now depend on (t, x) and (t, y) respectively) we compute

$$\begin{aligned} & \int_{\Omega_x \times \mathbb{R}_y^d} |(b(\bar{u}^c) - b(\bar{v}^c)) \mathcal{B}_y^{<r}[\chi_\Omega^y, \rho_\varepsilon]| \, dx \, dy \\ & \leq \frac{Lb(\bar{u}^c)}{2} \int_{\substack{x \in \Omega \\ y \in \mathbb{R}^d \\ |z| < r}} |x - y| |\chi_\Omega(y + z) - \chi_\Omega(y)| \\ & \quad \times |\rho_\varepsilon(x - y - z) - \rho_\varepsilon(x - y)| \, dx \, dy \, d\mu(z), \end{aligned} \quad (5.25)$$

where $Lb(\bar{u}^c)$ is a local Lipschitz constant of $b(\bar{u}^c)$. As the integrand of the previous integral is globally nonnegative, we expand the domain of integration to attain an upper bound. Ignoring the factor $Lb(\bar{u}^c)/2$, the previous integral is then bounded by

$$\begin{aligned} & \int_{\substack{x, y \in \mathbb{R}^d \\ |z| < r}} |x - y| |\chi_\Omega(y + z) - \chi_\Omega(y)| |\rho_\varepsilon(x - y - z) - \rho_\varepsilon(x - y)| \, dx \, dy \, d\mu(z) \\ & = \int_{\substack{x, y \in \mathbb{R}^d \\ |z| < r}} |x| |\chi_\Omega(y + z) - \chi_\Omega(y)| |\rho_\varepsilon(x - z) - \rho_\varepsilon(x)| \, dx \, dy \, d\mu(z) \\ & \leq |\chi_\Omega|_{\text{TV}(\mathbb{R}^d)} \int_{\substack{x \in \mathbb{R}^d \\ |z| < r}} |x| |\rho_\varepsilon(x - z) - \rho_\varepsilon(x)| |z| \, dx \, d\mu(z), \end{aligned}$$

where we recall that $|\chi_\Omega|_{\text{TV}(\mathbb{R}^d)} = |\partial\Omega|_{\mathcal{H}^{d-1}} < \infty$ (see the comment after Proposition 3.8). For the quantity $|\rho_\varepsilon(x - z) - \rho_\varepsilon(x)|$ we have the estimate

$$|\rho_\varepsilon(x - z) - \rho_\varepsilon(x)| \leq \frac{1}{\varepsilon^d} \left(2\|\rho\|_{L^\infty(\mathbb{R}^d)} \wedge \frac{|z|}{\varepsilon} \|\nabla\rho\|_{L^\infty(\mathbb{R}^d)} \right) \lesssim \frac{|z|}{\varepsilon^d(|z| + \varepsilon)},$$

where the last bound exploits that $(1 \wedge p) \leq 2p/(1 + p)$ when $p \geq 0$, giving the desired conclusion for $p = |z|/\varepsilon$. Moreover, the support of the said quantity is contained in $\{(x, z): |x| \wedge |x - z| < \varepsilon\}$; for fixed z , this set is of measure $\simeq \varepsilon^d$, and any x in it satisfies $|x| < |z| + \varepsilon$. Putting this together yields

$$\int_{x \in \mathbb{R}^d} |x| |\rho_\varepsilon(x - z) - \rho_\varepsilon(x)| \, dx \lesssim \varepsilon^d (|z| + \varepsilon) \frac{|z|}{\varepsilon^d (|z| + \varepsilon)} = |z|,$$

so that we may conclude

$$\int_{\substack{x \in \mathbb{R}^d \\ |z| < r}} |x| |\rho_\varepsilon(x - z) - \rho_\varepsilon(x)| |z| \, dx \, d\mu(z) \lesssim \int_{|z| < r} |z|^2 \, d\mu(z) \rightarrow 0, \quad r \rightarrow 0.$$

To summarize, we have just demonstrated that both terms on the right-hand side of (5.24) vanish when taking $\lim_{r \rightarrow 0} \limsup_{\varepsilon \rightarrow 0}$. For the term we subtracted on the left-hand side of (5.24) we use the relation between $\mathcal{B}^{<r}$ and $\mathcal{L}^{<r}$ from Proposition 3.1 to compute

$$\begin{aligned} \int_{\Omega_x \times \mathbb{R}_y^d} |b(u) - b(\bar{u}^c)| \mathcal{B}_y^{<r}[\chi_\Omega^y, \rho_\varepsilon] \, dx \, dy &= - \int_{\Omega_x \times \Omega_y} |b(u) - b(\bar{u}^c)| \mathcal{L}_y^{<r}[\rho_\varepsilon] \, dx \, dy \\ &= - \int_{\Omega_x} |b(u) - b(\bar{u}^c)| \mathcal{L}_x^{<r}[\varrho_\varepsilon] \, dx \\ &= \int_{\mathbb{R}_x^d} \mathcal{B}_x^{<r}[|b(u) - b(\bar{u}^c)|, \varrho_\varepsilon] \, dx, \end{aligned}$$

where ϱ_ε is as defined in (5.18) and where the compact support of $|b(u) - b(\bar{u}^c)|$ justified expanding the domain of integration from Ω_x to \mathbb{R}_x^d (so to shift $\mathcal{L}_x^{<r}$ over to $\mathcal{B}_x^{<r}$). All in all, analogous to (5.23), we see that the second integral in (5.21) can be written

$$\begin{aligned} & - \int_0^T \int_{\Omega_x \times \mathbb{R}_y^d} |b(u) - b(v)| \mathcal{B}_y^{<r}[\chi_\Omega^y, \rho_\varepsilon] \alpha \, dx \, dy \, dt \\ &= \int_{M_x} \mathcal{B}_x^{<r}[|b(u) - b(\bar{u}^c)|, \varrho_\varepsilon] \alpha \, dx \, dt + o(1), \end{aligned} \tag{5.26}$$

where the o -term is such that $\lim_{r \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} o(1) = 0$. Finally, combining (5.21) with (5.23) and (5.26) we conclude that

$$\begin{aligned} & \lim_{r \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \lim_{\epsilon \rightarrow 0} L_u^2 \\ &= \lim_{r \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \left(\int_{\mathcal{Q}_x} F(u, \bar{u}^c) \cdot \nabla_x \varrho_\varepsilon \alpha - \int_{M_x} \mathcal{B}_x^{<r}[|b(u) - b(\bar{u}^c)|, \varrho_\varepsilon] \alpha \right). \end{aligned}$$

Observe that $(\varrho_\varepsilon)_{\varepsilon>0}$ is a boundary layer sequence satisfying $\lim_{\varepsilon \rightarrow 0} \varrho_\varepsilon = \frac{1}{2}$ pointwise on $\partial\Omega$. By the weak trace result of Proposition 4.8 we conclude

$$\lim_{r \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \lim_{\varepsilon \rightarrow 0} L_u^2 = \frac{1}{2} \int_{\Gamma} \alpha \, d\nu_u,$$

where we used that $\lim_{r \rightarrow 0} \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \chi_\Omega] = 0$ in $L^1(M)$ as follows from Corollary 4.7. By analogous arguments, we also find that

$$\lim_{r \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \lim_{\varepsilon \rightarrow 0} L_v^2 = \frac{1}{2} \int_{\Gamma} \alpha \, d\nu_v.$$

(10) *The limits of L_u^3 and L_v^3 .* We will demonstrate that both limits vanish. Starting with L_u^3 , we begin as before by going to the diagonal in the time variable,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} L_u^3 &= \int_0^T \int_{\mathbb{R}_x^d \times \mathbb{R}_y^d} |b(u) - b(v)| \mathcal{B}_{x,y}^{<r} [\chi_\Omega^x, \chi_\Omega^y] \rho_\varepsilon \alpha \, dx \, dy \, dt \\ &\quad - \int_0^T \int_{\mathbb{R}_x^d \times \mathbb{R}_y^d} \mathcal{B}_y^{<r} [|b(u) - b(\bar{v}^c)|, \chi_\Omega^y] \chi_\Omega^x \rho_\varepsilon \alpha \, dx \, dy \, dt, \end{aligned} \quad (5.27)$$

where the time dependence of v and \bar{v}^c is now in t . For brevity, we again ignore α , the integral in t , and arguments in t .

The second integral in (5.27) is easily seen to vanish uniformly in ε as $r \rightarrow 0$. Indeed, as $b(u)$ depends on x and not y it follows by a triangle argument that

$$\begin{aligned} &\left| \int_{\mathbb{R}_x^d \times \mathbb{R}_y^d} \mathcal{B}_y^{<r} [|b(u) - b(\bar{v}^c)|, \chi_\Omega^y] \chi_\Omega^x \rho_\varepsilon \, dx \, dy \right| \\ &\leq \frac{1}{2} \int_{\substack{x,y \in \mathbb{R}^d \\ |z| < r}} |b(\bar{v}^c(y+z)) - b(\bar{v}^c(y))| |\chi_\Omega(y+z) - \chi_\Omega(y)| \rho_\varepsilon(x-y) \, dx \, dy \, d\mu(z) \\ &\leq \frac{L_b(\bar{v}^c)}{2} \int_{\substack{y \in \mathbb{R}^d \\ |z| < r}} |z| |\chi_\Omega(y+z) - \chi_\Omega(y)| \, dy \, d\mu(z) \\ &\leq \frac{L_b(\bar{v}^c)}{2} |\chi_\Omega|_{\text{TV}} \int_{|z| < r} |z|^2 \, d\mu(z) \rightarrow 0, \quad r \rightarrow 0, \end{aligned} \quad (5.28)$$

where $L_b(\bar{v}^c)$ denotes a local Lipschitz constant of \bar{v}^c . Thus, we turn our attention to the first integral in (5.27). We begin by a similar argument to that used for the second integral in (5.21): Observe that

$$\begin{aligned} &\left| \int_{\mathbb{R}_x^d \times \mathbb{R}_y^d} (|b(u) - b(v)| - |b(\bar{u}^c) - b(\bar{v}^c)|) \mathcal{B}_{x,y}^{<r} [\chi_\Omega^x, \chi_\Omega^y] \rho_\varepsilon \, dx \, dy \right| \\ &\leq \int_{\mathbb{R}_x^d \times \mathbb{R}_y^d} (|b(u) - b(\bar{u}^c)| + |b(v) - b(\bar{v}^c)|) \mathcal{B}_{x,y}^{<r} [\chi_\Omega^x, \chi_\Omega^y] \rho_\varepsilon \, dx \, dy. \end{aligned} \quad (5.29)$$

We now show that the right-hand side vanishes uniformly in ε as $r \rightarrow 0$. Focusing on the $|b(u) - b(\bar{u}^c)|$ term (which is supported in Ω_x) we have

$$\begin{aligned} & \int_{\mathbb{R}_x^d \times \mathbb{R}_y^d} |b(u) - b(\bar{u}^c)| |\mathcal{B}_{x,y}^{<r}[\chi_\Omega^x, \chi_\Omega^y]| \rho_\varepsilon \, dx \, dy \\ &= \frac{1}{2} \int_{\substack{x \in \Omega \\ x+z \in \Omega^c \\ |z| < r}} |b(u(x)) - b(\bar{u}^c(x))| \left(\int_{\mathbb{R}_y^d} |\chi_\Omega(y+z) - \chi_\Omega(y)| \rho_\varepsilon \, dy \right) dx \, d\mu(z) \\ &\leq \int_{\substack{x \in \Omega \\ x+z \in \Omega^c \\ |z| < r}} |b(u(x)) - b(\bar{u}^c(x))| \, dx \, d\mu(z) \rightarrow 0, \quad r \rightarrow 0, \end{aligned}$$

where the limit holds by the boundary integrability of Proposition 4.6. As a corresponding computation can be carried out for the $|b(v) - b(\bar{v}^c)|$ term, we conclude that the right-hand side of (5.29) vanishes when taking $\lim_{r \rightarrow 0} \limsup_{\varepsilon \rightarrow 0}$. Thus, we may replace $|b(u) - b(v)|$ with the more regular $|b(\bar{u}^c) - b(\bar{v}^c)|$ when seeking the limit of the first integral in (5.27). Moreover, by a similar argument to that used to prove Proposition 3.1, we may shift $\mathcal{B}_{x,y}^{<r}[\chi_\Omega^x, \cdot]$ from χ_Ω^y over to $|b(\bar{u}^c) - b(\bar{v}^c)| \rho_\varepsilon$ giving

$$\begin{aligned} & \int_{\mathbb{R}_x^d \times \mathbb{R}_y^d} |b(\bar{u}^c) - b(\bar{v}^c)| |\mathcal{B}_{x,y}^{<r}[\chi_\Omega^x, \chi_\Omega^y]| \rho_\varepsilon \, dx \, dy \\ &= \int_{\mathbb{R}_x^d \times \mathbb{R}_y^d} \mathcal{B}_{x,y}^{<r}[\chi_\Omega^x, |b(\bar{u}^c) - b(\bar{v}^c)| \rho_\varepsilon] \chi_\Omega^y \, dx \, dy. \end{aligned}$$

And by a triangle inequality argument, we find

$$\begin{aligned} & \int_{\mathbb{R}_x^d \times \Omega_y} |\mathcal{B}_{x,y}^{<r}[\chi_\Omega^x, |b(\bar{u}^c) - b(\bar{v}^c)| \rho_\varepsilon]| \, dx \, dy \\ &\leq \int_{\mathbb{R}_x^d \times \Omega_y} \int_{|z| < r} |\chi_\Omega(x+z) - \chi_\Omega(x)| |b(\bar{v}^c(y+z)) - b(\bar{v}^c(y))| \\ &\quad \times \rho_\varepsilon(x-y-z) \, d\mu(z) \, dx \, dy \\ &\quad + \int_{\mathbb{R}_x^d \times \Omega_y} \int_{|z| < r} |\chi_\Omega(x+z) - \chi_\Omega(x)| |b(\bar{u}^c) - b(\bar{v}^c)| \\ &\quad \times |\rho_\varepsilon(x-y-z) - \rho_\varepsilon(x-y)| \, d\mu(z) \, dx \, dy. \end{aligned}$$

The terms on the right-hand side are of a similar form to the ones from (5.28) and (5.25) respectively; we infer that both vanish uniformly in ε as $r \rightarrow 0$.

In conclusion, both integrals on the right-hand side of (5.27) vanish when taking $\lim_{r \rightarrow 0} \limsup_{\varepsilon \rightarrow 0}$, and so

$$\lim_{r \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \lim_{\epsilon \rightarrow 0} L_u^3 = 0.$$

By analogous arguments, we also obtain

$$\lim_{r \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \lim_{\epsilon \rightarrow 0} L_v^3 = 0.$$

(11) *Concluding.* Together with (5.16) and (5.17), these limits imply that

$$-\int_Q |u(t, x) - v(t, x)|\alpha'(t) \, dx \, dt \leq 0. \tag{5.30}$$

This in turn gives for a.e. $\tau \in (0, T)$ the L^1 -contraction

$$\int_\Omega |u(\tau, x) - v(\tau, x)| \, dx \leq \int_\Omega |u_0(x) - v_0(x)| \, dx.$$

Indeed, this standard implication follows here by letting $\alpha \rightarrow \chi_{(0,\tau)}$ pointwise while $\|\alpha'\|_{L^1((0,T))} \leq 2$; the initial L^1 -continuity of u and v (Lemma 2.5) and the Lebesgue differentiation theorem then gives the above inequality for a.e. $\tau \in (0, T)$. With this, uniqueness is established. ■

A. Entropy solutions are a generalized solution concept

Entropy solutions in the sense of Definition 2.2 naturally contain classical solutions, and they are more restrictive than very weak solutions.

Lemma A.1. *Assume (A $_\Omega$)–(A $_\mu$). Then*

- (a) *classical solutions of (1.1) are entropy solutions in the sense of Definition 2.2;*
- (b) *entropy solutions in the sense of Definition 2.2 satisfy*

$$\int_M (u \partial_t \varphi + f(u) \cdot \nabla \varphi - b(u) \mathcal{L}[\varphi]) \, dx \, dt = 0,$$

for all $\varphi \in C_c^\infty(Q)$, i.e. they are (very) weak solutions.

Proof. (a) Let us assume that $u, b(u) \in C_b^2([0, T] \times \mathbb{R}^d)$ and u solves (1.1) classically. It remains to check part (a) of Definition 2.2. To that end, multiply the PDE by $\text{sgn}_\epsilon^\pm(u - k)\varphi$, where

$$\text{sgn}_\epsilon^\pm(s) = \begin{cases} 0, & \pm s \leq 0, \\ \frac{s}{\epsilon}, & 0 \leq \pm s \leq \epsilon, \\ \pm 1, & \epsilon \leq \pm s, \end{cases}$$

and $0 \leq \varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$ satisfies (2.5). With the shorthand notation

$$(\psi)_\epsilon^\pm = \int_0^\psi \text{sgn}_\epsilon^\pm(\xi) \, d\xi,$$

for (real-valued) expressions ψ , we get after integrating the PDE over Q (and performing several chain rules, integrations by parts, and using (2.5)),

$$\begin{aligned}
 & - \int_Q ((u-k)_\epsilon^\pm \partial_t \varphi + \operatorname{sgn}_\epsilon^\pm(u-k)(f(u)-f(k)) \cdot \nabla \varphi) \, dx \, dt \\
 & \quad - \int_Q \mathcal{L}^{\geq r}[b(u)] \operatorname{sgn}_\epsilon^\pm(u-k) \varphi \, dx \, dt - \int_M (b(u)-b(k))_\epsilon^\pm \mathcal{L}^{< r}[\varphi] \, dx \, dt \\
 & = \int_\Omega (u_0 - k)_\epsilon^\pm \varphi(0, \cdot) \, dx + L_f \int_\Gamma (\bar{u}^c - k)_\epsilon^\pm \varphi \, d\sigma(x) \, dt \\
 & \quad - \int_\Gamma (\operatorname{sgn}_\epsilon^\pm(u-k)(f(u)-f(k)) \cdot \hat{n} + L_f(u-k)_\epsilon^\pm) \varphi \, d\sigma(x) \, dt \\
 & \quad + \int_M (\mathcal{L}^{< r}[b(u)] \operatorname{sgn}_\epsilon^\pm(u-k) \chi_\Omega - \mathcal{L}^{< r}[(b(u)-b(k))_\epsilon^\pm]) \varphi \, dx \, dt \\
 & \quad + \int_Q (f(u)-f(k)) \cdot (\nabla u)(\operatorname{sgn}_\epsilon^\pm)'(u-k) \varphi \, dx \, dt.
 \end{aligned}$$

We then send $\epsilon \rightarrow 0$ to obtain

$$\begin{aligned}
 & - \int_Q ((u-k)^\pm \partial_t \varphi + F^\pm(u, k) \cdot \nabla \varphi) \, dx \, dt \\
 & \quad - \int_M \mathcal{L}^{\geq r}[b(u)-b(k)] \operatorname{sgn}^\pm(u-k) \varphi \chi_\Omega \, dx \, dt - \int_M (b(u)-b(k))^\pm \mathcal{L}^{< r}[\varphi] \, dx \, dt \\
 & = \int_\Omega (u_0 - k)^\pm \varphi(0, \cdot) \, dx + L_f \int_\Gamma (\bar{u}^c - k)^\pm \varphi \, d\sigma(x) \, dt - \int_M \varphi \, d\mu^{\pm, r},
 \end{aligned}$$

where

$$\begin{aligned}
 \mu^{\pm, r} & := (\hat{n} \cdot F^\pm(u, k) + L_f(u-k)^\pm) \delta_{(t,x) \in \Gamma} \\
 & \quad + (\mathcal{L}^{< r}[(b(u)-b(k))^\pm] - \mathcal{L}^{< r}[b(u)-b(k)] \operatorname{sgn}^\pm(u-k) \chi_\Omega).
 \end{aligned}$$

The proof is complete if we can demonstrate that $\int_M \varphi \, d\mu^{\pm, r} \geq 0$. Clearly, the first part of $\mu^{\pm, r}$ is nonnegative since $|\hat{n} \cdot F^\pm(u, k)| \leq L_f(u-k)^\pm$, and since $\varphi \geq 0$ we thus infer that

$$\begin{aligned}
 \int_M \varphi \, d\mu^{\pm, r} & \geq \int_M (\mathcal{L}^{< r}[(b(u)-b(k))^\pm] - \mathcal{L}^{< r}[b(u)-b(k)] \operatorname{sgn}^\pm(u-k) \chi_\Omega) \varphi \, dx \, dt \\
 & = \int_Q (\mathcal{L}^{< r}[(b(u)-b(k))^\pm] - \mathcal{L}^{< r}[b(u)-b(k)] \operatorname{sgn}^\pm(u-k)) \varphi \, dx \, dt \\
 & \quad + \int_{M \setminus Q} \mathcal{L}^{< r}[(b(u)-b(k))^\pm] \varphi \, dx \, dt.
 \end{aligned}$$

The first term on the right-hand side is nonnegative because of Corollary C.2 (specifically its second part), while the second term is nonnegative since (2.5) gives us the identity

$$\begin{aligned} & \int_{M \setminus Q} \mathcal{L}^{<r} [(b(u) - b(k))^\pm] \varphi \, dx \, dt \\ &= \int_{M \setminus Q} \int_{|z| < r} (b(u(t, x + z)) - b(k))^\pm \varphi(t, x) \, d\mu(z) \, dx \, dt \geq 0, \end{aligned}$$

and so we are done.

(b) The proof follows by Definition 2.2, the convex inequality $\mathcal{L}^{\geq r}[\psi] \operatorname{sgn}^\pm \psi \leq \mathcal{L}^{\geq r}[\psi^\pm]$ (cf. Corollary C.2), and using that $(u - k^\mp)^\pm = \pm(u - k^\mp)$ when

$$k^\mp = \mp(\|u_0\|_{L^\infty(\Omega)} + \|u^c\|_{L^\infty(Q^c)})$$

and $k^- \leq u \leq k^+$ by Lemma 2.4. For $0 \leq \varphi \in C_c^\infty(Q)$ we then conclude from (2.6) that

$$\mp \int_Q (u \partial_t \varphi + f(u) \cdot \nabla \varphi) + \int_M b(u) \mathcal{L}[\varphi] \leq \mp \int_Q (k^\mp \partial_t \varphi + f(k^\mp) \cdot \nabla \varphi),$$

where the nonsingular operator $\mathcal{L}^{\geq r}$ has been shifted over to φ using that $\mathcal{L}^{\geq r}$ is self-adjoint (Proposition 3.1) followed by letting $r \rightarrow 0$. As φ has compact support in Q , the right-hand side is zero and thus

$$- \int_Q (u \partial_t \varphi + f(u) \cdot \nabla \varphi) \, dx \, dt + \int_M b(u) \mathcal{L}[\varphi] \, dx \, dt = 0. \quad (\text{A.1})$$

The general case when $\varphi \in C_c^\infty(Q)$ holds by picking $0 \leq \psi \in C_c^\infty(Q)$ such that $0 \leq \phi := \psi - \varphi$ and using that (A.1) holds for ϕ and ψ . ■

B. Proofs of basic properties of entropy solutions

We will prove the maximum principle and time continuity at $t = 0$ for entropy solutions.

Proof of Lemma 2.4. In the entropy inequality (2.6) for u we set k in the (+) case to $k^+ := \sup_\Omega u_0 \vee \sup_{Q^c} u^c$ and in the (-) case to $k^- := \inf_\Omega u_0 \wedge \inf_{Q^c} u^c$. Then the condition (2.5) is automatically satisfied for every $0 \leq \varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$. We then set $\varphi(t, x) = \theta(t)\phi(x)$ with $0 \leq \theta \in C_c^\infty([0, T])$ and with $0 \leq \phi \in C_c^\infty(\mathbb{R}^d)$, and where $\phi = 1$ in Ω and $\phi \leq 1$ in Ω^c . By sending $r \rightarrow \infty$ in (2.6) we find that

$$- \int_Q (u - k^\pm)^\pm \theta' \, dx \, dt - \int_M (b(u) - b(k^\pm))^\pm \mathcal{L}[\phi] \theta \, dx \, dt \leq 0.$$

As $(b(u) - b(k^\pm))^\pm = 0$ a.e. in Q^c and $\mathcal{L}[\phi] \leq 0$ in Q (ϕ attains its global maximum on every point in Q), the integrand of the second integral is clearly nonpositive and we conclude that

$$- \int_Q (u - k^\pm)^\pm \theta' \, dx \, dt \leq 0 \quad \Rightarrow \quad \int_Q (u - k^\pm)^\pm \, dx \, dt \leq 0,$$

where the implication follows from setting $\theta(t) = T - t$, which can be approximated by nonnegative elements of $C_c^\infty([0, T])$. This concludes the proof. ■

Proof of Lemma 2.5. We follow [84, Lemma 7.41]. Let $(\theta_\epsilon)_{\epsilon>0} \subset C_c^\infty([0, T])$ be a family of functions satisfying $0 \leq \theta_\epsilon(t) \leq \theta_\epsilon(0) = 1$, $\text{supp}(\theta_\epsilon) \subset [0, \epsilon]$, and $\theta_\epsilon(t) - \theta_{\epsilon'}(t) \geq 0$ when $\epsilon \geq \epsilon' > 0$. For any $0 \leq \beta \in C_c^\infty(\Omega)$, we set $\varphi(t, x) = (\theta_\epsilon - \theta_{\epsilon'})(t)\beta(x)$ in (2.6), add the inequalities for $(\cdot)^+$ and $(\cdot)^-$, and since all terms are bounded and $\theta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$, we observe that

$$\limsup_{\epsilon \rightarrow 0} \sup_{\epsilon' \in (0, \epsilon)} - \iint_Q |u(t, x) - k|\beta(x)(\theta'_\epsilon - \theta'_{\epsilon'})(t) \, dx \, dt \leq 0, \tag{B.1}$$

or equivalently,

$$\limsup_{\epsilon \rightarrow 0} \iint_Q |u(t, x) - k|\beta(x)(-\theta'_\epsilon)(t) \, dx \, dt \leq \liminf_{\epsilon' \rightarrow 0} \iint_Q |u(t, x) - k|\beta(x)(-\theta'_{\epsilon'})(t) \, dx \, dt.$$

In the same way, but with $\varphi(t, x) = \theta_\epsilon(t)\beta(x)$, we find that

$$\lim_{\epsilon \rightarrow 0} \int_0^\epsilon \left(\int_\Omega |u(t, x) - k|\beta(x) \, dx \right) (-\theta'_\epsilon(t)) \, dt \leq \int_\Omega |u_0(x) - k|\beta(x) \, dx, \tag{B.2}$$

where the limit on the left-hand side exists by (B.1). It then follows that

$$\text{ess lim sup}_{t \rightarrow 0^+} \int_\Omega |u(t, x) - k|\beta(x) \, dx \leq \int_\Omega |u_0(x) - k|\beta(x) \, dx. \tag{B.3}$$

Indeed, assume (B.3) is false: There is an $r > 0$ and a set $E \subset [0, T]$ with $\text{meas}\{E \cap (0, \delta)\} > 0$ for all $\delta > 0$, such that $\int_\Omega |u(t, x) - k|\beta(x) \, dx > \int_\Omega |u_0(x) - k|\beta(x) \, dx + r$ for $t \in E$. Let $\theta_\epsilon(t) = 1 - \int_0^t \psi(s) \, ds$ and $\psi(s) = \frac{1}{\text{meas } E_\epsilon} 1_{E_\epsilon} * \rho_\delta(t)$ where $\delta \in (0, \epsilon)$, $E_\epsilon := E \cap (0, \epsilon)$, $\rho_\delta = \frac{1}{\delta} \rho(\frac{\cdot}{\delta})$, and $0 \leq \rho \in C_c^\infty((0, T))$ with $\int_0^T \rho \, dt = 1$. Integration in time against $\theta'_\epsilon = -\psi$ and sending $\delta \rightarrow 0$ then gives for all $\epsilon > 0$,

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \int_0^\epsilon \left(\int_\Omega |u(t, x) - k|\beta(x) \, dx \right) (-\theta'_\epsilon(t)) \, dt \\ &= \frac{1}{\text{meas } E_\epsilon} \int_{E_\epsilon} \int_\Omega |u(t, x) - k|\beta(x) \, dx \, dt \\ &> \int_\Omega |u_0(x) - k|\beta(x) \, dx + r, \end{aligned}$$

which contradicts (B.2).

By approximation, (B.3) holds also when β is a characteristic function. Since k is arbitrary, it follows by linearity of integrals and density of simple functions in $L^1(\Omega)$, that for any $v_0 \in L^\infty(\Omega) \subset L^1(\Omega)$,

$$\text{ess lim sup}_{t \rightarrow 0^+} \int_\Omega |u(t, x) - v_0(x)| \, dx \leq \int_\Omega |u_0(x) - v_0(x)| \, dx.$$

Taking $v_0 = u_0$ and using $0 \geq \text{ess lim sup}[\dots] \geq \text{ess lim inf}[\dots] \geq 0$, the proof is complete. ■

C. Convex inequality for nonlocal operators

The following lemma is well known, and so we omit the proof.

Lemma C.1. *Assume $\eta: \mathbb{R} \rightarrow \mathbb{R}$ is convex. Then, for any real numbers v, w , we have*

$$\eta(v) - \eta(w) \geq \eta'(w)(v - w),$$

where η' is a subderivative of η , that is, $\eta'(w) \in [\eta'(w-), \eta'(w+)]$.

Corollary C.2. *Let \mathcal{L} be a Lévy operator and $\eta: \mathbb{R} \rightarrow \mathbb{R}$ be convex. For $\phi \in L^\infty(\mathbb{R}^d)$ we have*

$$\mathcal{L}[\phi]\eta'(\phi) \leq \mathcal{L}[\eta(\phi)] \quad \text{a.e. in } \mathbb{R}^d,$$

provided the expressions are well defined in $L^1_{\text{loc}}(\mathbb{R}^d)$. In the special case $\eta = (\cdot)^\pm$, and with $h: \mathbb{R} \rightarrow \mathbb{R}$ a nondecreasing function, then for any $\phi, \psi \in L^\infty(\mathbb{R}^d)$ we have

$$\mathcal{L}[h(\phi) - h(\psi)] \text{sgn}^\pm(\phi - \psi) \leq \mathcal{L}[(h(\phi) - h(\psi))^\pm] \quad \text{a.e. in } \mathbb{R}^d,$$

again, provided that the expressions are well defined in $L^1_{\text{loc}}(\mathbb{R}^d)$.

Proof. It suffices to prove the results for zero-order Lévy operators, where $\mu(\mathbb{R}^d) < \infty$, since the general case then follows from a truncation argument (by assumption, $\lim_{r \rightarrow 0} \mathcal{L}^{\geq r} = \mathcal{L}$ for the involved expressions). Thus, we assume \mathcal{L} is of zero order.

The first inequality is a direct application of Lemma C.1 and so we focus on the second one: one gets the desired bound after writing out $\mathcal{L}[h(\phi) - h(\psi)](x) \text{sgn}^\pm(\phi - \psi)(x)$ in full, and then use the bound and identity

$$\begin{aligned} (h(\phi(x+z)) - h(\psi(x+z))) \text{sgn}^\pm(\phi(x) - \psi(x)) &\leq (h(\phi(x+z)) - h(\psi(x+z)))^\pm, \\ (h(\phi(x)) - h(\psi(x))) \text{sgn}^\pm(\phi(x) - \psi(x)) &= (h(\phi(x)) - h(\psi(x)))^\pm. \end{aligned} \tag{C.1}$$

The first line in (C.1) follows from the two trivial implications

$$w \in [0, 1] \Rightarrow vw \leq (v)^+, \quad w \in [-1, 0] \Rightarrow vw \leq (v)^-,$$

valid for all $v \in \mathbb{R}$, while the second line in (C.1) follows from the fact that h is non-decreasing. ■

D. Additional identities for the bilinear operator

We show here that the expression $\mathcal{B}_{x+y}[\phi, \psi]$ is somewhat analogous to the expression

$$(\nabla_x \phi + \nabla_y \phi) \cdot (\nabla_x \psi + \nabla_y \psi),$$

and that, in particular, we need not introduce “cross-terms” for the diffusion in the doubling of variables argument as is needed in the local case (cf. [36, 71, 85, 87]).

Proposition D.1 (Cross-terms formula). *Assume (A_μ) . Let $\phi, \psi \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ be functions in (x, y) and let ψ have compact support. Using the operator notation from Section 2, we then have*

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{B}_{x+y}[\phi, \psi] \, dx \, dy = \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{B}_{x,x+y}[\phi, \psi] + \mathcal{B}_{y,x+y}[\phi, \psi] \, dx \, dy,$$

provided the expressions in the integrals are well defined in $L^1(\mathbb{R}^d \times \mathbb{R}^d)$.

Proof. Fix $r > 0$. Observe that the integrands in the following argument are absolutely integrable with respect to $dx \, dy \, d\mu(z)$ due to the compact support of ψ and the truncation of our operators; in particular we are free to deploy Fubini's theorem. We compute

$$\begin{aligned} & 2 \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{B}_{x,x+y}^{\geq r}[\phi, \psi] \, dx \, dy \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|z| \geq r} (\phi(x+z, y) - \phi(x, y))(\psi(x+z, y+z) - \psi(x, y)) \, d\mu(z) \, dx \, dy \\ &= \int_{|z| \geq r} \int_{\mathbb{R}^d \times \mathbb{R}^d} (\phi(x, y-z) - \phi(x-z, y-z))\psi(x, y) \, dx \, dy \, d\mu(z) \\ &\quad - \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|z| \geq r} (\phi(x+z, y) - \phi(x, y))\psi(x, y) \, d\mu(z) \, dx \, dy \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|z| \geq r} (\phi(x, y) - \phi(x-z, y-z))\psi(x, y) \, d\mu(z) \, dx \, dy \\ &\quad + \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|z| \geq r} (\phi(x, y-z) - \phi(x+z, y))\psi(x, y) \, d\mu(z) \, dx \, dy. \end{aligned}$$

By the change of variables $z \mapsto -z$ and the symmetry of $d\mu(z)$ we see that the first integral on the right-hand side coincides with

$$- \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{L}_{x+y}^{\geq r}[\phi]\psi \, dx \, dy = \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{B}_{x+y}^{\geq r}[\phi, \psi] \, dx \, dy,$$

where the identity follows from Proposition 3.1. By a similar computation for the $\mathcal{B}_{y,x+y}^{\geq r}$ operator, we conclude

$$\begin{aligned} & \int_{\mathbb{R}^d \times \mathbb{R}^d} (\mathcal{B}_{x,x+y}^{\geq r} + \mathcal{B}_{y,x+y}^{\geq r})[\phi, \psi] \, dx \, dy \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{B}_{x+y}^{\geq r}[\phi, \psi] \, dx \, dy \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|z| \geq r} (\phi(x, y-z) - \phi(x+z, y))\psi(x, y) \, d\mu(z) \, dx \, dy \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|z| \geq r} (\phi(x-z, y) - \phi(x, y+z))\psi(x, y) \, d\mu(z) \, dx \, dy. \end{aligned}$$

The last two integrals cancel out, as can be seen by substituting $z \mapsto -z$ in either, and so the result follows by letting $r \rightarrow 0$. ■

We also introduce a specific product rule, which is useful in the uniqueness proof.

Lemma D.2 (An auxiliary product rule). *Assume (A_μ) . Let $\phi \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ be a function in (x, y) , and let $\zeta^x, \zeta^y, \rho \in L^\infty(\mathbb{R}^d)$ be functions in x, y , and $x - y$ respectively. Further, let ζ^x and ζ^y have compact support. Using the operator notation from Section 2, we then have*

$$\begin{aligned} & \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{B}_{x, x+y}[\phi, \zeta^x \zeta^y] \rho \, dx \, dy \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{B}_x[\phi(\cdot, y), \zeta^x] \zeta^y \rho \, dx \, dy \\ & \quad + \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(\mathcal{B}_y[\zeta^y, \rho(x - \cdot)]) \zeta^x - \mathcal{B}_{x, y}[\zeta^x, \zeta^y] \rho \, dx \, dy, \end{aligned}$$

provided the expressions in the integrals are well defined in $L^1(\mathbb{R}^d \times \mathbb{R}^d)$.

Proof. Fix $r > 0$. The compact support of ζ^x, ζ^y ensures that the following integrals are well defined. Adding and subtracting $\zeta^x(x + z)\zeta^y(y)$ to the integrand, and performing a few changes of variables, we compute

$$\begin{aligned} & \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{B}_{x, x+y}^{\geq r}[\phi, \zeta^x \zeta^y] \rho \, dx \, dy \\ &= \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|z| \geq r} (\phi(x + z, y) - \phi(x, y)) \\ & \quad \times (\zeta^x(x + z)\zeta^y(y + z) - \zeta^x(x)\zeta^y(y)) \rho(x - y) \, d\mu(z) \, dx \, dy \\ &= \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|z| \geq r} (\phi(x + z, y) - \phi(x, y)) \\ & \quad \times (\zeta^x(x + z) - \zeta^x(x)) \zeta^y(y) \rho(x - y) \, d\mu(z) \, dx \, dy \\ & \quad + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|z| \geq r} (\phi(x + z, y) - \phi(x, y)) \\ & \quad \times (\zeta^y(y + z) - \zeta^y(y)) \zeta^x(x + z) \rho(x - y) \, d\mu(z) \, dx \, dy. \end{aligned}$$

The first integral on the right-hand side is $\int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{B}_{x'}^{\geq r}[\phi(\cdot, y), \zeta^x] \zeta^y \rho \, dx \, dy$. For the second integral, we split it up and perform a change of variables so to get

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|z| \geq r} (\phi(x + z, y) - \phi(x, y)) \\ & \quad \times (\zeta^y(y + z) - \zeta^y(y)) \zeta^x(x + z) \rho(x - y) \, d\mu(z) \, dx \, dy \\ &= \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|z| \geq r} \phi(x, y) (\zeta^y(y + z) - \zeta^y(y)) \zeta^x(x) \rho(x - z - y) \, d\mu(z) \, dx \, dy \\ & \quad - \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|z| \geq r} \phi(x, y) (\zeta^y(y + z) - \zeta^y(y)) \zeta^x(x + z) \rho(x - y) \, d\mu(z) \, dx \, dy. \end{aligned}$$

If we further add and subtract the integral of $\phi(x, y) (\zeta^y(y + z) - \zeta^y(y)) \zeta^x(x) \rho(x - y)$ to this right-hand side we get

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \phi \mathcal{B}_y^{\geq r}[\zeta^y, \rho(x - \cdot)] \zeta^x \, dx \, dy - \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi \mathcal{B}_{x, y}^{\geq r}[\zeta^x, \zeta^y] \rho \, dx \, dy,$$

where we used that $\rho(x - z - y) = \rho(x - (y + z))$. Letting $r \rightarrow 0$ gives the result. \blacksquare

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