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Critical local well-posedness of the nonlinear Schrödinger equation on the torus

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Abstract. In this paper, we study the local well-posedness of nonlinear Schrödinger equations on tori $\mathbb{T}^d$ at the critical regularity. We focus on cases where the nonlinearity $|u|^a u$ is nonalgebraic with small $a > 0$. We prove the local well-posedness for a wide range covering the mass-supercritical regime. Moreover, we supplementarily investigate the regularity of the solution map. In pursuit of lowering a, we prove a bilinear estimate for the Schrödinger operator on tori $\mathbb{T}^d$, which enhances previously known multilinear estimates. We design a function space adapted to the new bilinear estimate and a package of Strichartz estimates, which is not based on conventional atomic spaces.

1. Introduction

1.1. Statement of the problem and main results

The subject of this paper is the critical local well-posedness and ill-posedness of the Cauchy problem for the nonlinear Schrödinger equation (NLS) on periodic spaces $\mathbb{T}^d$,

$$\begin{cases} iu_t + \Delta u = \pm |u|^a u =: \mathcal{N}(u), \\ u(0) = u_0 \in H^s(\mathbb{T}^d), \end{cases} \tag{NLS}$$

where $u: \mathbb{R} \times \mathbb{T}^d \to \mathbb{C}$.

The nonlinearity $|u|^a u$ is of a single-power type for $a > 0$. When a is an even integer, the nonlinearity $|u|^a u$ is algebraic. Otherwise, $|u|^a u$ is said to be nonalgebraic. In this work, we are particularly interested in the case of nonalgebraic nonlinearity, especially when a is small. At a glance, for small nonalgebraic a, one can observe that the regularity of solutions has certain restrictions and anticipate that the solution map is less regular. However, it turns out that there is a genuine difficulty: *the nonlinear term $|u|^a u$ is not sufficiently decomposable*. For instance, if one tries to take a paraproduct decomposition, there is not enough summability from existing technology. In this work, we investigate these issues and overcome them by introducing new function spaces and bilinear estimates. For the negative direction, we also study limitations on the regularity of the solution map.

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In view of scaling considerations, the critical Sobolev regularity is

$$s := s_c = \frac{d}{2} - \frac{2}{a}. \tag{1.1}$$

Since we consider only the critical local problem in $H^{s_c}(\mathbb{T}^d)$, we simply denote $s = s_c$. We say (NLS) is mass-critical if $s = 0$ and energy-critical if $s = 1$.

Firstly, we state our main theorem, the critical local well-posedness of a wide range of NLS on $\mathbb{T}^d$.

Theorem 1.1. *Let $a > \frac{4}{d}$ (or equivalently, $s > 0$). Assume $s < 1 + a$. Then* (NLS) *is locally well posed in the critical Sobolev space $H^s(\mathbb{T}^d)$.*

For a technical statement of Theorem 1.1, see Proposition 4.6. Theorem 1.1 extends many existing results on the critical local well-posedness and covers a new regime of small a. This covers a wide range of the mass-supercritical regime $s > 0$. In particular, this includes all energy-critical cases, for which the result is new for dimensions $d \geq 5$. The restriction $s < 1 + a$ arises from the fact that $|u|^a u$ does not have regularities higher than $1 + a$ for smooth functions u in general.

Next we consider the regularity of the solution map. In Theorem 1.1, we know only that the flow map is continuous from $H^s(\mathbb{T}^d)$ to $C^0([0,T]; H^s(\mathbb{T}^d))$. Yet, for a narrower range of a, we show the Lipschitz regularity of the flow map.

Theorem 1.2. *Assume that*

$$a > \max\left\{\frac{4}{d}, 1\right\} \quad \text{and} \quad s < a.$$

Then (NLS) *is locally Lipschitz well posed in $H^s(\mathbb{T}^d)$.*

On the other hand, if a is even lower, one expects the solution map to be less regular. As a negative result, we show that when a is smaller than 1, the solution map fails to be locally Lipschitz continuous. More precisely, we have the failure of α-Hölder continuity.

Theorem 1.3. *Assume*

$$0 < a < 1 \quad \text{and} \quad 0 < s < 1 + \frac{1}{a}. \tag{1.2}$$

Then the solution map fails to be locally α-Hölder continuous in $H^s(\mathbb{T}^d)$ for each $\alpha > a$.

More explicitly, there is no radius $\varepsilon > 0$ and time $T > 0$ such that for every $u_0, v_0 \in H^s(\mathbb{T}^d)$ with $\|u_0\|_{H^s}, \|v_0\|_{H^s} < \varepsilon$, the corresponding solutions u and v to (NLS) *satisfy $\|u - v\|_{C^0 H^s([0,T]\times\mathbb{T}^d)} \lesssim \|u_0 - v_0\|_{H^s}^{\alpha}$.*

Remark 1.4. Our proof does not rely on number-theoretic arguments on frequencies. Thus, the proof works for irrational tori $\widetilde{\mathbb{T}}^d = \mathbb{R}^d/(\theta_1\mathbb{Z} \times \cdots \times \theta_d\mathbb{Z})$ with any $\theta_j > 0$. For simplicity, in this paper, we assume our domain is the square torus $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$.

Remark 1.5. In Theorem 1.1, one can derive the exponent restriction by using (1.1). When $d \leq 7$, $s < 1 + a$ is void, so it holds true for any $a > \frac{4}{d}$. When $d \geq 8$, there is an uncovered band:

$$\begin{cases} \dfrac{4}{d} < a < \dfrac{d - 2 - \sqrt{d^2 - 4d - 28}}{4} \\ \text{or} \\ a > \dfrac{d - 2 + \sqrt{d^2 - 4d - 28}}{4}. \end{cases}$$

Remark 1.6. In Theorem 1.2, the restriction on exponents for the Lipschitz continuity for each dimension is as follows:

$$\begin{aligned} &(d \leq 4) \quad a > \frac{4}{d}, && s > 0, \\ &(d = 5) \quad a > 1, && s > \frac{1}{2}, \\ &(d \geq 6) \quad a > \frac{d + \sqrt{d^2 - 32}}{4}, && s > \frac{d + \sqrt{d^2 - 32}}{4}. \end{aligned}$$

In particular, we note that the energy-critical case ($s = 1$) is Lipschitz well posed when $d \leq 5$. For $d = 3, 4$, the LWP was previously proved via a contraction mapping [10, 15].

Remark 1.7. In Theorem 1.3, the range of exponents of (1.2) for each dimension is as follows:

$$\begin{aligned} &(d \leq 8) \quad a < 1, && s < \frac{d}{2} - 2, \\ &(d \geq 9) \quad a < \frac{6}{d - 2}, && s < \frac{d}{6} + \frac{2}{3}. \end{aligned}$$

In particular, we note that the energy-critical case ($s = 1$) fails to be Lipschitz well-posed when $d \geq 7$. However, when $d = 6$ ($a = 1$), Lipschitz continuity of the solution map is inconclusive.

1.2. Previous works

Bourgain [6] obtained a range of scale-invariant Strichartz estimates on square tori with a certain amount of loss of regularity, and also $X^{s,b}$ spaces were first introduced. He used these to obtain several local and small data global well-posedness results for subcritical NLS on tori. The atomic spaces U^p and V^p have been successfully used as a tool for constructing function spaces adapted to critical dispersive equations. The U^p and V^p spaces were developed for the Schrödinger operator in [3, 16] and many others. Based on atomic structures, the critical function spaces X^s and Y^s for NLS on (partially) periodic domains were introduced in [11] and [10].

Based on the development of function spaces, several local and global well-posedness results of NLS on periodic domains were shown for algebraic cases. In [11], using the

X^s spaces and multilinear Strichartz estimates, Herr, Tataru, and Tzvetkov obtained the local well-posedness and small data global well-posedness of the energy-critical NLS in $H^1(\mathbb{R}^2 \times \mathbb{T}^2)$ and $H^1(\mathbb{R}^3 \times \mathbb{T})$ with arbitrary torus parts $\mathbb{T}^m$ (including irrational tori). In [10], the same authors showed the local well-posedness and small data global well-posedness of the energy-critical NLS in $H^1(\mathbb{T}^3)$ for rational tori. In [22], the author developed scaling-critical multilinear Strichartz estimates and proved results for a larger range of exponent a. In [8], new scaling-critical Strichartz estimates on irrational tori were proved. As an application, they proved the critical local well-posedness of NLS in several regimes of algebraic nonlinearities. This result was further enhanced in [20].

Afterward, Bourgain and Demeter [7] established a Strichartz estimate with an arbitrarily small loss of scale and regularity on general irrational tori as an application of their celebrated ℓ^2-decoupling result. For rational tori, this result can be strengthened to a scale-invariant version by the argument in [6]. For irrational tori, the corresponding scale-invariant Strichartz estimate was shown in [15]. As an application of this, they obtained the local well-posedness and small data global well-posedness result for energy-critical NLS on $\mathbb{T}^3$ and $\mathbb{T}^4$.

The large data global well-posedness of the energy-critical defocusing NLS in $H^1(\mathbb{T}^3)$ was shown in [12]. In [13], the same result was shown in $H^1(\mathbb{R} \times \mathbb{T}^3)$. For focusing equations, the large data global well-posedness of the energy-critical focusing NLS in $H^1(\mathbb{T}^4)$ was shown in [23]. The aforementioned well-posedness works for algebraic nonlinearities using multilinear estimates and is based on contraction mapping arguments.

When $|u|^a u$ is nonalgebraic, Lee [18] proved the well-posedness of H^s-critical NLS in $H^s(\mathbb{T}^3)$ for $a \geq 2$ (or equivalently, $s \geq 1/2$). One main new ingredient of [18] was the *Bony linearization* [5] for nonalgebraic nonlinearities. When a nonlinearity $f(u)$ has sufficient regularity, one takes a paraproduct decomposition of $f(u)$ in terms of u_N and $\partial f(u_{\leq N})$. For given $f: \mathbb{C} \to \mathbb{C}$ and $u: \mathbb{T}^d \to \mathbb{C}$, we write

$$\begin{aligned} f(u) &= \sum_{N \in 2^{\mathbb{N}}} f(P_{\leq N} u) - f(P_{\leq N/2} u) \\ &= \sum_{N \in 2^{\mathbb{N}}} \int_0^1 P_N u \partial_z f(P_{\leq N/2} u + \theta P_N u)\, d\theta \\ &\quad + \int_0^1 \overline{P_N u} \partial_{\bar{z}} f(P_{\leq N/2} u + \theta P_N u)\, d\theta. \end{aligned}$$

Lee [18] used a contraction mapping argument based on previously known multilinear estimates [10, 15] and the Bony linearization. The condition $a \geq 2$ was required for triple iterations of Bony linearizations.

1.3. A new estimate, a function space Z^s, and proofs of the main results

The main difficulty of our well-posedness results, Theorems 1.1 and 1.2, lies in the previously unresolved regime $a < 2$. Here we investigate the limitations of existing techniques for lower a, which stems from the linear level, and introduce our new main ingredients to overcome them: a bilinear Strichartz estimate and a function space Z^s.

To date, critical Strichartz estimates have been established on pure tori $\mathbb{T}^d$ only with a loss of regularity. To compensate for the regularity loss, multilinear estimates have been used. On $\mathbb{R}^d$, for dyadic frequencies $N \gg R$, we have the bilinear estimate

$$\|P_N e^{it\Delta}\phi P_R e^{it\Delta}\psi\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim N^{\frac{d-1}{2}} R^{-\frac{1}{2}} \|\phi\|_{L^2}\|\psi\|_{L^2}. \tag{1.3}$$

We do not expect an estimate like (1.3) on $\mathbb{T}^d$, even with a finite time cutoff. Indeed, a trivial choice $\psi \equiv 1$ gives a simple counterexample for (1.3) on $\mathbb{T}^d$. Still, the following version of the bilinear estimate was previously known:

Proposition ([10, 15]). *For $d \geq 3$, when $N_1 \geq N_2$, there exists $\delta > 0$ such that*

$$\|P_{N_1} u P_{N_2} v\|_{L^2_{t,x}(I\times\mathbb{T}^d)} \lesssim_I N_2^{\frac{d-2}{2}} \left(\frac{N_2}{N_1} + \frac{1}{N_2}\right)^{\delta} \|u\|_{Y^0}\|v\|_{Y^0}. \tag{1.4}$$

A key strength of (1.4) is the decay factor $\delta > 0$. The first proof of (1.4) used spacetime almost orthogonalities, requiring both u and v to be free evolutions, and was applicable to algebraic nonlinearities. A weaker version with $\delta = 0$ allows a simpler proof by partitioning the frequency domain $\mathbb{Z}^d$ into congruent cubes. When $a \geq 2$, such a weaker estimate is sufficient for the local well-posedness by paraproduct decompositions (see [15, 18]).

The decay factor $\delta > 0$, however, becomes crucial when $a < 2$. Decomposing the nonlinear term $|u|^a u$ into a product of a linear part and the rest, say, of the form

$$|u|^a u = u \times A = \sum_{N,R\in 2^{\mathbb{N}}} P_N u \times P_R A,$$

all frequency sizes of A contribute critically if one uses a bilinear estimate without decay on a high-low product. Since $H^s(\mathbb{T}^d)$ is ℓ^2-based, we expect A to lie in any ℓ^1-based critical Besov space only if $a \geq 2$, otherwise causing a logarithmic loss in the summation.

We extend (1.4) to general Sobolev regularities through a new approach. For nice functions u and A, and dyadic numbers $N \gtrsim R$, we show

$$\|\chi_{[-1,1]} \cdot P_N u \cdot P_R A\|_{(Z^0)'} \lesssim \|u\|_{Z^0}((N/R)^{-\sigma_1} + R^{-2\sigma_1})R^{\theta}\|A\|_{B^{\frac{1}{r_0}-\frac{1}{q_0}}_{r_0,r_0} L^{r_0}}, \tag{1.5}$$

where Z^0 and $(Z^0)'$ are the new function space of this paper and its spacetime dual norm, respectively, and σ_1, q_0, r_0, and θ denote the exponents defined in (3.24) and Lemma 3.9. (See (3.25) for the precise form of (1.5).) While (1.4) in [10] was shown by estimating almost orthogonalities on the Fourier side, we detect the decay factor $\sigma_1 > 0$ for (1.5) based on the Galilean structure and spacetime Besov regularities of the Schrödinger operator.

Relying crucially on (1.5), the proof of Theorem 1.1 proceeds as follows: In view of Theorem 1.3, the solution map is not Lipschitz continuous for small a, so we do not use a contraction mapping argument. Instead, we separately show the existence of solutions, a decay of high-frequency pieces, and a contraction-type estimate in a space of lower regularity. Using Bony linearizations and (1.5), we construct an a priori bound on a solution for a short time and show the local existence by taking a weak limit. For the continuity of the solution map, we further obtain extra a priori decay on the high-frequency part of a solution. Then the problem reduces to showing a bootstrapping estimate of the difference between two solutions, which follows immediately from (1.5).

For initial data with large H^s norms, we face an obstacle in the choice of function spaces. Earlier works on the critical well-posedness of NLS on $\mathbb{T}^d$ ([8, 10, 15, 20, 22]) used atomic-based norms X^s and Y^s. The estimates used in the proof of Theorem 1.1 could also be shown in terms of Y^s. However, if one uses an atomic-based norm such as Y^s, the norm of a free evolution does not shrink sufficiently on any short time interval, making the bootstrapping inequalities not obvious for large initial data.

For the regime $a > 2$, earlier authors resolved the issue by estimating the high-quency portion of u separately. More precisely, they showed bootstrap bounds on $\|P_{\geq N}u\|_{Y^1}$ for $N \gg 1$ by paraproduct decompositions on the nonlinear term $\mathcal{N}(u)$. Such decompositions require a certain regularity of $\mathcal{N}(u)$ (or equivalently, a high power a).

We construct a new function space Z^s on $\mathbb{R} \times \mathbb{T}^d$ adapted to conventional linear estimates, the bilinear Strichartz estimate (1.5), and the desired norm-shrinking property. More precisely, the Z^s space has the following favorable properties:

(1) Boundedness of the retarded dual Schrödinger propagator from $(Z^0)'$ to Z^0, (3.11);

(2) Strichartz embeddings into Sobolev spaces, (3.14) and (3.15);

(3) Shrinking of the norm to zero as we give shorter time cutoffs; for $u \in Z^s$, $\|\chi_{[0,T]}u\|_{Z^s} \to 0$ as $T \to 0$, (3.13).

Based on this new Z^s space, the proof of Theorem 1.1 works consistently for arbitrarily large initial data.

The proof of Theorem 1.2 is similar to that of Theorem 1.1 at the level of functional estimates. For the Lipschitz regularity of the solution map, we use a contraction mapping argument in Theorem 1.2. Although a conventional contraction argument is used, the main difficulty of Theorem 1.1 that requires (1.5) and Z^s spaces is still present for the regime $1 < a < 2$, and the machinery built for Theorem 1.1 is thoroughly used.

The negative counterpart of Theorem 1.2 is addressed in Theorem 1.3 by constructing an explicit counterexample. In particular, Theorem 1.3 shows that the main assumption $a > 1$ of Theorem 1.2, which is crucially required for a difference form for a contraction inequality, is indeed almost sharp. A key observation for the construction is an oscillating behavior of the frequency-localized Schrödinger kernel $e^{it\Delta}\delta_N$ on $\mathbb{T}$, (6.2). We show that the L^2 and L^∞ norms of $e^{it\Delta}\delta_N$ are comparable on a large set of times t, which implies that $e^{it\Delta}\delta_N$ mostly tends to oscillate rather than concentrate.

The rest of the paper is organized as follows: In Section 2 we provide preliminary materials, such as notation, Strichartz estimates, and atomic spaces. In Section 3 we define the function space Z^s and show related bilinear estimates. In Section 4 we provide the proof of Theorem 1.1. In Section 5 we show Theorem 1.2. In Section 6 we prove Theorem 1.3.

2. Preliminaries

2.1. Notation

We denote $A \lesssim B$ if $A \le CB$ for some constant C.

Given a set $E \subset \mathbb{R}^d$ or $\mathbb{T}^d$, we denote by χ_E the sharp cutoff function of E.

Fourier truncations. We handle functions of spacetime variables $f(t,x)$ and $f(x)$ for $x \in \mathbb{T}^d$ and $t \in \mathbb{R}$. We denote the Fourier transform (or the Fourier series) of f with respect to the associated variables x, t, and (t,x) by $\mathcal{F}_x f$, $\mathcal{F}_t f$, and $\mathcal{F}_{t,x} f$, respectively. For simplicity, we also denote the spatial Fourier transform by $\hat{f}$ and the spacetime Fourier transform by $\tilde{f}$.

We use frequency truncation operators. For spatial frequencies, we use sharp cutoffs. For time frequencies, we use smooth cutoffs. We denote by P_C the spatial frequency cutoff projection for a given set $C \subset \mathbb{Z}^d$; that is, P_C is the Fourier multiplier operator associated with the characteristic function χ_C. For most cases, we use the Littlewood–Paley projection. We denote the set of natural numbers by $\mathbb{N} = \{0\} \cup \mathbb{Z}_+$ and dyadic numbers by $2^{\mathbb{N}}$. For a dyadic number $N \in 2^{\mathbb{N}}$, we denote the Littlewood–Paley operators by

$$P_{\le N} := P_{[-N,N]^d} \quad \text{and} \quad P_N := P_{\le N} - P_{\le N/2},$$

where we set $P_{\le 1/2} := 0$. In particular, the cutoff $P_1 = P_{\le 1}$ contains the zero frequency mode. For simplicity, we denote $u_N = P_N u$ and $u_{\le N} = P_{\le N} u$ for $u\colon \mathbb{T}^d \to \mathbb{C}$.

For time Fourier projections, we use the superscript t; $P^t_{\le N}$ is a smooth time Littlewood–Paley operator. Let $\varphi\colon \mathbb{R} \to [0,\infty)$ be a smooth even bump function such that $\varphi|_{[-1,1]} \equiv 1$ and $\operatorname{supp}(\varphi) \subset [-\frac{11}{10}, \frac{11}{10}]$. For a dyadic number $N \in 2^{\mathbb{N}}$, we denote by $\varphi_N\colon \mathbb{R} \to [0,\infty)$ the function $\varphi_N(t) = \varphi(t/N)$. We denote by $P^t_{\le N}$ the Fourier multiplier operator induced by φ_N.

In Section 6 we will use a smooth cutoff for the spatial frequency truncation operator on $\mathbb{T}$. For a dyadic number $N \in 2^{\mathbb{N}}$, we denote by $\mathbb{P}_{\le N}$ the smooth Littlewood–Paley operator on $\mathbb{T}$, i.e., $\mathbb{P}_{\le N}$ denotes the Fourier multiplier operator induced by φ_N. We also denote by $\delta_N = \mathbb{P}_N \delta$ the function on $\mathbb{T}$ defined as $\mathcal{F}_x^{-1} \varphi_N$.

Paraproducts. We use paraproduct decompositions on spatial frequencies. Given functions u and v defined on either $\mathbb{T}^d$ or $\mathbb{R} \times \mathbb{T}^d$, we denote their paraproducts by

$$\pi_>(u,v) := \sum_{M \ge 32N} u_M v_N \quad \text{and} \quad \pi_<(u,v) := \sum_{N \ge 32M} u_M v_N,$$

where the summations are made over dyadic numbers. Similarly, we also use the notation $\pi_\ge(u,v) := \sum_{M \ge \frac{1}{16}N} u_M v_N$ and $\pi_\le(u,v) := \sum_{N \ge \frac{1}{16}M} u_M v_N$.

Interpolations. We use various function spaces for functions defined on $\mathbb{T}^d$ or $\mathbb{R} \times \mathbb{T}^d$, such as L^p-based spaces, atomic spaces, and the spaces X^s and Y^s generated from the atomic spaces. Each space is a Banach space and we denote the norm of a Banach space B by $\|f\|_B$.

We denote by B' the dual of B with respect to the inner product $\langle u, v\rangle := \int \bar{u}v$ over the domain $\mathbb{T}^d$ or $\mathbb{R} \times \mathbb{T}^d$.

A finite collection of Banach spaces $(B_1, \ldots, B_n)$ is said to be an interpolation tuple if $B_1, \ldots, B_n$ can be embedded simultaneously in a Hausdorff topological vector space. For an interpolation tuple $(B_1, \ldots, B_n)$, we define the intersection and the sum of Banach spaces $\bigcap_{j=1}^n B_j$ and $\sum_{j=1}^n B_j$ by the norms

$$\|u\|_{\bigcap_{j=1}^n B_j} := \max_j \|u\|_{B_j}$$

and

$$\|u\|_{\sum_{j=1}^n B_j} := \inf_{\substack{u_1+\cdots+u_n=u \\ u_j \in B_j}} \sum_j \|u_j\|_{B_j},$$

respectively.

We use conventional notation for interpolation spaces. Let $\theta \in (0, 1)$ and let (B_0, B_1) be an interpolation couple. The complex interpolation space between B_0 and B_1 of exponent θ is denoted by $[B_0, B_1]_\theta$.

Given $q \in [1, \infty]$, the real interpolation space between B_0 and B_1 of exponent θ and parameter q is denoted by $(B_0, B_1)_{\theta,q}$. (For more details, see, for example, [4].)

2.2. Function spaces

Here we collect well-known facts regarding function spaces.

Given $q \in [1, \infty]$ and a Banach space E defined on $\mathbb{T}^d$, the mixed norm $L^q E = L^q_t E$ is defined as

$$\|u\|_{L^q E} := \left(\int_{\mathbb{R}} \|u(t)\|_E^q \, dt\right)^{1/q}.$$

We omit the subscript t for simplicity of notation.

More generally, for $m \in \mathbb{N}$, we denote by $W^{m,q} E$ the norm

$$\|u\|_{W^{m,q} E} := \sum_{j=0}^m \|\partial_t^j u\|_{L^q E}.$$

Given $p \in (1, \infty)$ and $s \in \mathbb{R}$, we denote by $H^{s,p}(\mathbb{T}^d)$ the (fractional regularity) Sobolev space given by the norm $\|f\|_{H^{s,p}} = \|\mathcal{F}_x^{-1}(\hat{f}(\xi) \cdot \langle\xi\rangle^s)\|_{L^p(\mathbb{T}^d)}$, where $\langle\xi\rangle$ denotes $\sqrt{1 + |\xi|^2}$.

Given $s \in \mathbb{R}$, $p, q \in [1, \infty]$, and a Banach space E defined on $\mathbb{T}^d$, we define the (vector-valued) Besov space $B^s_{p,q} E = (B^s_{p,q})_t E_x$ as the dyadic summation of time frequency cutoffs,

$$\|u\|_{B^s_{p,q}E} := \Big(\sum_{N\in 2^{\mathbb{N}}} N^{qs}\|P^t_N u\|^q_{L^pE}\Big)^{1/q} + \|P^t_{\le 1}u\|_{L^pE}.$$

More generally, for a spacetime Banach space F of functions defined on $\mathbb{R}\times\mathbb{T}^d$, we denote by $\ell^q_{s;\tau}F$ the norm

$$\|u\|_{\ell^q_{s;\tau}F} := \Big(\sum_{N\in 2^{\mathbb{N}}} N^{qs}\|P^t_N u\|^q_{F}\Big)^{1/q} + \|P^t_{\le 1}u\|_{F}.$$

In particular, the time Besov space $B^s_{p,q}E$ is norm equivalent to $\ell^q_{s;\tau}L^pE$.

For spatial frequencies, we use the notation ℓ^q_s for Banach spaces on both $\mathbb{T}^d$ and $\mathbb{R}\times\mathbb{T}^d$. For Banach spaces E and F defined on $\mathbb{T}^d$ and $\mathbb{R}\times\mathbb{T}^d$, respectively, we define

$$\|u\|_{\ell^q_s E} := \Big(\sum_{N\in 2^{\mathbb{N}}} N^{qs}\|u_N\|^q_{E}\Big)^{1/q} + \|u_{\le 1}\|_{E}$$

and

$$\|u\|_{\ell^q_s F} := \Big(\sum_{N\in 2^{\mathbb{N}}} N^{qs}\|u_N\|^q_{F}\Big)^{1/q} + \|u_{\le 1}\|_{F}.$$

Unlike the Besov space notation, both ℓ^q_s and $\ell^q_{s;\tau}$ can be applied to a spacetime function space, so we use a subscript τ to distinguish them. When $s=0$, we omit the subscripts from ℓ^q_0 and $\ell^q_{0;\tau}$, denoting them by ℓ^q and ℓ^q_τ, respectively.

Proposition 2.1 ([1]). *We have the following embedding relations:*

(1) *For $s\in(0,1)$ and $p\in(1,\infty)$, we have*

$$\|u\|^p_{B^s_{p,p}(\mathbb{T}^d)} \sim \|u\|^p_{L^p} + \int_{\mathbb{T}^d\times\mathbb{T}^d}\Big(\frac{|u(x)-u(y)|}{|x-y|^s}\Big)^p\frac{d(x,y)}{|x-y|^d}. \tag{2.1}$$

Similarly, for a Banach space E on $\mathbb{T}^d$, we have

$$\|u\|^p_{B^s_{p,p}E} \sim \|u\|^p_{L^pE} + \int_{\mathbb{R}\times\mathbb{R}}\Big(\frac{\|u(t_1)-u(t_2)\|_E}{|t_1-t_2|^s}\Big)^p\frac{d(t_1,t_2)}{|t_1-t_2|}. \tag{2.2}$$

(2) *For $s\in\mathbb{R}$ and $p\in[2,\infty)$, we have*

$$\|u\|_{B^s_{p,p}(\mathbb{T}^d)} \lesssim \|u\|_{H^{s,p}(\mathbb{T}^d)} \lesssim \|u\|_{B^s_{p,2}(\mathbb{T}^d)}. \tag{2.3}$$

When $p\in(1,2]$, the opposite embedding relation holds.

Proof. Estimate (2.2) is introduced, for example, in [1, equation (5.8)]. Estimates (2.1) and (2.3) are known properties for $\mathbb{R}^d$; (2.1) is a special case of [1, equation (5.8)], and for (2.3), see, for example, [4, Theorem 6.4.4]. These results can be shown similarly on $\mathbb{T}^d$ via the Littlewood–Paley theory on $\mathbb{T}^d$. ∎

The proposition below provides facts for function-valued Besov spaces regarding embeddings, Bernstein inequalities, and interpolations.

Proposition 2.2 ([1, 2, 19]). *In this proposition, we denote by s_θ the number $s_\theta = (1-\theta)s_0 + \theta s_1$, where the numbers s_0 and s_1 are given in each corresponding statement and $\theta \in (0,1)$ is an arbitrary number. Similarly, we denote by p_θ and q_θ the numbers such that $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, respectively.*

Let E and E_j, $j = 0, 1$ be Banach spaces on $\mathbb{T}^d$. For a spacetime function $f : \mathbb{R} \times \mathbb{T}^d \to \mathbb{C}$, we have the following embedding relations:

- *For $1 \le p < \infty$ and $m \in \mathbb{N}$, we have*

$$\|f\|_{B^m_{p,\infty}E} \lesssim \|f\|_{W^{m,p}E} \lesssim \|f\|_{B^m_{p,1}E}. \tag{2.4}$$

- *For $1 \le \tilde{p} < p < \infty$ and $1 \le q \le \infty$, we have*

$$\|f\|_{L^{p,q}E} \lesssim \|f\|_{B^{1/\tilde{p}-1/p}_{\tilde{p},q}E}. \tag{2.5}$$

- *For $1 \le \tilde{p} < p < \infty$ and $M \in 2^{\mathbb{N}}$, we have*

$$\|P^t_M f\|_{L^p E} \lesssim M^{1/\tilde{p}-1/p}\|P^t_M f\|_{L^{\tilde{p}}E}. \tag{2.6}$$

- *For $1 \le \tilde{p} < p < \infty$, $1 \le q \le \infty$, and $s \in \mathbb{R}$, we have*

$$\|f\|_{B^s_{p,q}E} \lesssim \|f\|_{B^{s+1/\tilde{p}-1/p}_{\tilde{p},q}E}. \tag{2.7}$$

- *For $p, q \in (1,\infty)$ and $s \in \mathbb{R}$, assuming further that E' is separable, we have*

$$\|f\|_{(B^s_{p,q}E)'} \sim \|f\|_{B^{-s}_{p',q'}E'}. \tag{2.8}$$

- *For $p \in [1,\infty)$, $q_0, q_1, \eta \in [1,\infty]$, and $s_0, s_1 \in \mathbb{R}$ such that $s_0 \ne s_1$, we have*

$$\|f\|_{(B^{s_0}_{p,q_0}E, B^{s_1}_{p,q_1}E)_{\theta,\eta}} \sim \|f\|_{B^{s_\theta}_{p,\eta}E}. \tag{2.9}$$

- *For $p_0, p_1 \in [1,\infty)$, $q_0, q_1 \in [1,\infty]$, $s_0, s_1 \in \mathbb{R}$, and an interpolation couple (E_0, E_1), we have*

$$\|f\|_{[B^{s_0}_{p_0,q_0}E_0, B^{s_1}_{p_1,q_1}E_1]_\theta} \sim \|f\|_{B^{s_\theta}_{p_\theta,q_\theta}[E_0,E_1]_\theta}. \tag{2.10}$$

Proof. Estimate (2.4) is given in [2, equation (3.6)]; (2.5) is given in [19, Lemma 2.4 (1)]; (2.6) and (2.7) are direct consequences of (2.5); and (2.8) is given in [1, equation (5.22)]. In [1, Lemma 5.1], it is shown that $B^s_{p,q}E$ are retracts of $\ell^s_q(L^p E)$ with a common retraction, which implies (2.10) and (2.9) (see, for example, [4, Sections 6.4, 5.6]). ∎

Proposition 2.3. *Let E_j, $j = 1, 2, 3$ be Banach spaces on $\mathbb{T}^d$ satisfying the inequality*

$$\left|\int_{\mathbb{T}^d} fgh\,dx\right| \lesssim \|f\|_{E_1}\|g\|_{E_2}\|h\|_{E_3}. \tag{2.11}$$

Let $s_j \in \mathbb{R}$, $p_j \in (1,\infty)$, $q_j \in [1,\infty]$, $j = 1,2,3$ *be exponents satisfying the inequalities*

$$s_1 + s_2 + s_3 > 0, \quad s_2 + s_3 > 0, \quad \frac{1}{p_1} > s_1,$$

and the scaling conditions

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = s_1 + s_2 + s_3 + 1 \quad \textit{and} \quad \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1.$$

We have the estimate

$$\sum_{L \lesssim M \lesssim N} \left| \int_{\mathbb{R}\times\mathbb{T}^d} P_L^t f \cdot P_M^t g \cdot P_N^t h \, dx\, dt \right| \lesssim \|f\|_{B^{s_1}_{p_1,q_1} E_1} \|g\|_{B^{s_2}_{p_2,q_2} E_2} \|h\|_{B^{s_3}_{p_3,q_3} E_3}. \tag{2.12}$$

Proof. Since s_1 is of the lowest frequency and $s_1 + s_2 + s_3 > 0$, by increasing s_1 and decreasing $s_2 + s_3$, we may assume $s_1 > 0$ in advance.

Similarly, since the frequencies M and N are comparable and $\frac{1}{p_2} + \frac{1}{p_3} = s_2 + s_3 + s_1 + 1 - \frac{1}{p_1} > s_2 + s_3 > 0$, by perturbing s_2 and s_3 keeping $s_2 + s_3$ fixed, we may assume $\frac{1}{p_2} > s_2 > 0$ and $\frac{1}{p_3} > s_3 > 0$ in advance.

For each $j = 1,2,3$, by $\frac{1}{p_j} > s_j > 0$ and (2.5), we have the embedding $B^{s_j}_{p_j,q_j} E \hookrightarrow L^{\tilde{p}_j,q_j} E$, where $\tilde{p}_j$ is the exponent $\frac{1}{\tilde{p}_j} := \frac{1}{p_j} - s_j$. By scaling conditions, we have $\frac{1}{\tilde{p}_1} + \frac{1}{\tilde{p}_2} + \frac{1}{\tilde{p}_3} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1$, which implies (2.12). ∎

As a particular consequence, we have product rules for time Besov spaces.

Corollary 2.4. *Let* E_j, $j = 1,2,3$ *be Banach spaces satisfying* (2.11). *Let* $s_j \in \mathbb{R}$, $p_j \in (1,\infty)$, $q_j \in [1,\infty]$, $j = 1,2,3$ *be parameters such that* $s_1 + s_2 + s_3 > 0$, $\frac{1}{p_j} > s_j$, $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = s_1 + s_2 + s_3 + 1$, *and* $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1$.

(1) *Assume* $s_1 + s_2 > 0$, $s_1 + s_3 > 0$, *and* $s_2 + s_3 > 0$. *Then we have*

$$\|uv\|_{B^{-s_3}_{p_3',q_3'} E_3'} \lesssim \|u\|_{B^{s_1}_{p_1,q_1} E_1} \|v\|_{B^{s_2}_{p_2,q_2} E_2}. \tag{2.13}$$

(2) *Assume* $s_1 + s_2 > 0$ *and* $s_2 + s_3 > 0$. *Then we have*

$$\|\pi_{\le}(u,v)\|_{B^{-s_3}_{p_3',q_3'} E_3'} \lesssim \|u\|_{B^{s_1}_{p_1,q_1} E_1} \|v\|_{B^{s_2}_{p_2,q_2} E_2}. \tag{2.14}$$

Proof. Estimates (2.13) and (2.14) are direct consequences of (2.12) and dualities. For (2.13), the high-frequency terms can be either (u,v), $(u, \pi_{\le}(u,v))$, or $(v, \pi_{\le}(u,v))$, so we assume all of $s_1 + s_2 > 0$, $s_1 + s_3 > 0$, and $s_2 + s_3 > 0$. For (2.14), the high-frequency terms can be either (u,v) or $(v, \pi_{\le}(u,v))$, so we require only $s_1 + s_2 > 0$ and $s_2 + s_3 > 0$. ∎

Proposition 2.2, (2.13), and (2.14) can be shown similarly on (scalar-valued) Besov spaces on $\mathbb{T}^d$, unless they are L^1- or L^∞-based. This can be done via estimates on Littlewood–Paley convolution kernels on $\mathbb{T}^d$.

Next we state the fractional chain rule for Hölder continuous functions.

Lemma 2.5. *Let $F \in C^{0,\alpha}(\mathbb{C})$, $\alpha \in (0,1)$. Let $s \in (0,\alpha)$, $\sigma > 0$, and $p, p_1, p_2 \in (1,\infty)$ be exponents satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $(\alpha - \frac{s}{\sigma})p_1 > 1$. We have*

$$\|F(u)\|_{H^{s,p}} \lesssim \|u\|_{L^{(\alpha-\frac{s}{\sigma})p_1}}^{\alpha-\frac{s}{\sigma}} \cdot \|u\|_{H^{\sigma,\frac{s}{\sigma}p_2}}^{\frac{s}{\sigma}}. \tag{2.15}$$

Proof. In [21], (2.15) is proved for $\sigma < 1$. For $\sigma \geq 1$, we choose $\tilde{\sigma} \in (\frac{s}{\alpha}, 1)$ and let $\tilde{p}_1 := \frac{(\alpha-\frac{s}{\sigma})}{(\alpha-\frac{s}{\tilde{\sigma}})}p_1$ and $\tilde{p}_2 := \frac{1}{p} - \frac{1}{\tilde{p}_1}$. Since $\tilde{\sigma} < 1$, by complex interpolation, we have

$$\begin{aligned}
\|F(u)\|_{H^{s,p}} &\lesssim \|u\|_{L^{(\alpha-\frac{s}{\tilde{\sigma}})\tilde{p}_1}}^{\alpha-\frac{s}{\tilde{\sigma}}} \cdot \|u\|_{H^{\tilde{\sigma},\frac{s}{\tilde{\sigma}}\tilde{p}_2}}^{\frac{s}{\tilde{\sigma}}} \\
&\lesssim \|u\|_{L^{(\alpha-\frac{s}{\sigma})p_1}}^{\alpha-\frac{s}{\tilde{\sigma}}} \cdot \|u\|_{H^{\tilde{\sigma},\frac{s}{\tilde{\sigma}}\tilde{p}_2}}^{\frac{s}{\tilde{\sigma}}} \\
&\lesssim \|u\|_{L^{(\alpha-\frac{s}{\sigma})p_1}}^{\alpha-\frac{s}{\tilde{\sigma}}} \cdot \|u\|_{L^{(\alpha-\frac{s}{\sigma})p_1}}^{\frac{s}{\tilde{\sigma}}-\frac{s}{\sigma}} \|u\|_{H^{\sigma,\frac{s}{\sigma}p_2}}^{\frac{s}{\sigma}} \\
&\lesssim \|u\|_{L^{(\alpha-\frac{s}{\sigma})p_1}}^{\alpha-\frac{s}{\sigma}} \cdot \|u\|_{H^{\sigma,\frac{s}{\sigma}p_2}}^{\frac{s}{\sigma}}.
\end{aligned}$$

∎

A similar result on higher Hölder regularities can be deduced.

Lemma 2.6. *Let $\alpha \geq 1$ and $m \in \mathbb{Z}$. Let $F\colon \mathbb{C} \to \mathbb{C}$ be the function $F(z) := |z|^{\alpha-m}z^m$. Let $s \in [0,\alpha)$ and $p, p_1, p_2 \in (1,\infty)$ be exponents satisfying $\frac{1}{p} = \frac{\alpha-1}{p_1} + \frac{1}{p_2}$. Then, for $u\colon \mathbb{T}^d \to \mathbb{C}$, we have*

$$\|F(u)\|_{H^{s,p}} \lesssim \|u\|_{L^{p_1}}^{\alpha-1} \|u\|_{H^{s,p_2}}. \tag{2.16}$$

Proof. We use an induction on α. First, we focus on the case of $1 \leq \alpha \leq 2$. When $0 \leq s \leq 1$, (2.16) is the well-known fractional chain rule. Assume $1 < s < \alpha$. Define numbers $\tilde{q}$, r, $\tilde{r} \in (1,\infty)$ as $\frac{s}{\tilde{q}} = \frac{s-1}{p_1} + \frac{1}{p_2}$, $\frac{1}{r} = \frac{1}{p} - \frac{1}{p_2}$, and $\frac{1}{\tilde{r}} = \frac{1}{p} - \frac{1}{\tilde{q}}$, where we used $s < \alpha$. Using $\frac{1}{\tilde{r}} = \frac{\alpha+\frac{1}{s}-2}{p_1} + \frac{1-\frac{1}{s}}{p_2}$, we apply (2.15) to $\|(\nabla F)(u)\|_{H^{s-1,\tilde{r}}}$ with $\sigma = s$. Using Wirtinger derivatives, we have

$$\begin{aligned}
\|\nabla(F(u))\|_{H^{s-1,p}} &= \|\nabla u \cdot \partial_z F(u) + \overline{\nabla u} \cdot \partial_{\bar{z}} F(u)\|_{H^{s-1,p}} \\
&\lesssim \|\nabla u\|_{H^{s-1,p_2}} (\|\partial_z F(u)\|_{L^r} + \|\partial_{\bar{z}} F(u)\|_{L^r}) \\
&\quad + \|\nabla u\|_{L^{\tilde{q}}} (\|\partial_z F(u)\|_{H^{s-1,\tilde{r}}} + \|\partial_{\bar{z}} F(u)\|_{H^{s-1,\tilde{r}}}) \\
&\lesssim \|u\|_{H^{s,p_2}} \|u\|_{L^{p_1}}^{\alpha-1} + \|u\|_{L^{p_1}}^{\frac{s-1}{s}} \|u\|_{H^{s,p_2}}^{\frac{1}{s}} \cdot \|u\|_{L^{p_1}}^{\alpha-1-\frac{s-1}{s}} \|u\|_{H^{s,p_2}}^{\frac{s-1}{s}} \\
&\lesssim \|u\|_{L^{p_1}}^{\alpha-1} \|u\|_{H^{s,p_2}},
\end{aligned}$$

which implies

$$\|F(u)\|_{H^{s,p}} \lesssim \|F(u)\|_{L^p} + \|\nabla(F(u))\|_{H^{s-1,p}} \lesssim \|u\|_{L^{p_1}}^{\alpha-1} \|u\|_{H^{s,p_2}}.$$

Now we fix an integer $N \geq 2$ and assume that (2.16) holds when $\alpha \leq N$. We claim that (2.16) holds for $\alpha \leq N+1$ as well.

Define the numbers $q_k, r_k, \tilde{q}_k, \tilde{r}_k$, $0 \le k \le \lfloor s \rfloor - 1$ as $\frac{s}{q_k} = \frac{k}{p_1} + \frac{s-k}{p_2}$, $\frac{1}{r_k} = \frac{1}{p} - \frac{1}{q_k}$ and $\frac{s}{\tilde{q}_k} = \frac{s-1-k}{p_1} + \frac{1+k}{p_2}$, $\frac{1}{\tilde{r}_k} = \frac{1}{p} - \frac{1}{\tilde{q}_k}$. We have

$$\begin{aligned}
\|\nabla(F(u))\|_{H^{s-1,p}} &= \|\nabla u \cdot \partial_z F(u) + \overline{\nabla u} \cdot \partial_{\bar{z}} F(u)\|_{H^{s-1,p}} \\
&\lesssim \sum_{k=0}^{\lfloor s \rfloor -1} \|\nabla u\|_{H^{s-1-k,q_k}} (\|\partial_z F(u)\|_{H^{k,r_k}} + \|\partial_{\bar{z}} F(u)\|_{H^{k,r_k}}) \\
&\quad + \sum_{k=0}^{\lfloor s \rfloor -1} \|\nabla u\|_{H^{k,\tilde{q}_k}} (\|\partial_z F(u)\|_{H^{s-1-k,\tilde{r}_k}} + \|\partial_{\bar{z}} F(u)\|_{H^{s-1-k,\tilde{r}_k}}) \\
&\lesssim \|u\|_{L^{p_1}}^{\alpha-1} \|u\|_{H^{s,p_2}},
\end{aligned}$$

where we used complex interpolations to bound norms of ∇u, and used (2.15) and the induction hypothesis to bound norms of $\partial_z F(u)$ and $\partial_{\bar{z}} F(u)$.

It follows that

$$\|F(u)\|_{H^{s,p}} \lesssim \|F(u)\|_{L^p} + \|\nabla(F(u))\|_{H^{s-1,p}} \lesssim \|u\|_{L^{p_1}}^{\alpha-1} \|u\|_{H^{s,p_2}},$$

which finishes the proof by induction on N. ■

Next we propose a Sobolev–Slobodeckij version of the fractional Hölder inequality.

Lemma 2.7. *Fix $s \in (0,1)$, $p \in (1,\infty)$, $\alpha \in (0,1)$, and $F \in C^{0,\alpha}(\mathbb{C})$. For $u \in B^s_{p,p}(\mathbb{T}^d)$, we have*

$$\|F(u)\|_{B^{s\alpha}_{p/\alpha,p/\alpha}(\mathbb{T}^d)} \lesssim \|u\|^{\alpha}_{B^s_{p,p}(\mathbb{T}^d)}. \tag{2.17}$$

Proof. The proof naturally follows from the Hölder continuity of F:

$$\begin{aligned}
\|F(u)\|^{p/\alpha}_{B^{s\alpha}_{p/\alpha,p/\alpha}} &\sim \|F(u)\|^{p/\alpha}_{L^{p/\alpha}} + \int_{\mathbb{T}^d \times \mathbb{T}^d} \left(\frac{|F(u(x)) - F(u(y))|}{|x-y|^{s\alpha}} \right)^{p/\alpha} \frac{d(x,y)}{|x-y|^d} \\
&\lesssim \|u\|^p_{L^p} + \int_{\mathbb{T}^d \times \mathbb{T}^d} \left(\frac{|u(x) - u(y)|}{|x-y|^{s}} \right)^{p} \frac{d(x,y)}{|x-y|^d} \lesssim \|u\|^p_{B^s_{p,p}}.
\end{aligned}$$
■

We also have a spacetime version of the fractional chain rule for Besov spaces.

Lemma 2.8. *Let $s_0, s_1 > 0$ be exponents satisfying $2s_0 + s_1 < 1$. Fix $p \in (1,\infty)$, $\alpha \in (0,1)$, and a function $F \in C^{0,\alpha}(\mathbb{C})$. For $u \colon \mathbb{R} \times \mathbb{T}^d \to \mathbb{C}$, we have*

$$\|F(u)\|_{B^{s_0\alpha}_{p/\alpha,p/\alpha} B^{s_1\alpha}_{p/\alpha,p/\alpha}} \lesssim \|u\|^{\alpha}_{L^p B^{2s_0+s_1}_{p,p} \cap B^{s_0+s_1/2}_{p,p} L^p}. \tag{2.18}$$

Proof. From the assumptions $s_0, s_1 > 0$, $0 < 2s_0 + s_1 < 1$, and $\alpha \in (0,1)$, we have $s_0, s_1, s_0\alpha, s_1\alpha, 2s_0 + s_1, s_0 + s_1/2 \in (0,1)$. Thus, (2.1), (2.2), and (2.17) are applicable to each Besov space in (2.18).

By (2.2), we have

$$\begin{aligned}\|F(u)\|_{B^{s_0\alpha}_{p/\alpha,p/\alpha}B^{s_1\alpha}_{p/\alpha,p/\alpha}} &\lesssim \|F(u)\|_{L^{p/\alpha}B^{s_1\alpha}_{p/\alpha,p/\alpha}} \\ &\quad + \left(\int_{\mathbb{R}\times\mathbb{R}} \frac{\|F(u)(t,\cdot)-F(u)(s,\cdot)\|^{p/\alpha}_{B^{s_1\alpha}_{p/\alpha,p/\alpha}}}{|t-s|^{s_0 p}} \cdot \frac{d(t,s)}{|t-s|}\right)^{\alpha/p} \\ &= \mathrm{I}+\mathrm{II}.\end{aligned}$$

We estimate I using (2.17):

$$\mathrm{I} = \|F(u)\|_{L^{p/\alpha}B^{s_1\alpha}_{p/\alpha,p/\alpha}} \lesssim \|u\|^{\alpha}_{L^p B^{s_1}_{p,p}} \lesssim \|u\|^{\alpha}_{L^p B^{2s_0+s_1}_{p,p}}.$$

To estimate II, we further decompose II using (2.1):

$$\begin{aligned}\mathrm{II} &\lesssim \|F(u)\|_{B^{s_0\alpha}_{p/\alpha,p/\alpha}L^{p/\alpha}} \\ &\quad + \Bigg(\int_{\mathbb{R}\times\mathbb{R}} \frac{1}{|t-s|^{s_0 p}} \int_{\mathbb{T}^d\times\mathbb{T}^d} \frac{1}{|x-y|^{s_1 p}} \\ &\qquad \cdot |F(u)(t,x)-F(u)(s,x)-F(u)(t,y)+F(u)(s,y)|^{p/\alpha} \frac{d(x,y)}{|x-y|^d}\frac{d(t,s)}{|t-s|}\Bigg)^{\alpha/p} \\ &= \mathrm{II}_A + \mathrm{II}_B.\end{aligned}$$

We estimate II_A using an argument similar to (2.17):

$$\|F(u)\|_{B^{s_0\alpha}_{p/\alpha,p/\alpha}L^{p/\alpha}} \lesssim \|u\|^{\alpha}_{B^{s_0}_{p,p}L^p} \lesssim \|u\|^{\alpha}_{B^{s_0+s_1/2}_{p,p}L^p}.$$

We estimate II_B. For variables $t_0, t_1 \in \mathbb{R}$ and $x_0, x_1 \in \mathbb{T}^d$, let $c_j := |u(t_j,x_0)-u(t_j,x_1)|$ and $d_j := |u(t_0,x_j)-u(t_1,x_j)|$. Then we have

$$\begin{aligned}&|F(u)(t_0,x_0)-F(u)(t_0,x_1)-F(u)(t_1,x_0)+F(u)(t_1,x_1)|^{1/\alpha} \\ &\quad\lesssim \min\{\max\{c_0,c_1\},\max\{d_0,d_1\}\} \\ &\quad= \max_{i,j\in\{0,1\}} \min\{c_i,d_j\}. \end{aligned} \tag{2.19}$$

By (2.19) and the symmetry of (x,y) and (t,s) in integrals, we have

$$\begin{aligned}(\mathrm{II}_B)^{p/\alpha} &= \int_{\mathbb{R}\times\mathbb{R}}\int_{\mathbb{T}^d\times\mathbb{T}^d} |F(u)(t,x)-F(u)(s,x)-F(u)(t,y)+F(u)(s,y)|^{p/\alpha} \\ &\qquad \cdot \frac{d(x,y)}{|x-y|^{s_1 p+d}} \cdot \frac{d(t,s)}{|t-s|^{s_0 p+1}} \\ &\lesssim \int_{\mathbb{R}\times\mathbb{R}}\int_{\mathbb{T}^d\times\mathbb{T}^d} \min\{|u(t,x)-u(s,x)|,|u(t,x)-u(t,y)|\}^p \\ &\qquad \cdot \frac{d(x,y)}{|x-y|^{s_1 p+d}} \cdot \frac{d(t,s)}{|t-s|^{s_0 p+1}}\end{aligned}$$

$$\begin{aligned}
&\lesssim \int_{|t-s|\geq|x-y|^2} |u(t,x)-u(t,y)|^p \cdot \frac{d(x,y)}{|x-y|^{s_1 p+d}} \cdot \frac{d(t,s)}{|t-s|^{s_0 p+1}} \\
&\quad + \int_{|t-s|\leq|x-y|^2} |u(t,x)-u(s,x)|^p \cdot \frac{d(x,y)}{|x-y|^{s_1 p+d}} \cdot \frac{d(t,s)}{|t-s|^{s_0 p+1}} \\
&\lesssim \int_{\mathbb{R}} \int_{\mathbb{T}^d\times\mathbb{T}^d} |u(t,x)-u(t,y)|^p \cdot \frac{d(x,y)}{|x-y|^{(2s_0+s_1)p+d}} \cdot dt \\
&\quad + \int_{\mathbb{T}^d} \int_{\mathbb{R}\times\mathbb{R}} |u(t,x)-u(s,x)|^p \cdot \frac{d(t,s)}{|t-s|^{(s_0+s_1/2)p+1}} \cdot dx \\
&\lesssim \int_{\mathbb{R}} \int_{\mathbb{T}^d\times\mathbb{T}^d} \Big(\frac{|u(t,x)-u(t,y)|}{|x-y|^{2s_0+s_1}}\Big)^p \cdot \frac{d(x,y)}{|x-y|^d} \cdot dt \\
&\quad + \int_{\mathbb{R}\times\mathbb{R}} \int_{\mathbb{T}^d} \Big(\frac{|u(t,x)-u(s,x)|}{|t-s|^{s_0+s_1/2}}\Big)^p \cdot dx \cdot \frac{d(t,s)}{|t-s|} \\
&\lesssim \|u\|^p_{L^p B^{2s_0+s_1}_{p,p}} + \|u\|^p_{B^{s_0+s_1/2}_{p,p} L^p},
\end{aligned}$$

which finishes the proof of (2.18). ∎

Remark 2.9. The parameter 2 in (2.18) is replaceable. Indeed, for every $\lambda > 0$, we can show $\|F(u)\|_{B^{s_0\alpha}_{p/\alpha,p/\alpha} B^{s_1\alpha}_{p/\alpha,p/\alpha}} \lesssim \|u\|^\alpha_{L^p B^{\lambda s_0+s_1}_{p,p} \cap B^{s_0+s_1/\lambda}_{p,p} L^p}$ by the same argument. The choice $\lambda = 2$ is for the scaling of the Schrödinger operator.

2.3. Schrödinger operators, Strichartz estimates, and atomic spaces

We collect Strichartz estimates for linear Schrödinger operators on tori, properties of the atomic spaces, and Galilean invariance properties.

Schrödinger operators. For a function $\phi\colon \mathbb{T}^d \to \mathbb{C}$ and $t \in \mathbb{R}$, we denote by $e^{it\Delta}\phi$ the function such that

$$\widehat{e^{it\Delta}\phi}(\xi) = e^{-it|\xi|^2}\hat{\phi}(\xi).$$

For a function $f\colon \mathbb{R}\times\mathbb{T}^d \to \mathbb{C}$, we denote the retarded Schrödinger operator K^+ by

$$K^+ f(t) := \int_{-\infty}^t e^{i(t-s)\Delta} f(s)\, ds.$$

Strichartz estimates and atomic spaces. The following are the kernel estimate and the $L^4_{t,x}$-Strichartz estimate for the Schrödinger operator on $\mathbb{T}$, first shown in [6]:

Proposition 2.10 ([6, Lemma 3.18]). *On $\mathbb{T}$, for dyadic $N \in 2^{\mathbb{N}}$ and coprime integers l and m such that $1 \leq l < m < N$ and $|t - \frac{l}{m}| \leq \frac{1}{mN}$, we have*

$$|e^{it\Delta}\delta_N(t,x)| \lesssim \frac{N}{\sqrt{m}(1+N|t-\frac{l}{m}|^{1/2})}. \tag{2.20}$$

Proposition 2.11 ([6, equation (2.2)]). *On the domain* $[0, T] \times \mathbb{T}$, *where* $T > 0$, *for any function* $f \in L^{3/4}_{t,x} + L^1 L^2$, *we have*

$$\|K^+ f\|_{L^4_{t,x} \cap L^\infty L^2} \lesssim_T \|f\|_{L^{3/4}_{t,x} + L^1 L^2}. \tag{2.21}$$

The following is a scale-invariant Strichartz estimate for general tori, which is a main ingredient of the proof of Theorem 1.1. This was first shown for rational tori in [7]. For general tori, a subcritical version was first shown in [7] and was sharpened to the critical scale in [15].

Proposition 2.12 ([6,7,15]). *Fix* $p \in (\frac{2(d+2)}{d}, \infty)$. *Let* $\sigma = \frac{d}{2} - \frac{d+2}{p}$. *Fix a finite interval* $I \subset \mathbb{R}$. *For* $N \in 2^{\mathbb{N}}$, *we have*

$$\|P_N e^{it\Delta} f\|_{L^p_{t,x}(I \times \mathbb{T}^d)} \lesssim_{p,I} \|f\|_{H^\sigma(\mathbb{T}^d)}. \tag{2.22}$$

Next we recall the definition of atomic spaces U^p and V^p. Although we will not directly use U^p, V^p-structures to construct the function spaces for the well-posedness, we will still use their embedding properties. Here we collect facts relevant to them. For a general theory, we refer to [9, 10, 17].

Definition 2.13 (Atomic spaces, [10]). Let H be a separable Hilbert space. Let $\mathcal{Z}$ be the collection of finite nondecreasing sequences $\{t_k\}_{k=0}^K$ in $(-\infty, \infty]$. For $1 \le p < \infty$, we call $a\colon \mathbb{R} \to H$ a U^p-atom if a can be expressed as $a = \sum_{k=1}^K \chi_{[t_{k-1}, t_k)} \phi_k$, $\sum_{k=1}^K \|\phi_k\|_H^p = 1$. We define $U^p H$ as the space of all functions $u\colon \mathbb{R} \to H$ that can be represented as $u = \sum_{j=1}^\infty \lambda_j a_j$, where a_j is a U^p-atom for each $j \in \mathbb{N}$ and $\{\lambda_j\} \in \ell^1$ is a complex-valued sequence, equipped with the norm

$$\|u\|_{U^p H} := \inf\Big\{\sum_{j=1}^\infty |\lambda_j| : u = \sum_{j=1}^\infty \lambda_j a_j,\ \lambda_j \in \mathbb{C},\ a_j : U^p\text{-atom}\Big\}.$$

We define $V^p H$ as the space of all functions $u\colon \mathbb{R} \to H$ with $\|u\|_{V^p H} < \infty$, where the norm is defined as

$$\|u\|_{V^p H}^p := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \sum_{k=1}^K \|u(t_k) - u(t_{k-1})\|_H^p,$$

where the convention $u(\infty) = 0$ is used. Then we define $V^p_{\mathrm{rc}} H$ as the subspace of $V^p H$ of right-continuous functions $u\colon \mathbb{R} \to H$ satisfying $\lim_{t \to -\infty} u(t) = 0$. For simplicity of notation, we omit H in $U^p H$, $V^p H$, $V^p_{\mathrm{rc}} H$ when $H \simeq \mathbb{C}$. Based on this, we define the spaces $U^p_\Delta H$, $V^p_\Delta H$, $V^p_{\Delta,\mathrm{rc}} H$ as the images by the map $u \mapsto e^{it\Delta} u$ of $U^p H$, $V^p H$, $V^p_{\mathrm{rc}} H$, respectively.

We define Y^s as the space of $u\colon \mathbb{R} \times \mathbb{T}^d \to \mathbb{C}$ such that $\hat{u}(n)$ lies in V^2_{rc} for each $n \in \mathbb{Z}^d$ and

$$\|u\|_{Y^s}^2 := \sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2s} \|e^{it|n|^2} \widehat{u(t)}(n)\|_{V^2}^2 < \infty.$$

While the space Y^s is defined on the full domain $\mathbb{R} \times \mathbb{T}^d$, since Strichartz estimates such as (2.22) depend on the size of a time interval, we often need to restrict the space to a finite interval. Given a time interval I, the Y^s space corresponding to I, $Y^s(I)$, is the restriction of the Y^s space to the domain $I \times \mathbb{T}^d$.

In particular, in the proof of Theorem 1.1, we will always consider Y^s and all the other solution spaces localized on a short time interval containing 0 to avoid any issue with the interval size.

The space Y^s is used in [10, 15, 18]. Some well-known properties of such atomic spaces are the following propositions:

Proposition 2.14 ([10]). *Fix $s \in \mathbb{R}$. Fix a finite time interval I and the corresponding Y^s space. We have the following:*

(1) *Let A and B be disjoint subsets of $\mathbb{Z}^d$. We have*

$$\|P_{A\cup B}u\|_{Y^s}^2 = \|P_A u\|_{Y^s}^2 + \|P_B u\|_{Y^s}^2 \quad (\ell^2_\xi\text{-structure}).$$

(2) *For $q > 2$, we have*

$$U^2_\Delta H^s \hookrightarrow Y^s \hookrightarrow V^2_{\Delta,\mathrm{rc}} H^s \hookrightarrow U^q_\Delta H^s \hookrightarrow L^\infty H^s.$$

Proposition 2.15 (Strichartz estimates). *Fix a finite time interval I and the corresponding Y^s space. Fix $p \in (\frac{2(d+2)}{d}, \infty)$. Let $\sigma = \frac{d}{2} - \frac{d+2}{p}$. Denote the diameter of a set $S \subset \mathbb{Z}^d$ by* $\mathrm{diam}(S)$. *We have the following estimates:*

- *Let $C \subset \mathbb{Z}^d$ be a square cube. We have the estimate*

$$\|\chi_I P_C u\|_{L^p_{t,x}} \lesssim_{p,I} \langle \mathrm{diam}(C)\rangle^\sigma \|P_C u\|_{Y^0}. \tag{2.23}$$

- *For $s \in \mathbb{R}$ and a function $f: I \times \mathbb{T}^d \to \mathbb{C}$, we have*

$$\|K^+ f\|_{Y^s} \lesssim_I \|f\|_{(Y^{-s})'}. \tag{2.24}$$

Estimate (2.23) is a consequence of (2.22) used, for example, in [15]. It is obtained using the atomic structure and the Galilean invariance of the Y^0 norm.

Estimate (2.24) is a version of the $U^2 - V^2$ dual estimate; see [10].[1]

Galilean transforms. We denote the Galilean transform with a shift $\xi \in \mathbb{Z}^d$ by $I_\xi: \mathcal{S}'(\mathbb{R} \times \mathbb{T}^d) \to \mathcal{S}'(\mathbb{R} \times \mathbb{T}^d)$, where $\mathcal{S}'$ denotes the set of tempered distributions, which maps $u: \mathbb{R} \times \mathbb{T}^d \to \mathbb{C}$ to

$$I_\xi u(t,x) = e^{ix\cdot\xi - it|\xi|^2} u(t, x - 2t\xi).$$

We collect some elementary properties of the Galilean transforms. In particular, the Y^0 norm is invariant under the Galilean transforms.

[1] In other literature, such as [10, 15, 18], (2.24) is obtained from the duality between X^s and Y^s. Since $X^s \hookrightarrow Y^s$, we still have (2.24).

Proposition 2.16. *For $u \in \mathcal{S}'(\mathbb{R} \times \mathbb{T}^d)$ and $\xi, \eta \in \mathbb{Z}^d$, we have the following:*

- $(i\partial_t + \Delta) I_\xi u = I_\xi (i\partial_t + \Delta) u$.
- $I_\xi I_\eta u = I_{\xi+\eta} u$.
- *For each set $C \subset \mathbb{Z}^d$, we have $P_{C+\xi} I_\xi u = I_\xi P_C u$.*
- *The Y^0 norm is invariant under the Galilean transforms, i.e., $\|u\|_{Y^0} = \|I_\xi u\|_{Y^0}$.*
- *The spacetime Fourier transform of $I_\xi u$ can be written as*

$$\widetilde{I_\xi u}(\tau, n) = \tilde{u}(\tau + 2n \cdot \xi - |\xi|^2, n - \xi).$$

3. Z^s spaces

In this section we introduce our main function spaces $Z^s = Z^s(\mathbb{R} \times \mathbb{T}^d)$ for the local well-posedness theorem. We fix a sufficiently small number $\sigma > 0$ and the corresponding parameter p, depending only on d and s:

$$\sigma = \sigma(d, s) \ll 1 \quad \text{and} \quad p := \frac{d+2}{\frac{d}{2} - \sigma}, \tag{3.1}$$

where σ has to approach 0 as $d \to \infty$ or $s \to 0$. For example, it suffices to choose $\sigma = 10^{-10^{10^{10^{10^{d+1/s}}}}}$.

3.1. Strichartz estimates

In this subsection we prove that the Y^0 norm is stronger than certain time Besov spaces. For this, we start with a lemma that helps estimate time Besov norms of U^p-atoms.

Lemma 3.1. *Let E be a Banach space on $\mathbb{T}^d$. Let $q \in (1, \infty)$ and $\alpha \in (0, \frac{1}{q})$. Given a finite collection of disjoint intervals $I_1, \ldots, I_n \subset \mathbb{R}$ and functions $f_1, \ldots, f_n \in B^\alpha_{q,1} E$, we have*

$$\left\| \sum_{j=1}^n f_j \chi_{I_j} \right\|_{B^\alpha_{q,\infty} E} \lesssim_{\alpha,q} \left(\sum_{j=1}^n \|f_j\|^q_{B^\alpha_{q,1} E} \right)^{1/q}. \tag{3.2}$$

Here, χ_I is the sharp cutoff of I. We emphasize that (3.2) is independent of the choice of $\{I_j\}$.

Proof. Let $r = \frac{1-\alpha}{\frac{1}{q} - \alpha}$. Since the I_j are disjoint, we have

$$\begin{aligned} \left\| \sum_{j=1}^n f_j \chi_{I_j} \right\|_{B^0_{r,\infty} E} \lesssim \left\| \sum_{j=1}^n f_j \chi_{I_j} \right\|_{L^r E} &\lesssim \left(\sum_{j=1}^n \|f_j\|^r_{L^r E} \right)^{1/r} \\ &\lesssim \left(\sum_{j=1}^n \|f_j\|^r_{B^0_{r,1} E} \right)^{1/r}. \end{aligned} \tag{3.3}$$

For each j, let $\varphi_j^{(k)}$ be an approximation of the sharp cutoff χ_{I_j}, i.e., $\varphi_j^{(k)} = \chi_{I_j} * \phi_k$, where we set ϕ_k as $\phi_k(t) = k\phi(kt)$ with $\phi \in C_c^\infty(\mathbb{R})$ chosen as a fixed function such that $\int_{\mathbb{R}} \phi \, dt = 1$. Then we have $\|\partial_t \varphi_j^{(k)}\|_{L^1} + \|\varphi_j^{(k)}\|_{L^\infty} \lesssim 1$.

For each j, we have

$$\begin{aligned}\|\partial_t(f_j\varphi_j^{(k)})\|_{L^1E} &\le \|\partial_t f_j \cdot \varphi_j^{(k)}\|_{L^1E} + \|f_j \cdot \partial_t\varphi_j^{(k)}\|_{L^1E}\\ &\lesssim \|\partial_t f_j\|_{L^1E}\|\varphi_j^{(k)}\|_{L^\infty} + \|f_j\|_{L^\infty E}\|\partial_t\varphi_j^{(k)}\|_{L^1}\\ &\lesssim \|f_j\|_{W^{1,1}E}.\end{aligned}$$

Thus, by (2.4), we have

$$\Big\|\sum_{j=1}^n f_j\varphi_j^{(k)}\Big\|_{B^1_{1,\infty}E} \lesssim \Big\|\sum_{j=1}^n f_j\varphi_j^{(k)}\Big\|_{W^{1,1}E} \lesssim \sum_{j=1}^n \|f_j\|_{W^{1,1}E} \lesssim \sum_{j=1}^n \|f_j\|_{B^1_{1,1}E}.$$

For each $M \in 2^{\mathbb{N}}$, we have

$$\Big\|P^t_{\le M}\sum_{j=1}^n f_j\chi_{I_j}\Big\|_{B^1_{1,\infty}E} = \lim_{k\to\infty}\Big\|P^t_{\le M}\sum_{j=1}^n f_j\varphi_j^{(k)}\Big\|_{B^1_{1,\infty}E} \lesssim \sum_{j=1}^n \|f_j\|_{B^1_{1,1}E}.$$

Taking $M \to \infty$, we have $\sum_{j=1}^n f_j\chi_{I_j} \in B^1_{1,\infty}E$ and

$$\Big\|\sum_{j=1}^n f_j\chi_{I_j}\Big\|_{B^1_{1,\infty}E} \lesssim \sum_{j=1}^n \|f_j\|_{B^1_{1,1}E}. \tag{3.4}$$

Since $r = \frac{1-\alpha}{\frac{1}{q}-\alpha}$ implies $(1-\frac{1}{q})/(1-\frac{1}{r}) = 1-\alpha$, by a complex interpolation between (3.3) and (3.4), we have (3.2). ∎

The next lemma presents a Strichartz estimate in time Besov spaces. In the proof of the next lemma, we will use a real interpolation technique, which is also used in many contexts, e.g., [14]. We will first show the estimate with the spacetime frequency domain restricted to a fixed support. Such an estimate gives information only on Besov spaces of parameter ∞, i.e., $B^\alpha_{p,\infty}$. We will improve that estimate by using (2.9) with perturbed choices of exponents. This kind of technique will be frequently used in later proofs.

Let $\psi : \mathbb{R} \to [0,\infty)$ be a smooth even bump function such that $\psi|_{[-1,1]} \equiv 1$ and $\mathrm{supp}(\psi) \subset [-\frac{11}{10}, \frac{11}{10}]$.

Lemma 3.2. *Let σ and p be as in (3.1). Fix $\alpha \in (0, \frac{1}{p})$. Let $\beta = \sigma + 2\alpha$. For every $N \in 2^{\mathbb{N}}$ and $f \in L^1L^2$, we have*

$$\|\psi K^+ f_N\|_{B^\alpha_{p,1}L^p} \lesssim_{\alpha,\psi} N^\beta \|f\|_{L^1L^2}, \tag{3.5}$$

where $f_N = P_N f$.

Proof. Let $u = K^+ f_N$. We first claim that for $\tilde{\alpha} \in (0, \frac{1}{p})$ and $M \in 2^{\mathbb{N}}$, we have

$$\|P_M^t(\psi u)\|_{L^p L^p} \lesssim M^{-\tilde{\alpha}} N^{\sigma+2\tilde{\alpha}} \|f\|_{L^1 L^2}. \tag{3.6}$$

Once we have (3.6), we plug in $\alpha_0 \in (0, \alpha)$ and $\alpha_1 \in (\alpha, \frac{1}{p})$ to obtain

$$\|\psi u\|_{B^{\alpha_j}_{p,\infty} L^p} \lesssim N^{\sigma+2\alpha_j} \|f\|_{L^1 L^2}, \quad j = 0, 1. \tag{3.7}$$

Applying a real interpolation of parameter 1 to (3.7) gives (3.5) for $\alpha \in (0, \frac{1}{p})$.

Now we prove the claim (3.6). When $M \sim N^2$, (3.6) is merely (2.22). From now, we assume $M \nsim N^2$. With this assumption, we have

$$|\tau + |\xi|^2| \sim \max\{M, N^2\}.$$

Since $(i\partial_t + \Delta)u = f_N$, we have $(i\partial_t + \Delta)P_M^t(\psi u) = P_M^t(\psi f_N + i\psi_t u)$. Thus, we have

$$\begin{aligned}
\|P_M^t(\psi u)\|_{L^p_{t,x}} &\lesssim M^{\frac{1}{2}-\frac{1}{p}} N^{d(\frac{1}{2}-\frac{1}{p})} \|P_M^t(\psi u)\|_{L^2_{t,x}} \\
&\lesssim M^{\frac{1}{2}-\frac{1}{p}} N^{d(\frac{1}{2}-\frac{1}{p})} \left\| \frac{1}{\tau + |\xi|^2} \mathcal{F}_{t,x}(P_M^t(\psi f_N + i\psi_t u)) \right\|_{L^2_\tau \ell^2_\xi} \\
&\lesssim \frac{M^{\frac{1}{2}-\frac{1}{p}} N^{d(\frac{1}{2}-\frac{1}{p})}}{\max\{M, N^2\}} \|P_M^t(\psi f_N + i\psi_t u)\|_{L^2_{t,x}} \\
&\lesssim \frac{M^{1-\frac{1}{p}} N^{d(\frac{1}{2}-\frac{1}{p})}}{\max\{M, N^2\}} \|\psi f_N + i\psi_t u\|_{L^1 L^2} \\
&\lesssim M^{-\tilde{\alpha}} N^{\sigma+2\tilde{\alpha}} \|f\|_{L^1 L^2}.
\end{aligned}$$

Here we have $\frac{M^{1-\frac{1}{p}} N^{d(\frac{1}{2}-\frac{1}{p})}}{\max\{M,N^2\}} \lesssim M^{-\tilde{\alpha}} N^{\sigma+2\tilde{\alpha}}$ since the scales match as $2(1-\frac{1}{p}) + d(\frac{1}{2}-\frac{1}{p}) - 2 = \frac{d}{2} - \frac{d+2}{p} = \sigma = -2\tilde{\alpha} + (\sigma + 2\tilde{\alpha})$ and $-\tilde{\alpha} \in (-\frac{1}{p}, 0) \subset (-\frac{1}{p}, 1-\frac{1}{p})$. ■

Now we transfer Lemma 3.2 to an estimate regarding Y^0.

Lemma 3.3. *Let σ, p, α, and β be as in Lemma 3.2. For $u \in Y^0$, we have*

$$\left(\sum_{N \in 2^{\mathbb{N}}} N^{-2\beta} \|\psi u_N\|^2_{B^\alpha_{p,1} L^p} \right)^{1/2} = \|\psi u\|_{\ell^2_{-\beta} B^\alpha_{p,1} L^p} \lesssim \|u\|_{Y^0}. \tag{3.8}$$

Proof. Let $\alpha \in (0, \frac{1}{p})$. By (3.2) and (3.5), for $\{t_k\}_{k=1}^K \in \mathcal{Z}$, $N \in 2^{\mathbb{N}}$, and $\{\phi_k\}_{k=1}^K$ in $L^2(\mathbb{T}^d)$, we have

$$\left\| \psi P_N \sum_{j=1}^K e^{it\Delta} \phi_j \cdot \chi_{[t_{j-1}, t_j)} \right\|_{B^\alpha_{p,\infty} L^p} \lesssim_{p,\alpha} N^\beta \| \|\phi_j\|_{L^2} \|_{\ell^p_j}.$$

In other words, we have

$$\|\psi e^{it\Delta} f_N\|_{B^{\alpha}_{p,\infty}L^p} \lesssim_{p,\alpha} N^{\beta}$$

for every U^pL^2-atom $f = \sum_{j=1}^{K} \phi_j \cdot \chi_{[t_{j-1},t_j)}$, i.e., $\sum_{j=1}^{K} \|\phi_j\|_{L^2}^p = 1$. Using the embedding $Y^0 \hookrightarrow U^p_{\Delta}L^2$, we have

$$\|\psi u_N\|_{B^{\alpha}_{p,\infty}L^p} \lesssim N^{\beta}\|u_N\|_{Y^0}$$

for $u \in Y^0$. Then, by the real interpolation argument used in Lemma 3.2, we arrive at $\|\psi u_N\|_{B^{\alpha}_{p,1}L^p} \lesssim N^{\beta}\|u_N\|_{Y^0}$, which implies (3.8) due to $Y^0 = \ell^2 Y^0$. ∎

3.2. Definition of Z^s spaces

We now introduce a function space for the well-posedness problem. In the introduction, we explained a limitation of the conventional candidate, the Y^s space. Large data a priori bounds were obtained in [10, 15, 18] using an argument that works only when the nonlinearity $\mathcal{N}(u)$ is either algebraic or sufficiently regular, i.e., $a \geq 2$.

Here we overcome this difficulty by introducing a new function space Z^s. We hope to keep bringing up the favorable embedding properties of the Y^s space. Still, using an L^p_t-structure for the time variable, we make the Z^s norm shrink to zero as the length of the time interval goes to zero.

Indeed, we look for Z^s weaker than Y^s, i.e., $Z^s \hookleftarrow Y^s$. While the embedding $Z^s \hookleftarrow Y^s$ makes the retarded estimate $K^+ \colon (Z^{-s})' \to Z^s$ immediate, it also makes the bilinear estimate based on Z^s stronger than that based on Y^s. Thus we focus on the structure of the norm we use in our proof of the main bilinear estimate.

In Section 3.3 we will prove our bilinear estimate using the identity

$$\int_{\mathbb{R}\times\mathbb{T}^d} v\bar{w}A\,dx\,dt = \int_{\mathbb{R}\times\mathbb{T}^d} I_\xi v \cdot \overline{I_\xi w} \cdot J_\xi A\,dx\,dt \tag{3.9}$$

for $v, w, A\colon \mathbb{R}\times\mathbb{T}^d \to \mathbb{C}$ and $\xi \in \mathbb{Z}^d$, where J_ξ denotes the shear effect $J_\xi A(t,x) := A(t, x - 2t\xi)$. To prove the bilinear estimate, we will partition the spatial frequency domain into congruent cubes $C \subset \mathbb{Z}^d$, then apply (3.9) to each C with the shift ξ that translates the center of C to the origin. Combining spacetime regularities of $v, w, A\colon \mathbb{R}\times\mathbb{T}^d \to \mathbb{C}$ and the shear effect J_ξ will give an extra gain of decay that we mentioned in the introduction. To use the Galilean structure, frequency partitioning, and time Besov regularity, we are motivated to consider a norm of the form

$$\max_{R\in 2^{\mathbb{N}}} R^{-\beta} \big\| \|\psi P_{\leq R} I_{Rk} u\|_{B^{\alpha}_{p,1}L^p} \big\|_{\ell^2(k\in\mathbb{Z}^d)},$$

where α and β are some scaling-critical exponents.

Inspired by this observation, we design a new space Z^s.

Definition 3.4. Let σ and p be as in (3.1). We define $Z^s = Z^s(\mathbb{R} \times \mathbb{T}^d)$ as a Banach space given by the norm

$$\|u\|_{Z^s} = \max_{q \in [p, \frac{1}{\sigma}]} \| \|\psi u_N\|_{L^q L^r} \|_{\ell^2_{s-\sigma}(N \in 2^{\mathbb{N}})} + \max_{\alpha \in [\sigma, \frac{1}{p} - \sigma]} \Big\| \max_{\substack{R \in 2^{\mathbb{N}} \\ R \le 8N}} R^{-\beta} \big\| \|\psi P_{\le 8R} I_{Rk} u_N\|_{B^{\alpha}_{p,1} L^p} \big\|_{\ell^2(k \in \mathbb{Z}^d)} \Big\|_{\ell^2_s(N \in 2^{\mathbb{N}})}, \tag{3.10}$$

where scaling conditions $\frac{d}{r} = \frac{d}{2} - \sigma - \frac{2}{q}$ and $\beta = \sigma + 2\alpha$ are imposed.

Similarly, Z^0 is defined as (3.10) with s replaced by 0.

Remark 3.5. As seen in (3.5), the scaling conditions $\beta = \sigma + 2\alpha$ and $p = \frac{d+2}{\frac{d}{2} - \sigma}$ make the embedding $Y^0 \hookrightarrow \ell^2_{-\beta} B^{\alpha}_{p,1} L^p$ scaling invariant with the scale of $C^0 L^2$. For the same reason, Z^s and Z^0 spaces have scales identical to those of Y^s and Y^0, respectively.

In view of complex interpolations, $\|u\|_{Z^s}$ is equivalent to (3.10) with q and α at the endpoints of the intervals.

We collect elementary properties of the space Z^s.

Lemma 3.6. *We have the following properties:*

- *For a finite interval $I \subset \mathbb{R}$, we have the embedding*

$$\ell^2_s (Z^0)' = (Z^{-s})' \hookrightarrow (Y^{-s})' \xrightarrow{K^+} Y^s \hookrightarrow Z^s = \ell^2_s Z^0. \tag{3.11}$$

- *For a finite interval $I \subset \mathbb{R}$, we have*

$$\|u \cdot \chi_I\|_{Z^s} \lesssim \|u\|_{Z^s}. \tag{3.12}$$

  This estimate is uniform in the choice of I.

- *For $u \in Z^s$, we have*

$$\lim_{T \to 0^+} \|u \cdot \chi_{[0,T]}\|_{Z^s} = 0. \tag{3.13}$$

- *Let $q \in [p, \frac{1}{\sigma}]$ and r be parameters such that $\frac{2}{q} + \frac{d}{r} = \frac{d}{2} - \sigma$. We have*

$$\|\psi u\|_{L^q H^{s-\sigma, r}} \lesssim \|\psi u\|_{L^q B^{s-\sigma}_{r,2}} \lesssim \|\psi u\|_{\ell^2_{s-\sigma} L^q L^r} \lesssim \|u\|_{Z^s}. \tag{3.14}$$

- *Let $\alpha \in [\sigma, \frac{1}{p} - \sigma]$ and $\beta = \sigma + 2\alpha$. We have*

$$\|\psi u\|_{B^{\alpha}_{p,2} B^{s-\beta}_{p,2}} \lesssim \|\psi u\|_{\ell^2_{s-\beta} B^{\alpha}_{p,1} L^p} \lesssim \|u\|_{Z^s}. \tag{3.15}$$

Similar properties hold with s replaced by 0.

Proof. In (3.11) we show that Z^s is weaker than Y^s. In (3.15) we bring the Strichartz estimate (3.8) to Z^s. Most importantly, in (3.13), we show that the Z^s norm of a free evolution converges to 0 as the time cutoff shrinks.

(1) We show

$$Y^s \hookrightarrow Z^s. \tag{3.16}$$

Once we have (3.16), by duality and (2.24), (3.11) follows.

We decompose the equation for Z^s as follows:

$$\begin{aligned} \|u\|_{Z^s} = &\max_{\substack{q\in[p,\frac{1}{\sigma}]\\ \frac{d}{r}=\frac{d}{2}-\sigma-\frac{2}{q}}} \big\| \|\psi u_N\|_{L^qL^r} \big\|_{\ell^2_{s-\sigma}(N\in 2^{\mathbb{N}})} \\ &+ \max_{\substack{\alpha\in[\sigma,\frac{1}{p}-\sigma]\\ \beta=\sigma+2\alpha}} \Big\| \max_{\substack{R\in 2^{\mathbb{N}}\\ R\le 8N}} R^{-\beta} \big\| \|\psi P_{\le 8R} I_{Rk} u_N\|_{B^\alpha_{p,1}L^p} \big\|_{\ell^2(k\in\mathbb{Z}^d)} \Big\|_{\ell^2_s(N\in 2^{\mathbb{N}})} \\ = &\,\mathrm{I} + \mathrm{II}. \end{aligned}$$

We bound I first. We fix $q \in [p, \frac{1}{\sigma}]$ and $r \in (2,\infty)$ satisfying $\frac{2}{q} + \frac{d}{r} = \frac{d}{2} - \sigma$. Since the spacetime norms $L^\infty L^2$, $\ell^2_{-\sigma} L^\infty L^{(1/2-\sigma/d)^{-1}}$, $\ell^2_{-\sigma} L^p L^p$, $\ell^2_{-\sigma} L^q L^r$, and Y^0 have the same scale, a complex interpolation between $\|\psi u\|_{\ell^2_{-\sigma} L^\infty L^{(1/2-\sigma/d)^{-1}}} \lesssim \|\psi u\|_{\ell^2 L^\infty L^2} \lesssim \|u\|_{Y^0}$ and $\|\psi u\|_{\ell^2_{-\sigma} L^p L^p} \lesssim \|u\|_{Y^0}$ gives $\|\psi u\|_{\ell^2_{-\sigma} L^q L^r} \lesssim \|u\|_{Y^0}$. Therefore, we have

$$\mathrm{I} = \max_{\substack{q\in[p,\frac{1}{\sigma}]\\ \frac{d}{r}=\frac{d}{2}-\sigma-\frac{2}{q}}} \big\| \|\psi u_N\|_{L^qL^r} \big\|_{\ell^2_{s-\sigma}(N\in 2^{\mathbb{N}})} \lesssim \|u\|_{\ell^2_s Y^0} = \|u\|_{Y^s}.$$

To bound II, we fix $\alpha \in [\sigma, \frac{1}{p} - \sigma]$ and $R \in 2^{\mathbb{N}}$. Let $\beta = 2\alpha + \sigma$. By (3.8) and the Galilean invariance of the Y^0 norm, we have

$$\begin{aligned} R^{-\beta} \big\| \|\psi P_{\le 8R} I_{Rk} u_N\|_{B^\alpha_{p,1}L^p} \big\|_{\ell^2(k\in\mathbb{Z}^d)} &\lesssim \big\| \|P_{\le 8R} I_{Rk} u_N\|_{Y^0} \big\|_{\ell^2(k\in\mathbb{Z}^d)} \\ &\lesssim \big\| \|I_{Rk} P_{-Rk+[-8R,8R]^d} u_N\|_{Y^0} \big\|_{\ell^2(k\in\mathbb{Z}^d)} \\ &\lesssim \big\| \|P_{-Rk+[-8R,8R]^d} u_N\|_{Y^0} \big\|_{\ell^2(k\in\mathbb{Z}^d)} \\ &\lesssim \|u_N\|_{Y^0}, \end{aligned}$$

which implies $\mathrm{II} \lesssim \|u\|_{Y^0}$ immediately.

(2) In fact, the stability under time cutoffs holds true for any time Besov space. We prove a more general statement: For any Banach space E on $\mathbb{T}^d$, $\alpha \in (0, \frac{1}{p})$, $q \in [1,\infty]$, $f \in B^\alpha_{p,q}E$, and any interval $I \subset \mathbb{R}$, we have

$$\|f\chi_I\|_{B^\alpha_{p,q}E} \lesssim_{\alpha,p,q} \|f\|_{B^\alpha_{p,q}E}. \tag{3.17}$$

Once we have (3.17), (3.12) follows directly.

For $\alpha_0 \in (0,\alpha)$ and $\alpha_1 \in (\alpha, \frac{1}{p})$, by (3.2) we have

$$\|f\chi_I\|_{B^{\alpha_j}_{p,\infty}E} \lesssim \|f\|_{B^{\alpha_j}_{p,1}E}, \quad j = 0, 1. \tag{3.18}$$

Thus, applying a real interpolation of parameter q to (3.18) gives (3.17).

(3) We claim that for $N, R \in 2^{\mathbb{N}}$, $\alpha \in [\sigma, \frac{1}{p} - \sigma]$, $k \in \mathbb{Z}^d$, and $u \in Z^s$, we have

$$\lim_{T \to 0^+} \| P_{\leq 8R} I_{Rk} u_N \cdot \chi_{[0,T]} \|_{B^{\alpha}_{p,1} L^p} = 0. \tag{3.19}$$

Once we have (3.19), since we can deduce by (3.17) that

$$\begin{aligned} &\max_{\substack{q \in \{p, \frac{1}{\sigma}\} \\ \frac{d}{r} = \frac{d}{2} - \sigma - \frac{2}{q}}} \Big\| \sup_{T \in [0,1]} \| \psi u_N \cdot \chi_{[0,T]} \|_{L^q L^r} \Big\|_{\ell^2_{s-\sigma}(N \in 2^{\mathbb{N}})} \\ &\quad + \max_{\substack{\alpha \in \{\sigma, \frac{1}{p} - \sigma\} \\ \beta = \sigma + 2\alpha}} \Big\| \max_{\substack{R \in 2^{\mathbb{N}} \\ R \leq 8N}} R^{-\beta} \Big\| \sup_{T \in [0,1]} \| \psi P_{\leq 8R} I_{Rk} u_N \cdot \chi_{[0,T]} \|_{B^{\alpha}_{p,1} L^p} \Big\|_{\ell^2(k \in \mathbb{Z}^d)} \Big\|_{\ell^2_s(N \in 2^{\mathbb{N}})} \\ &< \infty, \end{aligned}$$

by the dominated convergence theorem, it follows that

$$\begin{aligned} &\lim_{T \to 0^+} \| u \cdot \chi_{[0,T]} \|_{Z^s} \\ &= \lim_{T \to 0^+} \max_{\substack{q \in \{p, \frac{1}{\sigma}\} \\ \frac{d}{r} = \frac{d}{2} - \sigma - \frac{2}{q}}} \Big\| \| \psi u_N \cdot \chi_{[0,T]} \|_{L^q L^r} \Big\|_{\ell^2_{s-\sigma}(N \in 2^{\mathbb{N}})} \\ &\quad + \lim_{T \to 0^+} \max_{\substack{\alpha \in \{\sigma, \frac{1}{p} - \sigma\} \\ \beta = \sigma + 2\alpha}} \Big\| \max_{\substack{R \in 2^{\mathbb{N}} \\ R \leq 8N}} R^{-\beta} \Big\| \| \psi P_{\leq 8R} I_{Rk} u_N \cdot \chi_{[0,T]} \|_{B^{\alpha}_{p,1} L^p} \Big\|_{\ell^2(k \in \mathbb{Z}^d)} \Big\|_{\ell^2_s(N \in 2^{\mathbb{N}})} \\ &= \max_{\substack{q \in \{p, \frac{1}{\sigma}\} \\ \frac{d}{r} = \frac{d}{2} - \sigma - \frac{2}{q}}} \Big\| \lim_{T \to 0^+} \| \psi u_N \cdot \chi_{[0,T]} \|_{L^q L^r} \Big\|_{\ell^2_{s-\sigma}(N \in 2^{\mathbb{N}})} \\ &\quad + \max_{\substack{\alpha \in \{\sigma, \frac{1}{p} - \sigma\} \\ \beta = \sigma + 2\alpha}} \Big\| \max_{\substack{R \in 2^{\mathbb{N}} \\ R \leq 8N}} R^{-\beta} \Big\| \lim_{T \to 0^+} \| \psi P_{\leq 8R} I_{Rk} u_N \cdot \chi_{[0,T]} \|_{B^{\alpha}_{p,1} L^p} \Big\|_{\ell^2(k \in \mathbb{Z}^d)} \Big\|_{\ell^2_s(N \in 2^{\mathbb{N}})} \\ &= 0. \end{aligned}$$

Now we show the claim (3.19). Let $f = P_{\leq 8R} I_{Rk} u_N$. Choose $\alpha_+ \in (\alpha, \frac{1}{p})$. Since $\|u\|_{Z^s} < \infty$, we have $\|f\|_{B^{\alpha_+}_{p,1} L^p} < \infty$. By (3.17), for a dyadic number $M \in 2^{\mathbb{N}}$, we have

$$\begin{aligned} \sup_{T \in [0,1]} \| P^t_{\geq M} (f \cdot \chi_{[0,T]}) \|_{B^{\alpha}_{p,1} L^p} &\lesssim M^{\alpha - \alpha_+} \sup_{T \in [0,1]} \| f \cdot \chi_{[0,T]} \|_{B^{\alpha_+}_{p,1} L^p} \\ &\lesssim M^{\alpha - \alpha_+} \| f \|_{B^{\alpha_+}_{p,1} L^p}. \end{aligned}$$

As a consequence, we have (3.19).

(4) Since $q, r \in (2, \infty)$, applying $\ell^2_{s-\sigma} L^q L^r \hookrightarrow L^q \ell^2_{s-\sigma} L^r = L^q B^{s-\sigma}_{r,2}$ to (3.10), we have (3.14).

(5) Now, since $p > 2$, we have $B^{\alpha}_{p,2} B^{s-\beta}_{p,2} = \ell^2_{\alpha;\tau} L^p_t \ell^2_{s-\beta} L^p_x \hookleftarrow \ell^2_{\alpha;\tau} \ell^2_{s-\beta} L^p_t L^p_x = \ell^2_{s-\beta} \ell^2_{\alpha;\tau} L^p_t L^p_x = \ell^2_{s-\beta} B^{\alpha}_{p,2} L^p$. Thus, we have

$$\| \psi u \|_{B^{\alpha}_{p,2} B^{s-\beta}_{p,2}} \lesssim \| \psi u \|_{\ell^2_{s-\beta} B^{\alpha}_{p,1} L^p} \lesssim \| N^{-\beta} \| \psi P_{\leq 8N} u_N \|_{B^{\alpha}_{p,1} L^p} \|_{\ell^2_s(N \in 2^{\mathbb{N}})} \lesssim \| u \|_{Z^s}. \quad \blacksquare$$

3.3. Bilinear estimates

In this subsection we prove the main bilinear estimate (3.25), as introduced in (1.5). To obtain the decay in (3.25), we use the identity for functions $f, g, A : \mathbb{R} \times \mathbb{T}^d \to \mathbb{C}$:

$$\int_{\mathbb{R}\times\mathbb{T}^d} f \bar{g} A \, dx \, dt = \int_{\mathbb{R}\times\mathbb{T}^d} I_\xi f \cdot \overline{I_\xi g} \cdot J_\xi A \, dx \, dt, \tag{3.20}$$

where

$$J_\xi A(t,x) := A(t, x - 2t\xi).$$

We first estimate $J_\xi A$ in Besov spaces.

Lemma 3.7. *For a dyadic number $N \in 2^{\mathbb{N}}$, an integer point $k \in \mathbb{Z}^d \setminus \{0\}$, and a function $A \in B^{\frac{1}{2}}_{1,1} L^1$, we have*

$$\|J_{Nk} A_N\|_{B^{-\frac{1}{8}}_{2,2} L^2} \lesssim N^{\frac{d}{2}-\frac{1}{4}} (N^{-\frac{1}{4}} + |k|^{-\frac{1}{8}}) \|A\|_{B^{\frac{1}{2}}_{1,1} L^1}. \tag{3.21}$$

Proof. We choose a coordinate vector e_j such that $\kappa = k \cdot e_j \sim |k|$. Let $k' = k - \kappa e_j$. Since $\widetilde{J_{Nk} A}(\tau, \xi) = \tilde{A}(\tau + 2Nk \cdot \xi, \xi)$, for a dyadic number $M \in 2^{\mathbb{N}}$, we have

$$\begin{aligned}
\|J_{Nk} P^t_M A_N\|^2_{B^{-\frac{1}{8}}_{2,2} L^2} &\lesssim \sum_{\xi \in \mathbb{Z}^d} \int_{\mathbb{R}} \langle \tau \rangle^{-\frac{1}{4}} |\mathcal{F}_{t,x}(J_{Nk} P^t_M A_N)(\tau, \xi)|^2 \, d\tau \\
&\lesssim \sum_{\xi \in \mathbb{Z}^d} \int_{\mathbb{R}} \langle \tau \rangle^{-\frac{1}{4}} |\mathcal{F}_{t,x}(P^t_M A_N)(\tau + 2Nk \cdot \xi, \xi)|^2 \, d\tau \\
&\lesssim \sum_{\xi \in [-N,N]^d} \int_{-2Nk\cdot\xi + [-10M, 10M]} \langle \tau \rangle^{-\frac{1}{4}} |\tilde{A}(\tau + 2Nk \cdot \xi, \xi)|^2 \, d\tau \\
&\lesssim \sum_{\xi \in [-N,N]^d} M \langle Nk \cdot \xi \rangle^{-\frac{1}{4}} \cdot \|\tilde{A}\|^2_{L^\infty_\tau \ell^\infty_\xi} \\
&\lesssim \sum_{\xi' \in [-N,N]^{d-1}} \sum_{n \in [-N,N]} M \langle Nk' \cdot \xi' + N\kappa n \rangle^{-\frac{1}{4}} \cdot \|\tilde{A}\|^2_{L^\infty_\tau \ell^\infty_\xi} \\
&\lesssim M N^{d-1} (1 + |\kappa|^{-\frac{1}{4}} N^{\frac{1}{2}}) \|A\|^2_{L^1_{t,x}} \\
&\lesssim M N^{d-\frac{1}{2}} (N^{-\frac{1}{2}} + |k|^{-\frac{1}{4}}) \|A\|^2_{L^1_{t,x}}.
\end{aligned}$$

Taking a square root, summing over $M \in 2^{\mathbb{N}}$, and applying the triangle inequality, we obtain (3.21), finishing the proof. ∎

Meanwhile, for $1 < r < q < \infty$, we have the embedding

$$\|J_{Nk} A_N\|_{B^0_{q,\infty} L^q} \lesssim \|J_{Nk} A_N\|_{L^q_{t,x}} = \|A_N\|_{L^q_{t,x}} \lesssim N^{d(\frac{1}{r}-\frac{1}{q})} \|A\|_{B^{\frac{1}{r}-\frac{1}{q}}_{r,r} L^r}. \tag{3.22}$$

Interpolating between (3.21) and (3.22), we have the following lemma.

Lemma 3.8. *Let Ω be the open tetrahedron whose vertices are $(\frac{1}{2}, 1, \frac{1}{8})$, $(0,1,0)$, $(1,1,0)$, and $(0,0,0)$. Let q, r, and ρ be parameters such that $(\frac{1}{q}, \frac{1}{r}, \rho)$ lies in Ω. For a dyadic number $N \in 2^{\mathbb{N}}$, an integer point $k \in \mathbb{Z}^d \setminus \{0\}$, and a function $A: \mathbb{R} \times \mathbb{T}^d \to \mathbb{C}$, we have*

$$\|J_{Nk} A_N\|_{B^{-\rho}_{q,\infty} L^q} \lesssim_{q,r,\rho} (N^{-2\rho} + |k|^{-\rho}) N^{d(\frac{1}{r}-\frac{1}{q})-2\rho} \|A\|_{B^{\frac{1}{r}-\frac{1}{q}}_{r,r} L^r}. \tag{3.23}$$

We choose $\sigma_1, \dots, \sigma_4 > 0$ satisfying

$$\sigma \ll \sigma_1 \ll \sigma_2 \ll \sigma_3 \ll \sigma_4 \ll 1. \tag{3.24}$$

For example, if we choose $\sigma = 10^{-10^{10^{10^{10^{d+1/s}}}}}$ for (3.1), we can choose the σ_j as

$$\sigma_1 = 10^{-10^{10^{10^{d+1/s}}}}, \quad \sigma_2 = 10^{-10^{10^{d+1/s}}}, \quad \sigma_3 = 10^{-10^{d+1/s}}, \quad \sigma_4 = 10^{-d-1/s}.$$

Lemma 3.9. *Let q_0, r_0, and θ be the exponents such that $\frac{1}{q_0} = \frac{2+\sigma_3}{d+2}$, $\frac{1}{r_0} = \frac{2+\sigma_3+\sigma_2}{d+2}$, and $\theta = \frac{2}{q_0} + \frac{d}{r_0} - 2$. For $A \in B^{\frac{1}{r_0}-\frac{1}{q_0}}_{r_0,r_0} B^{\theta}_{r_0,\infty}$, $u, v \in Z^0$, and frequencies $N, R \in 2^{\mathbb{N}}$ such that $N \geq 32R$, we have*

$$\begin{aligned}&\left|\int_{\mathbb{R}\times\mathbb{T}^d} \psi^2 u_N \bar{v} A_R \, dx\, dt\right| \\ &\qquad \lesssim \|u\|_{Z^0} \|v\|_{Z^0} (\langle N/R\rangle^{-\sigma_1} + R^{-2\sigma_1}) R^{\theta} \|A_R\|_{B^{\frac{1}{r_0}-\frac{1}{q_0}}_{r_0,r_0} L^{r_0}}.\end{aligned} \tag{3.25}$$

Proof. For simplicity of notation, we denote $r = r_0$ and $q = q_0$ in this proof. Let α and β be the parameters satisfying $\frac{1}{q} + \sigma_1 + 2(\frac{1}{p} - \alpha) = 1$ and $\beta = 2\alpha + \sigma$. Since $\int_{\mathbb{R}\times\mathbb{T}^d} \psi^2 u_N \overline{v_M} A_R \, dx\, dt$ is zero whenever $M > 4N$ or $M < \frac{1}{4}N$, (3.25) follows once we have for $M \sim N$ the estimate

$$\left|\int_{\mathbb{R}\times\mathbb{T}^d} \psi^2 u_N \overline{v_M} A_R \, dx\, dt\right| \lesssim \|u\|_{Z^0} \|v\|_{Z^0} (\langle N/R\rangle^{-\sigma_1} + R^{-2\sigma_1}) R^{\theta} \|A_R\|_{B^{\frac{1}{r}-\frac{1}{q}}_{r,r} L^r}.$$

Due to (3.20), we have

$$\int_{\mathbb{R}\times\mathbb{T}^d} v\bar{w} A \, dx\, dt = \int_{\mathbb{R}\times\mathbb{T}^d} I_{Rk} v \cdot \overline{I_{Rk} w} \cdot J_{Rk} A \, dx\, dt$$

for $v, w: \mathbb{R} \times \mathbb{T}^d \to \mathbb{C}$ and $k \in \mathbb{Z}^d$. We have

$$\begin{aligned}&\left|\int_{\mathbb{R}\times\mathbb{T}^d} \psi^2 u_N \overline{v_M} A_R \, dx\, dt\right| \\ &\quad = \left|\sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}\times\mathbb{T}^d} \psi^2 u_N \cdot \overline{P_{(-R,R]^d-2Rk} v_M} A_R \, dx\, dt\right| \\ &\quad = \left|\sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}\times\mathbb{T}^d} \psi^2 P_{[-8R,8R]^d-2Rk} u_N \cdot \overline{P_{(-R,R]^d-2Rk} v_M} A_R \, dx\, dt\right|\end{aligned}$$

$$= \Big| \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}\times\mathbb{T}^d} \psi I_{2Rk} P_{[-8R,8R]^d-2Rk} u_N \cdot \overline{\psi I_{2Rk} P_{(-R,R]^d-2Rk} v_M} J_{2Rk} A_R \, dx\, dt \Big|$$

$$= \Big| \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}\times\mathbb{T}^d} \psi P_{\le 8R} I_{2Rk} u_N \cdot \overline{\psi P_{(-R,R]^d} I_{2Rk} v_M} J_{2Rk} A_R \, dx\, dt \Big|.$$

Using $0 < \sigma_1 < \alpha$, $\frac{1}{q} + \sigma_1 + 2(\frac{1}{p} - \alpha) = 1$, and (2.13), we continue to estimate:

$$\lesssim \sum_{k\in\mathbb{Z}^d} \|\psi P_{\le 8R} I_{2Rk} u_N\|_{B^{\alpha}_{p,2}L^p} \|\psi P_{\le 8R} I_{2Rk} v_M\|_{B^{\alpha}_{p,2}L^p} \|J_{2Rk} A_R\|_{B^{-\sigma_1}_{q,\infty}L^{(p/2)'}}.$$

Again, using $(\frac{1}{q}, \frac{1}{r}, \sigma_1) \in \Omega$, $d(\frac{1}{r} + \frac{2}{p} - 1) - 2\sigma_1 = \theta - 2\beta$, $(p/2)' > q$, (3.23), and Bernstein estimates, we estimate

$$\begin{aligned}
&\lesssim \sum_{k\in\mathbb{Z}^d} \|\psi P_{\le 8R} I_{2Rk} u_N\|_{B^{\alpha}_{p,2}L^p} \|\psi P_{\le 8R} I_{2Rk} v_M\|_{B^{\alpha}_{p,2}L^p} \\
&\qquad \cdot R^{d(\frac{1}{r}+\frac{2}{p}-1)-2\sigma_1} (|k|^{-\sigma_1} + R^{-2\sigma_1}) \|A_R\|_{B^{\frac{1}{r}-\frac{1}{q}}_{r,r} L^r} \\
&\lesssim \sum_{k\in\mathbb{Z}^d} R^{-\beta} \|\psi P_{\le 8R} I_{2Rk} u_N\|_{B^{\alpha}_{p,2}L^p} R^{-\beta} \|\psi P_{\le 8R} I_{2Rk} v_M\|_{B^{\alpha}_{p,2}L^p} \\
&\qquad \cdot (|k|^{-\sigma_1} + R^{-2\sigma_1}) R^{\theta} \|A_R\|_{B^{\frac{1}{r}-\frac{1}{q}}_{r,r} L^r} \\
&\lesssim \|u_N\|_{Z^0} \|v_M\|_{Z^0} (\langle N/R\rangle^{-\sigma_1} + R^{-2\sigma_1}) \cdot R^{\theta} \|A_R\|_{B^{\frac{1}{r}-\frac{1}{q}}_{r,r} L^r},
\end{aligned}$$

where the last inequality holds since $P_{\le 8R} I_{2Rk} u_N$ is nonzero only if $|k| \gtrsim N/R$. This finishes the proof of (3.25). ∎

Lemma 3.10. *Let q_0, r_0, and θ be defined in Lemma 3.9. For $A \in B^{\frac{1}{r_0}-\frac{1}{q_0}}_{r_0,r_0} B^{\theta}_{r_0,\infty}$ and $u \in Z^0$, we have*

$$\Big| \int_{\mathbb{R}\times\mathbb{T}^d} \psi^2 |u|^2 A \, dx\, dt \Big| \lesssim \|u\|^2_{Z^0} \|A\|_{B^{\frac{1}{r_0}-\frac{1}{q_0}}_{r_0,r_0} B^{\theta}_{r_0,\infty}}. \tag{3.26}$$

Proof. Again, we denote $r = r_0$ and $q = q_0$ in this proof. We decompose the left-hand side of (3.26) into

$$\begin{aligned}
\Big| \int_{\mathbb{R}\times\mathbb{T}^d} \psi^2 |u|^2 A \, dx\, dt \Big| &\le \Big| \int_{\mathbb{R}\times\mathbb{T}^d} \psi^2 u \cdot \pi_{>}(\bar{u}, A) \, dx\, dt \Big| \\
&\quad + \Big| \int_{\mathbb{R}\times\mathbb{T}^d} \psi^2 u \cdot \pi_{\le}(\bar{u}, A) \, dx\, dt \Big|.
\end{aligned}$$

We first estimate $|\int_{\mathbb{R}\times\mathbb{T}^d} \psi^2 u \cdot \pi_{\le}(\bar{u}, A)\, dx\, dt|$. This is simply estimated using the Besov product rule (2.14) on $\mathbb{T}^d$ and the embedding property of Z^s (3.15). Let $\tilde{\alpha}$ and $\tilde{\beta}$ be the exponents such that $\frac{1}{q} + 2(\frac{1}{p} - \tilde{\alpha}) = 1$ and $\tilde{\beta} = 2\tilde{\alpha} + \sigma$. Since $\tilde{\alpha} \in [\sigma, \frac{1}{p} - \sigma]$ and $2\tilde{\beta} < \theta$, we have

$$\Big| \int_{\mathbb{R}\times\mathbb{T}^d} \psi^2 u \cdot \pi_{\le}(\bar{u}, A) \, dx\, dt \Big| \lesssim \|\psi u\|^2_{B^{\tilde{\alpha}}_{p,2} B^{-\tilde{\beta}}_{p,2}} \|A\|_{B^{\frac{1}{r}-\frac{1}{q}}_{r,r} B^{\theta}_{r,\infty}} \lesssim \|u\|^2_{Z^0} \|A\|_{B^{\frac{1}{r}-\frac{1}{q}}_{r,r} B^{\theta}_{r,\infty}}.$$

The main part of the proof is to estimate $|\int_{\mathbb{R}\times\mathbb{T}^d} \psi^2 u \cdot \pi_>(\bar{u}, A)\, dx\, dt|$. Since $\int_{\mathbb{R}\times\mathbb{T}^d} \psi^2 u_N \cdot \pi_>(\overline{u_M}, A_R)\, dx\, dt$ is zero whenever the dyadic frequencies $N, M \gg R$ are not comparable, by (3.25), we have

$$\left|\int_{\mathbb{R}\times\mathbb{T}^d} \psi^2 u \cdot \pi_>(\bar{u}, A_R)\, dx\, dt\right| \lesssim \sum_{N\geq 16R} \|u_N\|_{Z^0}^2(\langle N/R\rangle^{-\sigma_1} + R^{-2\sigma_1}) \cdot R^\theta \|A_R\|_{B_{r,r}^{\frac{1}{r}-\frac{1}{q}} L^r}.$$

Writing $A = \sum_{R\in 2^{\mathbb{N}}} A_R$, we have

$$\begin{aligned}\left|\int_{\mathbb{R}\times\mathbb{T}^d} \psi^2 u \cdot \pi_>(\bar{u}, A)\, dx\, dt\right| &\lesssim \sum_{\substack{R\in 2^{\mathbb{N}}\\ N\geq 16R}} \|u_N\|_{Z^0}^2(\langle N/R\rangle^{-\sigma_1} + R^{-2\sigma_1}) \cdot \|A\|_{B_{r,r}^{\frac{1}{r}-\frac{1}{q}} B_{r,\infty}^\theta} \\ &\lesssim \|u\|_{Z^0}^2 \|A\|_{B_{r,r}^{\frac{1}{r}-\frac{1}{q}} B_{r,\infty}^\theta},\end{aligned}$$

which yields the estimate of $|\int_{\mathbb{R}\times\mathbb{T}^d} \psi^2 u \cdot \pi_>(\bar{u}, A)\, dx\, dt|$. ■

As a consequence, we have the main estimate of this section.

Proposition 3.11. *Let $m \in \mathbb{Z}$. Let $\psi_1 : \mathbb{R} \to \mathbb{R}$ be a C_0^∞ bump function satisfying $\psi|_{\operatorname{supp}(\psi_1)} \equiv 1$. Then, for $u \in Z^s$ and $v \in Z^0$, we have*

$$\|v^* \cdot \psi_1 |u|^{a-m} u^m\|_{(Z^0)'} \lesssim \|v\|_{Z^0} \|u\|_{Z^s}^a, \tag{3.27}$$

where v^* denotes either v or $\bar{v}$.

Proof. In this proof, we use exponents $\hat{r}$, ζ, and η defined as $\frac{1}{\hat{r}} = \frac{1+\sigma_4}{d+2}$, $\zeta = \frac{1}{\hat{r}} - \frac{1}{2q_0}$, and $\eta = \frac{d}{\hat{r}} - \frac{d}{2r_0} + \frac{\theta}{2}$, respectively, where q_0, r_0, and θ are the exponents defined in Lemma 3.9.

First, we claim that for $u \in Z^0$ and $A \in B_{\hat{r},\hat{r}}^\zeta B_{\hat{r},\hat{r}}^\eta$, we have

$$\|\psi A u\|_{L_{t,x}^2} \lesssim \|u\|_{Z^0} \|A\|_{B_{\hat{r},\hat{r}}^\zeta B_{\hat{r},\hat{r}}^\eta}. \tag{3.28}$$

Since $\zeta > \frac{1}{r_0} - \frac{1}{q_0}$, $\eta > \theta$, and $\frac{1}{r_0} < \frac{2}{\hat{r}}$, by (2.13), we have

$$\||A|^2\|_{B_{r_0,r_0}^{\frac{1}{r_0}-\frac{1}{q_0}} B_{r_0,\infty}^\theta} \lesssim \|A\|_{B_{\hat{r},\hat{r}}^\zeta B_{\hat{r},\hat{r}}^\eta}^2. \tag{3.29}$$

By (3.26) and (3.29), we have $\int \psi^2 u\bar{u}|A|^2\, dx\, dt \lesssim \|u\|_{Z^0}^2 \|A\|_{B_{\hat{r},\hat{r}}^\zeta B_{\hat{r},\hat{r}}^\eta}^2$, which implies (3.28).

Next we claim that for $m \in \mathbb{Z}$, we have

$$\||\psi u|^{a/2-m} (\psi u)^m\|_{B_{\hat{r},\hat{r}}^\zeta B_{\hat{r},\hat{r}}^\eta} \lesssim \|u\|_{Z^s}^{a/2}. \tag{3.30}$$

Once we have (3.30), by (3.28), we obtain (3.27) as the following dual form:

$$\begin{aligned}
&\left|\int_{\mathbb{R}\times\mathbb{T}^d} \psi_1 |u|^{a-m} u^m v^* w \, dx \, dt\right| \\
&\quad \lesssim \|\psi|\psi u|^{a/2-m}(\psi u)^m v^*\|_{L^2_{t,x}} \cdot \|\psi|\psi u|^{a/2} w\|_{L^2_{t,x}} \\
&\quad \lesssim \||\psi u|^{a/2-m}(\psi u)^m\|_{B^{\zeta}_{\hat r,\hat r} B^{\eta}_{\hat r,\hat r}} \||\psi u|^{a/2}\|_{B^{\zeta}_{\hat r,\hat r} B^{\eta}_{\hat r,\hat r}} \|v\|_{Z^0} \|w\|_{Z^0} \\
&\quad \lesssim \|u\|^a_{Z^s} \|v\|_{Z^0} \|w\|_{Z^0}.
\end{aligned}$$

Since (2.18) holds only for functions of Hölder regularities $C^{0,\alpha}$, $\alpha \in (0,1)$, we first decompose the exponent $a/2$ into $a_1, \dots, a_k \in (0,1)$ satisfying

$$a_1 + \cdots + a_k = a/2 \tag{3.31}$$

and $a_j \gg \sigma_4$ for $j = 1, \dots, k$. We partition $|\psi u|^{a/2-m}(\psi u)^m$ into a product of k terms of the form

$$|\psi u|^{a/2-m}(\psi u)^m = \prod_{j=1}^{k} |\psi u|^{a_j - m_j} (\psi u)^{m_j}, \tag{3.32}$$

where the $m_j \in \mathbb{Z}$ are certain integers, then estimate each term using (2.18).

Choose $s_0 > 0$ satisfying $\sigma_4 \ll_{d,a_j,m_j} s_0 \ll_{d,a_j,m_j} 1$. Let r and s_1 be the exponents satisfying $\frac{a}{2}(\frac{1}{r} - s_0) = \frac{1}{\hat r} - \zeta$ and $\frac{a}{2}(\frac{1}{r} - \frac{s_1}{d}) = \frac{1}{\hat r} - \frac{\eta}{d}$.

Since $s - \sigma > 2s_0 + s_1$ and $r \in [p, \frac{1}{\sigma}]$, by (3.14) and (2.7), we have

$$\|\psi u\|_{L^r B^{2s_0+s_1}_{r,2}} \lesssim \|u\|_{Z^s}. \tag{3.33}$$

Choose $\alpha = \frac{1}{p} - \frac{1}{r} + s_0 + s_1/2$ and $\beta = 2\alpha + \sigma$. Since $\alpha \in [\sigma, \frac{1}{p} - \sigma]$, $\alpha > s_0 + s_1/2$, and $s - \beta > 0$, by (3.15), (2.5), and (2.7), we have

$$\|\psi u\|_{B^{s_0+s_1/2}_{r,2} L^r} \lesssim \|\psi u\|_{B^{\alpha}_{p,2} B^{s-\beta}_{p,2}} \lesssim \|u\|_{Z^s}. \tag{3.34}$$

Applying (2.13) and (2.18) to (3.32), since $r \geq 2$, $s_0 \cdot a_j > \zeta > 0$, and $s_1 \cdot a_j > \eta > 0$, by (3.33) and (3.34), we have

$$\begin{aligned}
\||\psi u|^{a/2-m}(\psi u)^m\|_{B^{\zeta}_{\hat r,\hat r} B^{\eta}_{\hat r,\hat r}} &\lesssim \prod_{j=1}^{k} \||\psi u|^{a_j-m_j}(\psi u)^{m_j}\|_{B^{s_0\cdot a_j}_{r/a_j, r/a_j} B^{s_1\cdot a_j}_{r/a_j, r/a_j}} \\
&\lesssim \|\psi u\|^{a/2}_{L^r B^{2s_0+s_1}_{r,2} \cap B^{s_0+s_1/2}_{r,2} L^r} \\
&\lesssim \|u\|^{a/2}_{Z^s},
\end{aligned}$$

which is just (3.30) and finishes the proof. ∎

4. Local well-posedness of (NLS)

In this section we prove Theorem 1.1, the local well-posedness of (NLS). We construct solutions in the Z^s space introduced in Section 3. Based on the key estimate, Proposition 3.11, we first prove the main nonlinear estimate (Lemma 4.4). On the way, we handle nonalgebraic nonlinear terms using the Bony linearization. Since we construct solutions by weak limits, we provide a separate argument for the continuous dependence. For this purpose, we use an enhanced form of nonlinear estimate (see (4.7)).

Fix a C_0^∞-function ψ_1 such that $\psi|_{\mathrm{supp}(\psi_1)} \equiv 1$ and $\psi_1 \equiv 1$ on some open interval containing 0.

4.1. Nonlinear estimates

In this subsection we propose some estimates on $\psi_1 \mathcal{N}(u)$, where $u: \mathbb{R} \times \mathbb{T}^d \to \mathbb{C}$. To handle the nonalgebraicity of $\mathcal{N}(u)$, we use a paraproduct technique known as the Bony linearization method. We decompose $\mathcal{N}(u)$ as a sum of

$$F^N := \mathcal{N}(u_{\le N}) - \mathcal{N}(u_{\le N/2})$$

over $N \in 2^{\mathbb{N}}$, then estimate $F_K^N = P_K F^N$ in terms of $N, K \in 2^{\mathbb{N}}$. (We use the conventional notation $u_{\le 1/2} = 0$.) Here, the following Bony linearization formula given in [18] is used:

$$F^N = u_N \int_0^1 \partial_z \mathcal{N}(u_{\le N/2} + \theta u_N)\, d\theta + \overline{u_N} \int_0^1 \partial_{\bar{z}} \mathcal{N}(u_{\le N/2} + \theta u_N)\, d\theta. \tag{4.1}$$

For simplicity of notation, we denote by $A^N: \mathbb{R} \times \mathbb{T}^d \to \mathbb{C}^2$, $N \in 2^{\mathbb{N}}$ the function

$$A^N := \left(\int_0^1 \partial_z \mathcal{N}(u_{\le N/2} + \theta u_N)\, d\theta, \int_0^1 \partial_{\bar{z}} \mathcal{N}(u_{\le N/2} + \theta u_N)\, d\theta \right)$$

and denote by $u \times A$ for $u: \mathbb{R} \times \mathbb{T}^d \to \mathbb{C}$ and $A = (A_1, A_2): \mathbb{R} \times \mathbb{T}^d \to \mathbb{C}^2$ the function

$$u \times A := u A_1 + \bar{u} A_2.$$

With this notation, (4.1) can be rewritten as $F^N = u_N \times A^N$.

The bilinear estimate (3.27) is transferred to the following estimate of F_K^N:

Lemma 4.1. *For $N, K \in 2^{\mathbb{N}}$, we have*

$$\|\psi_1 F_K^N\|_{(Z^{-s})'} \lesssim \|u\|_{Z^s}^a (K/N)^s \cdot \|u_N\|_{Z^s}. \tag{4.2}$$

Proof. It suffices to show $\|\psi_1 F^N\|_{(Z^0)'} \lesssim \|u_N\|_{Z^0} \|u\|_{Z^s}^a$ by frequency localization. By (3.27), we have

$$\begin{aligned}\|\psi_1 F^N\|_{(Z^0)'} &= \|u_N \times \psi_1 A^N\|_{(Z^0)'} \\ &\lesssim \int_0^1 \|u_N \times \psi_1(\partial_z \mathcal{N}(u_{\le N/2} + \theta u_N), \partial_{\bar{z}} \mathcal{N}(u_{\le N/2} + \theta u_N))\|_{(Z^0)'}\, d\theta\end{aligned}$$

$$\lesssim \int_0^1 \|u_N\|_{Z^0} \|u_{\leq N/2} + \theta u_N\|_{Z^s}^a \, d\theta$$
$$\lesssim \|u_N\|_{Z^0} \|u\|_{Z^s}^a,$$

which finishes the proof. ■

On the other hand, the fractional Hölder inequalities give the following estimate:

Lemma 4.2. *Let $\mu > 0$ and $\nu > 0$ be the exponents $\nu = 1 + a - s - \sigma_1$ and $\mu = (1 - \frac{2\sigma}{\sigma_1})(\nu + \sigma) + \frac{2\sigma}{\sigma_1}(-s + \sigma)$. For $N, K \in 2^{\mathbb{N}}$ satisfying $4N \leq K$, we have*

$$\|\psi_1 F_K^N\|_{(Z^{-s})'} \lesssim (N/K)^{\mu} N^{-\nu} \|u_{\leq N}\|_{Z^{s+\nu}} \|u\|_{Z^s}^a. \tag{4.3}$$

Proof. We will estimate $P_K(u_N \times A^N) = F_K^N$ in two ways: (i) applying (2.15) to A_M^N, $M \in 2^{\mathbb{N}}$ to estimate F_K^N; (ii) applying (2.16) to F_K^N directly. Then we will interpolate between (i) and (ii) to obtain (4.3).

Let q and r be the parameters satisfying $\frac{1+a}{q} = \frac{1}{p'}$ and $\frac{d}{r} = \frac{d}{2} - \sigma - \frac{2}{q}$. Let p_0 and p_1 be the parameters satisfying $\frac{1}{p_1'} - \frac{1}{r} - \frac{\sigma_1}{d} = a(\frac{1}{r} - \frac{s-\sigma}{d})$ and $\frac{1}{p_0'} - \frac{1}{r} = a(\frac{1}{r} - \frac{s-\sigma}{d})$.

(i) Since $\sigma_1 < a(s - \sigma)$ and $\sigma_1 < s - \sigma$, by either (2.15) or (2.16), for $M \in 2^{\mathbb{N}}$, we have

$$\begin{aligned}\|u_N \times A_M^N\|_{L^{p_1'}} &\lesssim \|u_N\|_{L^r} \|A_M^N\|_{L^{(\frac{1}{p_1'}-\frac{1}{r})^{-1}}} \\ &\lesssim N^{-s+\sigma} \|u_N\|_{H^{s-\sigma,r}} \cdot M^{-\sigma_1} \|A^N\|_{H^{\sigma_1,(\frac{1}{p_1'}-\frac{1}{r})^{-1}}} \\ &\lesssim N^{-s+\sigma} M^{-\sigma_1} \|u_N\|_{H^{s-\sigma,r}} \|u\|_{H^{s-\sigma,r}}^a.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|F_K^N\|_{L^{p_1'}} &\lesssim \sum_{M \in 2^{\mathbb{N}}} \|P_K(u_N \times A_M^N)\|_{L^{p_1'}} \\ &\lesssim \sum_{M \sim K} \|u_N \times A_M^N\|_{L^{p_1'}} \\ &\lesssim (N/K)^{-s+\sigma} K^{-s-\sigma_1+\sigma} \|u_N\|_{H^{s-\sigma,r}} \|u\|_{H^{s-\sigma,r}}^a \\ &\lesssim (N/K)^{-s+\sigma} K^{-s-\sigma_1+\sigma} N^{-\nu} \|u_{\leq N}\|_{H^{s-\sigma+\nu,r}} \|u\|_{H^{s-\sigma,r}}^a.\end{aligned} \tag{4.4}$$

(ii) Since $s + \nu < 1 + a$, by (2.16), we have

$$\begin{aligned}\|F_K^N\|_{L^{p_0'}} &\lesssim K^{-s-\nu} \|F_K^N\|_{H^{s+\nu,p_0'}} \\ &\lesssim K^{-s-\nu} \|u_{\leq N}\|_{H^{s+\nu,r}} \|u\|_{L^{a(\frac{1}{p_0'}-\frac{1}{r})^{-1}}}^a \\ &\lesssim K^{-s-\nu} N^{\sigma} \|u_{\leq N}\|_{H^{s-\sigma+\nu,r}} \|u\|_{H^{s-\sigma,r}}^a \\ &\lesssim (N/K)^{\nu+\sigma} K^{-s+\sigma} N^{-\nu} \|u_{\leq N}\|_{H^{s-\sigma+\nu,r}} \|u\|_{H^{s-\sigma,r}}^a.\end{aligned} \tag{4.5}$$

We interpolate between (4.4) and (4.5). Interpolating between (4.4) and (4.5) with exponent $2\sigma/\sigma_1$, we have

$$\|F_K^N\|_{L^{p'}} \lesssim (N/K)^{\mu} K^{-s-\sigma} N^{-\nu} \|u_{\le N}\|_{H^{s-\sigma+\nu,r}} \|u\|_{H^{s-\sigma,r}}^a .$$

Multiplying cutoffs and applying $L_t^{p'}$ norms to both sides, and using $\frac{1+a}{q} = \frac{1}{p'}$, $\|\psi u_{\le N}\|_{L^q H^{s-\sigma+\nu,r}} \lesssim \|u_{\le N}\|_{Z^{s+\nu}}$, and $\|\psi u_{\le N}\|_{L^q H^{s-\sigma,r}} \lesssim \|u_{\le N}\|_{Z^s}$, we have

$$\|\psi_1 F_K^N\|_{L_{t,x}^{p'}} \lesssim (N/K)^{\mu} K^{-s-\sigma} N^{-\nu} \|u_{\le N}\|_{Z^{s+\nu}} \|u\|_{Z^s}^a ,$$

which implies (4.3) due to (3.14). ∎

In the rest of this section, for $K, M, N \in 2^{\mathbb{N}}$, we denote

$$\begin{aligned} \beta_K^N &:= \|\psi_1 F_K^N\|_{(Z^{-s})'}^2, \\ \alpha_M &:= \|u_M\|_{Z^s}^2, \\ \alpha_\Omega &:= \|u\|_{Z^s}^2 = \sum_{M\in 2^{\mathbb{N}}} \alpha_M . \end{aligned}$$

In terms of α_N and β_K^N, (4.2) and (4.3) can be rewritten as

$$\beta_K^N \lesssim \alpha_\Omega^a (K/N)^{2s} \cdot \alpha_N$$

and

$$\beta_K^N \lesssim \alpha_\Omega^a (N/K)^{2\mu} \cdot \sum_{L\le N} (L/N)^{2\nu} \alpha_L, \quad 4N \le K,$$

respectively.

Combining (4.2) and (4.3), we obtain the following inequality:

Corollary 4.3. *There exists $\varepsilon_0 = \varepsilon_0(d,s,a) > 0$ (e.g., $\varepsilon_0 = \min\{s,\mu,2\nu\}/2$) such that, for each $K \in 2^{\mathbb{N}}$,*

$$\Big(\sum_{N\in 2^{\mathbb{N}}} \sqrt{\beta_K^N}\Big)^2 \lesssim \alpha_\Omega^a \cdot \sum_{N\in 2^{\mathbb{N}}} \max\Big\{\frac{N}{K}, \frac{K}{N}\Big\}^{-\varepsilon_0} \alpha_N . \tag{4.6}$$

Proof. Let μ and ν be the exponents defined in Lemma 4.2. Since $\sum_{N>K/4}(N/K)^{-s} + \sum_{N\le K/4}(K/N)^{-\mu} \lesssim 1$, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \Big(\sum_{N\in 2^{\mathbb{N}}} \sqrt{\beta_K^N}\Big)^2 &\lesssim \sum_{N>K/4} (N/K)^s \beta_K^N + \sum_{N\le K/4} (K/N)^{\mu} \beta_K^N \\ &\lesssim \alpha_\Omega^a \cdot \Big(\sum_{N>K/4} (K/N)^s \alpha_N + \sum_{N\le K/4} (N/K)^{\mu} \sum_{L\le N} (L/N)^{2\nu} \alpha_L\Big) \end{aligned}$$

$$\lesssim \alpha_\Omega^a \cdot \Big(\sum_{N>K/4} (K/N)^s \alpha_N + \sum_{L\le K/4} (L/K)^{\min\{\mu,2\nu\}-} \alpha_L \Big)$$

$$\lesssim \alpha_\Omega^a \cdot \sum_{N\in 2^{\mathbb{N}}} \max\Big\{\frac{N}{K}, \frac{K}{N}\Big\}^{-\varepsilon_0} \alpha_N,$$

which is (4.6).

Here, it suffices to choose $\varepsilon_0 = \min\{s, \mu, 2\nu\}/2$. ∎

Now we present the main inequality for the nonlinear term $\mathcal{N}(u)$.

Lemma 4.4. *Let ε_0 be as in Corollary 4.3. Fix $\varepsilon_1 \ll \varepsilon_0$. Let $\{\gamma_N\}_{N\in 2^{\mathbb{N}}}$ be a sequence of positive numbers such that $1 \le \frac{\gamma_{2N}}{\gamma_N} \le 1+\varepsilon_1$. We have*

$$\sum_{N\in 2^{\mathbb{N}}} \gamma_N \|\psi_1 P_N \mathcal{N}(u)\|^2_{(Z^{-s})'} \lesssim \|u\|^{2a}_{Z^s} \sum_{N\in 2^{\mathbb{N}}} \gamma_N \|u_N\|^2_{Z^s}. \tag{4.7}$$

In particular, plugging in $\gamma_N \equiv 1$ gives

$$\|\psi_1 \mathcal{N}(u)\|_{(Z^{-s})'} \lesssim \|u\|^{1+a}_{Z^s}. \tag{4.8}$$

A standard nonlinear estimate for constructing a solution is (4.8). We need an enhanced form, (4.7), for showing the continuous dependence of the solution map.

Proof of Lemma 4.4. The proof follows easily from (4.6). We have

$$\sum_{K\in 2^{\mathbb{N}}} \gamma_K \Big(\sum_{N\in 2^{\mathbb{N}}} \sqrt{\beta_K^N} \Big)^2 \lesssim \alpha_\Omega^a \cdot \sum_{K,N\in 2^{\mathbb{N}}} \max\Big\{\frac{N}{K}, \frac{K}{N}\Big\}^{-\varepsilon_0} \gamma_K \alpha_N$$

$$\lesssim \alpha_\Omega^a \cdot \sum_{N\in 2^{\mathbb{N}}} \gamma_N \alpha_N. \tag{4.9}$$

Since $\|\psi_1 P_K \mathcal{N} u\|_{(Z^{-s})'} \lesssim \sum_{N\in 2^{\mathbb{N}}} \sqrt{\beta_K^N}$, (4.9) implies (4.7). ∎

Remark 4.5. Due to the embedding between Y^s and Z^s, (3.11), (4.8) also implies the nonlinear estimate in Y^s space:

$$\|\psi_1 \mathcal{N}(u)\|_{(Y^{-s})'} \lesssim \|u\|^{1+a}_{Y^s}. \tag{4.10}$$

Although $Y^s \hookrightarrow Z^s$, we expect the proof of (4.8) to be almost at the same level as that of (4.10). This is because, in designing Z^s, we incorporated structures of Y^s required for the nonlinear estimate, such as the Galilean invariance, the frequency ℓ^2-basedness, and the Besov Strichartz estimates. We recall that the main benefit of working in the Z^s space is that $\|u \cdot \chi_{[0,T]}\|_{Z^s}$ shrinks as $T \to 0$, (3.13).

4.2. Proof of Theorem 1.1

In this subsection we finish the proof of Theorem 1.1. We restate it in a more precise technical form in the following proposition:

Proposition 4.6. *Let $a > \frac{4}{d}$ and $s < 1 + a$. Fix $\underline{u_0} \in H^s$. There exist an open interval $I = I(\underline{u_0}) \ni 0$ and $R = R(\underline{u_0}) > 0$ such that for every $u_0 \in H^s$ with $\|u_0 - \underline{u_0}\|_{H^s} < R$, there exists a unique solution $\Phi(u_0) := u \in C^0(I; H^s) \cap Y^s$ to*

$$\begin{cases} iu_t + \Delta u = \mathcal{N}(u) \cdot \chi_I, \\ u(0) = u_0, \end{cases} \tag{4.11}$$

and the solution map $\Phi: \{u_0 \in H^s \mid \|u_0 - \underline{u_0}\|_{H^s} < R\} \to C^0(I; H^s) \cap Y^s$ is continuous.

In Proposition 4.6, I depends on $\underline{u_0}$ to guarantee the smallness of $\|e^{it\Delta}\underline{u_0} \cdot \chi_I\|_{Z^s}$, for which our Z^s space plays a crucial role.

Proof of Proposition 4.6. We use nonlinear estimates based on Z^s. Using the embedding $Y^s \hookrightarrow Z^s$ we can show that the constructed solution lies in Y^s as well.

Construction. First, we show that such a map Φ exists. We construct such a solution $u = \Phi(u_0)$ by taking a weak limit of approximate solutions bounded in Y^s.

By (3.13), there exists an interval $I \ni 0$ such that $\|e^{it\Delta}\underline{u_0} \cdot \chi_I\|_{Z^s}$ is sufficiently small and $\psi_1|_I \equiv 1$. Choose $R \ll_{\underline{u_0},I} 1$. For $\lambda \geq 0$, we denote

$$P_{<2^\lambda} := \begin{cases} P_{\leq 2^{\lfloor \lambda \rfloor - 1}} + (\lambda - \lfloor \lambda \rfloor) P_{2^{\lfloor \lambda \rfloor}}, & \lambda \geq 1, \\ \lambda P_{\leq 1}, & 0 \leq \lambda < 1, \end{cases}$$

where $\lfloor \lambda \rfloor$ denotes the greatest integer $n \leq \lambda$.

Fix $\lambda \geq 0$. For $u_0 \in H^s$, let $u = u^{(\lambda)}$ be the strong solution of the following equation:

$$\begin{cases} iu_t + \Delta u = P_{<2^\lambda}(\mathcal{N}(u) \cdot \chi_I), \\ u(0) = u_0. \end{cases}$$

Let $v = u - e^{it\Delta}u_0$. By (4.8), (3.12), and (3.11), we have a bootstrap bound on $\|v\|_{Z^s}$:

$$\begin{aligned} \|v\|_{Z^s} &\lesssim \|v\|_{Y^s} \lesssim \|\psi_1 \mathcal{N}(u \cdot \chi_I)\|_{(Y^{-s})'} \lesssim \|\psi_1 \mathcal{N}(u \cdot \chi_I)\|_{(Z^{-s})'} \\ &\lesssim \|u \cdot \chi_I\|_{Z^s}^{1+a} \lesssim (\|v\|_{Z^s} + \|e^{it\Delta}u_0 \cdot \chi_I\|_{Z^s})^{1+a}. \end{aligned} \tag{4.12}$$

Assume $\|u_0 - \underline{u_0}\|_{H^s} < R$. Since $\|v^{(\lambda)}\|_{Z^s}$ is continuous on λ with $\|v^{(0)}\|_{Z^s} = 0$ and

$$\|e^{it\Delta}u_0 \cdot \chi_I\|_{Z^s} \lesssim \|e^{it\Delta}\underline{u_0} \cdot \chi_I\|_{Z^s} + \|u_0 - \underline{u_0}\|_{H^s} \ll 1,$$

we have $\sup_{\lambda \geq 0}\|v^{(\lambda)}\|_{Z^s} \ll 1$ and so $\sup_{\lambda \geq 0}\|v^{(\lambda)}\|_{Y^s} \ll 1$. By (3.14), (3.15), and (4.12), we have

$$\sup_{\lambda \geq 0}\|\psi u^{(\lambda)}\|_{B^\sigma_{p,2} B^{s-3\sigma}_{p,2} \cap L^{(2d+4)/(d-2s)}_{t,x}} \lesssim \sup_{\lambda \geq 0}\|u^{(\lambda)}\|_{Y^s} < \infty.$$

Since $\sigma > 0$, $s - 3\sigma > 0$, and $(2d+4)/(d-2s) > 1+a$, the embedding $B^{\sigma}_{p,2} B^{s-3\sigma}_{p,2} \cap L^{(2d+4)/(d-2s)}_{t,x} \hookrightarrow L^{1+a}_{t,x}$ is compact on the space of functions supported on $I \times \mathbb{T}^d$. Thus, we have a sequence $\{\lambda_n\}$ increasing to ∞ such that $u^{(\lambda_n)}$ converges to a function $u^{(\infty)}$ in $L^{1+a}_{t,x}(I \times \mathbb{T}^d)$. Let $u \in Y^s$ be the Duhamel solution of $iu_t + \Delta u = \mathcal{N}(u^{(\infty)}) \cdot \chi_I \in (Z^{-s})'$, $u(0) = u_0$. Since $u^{(\infty)}$ is a weak solution to (4.11), we have $P_{\le N} u = P_{\le N} u^{(\infty)}$ almost everywhere for every $N \in 2^{\mathbb{N}}$. Hence $u = u^{(\infty)} \in L^{1+a}_{t,x}(I \times \mathbb{T}^d)$ holds almost everywhere and so $u \in Y^s$ is a solution to (4.11), $u(0) = u_0$. Furthermore, for each $t_0 \in I$, by (3.13), we have continuity in time:

$$\begin{aligned}\limsup_{t_1 \to t_0} \|u(t_1) - u(t_0)\|_{H^s} &\lesssim \limsup_{t_1 \to t_0} \|\mathcal{N}(u(t+t_0)) \cdot \chi_{[0,t_1-t_0]}\|_{(Z^{-s})'} \\ &\lesssim \limsup_{t_1 \to t_0} \|u(t+t_0) \cdot \chi_{[0,t_1-t_0]}\|^{1+a}_{Z^s} = 0.\end{aligned}$$

Therefore, we obtained a strong solution $u \in C^0 H^s \cap Y^s$ to (4.11). We emphasize the estimate

$$\|u \cdot \chi_I\|_{Z^s} \lesssim \|e^{it\Delta} \underline{u_0} \cdot \chi_I\|_{Z^s} + \|v\|_{Z^s} \ll 1. \tag{4.13}$$

Estimate (4.13) will be used to show the continuity of the solution map Φ.

Uniqueness. Next we check that such a solution $u \in C^0 H^s \cap Y^s$ is unique. Let $u, v \in C^0 H^s \cap Y^s$ be two solutions to (NLS) with $u(0) = v(0) \in H^s$. We show that $u = v$ holds on a sufficiently small interval $I_1 \subset I$. Using (3.13), we freely shrink an open interval $I_1 \subset I$ containing 0 such that $\|u \cdot \chi_{I_1}\|_{Z^s}, \|v \cdot \chi_{I_1}\|_{Z^s} \ll 1$.

Let $w = (v - u) \cdot \chi_{I_1}$. By (3.27), we have

$$\begin{aligned}\|w\|_{Z^0} &\lesssim \|(\mathcal{N}(v) - \mathcal{N}(u))\chi_{I_1}\|_{(Z^0)'} \\ &\lesssim \bigg\| \psi_1 \cdot \bigg(w \int_0^1 \partial_z \mathcal{N}(u \cdot \chi_{I_1} + \theta w \cdot \chi_{I_1})\, d\theta \\ &\qquad + \bar{w} \int_0^1 \partial_{\bar{z}} \mathcal{N}(u \cdot \chi_{I_1} + \theta w \cdot \chi_{I_1})\, d\theta \bigg) \bigg\|_{(Z^0)'} \\ &\lesssim \|w\|_{Z^0} \int_0^1 \|u \cdot \chi_{I_1} + \theta w \cdot \chi_{I_1}\|^a_{Z^s}\, d\theta \\ &\lesssim \|w\|_{Z^0} (\|u \cdot \chi_{I_1}\|^a_{Z^s} + \|v \cdot \chi_{I_1}\|^a_{Z^s}).\end{aligned}$$

Thus, we have $\|w\|_{Z^0} = 0$, which implies $u = v$ on I_1.

Continuous dependence. So far, we have constructed the solution map $\Phi\colon \{u_0 \in H^s \mid \|u_0 - \underline{u_0}\|_{H^s} < R\} \to C^0 H^s \cap Y^s$, whose image is in a small neighborhood of 0 in Z^s when localized on I. Now we prove the continuity of the solution map Φ. For simplicity of notation, we show the continuity only at $\underline{u_0}$. The only information we use about $\underline{u_0}$ is (4.13), thus the continuity of Φ on a small neighborhood of $\underline{u_0}$ can be shown by the same argument. For this purpose, we show that

$$\limsup_{N_0 \to \infty} \limsup_{\delta \to 0} \sup_{\|u_0 - \underline{u_0}\|_{H^s} < \delta} \|P_{\ge N_0} \Phi(u_0)\|_{Y^s} = 0$$

and

$$P_{\le N}\Phi \text{ is Lipschitz for any } N \in 2^{\mathbb{N}}.$$

Fix $\varepsilon > 0$. There exist $N_0 \in 2^{\mathbb{N}}$ and a positive sequence $\{\gamma_N\}$ satisfying the condition of (4.7), $\sum_{N\in 2^{\mathbb{N}}} \gamma_N \|\underline{u_0}_N\|_{H^s}^2 \le \varepsilon^2$, and $\gamma_N = 1$ for every $N \ge N_0$. (Such a choice is possible since one can choose γ_N close to 0 for arbitrarily many N.) Fix $\delta \ll \varepsilon N_0^{-s}$. For $\|u_0 - \underline{u_0}\|_{H^s} < \delta$, by (4.7), $u = \Phi(u_0)$ satisfies

$$\begin{aligned}
\sum_{N\in 2^{\mathbb{N}}} \gamma_N \|u_N\|_{Y^s}^2 &\lesssim \sum_{N\in 2^{\mathbb{N}}} \gamma_N \|u_{0N}\|_{H^s}^2 + \sum_{N\in 2^{\mathbb{N}}} \gamma_N \|P_N \mathcal{N}(u)\cdot \chi_I\|_{(Z^{-s})'}^2 \\
&\lesssim \|u_0 - \underline{u_0}\|_{H^s}^2 + \sum_{N\in 2^{\mathbb{N}}} \gamma_N \|\underline{u_0}_N\|_{H^s}^2 \\
&\quad + \|u\cdot\chi_I\|_{Z^s}^{2a} \cdot \sum_{N\in 2^{\mathbb{N}}} \gamma_N \|u_N \cdot \chi_I\|_{Z^s}^2.
\end{aligned} \tag{4.14}$$

Since $\|u_N\cdot\chi_I\|_{Z^s}^2 \lesssim \|u_N\|_{Y^s}^2$, by (4.14) and (4.13), we have

$$\|u_{\ge N_0}\|_{Y^s}^2 \le \sum_{N\in 2^{\mathbb{N}}} \gamma_N \|u_N\|_{Y^s}^2 \lesssim \delta^2 + \varepsilon^2. \tag{4.15}$$

Let $\underline{u} = \Phi(\underline{u_0})$ and $w = u - \underline{u}$. By (3.27), we have

$$\begin{aligned}
&\|w - e^{it\Delta}(u_0 - \underline{u_0})\|_{Y^0} \\
&\quad \lesssim \|(\mathcal{N}(u) - \mathcal{N}(\underline{u}))\chi_I\|_{(Z^0)'} \\
&\quad \lesssim \Big\| \psi_1 \cdot \Big(w \int_0^1 \partial_z \mathcal{N}(\underline{u}\cdot\chi_I + \theta w\cdot\chi_I)\, d\theta \\
&\qquad\qquad + \bar{w} \int_0^1 \partial_{\bar{z}} \mathcal{N}(\underline{u}\cdot\chi_I + \theta w\cdot\chi_I)\, d\theta \Big) \Big\|_{(Z^0)'} \\
&\quad \lesssim \|w\|_{Z^0} \int_0^1 \|\underline{u}\cdot\chi_I + \theta w\cdot\chi_I\|_{Z^s}^a \, d\theta \\
&\quad \lesssim \|w\|_{Y^0}(\|u\cdot\chi_I\|_{Z^s}^a + \|\underline{u}\cdot\chi_I\|_{Z^s}^a).
\end{aligned}$$

Since $\|u\cdot\chi_I\|_{Z^s}^a, \|\underline{u}\cdot\chi_I\|_{Z^s}^a \ll 1$, we have

$$\|w\|_{Y^0} \lesssim \|u_0 - \underline{u_0}\|_{L^2},$$

which implies

$$\|P_{\le N_0}(u - \underline{u})\|_{Y^s}^2 \lesssim N_0^{2s} \|u_0 - \underline{u_0}\|_{L^2}^2 \lesssim N_0^{2s}\delta^2. \tag{4.16}$$

Combining (4.15) and (4.16), we have

$$\|\Phi(u_0) - \Phi(\underline{u_0})\|_{Y^s} = \|u - \underline{u}\|_{Y^s} \lesssim \varepsilon,$$

finishing the proof of Proposition 4.6. ∎

5. Proof of Theorem 1.2

In this section we prove Theorem 1.2. Throughout this section we fix exponents d, a, and s such that

$$0 < s < a \quad \text{and} \quad 1 < a.$$

Note that $\frac{4}{d} < a$ is equivalent to $0 < s$.

The proof of Theorem 1.2 follows from a nonlinear estimate of the difference between solutions:

Lemma 5.1. *For $u, v \in Z^s$, we have*

$$\|\psi_1 \partial_z \mathcal{N}(u) \cdot v + \psi_1 \partial_{\bar{z}} \mathcal{N}(u) \cdot \bar{v}\|_{(Z^{-s})'} \lesssim \|u\|_{Z^s}^a \|v\|_{Z^s}. \tag{5.1}$$

We first show that Lemma 5.1 implies Theorem 1.2. Fix $\underline{u_0} \in H^s$. Let I and $R > 0$ be defined in Proposition 4.6, chosen to be small enough so that (4.13) also holds. For $u_0, v_0 \in H^s$ satisfying $\|u_0 - \underline{u_0}\|_{H^s}, \|v_0 - \underline{u_0}\|_{H^s} < R$, let $w = v - u$ and $w_0 = v_0 - u_0$. We have

$$\begin{aligned}\|w\|_{Y^s} &\lesssim \|w_0\|_{H^s} + \int_0^1 \|(\partial_z \mathcal{N}(u + \theta w) \cdot w + \partial_{\bar{z}} \mathcal{N}(u + \theta w) \cdot \bar{w}) \cdot \chi_I\|_{(Z^{-s})'}\, d\theta \\ &\lesssim \|w_0\|_{H^s} + \sup_{\theta \in [0,1]} \|(u + \theta w) \cdot \chi_I\|_{Z^s}^a \|w \cdot \chi_I\|_{Z^s}.\end{aligned}$$

This implies $\|w\|_{Y^s} \lesssim \|w_0\|_{H^s}$, which completes the proof of Theorem 1.2.

Lemma 5.1 is based on modifications of (3.27), (4.2), (4.3), and (4.6), allowing one linear term of u to be replaced by that of another function v. To handle the difference form, we apply fractional Hölder inequalities to the linearization form $\partial_z \mathcal{N} u \cdot v$. Here we require $s < a$ instead of $s < 1 + a$. To mimick the proof of (3.27), we require that at least one factor in (3.31), say a_k, is equal to $1/2$. For this, we need $a > 1$.

Proof of Lemma 5.1. We show the dual form of (5.1) regarding $\partial_z \mathcal{N}(u) \cdot v$:

$$\left| \int \psi_1 \partial_z \mathcal{N}(u) \cdot v \cdot \bar{w}\, dx\, dt \right| \lesssim \|u\|_{Z^s}^a \|v\|_{Z^s} \|w\|_{Z^{-s}}. \tag{5.2}$$

The conjugate term $\partial_{\bar{z}} \mathcal{N}(u) \cdot \bar{v}$ can be dealt with similarly.

For $N \in 2^{\mathbb{N}}$, we denote by $G^N : \mathbb{R} \times \mathbb{T}^d \to \mathbb{C}$ and $B^N : \mathbb{R} \times \mathbb{T}^d \to \mathbb{C}^2$ the functions

$$G^N := \partial_z \mathcal{N}(u_{\le N}) - \partial_z \mathcal{N}(u_{\le N/2})$$

and

$$B^N := \left(\int_0^1 \partial_{zz} \mathcal{N}(u_{\le N/2} + \theta u_N)\, d\theta, \int_0^1 \partial_{z\bar{z}} \mathcal{N}(u_{\le N/2} + \theta u_N)\, d\theta \right).$$

With this notation, we have $G^N = u_N \times B^N$. For $N = 1$, we use the conventional notation $u_{\le 1/2} = 0$.

We estimate the terms

$$
\begin{aligned}
&\left|\int \psi_1 \partial_z \mathcal{N}(u)\cdot v\cdot \overline{w}\,dx\,dt\right| \\
&\quad\le \left|\sum_{L\in 2^{\mathbb{N}}}\int \psi_1 \partial_z \mathcal{N}(u)\cdot v_{\ge L/8}\cdot \overline{w_L}\,dx\,dt\right| \\
&\qquad+\left|\sum_{L\in 2^{\mathbb{N}}}\int \psi_1 (\partial_z \mathcal{N}(u)-\partial_z \mathcal{N}(u_{\le L/16}))\cdot v_{\le L/16}\cdot \overline{w_L}\,dx\,dt\right| \\
&\qquad+\left|\sum_{L\in 2^{\mathbb{N}}}\int \psi_1 \partial_z \mathcal{N}(u_{\le L/16})\cdot v_{\le L/16}\cdot \overline{w_L}\,dx\,dt\right| \\
&\quad= \mathrm{I}+\mathrm{II}+\mathrm{III}
\end{aligned}
$$

separately.

(1) *Estimate of* I. Denote by $K, L, M, N \in 2^{\mathbb{N}}$ dyadic numbers. By (3.27), we have

$$
\left|\int \psi_1 \partial_z \mathcal{N}(u)\cdot v_M\cdot \overline{w_L}\,dx\,dt\right| \lesssim \|u\|_{Z^s}^{a}\|v_M\|_{Z^0}\|w_L\|_{Z^0},
$$

which implies

$$
\begin{aligned}
\mathrm{I} &\le \sum_{\substack{L,M\in 2^{\mathbb{N}}\\ L\le 8M}}\left|\int \psi_1 \partial_z \mathcal{N}(u)\cdot v_M\cdot \overline{w_L}\,dx\,dt\right| \\
&\lesssim \|u\|_{Z^s}^{a}\cdot \sum_{\substack{L,M\in 2^{\mathbb{N}}\\ L\le 8M}} (L/M)^s\|v_M\|_{Z^s}\|w_L\|_{Z^{-s}} \\
&\lesssim \|u\|_{Z^s}^{a}\|v\|_{Z^s}\|w\|_{Z^{-s}}.
\end{aligned}
\tag{5.3}
$$

(2) *Estimate of* II. Since $a/2 > 1/2$, mimicking the proof of (3.27) with the choice $a_k = 1/2$ gives the estimate

$$
\left|\int \psi_1 (u_N\times B^N)\cdot v_{\le L/16}\cdot \overline{w_L}\,dx\,dt\right| \lesssim \|u\|_{Z^s}^{a-1}\|u_N\|_{Z^0}\|v_{\le L/16}\|_{Z^s}\|w_L\|_{Z^0}.
$$

This implies

$$
\begin{aligned}
\mathrm{II} &\le \sum_{\substack{L,N\in 2^{\mathbb{N}}\\ L\le 8N}}\left|\int \psi_1 (u_N\times B^N)\cdot v_{\le L/16}\cdot \overline{w_L}\,dx\,dt\right| \\
&\lesssim \|u\|_{Z^s}^{a-1}\|v\|_{Z^s}\cdot \sum_{\substack{L,N\in 2^{\mathbb{N}}\\ L\le 8N}} (L/N)^s\|u_N\|_{Z^s}\|w_L\|_{Z^{-s}} \\
&\lesssim \|u\|_{Z^s}^{a}\|v\|_{Z^s}\|w\|_{Z^{-s}}.
\end{aligned}
\tag{5.4}
$$

(3) *Estimate of* III. To estimate III, we estimate $|\int \psi_1 G^N \cdot v_{\leq L/16} \cdot \overline{w_L}\, dx\, dt|$ assuming $L \geq 16N$. Let q, r, p_0, and p_1 be the parameters defined in the proof of (4.3). Choose $\mu > 0$ and $\nu > 0$ as the exponents $\nu = a - s - \sigma_1$ and $\mu = (1 - \frac{2\sigma}{\sigma_1})(\nu + \sigma) + \frac{2\sigma}{\sigma_1}(-s + \sigma)$ (recall that $\sigma \ll \sigma_1 \ll 1$ in (3.24)).

(i) Since $\sigma_1 < \min\{(a-1)(s-\sigma), s-\sigma\}$, by either (2.15) or (2.16), for $M \in 2^{\mathbb{N}}$ we have

$$\begin{aligned}
&\|(u_N \times B_M^N) \cdot v_{\leq L/16}\|_{L^{p_1'}} \\
&\quad\lesssim \|u_N\|_{L^r} \cdot \|v_{\leq L/16}\|_{H^{s-\sigma,r}} \cdot \|B_M^N\|_{L^{(\frac{1}{p_1'} - \frac{2}{r} + \frac{s-\sigma}{d})^{-1}}} \\
&\quad\lesssim N^{-s+\sigma}\|u_N\|_{H^{s-\sigma,r}} \cdot \|v_{\leq L/16}\|_{H^{s-\sigma,r}} \cdot M^{-\sigma_1}\|B_M^N\|_{H^{\sigma_1,(\frac{1}{p_1'} - \frac{2}{r} + \frac{s-\sigma}{d})^{-1}}} \\
&\quad\lesssim N^{-s+\sigma} M^{-\sigma_1}\|u_N\|_{H^{s-\sigma,r}} \cdot \|v_{\leq L/16}\|_{H^{s-\sigma,r}} \cdot \|u\|_{H^{s-\sigma,r}}^{a-1}.
\end{aligned}$$

Since $N \leq L/16$, we have

$$\begin{aligned}
&\|P_L(G^N \cdot v_{\leq L/16})\|_{L^{p_1'}} \\
&\quad\lesssim \sum_{M \in 2^{\mathbb{N}}} \|P_L((u_N \times B_M^N) \cdot v_{\leq L/16})\|_{L^{p_1'}} \\
&\quad\lesssim \sum_{M \sim L} \|P_L((u_N \times B_M^N) \cdot v_{\leq L/16})\|_{L^{p_1'}} \\
&\quad\lesssim (N/L)^{-s+\sigma} L^{-s-\sigma_1+\sigma}\|u_N\|_{H^{s-\sigma,r}} \cdot \|v_{\leq L/16}\|_{H^{s-\sigma,r}} \cdot \|u\|_{H^{s-\sigma,r}}^{a-1} \\
&\quad\lesssim (N/L)^{-s+\sigma} L^{-s-\sigma_1+\sigma} N^{-\nu}\|u_{\leq N}\|_{H^{s-\sigma+\nu,r}} \cdot \|v_{\leq L/16}\|_{H^{s-\sigma,r}} \cdot \|u\|_{H^{s-\sigma,r}}^{a-1}. \qquad (5.5)
\end{aligned}$$

(ii) Since $s + \nu < a$, by (2.16), we have

$$\begin{aligned}
&\|P_L(G^N \cdot v_{\leq L/16})\|_{L^{p_0'}} \\
&\quad\lesssim L^{-s-\nu}\|G^N\|_{H^{s+\nu,(\frac{1}{p_0'} - \frac{1}{r} + \frac{s-\sigma}{d})^{-1}}} \cdot \|v_{\leq L/16}\|_{H^{s-\sigma,r}} \\
&\quad\lesssim L^{-s-\nu}\|u_{\leq N}\|_{H^{s+\nu,r}} \cdot \|u\|_{L^{a(\frac{1}{p_0'} - \frac{1}{r})^{-1}}}^{a-1} \cdot \|v_{\leq L/16}\|_{H^{s-\sigma,r}} \\
&\quad\lesssim L^{-s-\nu} N^{\sigma}\|u_{\leq N}\|_{H^{s-\sigma+\nu,r}} \cdot \|u\|_{H^{s-\sigma,r}}^{a-1} \cdot \|v_{\leq L/16}\|_{H^{s-\sigma,r}} \\
&\quad\lesssim (N/L)^{\nu+\sigma} L^{-s+\sigma} N^{-\nu}\|u_{\leq N}\|_{H^{s-\sigma+\nu,r}} \cdot \|u\|_{H^{s-\sigma,r}}^{a-1} \cdot \|v_{\leq L/16}\|_{H^{s-\sigma,r}}. \qquad (5.6)
\end{aligned}$$

Interpolating between (5.5) and (5.6) with the exponent $2\sigma/\sigma_1$, we have

$$\begin{aligned}
&\|P_L(G^N \cdot v_{\leq L/16})\|_{L^{p'}} \\
&\quad\lesssim (N/L)^{\mu} L^{-s-\sigma} N^{-\nu}\|u_{\leq N}\|_{H^{s-\sigma+\nu,r}} \cdot \|u\|_{H^{s-\sigma,r}}^{a-1} \cdot \|v_{\leq L/16}\|_{H^{s-\sigma,r}}.
\end{aligned}$$

Multiplying cutoffs and applying $L_t^{p'}$ norms to both sides, then using (3.14), we have

$$\|\psi_1 P_L(G^N \cdot v_{\leq L/16})\|_{(Z^{-s})'} \lesssim (N/L)^{\mu} N^{-\nu}\|u_{\leq N}\|_{Z^{s+\nu}}\|u\|_{Z^s}^{a-1}\|v\|_{Z^s}.$$

Since $\sum_{N\le L/16}(N/L)^{\mu}\lesssim 1$, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\Big\| \sum_{\substack{L,N\in 2^{\mathbb{N}}\\ L\ge 16N}} \int P_L(\psi_1 G^N \cdot v_{\le L/16}) \Big\|_{(Z^{-s})'}^2 \\
&\lesssim \sum_{L\in 2^{\mathbb{N}}} \Big(\sum_{N\le L/16} \|\psi_1 P_L(G^N\cdot v_{\le L/16})\|_{(Z^{-s})'} \Big)^2 \\
&\lesssim \|u\|_{Z^s}^{2a-2}\|v\|_{Z^s}^2 \sum_{L\in 2^{\mathbb{N}}} \Big(\sum_{N\le L/16} (N/L)^{\mu} N^{-\nu} \|u_{\le N}\|_{Z^{s+\nu}} \Big)^2 \\
&\lesssim \|u\|_{Z^s}^{2a-2}\|v\|_{Z^s}^2 \sum_{L\in 2^{\mathbb{N}}} \sum_{N\le L/16} (N/L)^{\mu} N^{-2\nu} \|u_{\le N}\|_{Z^{s+\nu}}^2 \\
&\lesssim \|u\|_{Z^s}^{2a-2}\|v\|_{Z^s}^2 \sum_{L\in 2^{\mathbb{N}}} \sum_{N\le L/16} \sum_{K\le N} (N/L)^{\mu} (K/N)^{2\nu} \|u_K\|_{Z^s}^2 \\
&\lesssim \|u\|_{Z^s}^{2a-2}\|v\|_{Z^s}^2 \sum_{\substack{K,L\in 2^{\mathbb{N}}\\ L\ge 16K}} (K/L)^{\min\{\mu,2\nu\}-} \|u_K\|_{Z^s}^2 \\
&\lesssim \|u\|_{Z^s}^{2a}\|v\|_{Z^s}^2,
\end{aligned}$$

which implies

$$\mathrm{III} \le \Big| \sum_{\substack{L,N\in 2^{\mathbb{N}}\\ L\ge 16N}} \int \psi_1 G^N \cdot v_{\le L/16}\cdot \overline{w_L}\,dx\,dt \Big| \lesssim \|u\|_{Z^s}^a \|v\|_{Z^s}\|w\|_{Z^{-s}}. \tag{5.7}$$

Combining (5.3), (5.4), and (5.7), we have (5.2), finishing the proof of Lemma 5.1. ∎

6. Proof of Theorem 1.3

In this section we prove the Hölder ill-posedness of (NLS), Theorem 1.3. For the proof, we construct an explicit counterexample. Although Theorem 1.3 is stated on $\mathbb{T}^d$, the counterexample we construct relies on one spatial variable x_1. Hence, we perform the construction on $\mathbb{T}$. Indeed, by considering initial data $u_0\in H^s(\mathbb{T}^d)$ of the form $u_0(x_1,\dots,x_d) = u_{01}(x_1)$, $u_{01}\in H^s(\mathbb{T})$, one can deduce the result in Theorem 1.3 from that with the domain replaced by $\mathbb{T}$. Here, s and a are no longer related, i.e., we do not require (1.1) in the rest of this section.

Reducing to the one-dimensional problem, we show the following statement:

Proposition 6.1. *Assume $0<a<1$ and $0<s<1+\frac{1}{a}$. The solution map to*

$$\begin{cases} iu_t + u_{xx} = \mathcal{N}(u), \\ u(0) = u_0 \in H^s(\mathbb{T}), \end{cases} \tag{6.1}$$

is not locally α-Hölder continuous in $H^s(\mathbb{T})$ for each $\alpha > a$.

The proof of Proposition 6.1 does not depend on the sign of the nonlinearity. For simplicity, we assume that $\mathcal{N}(u)$ is defocusing, i.e., $\mathcal{N}(u) = |u|^a u$.

We construct an explicit pair of initial data $u_{\lambda,0}, u_{2\lambda,0} \in H^s(\mathbb{T})$ which contradicts the Hölder continuity in Proposition 6.1. Denoting by $u_\lambda, u_{2\lambda} \in C^0H^s([0,T]\times\mathbb{T})$ the solutions to (6.1) and letting $w = u_{2\lambda} - u_\lambda$, we expand $\mathbb{P}_N\partial_x w(T,0)$, $N \gg 1$ in a Duhamel form, expecting that $|\mathbb{P}_N\partial_x w(T,0)|$ exceeds the bound required by such a Hölder assumption.

In the Duhamel expansion, we have a term of u, $\partial_x u$, and w (6.12), which is nonoscillating if we plug $\partial_x u_\lambda = \partial_x u_{2\lambda} = Ce^{i(t-T)\Delta}\delta_N$ and positive reals $u_\lambda, u_{2\lambda}, w \gtrsim \lambda$. Inspired by this observation, we choose the initial data $u_{\lambda,0}$ and $u_{2\lambda,0}$ as functions such that $\frac{1}{2\pi}\int_{\mathbb{T}} u_{\lambda,0}\,dx = \lambda$, $\frac{1}{2\pi}\int_{\mathbb{T}} u_{2\lambda,0}\,dx = 2\lambda$, and $\partial_x u_{\lambda,0} = \partial_x u_{2\lambda,0} = \theta e^{-iT\Delta}\delta_N$. The parameters λ and θ are chosen as constants such that u_λ and $u_{2\lambda}$ are small perturbations of free evolutions and (6.12) dominates the rest in the expansion of $\mathbb{P}_N\partial_x w(T,0)$.

To avoid a small set of irregular high peaks of u generating a large error, we partition the set $[0,T]\times\mathbb{T}^d$ into $E\cup E^c$, where E is the set of points (t,x) at which $u_\rho - e^{it\Delta}u_{\rho,0}$ and $\partial_x(u_\rho - e^{it\Delta}u_{\rho,0})$ are both small for $\rho = \lambda, 2\lambda$. The Duhamel integral is estimated on E and E^c separately.

Before proving Proposition 6.1, we show a preliminary fact on the Schrödinger kernel $e^{it\Delta}\delta_N$, $N \in 2^{\mathbb{N}}$ on $\mathbb{T}$. The following lemma states that for each dyadic $N \in 2^{\mathbb{N}}$, the L^2 norm and the L^∞ norm of $e^{it\Delta}\delta_N$ are comparable on a large set of times t:

Lemma 6.2. *For $T > 0$, there exists a constant C depending only on T such that for every dyadic $N \in 2^{\mathbb{N}}$, we have the estimate*

$$m\big(\{t \in [0,T] \mid \|e^{it\Delta}\delta_N\|_{L^\infty} \le C\sqrt{N}\}\big) \gtrsim_T 1, \tag{6.2}$$

where $m(\cdot)$ denotes the (Lebesgue) measure of a set.

Proof. Let $\mathcal{A}$ be the collection of coprime integer pairs (l,m) such that $\frac{N}{10} \le m \le N$ and $0 < \frac{l}{m} < T$. Let $\mathcal{I}$ be the collection of intervals $(\frac{l}{m} - \frac{1}{mN}, \frac{l}{m} + \frac{1}{mN})$, $(l,m) \in \mathcal{A}$. Denote by $\mathcal{T}$ the union $\mathcal{T} = \bigcup_{I\in\mathcal{I}} I$.

Once we have $m(\mathcal{T}) \gtrsim_T 1$, we have (6.2). For $t \in \mathcal{T}$, by the kernel estimate (2.20), we have $\|e^{it\Delta}\delta_N\|_{L^\infty} \le C\sqrt{N}$ for some universal constant $C = C(T)$, which implies (6.2).

We show the estimate $m(\mathcal{T}) \gtrsim_T 1$. For two pairs of coprime integers (l_1,m_1) and (l_2,m_2), we have

$$\Big|\frac{l_1}{m_1} - \frac{l_2}{m_2}\Big| = \Big|\frac{l_1m_2 - l_2m_1}{m_1m_2}\Big| \ge \frac{1}{m_1m_2}.$$

Thus, no more than 100 members of $\mathcal{I}$ can intersect at a common point. Since the length of each interval in $\mathcal{I}$ is comparable to $\frac{1}{N^2}$, we are done once we have $\#\mathcal{A} \gtrsim N^2$. Since $\sum_{k=2}^{\infty}\frac{1}{k^2} < 1$, the number of such points is indeed comparable to N^2, finishing the proof of (6.2). ∎

Since $e^{it\Delta}\delta_N$ is localized on frequencies comparable to N, (6.2) indicates an oscillating behavior of $e^{it\Delta}\delta_N$ for such times t. As a particular consequence, we have

$|e^{it\Delta}\delta_N(t,x)| \gtrsim_T \sqrt{N}$ on a large set of points $(t,x) \in [0,T] \times \mathbb{T}$, which is stated in the following corollary:

Corollary 6.3. *For $T > 0$, there exists a constant $\varepsilon > 0$ depending only on T such that for every dyadic $N \in 2^{\mathbb{N}}$, we have the estimate*

$$m\big(\{(t,x) \in [0,T] \times \mathbb{T} \bigm| |e^{it\Delta}\delta_N(x)| \geq \varepsilon\sqrt{N}\}\big) \gtrsim_T 1. \tag{6.3}$$

Proof. For each $t \in [0,T]$ such that $\|e^{it\Delta}\delta_N\|_{L^\infty} \sim \sqrt{N} \sim \|e^{it\Delta}\delta_N\|_{L^2}$, $|e^{it\Delta}\delta_N(x)| \gtrsim \sqrt{N}$ is attained on a set of points $x \in \mathbb{T}$ of measure comparable to 1. Thus, (6.2) implies (6.3). ■

Using (6.3), we prove Proposition 6.1 as follows:

Proof of Proposition 6.1. Since Proposition 6.1 is stronger with lower α, we may assume $0 < \alpha - a \ll_{a,s} 1$ in advance.

Fix $T > 0$. We use the Wirtinger derivatives with the convention $u^\gamma \bar{u}^\gamma := |u|^{2\gamma}$ for $\gamma \in \mathbb{R}$.

Fix a number $\varepsilon_0 \ll_{a,s,T} 1$. Let $N \in 2^{\mathbb{N}}$ be a large dyadic number. Let $\lambda, \kappa, \iota \ll 1$ be the numbers

$$\begin{aligned} \lambda &= N^{-\frac{s}{1-\alpha+2a}+\sigma_1}, \\ \kappa &= \lambda^{\alpha-a} \cdot N^{\sigma}, \\ \iota &= N^{-\sigma}, \end{aligned} \tag{6.4}$$

where σ and σ_1 are the numbers chosen in (3.24).

For $\rho \in [\lambda, 2\lambda]$, let $u_{\rho,0} \in H^\infty(\mathbb{T})$ be the unique function such that

$$\partial_x u_{\rho,0} = \kappa N^{\frac{1}{2}-s} e^{-iT\Delta}\delta_N$$

and

$$\frac{1}{2\pi}\int_{\mathbb{T}} u_{\rho,0}\, dx = \rho.$$

Denote by u_λ and $u_{2\lambda}$ the solutions to (6.1) on $[0,T]$ with initial data $u_{\lambda,0}$ and $u_{2\lambda,0}$, respectively. For $\rho \in (\lambda, 2\lambda)$, denote by $u_\rho \colon [0,T] \times \mathbb{T} \to \mathbb{C}$ the function $u_\rho = \frac{2\lambda-\rho}{\lambda}u_\lambda + \frac{\rho-\lambda}{\lambda}u_{2\lambda}$. We also denote $w = u_{2\lambda} - u_\lambda$.

First, we collect some estimates on u_λ and $u_{2\lambda}$. For $\rho = \lambda$ and $\rho = 2\lambda$, by $0 < s < 1 + \frac{1}{a}$ and $a < \alpha$, we have $\|u_{\rho,0}\|_{L^2} \lesssim \kappa N^{-s} + \lambda \sim \lambda$ and $\|u_{\rho,0}\|_{H^1} \lesssim \kappa N^{1-s} + \lambda \sim \kappa N^{1-s}$. Thus, a contraction mapping argument using (2.21) gives

$$\|u_\rho\|_{C^0L^2 \cap L^4L^4} \lesssim \|u_{\rho,0}\|_{L^2} \lesssim \lambda, \tag{6.5}$$

$$\|u_\rho - e^{it\Delta}u_{\rho,0}\|_{C^0L^2 \cap L^4L^4} \lesssim \|u_\rho\|_{L^4L^4}^{1+a} \lesssim \lambda^{1+a}, \tag{6.6}$$

$$\|u_\rho\|_{C^0H^1 \cap L^4W^{1,4}} \lesssim \|u_{\rho,0}\|_{H^1} \lesssim \kappa N^{1-s}, \tag{6.7}$$

and

$$\|u_\rho - e^{it\Delta}u_{\rho,0}\|_{C^0H^1\cap L^4W^{1,4}} \lesssim \|u_\rho\|^a_{L^4L^4}\|u_\rho\|_{L^4W^{1,4}} \lesssim \lambda^a\kappa N^{1-s}. \tag{6.8}$$

If Proposition 6.1 were false on $[0,T]$, since

$$\|u_{\lambda,0}\|_{H^s}, \|u_{2\lambda,0}\|_{H^s} \lesssim \kappa + \lambda \ll 1,$$

we would have

$$|\mathbb{P}_N\partial_x w(T,0)| \lesssim N^{\frac{3}{2}-s}\|w\|_{C^0H^s} \lesssim N^{\frac{3}{2}-s}\|u_{2\lambda,0} - u_{\lambda,0}\|^\alpha_{H^s} \lesssim \lambda^\alpha N^{\frac{3}{2}-s}.$$

Thus, showing the following leads to a contradiction:

$$|\mathbb{P}_N\partial_x w(T,0)| \gg \lambda^\alpha N^{\frac{3}{2}-s}. \tag{6.9}$$

Since $w(0) = \lambda$ and

$$(i\partial_t + \partial_{xx})w = \mathcal{N}(u_{2\lambda}) - \mathcal{N}(u_\lambda),$$

we have

$$\begin{aligned} i\mathbb{P}_N\partial_x w(T,0) &= \int_{[0,T]\times\mathbb{T}} \overline{e^{i(t-T)\Delta}\delta_N}\,\partial_x(\mathcal{N}(u_{2\lambda}) - \mathcal{N}(u_\lambda))\,dx\,dt \\ &= \int_E \overline{e^{i(t-T)\Delta}\delta_N}(A+B)\,dx\,dt \\ &\quad + \int_{E^c} \overline{e^{i(t-T)\Delta}\delta_N}(\partial_x\mathcal{N}(u_{2\lambda}) - \partial_x\mathcal{N}(u_\lambda))\,dx\,dt, \end{aligned}$$

where $E \subset [0,T]\times\mathbb{T}$ is the set of $(t,x) \in [0,T]\times\mathbb{T}$ such that for $\rho = \lambda, 2\lambda$,

$$\left|\frac{u_\rho}{\rho} - 1\right| \le \varepsilon_0, \tag{6.10}$$

$$|\partial_x u_\rho - \kappa N^{\frac{1}{2}-s}e^{i(t-T)\Delta}\delta_N| \le \iota\kappa N^{1-s} \tag{6.11}$$

are satisfied, and the terms A and B denote

$$\begin{aligned} A &= \frac{1}{\lambda}\int_\lambda^{2\lambda} \frac{a}{2}\Big(\frac{a}{2}+1\Big)\big(u_\rho^{\frac{a}{2}-1}\bar{u}_\rho^{\frac{a}{2}}w + u_\rho^{\frac{a}{2}}\bar{u}_\rho^{\frac{a}{2}-1}\overline{w}\big)\partial_x u_\rho\,d\rho \\ &\quad + \frac{1}{\lambda}\int_\lambda^{2\lambda}\Big(\frac{a}{2}\Big(\frac{a}{2}+1\Big)u_\rho^{\frac{a}{2}}\bar{u}_\rho^{\frac{a}{2}-1}w + \frac{a}{2}\Big(\frac{a}{2}-1\Big)u_\rho^{\frac{a}{2}+1}\bar{u}_\rho^{\frac{a}{2}-2}\overline{w}\Big)\overline{\partial_x u_\rho}\,d\rho \end{aligned}$$

and

$$B = \frac{1}{\lambda}\int_\lambda^{2\lambda}\Big(\frac{a}{2}+1\Big)u_\rho^{\frac{a}{2}}\bar{u}_\rho^{\frac{a}{2}}\partial_x w + \frac{a}{2}u_\rho^{\frac{a}{2}+1}\bar{u}_\rho^{\frac{a}{2}-1}\overline{\partial_x w}\,d\rho.$$

We show the following claims:

$$\left|\int_E \overline{e^{i(t-T)\Delta}\delta_N}A\,dx\,dt\right| \gg \lambda^\alpha N^{\frac{3}{2}-s}, \tag{6.12}$$

$$\left|\int_E \overline{e^{i(t-T)\Delta}\delta_N}B\,dx\,dt\right| \ll \lambda^\alpha N^{\frac{3}{2}-s}, \tag{6.13}$$

and

$$\left| \int_{E^c} \overline{e^{i(t-T)\Delta}\delta_N} (\partial_x \mathcal{N}(u_{2\lambda}) - \partial_x \mathcal{N}(u_\lambda))\, dx\, dt \right| \ll \lambda^\alpha N^{\frac{3}{2}-s}. \tag{6.14}$$

Once we have (6.12), (6.13), and (6.14), we have (6.9) immediately, finishing the proof.

(1) *Proof of* (6.12). For $\rho = \lambda$ and $\rho = 2\lambda$, by (6.6), we have

$$\begin{aligned} \|u_\rho - \rho\|_{L^4 L^4} &\lesssim \|e^{it\Delta} u_{\rho,0} - \rho\|_{L^4 L^4} + \|u_\rho - e^{it\Delta} u_{\rho,0}\|_{L^4 L^4} \\ &\lesssim \kappa N^{-s} + \lambda^{1+a} \lesssim \lambda^{1+a}, \end{aligned}$$

which implies

$$m\big(\{(t,x) \in [0,T] \times \mathbb{T} \mid |u_\rho - \rho| > \varepsilon_0 \rho\}\big) \lesssim \left(\frac{\lambda^{1+a}}{\varepsilon_0 \rho}\right)^4 \sim \lambda^{4a}. \tag{6.15}$$

Similarly, by (6.8), we have

$$\|\partial_x u_\rho - \kappa N^{\frac{1}{2}-s} e^{i(t-T)\Delta} \delta_N\|_{L^4 L^4} \lesssim \lambda^a \kappa N^{1-s},$$

which implies

$$\begin{aligned} m\big(\{(t,x) \in [0,T] \times \mathbb{T} \mid |\partial_x u_\rho - \kappa N^{\frac{1}{2}-s} e^{i(t-T)\Delta} \delta_N| > \iota \kappa N^{1-s}\}\big) &\lesssim \left(\frac{\lambda^a \kappa N^{1-s}}{\iota \kappa N^{1-s}}\right)^4 \\ &= \frac{\lambda^{4a}}{\iota^4}. \end{aligned} \tag{6.16}$$

Combining (6.15) and (6.16), since $\iota \ll 1$, we have

$$m(E^c) \lesssim \frac{\lambda^{4a}}{\iota^4}.$$

For $(t,x) \in E$, by (6.10) and (6.11), we have

$$\begin{aligned} \mathrm{Re}[\overline{e^{i(t-T)\Delta}\delta_N} A] \geq \frac{a^2}{100} \lambda^a \kappa N^{\frac{1}{2}-s} |e^{i(t-T)\Delta}\delta_N|^2 \\ - 100 \lambda^a \iota \kappa N^{1-s} |e^{i(t-T)\Delta}\delta_N|. \end{aligned}$$

Since $m(E^c) \ll 1$, by (6.3), we have

$$\int_E |e^{i(t-T)\Delta}\delta_N|^2\, dx\, dt \sim N.$$

Thus, by $\iota \ll 1$, we have

$$\left| \int_E \overline{e^{i(t-T)\Delta}\delta_N} A\, dx\, dt \right| \gtrsim \lambda^a \kappa N^{\frac{3}{2}-s} \gg \lambda^\alpha N^{\frac{3}{2}-s}.$$

(2) *Proof of* (6.13). By (6.5) and (6.8), we have

$$\begin{aligned}\left|\int_E \overline{e^{i(t-T)\Delta}\delta_N} B\,dx\,dt\right| &\lesssim N^{\frac{1}{2}}\|B\|_{L^1L^2}\\ &\lesssim N^{\frac{1}{2}} \sup_{\rho\in[\lambda,2\lambda]} \|u_\rho^a\|_{L^4L^4}\|\partial_x w\|_{L^4L^4}\\ &\lesssim N^{\frac{1}{2}} \sup_{\rho\in[\lambda,2\lambda]} \|u_\rho\|_{L^4L^4}^a\|w-\lambda e^{it\Delta}1\|_{L^4W^{1,4}}\\ &\lesssim \lambda^{2a}\kappa N^{\frac{3}{2}-s} \ll \lambda^\alpha N^{\frac{3}{2}-s}.\end{aligned}$$

(3) *Proof of* (6.14). For $\rho = \lambda$ and $\rho = 2\lambda$, we have

$$\begin{aligned}\left|\int_{E^c} \overline{e^{i(t-T)\Delta}\delta_N}\,\partial_x(\mathcal{N}(u_\rho))\,dx\,dt\right| &\lesssim m(E^c)^{\frac{1}{4}}\|\overline{e^{i(t-T)\Delta}\delta_N}\,\partial_x(\mathcal{N}(u_\rho))\|_{L^{4/3}L^{4/3}}\\ &\lesssim m(E^c)^{\frac{1}{4}}\|e^{i(t-T)\Delta}\delta_N\|_{L^4L^4}\|u_\rho\|_{L^4L^4}^a\|u_\rho\|_{L^4W^{1,4}}\\ &\lesssim m(E^c)^{\frac{1}{4}}\cdot N^{\frac{1}{2}}\lambda^a\kappa N^{1-s}\\ &\lesssim \lambda^{2a}\iota^{-1}\kappa N^{\frac{3}{2}-s} \ll \lambda^\alpha N^{\frac{3}{2}-s}.\end{aligned}$$

Combining (6.12), (6.13), and (6.14), we have (6.9), which finishes the proof. ∎

Appendix

The following is a list of parameters used in Sections 3 and 6, rewritten in terms of the σ_j:

Relation	First appearance
$s = \frac{d}{2} - \frac{2}{a}$	(1.1)
$\frac{1}{p} = \frac{\frac{d}{2}-\sigma}{d+2}$	(3.1)
$\sigma \ll \sigma_1 \ll \sigma_2 \ll \sigma_3 \ll \sigma_4 \ll 1$	(3.24)
$\frac{1}{q_0} = \frac{2+\sigma_3}{d+2}$	Lemma 3.9
$\frac{1}{r_0} = \frac{2+\sigma_3+\sigma_2}{d+2}$	Lemma 3.9
$\theta = \frac{2}{q_0} + \frac{d}{r_0} - 2 = \sigma_3 + \frac{d\sigma_2}{d+2}$	Lemma 3.9
$\frac{1}{\hat{r}} = \frac{1+\sigma_4}{d+2}$	Proof of Proposition 3.11
$\zeta = \frac{1}{\hat{r}} - \frac{1}{2q_0} = \frac{\sigma_4-\sigma_3/2}{d+2}$	Proof of Proposition 3.11
$\eta = \frac{d}{\hat{r}} - \frac{d}{2r_0} + \frac{\theta}{2} = \frac{d}{d+2}\sigma_4 + \frac{1}{d+2}\sigma_3$	Proof of Proposition 3.11
$\lambda = N^{-\frac{s}{1-\alpha+2a}+\sigma_1}$	(6.4)
$\kappa = \lambda^{\alpha-a}\cdot N^\sigma$	(6.4)
$\iota = N^{-\sigma}$	(6.4)

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