Ann. Inst. H. Poincaré D
Comb. Phys. Interact. 13 (2026), 81–163
DOI 10.4171/AIHPD/194

Combinatorial QFT on graphs: First quantization formalism

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Abstract. We study a combinatorial model of the quantum scalar field with polynomial potential on a graph. In the first quantization formalism, the value of a Feynman graph is given by a sum over maps from the Feynman graph to the spacetime graph (mapping edges to paths). This picture interacts naturally with Atiyah–Segal-like cutting-gluing of spacetime graphs. In particular, one has combinatorial counterparts of the known gluing formulae for Green's functions and (zeta-regularized) determinants of Laplacians.

1. Introduction

In this paper, we study a combinatorial model of the quantum massive scalar field with polynomial potential on a spacetime given by a graph X. Our motivation to do so was the study of the first quantization formalism, that we recall in Section 1.1 below, and, in particular, its interplay with locality, i.e., cutting and gluing of the spacetime manifold. At the origin is the Feynman–Kac formula (4) for the Green's function of the kinetic operator. In case the spacetime is a graph, this formula has a combinatorial analog given by summing over paths with certain weights (see Section 5). These path sums interact very naturally with cutting and gluing, in a mathematically rigorous way, see Theorem 3.7 and its proof from path sum formulae (see Section 5.5).

A second motivation to study this model was the notion of functorial QFTs with a source a Riemannian cobordism. Few examples of functorial QFTs out of Riemannian cobordism categories exist, for instance, [10, 11, 15]. In this paper, we define a graph cobordism category and show that the combinatorial model defines a functor to the category of Hilbert spaces (see Section 2.1).

Finally, one can use this discrete toy model to approximate the continuum theory, which in this paper we do only in easy one-dimensional examples (see Section 3.4). We think that the results derived in this paper will be helpful to study the interplay between renormalization and locality in higher dimensions (the two-dimensional case was discussed in detail in [11]).

Mathematics Subject Classification 2020: 81T25 (primary); 81T18 (secondary).
Keywords: functorial QFT, combinatorial QFT, first quantization formalism, scalar field with polynomial potential, perturbative path integral, Feynman diagrams.

1.1. Motivation: first quantization formalism

We outline the idea of the first quantization picture in QFT in the example of the interacting scalar field[1].

Consider the scalar field theory on a Riemannian n-manifold M perturbed by a polynomial potential $p(\phi) = \sum_{k\geq 3} \frac{p_k}{k!}\phi^k$, defined by the action functional

$$S(\phi) = \int_M \left(\frac{1}{2}\phi(\Delta + m^2)\phi + p(\phi)\right) d^n x. \tag{1}$$

Here, $\phi \in C^\infty(M)$ is the field, Δ is the Laplacian determined by the metric, $m > 0$ is the mass parameter and $d^n x$ denotes the metric volume element.

The partition function is formally given by a (mathematically ill-defined) functional integral understood perturbatively as a sum over Feynman graphs Γ (see footnote[2]),

$$Z_M = \text{“}\int_{C^\infty(M)} \mathcal{D}\phi\, e^{-\frac{1}{\hbar}S(\phi)}\text{”} = \left(\det{}^\zeta(\Delta + m^2)\right)^{-\frac{1}{2}} \cdot \sum_\Gamma \Phi_\Gamma. \tag{2}$$

Here, $\det^\zeta$ is the functional determinant in zeta function regularization. The weight Φ_Γ of a Feynman graph is the product of Green's functions $G(x, y)$ of the kinetic operator $\Delta + m^2$ associated with the edges of Γ, integrated over ways to position vertices of Γ at points of M (times the vertex factors, a symmetry factor and a loop-counting factor):

$$\Phi_\Gamma = \frac{\hbar^{|E|-|V|}}{|\operatorname{Aut}(\Gamma)|} \int_{M^{\times V}} d^n x_1 \cdots d^n x_{|V|} \prod_{(u,v)\in E} G(x_u, x_v) \cdot \prod_{v\in V}(-p_{\mathrm{val}(v)}). \tag{3}$$

Here, V, E are the set of vertices and the set of edges of Γ, respectively.

Next, one can understand the kinetic operator $\Delta + m^2 =: \widehat{H}$ as a quantum Hamiltonian of an auxiliary quantum mechanical system with Hilbert space $L^2(M)$. Then, one can write the Green's function $G(x, y)$ as the evolution operator of this auxiliary system integrated over the time of evolution:

$$G(x, y) = \int_0^\infty dt \langle x| e^{-t\widehat{H}} |y\rangle.$$

Replacing the evolution operator (a.k.a. heat kernel) with its Feynman–Kac path integral representation, one has

$$G(x, y) = \int_0^\infty dt \int_{\gamma:[0,t]\to M,\, \gamma(0)=x,\, \gamma(t)=y} \mathcal{D}\gamma\, e^{-S^{1q}(\gamma)}. \tag{4}$$

[1] We refer the reader to the inspiring exposition of this idea in [7, Section 3.2].

[2] In this discussion, we will ignore the issue of divergencies and renormalization.

Here, the inner integral is over paths γ on M parameterized by the interval $[0, t]$, starting at y and ending at x; the auxiliary ("first quantization") action in the exponent is

$$S^{1q}(\gamma) = \int_0^t d\tau \left(\frac{\dot{\gamma}^2}{4} + m^2 - \frac{1}{6} R(\gamma) \right), \tag{5}$$

where R is the scalar curvature of the metric on M; $\dot{\gamma}^2 := g_{\gamma(\tau)}(\dot{\gamma}, \dot{\gamma})$ is the square norm of the velocity vector $\dot{\gamma} \in T_{\gamma(\tau)} M$ of the path with respect to the metric g on M (see footnote[3]).

Plugging the integral representation (4) of Green's function into (3), one obtains the following integral formula for the weight of a Feynman graph:

$$\Phi_\Gamma = \frac{\hbar^{|E|-|V|}}{|\mathrm{Aut}(\Gamma)|} \int_{0<t_1,\ldots,t_{|E|}<\infty} dt_1 \cdots dt_{|E|} \int_{\boldsymbol{\gamma}: \Gamma_{t_1,\ldots,t_{|E|}} \to M} \mathcal{D}\boldsymbol{\gamma}\, e^{-S^{1q}(\boldsymbol{\gamma})} \cdot \prod_{v \in V} (-p_{\mathrm{val}(v)}). \tag{6}$$

Here, $\Gamma_{t_1,\ldots,t_{|E|}}$ is the graph Γ seen as a metric graph with t_e the length of edge e. The outer integral is over metrics on Γ, the inner (path) integral is over maps $\boldsymbol{\gamma}$ of Γ to M, sending vertices to points of M and edges to paths connecting those points; $S^{1q}(\boldsymbol{\gamma})$ is understood as a sum of expressions in the right-hand side of (5) over edges of Γ.

We refer to the formula (6), representing the weight of a Feynman graph via an integral over maps $\Gamma \to M$ (or, equivalently, as a partition function of an auxiliary 1d sigma model on the graph Γ with target M), as the "first quantization formula."[4]

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[3]The action (5) can be obtained from the short-time asymptotics (Seeley–DeWitt expansion) of the heat kernel

$$\kappa(x, y; t) = \langle x | e^{-t\widehat{H}} | y \rangle \underset{t \to 0}{\sim} (4\pi t)^{-\frac{n}{2}} e^{-\frac{d(x,y)^2}{4t}} (1 + b_2(x, y)t + b_4(x, y)t^2 + \cdots),$$

with b_{2k} smooth functions on $M \times M$ (in particular, on the diagonal), with $b_2(x, x) = -m^2 + \frac{1}{6} R(x)$; $d(x, y)$ is the geodesic distance on M, see, e.g., [18]. One then has

$$\begin{aligned} &\kappa(x, y; t) \\ &= \lim_{N \to \infty} \int_{M^{\times(N-1)}} d^n x_{N-1} \cdots d^n x_1 \kappa(x_N = x, x_{N-1}; \delta t) \cdots \kappa(x_2, x_1; \delta t) \kappa(x_1, x_0 = y; \delta t) \\ &= \lim_{N \to \infty} \int \prod_{j=1}^{N-1} ((4\pi \delta t)^{-\frac{n}{2}} dx_j) e^{-\sum_{j=1}^N \left(\frac{d(x_j, x_{j-1})^2}{4(\delta t)^2} + (m^2 - \frac{1}{6} R(x_j)) \right) \delta t}. \end{aligned}$$

In the right-hand side one recognizes the path integral of (4) written as a limit of finite-dimensional integrals (cf., [9]). We denoted $\delta t = t/N$.

[4]As opposed to the functional integral (2) – the "second quantization formula."

Remark 1.1. It is known that $\frac{1}{6}R$ appears in the quantum Hamiltonian of the quantum mechanical system of a free particle on a closed Riemannian manifold M; see, for example, [1,20]. Here, the difference is that $\frac{1}{6}R$ is introduced in the classical action (5) so that $\Delta + m^2$ is the quantum Hamiltonian.

Remark 1.2. One can absorb the determinant factor in the right-hand side of (2) into the sum over graphs, if we extend the set of graphs Γ to allow them to have circle connected components (with no vertices), with the rule

$$\begin{aligned}\Phi_{S^1} &= -\frac{1}{2}\log\det{}^{\zeta}(\Delta + m^2) = \frac{1}{2}\int_0^\infty \frac{dt}{t}\,\mathrm{tr}\big(e^{-t\hat{H}}\big)\\ &= \frac{1}{2}\int_0^\infty \frac{dt}{t}\int_{\gamma: S^1_t\to M} \mathcal{D}\gamma\, e^{-S^{1q}(\gamma)},\end{aligned} \tag{7}$$

where the integral in t is understood in zeta-regularized sense;

$$S^1_t = \mathbb{R}/t\mathbb{Z}$$

is the circle of perimeter t.

1.1.1. Version with 1d gravity. Another way to write the formula (4) is to consider paths γ parameterized by the standard interval $I = [0,1]$ (with coordinate σ) and introduce an extra field – the metric $\xi = \underline{\xi}(\sigma)(d\sigma)^2$ on I:

$$G(x,y) = \int_{(\mathrm{Met}(I)\times\mathrm{Map}(I,M)_{x,y})/\,\mathrm{Diff}(I)} \mathcal{D}\xi\,\mathcal{D}\gamma\, e^{-\bar{S}^{1q}(\gamma,\xi)}. \tag{8}$$

Here, $\mathrm{Map}(I,M)_{x,y}$ is the space of paths $\gamma\colon I\to M$ from x to y; the exponent in the integrand is

$$\begin{aligned}\bar{S}^{1q}(\gamma,\xi) &= \int_I \left(\frac{1}{4}(\xi^{-1}\otimes\gamma^* g)(d\gamma,d\gamma) + m^2 - \frac{1}{6}R(\gamma)\right) d\,\mathrm{vol}_\xi\\ &= \int_0^1 \left(\frac{\dot{\gamma}^2}{4\underline{\xi}} + m^2 - \frac{1}{6}R(\gamma)\right)\sqrt{\underline{\xi}}\,d\sigma,\end{aligned} \tag{9}$$

with $d\,\mathrm{vol}_\xi$ the Riemannian volume form of I induced by ξ. Note that the action (9) is invariant under diffeomorphisms of I. One can gauge-fix this symmetry by requiring that the metric ξ is constant on I, then one is left with integration over the length t of I with respect to the constant metric; this reduces the formula (8) back to (4).

In (8), the Green's function of the original theory on M is understood in terms of a 1d sigma-model on I with target M coupled to 1d gravity. For a Feynman graph, similarly to (6), one has

$$\Phi_\Gamma = \hbar^{|E|-|V|}\int_{(\mathrm{Met}(\Gamma)\times\mathrm{Map}(\Gamma,M))/\,\mathrm{Diff}(\Gamma)} \mathcal{D}\boldsymbol{\xi}\,\mathcal{D}\boldsymbol{\gamma}\, e^{-\bar{S}^{1q}(\boldsymbol{\gamma},\boldsymbol{\xi})}\prod_{v\in V}(-p_{\mathrm{val}(v)}), \tag{10}$$

the partition function of 1d sigma model on the Feynman graph Γ coupled to 1d gravity on Γ; $\overline{S}^{1q}(\boldsymbol{\gamma}, \boldsymbol{\xi})$ is understood as a sum of terms (9) over the edges of Γ (see footnote[5]).

Formula (8) may be viewed as a 1d toy model for a correlator in perturbative string theory, which is a correlator in a 2d conformal field theory (a sigma model), integrated over the moduli space of conformal structures on the worldsheet surface. I.e., one has a sigma model coupled to the 2d gravity on the worldsheet.

1.1.2. Heuristics on locality in the first quantization formalism. Suppose that we have a decomposition $M = M_1 \cup_Y M_2$ of M into two Riemannian manifolds M_i, with common boundary Y. Then, the locality of quantum field theory – or, a fictional "Fubini theorem" for the (also fictional) functional integral – suggests a gluing formula

$$Z_M = \text{“}\int_{C^\infty(Y)} \mathcal{D}\phi_Y \, Z_{M_1}(\phi_Y) Z_{M_2}(\phi_Y),\text{”} \tag{11}$$

where Z_{M_i} is a functional of $C^\infty(Y)$, again formally given by a functional integral understood as a sum over Feynman graphs[6],

$$\begin{aligned} Z_{M_i}(\phi_Y) &= \text{“}\int_{\substack{\phi \in C^\infty(M) \\ \phi|_Y = \phi_Y}} \mathcal{D}\phi \, e^{-\frac{1}{\hbar} S(\phi)}\text{”} \\ &= \left(\det{}^\zeta(\Delta_{M_i,Y} + m^2)\right)^{-\frac{1}{2}} \cdot \sum_\Gamma \Phi_\Gamma(\phi_Y), \end{aligned}$$

where we are putting Dirichlet boundary conditions on the kinetic operator. Feynman graphs now have bulk and boundary vertices, $V = V^{\text{bulk}} \sqcup V^\partial$, where boundary vertices are required to be univalent. The set of edges then decomposes as

$$E = \bigsqcup_{i=0}^{2} E_i,$$

[5] If one thinks of the quotient by $\mathrm{Diff}(\Gamma)$ in (10) as a stack, one can see the symmetry factor $\frac{1}{|\mathrm{Aut}(\Gamma)|}$ as implicitly contained in the integral. To see that, one should think of the quotient by $\mathrm{Diff}(\Gamma)$ as first a quotient by the connected component of the identity map $\mathrm{Diff}_0(\Gamma)$ and then a quotient by the mapping class group $\pi_0 \mathrm{Diff}(\Gamma) = \mathrm{Aut}(\Gamma)$.

One can interpret the $1/t$ factor in the right-hand side of (7) in a similar fashion: $\mathrm{Diff}(S^1)$ splits as $\mathrm{Diff}(S^1, \mathrm{pt}) \times S^1$ – diffeomorphisms preserving a marked point on S^1, plus an extra factor S^1 corresponding to rigid rotations (moving the marked point). It is that extra S^1 factor that leads to the factor $1/t = 1/\mathrm{vol}(S^1)$ in the integration measure over t. The factor $1/2$ in (7) comes by the previous mechanism from the mapping class group $\mathbb{Z}_2$ of S^1 (orientation preserving/reversing diffeomorphisms up to isotopy).

[6] Again, for the purpose of this motivational section we are not discussing the problem of divergencies and renormalization. For $n = \dim M = 2$, a precise definition of all involved objects and a proof of the gluing formula (11) can be found in [11].

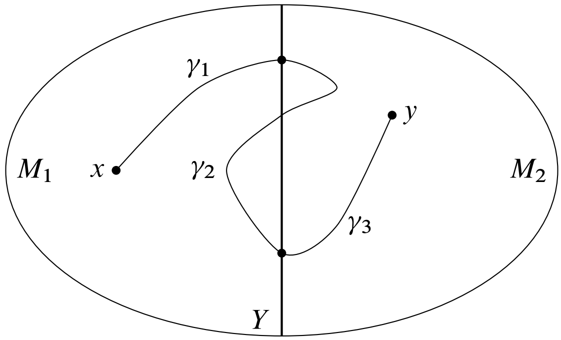

Figure 1. Decomposing a path $\gamma = \gamma_1 * \gamma_2 * \gamma_3$.

where edges in E_i have i endpoints in V^∂. The weight of a Feynman graph then is

$$\Phi_\Gamma(\phi_Y) = \frac{\hbar^{|E|-|V|}}{|\operatorname{Aut}(\Gamma)|} \int_{M_i^{\times V^{\text{bulk}}}} d^n x_1 \cdots d^n x_{|V^{\text{bulk}}|} \int_{Y^{V^\partial}} d^{n-1} y_1 \cdots d^{n-1} y_{|V^\partial|}$$
$$\cdot \prod_{v \in V^{\text{bulk}}} (-p_{\text{val}(v)}) \prod_{w \in V^\partial} \phi_Y(y_w) \prod_{(u,v) \in E_0} G_{M_i,Y}(x_u, x_v)$$
$$\cdot \prod_{(u,v) \in E_1} E_{Y,M_i}(x_u, y_v) \prod_{(u,v) \in E_2} -\mathrm{DN}_{Y,M_i}(y_u, y_v),$$

where $G_{M_i,Y}$ denotes the Green's function of the operator with Dirichlet boundary conditions, $E_{Y,M_i}(x, y) = \partial_{n_y} G(x, y)$ is the normal derivative of the Green's function at a boundary point $y \in Y$, and DN_{Y,M_i} is the Dirichlet-to-Neumann operator associated to the kinetic operator (see Section 3.4 for details).

Let us sketch an interpretation of the gluing formula for the Green's function from the standpoint of the first quantization formalism. Let $x \in M_1$, $y \in M_2$ and consider a path $\gamma\colon [0, t] \to M$ with $\gamma(0) = x$ and $\gamma(t) = y$. Then, the decomposition $M = M_1 \cup_Y M_2$ induces a decomposition $\gamma = \gamma_1 * \gamma_2 * \gamma_3$ as follows ("$*$" means concatenation of paths). Let $t_0 = 0$, $t_1 = \min\{t, \gamma(t) \in Y\}$ and $t_2 = \max\{t, \gamma(t) \in Y\}$ and $t_3 = t$, then $\gamma_i = \gamma|_{[t_{i-1},t_i]}$. This gives a decomposition

$$P_M(x, y) = \bigsqcup_{u,v \in Y} P'_{M_1}(x, u) \times P_M(u, v) \times P'_{M_2}(v, y),$$

where we have introduced the notation $P_M(x, y)$ for the set of all paths from x to y (of arbitrary length) and $P'_{M_i}(x, u)$ for the set of all paths starting at $x \in M_i$ and ending at $u \in Y$ and not intersecting Y in between. See Figure 1.

Paths of a specific length t will be denoted by $P^t_M(x, y)$, or $(P'_M)^t(x, u)$. Assuming a Fubini theorem for the path measure $\mathcal{D}\gamma$, additivity of the action suggests that we could rewrite (4) as

$$
\begin{aligned}
&G(x,y)\\
&\text{“}=\text{”}\int_{Y\times Y} d^{n-1}u\, d^{n-1}v \int_0^\infty dt_1 \int_{\gamma_1\in (P'_{M_1})^{t_1}(x,u)} \mathcal{D}\gamma_1\, e^{-S^{1q}(\gamma_1)}\\
&\quad\times \int_0^\infty dt_2 \int_{\gamma_2\in P_M^{t_2}(u,v)} \mathcal{D}\gamma_2\, e^{-S^{1q}(\gamma_2)} \int_0^\infty dt_3 \int_{\gamma_3\in (P_{M_2})^{t_3}(v,y)} \mathcal{D}\gamma_3\, e^{-S^{1q}(\gamma_1)}.
\end{aligned}
$$

Comparing with the gluing formula for the Green's function[7]

$$
G(x,y)=\int_{Y\times Y} d^{n-1}u\, d^{n-1}v\, E_{Y,M_1}(x,u)\, \varkappa_{Y,M}(u,v)\, E_{Y,M_2}(v,y),
$$

with $\varkappa_{Y,M}=(\mathrm{DN}_{Y,M_1}+\mathrm{DN}_{Y,M_2})^{-1}$ the inverse of the "total" Dirichlet-to-Neumann operator, suggests the following path integral formulae for the extension operator and $\varkappa$:

$$
E_{Y,M_i}(x,u)=\int_0^\infty dt \int_{\gamma\in (P'_{M_i})^{t}(x,u)} \mathcal{D}\gamma\, e^{-S^{1q}(\gamma)}, \tag{12}
$$

$$
\varkappa_{Y,M}(u,v)=\int_0^\infty dt \int_{\gamma\in P_M^{t}(u,v)} \mathcal{D}\gamma\, e^{-S^{1q}(\gamma)}. \tag{13}
$$

The results of our paper[8] actually suggest also the following path integral formula for the Dirichlet-to-Neumann operator:

$$
\mathrm{DN}_{Y,M_i}(u,v)=\int_0^\infty dt \int_{\gamma\in (P''_{M_i})^{t}(u,v)} \mathcal{D}\gamma\, e^{-S^{1q}(\gamma)}. \tag{14}
$$

Here, $(P''_{M_i})^t(u,v)$ is the set of all paths from $\gamma\colon [0,t]\to M_i$ from $u\in Y$ to $v\in Y$ such that $\gamma(\tau)\notin Y$ for all $0<\tau<t$.

Assuming this formula, we have again a "first quantization formula" for weights of Feynman graphs with boundary vertices

$$
\begin{aligned}
\Phi_\Gamma(\phi_Y)=\frac{\hbar^{|E|-|V|}}{|\operatorname{Aut}(\Gamma)|}&\int_{0<t_1,\ldots,t_{|E|}<\infty} dt_1\cdots dt_{|E|}\\
&\times\int_{\boldsymbol{\gamma}:\Gamma_{t_1,\ldots,t_{|E|}}\to M_i} \mathcal{D}\boldsymbol{\gamma}\, e^{-S^{1q}(\boldsymbol{\gamma})} \prod_{v\in V^{\mathrm{bulk}}} (-p_{\mathrm{val}(v)}) \prod_{v^\partial\in V^\partial} \phi_Y(\boldsymbol{\gamma}(v^\partial)).
\end{aligned} \tag{15}
$$

[7] See [11, Proposition 4.2] and Section 3.4.1 of the present paper for details.

[8] Another reason to guess that formula is the fact that the integral kernel of the Dirichlet-to-Neumann operator is given by a symmetric normal derivative of the Green's function $\mathrm{DN}_{Y,M_i}(u,v)=-\partial_{n_u}\partial_{n_v}G_{M_i,Y}$ (in a regularized sense – see [11, Remark 3.4]), and formula (12) for the first normal derivative of the Green's function.

Here, notation is as in (6), the only additional condition is that $\boldsymbol{\gamma}$ respects the type of edges in Γ, that is, for all $x \in \Gamma_{t_1,\dots,t_{|E|}}$ we have $\boldsymbol{\gamma}(x) \in Y$ if and only if $x \in V^\partial$.

1.2. QFT on a graph: A guide to the paper

In this paper, we study a toy ("combinatorial" or "lattice") version of the scalar field theory (1), where the spacetime manifold M is replaced by a graph X, the scalar field ϕ is a function on the vertices of X and the Laplacian in the kinetic operator is replaced by the graph Laplacian Δ_X. I.e., the model is defined by the action

$$S_X(\phi) = \sum_{v \in V_X} \frac{1}{2}\phi(v)((\Delta_X + m^2\mathrm{Id})\phi)(v) + p(\phi(v)), \tag{16}$$

where V_X is the set of vertices of X and p is the interaction potential (a polynomial of ϕ), as before.

This model has the following properties.

(i) The "functional integral" over the space of fields is a finite-dimensional convergent integral (Section 2).

(ii) The functional integral can be expanded in Feynman graphs, giving an asymptotic expansion of the nonperturbative partition function in powers of $\hbar$ (Section 4).

(iii) Partition functions are compatible with unions of graphs over a subgraph ("gluing") – we see this as a graph counterpart of Atiyah–Segal functorial picture of QFT with compatibility with respect to cutting-gluing n manifolds along closed $(n-1)$-submanifolds. This functorial property of the graph QFT can be proven

(a) directly from the functional integral perspective (by a Fubini theorem argument) – Section 2.1,

(b) at the level of Feynman graphs (Section 4.2).

The proof of functoriality at the level of Feynman graphs relies on the "gluing formulae" describing the behavior of Green's functions and determinants with respect to gluing of spacetime graphs (Section 3.3). These formulae are a combinatorial analog of known gluing formulae for Green's functions and zeta-regularized functional determinants on manifolds (Section 3.4).

(iv) The Green's function on a graph X can be written as a sum over paths (Section 5, in particular, Table 3), giving an analog of the formula (4); similarly, the determinant can be written as a sum over closed paths, giving an analog of (7). This leads to a "first-quantization" representation of Feynman graphs, as a sum over maps $\Gamma \to X$, sending vertices of Γ to vertices of X and sending edges of Γ to paths on X (connecting the images of the incident vertices) – Section 6. This yields a graph counterpart of the continuum first quantization formula (6).

(v) There are path sum formulae for the combinatorial extension (or "Poisson") operators and Dirichlet-to-Neumann operators (Section 5.4, see, in particular, Table 4), analogous to the path integral formulae (12) and (14).

(vi) First quantization perspective gives a visual interpretation of the gluing formula for Green's functions and determinants on a graph $X = X' \cup_Y X''$ in terms of cutting the path into portions spent in X' or in X'' (Section 5.5), and likewise an interpretation of the cutting-gluing of Feynman graphs (Section 6.3).

1.3. Comparison to existing literature

Let us briefly discuss closely related works.

The thesis [19] also discusses path sum formulae for the propagator and the determinant and gluing formulae derived from them; however, it uses a different model "bridge gluing" for gluing of graphs[9]. In particular, our gluing formulae are different and more adapted to combinatorial functorial QFT, with Y playing the role of "boundary" of the graphs X_i. Moreover, the path sum results derived in this paper go beyond the results of [19] in several ways. Firstly, in [19] formulae for non-regular graphs are only given in terms of hesitant paths[10], whereas we give formulae both in terms of hesitant and non-hesitant paths for not necessarily regular graphs. Secondly, in [19], there is no notion of the combinatorial Dirichlet-to-Neumann and extension operators. Their presence renders the combinatorial gluing formulae simple and explicit.

A free (Gaussian) version of the combinatorial model is considered in [17]. The main focus of [17] is to provide an explicit construction of functorial QFT and establish a BFK-like gluing formula for the determinant of Laplace operators using quantum field theoretic methods. In our work, we consider an interacting theory by adding polynomial potentials to the free theory. We establish the gluing formula for propagators and utilize it to construct interacting combinatorial functorial QFTs. In this sense, our work extends the free theory considered in [17].

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[9] In bridge gluing, graphs $\Gamma = (V_\Gamma, E_\Gamma)$ are decomposed as $V_\Gamma = V_{\Gamma_1} \sqcup V_{\Gamma_2}$, where Γ_1 and Γ_2 are disjoint full subgraphs, then $E_\Gamma = E_{\Gamma_1} \sqcup E_{\Gamma_2} \sqcup I(\Gamma)$ with $I(\Gamma)$ the remaining edges called "bridges". We use "interface gluing," where $V_X = V_{X_1} \cup_{V_Y} V_{X_2}$ for full subgraphs X_1 and X_2 which intersect along a subgraph Y.

[10] See Section 5.2.1 for our definitions of paths and hesitant paths.

Two-dimensional massive scalar field theories have been studied in functorial QFT context in [11, 15]. In particular, [15] shows that the (non-perturbative) partition function of an interacting theory with polynomial potentials satisfies a gluing formula, leading to the construction of functorial QFTs. In [11], similar results are established for partition functions derived from perturbative quantization. In the current work, we establish a combinatorial analog of these results.

Notations

We will be using the following notations throughout the text.

Symbol	**Description**
X	Spacetime graph
Y or Y_i	A subgraph $Y \subset X$ (or several subgraphs $Y_i \subset X$) of the spacetime graph
Δ_X	Laplacian on X
K_X	Kinetic operator on X, $K_X = \Delta_X + m^2$
Γ	Feynman graph
$G(x, y)$	The propagator or Green's function of the kinetic operator K_X, integral kernel (matrix) of $G = K_X^{-1}$
Z_X	The partition function on X
Z_X^{pert}	The perturbative partition function on X
$E_{Y,X}$	The extension operator (also known as Poisson operator): extends a field ϕ_Y into bulk X as a solution of Dirichlet problem
$\text{DN}_{Y,X}$	Dirichlet-to-Neumann operator
S_X	Action functional on the space of fields on X
S_X^{1q}	First quantized action functional
$\det^\zeta A$	Zeta-regularized determinant of operator A
$P_X(u, v)$	The set of paths in X joining u to v
$\Pi_X(u, v)$	The set of h-paths in X joining u to v
P_X^Γ	The set of edge-to-path maps from Γ to X
Π_X^Γ	The set of edge-to-h-path maps from Γ to X
$\deg(\tilde{\gamma})$	The number of jumps in an h-path $\tilde{\gamma}$
$l(\gamma)$	The length of a path (or h-path) γ
$h(\tilde{\gamma})$	The number of hesitations of an h-path $\tilde{\gamma}$
$s(\tilde{\gamma})$	Weight of an h-path, $s(\tilde{\gamma}) = m^{-2l(\tilde{\gamma})}(-1)^{h(\tilde{\gamma})}$
$w(\gamma)$	Weight of a path, $w(\gamma) = \prod_{v \in V(\gamma)} \frac{1}{m^2 + \text{val}(v)}$

2. Scalar field theory on a graph

Let X be a finite graph. Consider the toy field theory on X, where fields are real-valued functions $\phi(v)$ on the set of vertices V_X, i.e., the space of fields is the space of 0-cochains on X seen as a 1-dimensional CW complex,

$$F_X = C^0(X).$$

We define the action functional as

$$\begin{aligned} S_X(\phi) &= \frac{1}{2}(d\phi, d\phi) + \left\langle \mu, \frac{m^2}{2}\phi^2 + p(\phi) \right\rangle \\ &= \frac{1}{2}(\phi, (\Delta_X + m^2)\phi) + \langle \mu, p(\phi)\rangle \\ &= \sum_{e \in E_X} \frac{1}{2}(\phi(v_e^1) - \phi(v_e^0))^2 + \sum_{v \in V_X} \left(\frac{m^2}{2}\phi(v)^2 + p(\phi(v)) \right). \end{aligned} \tag{17}$$

Here,

- $d: C^0(X) \to C^1(X)$ is the cellular coboundary operator (we assume that 0 cells carry + orientation and 1-cells carry some orientation – the model does not depend on this choice).
- $(\,,\,): \mathrm{Sym}^2\, C^k(X) \to \mathbb{R}$ for $k = 0, 1$ is the standard metric, in which the cell basis is orthonormal.
- $\langle\,,\,\rangle$ is the canonical pairing of chains and cochains; μ is the 0-chain given by the sum of all vertices with coefficient 1 (see footnote[11]).
- $m > 0$ is the fixed "mass" parameter.
- $p(\phi)$ is a fixed polynomial ("potential"),

$$p(\phi) = \sum_{k \geq 3} \frac{p_k}{k!}\phi^k. \tag{18}$$

More generally, $p(\phi)$ can be a real analytic function. We will assume that $\frac{m^2}{2}\phi^2 + p(\phi)$ has a unique absolute minimum at $\phi = 0$ and that it grows sufficiently fast[12] at $\phi \to \pm\infty$, so that the integral (19) converges measure-theoretically.

---

[11]The 0-chain μ is an analog of the volume form on the spacetime manifold in our model. If we want to consider the field theory on X as a lattice approximation of a continuum field theory, we would need to scale the metric $(,)$ and the 0-chain μ appropriately with the mesh size. Additionally, one would need to add mesh-dependent counterterms to the action in order to have finite limits for the partition function and correlators.

[12]Namely, we want the integral $\int_{\mathbb{R}} d\phi\, e^{-\frac{1}{\hbar}(\frac{m^2}{2}\phi^2 + p(\phi))}$ to converge for any $\hbar > 0$.

- $\Delta_X = d^T d : C^0(X) \to C^0(X)$ is the graph Laplace operator on 0-cochains, where $d^T : C^1(X) \to C^0(X)$ is the dual (transpose) map to the coboundary operator (in the construction of the dual, one identifies chains and cochains using the standard metric). The matrix elements of Δ_X in the cell basis, for X a simple graph (i.e., without double edges and short loops), are

$$(\Delta_X)_{uv} = \begin{cases} \mathrm{val}(v) & \text{if } u = v, \\ -1 & \text{if } u \neq v \text{ and } u \text{ is connected to } v \text{ by an edge}, \\ 0 & \text{otherwise}, \end{cases}$$

where $\mathrm{val}(v)$ is the valence of the vertex v. More generally, for X not necessarily simple, one has

$$(\Delta_X)_{uv} = \begin{cases} \mathrm{val}(v) - 2 \cdot \#\{\text{short loops } v \to v\} & \text{if } u = v, \\ -\#\{\text{edges } u \to v\} & \text{if } u \neq v. \end{cases}$$

We will be interested in the partition function

$$Z_X = \int_{F_X} D\phi \, e^{-\frac{1}{\hbar} S_X}, \tag{19}$$

where

$$D\phi = \prod_{v \in V_X} \frac{d\phi(v)}{\sqrt{2\pi\hbar}}$$

is the "functional integral measure" on the space of fields F_X (in this case, just the Lebesgue measure on a finite-dimensional space); $\hbar > 0$ is the parameter of quantization – the "Planck constant."[13] The integral in the right-hand side of (19) is absolutely convergent. One can also consider correlation functions

$$\langle \phi(v_1) \cdots \phi(v_n) \rangle = \frac{1}{Z_X} \int_{F_X} D\phi \, e^{-\frac{1}{\hbar} S_X} \phi(v_1) \cdots \phi(v_n). \tag{20}$$

Remark 2.1. We stress that in this section we consider the *nonperturbative* partition functions/correlators and $\hbar$ is to be understood as an actual positive number, unlike in the setting of perturbation theory (Section 4), where $\hbar$ becomes a formal parameter.

Remark 2.2. In this paper, we use the Euclidean QFT convention for our (toy) functional integrals, with the integrand $e^{-\frac{1}{\hbar} S}$ instead of $e^{\frac{i}{\hbar} S}$, in order to have a better measure-theoretic convergence situation. The first convention leads to absolutely convergent integrals, whereas the second leads to conditionally convergent oscillatory integrals.

---

[13]Or one can think of $\hbar$ as "temperature" if one thinks of (19) as a partition function of statistical mechanics with S the energy of a state ϕ.

2.1. Functorial picture

One can interpret our model in the spirit of Atiyah–Segal functorial picture of QFT, as a (symmetric monoidal) functor

$$\mathrm{GraphCob} \xrightarrow{(\mathcal{H},Z)} \mathrm{Hilb} \tag{21}$$

from the spacetime category[14] of graph cobordisms to the category of Hilbert spaces and Hilbert–Schmidt operators.

Here, in the source category GraphCob is as follows.

- The objects are graphs Y.
- A morphism from Y_{in} to Y_{out} is a graph X which contains Y_{in} and Y_{out} as disjoint subgraphs. We will write $Y_{\mathrm{in}} \xrightarrow{X} Y_{\mathrm{out}}$ and refer to Y_{in}, Y_{out} as "ends" (or "boundaries") of X, or we will say that X is a "graph cobordism" between Y_{in} and Y_{out}.
- The composition is given by unions of graphs with the out-end of one cobordism identified with the in-end of the subsequent one:

$$\big(Y_3 \xleftarrow{X''} Y_2\big) \circ \big(Y_2 \xleftarrow{X'} Y_1\big) = Y_3 \xleftarrow{X} Y_1, \tag{22}$$

  where

$$X = X' \bigcup_{Y_2} X''.$$

- The monoidal structure is given by disjoint unions of graphs.

All graphs are assumed to be finite. As defined, GraphCob does not have unit morphisms (as usual for spacetime categories in non-topological QFTs); by abuse of language, we still call it a category.

The target category Hilb has as its objects Hilbert spaces $\mathcal{H}$ over $\mathbb{C}$ (see footnote[15]); the morphisms are Hilbert–Schmidt operators; the composition is composition of operators. The monoidal structure is given by tensor products (of Hilbert spaces and of operators).

The functor (21) is constructed as follows. For an end-graph $Y \in \mathrm{Ob}(\mathrm{GraphCob})$, the associated vector space is

$$\mathcal{H}_Y = L^2(C^0(Y)), \tag{23}$$

the space of complex-valued square-integrable functions on the vector space $C^0(Y) = \mathbb{R}^{V_Y}$.

[14] This terminology is taken from [16].

[15] Alternatively (since we do not put i in the exponent in the functional integral), one can consider Hilbert spaces over $\mathbb{R}$.

For a graph cobordism $Y_{\text{in}} \xrightarrow{X} Y_{\text{out}}$, the associated operator $Z_X\colon \mathcal{H}_{Y_{\text{in}}} \to \mathcal{H}_{Y_{\text{out}}}$ is

$$Z_X\colon \Psi_{\text{in}} \mapsto \left(\Psi_{\text{out}}\colon \phi_{\text{out}} \mapsto \int_{F_{Y_{\text{in}}}} D\phi_{\text{in}}\, \langle \phi_{\text{out}} | Z_X | \phi_{\text{in}} \rangle \Psi_{\text{in}}(\phi_{\text{in}})\right) \tag{24}$$

with the integral kernel

$$\langle \phi_{\text{out}} | Z_X | \phi_{\text{in}} \rangle := \int_{F_X^{\phi_{\text{in}},\phi_{\text{out}}}} [D\phi]^{\phi_{\text{in}},\phi_{\text{out}}}\, e^{-\frac{1}{\hbar}(S_X(\phi) - \frac{1}{2} S_{Y_{\text{in}}}(\phi_{\text{in}}) - \frac{1}{2} S_{Y_{\text{out}}}(\phi_{\text{out}}))}. \tag{25}$$

Here,

- $F_X^{\phi_{\text{in}},\phi_{\text{out}}}$ is the space of fields on X subject to boundary conditions $\phi_{\text{in}}, \phi_{\text{out}}$ imposed on the ends, i.e., it is the fiber of the evaluation-at-the-ends map
$$F_X \to F_{Y_{\text{in}}} \times F_{Y_{\text{out}}}$$
over the pair $(\phi_{\text{in}}, \phi_{\text{out}})$.
- The measure
$$[D\phi]^{\phi_{\text{in}},\phi_{\text{out}}} = \prod_{v \in V_X \setminus (V_{Y_{\text{in}}} \sqcup V_{Y_{\text{out}}})} \frac{d\phi(v)}{\sqrt{2\pi\hbar}}$$
stands for the "conditional functional measure" on fields subject to boundary conditions.

We will also call the expression (25) the partition function on the graph X "relative" to the ends Y_{in}, Y_{out}, or just the partition function relative to the "boundary" subgraph

$$Y = Y_{\text{in}} \sqcup Y_{\text{out}},$$

if the distinction between "in" and "out" is irrelevant. In the latter case, we will use the notation $Z_{X,Y}(\phi_Y)$, with

$$\phi_Y = (\phi_{\text{in}}, \phi_{\text{out}}).$$

Proposition 2.3. *The assignment* (23), (24) *is a functor of monoidal categories.*

Proof. The main point to check is that composition is mapped to composition. It follows from the Fubini theorem, the locality of the integration measure (that it is a product over vertices of local measures) and additivity of the action:

$$S_X(\phi) = S_{X'}(\phi|_{X'}) + S_{X''}(\phi|_{X''}) - S_{Y_2}(\phi|_{Y_2}) \tag{26}$$

in the notations of (22). Indeed, it suffices to prove that

$$\int_{F_{Y_2}} D\phi_2\, \langle \phi_3 | Z_{X''} | \phi_2 \rangle \langle \phi_2 | Z_{X'} | \phi_1 \rangle \overset{!}{=} \langle \phi_3 | Z_X | \phi_1 \rangle \tag{27}$$

Combinatorial QFT	Continuum QFT
graph X	closed spacetime n-manifold M;
field $\phi\colon V_X \to \mathbb{R}$	scalar field $\phi \in C^\infty(M)$;
action (17)	action $S(\phi) = \int_M \frac{1}{2} d\phi \wedge * d\phi + (\frac{m^2}{2}\phi^2 + p(\phi)) d\,\mathrm{vol}$
	$= \int_M (\frac{1}{2}\phi(\Delta + m^2)\phi + p(\phi)) d\,\mathrm{vol}$;
partition function (19)	functional integral on a closed manifold;
graph cobordism $Y_{\mathrm{in}} \xrightarrow{X} Y_{\mathrm{out}}$	n-manifold M with in/out-boundaries
	being closed $(n-1)$-manifolds $\gamma_{\mathrm{in}}, \gamma_{\mathrm{out}}$;
gluing/cutting of graph cobordisms	gluing/cutting of smooth n-cobordisms;
matrix element (25)	functional integral
	with boundary conditions $\phi_{\mathrm{in}}, \phi_{\mathrm{out}}$.

Table 1. Comparison between toy model and continuum QFT.

– again, we are considering the gluing of graph cobordisms as in (22). The left-hand side is

$$\int_{F_{Y_2}} D\phi_2 \int_{F_{X'}^{\phi_1,\phi_2}} [D\phi']^{\phi_1,\phi_2} \int_{F_{X''}^{\phi_2,\phi_3}} [D\phi'']^{\phi_2,\phi_3}$$

$$\times \exp\Bigg(-\frac{1}{\hbar}\Bigg(\underbrace{\Big(-\frac{1}{2}S_{Y_1}(\phi_1) + S_{X'}(\phi') - \frac{1}{2}S_{Y_2}(\phi_2)\Big) + \Big(-\frac{1}{2}S_{Y_2}(\phi_2) + S_{X''}(\phi''\Big)}_{S_X(\phi)}$$

$$-\frac{1}{2}S_{Y_3}(\phi_3)\Big)\Bigg)\Bigg)$$

$$= \int_{F_X^{\phi_1,\phi_3}} [D\phi]^{\phi_1,\phi_3} e^{-\frac{1}{\hbar}(S_X(\phi) - \frac{1}{2}S_{Y_1}(\phi_1) - \frac{1}{2}S_{Y_2}(\phi_3))},$$

which proves (27). Here, we understood that ϕ is a field on the glued cobordism X restricting to ϕ', ϕ'' on X', X'', respectively. Compatibility with disjoint unions is obvious by construction. ■

Remark 2.4. One can interpret the correlator (20) as the partition function of X seen as a cobordism $\{v_1, \ldots, v_n\} \xrightarrow{X} \varnothing$ applied to the state $\phi(v_1) \otimes \cdots \otimes \phi(v_n) \in \mathcal{H}_{\{v_1,\ldots,v_n\}}$.

Remark 2.5. The combinatorial model we are presenting is intended to be an analog (toy model) of the continuum QFT, according to the dictionary in Table 1.

When we want to emphasize that a graph X is not considered as a cobordism (or equivalently X is seen as a cobordism $\varnothing \xrightarrow{X} \varnothing$), we will call X a “closed” graph (by analogy with closed manifolds).

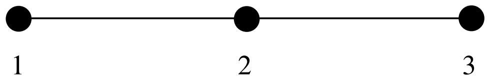


Figure 2. A line graph on 3 vertices.

3. Gaussian theory

3.1. Gaussian theory on a closed graph

Consider the free case of the model (17), with the interaction $p(\phi)$ set to zero. The action is quadratic

$$S_X(\phi) = \frac{1}{2}(\phi, K_X\phi), \tag{28}$$

where the kinetic operator is

$$K_X := \Delta_X + m^2;$$

it is a positive self-adjoint operator on F_X. Let us denote its inverse

$$G_X := (K_X)^{-1},$$

the "Green's function" or "propagator;" we will denote the matrix elements of G_X in the basis of vertices by $G_X(u, v)$ for $u, v \in V_X$.

The partition function (19) for a closed graph X is the Gaussian integral

$$Z_X = \int_{F_X} D\phi\, e^{-\frac{1}{\hbar}S_X(\phi)} = \det(K_X)^{-\frac{1}{2}}. \tag{29}$$

The correlator (20) is given by Wick's lemma, as a moment of the Gaussian measure

$$\langle\phi(v_1)\cdots\phi(v_{2m})\rangle = \hbar^m \sum_{\text{partitions } \{1,\ldots,2m\}=\bigcup_{i=1}^m \{a_i,b_i\}} G_X(v_{a_1}, v_{b_1})\cdots G_X(v_{a_m}, v_{b_m}).$$

3.1.1. Examples.

Example 3.1. Consider the graph X shown in Figure 2.

The kinetic operator is

$$K_X = \begin{pmatrix} 1+m^2 & -1 & 0 \\ -1 & 2+m^2 & -1 \\ 0 & -1 & 1+m^2 \end{pmatrix}.$$

Its determinant is

$$\det K_X = m^2(1+m^2)(3+m^2), \tag{30}$$

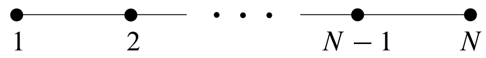


Figure 3. A line graph of length N.

and the inverse is

$$G_X = \frac{1}{m^2(1+m^2)(3+m^2)} \begin{pmatrix} 1+3m^2+m^4 & 1+m^2 & 1 \\ 1+m^2 & (1+m^2)^2 & 1+m^2 \\ 1 & 1+m^2 & 1+3m^2+m^4 \end{pmatrix}. \tag{31}$$

Example 3.2. Consider the line graph of length N as shown in Figure 3.

The kinetic operator is the tridiagonal matrix

$$K_X = \begin{pmatrix} 1+m^2 & -1 & & & \\ -1 & 2+m^2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2+m^2 & -1 \\ & & & -1 & 1+m^2 \end{pmatrix}. \tag{32}$$

The matrix elements of its inverse are

$$G_X(i,j) = \frac{\cosh \beta(N - |i-j|) + \cosh \beta(N+1-i-j)}{2 \sinh \beta \sinh \beta N}, \quad 1 \le i,\ j \le N, \tag{33}$$

where β is related to m by

$$\sinh \frac{\beta}{2} = \frac{m}{2}. \tag{34}$$

The determinant is

$$\det K_X = 2 \tanh \frac{\beta}{2} \sinh \beta N. \tag{35}$$

We refer to Appendix A for the details of the computation.

Example 3.3. Consider the circle graph with N vertices as shown in Figure 4.

The kinetic operator is

$$K_X = \begin{pmatrix} 2+m^2 & -1 & & & -1 \\ -1 & 2+m^2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2+m^2 & -1 \\ -1 & & & -1 & 2+m^2 \end{pmatrix}.$$

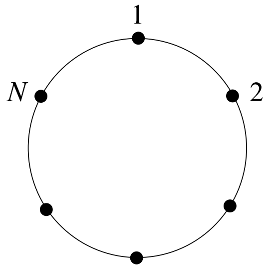


Figure 4. A circle graph with N vertices.

(We are only writing the nonzero entries.) Its inverse is given by

$$G_X(i,j) = \frac{\cosh \beta(\frac{N}{2} - |i-j|)}{2\sinh\beta \sinh\frac{\beta N}{2}}, \quad 1 \leq i,\ j \leq N. \tag{36}$$

Here, β is as in (34). The determinant is

$$\det K_X = 4\sinh^2 \frac{\beta N}{2}. \tag{37}$$

For instance, for $N = 3$, we obtain

$$G_X = \frac{1}{m^2(m^2+3)} \begin{pmatrix} m^2+1 & 1 & 1 \\ 1 & m^2+1 & 1 \\ 1 & 1 & m^2+1 \end{pmatrix} \tag{38}$$

and

$$\det K_X = m^2(m^2+3)^2. \tag{39}$$

3.2. Gaussian theory relative to the boundary

Consider the Gaussian theory on a graph X with "boundary subgraph" $Y \subset X$.

3.2.1. Dirichlet problem. Consider the following "Dirichlet problem." For a fixed field configuration on the boundary $\phi_Y \in F_Y$, we are looking for a field configuration on X, $\phi \in F_X$ such that

$$\phi|_Y = \phi_Y, \tag{40}$$

$$(K_X\phi)(v) = 0 \quad \text{for all } v \in V_X \setminus V_Y. \tag{41}$$

Equivalently, we are minimizing the action (28) on the fiber $F_X^{\phi_Y}$ of the evaluation-on-Y map $F_X \to F_Y$ over ϕ_Y. The solution exists and is unique due to convexity and nonnegativity of S_X.

Let us write the inverse of K_X as a 2×2 block matrix according to partition of vertices of X into (1) not belonging to Y ("bulk vertices") or (2) belonging to Y ("boundary vertices"):

$$(K_X)^{-1} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right). \tag{42}$$

Note that this matrix is symmetric, so A and D are symmetric and $C = B^T$.

Then, we can write the solution of the Dirichlet problem as follows: (41) implies $K_X \phi = \binom{0}{\xi}$ for some $\xi \in F_Y$. Hence,

$$\phi = (K_X)^{-1} \begin{pmatrix} 0 \\ \xi \end{pmatrix} = \begin{pmatrix} B\xi \\ D\xi \end{pmatrix} \underset{(40)}{=} \begin{pmatrix} B\xi \\ \phi_Y \end{pmatrix}.$$

Therefore, $\xi = D^{-1}\phi_Y$ and the solution of the Dirichlet problem is

$$\phi = \begin{pmatrix} BD^{-1}\phi_Y \\ \phi_Y \end{pmatrix}. \tag{43}$$

3.2.2. Dirichlet-to-Neumann operator. Note also that the evaluation of the action S_X on the solution of the Dirichlet problem is

$$\begin{aligned} S_X(\phi) = \frac{1}{2}(\phi, K_X\phi) &= \frac{1}{2}\left(\begin{pmatrix} BD^{-1}\phi_Y \\ \phi_Y \end{pmatrix}, \begin{pmatrix} 0 \\ \xi \end{pmatrix} \right) \\ &= \frac{1}{2}(\phi_Y, \xi) = \frac{1}{2}(\phi_Y, D^{-1}\phi_Y). \end{aligned} \tag{44}$$

The map sending ϕ_Y to the corresponding ξ (i.e., the kinetic operator evaluated on the solution of the Dirichlet problem) is a combinatorial analog of the Dirichlet-to-Neumann operator[16]. We will call the operator $\mathrm{DN}_{Y,X} := D^{-1}\colon F_Y \to F_Y$ the (combinatorial) Dirichlet-to-Neumann operator[17].

Recall (see, e.g., [11]) that in the continuum setting, the action of the free massive scalar field on a manifold with boundary, evaluated on a classical solution with Dirichlet boundary condition ϕ_∂ is $\int_{\partial X} \frac{1}{2}\phi_\partial\, \mathrm{DN}(\phi_\partial)$. Comparing with (44) reinforces the idea that is reasonable to call D^{-1} the Dirichlet-to-Neumann operator.

[16] Recall that in the continuum setting, for X a manifold with boundary, the Dirichlet-to-Neumann operator $\mathrm{DN}\colon C^\infty(\partial X) \to C^\infty(\partial X)$ maps a smooth function ϕ_∂ to the normal derivative $\partial_n \phi(x)$ on ∂X of the solution ϕ of the Helmholtz equation $(\Delta + m^2)\phi = 0$ subject to Dirichlet boundary condition $\phi|_\partial = \phi_\partial$.

[17] We put the subscripts in $\mathrm{DN}_{Y,X}$ to emphasize that we are extending ϕ_Y into X as a solution of (41). When we will discuss gluing, the same Y can be a subgraph of two different graphs X', X''; then it is important into which graph we are extending ϕ_Y.

We will denote the operator BD^{-1} appearing in (43) by

$$E_{Y,X} = BD^{-1}, \tag{45}$$

the "extension" operator (extending ϕ_Y into the bulk of X as a solution of the Dirichlet problem)[18].

3.2.3. Partition function and correlators (relative to a boundary subgraph). Let us introduce a notation for the blocks of the matrix K_X corresponding to splitting of the vertices of X into bulk and boundary vertices, similarly to (42):

$$K_X = \left(\begin{array}{c|c} \widehat{A} = K_{X,Y} & \widehat{B} \\ \hline \widehat{C} & \widehat{D} \end{array} \right). \tag{46}$$

In our notations, the subscript X, Y (as in $K_{X,Y}$, $G_{X,Y}$, $Z_{X,Y}$) stands for an object on X relative to Y[19]. On the other hand, the subscript Y, X (as in $\mathrm{DN}_{Y,X}$, $E_{Y,X}$) refers to an object related to extending a field on Y to a classical solution in the "bulk" X.

The partition function relative to Y (cf. (25)) is again given by a Gaussian integral:

$$\begin{aligned} Z_{X,Y}(\phi_Y) &= \int_{F_X^{\phi_Y}} [D\phi]^{\phi_Y} e^{-\frac{1}{\hbar}(S_X(\phi)-\frac{1}{2}S_Y(\phi_Y))} \\ &= \det(K_{X,Y})^{-\frac{1}{2}} e^{-\frac{1}{2\hbar}(\phi_Y,(\mathrm{DN}_{Y,X}-\frac{1}{2}K_Y)\phi_Y)}. \end{aligned} \tag{47}$$

The normalized correlators (depending on the boundary field ϕ_Y) are as follows.

- 1-point correlator:[20]

$$\langle \phi(v) \rangle_{\phi_Y} = (E_{Y,X}\phi_Y)(v), \quad v \in X \setminus Y. \tag{48}$$

- Centered $2m$-point correlator:

$$\begin{aligned} &\langle \delta\phi(v_1) \cdots \delta\phi(v_{2m}) \rangle_{\phi_Y} \\ &= \hbar^m \sum_{\text{partitions } \{1,\ldots,2m\} = \bigcup_{i=1}^m \{a_i, b_i\}} G_{X,Y}(v_{a_1}, v_{b_1}) \cdots G_{X,Y}(v_{a_m}, v_{b_m}), \\ &\qquad v_1, \ldots, v_{2m} \in X \setminus Y. \end{aligned} \tag{49}$$

Here,

- $\delta\phi(v) := \phi(v) - \langle \phi(v) \rangle_{\phi_Y} = \phi(v) - (E_{Y,X}\phi_Y)(v)$ is the fluctuation of the field with respect to its average;

[18] In [17], this operator is called the *Poisson operator*.

[19] I.e., we think of (X, Y) as a *pair* of 1-dimensional CW complexes, where "pair" has the same meaning as in, e.g., the long exact sequence in cohomology of a pair.

[20] When specifying that a vertex v is in $V_X \setminus V_Y$ we will use a shorthand and write $v \in X \setminus Y$.

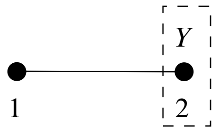


Figure 5. A graph with two vertices relative to one vertex.

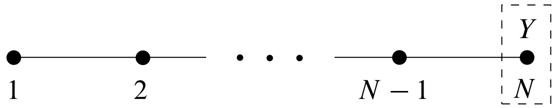


Figure 6. A line graph relative to one endpoint.

  - $G_{X,Y} := (K_{X,Y})^{-1}$ is the "propagator with Dirichlet boundary condition on Y" (or "propagator relative to Y").

- Non-centered correlators follow from (49), e.g.,

$$\langle \phi(v_1)\phi(v_2)\rangle_{\phi_Y} = \hbar\, G_{X,Y}(v_1, v_2) + (E_{Y,X}\phi_Y)(v_1) \cdot (E_{Y,X}\phi_Y)(v_2). \tag{50}$$

3.2.4. Examples.

Example 3.4. Consider the graph X shown in Figure 5, relative to the subgraph Y consisting solely of the vertex 2.

The full kinetic operator is

$$K_X = \left(\begin{array}{c|c} 1+m^2 & -1 \\ \hline -1 & 1+m^2 \end{array}\right),$$

and the relative version is its top left block, $K_{X,Y} = 1 + m^2$. The relative propagator is $G_{X,Y} = K_{X,Y}^{-1} = \frac{1}{1+m^2}$. The inverse of the full kinetic operator is

$$K_X^{-1} = \frac{1}{m^2(2+m^2)}\left(\begin{array}{c|c} 1+m^2 & 1 \\ \hline 1 & 1+m^2 \end{array}\right).$$

The DN operator is the inverse of the bottom right block: $\mathrm{DN}_{Y,X} = \frac{m^2(2+m^2)}{1+m^2}$ and the extension operator (45) is $E_{Y,X} = \frac{1}{1+m^2}$.

In particular, the relative partition function is

$$Z_{X,Y}(\phi_Y) = (1+m^2)^{-\frac{1}{2}}\, e^{-\frac{1}{2\hbar}\left(\frac{m^2(2+m^2)}{1+m^2} - \frac{m^2}{2}\right)\phi_Y^2}.$$

Example 3.5. Consider the line graph of length N relative to the subgraph consisting of the right endpoint $Y = \{N\}$ (see Figure 6).

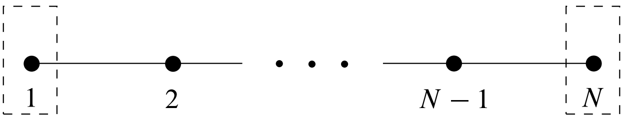


Figure 7. A line graph relative to both endpoints.

The relative propagator is

$$G_{X,Y}(i,j) = \frac{\sinh\beta(N-\frac{1}{2}-|i-j|)+\sinh\beta(N+\frac{1}{2}-i-j)}{2\sinh\beta\cosh\beta(N-\frac{1}{2})}, \quad 1\le i,\ j\le N-1,$$

with β as in (34). The DN operator is the inverse of the (N,N) block (element) of the absolute propagator (33),

$$\mathrm{DN}_{Y,X} = \frac{2\sinh\frac{\beta}{2}\sinh\beta N}{\cosh\beta(N-\frac{1}{2})}.$$

The extension operator is

$$E_{Y,X}(i,N) = \frac{\cosh\beta(i-\frac{1}{2})}{\cosh\beta(N-\frac{1}{2})}, \quad 1\le i\le N-1,$$

and the determinant is

$$\det K_{X,Y} = \frac{\cosh\beta(N-\frac{1}{2})}{\cosh\frac{\beta}{2}}. \tag{51}$$

Example 3.6. Consider again the line graph, but now relative to both left and right endpoints, see Figure 7.

Then, we have

$$G_{X,Y}(i,j) = \frac{\cosh\beta(N-1-|i-j|)-\cosh\beta(N+1-i-j)}{2\sinh\beta\sinh\beta(N-1)}, \quad 2\le i,\ j\le N-1, \tag{52}$$

$$\mathrm{DN}_{Y,X} = \frac{2\sinh\frac{\beta}{2}}{\sinh\beta(N-1)}\begin{pmatrix} \cosh\beta(N-\frac{1}{2}) & -\cosh\frac{\beta}{2} \\ -\cosh\frac{\beta}{2} & \cosh\beta(N-\frac{1}{2}) \end{pmatrix}, \tag{53}$$

$$E_{Y,X}(i,1) = \frac{\sinh\beta(N-i)}{\sinh\beta(N-1)}, \quad E_{Y,X}(i,N) = \frac{\sinh\beta(i-1)}{\sinh\beta(N-1)}, \tag{54}$$

$$\det K_{X,Y} = \frac{\sinh\beta(N-1)}{\sinh\beta}. \tag{55}$$

3.3. Gluing in Gaussian theory: Gluing of propagators and determinants

3.3.1. Cutting a closed graph. Consider a closed graph $X = X' \bigcup_Y X''$ obtained from graphs X', X'' by gluing along a common subgraph $X' \supset Y \subset X''$.

Theorem 3.7. (a) *The propagator on X is expressed in terms the data (propagators, DN operators, extension operators) on X', X'' relative to Y as follows.*

For both vertices $v_1, v_2 \in X'$,

$$G_X(v_1, v_2) = G_{X',Y}(v_1, v_2) + \sum_{u_1,u_2 \in Y} E_{Y,X'}(v_1, u_1)\,\mathrm{DN}^{-1}_{Y,X}(u_1, u_2) E_{Y,X'}(v_2, u_2). \tag{56}$$

For both vertices in X'', the formula is similar. Here, the total DN operator is

$$\mathrm{DN}_{Y,X} = \mathrm{DN}_{Y,X'} + \mathrm{DN}_{Y,X''} - K_Y. \tag{57}$$

Also, we assume by convention that $G_{X',Y}(v_1, v_2) = 0$ if either of v_1, v_2 is in Y. We also set $E_{Y,X'}(u, v) = \delta_{u,v}$ if $u, v \in Y$.

For $v_1 \in X'$, $v_2 \in X''$,

$$G_X(v_1, v_2) = \sum_{u_1,u_2 \in Y} E_{Y,X'}(v_1, u_1)\,\mathrm{DN}^{-1}_{Y,X}(u_1, u_2) E_{Y,X''}(v_2, u_2) \tag{58}$$

and similarly for $v_1 \in X''$, $v_2 \in X'$.

(b) *The determinant of K_X is expressed in terms of the data on X', X'' relative to Y as follows.*

$$\det K_X = \det(K_{X',Y}) \det(K_{X'',Y}) \det(\mathrm{DN}_{Y,X}). \tag{59}$$

We will give three proofs of these gluing formulae.

(1) From Fubini theorem for the “functional integral” (QFT/second quantization approach).

(2) From inverting a 2×2 block matrix via Schur complement and Schur’s determinant formula.

(3) From path counting (first quantization approach) – later, in Section 5.5.

3.3.2. Proof 1 (“functional integral approach”). First, consider the partition function on X relative to Y:

$$\begin{aligned} Z_{X,Y}(\phi_Y) &= \int_{F_X^{\phi_Y}} [D\phi]^{\phi_Y} e^{-\frac{1}{\hbar}(S_X(\phi) - \frac{1}{2} S_Y(\phi_Y))} \underset{(26)}{=} \\ &= \int_{F_{X'}^{\phi_Y}} [D\phi']^{\phi_Y} \int_{F_{X''}^{\phi_Y}} [D\phi'']^{\phi_Y} e^{-\frac{1}{\hbar}(S_{X'}(\phi') + S_{X''}(\phi'') - \frac{3}{2} S_Y(\phi_Y))} \\ &= (\det K_{X',Y})^{-\frac{1}{2}} (\det K_{X'',Y})^{-\frac{1}{2}} e^{-\frac{1}{2\hbar}(\phi_Y, (\mathrm{DN}_{Y,X'} + \mathrm{DN}_{Y,X''} - \frac{3}{2} K_Y)\phi_Y)}. \end{aligned} \tag{60}$$

Comparing the right-hand side with (47) as functions of $\hbar$, we obtain the formula (57) for the total DN operator and the relation for determinants

$$\det K_{X,Y} = \det K_{X',Y} \cdot \det K_{X'',Y}.$$

The partition function on X can be obtained by integrating (60) over the field on the "gluing interface" Y:

$$\begin{aligned} Z_X &= \int_{F_X} D\phi\, e^{-\frac{1}{\hbar}S_X(\phi)} = \int_{F_Y} D\phi_Y\, e^{\frac{1}{2\hbar}S_Y(\phi_Y)} Z_{X,Y}(\phi_Y) \\ &= \int_{F_Y} D\phi_Y (\det K_{X,Y})^{-\frac{1}{2}} e^{-\frac{1}{2\hbar}(\phi_Y, \mathrm{DN}_{Y,X}\phi_Y)} \\ &= (\det K_{X,Y})^{-\frac{1}{2}} (\det \mathrm{DN}_{Y,X})^{-\frac{1}{2}}. \end{aligned}$$

Comparing the right-hand side with (29), we obtain the gluing formula for determinants (59).

Next, we prove the gluing formula for propagators thinking of them as 2-point correlation functions. We denote by $\langle\!\langle \cdots \rangle\!\rangle$ correlators not normalized by the partition function. Consider the case $v_1, v_2 \in X'$. We have

$$\begin{aligned} &\underbrace{\langle\!\langle \phi(v_1)\phi(v_2) \rangle\!\rangle}_{\hbar\, G_X(v_1,v_2)\cdot Z_X} \\ &= \int_{F_X} D\phi\, e^{-\frac{1}{\hbar}S_X(\phi)} \phi(v_1)\phi(v_2) \\ &= \int_{F_Y} D\phi_Y \underbrace{\int_{F_{X'}^{\phi_Y}} [D\phi']^{\phi_Y} \phi'(v_1)\phi'(v_2) e^{-\frac{1}{\hbar}(S_{X'}(\phi') - \frac{1}{2}S_Y(\phi_Y))}}_{\langle\!\langle \phi(v_1)\phi(v_2)\rangle\!\rangle^{X'}_{\phi_Y} \underset{(50)}{=} Z_{X',Y}(\phi_Y)\cdot(\hbar\, G_{X',Y}(v_1,v_2) + (E_{Y,X'}\phi_Y)(v_1)\cdot(E_{Y,X'}\phi_Y)(v_2))} \\ &\quad \cdot \underbrace{\int_{F_{X''}^{\phi_Y}} [D\phi'']^{\phi_Y} e^{-\frac{1}{\hbar}(S_{X''}(\phi'') - \frac{1}{2}S_Y(\phi_Y))}}_{Z_{X'',Y}(\phi_Y)} \\ &= \int_{F_Y} D\phi_Y (\det K_{X',Y})^{-\frac{1}{2}} (\det K_{X'',Y})^{-\frac{1}{2}} e^{-\frac{1}{2\hbar}(\phi_Y, \mathrm{DN}_{Y,X}\phi_Y)} (\hbar\, G_{X',Y}(v_1, v_2) \\ &\quad + \sum_{u_1,u_2 \in Y} E_{Y,X'}(v_1,u_1)\phi_Y(u_1)\phi_Y(u_2)E_{Y,X'}(v_2,u_2)) \\ &= Z_X \cdot \hbar (G_{X',Y}(v_1,v_2) + \sum_{u_1,u_2\in Y} E_{Y,X'}(v_1,u_1)\, \mathrm{DN}_{Y,X}^{-1}(u_1,u_2) E_{Y,X'}(v_2,u_2)). \end{aligned}$$

This proves the gluing formula (56).

Finally, consider the case $v_1 \in X'$, $v_2 \in X''$. By a similar computation we find

$$\underbrace{\langle\langle \phi(v_1)\phi(v_2)\rangle\rangle}_{\hbar\, G_X(v_1,v_2)\cdot Z_X} = \int_{F_Y} D\phi_Y \underbrace{\int_{F_{X'}^{\phi_Y}} [D\phi']^{\phi_Y} \phi'(v_1) e^{-\frac{1}{\hbar}(S_{X'}(\phi')-\frac{1}{2}S_Y(\phi_Y))}}_{=\langle\langle \phi(v_1)\rangle\rangle_{\phi_Y}^{X'} \underset{(48)}{=} Z_{X',Y}(\phi_Y)\cdot(E_{Y,X'}\phi_Y)(v_1)}$$

$$\cdot \underbrace{\int_{F_{X''}^{\phi_Y}} [D\phi'']^{\phi_Y} \phi''(v_2) e^{-\frac{1}{\hbar}(S_{X''}(\phi'')-\frac{1}{2}S_Y(\phi_Y))}}_{=\langle\langle \phi(v_2)\rangle\rangle_{\phi_Y}^{X''} = Z_{X'',Y}(\phi_Y)\cdot(E_{Y,X''}\phi_Y)(v_2)}$$

$$= \int_{F_Y} D\phi_Y (\det K_{X',Y})^{-\frac{1}{2}} (\det K_{X'',Y})^{-\frac{1}{2}} e^{-\frac{1}{2\hbar}(\phi_Y, \mathrm{DN}_{Y,X}\phi_Y)}$$

$$\cdot \sum_{u_1,u_2\in Y} E_{Y,X'}(v_1,u_1)\phi_Y(u_1)\phi_Y(u_2)E_{Y,X''}(v_2,u_2)$$

$$= Z_X \cdot \hbar \sum_{u_1,u_2\in Y} E_{Y,X'}(v_1,u_1)\, \mathrm{DN}_{Y,X}^{-1}(u_1,u_2) E_{Y,X''}(v_2,u_2).$$

This proves (58).

3.3.3. Proof 2 (Schur complement approach). Let us introduce the notations

$$\bar{G}_{X,Y} = \left(\begin{array}{c|c} G_{X,Y} & 0 \\ \hline 0 & 0 \end{array}\right), \quad \bar{E}_{Y,X} = \begin{pmatrix} E_{Y,X} \\ \mathrm{id} \end{pmatrix} \tag{61}$$

for the extension of the propagator on X relative to Y by zero to vertices of Y and the extension of the extension operator by identity to vertices of Y (the blocks correspond to vertices of $X \setminus Y$ and vertices of Y, respectively)[21]. Using these notations, gluing formulae (56) and (58) for the propagator can be jointly expressed as

$$G_X \overset{!}{=} \bar{G}_{X,Y} + \bar{E}_{Y,X}\, \mathrm{DN}_{Y,X}^{-1}\, \bar{E}_{Y,X}^T. \tag{62}$$

The right-hand side here is

$$\left(\begin{array}{c|c} G_{X,Y} + E\ \mathrm{DN}^{-1} E^T & E\ \mathrm{DN}^{-1} \\ \hline \mathrm{DN}^{-1} E^T & \mathrm{DN}^{-1} \end{array}\right) = \left(\begin{array}{c|c} \hat{A}^{-1} + BD^{-1}DD^{-1}C & BD^{-1}D \\ \hline DD^{-1}C & D \end{array}\right)$$
$$= \left(\begin{array}{c|c} \hat{A}^{-1} + BD^{-1}C & B \\ \hline C & D \end{array}\right).$$

[21] Note that one can further refine the block decompositions (61) according to partitioning of vertices in $X \setminus Y$ into those in $X' \setminus Y$ and those in $X'' \setminus Y$. Then, the block $G_{X,Y}$ becomes $\begin{pmatrix} G_{X',Y} & 0 \\ 0 & G_{X'',Y} \end{pmatrix}$ and the block $E_{Y,X}$ becomes $\begin{pmatrix} E_{Y,X'} \\ E_{Y,X''} \end{pmatrix}$.

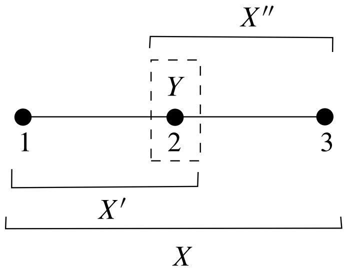


Figure 8. Gluing two line graphs into a longer line graph.

Here, we are suppressing the subscript Y, X for E and DN; notations for the blocks are as in (42) and (46). So, the only part to check is that the $(1,1)$ block above is A. It is a consequence of the inversion formula for 2×2 block matrices, which, in particular, asserts that the $(1,1)$ block $\hat{A}$ of the matrix K_X inverse to G_X is the inverse of the Schur complement of the $(2,2)$ block in G_X, i.e.,

$$\hat{A}^{-1} = A - BD^{-1}C.$$

This finishes the proof of the gluing formula for propagators (62).

Schur's formula for a determinant of a block 2×2 matrix applied to (42) yields

$$\det K_X^{-1} = \det \underbrace{D}_{\mathrm{DN}^{-1}} \cdot \det(\underbrace{A - BD^{-1}C}_{\hat{A}^{-1} = K_{X,Y}^{-1}}),$$

and thus,

$$\det K_X = \det \mathrm{DN} \cdot \det K_{X,Y} = \det \mathrm{DN} \cdot \det K_{X',Y} \cdot \det K_{X'',Y}.$$

In the last equality, we used the fact that $K_{X,Y}$ is block-diagonal, with blocks corresponding to $X' \setminus Y$ and $X'' \setminus Y$. This proves the gluing formula for determinants.

3.3.4. Examples.

Example 3.8. Consider the gluing of two line graphs of length 2, X', X'' over a common vertex Y into a line graph X of length 3 as pictured in Figure 8.

The data of the constituent graphs X', X'' relative to Y was computed in Example 3.4. We assemble the data on the glued graph X using the gluing formulae of Theorem 3.7. We have

$$\begin{aligned}\underbrace{\mathrm{DN}_{Y,X}}_{=\mathrm{DN}_{Y,X}(2,2)} &= \mathrm{DN}_{Y,X'} + \mathrm{DN}_{Y,X''} - K_Y \\ &= \frac{m^2(2+m^2)}{1+m^2} + \frac{m^2(2+m^2)}{1+m^2} - m^2 = \frac{m^2(3+m^2)}{1+m^2}.\end{aligned}$$

For the propagator we have, e.g.,

$$\begin{aligned} G_X(1,1) &= G_{X',Y}(1,1) + E_{Y,X'}(1,2)\, \mathrm{DN}_{Y,X}^{-1}(2,2) E_{Y,X'}(1,2) \\ &= \frac{1}{1+m^2} + \frac{1}{1+m^2} \cdot \frac{1+m^2}{m^2(3+m^2)} \cdot \frac{1}{1+m^2} \end{aligned}$$

and

$$G_X(1,3) = E_{Y,X'}(1,2)\, \mathrm{DN}_{Y,X}^{-1}(2,2) E_{Y,X''}(3,2) = \frac{1}{1+m^2} \cdot \frac{1+m^2}{m^2(3+m^2)} \cdot \frac{1}{1+m^2},$$

which agrees with the 1-1 entry and 1–3 entry in (31), respectively.

For the gluing of determinants, we have

$$\det K_{X',Y} \cdot \det K_{X'',Y} \cdot \det \mathrm{DN}_{Y,X} = (1+m^2)\cdot(1+m^2)\cdot \frac{m^2(3+m^2)}{1+m^2},$$

which agrees with (30).

Example 3.9. Consider the circle graph X with N vertices presented as a gluing by the two endpoints of two line graphs X', X'' of lengths N', N'', respectively, with $N = N' + N'' - 2$, see Figure 9.

One can then use the gluing formulae of Theorem 3.7 to recover the propagator and the determinant on the circle graph (cf., Example 3.3) from the data for line graphs relative to the endpoints (cf., Example 3.6). E.g., for the determinant, we have

$$\begin{aligned} &\underbrace{\frac{\sinh \beta(N'-1)}{\sinh \beta}}_{\det K_{X',Y}} \cdot \underbrace{\frac{\sinh \beta(N''-1)}{\sinh \beta}}_{\det K_{X'',Y}} \cdot \det\left(\mathrm{DN}_{Y,X'} + \mathrm{DN}_{Y,X''} - \begin{pmatrix} m^2 & 0 \\ 0 & m^2 \end{pmatrix}\right) \\ &= \underbrace{4\sinh^2 \frac{\beta N}{2}}_{\det K_X}. \end{aligned}$$

Here, the 2×2 matrices $\mathrm{DN}_{Y,X'}$, $\mathrm{DN}_{Y,X''}$ are given by (53), with N replaced by N', N'', respectively.

3.3.5. General cutting/gluing of cobordisms. Consider the gluing of graph cobordisms (22),

$$Y_1 \xrightarrow{X'} Y_2 \xrightarrow{X''} Y_3 = Y_1 \xrightarrow{X} Y_3.$$

Let us introduce the following shorthand notations.

- DN operators: $\mathrm{DN}_{ij,A} := \mathrm{DN}_{Y_i \sqcup Y_j, A}$, with $i, j \in \{1,2,3\}$ and $A \in \{X', X'', X\}$. Also, by $(\mathrm{DN}_{ij,A})_{kl}$ we will denote (Y_k, Y_l) block in $\mathrm{DN}_{ij,A}$.
- "Interface" DN operator: $\mathrm{DN}_{\mathrm{int}} := (\mathrm{DN}_{Y_1\sqcup Y_2 \sqcup Y_3, X})_{(Y_2,Y_2)\,\text{block}}$.

- Extension operators: $E_{ij,A} = E_{Y_i \sqcup Y_j, A}$. We will also denote its (A, Y_k) block by $(E_{ij,A})_k$.
- Propagators: $G_{A,ij} := G_{A, Y_i \sqcup Y_j}$.

One has the following straightforward generalization of Theorem 3.7 to the case of possibly nonempty Y_1, Y_3.

Theorem 3.10. *The data of the Gaussian theory on the glued cobordism* $Y_1 \xrightarrow{X} Y_3$ *can be computed from the data of the constituent cobordisms* $Y_1 \xrightarrow{X'} Y_2$, $Y_2 \xrightarrow{X''} Y_3$ *as follows.*

(a) *Glued DN operator* $\mathrm{DN}_{13,X}$:

$$\begin{pmatrix} \begin{matrix} (\mathrm{DN}_{12,X'})_{11} \\ -(\mathrm{DN}_{12,X'})_{12}\,\mathrm{DN}_{\mathrm{int}}^{-1}\,(\mathrm{DN}_{12,X'})_{21} \end{matrix} & -(\mathrm{DN}_{12,X'})_{12}\,\mathrm{DN}_{\mathrm{int}}^{-1}\,(\mathrm{DN}_{23,X''})_{23} \\ -(\mathrm{DN}_{23,X''})_{32}\,\mathrm{DN}_{\mathrm{int}}^{-1}\,(\mathrm{DN}_{12,X'})_{21} & \begin{matrix} (\mathrm{DN}_{23,X''})_{33} \\ -(\mathrm{DN}_{23,X''})_{32}\,\mathrm{DN}_{\mathrm{int}}^{-1}\,(\mathrm{DN}_{23,X''})_{23} \end{matrix} \end{pmatrix}. \tag{63}$$

The blocks correspond to vertices of Y_1 *and* Y_3. *The interface DN operator here is*

$$\mathrm{DN}_{\mathrm{int}} = (\mathrm{DN}_{12,X'})_{22} + (\mathrm{DN}_{23,X''})_{22} - K_{Y_2}. \tag{64}$$

(b) *Extension operator* $E_{13,X}$:

$$\begin{pmatrix} \begin{matrix} (E_{12,X'})_1 - (E_{12,X'})_2\,\mathrm{DN}_{\mathrm{int}}^{-1}\,(\mathrm{DN}_{12,X'})_{21} \\ -\mathrm{DN}_{\mathrm{int}}^{-1}\,(\mathrm{DN}_{12,X'})_{21} \end{matrix} & \begin{matrix} -(E_{12,X'})_2\,\mathrm{DN}_{\mathrm{int}}^{-1}\,(\mathrm{DN}_{23,X''})_{23} \\ -\mathrm{DN}_{\mathrm{int}}^{-1}\,(\mathrm{DN}_{23,X''})_{23} \end{matrix} \\ -(E_{23,X''})_2\,\mathrm{DN}_{\mathrm{int}}^{-1}\,(\mathrm{DN}_{12,X'})_{21} & \begin{matrix} (E_{23,X''})_3 \\ -(E_{23,X''})_2\,\mathrm{DN}_{\mathrm{int}}^{-1}\,(\mathrm{DN}_{23,X''})_{23} \end{matrix} \end{pmatrix}. \tag{65}$$

Here, horizontally, the blocks correspond to vertices of Y_1, Y_3*; vertically – to vertices of* $X' \setminus (Y_1 \sqcup Y_2)$, Y_2 *and* $X'' \setminus (Y_2 \sqcup Y_3)$.

(c) *Determinant*:

$$\det K_{X, Y_1 \sqcup Y_3} = \det K_{X', Y_1 \sqcup Y_2} \cdot \det K_{X'', Y_2 \sqcup Y_3} \cdot \det \mathrm{DN}_{\mathrm{int}}. \tag{66}$$

Propagator:

- *For* $v_1, v_2 \in X'$,

$$\begin{aligned} G_{X,13}(v_1, v_2) &= G_{X',12}(v_1, v_2) \\ &\quad + \sum_{u_1, u_2 \in Y} E_{12,X'}(v_1, u_1)\,\mathrm{DN}_{\mathrm{int}}^{-1}(u_1, u_2) E_{12,X'}(v_2, u_2) \end{aligned} \tag{67}$$

and similarly for $v_1, v_2 \in X''$.

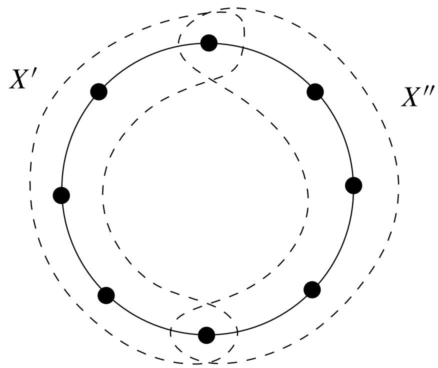


Figure 9. Gluing a circle from two intervals.

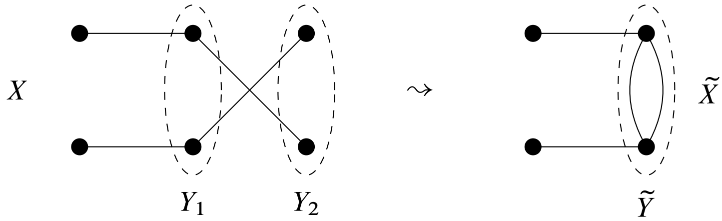


Figure 10. An example of self-gluing.

(d) *For* $v_1 \in X'$, $v_2 \in X''$:

$$G_{X,13}(v_1, v_2) = \sum_{u_1,u_2 \in Y} E_{12,X'}(v_1, u_1)\,\mathrm{DN}_{\mathrm{int}}^{-1}(u_1, u_2) E_{23,X''}(v_2, u_2) \tag{68}$$

and similarly for $v_1 \in X''$, $v_2 \in X'$.

3.3.6. Self-gluing and trace formula. As another generalization of Theorem 3.7, one can consider the case of a graph X relative to a subgraph Y that admits a decomposition $Y = Y_1 \sqcup Y_2$, where Y_1 and Y_2 are isomorphic graphs. Then, specifying a graph isomorphism $f: Y_1 \to Y_2$, we can glue Y_1 to Y_2 using f to form a new graph $\widetilde{X}$ with a distinguished subgraph $\widetilde{Y}$[22]. We have $\widetilde{Y} \cong Y_1 \cong Y_2$ if and only if there are no edges between Y_1 and Y_2. See Figure 10.

Then, one has the following relation between the Dirichlet-to-Neumann operators of Y relative to X and $\widetilde{Y}$ relative to $\widetilde{X}$.

[22]In the setting of Theorem 3.7, we have $X = X' \sqcup X''$, and there are no edges between X' and X''. In the following discussion we will suppress f but remark that in principle the glued graphs $\widetilde{X}$ and $\widetilde{Y}$ do depend on f.

Proposition 3.11. *Let $\phi \in C^0(Y_1) \simeq C^0(Y_2) \simeq C^0(\widetilde{Y})$, then*[23]

$$\left((\phi,\phi), \left(\mathrm{DN}_{Y,X} - \frac{1}{2}K_Y\right)\begin{pmatrix}\phi\\ \phi\end{pmatrix}\right) = (\phi, \mathrm{DN}_{\widetilde{Y},\widetilde{X}}\,\phi) - \left(\phi, \left(\frac{1}{2}K_{\widetilde{Y}} - \frac{1}{2}K_{Y_1}\right)\phi\right). \tag{69}$$

Equivalently,

$$\left((\phi,\phi), \left(\mathrm{DN}_{Y,X} - \frac{1}{2}K_{Y_1} - \frac{1}{2}K_{Y_2}\right)\begin{pmatrix}\phi\\ \phi\end{pmatrix}\right) = (\phi, \mathrm{DN}_{\widetilde{Y},\widetilde{X}}\,\phi). \tag{70}$$

Proof. We have

$$S_X = S_{X,Y} + S_{Y_1} + S_{Y_2} + S_{Y_1,Y_2},$$

where the first term contains contributions to the action from vertices in $X \setminus Y$ and edges with at least one vertex in $X \setminus Y$, while the last term contains just contributions from edges between Y_1 and Y_2. Evaluating on the subspace of fields $F_X^{(\phi,\phi)}$ that agree on Y_1 and Y_2, we get

$$S_X|_{F_X^{(\phi,\phi)}} = S_{X,Y} + 2S_{Y_1}(\phi) + S_{Y_1,Y_2}(\phi)$$

and

$$S_X - \frac{1}{2}S_Y = S_{X,Y} + S_{Y_1} + \frac{1}{2}S_{Y_1,Y_2}.$$

On the other hand, we have

$$\begin{aligned} S_{\widetilde{X}} - \frac{1}{2}S_{\widetilde{Y}}|_{F^{\phi}_{\widetilde{X}}} &= S_{\widetilde{X},\widetilde{Y}} + \frac{1}{2}S_{\widetilde{Y}}(\phi) = S_{X,Y} + \frac{1}{2}S_{Y_1}(\phi) + \frac{1}{2}S_{Y_1,Y_2}(\phi) \\ &= S_X - \frac{1}{2}S_Y(\phi) - \frac{1}{2}S_{Y_1}. \end{aligned}$$

Therefore,

$$Z_{X,Y}((\phi,\phi)) = Z_{\widetilde{X},\widetilde{Y}}(\phi)e^{-\frac{1}{2\hbar}S_{Y_1}(\phi)}.$$

Noticing that the relative operators agree with

$$K_{X,Y} = K_{\widetilde{X},\widetilde{Y}}$$

and using (47), we obtain (69). To see (70), notice that the difference

$$K_Y - K_{Y_1} - K_{Y_2} = K_{Y_1,Y_2} = K_{\widetilde{Y}} - K_{Y_1},$$

so adding $\frac{1}{2}K_{Y_1,Y_2}$ to (69), we obtain (70). ∎

[23] Below we are identifying using f to identify $V(Y_1)$ and $V(Y_2)$, and then also ϕ and $(f^{-1})^*\phi$.

Corollary 3.12. *We have the following trace formula:*

$$Z_{\widetilde{X}} = \int_{F_{Y_1}} [D\phi]\langle\phi|Z_{X,Y_1,Y_2}|\phi\rangle. \tag{71}$$

Proof. We have

$$\begin{aligned}\langle\phi|Z_{X,Y_1,Y_2}|\phi\rangle &= \det K_{X,Y}^{-\frac{1}{2}} e^{-\frac{1}{2\hbar}((\phi,\phi),(\mathrm{DN}_{Y,X}-\frac{1}{2}K_{Y_1}-\frac{1}{2}K_{Y_2})(\phi,\phi))} \\ &= \det K_{\widetilde{X},\widetilde{Y}}^{-\frac{1}{2}} e^{-\frac{1}{2\hbar}((\phi,\mathrm{DN}_{\widetilde{Y},\widetilde{X}})\phi)}.\end{aligned}$$

Integrating over ϕ, we obtain the result. ■

Example 3.13 (Gluing a circle graph from a line graph). For the line graph L_3 relative to both endpoints,

$$DN_{Y,X} = \frac{m^2+3}{m^2+2}\begin{pmatrix} m^2+1 & -1 \\ -1 & m^2+1\end{pmatrix}.$$

In this case, we have $K_{Y_1} = K_{Y_2} = m^2$ and

$$\begin{pmatrix}1 & 1\end{pmatrix}\frac{m^2+3}{m^2+2}\begin{pmatrix} m^2+1 & -1 \\ -1 & m^2+1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \frac{2m^2(m^2+3)}{m^2+2},$$

which implies

$$\frac{2m^2(m^2+3)}{m^2+2} - \underbrace{m^2}_{-\frac{1}{2}K_{Y_1}-\frac{1}{2}K_{Y_2}} = \frac{m^2(m^2+4)}{m^2+2}.$$

Here, $m^2 = K_{Y_1}$. On the other hand, $\widetilde{X} = C_2$ is a circle graph with $\widetilde{Y}$ a point, and we have

$$K_X = \begin{pmatrix} m^2+2 & -2 \\ -2 & m^2+2\end{pmatrix}, \quad G_X = \frac{1}{m^2(m^2+4)}\begin{pmatrix} m^2+2 & 2 \\ 2 & m^2+2\end{pmatrix},$$

therefore the corresponding Dirichlet-to-Neumann operator is

$$\mathrm{DN}_{\widetilde{Y},\widetilde{X}} = \frac{m^2(m^2+4)}{m^2+2},$$

as predicted by Proposition 3.11. The relative determinant $K_{Y,X}$ is m^2+2 so that the trace formula becomes

$$\int_{F_{Y_1}} [D\phi]\langle\phi|Z_{X,Y_1,Y_2}|\phi\rangle = (m^2+2)^{-\frac{1}{2}}\left(\frac{m^2(m^2+4)}{m^2+2}\right)^{-\frac{1}{2}} = (m^2(m^2+4))^{-\frac{1}{2}} = Z_{C_2}.$$

Similarly, for the line graph of length N relative to both endpoints, the Dirichlet-to-Neumann operator is given by (53) and we have

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathrm{DN}_{Y,X} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{4\sinh\frac{\beta}{2}(\cosh\beta(N-\frac{1}{2}) - \cosh\frac{\beta}{2})}{\sinh\beta(N-1)}.$$

On the other hand, the Dirichlet-to-Neumann operator of $\widetilde{X} = C_{N-1}$ relative to a single vertex is

$$\mathrm{DN}_{\widetilde{Y},\widetilde{X}} = 2\sinh\beta\tanh\beta\frac{N-1}{2}.$$

Then, one can check that

$$\frac{4\sinh\frac{\beta}{2}(\cosh\beta(N-\frac{1}{2}) - \cosh\frac{\beta}{2})}{\sinh\beta(N-1)} - m^2 = 2\sinh\beta\tanh\beta\frac{N-1}{2}.$$

Remark 3.14. There is of course also a common generalization of Theorem 3.10 and Proposition 3.11, where we have several boundary components and are allowed to sew any two isomorphic components together, we leave this statement to the imagination of the reader.

3.4. Comparison to continuum formulation

In this subsection, we compare of results of Sections 3.2 and 3.3 to the continuum counterparts for a free scalar theory on a Riemannian manifold. For details on the latter, we refer to [11].

Consider the free scalar theory on a closed Riemannian manifold M defined by the action

$$S(\phi) = \int_M \frac{1}{2} d\phi \wedge * d\phi + \frac{m^2}{2}\phi^2 d\,\mathrm{vol} = \int_M \frac{1}{2}\phi(\Delta + m^2)\phi\, d\,\mathrm{vol},$$

where $\phi \in C^\infty(M)$ is the scalar field, $m > 0$ is the mass, $*$ is the Hodge star associated with the metric, d vol is the metric volume form and Δ is the metric Laplacian.

The partition function is defined to be

$$Z = \text{``}\int D\phi\, e^{-\frac{1}{\hbar}S(\phi)}\text{''} := \left(\det{}^{\zeta}(\Delta + m^2)\right)^{-\frac{1}{2}},$$

where $\det^\zeta$ stands for the functional determinant understood in the sense of zeta-regularization. Correlators are given by Wick's lemma in terms of the Green's function $G(x, y) \in C^\infty(M \times M \setminus \mathrm{Diag})$ of the operator $\Delta + m^2$.

Next, if M is a compact Riemannian manifold with boundary ∂M, one can impose Dirichlet boundary condition $\phi|_{\partial M} = \phi_\partial$ – a fixed function on ∂M (thus, fluctuations of fields are zero on the boundary). The unique solution of the Dirichlet boundary value problem on M,

$$(\Delta + m^2)\phi = 0, \quad \phi|_\partial = \phi_\partial,$$

can be written as

$$\phi(x) = \int_{\partial M} \partial^n_y G_D(x, y)\, \phi_\partial(y) d\, \mathrm{vol}^\partial_y\,. \tag{72}$$

Here,

- $d\,\mathrm{vol}^\partial$ is the Riemannian volume form on ∂M (with respect to the induced metric from the bulk).
- $G_D \in C^\infty(M \times M \setminus \mathrm{Diag})$ is the Green's function for the operator $\Delta + m^2$ with Dirichlet boundary condition.
- ∂^n stands for the normal derivative at the boundary. In particular, for $x \in M$, $y \in \partial M$,

$$\partial^n_y G_D(x, y) = \frac{\partial}{\partial t}\Big|_{t=0} G_D(x, \tilde{y}_t), \tag{73}$$

  where $\tilde{y}_t$, $t \geq 0$ is a curve in M starting at $\tilde{y}_0 = y$ with initial velocity being the inward unit normal to the boundary.

  Then, on a manifold with boundary one has the partition function

$$\begin{aligned} Z(\phi_\partial) &= \text{“}\int_{\phi|_\partial = \phi_\partial} D\phi\, e^{-\frac{1}{\hbar}S(\phi)}\text{”} \\ &= \left(\det{}^\zeta_D(\Delta + m^2)\right)^{-\frac{1}{2}} e^{-\frac{1}{2\hbar}\int_{\partial M} \phi_\partial\, \mathrm{DN}(\phi_\partial) d\,\mathrm{vol}^\partial} \\ &= \left(\det{}^\zeta_D(\Delta + m^2)\right)^{-\frac{1}{2}} e^{\frac{1}{2\hbar}\int_{\partial M \times \partial M} \phi_\partial(x)\, \partial^n_x \partial^n_y G_D(x,y)\, \phi_\partial(y) d\,\mathrm{vol}^\partial_x\, d\,\mathrm{vol}^\partial_y}. \end{aligned} \tag{74}$$

Here, in the determinant in the right-hand side, $\Delta + m^2$ is understood as acting on smooth functions on M vanishing on ∂M (which we indicate by the subscript D for “Dirichlet boundary condition”); DN: $C^\infty(\partial M) \to C^\infty(\partial M)$ is the Dirichlet-to-Neumann operator (see footnote 16). The integral kernel of the DN operator is $-\partial^n_x \partial^n_y G_D(x, y)$.

The integral in the exponential in the last line of equation (74) contains a non-integrable singularity on the diagonal and has to be appropriately regularized, cf., [11, Remark 3.4].

Correlators on a manifold with boundary are the following.

- One-point correlator:

$$\langle \phi(x) \rangle_{\phi_\partial} = \int_{\partial M} \partial^n_y G_D(x, y) \phi_\partial(y) d\, \mathrm{vol}^\partial_y\,.$$

Scalar theory on a graph X relative to subgraph Y	Scalar theory on a Riemannian manifold M with boundary ∂M
propagator $G_{X,Y}$	$G_D(x,y)$ with $x,y\in M$
extension operator $E_{Y,X}$	$\partial_y^n G_D(x,y)$ with $x\in M$, $y\in\partial M$
DN operator $\mathrm{DN}_{Y,X}$	DN or $-\partial_x^n\partial_y^n G_D(x,y)$ with $x,y\in\partial M$
$\det K_{X,Y}$	$\det_D^\zeta(\Delta+m^2)$

Table 2. Comparing the toy model and continuum theory, continued.

- Centered two-point correlator:
$$\langle\delta\phi(x)\delta\phi(y)\rangle_{\phi_\partial}=\hbar\, G_D(x,y),$$
where
$$\delta\phi(x){:}=\phi(x)-\langle\phi(x)\rangle_{\phi_\partial}.$$
- k-point centered correlators are given by Wick's lemma.

When more detailed notations of the manifolds involved is needed, instead of G_D we will write $G_{M,\partial M}$ (and similarly for $\det_D^\zeta$) and instead of DN we will write $\mathrm{DN}_{\partial M,M}$.

Continuing the dictionary of Remark 2.5 to free scalar theory on graphs versus Riemannian manifolds (see Table 2).

Here, in the right column we are writing the integral kernels of operators instead of operators themselves.

3.4.1. Gluing formulae for Green's functions and determinants. Assume that M is a closed Riemannian manifold cut be a closed codimension 1 submanifold γ into two pieces, M' and M''. Then, one can recover the Green's function for the operator $\Delta+m^2$ on M from Green's functions on M' and M'', both with Dirichlet condition on γ, as follows[24].

- For $x,y\in M'$, one has
$$G_M(x,y)=G_{M',\gamma}(x,y)\\ +\int_\gamma d\,\mathrm{vol}_w^\gamma\int_\gamma d\,\mathrm{vol}_z^\gamma\;\partial_w^n G_{M',\gamma}(x,w)\varkappa(w,z)\partial_z^n G_{M',\gamma}(z,y). \tag{75}$$

[24] This result is originally due to [3], see also [11].

Here, $\varkappa(w,z)$ is the integral kernel of the inverse of the total (or "interface") DN operator

$$\mathrm{DN}_{\mathrm{int}} := \mathrm{DN}_{\gamma,M'} + \mathrm{DN}_{\gamma,M''}\,. \tag{76}$$

For $x, y \in M''$, one has a similar formula to (75).

- For $x \in M'$, $y \in M''$, one has

$$G_M(x,y) = \int_\gamma d\,\mathrm{vol}^\gamma_w \int_\gamma d\,\mathrm{vol}^\gamma_z\ \partial^n_w G_{M',\gamma}(x,w)\varkappa(w,z)\partial^n_z G_{M'',\gamma}(z,y), \tag{77}$$

and similarly if $x \in M''$, $y \in M'$.

In the case $\dim M = 2$, the zeta-regularized determinants satisfy a remarkable Mayer–Vietoris type gluing formula due to Burghelea–Friedlander–Kappeler [2],

$$\det{}^\zeta_M(\Delta + m^2) = \det{}^\zeta_{M',\gamma}(\Delta + m^2)\det{}^\zeta_{M'',\gamma}(\Delta + m^2)\det{}^\zeta_\gamma(\mathrm{DN}_{\mathrm{int}}). \tag{78}$$

This formula also holds for higher even dimensions provided that the metric near the cut γ is of warped product type (this is a result of Lee [12]). In odd dimensions, under a similar assumption, the formula is known to hold up to a multiplicative constant known explicitly in terms of the metric on the cut.

Note that formulae (75), (77) have the exact same structure as formulae (56), (58) for gluing of graph propagators[25]. Likewise, the gluing formulae for determinants in the continuum setting (78) and in graph setting (59) have the same structure.

One can also allow the manifold M to have extra boundary components disjoint from the cut, i.e., to consider M as a composition of two cobordisms $\gamma_1 \xrightarrow{M'} \gamma_2$, $\gamma_2 \xrightarrow{M''} \gamma_3$. One then has the corresponding gluing formulae which have the same structure as the formulae of Theorem 3.10. In particular, one has a gluing formula for continuum DN operators (see [15]) similar to the formula (63) in the graph setting.

3.4.2. Example: Continuum limit of line and circle graphs. The action of the continuum theory on an interval $[0, L]$ evaluated on a smooth field $\phi \in C^\infty([0, L])$ can be seen as a limit of Riemann sums

$$S(\phi) = \lim_{N\to\infty} \sum_{i=2}^N \frac{\varepsilon_N}{2}\left(\frac{\phi(i\varepsilon_N) - \phi((i-1)\varepsilon_N)}{\varepsilon_N}\right)^2 + \sum_{i=1}^N \varepsilon_N \frac{m^2}{2}\phi(i\varepsilon_N)^2,$$

where in the right-hand side we denoted $\varepsilon_N = L/N$. The right-hand side can be seen as the action of the graph theory on a line graph with $N = L/\varepsilon$ vertices, where the

[25] One small remark is that the continuum formula for the interface DN operator (76) is similar to (57), except for the $-K_Y$ term in the left-hand side which is specific to the graph setting and disappears in the continuum limit.

mass is scaled as $m \mapsto \varepsilon m$ and then the kinetic operator is scaled as $K \mapsto \varepsilon^{-1} K$ (and thus the propagator scales as $G \to \varepsilon G$), where we consider the limit $\varepsilon \to 0$ (we are approximating the interval by a portion of a 1d lattice and taking the lattice spacing to zero).

Applying the scaling above to the formulae of Example 3.6, we obtain the following for the propagator (52):

$$G_{\text{graph}}(x, y) \underset{\varepsilon \to 0}{\sim} \frac{\cosh m(L - |x - y|) - \cosh m(L - x - y)}{\sinh mL},$$

where we denoted $x = i\varepsilon$, $y = j\varepsilon$ – we think of i, j as scaling with ε so that x, y remain fixed. The right-hand side above is the Green's function for the operator $\Delta + m^2$ on an interval $[0, L]$ with Dirichlet boundary conditions at the endpoints[26]. For the DN operator (53), we obtain

$$\text{DN}_{\text{graph}} \underset{\varepsilon \to 0}{\sim} \frac{m}{\sinh mL} \begin{pmatrix} \cosh mL & -1 \\ -1 & \cosh mL \end{pmatrix}.$$

The right-hand side is the correct DN operator of the continuum theory on the interval.

For the determinant (55), we have

$$\det K_{\text{graph}} \underset{\varepsilon \to 0}{\sim} \varepsilon^{-N} \frac{\sinh mL}{m}.$$

For comparison, the zeta-regularized determinant on the interval is

$$\det{}^{\zeta}_{D}(\Delta + m^2) = \frac{2 \sinh mL}{m}.$$

It differs from the graph result by a scaling factor ε^N and an extra factor 2 which exhibits a discrepancy between the two regularizations of the functional determinant – lattice vs. zeta regularization.

Remark 3.15. One can similarly consider the continuum limit for the line graph of Example 3.2, without Dirichlet condition at the endpoints. Its continuum counterpart is the theory on an interval $[0, L]$ with *Neumann* boundary conditions at the endpoints, cf., Appendix A. Likewise, in the continuum limit for line graphs relative to one endpoint (Example 3.5), one recovers the continuum theory with Dirichlet condition at one endpoint and Neumann condition at the other.

For example, the zeta-determinant for Neumann condition at both ends is $\det{}^{\zeta}_{N-N}(\Delta + m^2) = 2m \sinh mL$. For Dirichlet condition at one end and Neumann at the other hand, one has

$$\det{}^{\zeta}_{D-N}(\Delta + m^2) = 2 \cosh mL.$$

[26]For the formulae pertaining to the continuum theory on an interval, see, e.g., [11, Appendix A.1].

These formulae are related to the continuum limit of the discrete counterparts ((35) for Neumann-Neumann and (51) for Neumann–Dirichlet boundary conditions) in the same way as in Dirichlet-Dirichlet case (by scaling with ε^N and an extra factor of 2).

In the same vein, we can consider a circle of length L as a limit of circle graphs (Example 3.3) with spacing ε. Then, in the scaling limit, from (36), we have

$$G_{\text{graph}}(x,y) \underset{\varepsilon\to 0}{\sim} \frac{\cosh m(\frac{L}{2}-|x-y|)}{2m\sinh\frac{mL}{2}},$$

where the right-hand side coincides with the continuum Green's function on a circle. For the determinant (37), we have

$$\det K_{\text{graph}} \underset{\varepsilon\to 0}{\sim} \varepsilon^{-N} 4\sinh^2\frac{mL}{2}. \tag{79}$$

For comparison, the corresponding zeta-regularized functional determinant is

$$\det{}^{\zeta}(\Delta+m^2) = 4\sinh^2\frac{mL}{2},$$

which coincides with the right-hand side of (79) up to the scaling factor ε^N.

4. Interacting theory via Feynman diagrams

Consider scalar field theory on a closed graph X defined by the action (17) – the perturbation of the Gaussian theory by an interaction potential $p(\phi)$:

$$S_X(\phi) = \underbrace{\frac{1}{2}(\phi, K_X\phi)}_{S_X^0(\phi)} + \underbrace{\langle\mu, p(\phi)\rangle}_{S_X^{\text{int}}(\phi)}.$$

We recall from Section 2 that $p(\phi)$ is a fixed polynomial or, more generally, a real analytic function. The assumptions on $p(\phi)$ are that

$$\frac{m^2}{2}\phi^2 + p(\phi)$$

has a unique absolute minimum at $\phi = 0$ and that it grows sufficiently fast at $\phi \to \pm\infty$, namely, we require the integral

$$\int_{\mathbb{R}} d\phi\, e^{-\frac{1}{\hbar}(\frac{m^2}{2}\phi^2+p(\phi))}$$

to converge for any $\hbar > 0$.

The partition function (19) can be computed by perturbation theory – the Laplace method for the $\hbar \to 0$ asymptotics of the integral, with corrections given by Feynman diagrams (see, e.g., [8]):

$$Z_X = \int_{F_X} D\phi\, e^{-\frac{1}{\hbar}(S_X^0(\phi)+S_X^{\mathrm{int}}(\phi))} = \langle\!\langle e^{-\frac{1}{\hbar}S_X^{\mathrm{int}}(\phi)} \rangle\!\rangle^0$$
$$= \Big\langle\!\!\Big\langle \sum_{n\geq 0} \frac{(-1)^n}{\hbar^n n!} \sum_{v_1,\ldots,v_n\in V_X} p(\phi(v_1))\cdots p(\phi(v_n)) \Big\rangle\!\!\Big\rangle^0 \underset{\hbar\to 0}{\sim} \det(K_X)^{-\frac{1}{2}} \sum_\Gamma \frac{\hbar^{-\chi(\Gamma)}}{|\operatorname{Aut}(\Gamma)|}\Phi_{\Gamma,X}. \tag{80}$$

Here,

- $\langle\!\langle \cdots \rangle\!\rangle^0$ stands for the non-normalized correlator in the Gaussian theory.
- The sum in the right-hand side is over finite (not necessarily connected) graphs Γ (Feynman graphs) with vertices of valence ≥ 3; $\chi(\Gamma)\leq 0$ is the Euler characteristic of a graph; $|\operatorname{Aut}(\Gamma)|$ – the order of the automorphism group of the graph.
- The weight of the Feynman graph is[27]

$$\Phi_{\Gamma,X} = \sum_{f:V_\Gamma\to V_X} \prod_{\mathsf{v}\in V_\Gamma} (-p_{\mathrm{val}(\mathsf{v})}) \cdot \prod_{(\mathsf{u},\mathsf{v})\in E_\Gamma} G_X(f(\mathsf{u}), f(\mathsf{v})), \tag{81}$$

  where p_k are the coefficients of the potential (18); G_X is the Green's function of the Gaussian theory. The sum over f here – the sum over $|V_\Gamma|$-tuples of vertices of the spacetime graph X – is a graph QFT analog of the configuration space integral formula for the weight of a Feynman graph (cf., e.g., [11]).

The expression in the right-hand side of (80) is a power series in nonnegative powers in $\hbar$, with finitely many graphs contributing at each order[28]. We denote the right-hand side of (80) by Z_X^{pert} – the perturbative partition function. It is an asymptotic expansion of the measure-theoretic integral in the left-hand side of (80) – the nonperturbative partition function.

4.1. Version relative to a boundary subgraph

Let X be graph and $Y\subset X$ a "boundary" subgraph. We have the following relative version of the perturbative expansion (80):

$$Z_{X,Y}(\phi_Y) = \int_{F_X^{\phi_Y}} [D\phi]^{\phi_Y} e^{-\frac{1}{\hbar}(S_X(\phi)-\frac{1}{2}S_Y(\phi_Y))} = e^{-\frac{1}{2\hbar}S_Y^{\mathrm{int}}(\phi_Y)} \langle\!\langle e^{-\frac{1}{\hbar}S_{X\setminus Y}^{\mathrm{int}}(\phi)} \rangle\!\rangle^0_{\phi_Y}$$
$$\underset{\hbar\to 0}{\sim} \det(K_{X,Y})^{-\frac{1}{2}} e^{-\frac{1}{2\hbar}((\phi_Y,(\mathrm{DN}_{Y,X}-\frac{1}{2}K_Y)\phi_Y)+S_Y^{\mathrm{int}}(\phi_Y))} \sum_\Gamma \frac{\hbar^{-\chi(\Gamma)}}{|\operatorname{Aut}(\Gamma)|}\Phi_{\Gamma,(X,Y)}(\phi_Y). \tag{82}$$

[27] We are using sans-serif font to distinguish vertices of the Feynman graph u, v from the vertices of the spacetime graph u, v.

[28] This is due to the restriction that valencies in Γ are ≥ 3, which is in turn due to the assumption that $p(\phi)$ contains only terms of degree ≥ 3.

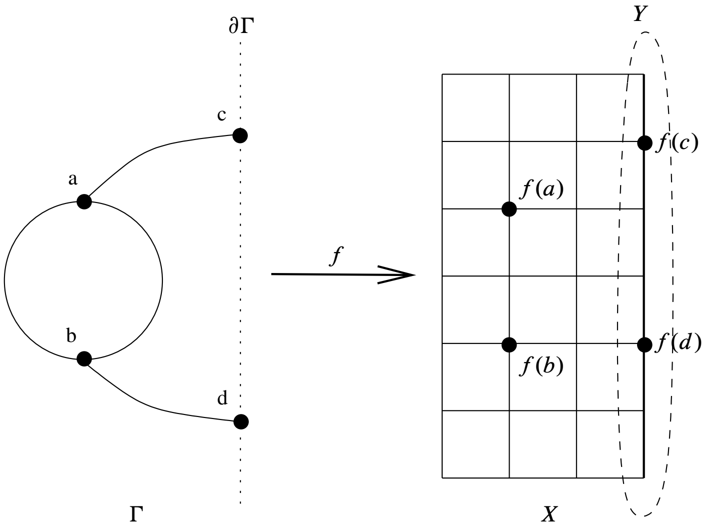


Figure 11. A Feynman graph with boundary vertices and a map contributing to its Feynman weight.

Here, the sum is over Feynman graphs Γ with vertices split into two subsets – "bulk" vertices V_Γ^{bulk} and "boundary" vertices V_Γ^∂ – with bulk vertices of valence ≥ 3 and univalent boundary vertices. In graphs Γ we are not allowing edges connecting two boundary vertices (while bulk-bulk and bulk-boundary edges are allowed). The weight of a Feynman graph is a polynomial in the boundary field ϕ_Y:

$$\begin{aligned}\Phi_{\Gamma,(X,Y)}(\phi_Y) &= \sum_{\substack{f\colon V_\Gamma^{\text{bulk}} \to V_X \setminus V_Y \\ V_\Gamma^\partial \to V_Y}} \prod_{\mathsf{v}\in V_\Gamma^{\text{bulk}}} (-p_{\text{val}(\mathsf{v})}) \cdot \prod_{\mathsf{u}^\partial \in V_\Gamma^\partial} \phi(f(\mathsf{u}^\partial)) \cdot \\ &\cdot \prod_{(\mathsf{u},\mathsf{v})\in E_\Gamma^{\text{bulk-bulk}}} G_{X,Y}(f(\mathsf{u}), f(\mathsf{v})) \cdot \prod_{(\mathsf{u}^\partial,\mathsf{v})\in E_\Gamma^{\text{bdry-bulk}}} E_{Y,X}(f(\mathsf{u}^\partial), f(\mathsf{v})). \end{aligned} \tag{83}$$

The sum over f here can be seen as a sum over tuples of bulk and boundary vertices in X. Similarly to (81), it is a graph QFT analog of a configuration space integral formula for the Feynman diagrams in the interacting scalar field theory on manifolds with boundary (cf., [11]), where one is integrating over configurations of n bulk points and m boundary points on the spacetime manifold.

We will denote the right-hand side of (82) by $Z_{X,Y}^{\text{pert}}(\phi_Y)$.

Example 4.1. Figure 11 is an example of a map f contributing to the Feynman weight (83).

The full Feynman weight of the graph on the left is

$$\Phi_{\Gamma,(X,Y)}(\phi_Y) = \sum_{a,b\in V_X\setminus V_Y,\, c,d\in V_Y} (p_3)^2 G_{X,Y}(a,b)^2 E_{Y,X}(c,a) E_{Y,X}(d,b)\, \phi(c)\phi(d),$$

where we denoted $a = f(\mathsf{a})$, $b = f(\mathsf{b})$, $c = f(\mathsf{c})$, $d = f(\mathsf{d})$.

Remark 4.2. (i) By the standard argument, due to multiplicativity of Feynman weights with respect to disjoint unions of Feynman graphs, the sum over graphs Γ in equations (80) and (82) can be written as the exponential of the sum over connected Feynman graphs, $\sum_\Gamma \cdots = e^{\sum_{\Gamma\,\text{connected}}\cdots}$.

(ii) One can rewrite the right-hand side of (82) without the DN operator in the exponent in the prefactor, but instead allowing graphs Γ with boundary-boundary edges. The latter contribute extra factors $-\mathrm{DN}_{Y,X}(\mathsf{u}^\partial,\mathsf{v}^\partial)$ in the Feynman weight (83).

(iii) Unlike the case of closed X, the sum over Γ in the right-hand side of (82) generally contributes infinitely many terms to each nonnegative order in $\hbar$ (for instance, in the order $O(\hbar^0)$, one has 1-loop graphs formed by trees connected to a cycle). However, there are finitely many graphs contributing to a given order in $\hbar$, in any fixed polynomial degree in ϕ_Y. Moreover, one can introduce a rescaled boundary field η_Y so that

$$\phi_Y = \sqrt{\hbar}\,\eta_Y. \tag{84}$$

Then, (82) expressed as a function of η_Y is a power series in nonnegative half-integer powers of $\hbar$, with finitely many graphs contributing at each order[29].

4.2. Cutting/gluing of perturbative partition functions via cutting/gluing of Feynman diagrams

As in Section 3.3.1, consider a closed graph $X = X' \bigcup_Y X''$ obtained from graphs X', X'' by gluing along a common subgraph $X' \supset Y \subset X''$ (but now we consider the interacting scalar QFT).

As we know from Proposition 2.3, the nonperturbative partition functions satisfy the gluing formula

$$Z_X = \int_{F_Y} D\phi_Y\, Z_{X',Y}(\phi_Y) Z_{X'',Y}(\phi_Y).$$

[29]The power of $\hbar$ accompanying a graph is $\hbar^{|E_\Gamma|-|V_\Gamma^{\text{bulk}}|-\frac{1}{2}|V_\Gamma^\partial|}$, i.e., one can think that with this normalization of the boundary field, boundary vertices contribute $1/2$ instead of 1 to the Euler characteristic of a Feynman graph.

We also note that the rescaling (84) is rather natural, as the expected magnitude of fluctuations of ϕ_Y around zero is $O(\sqrt{\hbar})$.

Replacing both sides with their expansions (asymptotic series) in $\hbar$, we have the gluing formula for the perturbative partition functions:

$$Z_X^{\text{pert}} = \int_{F_Y} D\phi_Y \, Z_{X',Y}^{\text{pert}}(\phi_Y) Z_{X'',Y}^{\text{pert}}(\phi_Y). \tag{85}$$

This latter formula admits an independent proof in the language of Feynman graphs which we will sketch here (adapting the argument of [11]).

Consider "decorations" of Feynman graphs Γ for the theory on X by the following data.

- Each vertex v of Γ is decorated by one of three symbols $\{X', X'', Y\}$, meaning that in the Feynman weight $f(\mathsf{v})$ is restricted to be in $V_{X'} \setminus V_Y$, in $V_{X''} \setminus V_Y$, or in V_Y, respectively.
- Each edge $e = (\mathsf{u}, \mathsf{v})$ of Γ is decorated by either u or c ("uncut" or "cut"), corresponding to the splitting of the Green's function on X in Theorem 3.7:

  $$G_X(f(\mathsf{u}), f(\mathsf{v})) = G_X^u(f(\mathsf{u}), f(\mathsf{v})) + G_X^c(f(\mathsf{u}), f(\mathsf{v})).$$

  Here,

  - if u, v are both decorated with X', the "uncut" term is $G_X^u := G_{X',Y}$. Similarly, if u, v are both decorated with X'', $G_X^u := G_{X'',Y}$. For all other decorations of u, v, $G_X^u := 0$. Because of this, we will impose a selection rule: u-decoration is only allowed for $X' - X'$ or $X'' - X''$ edges,
  - the "cut" term is

    $$G_X^c(f(\mathsf{u}), f(\mathsf{v})) := \sum_{w_1, w_2 \in Y} E_{Y,\alpha}(f(\mathsf{u}), w_1) \, \mathrm{DN}_{Y,X}^{-1}(w_1, w_2) E_{Y,\beta}(f(\mathsf{v}), w_2),$$

    where α, β are the decorations of u, v (and we understand $E_{Y,Y}$ as identity operator).

Let $\mathrm{Dec}(\Gamma)$ denote the set of all possible decorations of a Feynman graph Γ. Theorem 3.7 implies that for any Feynman graph Γ its weight splits into the contributions of its possible decorations:

$$\Phi_{\Gamma,X} = \sum_{\Gamma^{\text{dec}} \in \mathrm{Dec}(\Gamma)} \Phi_{\Gamma^{\text{dec}},X},$$

where in the summand on the right-hand side, we have restrictions on images of vertices of Γ as prescribed by the decoration, and we only select either cut or uncut piece of each Green's function.

Thus, the left-hand side of (85) can be written as

$$Z_X^{\text{pert}} = \det(K_X)^{-\frac{1}{2}} \cdot \sum_{\Gamma^{\text{dec}}} \frac{\hbar^{-\chi(\Gamma)}}{|\mathrm{Aut}(\Gamma^{\text{dec}})|} \Phi_{\Gamma^{\text{dec}},X}, \tag{86}$$

where on the right we are summing over all Feynman graphs with all possible decorations.

The right-hand side of (85) is

$$\begin{aligned}&\left\langle\!\!\left\langle \det(K_{X',Y})^{-\frac{1}{2}}\det(K_{X'',Y})^{-\frac{1}{2}}e^{-\frac{1}{\hbar}S_Y^{\mathrm{int}}(\phi_Y)}\right.\right.\\&\left.\left.\quad\cdot\sum_{\Gamma',\Gamma''}\frac{\hbar^{-\chi(\Gamma'\cup\Gamma'')}}{|\mathrm{Aut}(\Gamma'\cup\Gamma'')|}\Phi_{\Gamma',(X',Y)}(\phi_Y)\Phi_{\Gamma'',(X'',Y)}(\phi_Y)\right\rangle\!\!\right\rangle_Y\\&=\det(K_X)^{-\frac{1}{2}}\left\langle e^{-\frac{1}{\hbar}S_Y^{\mathrm{int}}(\phi_Y)}\sum_{\Gamma',\Gamma''}\frac{\hbar^{-\chi(\Gamma'\cup\Gamma'')}}{|\mathrm{Aut}(\Gamma'\cup\Gamma'')|}\Phi_{\Gamma',(X',Y)}(\phi_Y)\Phi_{\Gamma'',(X'',Y)}(\phi_Y)\right\rangle_Y,\end{aligned}\tag{87}$$

where

$$\langle\!\langle\cdots\rangle\!\rangle_Y := \int_{F^Y} D\phi_Y\, e^{-\frac{1}{2\hbar}(\phi_Y,\mathrm{DN}_{Y,X}\,\phi_Y)}\cdots$$

is the non-normalized Gaussian average with respect to the total DN operator; $\langle\cdots\rangle_Y$ is the corresponding normalized average.

The correspondence between (86) and (87) is as follows. Consider a decorated graph Γ^{dec} and form out of it subgraphs Γ', Γ'' in the following way. Let us cut every cut edge in Γ (except $Y-Y$ edges) into two, introducing two new boundary vertices. Then, we collapse every edge between a newly formed vertex and a Y-vertex. Γ' is the subgraph of Γ formed by vertices decorated by X' and uncut edges between them, and those among the newly formed boundary vertices which are connected to an X'-vertex by an edge; Γ'' is formed similarly.

Then, the contribution of Γ^{dec} to (86) is equal to the contribution of a particular Wick pairing for the term in (87) corresponding to the induced pair of graphs Γ', Γ'', and picking a term in the Taylor expansion of $e^{-\frac{1}{\hbar}S_Y^{\mathrm{int}}(\phi_Y)}$ corresponding to Y-vertices in Γ^{dec}. The sum over all decorated Feynman graphs in (86) recovers the sum over all pairs Γ', Γ'' and all Wick contractions in (87). This shows Feynman-graphwise the equality of (86) and (87). One can also check that the combinatorial factors work out similarly to the argument in [11, Lemma 6.10].

Example 4.3. Figure 12 is an example of a decorated Feynman graph on X (on the left; vertex decorations X', Y, X'' are according to the labels in the bottom) and the corresponding contribution to (87) on the right.

Dashed edges on the right denote the Wick pairing for $\langle\cdots\rangle_Y$ and are decorated with $\mathrm{DN}_{Y,X}^{-1}$. Circle vertices are the boundary vertices of graphs Γ', Γ'' or equivalently the vertices formed by cutting the c-edges of Γ^{dec}.

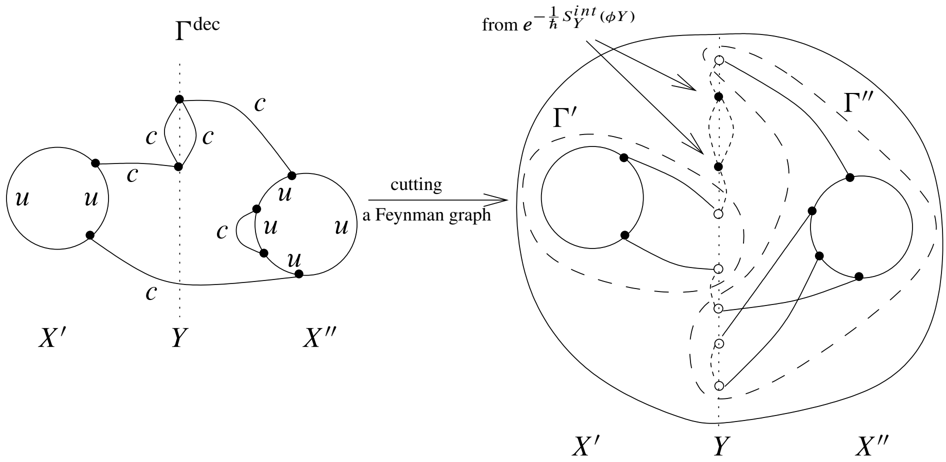


Figure 12. An example of cutting a Feynman graph.

5. Path sum formulae for the propagator and determinant (Gaussian theory in the first quantization formalism)

5.1. Quantum mechanics on a graph

Following the logic of Section 1.1, we now want to understand the kinetic operator $\Delta_X + m^2$ of the second quantized theory as the Hamiltonian of an auxiliary quantum mechanical system – a quantum particle on the graph X[30]. The space of states $\mathcal{H}_X$ for graph quantum mechanics on X is $\mathbb{C}^V$, i.e., the space of $\mathbb{C}$-valued functions on V. The graph Schrödinger equation[31] on X is

$$\frac{\partial}{\partial t}\left|\psi(t,v)\right\rangle(t,v) = -(\Delta_X + m^2)\left|\psi(t,v)\right\rangle, \tag{88}$$

where $|\psi(t,v)\rangle$ is a (time-dependent) state, i.e., a vector in $\mathbb{C}^V$. The explicit solutions to (88) are given by

$$|\psi(t_f)\rangle = e^{-(t_f-t_0)(\Delta_X+m^2)}\,|\psi(t_0)\rangle\,. \tag{89}$$

One can explicitly solve equation (89) by summing over certain paths on X, see equations (96), (94), and (112) below, in a way reminiscent of Feynman's path integral[32].

[30] This model of quantum mechanics on a graph – as a model for the interplay between the operator and path integral formalisms – was considered in [13, 14], see also [4].

[31] Here, we are talking about the Wick-rotated Schrödinger equation (i.e., describing quantum evolution in imaginary time), or equivalently the heat equation.

[32] This analogy is discussed in more detail in [13, 14].

This graph quantum mechanics is the first step of our first quantization approach to QFT on a graph.

5.2. Path sum formulae on closed graphs

5.2.1. Paths and h-paths in graphs. We start with some terminology. A *path* γ from a vertex u to a vertex v of a graph X is a sequence

$$\gamma = (u = v_0, e_0, v_1, \ldots, e_{k-1}, v_k = v),$$

where v_i are vertices of X and e_i is an edge between v_i and v_{i+1}[33]. We denote $V(\gamma)$ the ordered collection $(v_0, v_1, \ldots, v_k)$ of vertices of γ. We call $l(\gamma) = k$ the length of the path, and denote $P_X^k(u, v)$ the set of paths in X of length k from a vertex u to a vertex v. We denote by $P_X(u, v) = \cup_{k=0}^{\infty} P_X^k(u, v)$ the set of paths of any length from u to v. We also denote by $P_X^k = \cup_{u,v \in X} P_X^k(u, v)$, and $P_X = \cup_{k=0}^{\infty} P_X^k$ the sets of paths between any two vertices of X.

Below, we will also need a variant of this notion that we call *hesitant paths*[34]. Namely, a *hesitant path* (or "h-path") from a vertex u to a vertex v is a sequence

$$\tilde{\gamma} = (u = v_0, e_0, v_1, \ldots, e_{k-1}, v_k = v),$$

but now we allow the possibility that $v_{i+1} = v_i$, in which case e_i is allowed to be any edge starting at $v_i = v_{i+1}$. In this case, we say that $\tilde{\gamma}$ *hesitates* at step i. If $v_{i+1} \neq v_i$, then we say that $\tilde{\gamma}$ jumps at step i. As before, we say that such a path has length $l(\tilde{\gamma}) = k$, and we introduce the notion of the *degree* of a h-path as[35]

$$\deg(\tilde{\gamma}) = |\{i | v_i \neq v_{i+1},\ 0 \leq i \leq l(\gamma) - 1\}|, \tag{90}$$

i.e., the degree is the number of jumps of a h-path. We denote by

$$h(\tilde{\gamma}) = |\{i | v_i = v_{i+1},\ 0 \leq i \leq l(\gamma) - 1\}|$$

[33]For simplicity of the exposition, we assume that the graph X has no short loops. The generalization allowing short loops is straightforward: in the definition of a path and h-path, the edges traversed e_i are not allowed to be short loops (and in the formulae involving the valence of a vertex, it should be replaced with valence excluding the contribution of short loops). This is ultimately due to the fact that short loops do not contribute to the graph Laplacian Δ_X.

[34]Hesitant paths appear naturally from the factorization $\Delta = d^T d$, see [19]. Consequently, the signed count of hesitant paths gives the coefficients in the power series expansions of the Green's function and other objects in the variable m^{-2}. Moreover, they can be used to prove formulae for those objects in terms of ordinary paths through a resumming procedure. On non-regular graphs, the ordinary path sum formulae are not power series, while the hesitant path sums always are.

[35]Diverting slightly from the notation of [19].

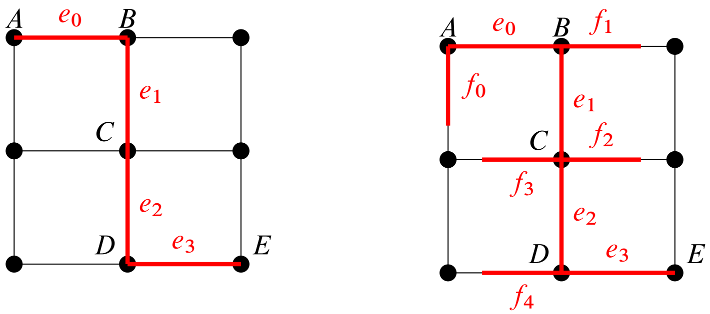


Figure 13. Left: The path $\gamma = (A, e_0, B, e_1, C, e_2, D, e_3, E)$ from A to E with length $l(\gamma) = 4$. Right: the h-path $\widetilde{\gamma} = (A, f_0, A, e_0, B, f_1, B, e_1, C, f_2, C, f_3, C, e_2, D, f_4, D, e_3, E)$ with $l(\widetilde{\gamma}) = 9$, $\deg(\widetilde{\gamma}) = 4$, $h(\widetilde{\gamma}) = 5$ and $P(\widetilde{\gamma}) = \gamma$.

the number of hesitations of $\widetilde{\gamma}$. Obviously,

$$l(\widetilde{\gamma}) = \deg(\widetilde{\gamma}) + h(\widetilde{\gamma}).$$

We denote the set of h-paths from u to v by $\Pi_X(u, v)$, and the set of length k hesitant paths by $\Pi_X^k(u, v)$. There is an obvious concatenation operation

$$\begin{aligned} \Pi_X^k(u, v) \times \Pi_X^l(v, w) &\to \Pi_X^{k+l}(u, w), \\ (\widetilde{\gamma}_1, \widetilde{\gamma}_2) &\mapsto \widetilde{\gamma}_1 * \widetilde{\gamma}_2. \end{aligned} \tag{91}$$

Observe that for every h-path $\widetilde{\gamma}$ there is a usual (“non-hesitant”) path γ of length

$$l(\gamma) = \deg(\widetilde{\gamma})$$

given by simply forgetting repeated vertices, giving a map $P\colon \Pi(u, v) \twoheadrightarrow P(u, v)$. See Figure 13.

A (hesitant) path is called *closed* if $v_k = v_0$, i.e., the first and last vertex agree. The cyclic group C_k acts on closed paths of length k by shifting the vertices and edges. We call the orbits of this group action *cycles* (i.e., closed paths without a preferred start- or end-point), and denote them by Γ_X for equivalence classes of h-paths, and C_X for equivalence classes of regular paths. A cycle $[\widetilde{\gamma}]$ is called *primitive* if its representatives have trivial stabilizer under this group action. Equivalently, this means that there is no $k > 1$ and $\widetilde{\gamma}'$ such that

$$\widetilde{\gamma} = \underbrace{\widetilde{\gamma}' * \widetilde{\gamma}' * \cdots * \widetilde{\gamma}'}_{k \text{ times}},$$

i.e., the cycle is traversed exactly once. In general, the order of the stabilizer of $\widetilde{\gamma}$ is precisely the number of traverses. We will denote this number by $t(\widetilde{\gamma})$. Obviously, it is well-defined on cycles.

Example 5.1. In the $N = 3$ circle graph, the two closed paths $\widetilde{\gamma}_1 = (1, (12), 2, (23), 3, (31), 1)$ and $\widetilde{\gamma}_2 = (2, (23), 3, (31), 1, (12))$ define the same primitive cycle, while the closed path $\widetilde{\gamma}_3 = (1, (31), 3, (23), 2, (12), 1)$ defines a different cycle (since the graph is traversed in a different order). The closed (hesitant) path $\widetilde{\gamma}_4 = (1, (12), 1, (12), 1)$ is not primitive, since $\gamma_4 = (1, (12), 1) * (1, (12), 1)$.

5.2.2. h-path formulae for heat kernel, propagator and determinant. It is a simple observation that

$$p_X^k(u, v) := |P_X^k(u, v)| = \langle u|(A_X)^k|v\rangle, \tag{92}$$

where A_X denotes the adjacency matrix of the graph X, $|v\rangle$ denotes the state which is 1 at v and vanishes elsewhere, and

$$\langle u|A|v\rangle = A_{uv}$$

denotes the (u, v)-matrix element of the operator A (in the bracket notation for the quantum mechanics on X). We consider the heat operator

$$e^{-t\Delta_X} : C^0(X) \to C^0(X), \tag{93}$$

which is the propagator of the quantum mechanics on the graph X (89).

Suppose that X is regular, i.e., all vertices have the same valence n. Then, $\Delta_X = n \cdot I - A_X$ and (92) implies that the heat kernel $\langle u|e^{-t\Delta_X}|v\rangle$ is given by

$$\langle u|e^{-t\Delta_X}|v\rangle = e^{-tn} \sum_{k=0}^{\infty} \frac{t^k}{k!} p_X^k(u, v) = e^{-tn} \sum_{\gamma \in P_X(u,v)} \frac{t^{l(\gamma)}}{l(\gamma)!}. \tag{94}$$

One can think of the right-hand side as a discrete analog of the Feynman path integral formula, where one is integrating over all paths (see [13]).

For a general graph, one can derive a formula for the heat kernel in terms of h-paths, by using the formula $\Delta = d^T d$. Namely, one has (see [19])

$$\langle u|\Delta_X^k|v\rangle = \sum_{\widetilde{\gamma} \in \Pi_X^k(u,v)} (-1)^{\deg(\widetilde{\gamma})}. \tag{95}$$

This implies the following formula for the heat kernel:

$$\langle u|e^{-t\Delta_X}|v\rangle = \sum_{k=0}^{\infty} \frac{t^k}{k!}(-1)^k \sum_{\widetilde{\gamma} \in \Pi_X^k(u,v)} (-1)^{\deg(\widetilde{\gamma})} = \sum_{\widetilde{\gamma} \in \Pi_X(u,v)} \frac{t^{l(\widetilde{\gamma})}}{l(\widetilde{\gamma})!}(-1)^{h(\widetilde{\gamma})}. \tag{96}$$

Here, we have used that

$$l(\widetilde{\gamma}) + \deg(\widetilde{\gamma}) = h(\widetilde{\gamma}) \bmod 2.$$

Then, we have the following h-path sum formula for the Green's function.

Lemma 5.2 ([19]). *The Green's function G_X is given by*

$$\begin{aligned} \langle u|G_X|v\rangle &= m^{-2}\sum_{k=0}^{\infty}(m^{-2})^k \sum_{\tilde{\gamma}\in\Pi_X^k}(-1)^{h(\tilde{\gamma})} \\ &= m^{-2}\sum_{\tilde{\gamma}\in\Pi_X(u,v)}(m^{-2})^{l(\tilde{\gamma})}(-1)^{h(\tilde{\gamma})}. \end{aligned} \tag{97}$$

Proof. By expanding

$$m^2 G_X = (m^{-2}K_X)^{-1} = (1+m^{-2}\Delta_X)^{-1}$$

in powers of m^{-2} using the geometric series[36], we obtain

$$\langle u|(1+m^{-2}\Delta_X)^{-1}|v\rangle = \sum_{\tilde{\gamma}\in\Pi_X(u,v)}(-m^{-2})^{l(\tilde{\gamma})}(-1)^{\deg(\tilde{\gamma})}, \tag{98}$$

which proves (97). Alternatively, one can prove (97) by integrating the heat kernel

$$e^{-tK_X} = e^{-tm^2}e^{-t\Delta_X}$$

for K_X over the time parameter t and using the Gamma function identity

$$\int_0^\infty dt\, e^{-tm^2}\frac{t^k}{k!} = (m^{-2})^{k+1}. \qquad \blacksquare$$

In equation (97), we see two slightly different ways of interpreting the path sum formula. In the middle we see that when expanding in powers of m^2, the coefficient of $m^{-2(k+1)}$ is given by a signed count of h-path of length k, and that the sign is determined by the number of hesitations. On the right-hand side we interpret the propagator as a weighted sum over all h-paths, in accordance with the first quantization picture.

We have the following formula for the determinant of the kinetic operator (normalized by $1/m^2$) in terms of closed h-paths or h-cycles.

Lemma 5.3. *The determinant of K_X/m^2 is given by*

$$\begin{aligned} \log\det\left(\frac{K_X}{m^2}\right) &= -\sum_{v\in X}\sum_{\tilde{\gamma}\in\Pi_X^{\geq 1}(v,v)}\frac{(m^{-2})^{l(\tilde{\gamma})}}{l(\tilde{\gamma})}(-1)^{h(\tilde{\gamma})} \\ &= -\sum_{[\tilde{\gamma}]\in\Gamma_X^{\geq 1}}\frac{(m^{-2})^{l(\tilde{\gamma})}}{t(\tilde{\gamma})}(-1)^{h(\tilde{\gamma})}. \end{aligned} \tag{99}$$

[36] This series converges absolutely if the operator norm of $m^{-2}\Delta_X$ is less that one, or equivalently $m^2 > \lambda_{\max}(\Delta_X)$, i.e., m^2 is larger than the largest eigenvalues of Δ_X.

The first equation in (99) appeared in [19], with a different proof.

Proof. Expand[37]

$$\log\det\left(\frac{K_X}{m^2}\right) = \operatorname{tr}\log(1+m^{-2}\Delta_X) = -\sum_{v\in X}\sum_{k=0}^{\infty}\frac{(-m^{-2})^k}{k}\langle v|\Delta_X^k|v\rangle,$$

which implies (99). ∎

Note that in the expression in the middle of (99), we are summing over h-paths of length at least 1 with a fixed starting point. To obtain the right-hand side, we sum over orbits of the group action of C_k on closed paths of length k, the size of the orbit of $\tilde{\gamma}$ is exactly $l(\tilde{\gamma})/t(\tilde{\gamma})$.

Remark 5.4. Both h-paths and paths form monoids with respect to concatenation, with P a monoid homomorphism. A map s from a monoid to $\mathbb{R}$ or $\mathbb{C}$ is called *multiplicative* if it is a homomorphism of monoids, i.e.,

$$s(\tilde{\gamma}_1 * \tilde{\gamma}_2) = s(\tilde{\gamma}_1)s(\tilde{\gamma}_2). \tag{100}$$

Notice that in the path sum expression for the propagator (97), we are summing over h-paths $\tilde{\gamma}$ with the weight

$$s(\tilde{\gamma}) := (m^{-2})^{l(\tilde{\gamma})}(-1)^{h(\tilde{\gamma})}. \tag{101}$$

Below it will be important that this weight is in fact multiplicative, which is obvious from the definition.

Remark 5.5. Using multiplicativity of s, we can resum over iterates of primitive cycles to rewrite the right-hand side of (99):

$$\log\det\left(\frac{K_X}{m^2}\right) = -\sum_{\substack{[\tilde{\gamma}]\in\Gamma_X^{\geq 1}\\ \tilde{\gamma}\text{ primitive}}}\sum_{k\geq 1}\frac{s(\tilde{\gamma})^k}{k} = \sum_{\substack{[\tilde{\gamma}]\in\Gamma_X^{\geq 1}\\ \tilde{\gamma}\text{ primitive}}}\log\left(1-m^{-2l(\tilde{\gamma})}(-1)^{h(\tilde{\gamma})}\right).$$

5.2.3. Resumming h-paths. Path sum formulae for propagator and determinant. Summing over the fibers of the map $P\colon \Pi_X(u,v) \twoheadrightarrow P_X(u,v)$, we can rewrite (98) as a path sum formula as follows.

Lemma 5.6. *If* $m^2 > \operatorname{val}(v)$ *for all* $v \in X$, *we have*

$$\langle u|(1+m^{-2}\Delta_X)^{-1}|v\rangle = m^2\sum_{\gamma\in P_X(u,v)}\prod_{v_i\in V(\gamma)}\frac{1}{m^2+\operatorname{val}(v_i)}. \tag{102}$$

[37] Again, this power series converges absolutely for $m^2 > \lambda_{\max}(\Delta_X)$.

Proof. For a path $\gamma \in P_X^k(u,v)$, the fiber $P^{-1}(\gamma)$ consists of h-paths $\widetilde{\gamma}$ which hesitate an arbitrary number j_i of times at every vertex v_i in $V(\gamma)$. For each vertex v_i, there are $\mathrm{val}(v_i)^{j_i}$ possibilities for a path to hesitate j_i times at v_i. The length of such a h-path is $l(\widetilde{\gamma}) = k + j_0 + \cdots + j_k$ and its degree is $\deg(\widetilde{\gamma}) = k$; hence, we can rewrite equation (98) as follows:

$$\begin{aligned}
&\sum_{\widetilde{\gamma}\in\Pi_X(u,v)} (-m^{-2})^{l(\widetilde{\gamma})}(-1)^{\deg(\widetilde{\gamma})} \\
&\quad = \sum_{k=0}^{\infty} \sum_{\gamma\in P_X^k(u,v)} \sum_{j_0,\ldots,j_k=0}^{\infty} \mathrm{val}(v_0)^{j_0}\cdot\cdots\cdot \mathrm{val}(v_k)^{j_k}(-m^{-2})^{k+j_0+\cdots+j_k}(-1)^k \\
&\quad = \sum_{k=0}^{\infty} \sum_{\gamma\in P_X^k(u,v)} (m^{-2})^k \sum_{j_0,\ldots,j_k}^{\infty} \mathrm{val}(v_0)^{j_0}\cdot\cdots\cdot \mathrm{val}(v_k)^{j_k}(-m^{-2})^{j_0+\cdots+j_k} \\
&\quad = \sum_{\gamma\in P_X(u,v)} m^2 \prod_{v_i\in V(\gamma)} \frac{m^{-2}}{1+m^{-2}\cdot\mathrm{val}(v_i)}. \qquad \blacksquare
\end{aligned}$$

Corollary 5.7. *The Green's function of the kinetic operator has the expression*

$$\langle u|G_X|v\rangle = \sum_{\gamma\in P_X(u,v)} \prod_{v_i\in V(\gamma)} \frac{1}{m^2+\mathrm{val}(v_i)}. \tag{103}$$

In particular, if X is regular of degree n, then

$$\langle u|G_X|v\rangle = \sum_{\gamma\in P_X(u,v)} \left(\frac{1}{m^2+n}\right)^{l(\gamma)+1} = \frac{1}{m^2+n}\sum_{k=0}^{\infty} p_X^k(u,v)(m^2+n)^{-k}. \tag{104}$$

To derive a path sum formula for the determinant, we use a slightly different idea, that also provides an alternative proof of the resummed formula for the propagator. Consider the operator Λ which acts on $C^0(X)$ diagonally in the vertex basis and sends $|v\rangle \mapsto (m^2+\mathrm{val}(v))|v\rangle$, that is,

$$\Lambda = \mathrm{diag}(m^2+\mathrm{val}(v_1),\ldots,m^2+\mathrm{val}(v_N))$$

in the basis of $C^0(X)$ corresponding to an enumeration $v_1,\ldots,v_N$ of the vertices of X. Then, consider the "normalized" kinetic operator

$$\widetilde{K}_X = \Lambda^{-1}K_X = I - \Lambda^{-1}A_X, \tag{105}$$

with A_X the adjacency matrix of the graph. Then, we have the simple generalization of the observation that matrix elements of the kth power of the adjacency matrix A_X

count paths of length k (see (92)), namely, matrix elements of $(\Lambda^{-1}A_X)^k\Lambda^{-1}$ count paths weighted with

$$w(\gamma) := \prod_{v\in V(\gamma)} \frac{1}{m^2+\mathrm{val}(v)}. \tag{106}$$

Then, we immediately obtain that

$$\begin{aligned}\langle u|G_X|v\rangle = \langle u|\widetilde{K}_X^{-1}\Lambda^{-1}|v\rangle &= \sum_{k=0}^{\infty}\langle u|(\Lambda^{-1}A_X)^k\Lambda^{-1}|v\rangle \\ &= \sum_{k=0}^{\infty}\sum_{\gamma\in P_X^k(u,v)} w(\gamma),\end{aligned} \tag{107}$$

which is (103).

For the determinant, we have the following statement.

Proposition 5.8. *The determinant of the normalized kinetic operator has the expansions*

$$\log\det\widetilde{K}_X = -\sum_{v\in X}\sum_{k=0}^{\infty}\sum_{\gamma\in P_X^k(v,v)}\frac{w'(\gamma)}{l(\gamma)} \tag{108}$$

$$= -\sum_{[\gamma]\in C_X^{\geq 1}}\frac{w'(\gamma)}{t(\gamma)}, \tag{109}$$

where for a closed path $\gamma\in P_X^k(v,v)$, $w'(\gamma)=w(\gamma)\cdot(m^2+\mathrm{val}(v))$ (see footnote[38]).

Proof. To see (108), we simply observe that

$$\log\det\widetilde{K}_X = \mathrm{tr}\log(1-\Lambda^{-1}A_X) = -\mathrm{tr}\sum_{k=0}^{\infty}\frac{(\Lambda^{-1}A_X)^k}{k} = -\sum_{v\in V}\sum_{k=0}^{\infty}\sum_{\gamma\in P_X^k(v,v)}\frac{w'(\gamma)}{k}.$$

To see the second formula (109), one sums over orbits of the cyclic group action on closed paths. ∎

In particular, for regular graphs we obtain a formula also derived in [19].

Corollary 5.9. *If X is a regular graph, then we have*

$$\log\det\widetilde{K}_X = -\sum_{k=1}^{\infty}\sum_{[\gamma]\in C_X^k}\frac{(m^2+n)^{-k}}{t(\gamma)}. \tag{110}$$

[38]Note that this is well-defined on a cycle, we are simply taking the product over all vertices in the path but without repeating the one corresponding to start-and endpoint.

Another corollary is the following first quantization formula for the partition function.

Theorem 5.10 (First quantization formula for Gaussian theory on closed graphs). *The partition function of the Gaussian theory on a closed graph can be expressed by*

$$\log Z_X = \frac{1}{2}\Bigg(\sum_{[\gamma]\in C_X^{\geq 1}} \frac{w'(\gamma)}{t(\gamma)} - \sum_{v\in X} \log(m^2 + \mathrm{val}(v)) \Bigg). \tag{111}$$

I.e., the logarithm of the partition function is given, up to the "normalization" term $-\sum_{v\in X}\log(m^2+\mathrm{val}(v))$, by summing over all cycles of length at least 1, dividing by automorphisms coming from orientation reversing and multiple traversals.

Proof. We have

$$\log Z_X = -\frac{1}{2}\log\det K_X = -\frac{1}{2}(\log\det \widetilde{K}_X + \log\det\Lambda),$$

from where the theorem follows by Proposition 5.8. ■

Remark 5.11. Notice that the weight $w(\gamma)$ of the resummed formula (103) is not multiplicative: if $\gamma_1 \in P_X(u,v)$ and $\gamma_2 \in P_X(v,w)$ then

$$\prod_{v_i\in\gamma_1}\frac{1}{m^2+\mathrm{val}(v_i)}\prod_{v_i\in\gamma_2}\frac{1}{m^2+\mathrm{val}(v_i)} = \frac{1}{m^2+\mathrm{val}(v)}\prod_{v_i\in\gamma_1*\gamma_2}\frac{1}{m^2+\mathrm{val}(v_i)},$$

since on the left-hand side the vertex v appears twice.

Remark 5.12. The sum over k in (107) and (108) is absolutely convergent for any $m^2>0$. The reason is that the matrix $a=\Lambda^{-1}A_X$ has spectral radius smaller than 1 for $m^2>0$. This in turn follows from Perron–Frobenius theorem: Since a is a nonnegative matrix, its spectral radius $\rho(a)$ is equal to its largest eigenvalue (also known as Perron–Frobenius eigenvalue), which in turn is bounded by the maximum of the row sums of a (see footnote[39]). The sum of entries on the vth row of a is $\frac{\mathrm{val}(v)}{m^2+\mathrm{val}(v)}<1$, which implies that $\rho(a)<\max_v\frac{\mathrm{val}(v)}{m^2+\mathrm{val}(v)}<1$.

In particular resummation from h-path-sum formula to a path-sum formula extends the absolute convergence region from $m^2>\lambda_{\max}(\Delta_X)$ to $m^2>0$. See Table 3.

[39]For any matrix A with entries a_{ij}, λ an eigenvalue of A and x an eigenvector for λ, we have

$$|\lambda| = \frac{\|\lambda x\|_\infty}{\|x\|_\infty} \leq \sup_{\|y\|_\infty=1}\|Ay\|_\infty = \max_i\sum_j |a_{ij}|.$$

Here, $\|x\|_\infty=\max_i|x_i|$ denotes the maximum norm of a vector x.

Object	h-path sum	path sum
$\langle u\|G_X\|v\rangle$	$m^{-2}\sum_{\tilde{\gamma}\in\Pi_X(u,v)} s(\tilde{\gamma})$ (equation (98))	$\sum_{\gamma\in P_X(u,v)} w(\gamma)$ (equation (103))
$\log\det m^{-2}K_X$	$-\sum_{[\tilde{\gamma}]\in\Gamma_X^{\geq 1}} \frac{s(\tilde{\gamma})}{t(\tilde{\gamma})}$ (equation (99))	
$\log\det \tilde{K}_X$		$-\sum_{[\gamma]\in C_X^{\geq 1}} \frac{w'(\gamma)}{t(\gamma)}$ (equation (109))

Table 3. Summary of path sum formulae, closed case.

5.2.4. Aside: path sum formulae for the heat kernel and the propagator – "1d gravity" version. There is the following generalization of the path sum formula (94) for the heat kernel for a not necessarily regular graph X.

Proposition 5.13. *One has*

$$\langle u|e^{-t\Delta_X}|v\rangle = \sum_{\gamma\in P_X(u,v)} W(\gamma;t), \tag{112}$$

where the t-dependent weight for a path γ of length k is given by an integral over a standard k-simplex of size t:

$$W(\gamma;t) = \int_{\substack{t_0,\ldots,t_k>0\\ t_0+\cdots+t_k=t}} dt_1\cdots dt_k\, e^{-\sum_{i=0}^k t_i\,\mathrm{val}(v_i)}, \tag{113}$$

where we denoted $v_0,\ldots,v_k$ the vertices along the path.

Proof. To prove this result, note that the Green's function G_X as a function of m^2 is the Laplace transform L of the heat kernel $e^{-t\Delta_X}$ as a function of t. Thus, one can recover the heat kernel as the inverse Laplace transform L^{-1} of G_X. Applying L^{-1} to (103) termwise, we obtain (112), and (113) (note that the product of functions $\frac{1}{m^2+\mathrm{val}(v)}$ is mapped by L^{-1} to the convolution of functions $L^{-1}(\frac{1}{m^2+\mathrm{val}(v)}) = e^{-t\,\mathrm{val}(v)}$). ■

As a function of t, the weight (113) is a certain polynomial in t and e^{-t} with rational coefficients (depending on the sequence of valences $\mathrm{val}(v_i)$). If all valences along γ are the same (e.g., if X is regular), then the integral over the simplex evaluates to $W(\gamma;t) = \frac{t^k}{k!}e^{-t\cdot\mathrm{val}}$ – same as the weight of a path in (94).

Note also that integrating (113) (multiplied by $e^{-m^2 t}$) in t, we obtain an integral expression for the weight (106) of a path in the path sum formula for the Green's function:

$$w(\gamma) = \int_{t_0,\ldots,t_k>0} dt_0\cdots dt_k\, e^{-\sum_{i=0}^k t_i(\mathrm{val}(v_i)+m^2)}. \tag{114}$$

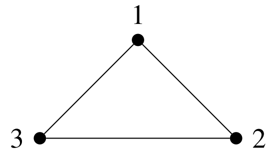


Figure 14. The $N = 3$ circle graph.

Here, unlike (113) the integral is over $\mathbb{R}_+^{k+1}$, not over a k-simplex.

Observe that the resulting formula for the Green's function:

$$\langle u|G_X|v\rangle = \sum_{\gamma \in P_X(u,v)} \int_{t_0,\ldots,t_k>0} dt_0 \cdots dt_k \, e^{-\sum_{i=0}^k t_i(\mathrm{val}(v_i)+m^2)} \tag{115}$$

bears close resemblance to the first quantization formula (8), where the proper times $t_0, \ldots, t_k$ should be though of as parametrizing the worldline metric field ξ (and the path γ is the field of the "1d sigma model")[40]. We imagine the particle moving on X along γ, spending time t_i at the ith vertex and making instantaneous jumps between the vertices, with the "action functional"

$$\overline{S}^{1q}(\gamma, \{t_i\}) = \sum_{i=0}^{k} t_i(\mathrm{val}(v_i) + m^2).$$

5.3. Examples

5.3.1. Circle graph, $N = 3$. Consider again the circle graph of Example 3.3 for $N = 3$ (Figure 14).

Counting h-paths from 1 to 2, we see that there are no paths of length 0, a unique path $(1, (12), 2)$ of length 1, and 5 paths of length 2:

$$(1, (13), 3, (23), 2),$$
$$(1, (13), 1, (12), 2),$$
$$(1, (12), 1, (12), 2),$$
$$(1, (12), 2, (12), 2),$$
$$(1, (12), 2, (23), 2).$$

[40]More explicitly, one can think of the worldline as a standard interval $[0, 1]$ subdivided into k sub-intervals by points $p_0 = 0 < p_1 < \cdots < p_{k-1} < p_k = 1$ (we think of p_i as moments when the particle jumps to the next vertex). Then, one can think of $\{t_i\}$ as moduli of metrics ξ on $[0, 1]$ modulo diffeomorphisms of $[0, 1]$ relative to (fixed at) the points $p_0, p_1, \ldots, p_k$.

The first one comes with a + sign, since it has no hesitations, the other 4 paths hesitate once at either 1 or 2 and come with a minus sign, the overall count is therefore -3. Counting paths beyond that is already quite hard. Looking at the Greens' function, we have

$$\begin{aligned}\langle 1|G_X|2\rangle &= \frac{1}{m^2(m^2+3)} = m^{-4}\left(\frac{1}{1+3m^{-2}}\right) = m^{-4}\sum_{k\geq 0}(-3m^{-2})^k \\ &= m^{-2}(0\cdot m^0 + 1\cdot m^{-2} - 3m^{-4} + 9m^{-6} + \cdots).\end{aligned}$$

Since the circle graph is regular, we can count paths from u to v by expanding in the parameter $\alpha^{-1} = \frac{1}{m^2+2}$. Here, we observe that

$$\begin{aligned}\langle 1|G_X|2\rangle &= \frac{1}{(\alpha-2)(\alpha+1)} = \frac{1}{3\alpha}\left(\frac{1}{1-2\alpha^{-1}} - \frac{1}{1+\alpha^{-1}}\right) \\ &= \frac{1}{3\alpha}\sum_{k\geq 0}(2^k - (-1)^k)\alpha^{-k} \\ &= \alpha^{-1}(0\cdot\alpha^0 + 1\cdot\alpha^{-1} + 1\alpha^{-2} + 3\alpha^{-3} + 5\alpha^{-4} + 11\alpha^{-5} + \cdots),\end{aligned}$$

and one can count explicitly that there is no path of length zero, a unique path (12) of length 1, a unique path (132) of length 2, 3 paths (1212), (1312), (1232) of length 3, 5 paths (12312), (13212), (13132), (12132), (13232) of length 4, and so on[41]. Similarly, we could expand

$$\begin{aligned}\langle 1|G_X|1\rangle &= \frac{m^2+1}{m^2(m^2+3)} = m^{-2}(1+m^{-2})\left(\frac{1}{1+3m^{-2}}\right) \\ &= m^{-2}\left(1 + (-2)\sum_{k\geq 1}(-3)^{k-1}m^{-2k}\right) \\ &= m^{-2}(1\cdot m^0 + (-2)\cdot m^{-2} + 6\cdot m^{-4} + (-18)\cdot m^{-6} + \cdots),\end{aligned}$$

which counts h-paths from vertex 1 to itself: a single paths of length 0, 2 length 1 paths which hesitate once at 1, two length 2 paths with 0 hesitations and 4 length 2 paths with 2 hesitations, and so on. In terms of $\alpha = m^2 + 2$, we get

$$\begin{aligned}\langle 1|G_X|1\rangle &= \frac{\alpha-1}{(\alpha-2)(\alpha+1)} \\ &= \alpha^{-1}\sum_{k\geq 0}\frac{2^k + 2(-1)^k}{3}\alpha^{-k} \\ &= \alpha^{-1}(1\cdot\alpha^0 + 0\cdot\alpha^{-1} + 2\cdot\alpha^{-2} + 2\cdot\alpha^{-3} + 6\cdot\alpha^{-4} + \cdots),\end{aligned}$$

[41] For brevity, here we just denote a path by its ordered collection of vertices, which determines the edges that are traversed.

where we recognize the path counts from 1 to itself: A unique path (1) of length 0, no paths of length 1, two paths (121), (131) of length 2, 2 paths (1231), (1321) of length 3, and so on.

The determinant is

$$\det K_X = m^2(m^2+3)^2,$$

so we have

$$\log\det m^{-2}K_X = 2\log(1+3m^{-2}) = -2\sum_{k\geq 1}\frac{(-3m^{-2})^k}{k} = \\ = 6m^{-2} - 9m^{-4} + 18m^{-6} - \frac{81}{2}m^{-8} + \cdots$$

and we can see that rational numbers appear, because we are either counting paths with $\frac{1}{l(\tilde{\gamma})}$, or cycles with $\frac{1}{t(\tilde{\gamma})}$. Let us verify the cycle count for the first two powers of m^2. Indeed, there is a total of 6 cycles of length 1 that hesitate once, of the form $(1,(12),1)$, and similar. At length 2, there are 3 closed cycles that do no hesitate, of the form $(1,(12),2,(12),1)$. Then, there are three cycles that hesitate twice and are of the form $(1,(12),1,(31),1)$ (they visit both edges starting at a vertex). Moreover, at every vertex we have the cycles of the form $(1,(12),1,(12),1)$. There are a total of 6 such cycles, however, they come with a factor of 1/2 because those are traversed twice! Overall, we obtain that

$$3+3+\frac{1}{2}\cdot 6 = 9$$

cycles (they all come with the same $+$ sign).

Finally, we can count cycles in X by expanding the logarithm of the determinant in powers of α:

$$\begin{aligned}\log\det\widetilde{K}_X &= \log\frac{(\alpha-2)(\alpha+1)^2}{\alpha^3} = \log 1 - 2\alpha^{-1} + 2\log 1 + \alpha^{-1} \\ &= -\sum_{k\geq 1}\frac{(2\alpha^{-1})^k}{k} - 2\sum_{k\geq 1}\frac{(-\alpha)^{-k}}{k} \\ &= -\sum_{k\geq 1}\frac{2^k + 2(-1)^k}{k}\alpha^{-k} \\ &= -\left(0\cdot\alpha^{-1} + 3\cdot\alpha^{-2} + 2\alpha^{-3} + \frac{9}{2}\alpha^{-4} + \cdots\right).\end{aligned} \tag{116}$$

Counting cycles we see there are 0 cycles of length 1, 3 cycles of length 2, namely, (121), (131), (232), 2 cycles of length 3, namely, (1231), (1321). There are 3 primitive cycles of length 4 (those of the form (12131) and similar), and 3 cycles which are traversed twice ((12121) and similar), which gives $3+\frac{3}{2}=\frac{9}{2}$.

5.3.2. Line graph, $N = 3$. Consider the line graph of Example 3.1. For instance, we have

$$\begin{aligned}\langle 1|G_X|3\rangle &= \frac{1}{m^2(1+m^2)(3+m^2)} = m^{-2}\frac{1}{2}\left(\frac{1}{1+m^2} - \frac{1}{3+m^2}\right) \\ &= \frac{m^{-4}}{2}\sum_{k=0}^{\infty}(-m^{-2})^k - (-3m^{-2})^{-k} = \frac{m^{-4}}{2}\sum_{k=0}^{\infty}((-1)^k - (-3)^k)m^{-2k} \\ &= m^{-2}(0\cdot m^0 + 0\cdot m^{-2} + 1\cdot m^{-4} - 4\cdot m^{-6} + 13m^{-8} + \cdots)\end{aligned}$$

and indeed we can observe there are no h-paths from 1 to 3 of length 0 and 1, and there is a unique path γ of length 2. At length 3, there are 4 different h-paths whose underlying path is γ and who hesitate exactly once, there are a total of $1 + 2 + 1$ possibilities to do so. At the next order, there are a total of 11 possibilities for γ to hesitate twice, and two new paths of length 4 appear, explaining the coefficient 13.

The path sum (103) becomes

$$\langle 1|G_X|3\rangle = \frac{1}{(1+m^2)^2(2+m^2)} + \frac{2}{(1+m^2)^3(2+m^2)^2} + \cdots .$$

Here, the numerator 1 corresponds to the single path of length 2, (123); the numerator 2 corresponds to the two paths of length 4, (12123), (12323). In fact, there are exactly 2^{l-1} paths $1 \to 3$ of length $2l$ for each $l \geq 1$, and along these paths the 1-valent vertices (endpoints) alternate with the 2-valent (middle) vertex, resulting in

$$\langle 1|G_X|3\rangle = \sum_{l\geq 1}\frac{2^{l-1}}{(1+m^2)^{l+1}(2+m^2)^l}.$$

For the determinant

$$\det K_X = m^2(m^2+1)(m^2+3),$$

we can give the hesitant cycles expansion

$$\begin{aligned}\log\det m^{-2}K_X &= \log(1+m^{-2}) + \log(1+3m^{-2}) \\ &= -\sum_{k\geq 1}\frac{(-1)^k + (-3)^k}{k}m^{-2k} \\ &= -\left(-4m^{-2} + 5m^{-4} + \frac{28}{3}m^{-6} + \cdots\right).\end{aligned}$$

Here, the first 4 is given by the four hesitant cycles of length 1. At length 2, we have the 4 iterates of length 1 hesitant cycles, contributing 2, a new hesitant cycle that hesitates twice at 2 (in different directions), and 2 regular cycles of length 2, for a total of $4 \cdot \frac{1}{2} + 1 + 2 = 5$. For the path sum we have

$$\begin{aligned} -\log\det \widetilde{K}_X &= -\log \frac{m^2(m^2+3)}{(m^2+1)(m^2+2)} \\ &= -\log\left(1 - \frac{2}{(m^2+1)(m^2+2)}\right) = \sum_{k\geq 1} \frac{2^k}{k}(m^2+1)^{-k}(m^2+2)^{-k}, \end{aligned}$$

which means there are $2^k/k$ cycles (counted with $1/t(\gamma)$) of length $2k$. For instance, there are 2 cycles of length 2, namely, (121) and (232). There is a unique primitive length 4 cycle, namely, (12321), and the two non-primitive cycles (12121), (23232), which contribute $k = \frac{1}{2}$, so we obtain $1 + 2\frac{1}{2} = 2$. There are 2 primitive length 6 cycles, namely, (1232321), (1212321), and the two non-primitive cycles (1212121), (2323232), contributing $\frac{1}{3}$ each, for a total of $2 + \frac{2}{3} = \frac{8}{3}$. At length 8 there are 3 new primitive cycles, the iterate of the length 4 cycle and the iterates of the 2 length 2 cycles for a total of $3 + \frac{1}{2} + 2 \cdot \frac{1}{4} = 4 = 2^4/4$.

5.4. Relative versions

In this section, we will study path-sum formulae for a graph X relative to a boundary subgraph Y. We will then give a path-sum proof of the gluing formula (Theorem 3.7) in the case of a closed graph presented as a gluing of subgraphs over Y. The extension to gluing of cobordisms is straightforward but notationally tedious.

5.4.1. h-path formulae for Dirichlet propagator, extension operator, Dirichlet-to-Neumann operator. In this section, we consider the path sum versions of the objects introduced in Section 3.2. Remember that, for a graph X and a subgraph Y, we have the notations (42):

$$(K_X)^{-1} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

and (46):

$$K_X = \left(\begin{array}{c|c} \widehat{A} = K_{X,Y} & \widehat{B} \\ \hline \widehat{C} & \widehat{D} \end{array}\right).$$

We are interested in the following objects.

- The propagator with Dirichlet boundary conditions on Y, $G_{X,Y} = K_{X,Y}^{-1}$ (cf., Section 3.2.3).
- The determinant of the kinetic operator $K_{X,Y}$ with Dirichlet boundary on Y (cf., Section 3.2.3).
- The combinatorial Dirichlet-to-Neumann operator $\mathrm{DN}_{Y,X} = D^{-1}\colon F_Y \to F_Y$ (cf., Section 3.2.2).
- The extension operator $E_{Y,X} = BD^{-1}\colon F_Y \to F_X$ (cf., equation (45)).

Propagator with Dirichlet boundary conditions. For u, v two vertices of $X \setminus Y$, let us denote by $\Pi_{X,Y}(u,v)$ the set of h-paths from u to v that contain no vertices in Y (but they may contain edges between $X \setminus Y$ and Y), and $\Pi^k_{X,Y}(u,v)$ the subset of such paths that have length k. Then, we have the formula [19]

$$\langle u|\Delta^k_{X,Y}|v\rangle = \sum_{\tilde{\gamma}\in\Pi^k_{X,Y}(u,v)} (-1)^{\deg(\tilde{\gamma})}. \tag{117}$$

In exactly the same manner as in the previous subsection, we can then prove

$$\langle u|(1+m^{-2}\Delta_{X,Y})^{-1}|v\rangle = \sum_{\tilde{\gamma}\in\Pi_{X,Y}(u,v)} s(\tilde{\gamma}), \tag{118}$$

and therefore,

$$\langle u|G_{X,Y}|v\rangle = m^{-2} \sum_{\tilde{\gamma}\in\Pi_{X,Y}(u,v)} s(\tilde{\gamma}). \tag{119}$$

Remark 5.14. In (119) and other formulae (120), (122), and (125) for other relative objects below, it is important to allow edges between X and Y in the hesitant paths. Not allowing those edges, e.g., in (119) would result in the sum over $\Pi_{X\setminus Y}(u,v)$ (hesitant paths in the full subgraph of X on the vertices $V_X \setminus V_Y$), which yields the Green's function of $X \setminus Y$ considered as a closed graph

$$m^{-2} \sum_{\tilde{\gamma}\in\Pi_{X\setminus Y}(u,v)} s(\tilde{\gamma}) = \langle u|G_{X\setminus Y}|v\rangle.$$

Determinant of relative kinetic operator. In the same fashion, we obtain the formula

$$\log\det\left(\frac{K_{X,Y}}{m^2}\right) = -\sum_{\tilde{\gamma}\in C^{\geq 1}_{X,Y}} \frac{s(\tilde{\gamma})}{t(\tilde{\gamma})}, \tag{120}$$

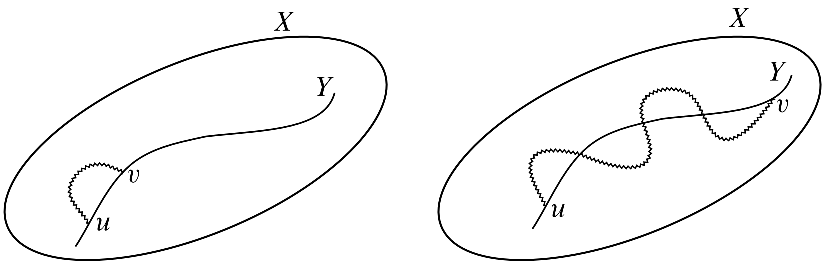
where we have introduced the notation $C^{\geq 1}_{X,Y}$ for cycles corresponding to closed h-paths in $X \setminus Y$ that may use edges between $X \setminus Y$ and Y.

Dirichlet-to-Neumann operator. Notice also that as a submatrix of K_X^{-1}, we have the following path sums for D (here $u, v \in Y$):

$$\langle u|D|v\rangle = m^{-2} \sum_{\tilde{\gamma}\in\Pi_X(u,v)} s(\tilde{\gamma}). \tag{121}$$

For $u, v \in Y$, we introduce the notation $\Pi''_{X,Y}(u,v)$ to be those h-paths from u to v containing exactly two vertices in Y, i.e., the start- and end-points. We define the operator $D'\colon C^0(Y) \to C^0(Y)$ given by summing over such paths (see Figure 15a)

$$\langle u|D'|v\rangle := \sum_{\tilde{\gamma}\in\Pi''_{X,Y}(u,v)} s(\tilde{\gamma}). \tag{122}$$



(a) h-paths in $\Pi''_X(u,v)$, contributing to $\langle u|D'|v\rangle$. **(b)** h-paths contributing to $\langle u|(D')^k|v\rangle$.

Figure 15. h-paths contributing to D' and $(D')^k$.

Notice that $\langle u|(D')^k|v\rangle$ is given by summing over paths which cross the interface Y exactly $k-1$ times between the start- and the end-point (see Figure 15b). Since the summand is multiplicative, we can therefore rewrite D as

$$D = m^{-2}\sum_{k\geq 0}(D')^k = m^{-2}(I-D')^{-1}. \tag{123}$$

Therefore, the Dirichlet-to-Neumann operator is given by the formula

$$\mathrm{DN}_{Y,X} = D^{-1} = m^2(I-D'). \tag{124}$$

Extension operator. Finally, we give a path sum formula for the extension operator. To do so we introduce the notation $\Pi'_{X,Y}(u,v)$ for h-paths that start at a vertex $u \in X\setminus Y$, end at a vertex $v\in Y$, and contain only a single vertex on Y, i.e., the end-point.

Lemma 5.15. *The extension operator can be expressed as*

$$BD^{-1}(u,v) = E_{Y,X}(u,v) = \sum_{\tilde\gamma\in\Pi'_{X,Y}(u,v)} s(\tilde\gamma). \tag{125}$$

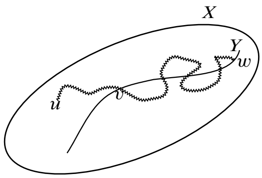
Proof. We will prove that composing with D we obtain B. Indeed, denote the right-hand side of equation (125) by $\widetilde{B}$. Then, using the h-path sum expression for D (121) we obtain

$$\widetilde{B}D = m^{-2}\sum_{v\in Y}\Big(\sum_{\tilde\gamma\in\Pi'_{X,Y}(u,v)} s(\tilde\gamma)\Big)\Big(\sum_{\tilde\gamma\in\Pi_X(v,w)} s(\tilde\gamma)\Big).$$

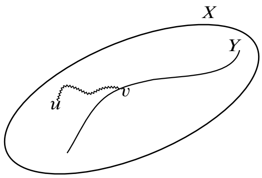
Using multiplicativity, we can rewrite this as

$$m^{-2}\sum_{(\tilde\gamma_1,\tilde\gamma_2)\in\bigsqcup_{v\in Y}\Pi'_{X,Y}(u,v)\times\Pi_X(v,w)} s(\tilde\gamma_1 * \tilde\gamma_2).$$

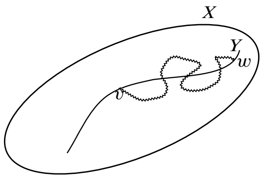
Now, the argument finishes by observing that any h-path $\tilde\gamma$ from a vertex u in $X\setminus Y$ to a vertex w in Y can be decomposed as follows. Let $v\in Y$ be the first vertex of Y that



(a) An h-path in $\Pi(u,w)$ contributing to $\langle u|B|w\rangle$.



(b) An h-path in $\Pi'(u,v)$ contributing to $\langle u|E_{Y,X}|v\rangle$.



(c) An h-path in $\Pi(v,w)$ contributing to $\langle v|D|w\rangle$.

Figure 16. Paths contributing to B (left) can be decomposed into paths contributing to $E_{Y,X}$ (middle) and paths contributing to D (right), proving that $B = E_{Y,X}D$.

appears in $\tilde{\gamma}$ and denote by $\tilde{\gamma}_1$ the part of the path before v, and by $\tilde{\gamma}_2$ the rest. Then, $\tilde{\gamma} = \tilde{\gamma}_1 * \tilde{\gamma}_2$ and $\tilde{\gamma}_1 \in \Pi'_X(u,v)$, see Figure 16. This decomposition is the inverse of the composition map

$$\begin{aligned}\bigsqcup_{v\in Y} \Pi'_{X,Y}(u,v) \times \Pi_X(v,w) &\to \Pi_X(u,w),\\ (\tilde{\gamma}_1,\tilde{\gamma}_2) &\mapsto \tilde{\gamma}_1 * \tilde{\gamma}_2,\end{aligned}$$

which is therefore a bijection. In particular, we can rewrite the expression above as

$$m^{-2} \sum_{\tilde{\gamma}\in\Pi_X(u,w)} s(\tilde{\gamma}) = B(u,w).$$

We conclude that $\tilde{B} = BD^{-1}$. ∎

5.4.2. Resumming h-paths. In the relative case, for any path γ we use the notation

$$w_{X,Y}(\gamma) = \prod_{v\in V(\gamma)\setminus V(Y)} \frac{1}{m^2 + \mathrm{val}_X(v)}, \tag{126}$$

where for a vertex $v \in X \setminus Y$, we put the subscript X on $\mathrm{val}_X(v)$ to emphasize we are considering its valence in X, i.e., we are counting all edges in X incident to v regardless if they end on Y or not[42]. Then, we have the following path sum formulae for the relative objects.

[42] Similarly to Remark 5.14, if we instead use the weights

$$w_{X\setminus Y}(\gamma) = \prod_{v\in V(\gamma)\setminus V(Y)} \frac{1}{m^2 + \mathrm{val}_{X\setminus Y}(v)},$$

we end up with the corresponding formulae for the *closed* graph $X \setminus Y$, instead of the formulae for X relative to Y.

Proposition 5.16. *The propagator with Dirichlet boundary condition can be expressed as*

$$\langle u|G_{X,Y}|v\rangle = \sum_{\gamma\in P_{X\setminus Y}(u,v)} w_{X,Y}(\gamma). \tag{127}$$

Here, the sum is over paths involving only vertices in $X \setminus Y$.

Similarly, for the extension operator we have

$$\langle u|E_{Y,X}|v\rangle = \sum_{\gamma\in P'_{X,Y}(u,v)} w_{X,Y}(\gamma), \tag{128}$$

where $P'_{X,Y}(u,v)$ denotes paths in $X \setminus Y$ from u to v with exactly one vertex (i.e., the endpoint) in Y. Finally, the operator D' appearing in the Dirichlet-to-Neumann operator can be written as

$$\langle u|D'|v\rangle = -m^{-2}\,\mathrm{val}(v)\delta_{uv} + m^{-2} \sum_{\gamma\in P''_{X,Y}(u,v)} w_{X,Y}(\gamma), \tag{129}$$

where $P''_{X,Y}(u,v)$ denotes paths in X with exactly two (i.e., start- and endpoint) vertices in Y. In particular, the Dirichlet-to-Neumann operator is

$$\langle u|\,\mathrm{DN}_{Y,X}\,|v\rangle = (m^2 + \mathrm{val}(v))\delta_{uv} - \sum_{P''_{X,Y}(u,v)} w_{X,Y}(\gamma). \tag{130}$$

Proof. Equation (127) is proved with a straightforward generalization of the arguments in the previous section. For equation (128), notice that because of the final jump there is an additional factor of m^{-2}. For the Dirichlet-to-Neumann operator, we have the initial and final jumps contributing a factor of $-m^{-2}$. In the case $u = v$, the contribution of the h-paths which simply hesitate once at v have to be taken into account separately and result in the first term in (129). Finally, (130) follows from (129) and

$$\mathrm{DN}_{Y,X} = m^2(I - D'). \qquad \blacksquare$$

We also have a similar statement for the determinant. For this, we introduce the normalized relative kinetic operator

$$\widetilde{K}_{X,Y} = \Lambda^{-1}_{X,Y} K_{X,Y} = I - \Lambda^{-1}_{X,Y} A_{X\setminus Y},$$

where $\Lambda_{X,Y}$ is the diagonal matrix whose entries are $m^2 + \mathrm{val}_X(v)$. For a closed path $\gamma \in P_{X\setminus Y}(v,v)$, we introduce the notation

$$w'_{X,Y}(\gamma) = (m^2 + \mathrm{val}_X)(v) \prod_{w\in V(\gamma)} \frac{1}{m^2 + \mathrm{val}_X(w)}.$$

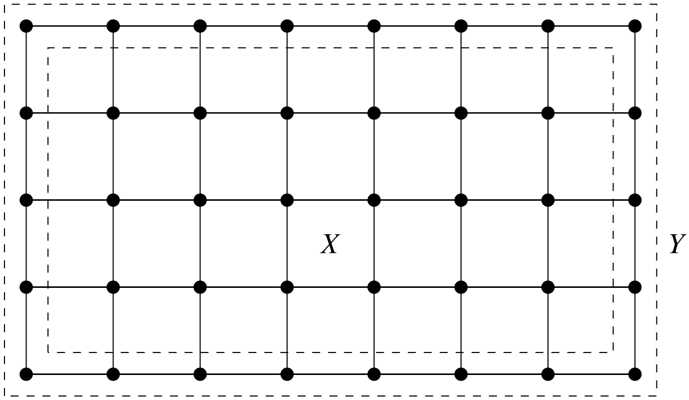


Figure 17. A graph X regular of degree $n = 4$ relative to a subgraph Y.

Proposition 5.17. *The determinant of the normalized relative kinetic operator is*

$$\begin{aligned}\log\det \widetilde{K}_{X,Y} &= -\sum_{v\in X\setminus Y}\sum_{k=1}^{\infty}\sum_{\gamma\in P^k_{X\setminus Y}(v,v)} \frac{w'_{X,Y}(\gamma)}{k} \\ &= -\sum_{[\gamma]\in C^{\geq 1}_{X\setminus Y}} \frac{w'_{X,Y}(\gamma)}{t(\gamma)}.\end{aligned} \tag{131}$$

Proof. Again, simply notice that

$$\log\det \widetilde{K}_{X,Y} = \operatorname{tr}\log(1 - \Lambda^{-1}_{X,Y} A_{X\setminus Y}).$$

Then, the argument is the same as in the proof of Proposition 5.8 above. ■

In the relative case, we are counting paths in $X \setminus Y$, but weighted according to the valence of vertices in X. This motivates the following definition.

Definition 5.18. We say that a graph X is regular of degree n relative to a subgraph Y if all vertices $v \in X \setminus Y$ have the same valence n in X, i.e.,

$$\mathrm{val}_X(v) = n \quad \forall v \in V(X \setminus Y).$$

If X is regular, then X is regular relative to any subgraph $Y \subset X$. An important class of examples are the line graphs X of Example 3.4 with Y both boundary vertices, or more generally rectangular graphs or their higher-dimensional counterparts with Y given by the collection of boundary vertices. See Figure 17.

For graphs regular rel. a subgraph, the path sums of Proposition 5.16 simplify to power series in $(m^2 + n)^{-1}$, with n the degree of (X, Y).

Corollary 5.19. *Suppose X is regular rel. Y, then we have the following power series expansions for relative propagator, extension operator, Dirichlet-to-Neumann operator, and determinant:*

$$\langle u|G_{X,Y}|v\rangle = \frac{1}{m^2+n}\sum_{k=0}^{\infty} p^k_{X\setminus Y}(u,v)(m^2+n)^{-k}, \tag{132}$$

$$\langle u|E_{Y,X}|v\rangle = \sum_{k=1}^{\infty}(p'^k_{X,Y})(u,v)(m^2+n)^{-k}, \tag{133}$$

$$\langle u|\operatorname{DN}_{Y,X}|v\rangle = (m^2+\operatorname{val}(v))\delta_{uv} - \sum_{k=2}^{\infty}(p''^k_{X,Y})(m^2+n)^{-k+1}, \tag{134}$$

$$\log\det \widetilde{K}_{X,Y} = -\sum_{k=1}^{\infty}\sum_{[\gamma]\in C^k_X}\frac{(m^2+n)^{-k}}{t(\gamma)}. \tag{135}$$

Again, we can collect our findings in the following first quantization formula for the partition function.

Theorem 5.20. *The logarithm of the partition function of the Gaussian theory relative to a subgraph Y is*

$$\begin{aligned}
&\hbar\log Z_{X,Y}(\phi_Y)\\
&\quad= -\frac{1}{2}\sum_{u,v\in Y}\phi_Y(u)\phi_Y(v)\\
&\qquad\cdot\left(\left(\frac{m^2}{2}+\operatorname{val}_X(v)-\frac{1}{2}\operatorname{val}_Y(v)\right)\delta_{uv}+\frac{1}{2}(A_Y)_{uv}-\sum_{\gamma\in P''_{X,Y}(u,v)} w_{X,Y}(\gamma)\right)\\
&\qquad+\frac{\hbar}{2}\left(\sum_{[\gamma]\in C^{\geq 1}_{X\setminus Y}}\frac{w'_{X,Y}(\gamma)}{t(\gamma)}-\sum_{v\in X}\log(m^2+\operatorname{val}(v))\right).
\end{aligned} \tag{136}$$

In (136), we are summing over all connected Feynman diagrams with no bulk vertices: boundary-boundary edges in the last term of the second line of the right-hand side at order $\hbar^0$ (together with the diagonal terms and $\frac{1}{2}(A_Y)_{uv}$, they sum up to $\operatorname{DN}_{Y,X} - \frac{1}{2}K_Y$) and "1-loop graphs" (cycles) on the third line at order $\hbar^1$.

5.4.3. Examples.

Example 5.21. Consider the graph X in Figure 18, with Y the subgraph consisting of the single vertex on the right. Then, the set $\Pi_{X,Y}$ consists exclusively of iterates of the path which hesitates once along the single edge at 1, $\widetilde{\gamma} = (1,(12),1)$. Therefore,

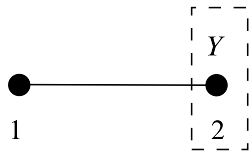


Figure 18. The 2-vertex line graph.

we obtain

$$\langle 1|G_{X,Y}|1\rangle = m^{-2}\sum_{k=0}^{\infty}(-m^{-2})^k = \frac{m^{-2}}{1+m^{-2}} = \frac{1}{1+m^2}.$$

Alternatively, we can obtain this from the path sum formula (127) by noticing there is a single (constant) path from 1 to 1 in $X \setminus Y$. For the determinant, we obtain

$$\log\det K_{X,Y}/m^2 = -\sum_{k\geq 1}\frac{(-m^{-2})^k}{k} = \log(1+m^{-2}) = \log\frac{1+m^2}{m^2}.$$

h-paths in $\Pi''_{X,Y}(2,2)$ are either $(2,(12),2)$ or of the form $(2,(12),1,(12),1,\ldots,1,(12),2)$ – i.e., jump from 2 to 1, hesitate k times and jump back – and therefore the operator D' is given by

$$D' = -m^{-2} + \sum_{k\geq 0}(m^{-2})^{k+1}(-1)^k = -m^{-2} + \frac{m^{-2}}{1+m^{-2}} = \frac{-1}{m^2+1}.$$

Alternatively, one can just notice there is a unique path in $P''_{X,Y}(2,2)$, namely, (212), and use formula (130). Therefore, the Dirchlet-to-Neumann operator is

$$\mathrm{DN}_{Y,X} = m^2\left(1 - \frac{-1}{m^2+1}\right) = \frac{m^2(2+m^2)}{1+m^2}.$$

Finally, h-paths in $\Pi'_{X,Y}(1,2)$ are only those that hesitate k times at 1 before eventually jumping to 2, and therefore the extension operator is

$$\langle 1|E_{Y,X}|2\rangle = m^{-2}\sum_{k=0}^{\infty}(-m)^{-2} = m^{-2}\frac{1}{1+m^{-2}} = \frac{1}{1+m^2},$$

alternatively, this follows directly from formula (128), because $P'_{X,Y}(1,2) = \{(12)\}$.

Example 5.22. Consider X the $N = 4$ line graph with Y both endpoints (1 and 4). Then, X is regular rel. Y, of degree 2, and we can count paths $X \setminus Y$ easily, namely, we have

$$p^k_{X\setminus Y}(2,2) = p^k_{X\setminus Y}(3,3) = \begin{cases} 1, & k \text{ even},\\ 0, & k \text{ odd}\end{cases}$$

Object	h-path sum	path sum
$\langle u\|G_{X,Y}\|v\rangle$	$m^{-2}\sum_{\tilde{\gamma}\in\Pi_{X,Y}(u,v)} s(\tilde{\gamma})$ (equation (119))	$\sum_{\gamma\in P_{X\setminus Y}(u,v)} w_{X,Y}(\gamma)$ (equation (127))
$\log\det\frac{K_{X,Y}}{m^2}$	$-\sum_{[\tilde{\gamma}]\in\Gamma^{\geq 1}_{X,Y}}\frac{s(\tilde{\gamma})}{t(\tilde{\gamma})}$ (equation (120))	
$\log\det\tilde{K}_{X,Y}$		$-\sum_{[\gamma]\in C^{\geq 1}_{X,Y}}\frac{w'_{X,Y}(\gamma)}{t(\gamma)}$ (equation (131))
$\langle u\|E_{Y,X}\|v\rangle$	$\sum_{\tilde{\gamma}\in\Pi'_{X,Y}(u,v)} s(\tilde{\gamma})$ (equation (125))	$\sum_{\gamma\in P'_{X,Y}(u,v)} w_{X,Y}(\gamma)$ (128)
$\langle u\|\,\mathrm{DN}_{Y,X}\,\|v\rangle$	$m^2\delta_{uv}-$	$(m^2+\mathrm{val}(v))\delta_{uv}-$
	$-m^2\sum_{\tilde{\gamma}\in\Pi''_{X,Y}(u,v)} s(\tilde{\gamma})$ (equation (124))	$-\sum_{P''_{X,Y}(u,v)} w_{X,Y}(\gamma)$ (equation (130))

Table 4. Summary of path sum formulae, relative case.

and

$$p^k_{X\setminus Y}(2,3)=p^k_{X\setminus Y}(3,2)=\begin{cases}0, & k \text{ even},\\ 1, & k \text{ odd},\end{cases}$$

Therefore, the relative Green's function is

$$G_{X,Y}(2,2)=\frac{1}{m^2+2}\sum_{k=0}^{\infty}\frac{1}{(m^2+2)^{2k}}=\frac{1}{m^2+2}\cdot\frac{1}{1-\frac{1}{(m^2+2)^2}}=\frac{m^2+2}{(m^2+1)(m^2+3)}$$

and

$$G_{X,Y}(2,3)=\frac{1}{m^2+2}\sum_{k=0}^{\infty}\frac{1}{(m^2+2)^{2k+1}}=\frac{1}{(m^2+1)(m^2+3)},$$

in agreement with (52).

As for the determinant, notice there is a unique cycle of length 2, all other cycles are iterates of this one, therefore, the logarithm of the normalized determinant is given by

$$\log\det\tilde{K}_{X,Y}=-\sum_{k=1}^{\infty}\frac{(m^2+2)^{-2k}}{k}=\log(1-(m^2+2)^{-2})$$

and the determinant is then

$$\det K_{X,Y}=(m^2+1)(m^2+3),$$

in agreement with (55).

For an example of the extension operator, notice that $(p')^k_{X,Y}(2,1)$ is 1 for odd k and 0 for even k, and therefore

$$\langle 2|E_{Y,X}|1\rangle=\sum_{k=0}^{\infty}(m^2+2)^{-(2k+1)}=\frac{m^2+2}{(m^2+1)(m^2+3)}$$

and similarly $(p')^k_{X,Y}(3,1) = 1$ for $k \geq 2$ even and 0 for odd k, and therefore

$$\langle 3|E_{Y,X}|1\rangle = \sum_{k=0}^{\infty}(m^2+2)^{-(2k+2)} = \frac{1}{(m^2+1)(m^2+3)},$$

in agreement with (54). Finally, we can compute the matrix elements of the Dirichlet-to-Neumann operator: we have $(p'')^k(1,1) = 1$ for even $k \geq 2$ and it vanishes for odd k, therefore

$$\langle 1|\,\mathrm{DN}_{Y,X}\,|1\rangle = m^2+1-\sum_{k=1}^{\infty}(m^2+2)^{-2k+1} = m^2+1-\frac{m^2+2}{(m^2+1)(m^2+3)}.$$

Similarly, $(p'')^k(1,4)$ vanishes for even k and is 1 for odd $k \geq 3$, and therefore

$$\langle 1|\,\mathrm{DN}_{Y,X}\,|4\rangle = -\sum_{k=1}^{\infty}\frac{1}{(m^2+2)^{2k}} = -\frac{1}{(m^2+1)(m^2+3)}.$$

These formulae agree with (53).

5.5. Gluing formulae from path sums

In this section we prove Theorem 3.7 from the path sum formulae presented in this chapter. The main observation in this proof is a decomposition of h-paths in X with respect to a subgraph Y.

Lemma 5.23. *Let $u, v \in X$, then we have a bijection*

$$\begin{aligned}&\Pi_X(u,v) \longleftrightarrow \\ &\Pi_{X,Y}(u,v) \sqcup \bigsqcup_{w_1,w_2\in Y} \Pi'_{X,Y}(u,w_1)\times\Pi_X(w_1,w_2)\times\Pi'_{X,Y}(w_2,v),\end{aligned} \tag{137}$$

where $\Pi_{X,Y}(u,v)$ denotes h-paths in X that contain no vertices in Y (but they may contain edges between $X \setminus Y$ and Y) and $\Pi'_{X,Y}(u,w)$, for either u or w in Y, denote h-paths containing exactly one vertex in Y, namely, the initial or final one[43].

Proof. One may decompose $\Pi_X(u,v)$ into paths containing no vertex in Y and those containing at least one vertex in Y. The former are precisely $\Pi_{X,Y}(u,v)$. If $\tilde{\gamma}$ is an element of the latter, let w_1 be the first vertex in $\tilde{\gamma}$ in Y and w_2 the last vertex in $\tilde{\gamma}$ in Y. Splitting $\tilde{\gamma}$ at w_1 and w_2 gives the map from left to right. The inverse map is given by composition of h-paths. See also Figure 19. ■

[43] It is possible to have $u = w \in Y$, in which case there $\Pi'_X(w,w)$ contains only the 1-element path.

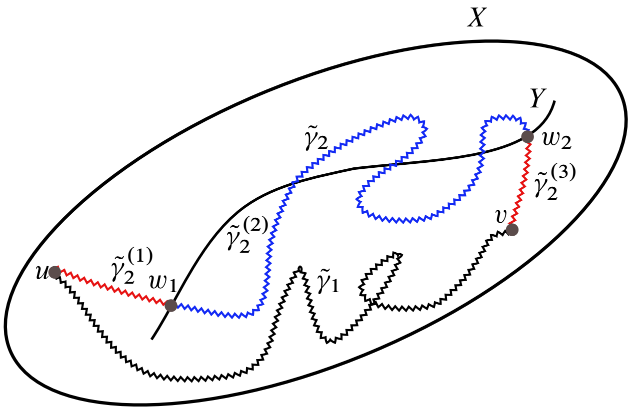


Figure 19. h-paths from u to v fall in two categories. Those not containing vertices in Y (e.g., $\tilde\gamma_1$) are paths in $\Pi_{X,Y}(u,v)$, while those intersecting Y (e.g., $\tilde\gamma_2$) can be decomposed in paths in $\tilde\gamma_2^{(1)} \in \Pi'_{X,Y}(u,w_1)$ and $\tilde\gamma_2^{(3)} \in \Pi'_{X,Y}(w_2,v)$ (red) and a path $\tilde\gamma_2^{(2)} \in \Pi_X(w_1,w_2)$, where w_1, w_2 are the first and last vertices in $\tilde\gamma_2$ contained in Y.

For the gluing formula for the determinant, we will also require the following observation on counting of closed paths.

Lemma 5.24. *Denote by $\Gamma^{\geq 1,(k)}_{X,Y}$ the set of h-cycles $[\tilde\gamma]$ in X of length $l(\tilde\gamma) \geq 1$ that intersect Y exactly k times, with $k \geq 1$. Then, concatenation of paths*

$$\bigsqcup_{w_1,w_2,\ldots,w_k \in Y} \Pi''_X(w_1,w_2) \times \Pi''_X(w_2,w_3) \times \cdots \times \Pi''_X(w_k,w_1) \to \Gamma^{\geq 1,(k)}_{X,Y} \quad (138)$$

is surjective, and a cycle $[\tilde\gamma]$ has precisely $k/t(\tilde\gamma)$ preimages.

Proof. For a cycle $\tilde\gamma \in \Gamma^{\geq 1,(k)}_{X,Y}$, denote by $w_1,\ldots,w_k$ the intersection points with Y and $\tilde\gamma^{(i)}$ the segment of $\tilde\gamma$ between w_{i+1} and w_i (here we set $w_{k+1} = w_1$). See Figure 20. Then, obviously $\tilde\gamma$ is the concatenation of the $\tilde\gamma^{(i)}$, so concatenation is surjective. On the other hand a k-tuple of paths concatenates to the same closed path if and only if they are related to each other by a cyclic shift (this corresponds to a cyclic shift of the labeling of the intersection points). They are precisely $k/t(\tilde\gamma)$ such shifts. ■

Recall that D' is the operator given by summing the weight $s(\tilde\gamma) = (m^{-2})^{l(\tilde\gamma)} (-1)^{h(\tilde\gamma)}$ over paths starting and ending on Y without intersecting Y in between (equation (122)).

Corollary 5.25. *We have that*

$$\operatorname{tr}(D')^k = k \sum_{\tilde\gamma \in \Gamma^{\geq 1,(k)}_{X,Y}} \frac{(m^{-2})^{l(\tilde\gamma)}(-1)^{h(\tilde\gamma)}}{t(\tilde\gamma)}. \quad (139)$$

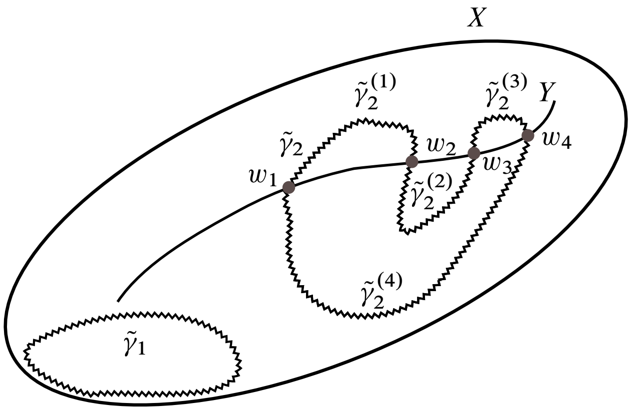


Figure 20. Cycles in X either do not intersect Y (like $\tilde{\gamma}_1$) and such that intersect Y k times (in the case of $\tilde{\gamma}_2$, $k = 4$). Such paths can be decomposed into k h-paths $\tilde{\gamma}^{(i)}$ in $\Pi''_{X,Y}(w_i, w_{i+1})$ in k different ways, corresponding to cyclic shift of the labels of w_i's.

Proof. The statement follows by summing the weight $s(\tilde{\gamma})$ over the left-hand side and right-hand side of (138) in Lemma 5.24, using multiplicativity of $s(\tilde{\gamma})$ in the left-hand side and with multiplicity $k/t(\tilde{\gamma})$ in the right-hand side (corresponding to the count of preimages of the map (138)). ∎

h-path sum proof of Theorem 3.7. We first prove the gluing formula

$$\langle u|G_X|v\rangle = \langle u|G_{X,Y}|v\rangle + \sum_{v_1,v_2\in Y} \langle u|E_{Y,X}|v_1\rangle\langle v_1|\,\mathrm{DN}_{Y,X}^{-1}\,|v_2\rangle\langle v_2|E_{Y,X}|v\rangle.$$

Applying the decomposition of $\Pi_X(u,v)$ (137), and using multiplicativity of the weight $s(\tilde{\gamma}) = (m^{-2})^{l(\tilde{\gamma})}(-1)^{h(\tilde{\gamma})}$, we get

$$\begin{aligned} &G_X(u,v) \\ &= m^{-2} \sum_{\tilde{\gamma}\in\Pi_{X,Y}(u,v)} s(\tilde{\gamma}) \\ &\quad + \sum_{w_1,w_2\in Y} \Big(\sum_{\tilde{\gamma}_1\in\Pi'_{X,Y}(u,w_1)} s(\tilde{\gamma})\Big)\Big(m^{-2}\sum_{\tilde{\gamma}_2\in\Pi_X(w_1,w_2)} s(\tilde{\gamma})\Big)\Big(\sum_{\tilde{\gamma}_1\in\Pi'_{X,Y}(w_2,v)} s(\tilde{\gamma})\Big). \end{aligned} \tag{140}$$

The first term is $G_{X,Y}$ by equation (119). In the second term, we recognize the path sum expressions (125) for the extension operator and (121) for the operator D, which is the inverse of the total Dirichlet-to-Neumann operator. This completes the proof of the gluing formula for the propagator.

Next, we prove the gluing formula for the determinant

$$\det(K_X) = \det(K_{X,Y}) \det(\mathrm{DN}_{Y,X}).$$

Dividing both sides by m^{2N}, where N is the number of vertices in X, this is equivalent to

$$\det(m^{-2} K_X) = \det(m^{-2} K_{X,Y}) \det(m^{-2}\, \mathrm{DN}_{Y,X}).$$

Taking negative logarithms and using $\log\det = \operatorname{tr}\log$, this is equivalent to

$$-\log\det(1 + m^{-2}\Delta_X) = -\log\det(1 + m^{-2}\Delta_{X,Y}) - \operatorname{tr}\log(I - D'), \tag{141}$$

where we have used that $\mathrm{DN}_{Y,X} = m^2(I - D')$.

We claim that equation (141) can be proven by summing over paths. Indeed, the left-hand side is given by summing over closed h-paths in X. We decompose them into paths which do not intersect Y, and those that do. From the former we obtain $-\log\det K_{X,Y}/m^2$ by equation (120). Decompose the latter set into paths that intersect Y exactly k times, previously denoted by $C^{\geq 1,(k)}_{X,Y}$. By Lemma 5.24, when summing over those paths we obtain precisely $\operatorname{tr}(D')^k/k$. Summing over k, we obtain

$$\operatorname{tr}\sum_{k\geq 1}(D')^k/k = -\operatorname{tr}\log(I - D'),$$

which proves the gluing formula for the determinant. ∎

Path sum proof of Theorem 3.7. In the proof above we used the h-path expansions, but of course one could have equally well used the formulae in terms of paths. To prove the gluing formula for the Green's function

$$\langle u|G_X|v\rangle = \langle u|G_{X,Y}|v\rangle + \sum_{v_1,v_2\in Y}\langle u|E_{Y,X}|v_1\rangle\langle v_1|\,\mathrm{DN}^{-1}_{Y,X}\,|v_2\rangle\langle v_2|E_{Y,X}|v\rangle$$

in terms of path counts, notice that a path crossing Y can again be decomposed into a path from X to Y, then a path from Y to Y and another path from Y to X. The weight $w(\gamma) = \prod_{v\in V(\gamma)}(m^2 + \mathrm{val}(v))^{-1}$ is distributed by among those three paths by taking the vertices on Y to the $Y - Y$ path: In this way, when summing over all paths from the $Y - Y$ paths we obtain precisely the operator $D = \mathrm{DN}^{-1}_{Y,X}$ (this is a submatrix of G_X and hence the weights of paths *include* start and end vertices) while from the other parts we obtain the extension operator $E_{Y,X}$ (where weights of paths *do not* include the vertex on Y).

Next, we consider the gluing formula for the determinant

$$\det K_X = \det K_{X,Y} \det \mathrm{DN}_{Y,X}\,.$$

Dividing both sides by

$$\det \Lambda_X = \prod_{v\in X}(m^2+\mathrm{val}(v)) = \prod_{v\in X\setminus Y}(m^2+\mathrm{val}_X(v))\prod_{v\in Y}(m^2+\mathrm{val}_X(v))$$
$$= \det \Lambda_{X,Y} \det \Lambda_Y,$$

this is equivalent to

$$\det \widetilde{K}_X = \det \widetilde{K}_{X,Y} \det \Lambda_Y^{-1}\,\mathrm{DN}_{Y,X}.$$

Taking logarithms and using the formulae (108) and (131) for logarithms of determinants of kinetic operators, we get

$$\log\det \widetilde{K}_X - \log\det \widetilde{K}_{X,Y} = -\sum_{[\gamma]\in C_X^{\geq 1}, V(\gamma)\cap Y\neq\varnothing} \frac{w'_X(\gamma)}{t(\gamma)}, \tag{142}$$

where on the right-hand side we are summing over cycles in X that intersect Y. We therefore want to show that the sum on the r.h.s. of (142) equals $\log\det \Lambda_Y^{-1}\,\mathrm{DN}_{Y,X}$. From (130), we have that $\mathrm{DN}_{Y,X} = \Lambda_Y - D''$, where we introduced the auxiliary operator $D''\colon C^0(Y)\to C^0(Y)$ with matrix elements

$$\langle u|D''|v\rangle = \sum_{\gamma\in P''_X(u,v)} w_{X,Y}(\gamma).$$

Then,

$$-\log\det\Lambda_Y^{-1}\,\mathrm{DN}_{Y,X} = -\operatorname{tr}\log(I-\Lambda_Y^{-1}D'') = \sum_{k\geq 1}\operatorname{tr}\frac{(\Lambda_Y^{-1}D'')^k}{k}. \tag{143}$$

Notice that $\operatorname{tr}(\Lambda_Y^{-1}D'')^k$ is given by summing over closed paths γ that intersect Y exactly k times, with the weight $w'(\gamma)$: the factor $(m^2+\mathrm{val}(v))^{-1}$ in $w'(\gamma)$ for vertices not on Y comes from D'' (recall that $w_{X,Y}(\gamma)$ does not contain factors for vertices on Y), and from Λ_Y^{-1}, for $v\in Y$. By a combinatorial argument analogous to Lemma 5.24, every cycle appears in this way exactly $\frac{k}{t(\gamma)}$ times. Therefore, the sum on the right-hand side of (142) equals the sum on the r.h.s. of equation (143), which finishes the proof. ∎

6. Interacting theory: first quantization formalism

In this section, we extend the path sum formulae to the interacting theory. In this language, weights of Feynman graphs are given by summing over all possible maps from a Feynman graph to a spacetime graph, where edges are mapped to paths. We also analyze the gluing formula in terms of path sums.

6.1. Closed graphs

We first consider the case of closed graphs.

6.1.1. Edge-to-path maps. Let Γ and X be graphs. Recall that by P_X we denote the set of all paths in X, and by Π_X the set of h-paths in X.

Definition 6.1. An *edge-to-path map* $F = (F_V, F_P)$ from Γ to X is a pair of maps $F_V\colon V_\Gamma \to V_X$ and $F_P\colon E(\Gamma) \to P_X$ such that for every edge $e = (\mathsf{u}, \mathsf{v})$ in Γ, we have

$$F_P(e) \in P_X(F_V(\mathsf{u}), F_V(\mathsf{v})).$$

The set of edge-to-path maps is denoted by P_X^Γ.

Equivalently, an edge-to-path map is a lift of a map $F_V\colon V_\Gamma \to V_X$ to the fibrations $E_\Gamma \to V_\Gamma \times V_\Gamma$ and $P_X \to V_X \times V_X$:

$$\begin{array}{ccc} E_\Gamma & \xrightarrow{F_P} & P_X \\ \downarrow & & \downarrow \\ V_\Gamma \times V_\Gamma & \xrightarrow{F_V \times F_V} & V_X \times V_X \end{array}$$

Similarly, we define an *edge-to-h-path map* as a lift of a map $F_V\colon V_\Gamma \to V_X$ to the fibrations $E_\Gamma \to V_\Gamma \times V_\Gamma$, $\Pi_X \to V_X \times V_X$:

$$\begin{array}{ccc} E_\Gamma & \xrightarrow{F_\Pi} & \Pi_X \\ \downarrow & & \downarrow \\ V_\Gamma \times V_\Gamma & \xrightarrow{F_V \times F_V} & V_X \times V_X \end{array}$$

The set of such maps is denoted by Π_X^Γ. Alternatively, an edge-to-path map can be thought of as labeling of Γ, where we label vertices in Γ by vertices of X and edges in Γ by *paths* in X.

6.1.2. Feynman weights. Suppose that Γ is a Feynman graph appearing in the perturbative partition function on a closed graph, with weight given by (81). By the results of the previous section, we have the following first quantization formula, a combinatorial analog of the first quantization formula (6).

Proposition 6.2. *The weight of the Feynman graph Γ has the path sum expression*

$$\Phi_{\Gamma,X} = \prod_{\mathsf{v}\in V_\Gamma} (-p_{\mathrm{val}(\mathsf{v})}) \sum_{F\in P_X^\Gamma} \prod_{e\in E_\Gamma} w(F_P(e)) \tag{144}$$

$$= \prod_{\mathsf{v}\in V_\Gamma} (-p_{\mathrm{val}(\mathsf{v})}) \sum_{F\in \Pi_X^\Gamma} \prod_{e\in E_\Gamma} s(F_\Pi(e)), \tag{145}$$

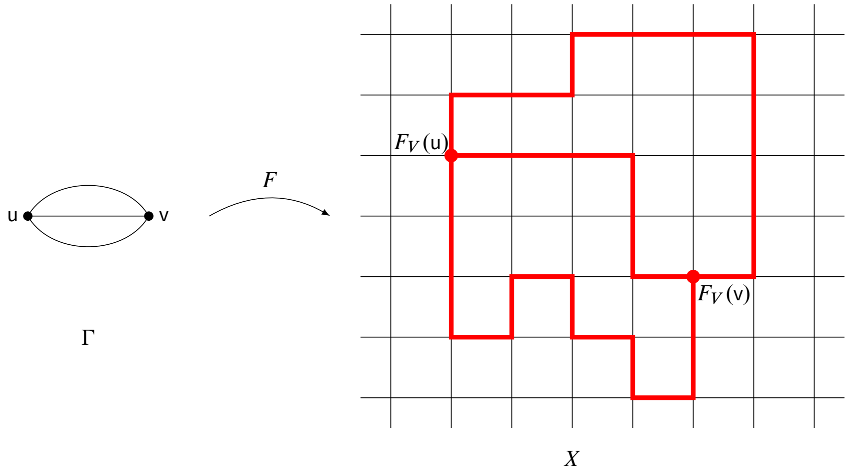


Figure 21. An example of an edge-to-path map.

where in (144) *we are summing over all edge-to-path maps from* Γ *to* X*, and in* (145) *we are summing over all edge-to-h-path maps.*

Figure 21 contains an example of an edge-to-path map from Γ the Θ-graph to a grid X.

We then have the following expression of the perturbative partition function.

Corollary 6.3. *The perturbative partition function of* X *is given in terms of edge-to-paths maps as*

$$Z_X^{\text{pert}} = \det(K_X)^{-\frac{1}{2}} \sum_{\Gamma} \frac{\hbar^{-\chi(\Gamma)}}{\text{Aut}(\Gamma)} \sum_{F \in P_X^\Gamma} \prod_{\mathsf{v} \in V_\Gamma} (-p_{\text{val}(\mathsf{v})}) \prod_{e \in E_\Gamma} w(F_P(e)). \tag{146}$$

We can reformulate this as the following "first quantization formula".

Corollary 6.4. *The logarithm of the perturbative partition function has the expression:*

$$\begin{aligned} \log Z_X^{\text{pert}} &= \frac{1}{2} \sum_{[\gamma] \in C_X^{\geq 1}} \frac{w'(\gamma)}{t(\gamma)} + \sum_{\Gamma^{\text{conn}}} \frac{\hbar^{-\chi(\Gamma)}}{\text{Aut}(\Gamma)} \sum_{F \in P_X^\Gamma} \prod_{\mathsf{v} \in V_\Gamma} (-p_{\text{val}(\mathsf{v})}) \prod_{e \in E_\Gamma} w(F_P(e)) \\ &\quad - \frac{1}{2} \sum_{v \in X} \log(m^2 + \text{val}(v)). \end{aligned} \tag{147}$$

Here, Γ^{conn} *stands for connected Feynman graphs.*

We remark that in the second line of (147), one can interpret the first term as coming from an analog of edge-to-path maps for the circle, divided by automorphisms of such maps (the factor of 2 comes from orientation reversal). In this sense, the second line can be interpreted as the partition function of a 1d sigma model with target X. The term in the third line should be interpreted as a normalizing constant.

6.2. Relative version

Now, we let X be a graph and Y a subgraph, and consider the interacting theory on X relative to Y. Recall that in the relative case, Feynman graphs Γ have vertices split into bulk and boundary vertices, with Feynman weight given by (83). Bulk vertices have valence at least 3, while boundary vertices are univalent. Again, we do not want to allow boundary-boundary edges. Edge-to-path maps now additionally have to respect the type of edge: bulk-bulk edges are mapped to paths in $P_{X \setminus Y}$ and bulk-boundary edges are mapped to paths in $P'_{X,Y}$. We collect this in following technical definition:

Definition 6.5. Let Γ be a graph with $V(\Gamma) = V_\Gamma^{\text{bulk}} \sqcup V_\Gamma^{\partial}$, such that $\text{val}(v) \geq 3$ for all $v \in V_\Gamma^{\text{bulk}}$ and $\text{val}(v^\partial) = 1$ for all $v^\partial \in V_\Gamma^\partial$. Denote by the induced decomposition of edges by

$$E(\Gamma) = E_\Gamma^{\text{bulk-bulk}} \sqcup E_\Gamma^{\text{bulk-bdry}} \sqcup E_\Gamma^{\text{bdry-bdry}}.$$

Let X be a graph and $Y \subset X$ be a subgraph. Then, a *relative edge-to-path map* (resp., *relative edge-to-h-path map*) is a pair $F = (F_V, F_P)$ (resp., $F = (F_V, F_\Pi)$), where $F_V : V(\Gamma) \to V(X)$ and $F_P : E(\Gamma) \to P_X$ (resp., $F_\Pi : E(\Gamma) \to \Pi_X$) such that

- F_V respects the vertex decompositions, i.e., $F_V(V_\Gamma^{\text{bulk}}) \subset V(X) \setminus V(Y)$ and $F_V(V_\Gamma^{\text{bdry}}) \subset V(Y)$,
- F_E (resp., F_Π) is a lift of F_V, i.e., for all edges $e = (\mathsf{u}, \mathsf{v}) \in E(\Gamma)$ we have $F_P(e) \in P_X(F_V(\mathsf{u}), F_V(\mathsf{v}))$ (resp., $F_\Pi(e) \in \Pi_X(F_V(\mathsf{u}), F_V(\mathsf{v}))$),
- F_P (resp., F_Π) respects the edge decompositions, i.e., $F_P(E_\Gamma^{\text{bulk-bulk}}) \subset P_{X \setminus Y}$, $F_P(E_\Gamma^{\text{bulk-bdry}}) \subset P'_{X,Y}$, and similarly for F_Π.

The set of relative edge-to-(h-)path maps is denoted by $P_{X,Y}^\Gamma$ (resp., $\Pi_{X,Y}^\Gamma$).

Figure 22 contains an example of a relative edge-to-path map from Γ a Feynman graph with boundary vertices to a grid X relative to a subgraph Y.

We can now express the weight of a Feynman graph with boundary vertices as a sum over relative edge-to-path maps – the combinatorial analog of the first quantization formula (15).

Proposition 6.6. *Suppose that Γ is a Feynman graph with boundary vertices and $\phi \in C^0(Y)$. Then, the Feynman weight $\Phi_{\Gamma,(X,Y)}(\phi_Y)$ can be expressed by summing*

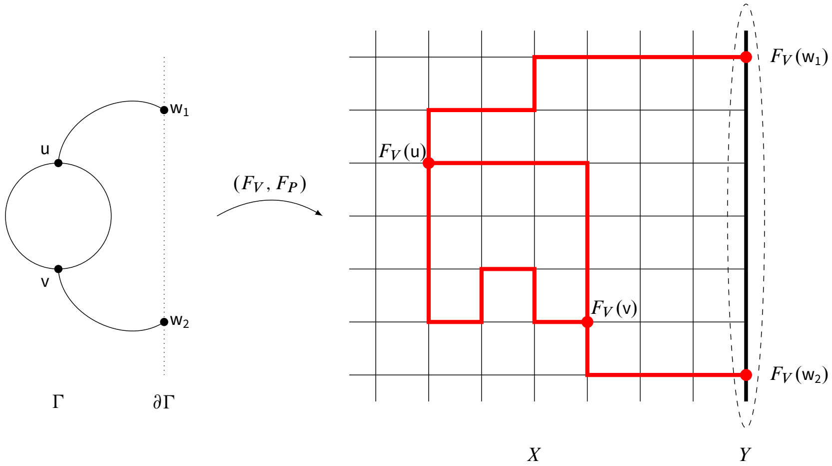


Figure 22. An example of a relative edge-to-path map.

over relative edge-to-path maps as

$$\Phi_{\Gamma,(X,Y)}(\phi_Y) = \sum_{F \in P^{\Gamma}_{X,Y}} \prod_{\mathsf{v} \in V^{\text{bulk}}_{\Gamma}} (-p_{\text{val}(\mathsf{v})}) \prod_{\mathsf{v}^{\partial} \in V^{\partial}_{\Gamma}} \phi_Y(F_V(\mathsf{v}^{\partial})) \cdot \prod_{e \in E_{\Gamma}} w_{X,Y}(F_P(e)). \tag{148}$$

In terms of h-paths, the expression is

$$\Phi_{\Gamma,(X,Y)}(\phi_Y) = (m^2)^{-\#E^{\text{bulk-bulk}}} \sum_{F \in \Pi^{\Gamma}_{X,Y}} \prod_{\mathsf{v} \in V^{\text{bulk}}_{\Gamma}} (-p_{\text{val}(\mathsf{v})}) \prod_{\mathsf{v}^{\partial} \in V^{\partial}_{\Gamma}} \phi_Y(F_V(\mathsf{v}^{\partial})) \cdot \prod_{e \in E_{\Gamma}} s(F_{\Pi}(e)). \tag{149}$$

Proof. In (148), we are using the path sum formulae (127), (128). Similarly, to see (149), we are using the relative h-path sums (119), (125), and notice that every bulk-bulk Green's function comes with an additional power of m^{-2}. ∎

We immediately obtain the following formula for the partition function.

Proposition 6.7. *The relative perturbative partition function can be expressed as*

$$Z^{\text{pert}}_{X,Y}(\phi) = \det(K_{X,Y})^{-\frac{1}{2}} \cdot e^{-\frac{1}{2\hbar}((\phi_Y, (\text{DN}_{Y,X} - \frac{1}{2}K_Y)\phi_Y) - S^{\text{int}}_Y(\phi_Y))} \cdot \sum_{\Gamma} \frac{\hbar^{-\chi(\Gamma)}}{|\operatorname{Aut}(\Gamma)|} \sum_{F \in P^{\Gamma}_{X,Y}} \prod_{\mathsf{v} \in V^{\text{bulk}}_{\Gamma}} (-p_{\text{val}(\mathsf{v})}) \prod_{\mathsf{v}^{\partial} \in V^{\partial}_{\Gamma}} \phi_Y(F_V(\mathsf{v}^{\partial})) \cdot \prod_{e \in E_{\Gamma}} w_{X,Y}(F_P(e)). \tag{150}$$

Remark 6.8. As in Remark 4.2, the Dirichlet-to-Neumann operator in the exponent of (150) could be expanded in terms of Feynman diagrams with boundary-boundary edges. An edge-to-path map F should map such a boundary-boundary edge $e = (\mathsf{u}^\partial, \mathsf{v}^\partial)$ either to a path $\gamma \in P''_X$ (which is weighted with $w_{X,Y}(\gamma)$) or, in the case, where $F_V(\mathsf{u}^\partial) = F_V(\mathsf{v}^\partial)$, possibly to the constant path (v^∂) (which is then weighted with $-(m^2 + \mathrm{val}(\mathsf{v}^\partial))$).

Equivalently, one has the following expression for the logarithm of the relative perturbative partition function:

$$\begin{aligned}
&\log Z^{\mathrm{pert}}_{X,Y}(\phi) \\
&= \frac{1}{2\hbar}\Bigg(\sum_{u,v\in Y}\phi_Y(u)\phi_Y(v)\cdot\Big(-(m^2+\mathrm{val}_X(v))\delta_{uv}+\sum_{\gamma\in P''_{X,Y}(u,v)} w_{X,Y}(\gamma)\Big) \\
&\qquad\qquad + \frac{1}{2}(\phi_Y, K_Y\phi_Y) + \sum_{v\in Y} p(\phi_Y(v))\Bigg) \\
&\quad + \frac{1}{2}\Bigg(\sum_{[\gamma]\in C^{\geq 1}_{X\setminus Y}} \frac{w'_{X,Y}(\gamma)}{t(\gamma)} - \sum_{v\in X}\log(m^2+\mathrm{val}(v))\Bigg) \\
&\quad + \sum_{\Gamma^{\mathrm{conn}}} \frac{\hbar^{-\chi(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \sum_{F\in P^\Gamma_{X,Y}} \prod_{\mathsf{v}\in V^{\mathrm{bulk}}_\Gamma} (-p_{\mathrm{val}(\mathsf{v})}) \prod_{\mathsf{v}^\partial\in V^\partial_\Gamma} \phi_Y(F_V(\mathsf{v}^\partial))\cdot \prod_{e\in E_\Gamma} w_{X,Y}(F_P(e)).
\end{aligned} \tag{151}$$

This generalizes the results (147) and (136) to relative interacting case.

6.3. Cutting and gluing

The goal of this section is to provide a sketch of a proof of the gluing of perturbative partition functions (85) by counting paths. Suppose that $X = X' \cup_Y X''$ and $F \in P^\Gamma_X$ is an edge-to-path map from a Feynman graph Γ to X[44]. Then, the decomposition $X = X' \cup_Y X''$ induces a decoration of Γ, as in Section 4.2. Namely, we decorate a vertex $\mathsf{v} \in V_\Gamma$ with $\alpha \in \{X', Y, X''\}$ if $F_V(\mathsf{v}) \in \alpha$, and we decorate an edge e with c if and only if the path $F_P(e)$ contains a vertex in Y. See Figure 23.

Recall that from a decorated graph, we can form two new graphs X' and X'' with boundary vertices. Given and edge-to-path map F and its induced decoration of Γ, we can define two new relative edge-to-path maps (F'_V, F'_P) and (F''_V, F''_P) for the new graphs X' and X'' as follows. The map F'_V is simply the restriction of F_V to vertices

[44] Again, for notational simplicity we consider only the case, where X is closed, with the generalization to cobordisms notationally tedious but straightforward.

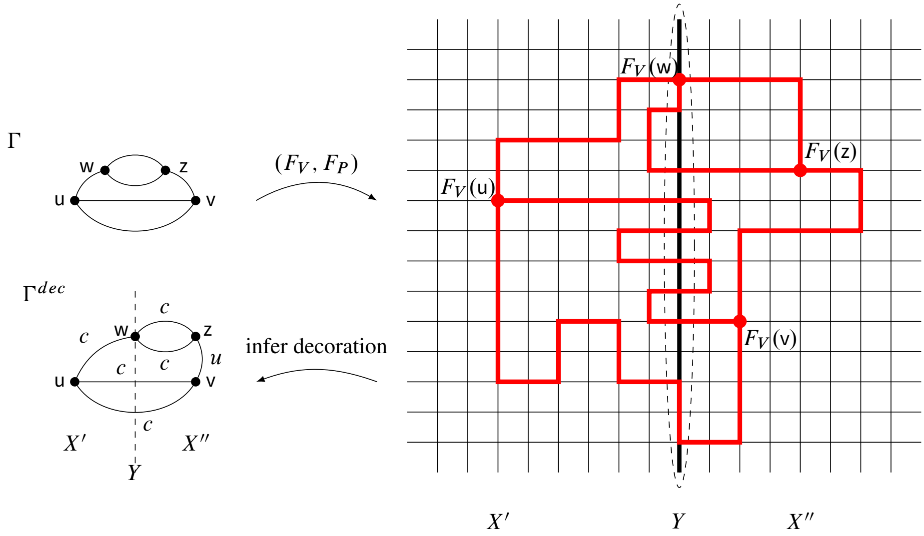


Figure 23. Inferring a decoration of Γ from cutting an edge-to-path map $\Gamma \to X' \cup_Y X''$.

colored X'. For edges labeled u, $F'_P(e) = F_P(e)$. For a bulk-boundary edge in X', $F'_P(e)$ is the segment of the path $F_P(\tilde{e})$ of the corresponding edge $\tilde{e}$ in Γ (that was necessarily labeled c) up to (and including) the first vertex in Y. The construction of (F''_V, F''_P) is similar, as is the extension to edge-to-hesitant-path maps. The definition of Γ', Γ'' ensures that (F'_V, F'_P) and (F''_V, F''_P) are well-defined relative edge-to-path maps. For example, from the edge-to-path map in Figure 23, one obtains the two edge-to-path maps in Figure 24.

Notice that in the process of creating the cut edge-to-path-maps we are forgetting about the parts of the paths between the first and the last crossing of Y, as well as the vertices labelled with Y. This information is encoded in the Dirichlet-to-Neumann operator and the interacting term S_Y^{int}, respectively. Integrating the product of a pair of relative edge-to-path maps appearing in the product $Z^{\text{pert}}_{X',Y}(\phi_Y) Z^{\text{pert}}_{X'',Y}(\phi_Y)$ over ϕ_Y, two things happen.

- An arbitrary number of vertices on Y is created (due to the factor of $e^{-\frac{1}{\hbar} S_Y^{\text{int}}}(\phi_Y)$).
- All vertices on Y (the new boundary vertices and those coming from the relative edge-to-path maps) are connected by the inverse D of total Dirichlet-to-Neumann operators.

In this way, we obtain all edge-to-path maps that give rise to this pair of relative edge-to-path maps. This provides a sketch of an alternative proof of the gluing formula for perturbative partition functions using the first quantization formalism, i.e., path sums.

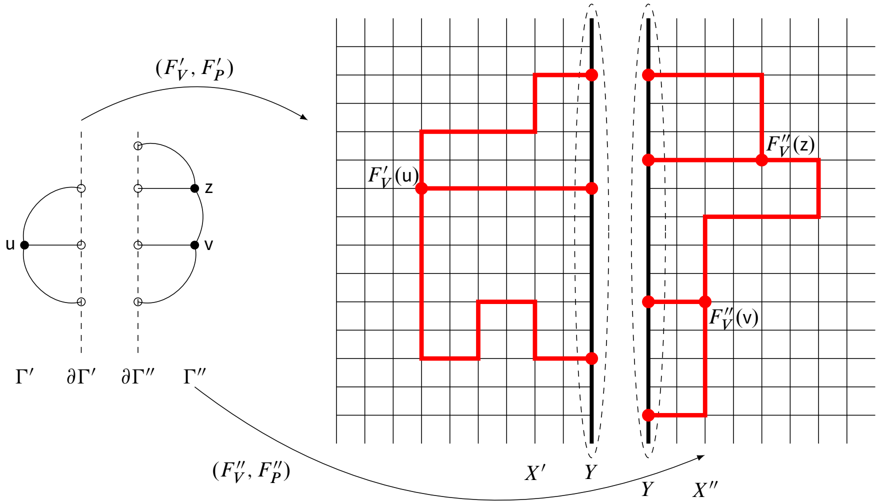


Figure 24. Relative edge-to-path maps (F'_V, F'_P) from Γ' to (X', Y) and (F''_V, F''_P) from Γ'' to (X'', Y) arising from the cutting of the Feynman graph Γ^{dec} in Figure 23.

7. Conclusion and outlook

In this paper, we analyzed a combinatorial toy model for massive scalar QFT, where the spacetime manifold is a graph. We focused on incarnations of locality – the behaviour under cutting and gluing – and the interplay with the first quantization formalism. In particular, we showed that the convergent functional integrals naturally define a functor with source a graph cobordism category and target the category of Hilbert spaces. We discussed the perturbative theory – the $\hbar \to 0$ limit – and its behaviour under cutting and gluing. Finally, we analyzed the theory in the first quantization formalism, where all objects have expressions in terms of sums over paths (or h-paths) in the spacetime graph. We showed that cutting and gluing interacts naturally with those path sums. Below we outline several promising directions for future research.

- *Continuum limit, renormalization.* In this paper, we discussed the behaviour of the theory in the continuum limit for line graphs only. However, our toy model is in principle dimension agnostic and makes sense on lattice graphs of any dimension, and one can take a similar continuum limit there. However, in the interacting theory in dimension $d \geq 2$, one has to take into account the issue of divergencies and renormalization[45]. It will be interesting to see how this problem manifests

[45]In $d = 2$, the only divergent subgraphs are short loops, but already those interact nontrivially with cutting and gluing – see [11].

itself in the continuum limit of a higher-dimensional lattice graph, and whether this approach will be helpful in defining a renormalized massive scalar QFT with cutting and gluing.

Another interesting question in the continuum limit is to recover the first quantization path measure (in particular, the action functionals (5), (9)) from a limit of our weight system on paths on a dense lattice graph.

- *Massless limit.* Another interesting problem is the study of the limit $m \to 0$. In this limit, the kinetic operator becomes degenerate if no boundary conditions are imposed, and extra work is needed to make sense of theory[46]. This will be particularly interesting in the case of two-dimensional lattice graphs, where the massless limit of the continuum theory is a conformal field theory (in the free case, $p(\phi) = 0$), thus the massless limit of our toy model is a discrete model for this CFT. We also remark that while the h-path formulae do not interact well with the $m \to 0$ limit, since they are expansions in m^{-2}, for path sums the weight of a path at $m = 0$ is

$$w(\gamma) = \prod_{v \in V(\Gamma)} \frac{1}{\mathrm{val}(v)},$$

i.e., the weight of the path is the probability of a random walk on the graph, where at every vertex, the walk can continue along all adjacent edges with probability $1/\mathrm{val}(v)$.

- *Gauge theories on cell complexes.* Finally, it will be interesting to study in a similar fashion (including first quantization formalism) gauge theories (e.g., p-form electrodynamics, Yang–Mills or AKSZ theories) on a cell complex, with gauge fields (and ghosts, higher ghosts and antifields) becoming cellular cochains[47].

A. Green's function and det K_X for a line graph

In this appendix, we will prove formulae (33), (35) from Example 3.2.

A.1. Green's function

Recall that in the continuum setting, the Green's function $G(x, y)$ for Helmholtz operator $\Delta + m^2$ on an interval $[a, b]$ (with Dirichlet or Neumann boundary conditions)

[46] In this case, it is natural to formulate the perturbative quantum answers in terms of effective actions of the zero-mode of the field ϕ; it might be natural here to employ the BV-BFV formalism [5] combining effective actions with cutting-gluing.

[47] Second quantization formalism for (abelian and non-abelian) BF theory on a cell complex is developed in [6]. What we propose here is a generalization to other models, possibly involving metric on cochains, and a focus on the path-sum approach.

can be found by fixing y and sewing the solutions $G_{1,2}(x, y)$ of Helmholtz equation $(\Delta + m^2)G(x, y) = 0$ for $x \in [a, y]$ and $x \in [y, b]$, imposing continuity and jump -1 of the derivative at $x = y$:

$$G_1(y-0, y) = G_2(y+0, y), \quad \left.\frac{\partial}{\partial x}\right|_{x=y+0} G_2(x, y) - \left.\frac{\partial}{\partial x}\right|_{x=y-0} G_1(x, y) = -1.$$

The computation below can be seen as a finite difference counterpart of this sewing argument.

For the line graph of Figure 3, we can write the equation $K_X G = \mathrm{Id}$ on the Green's function, with K_X given by (32), as

$$-G(i-1, j) + (2+m^2)G(i, j) - G(i+1, j) = \delta_{ij} \tag{152}$$

with $1 \leq j \leq N$ fixed and with $1 \leq i \leq N$ the discrete variable. Here, for convenience we introduced

$$G(0, j) := G(1, j), \tag{153}$$
$$G(N+1, j) := G(N, j). \tag{154}$$

Let us split equation (152) into three ranges for $i - i < j, i = j$ and $i > j$:

$$-G(i-1, j) + (2+m^2)G(i, j) - G(i+1, j) = 0 \quad \text{for } 1 \leq i < j, \tag{155}$$
$$-G(j-1, j) + (2+m^2)G(j, j) - G(j+1, j) = 1, \tag{156}$$
$$-G(i-1, j) + (2+m^2)G(i, j) - G(i+1, j) = 0 \quad \text{for } j < i \leq N. \tag{157}$$

Homogeneous equations (155), (157) are satisfied by any linear combination of $e^{\beta i}$ and $e^{-\beta i}$, where $\beta > 0$ is determined from

$$-e^{-\beta} + (2+m^2) - e^{\beta} = 0 \Leftrightarrow \cosh\beta = 1 + \frac{m^2}{2} \underset{\text{using } \beta>0}{\Leftrightarrow} \sinh\frac{\beta}{2} = \frac{m}{2}.$$

Thus, we have an ansatz for a solution of (155) and (157):

$$G(i, j) = A_+ e^{\beta i} + A_- e^{-\beta i} \quad \text{for } 0 \leq i \leq j, \tag{158}$$
$$G(i, j) = B_+ e^{\beta i} + B_- e^{-\beta i} \quad \text{for } j \leq i \leq N+1. \tag{159}$$

Here, the coefficients $A_\pm$, $B_\pm$ can depend on j (but not on i). Note that the range of i in the ansatz (158), (159) is increased by ± 1 with respect to the range in the equations (155), (157) so that they hold in their range.

Next, note that "Neumann boundary condition" (153) implies a relation between A_+ and A_-:

$$G(1, j) - G(0, j) = A_+(e^{\beta} - 1) + A_-(e^{-\beta} - 1) = 0 \Leftrightarrow A_- = e^{\beta} A_+ \tag{160}$$

$$\underset{(158)}{\Rightarrow} G(i, j) = A_+ e^{\beta/2} 2\cosh\beta\left(i - \frac{1}{2}\right) \quad \text{for } 0 \leq i \leq j. \tag{161}$$

Likewise, the "Neumann boundary condition" (154) implies a relation between B_+ and B_-:

$$G(N+1, j) - G(N, j)$$
$$= B_+ e^{\beta N}(e^{\beta} - 1) + B_- e^{-\beta N}(e^{-\beta - 1}) = 0 \Leftrightarrow B_- = e^{\beta(2N+1)} B_+ \tag{162}$$
$$\underset{(159)}{\Rightarrow} G(i, j) = B_+ e^{\beta(N+\frac{1}{2})} 2\cosh\beta\left(N + \frac{1}{2} - i\right) \quad \text{for } j \le i \le N+1. \tag{163}$$

Comparing (161) and (162) for $i = j$ yields a "continuity condition," tantamount to a relation between A_+ and B_+:

$$A_+ e^{\beta/2} 2\cosh\beta\left(j - \frac{1}{2}\right) = B_+ e^{\beta(N+\frac{1}{2})} 2\cosh\beta\left(N + \frac{1}{2} - j\right). \tag{164}$$

Finally, (158) ("derivative jump condition") yields

$$1 = -G(j-1, j) + (2 + m^2)G(j, j) - G(j+1, j)$$
$$= -A_+ e^{\beta/2}\cosh\beta\left(j - \frac{3}{2}\right) + 2\cosh\beta A_+ e^{\beta_2} 2\cosh\beta\left(j - \frac{1}{2}\right)$$
$$- B_+ e^{\beta(N+\frac{1}{2})} 2\cosh\beta\left(N - \frac{1}{2} - j\right)$$
$$\underset{(164)}{=} \frac{2e^{\beta/2} A_+}{\cosh\beta(N + \frac{1}{2} - j)} \sinh\beta \sinh\beta N$$
$$\Rightarrow A_+ = \frac{\cosh\beta(N + \frac{1}{2} - j)}{2e^{\beta/2}\sinh\beta\sinh\beta N}. \tag{165}$$

Substituting this value of A_+ into (161) and the corresponding B_+ into (162), we finally obtain the Green's function

$$G(i, j) = \frac{\cosh\beta(i - \frac{1}{2})\cosh\beta(N + \frac{1}{2} - j)}{\sinh\beta\sinh\beta N} \quad \text{for } 1 \le i \le j, \tag{166}$$

$$G(i, j) = \frac{\cosh\beta(j - \frac{1}{2})\cosh\beta(N + \frac{1}{2} - i)}{\sinh\beta\sinh\beta N} \quad \text{for } j \le i \le N. \tag{167}$$

An equivalent way to write the result is

$$G(i, j) = \frac{\cosh\beta(N - |i - j|) + \cosh\beta(N + 1 - i - j)}{\sinh\beta\sinh\beta N}. \tag{168}$$

This finishes the proof of (33).

A.2. Determinant

The derivative of $\det K_X$ in m^2 can be expressed as the trace of the Green's function:

$$\frac{d}{dm^2}\log\det K_X = \operatorname{tr} K_X^{-1}\underbrace{\frac{d}{dm^2}K_X}_{\mathrm{Id}} = \operatorname{tr} G = \sum_{i=1}^{N} G(i,i)$$
$$= \sum_{i=1}^{N}\frac{\cosh\beta N + \cosh\beta(N+1-2i)}{2\sinh\beta\sinh\beta N}$$
$$= \frac{1}{2\sinh\beta\sinh\beta N}\Big(N\cosh\beta N + \underbrace{\big(e^{\beta(N-1)} + e^{\beta(N-3)} + \cdots + e^{-\beta(N-1)}\big)}_{\frac{\sinh\beta N}{\sinh\beta}}\Big). \tag{169}$$

Next, switching to the derivative of the determinant in β, we obtain

$$\frac{d}{d\beta}\log\det K_X = \underbrace{\frac{dm^2}{d\beta}}_{2\sinh\beta}\frac{d}{dm^2}\log\det K_X = N\coth\beta N + \frac{1}{\sinh\beta}. \tag{170}$$

Integrating in β and exponentiating, we obtain

$$\det K_X = C_N \tanh\frac{\beta}{2}\sinh\beta N. \tag{171}$$

The constant C_N (independent of β) can be fixed, e.g., from $m^2 \to \infty$ asymptotics: $\det K_X$ is a polynomial of degree N in m^2 with top coefficient 1; hence,

$$\det K_X \underset{\beta\to\infty}{\sim} e^{\beta N}. \tag{172}$$

On the other hand, we have

$$\text{right-hand side of (171)} \underset{\beta\to\infty}{\sim} \frac{C_N}{2}e^{\beta N},$$

which implies that $C_N = 2$, thus proving the claimed result (35) for the determinant.

Acknowledgments. We thank Olga Chekeres, Andrey Losev, and Donald R. Youmans for inspiring discussions on the first quantization approach in QFT. We also thank the anonymous referee for useful comments. IC, PM, and KW would like to thank the Galileo Galilei Institute, where part of the work was completed, for hospitality.

Funding. The work of K. Wernli was supported by the ERCSyG project, Recursive and Exact New Quantum Theory (ReNewQuantum) which received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program under grant agreement no. 810573.

References

[1] L. Andersson and B. K. Driver, Finite-dimensional approximations to Wiener measure and path integral formulas on manifolds. *J. Funct. Anal.* **165** (1999), no. 2, 430–498 Zbl 0943.58024 MR 1698956

[2] D. Burghelea, L. Friedlander, and T. Kappeler, Meyer–Vietoris type formula for determinants of elliptic differential operators. *J. Funct. Anal.* **107** (1992), no. 1, 34–65 Zbl 0759.58043 MR 1165865

[3] G. Carron, Déterminant relatif et la fonction Xi. *Amer. J. Math.* **124** (2002), no. 2, 307–352 Zbl 1014.34015 MR 1890995

[4] B. Casiday, I. Contreras, T. Meyer, S. Mi, and E. Spingarn, Laplace and Dirac operators on graphs. *Linear Multilinear Algebra* **72** (2024), no. 2, 325–365 Zbl 07808509 MR 4703574

[5] A. S. Cattaneo, P. Mnev, and N. Reshetikhin, Perturbative quantum gauge theories on manifolds with boundary. *Comm. Math. Phys.* **357** (2018), no. 2, 631–730 Zbl 1390.81381 MR 3767705

[6] A. S. Cattaneo, P. Mnev, and N. Reshetikhin, A cellular topological field theory. *Comm. Math. Phys.* **374** (2020), no. 2, 1229–1320 Zbl 1514.81235 MR 4072239

[7] R. Dijkgraaf, Les houches lectures on fields, strings and duality. 1997, arXiv:hep-th/9703136

[8] P. Etingof, Mathematical ideas and notions of quantum field theory. 2002

[9] R. P. Feynman and A. R. Hibbs, *Quantum mechanics and path integrals*. McGraw-Hill, New York, 1965 Zbl 0176.54902

[10] S. Kandel, Functorial quantum field theory in the Riemannian setting. *Adv. Theor. Math. Phys.* **20** (2016), no. 6, 1443–1471 Zbl 1431.81153 MR 3607061

[11] S. Kandel, P. Mnev, and K. Wernli, Two-dimensional perturbative scalar QFT and Atiyah–Segal gluing. *Adv. Theor. Math. Phys.* **25** (2021), no. 7, 1847–1952 Zbl 1507.81129 MR 4456089

[12] Y. Lee, Mayer–Vietoris formula for the determinant of a Laplace operator on an even-dimensional manifold. *Proc. Amer. Math. Soc.* **123** (1995), no. 6, 1933–1940 Zbl 0830.58031 MR 1254845

[13] P. Mnëv, Discrete path integral approach to the Selberg trace formula for regular graphs. *Comm. Math. Phys.* **274** (2007), no. 1, 233–241 Zbl 1142.58018 MR 2318854

[14] P. Mnev, Graph quantum mechanics. 2016, contribution to *Jahrbuch der Max-Planck-Gesellschaft*

[15] D. Pickrell, $P(\phi)_2$ quantum field theories and Segal's axioms. *Comm. Math. Phys.* **280** (2008), no. 2, 403–425 Zbl 1155.81027 MR 2395476

[16] N. Reshetikhin, Lectures on quantization of gauge systems. In *New paths towards quantum gravity. extended proceedings of the summer school, Holbæk, Denmark, May 12–16, 2008*, pp. 125–190, Lecture Notes in Phys. 807, Springer, Berlin, 2010 Zbl 1235.83004

[17] N. Reshetikhin and B. Vertman, Combinatorial quantum field theory and gluing formula for determinants. *Lett. Math. Phys.* **105** (2015), no. 3, 309–340 Zbl 1312.58011 MR 3312509

[18] D. V. Vassilevich, Heat kernel expansion: user's manual. *Phys. Rep.* **388** (2003), no. 5-6, 279–360 Zbl 1042.81093 MR 2017528

[19] S. D. Vecchio, *Path sum formulae for propagators on graphs, gluing and continuum limit*. Master's thesis, ETH Zürich, Switzerland, 2012

[20] N. M. J. Woodhouse, *Geometric quantization*. 2nd edn., Oxford Math. Monogr., The Clarendon Press, Oxford University Press, New York, 1992 Zbl 0747.58004 MR 1183739

Communicated by Adrian Tanasă

Received 27 August 2023; revised 16 April 2024.

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