

Profinite non-rigidity of arithmetic groups

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Abstract. We show that for a typical higher-rank arithmetic lattice Γ , there exist finite index subgroups Γ_1 and Γ_2 such that $\Gamma_1 \not\cong \Gamma_2$ while $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$. But there are exceptions to that rule.

1. Introduction

Let Γ be a finitely generated residually finite group. We say that Γ is profinitely rigid if whenever $\widehat{\Lambda} \cong \widehat{\Gamma}$ for some finitely generated residually finite group Λ , then $\Lambda \cong \Gamma$. Here, $\widehat{\Gamma}$ (resp. $\widehat{\Lambda}$) denotes the profinite completion of Γ (resp. Λ).

Up until recently, the only profinite rigid groups were “small” (i.e., without non-abelian free subgroups). Recently, in a groundbreaking work, Bridson, McReynolds, Reid, and Spitler gave first examples of “big” groups which are profinitely rigid, among them are some fundamental groups of hyperbolic 3-manifolds [2] and some triangle groups [3].

Arithmetic subgroups of semisimple Lie groups need not be profinitely rigid [1, 7]. A well-known open problem asks:

For $n \geq 2$, is $SL_n(\mathbb{Z})$ profinitely rigid?

While we will not answer this question, we will show that there are finite index subgroups of these groups (at least when $n \geq 3$) which are not profinitely rigid. In fact, we will show a much more general result.

Theorem 1.1 (Main theorem). *Let k be a number field and \mathbf{G} be a connected, simply connected, absolutely almost simple k -linear algebraic group of high ∞ -rank such that $\mathbf{G}(k)$ satisfies the congruence subgroup property and $\Gamma \subseteq \mathbf{G}(k)$ an arithmetic subgroup:*

- (1) *Unless \mathbf{G} is of type G_2 , F_4 , or E_8 and $k = \mathbb{Q}$, Γ has infinitely many pairs of finite index subgroups Γ_1 and Γ_2 which are not isomorphic but their profinite completions are.*
- (2) *The exceptional cases are truly exceptional and in these cases, there are no such pairs at all. In fact, if $\Gamma_1, \Gamma_2 \subseteq \mathbf{G}(\mathbb{Q})$ are arithmetic subgroups with isomorphic profinite completions, then Γ_1 and Γ_2 are isomorphic.*

Note that we show that each such Γ has a finite index subgroup Γ_1 which is not profinitely rigid by showing that $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$ for some Γ_2 commensurable to it. This complements results of [1, 7, 8] which give examples of non-commensurable arithmetic groups which are profinitely isomorphic.

To illustrate our methods, let us now present them only for $\Gamma = SL_4(\mathbb{Z})$ (Methods A and B) and $\Gamma = SL_2(\mathbb{Z}[\sqrt{2}])$ (Method C).

Method A: Using the center of the simply connected form: Let $2 \neq p, q$ be two different primes, and let Λ be the principal congruence subgroup corresponding to pq . Let ρ_p be the element of $\widehat{SL}_4(\mathbb{Z}) \cong \prod_r SL_4(\mathbb{Z}_r)$ which is 1 at the places $r \neq p$ and -1 at the place p , similarly define ρ_q . Define $\Gamma_1 := \langle \widehat{\Lambda}, \rho_p \rangle \cap SL_4(\mathbb{Z})$ and $\Gamma_2 := \langle \widehat{\Lambda}, \rho_q \rangle \cap SL_4(\mathbb{Z})$. Then $\widehat{\Gamma}_1 \cong \widehat{\Lambda} \times \mathbb{Z}/2\mathbb{Z} \cong \widehat{\Gamma}_2$ but Γ_1 and Γ_2 cannot be isomorphic. For details see Theorem 3.1.

Method B: Using a non-trivial Dynkin automorphism: Let $2, 3 \neq p, q$ be two different primes. Consider the following maximal parabolic subgroups of $SL_4(\mathbb{Z}/p\mathbb{Z})$:

$$P_{1,p} := \left\{ \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix} \right\}, \quad P_{2,p} := \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ & & & * \end{pmatrix} \right\}.$$

Similarly define the maximal parabolic subgroups $P_{1,q}$ and $P_{2,q}$ of $SL_4(\mathbb{Z}/q\mathbb{Z})$. Now, let Γ_1 be the congruence subgroup corresponding to $P_{1,p} \bmod p$, $P_{1,q} \bmod q$ and is trivial mod 3, and Γ_2 to be the congruence subgroup corresponding to $P_{1,p} \bmod p$, $P_{2,q} \bmod q$ and is trivial mod 3. Then the profinite completions of Γ_1 and Γ_2 are isomorphic via the automorphism of $\widehat{SL}_4(\mathbb{Z})$ which is the non-trivial Dynkin automorphism at the place q and the identity elsewhere, but they themselves cannot be isomorphic. For details see Theorem 4.1.

Method C: Using the number field: Let $2 \neq p, q$ be two different primes such that 2 is a square in \mathbb{Q}_p and \mathbb{Q}_q , hence p and q split completely in $\mathbb{Q}[\sqrt{2}]$. Set $\mathcal{O} = \mathbb{Z}[\sqrt{2}]$ and $\mathfrak{p}_1, \mathfrak{p}_2|p$, $\mathfrak{q}_1, \mathfrak{q}_2|q$ to be the primes lying over p and q , respectively. Let Γ_1 be the principal congruence subgroup corresponding to $\mathfrak{p}_1\mathfrak{q}_1$, and Γ_2 be the principal congruence subgroup corresponding to $\mathfrak{p}_2\mathfrak{q}_1$. As $SL_2(\mathcal{O})$ has trivial congruence kernel, $\widehat{SL}_2(\mathcal{O}) \cong (\prod_{l \neq p,q} SL_2(\mathcal{O}_l)) \times \prod_{i=1}^2 (SL_2(\mathcal{O}_{\mathfrak{p}_i}) \times SL_2(\mathcal{O}_{\mathfrak{q}_i}))$, where for a prime $l \neq p, q$, \mathcal{O}_l is the completion of \mathcal{O} with respect to the primes lying over l . Then the profinite completions, $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_2$, are isomorphic via the automorphism of $\widehat{SL}_2(\mathbb{Z}[\sqrt{2}])$ which is the transposition of the places \mathfrak{p}_1 and \mathfrak{p}_2 . But Γ_1 and Γ_2 cannot be isomorphic. For details see Theorem 3.2.

The main theorem is proved by generalizing the above methods to more general arithmetic lattices. (In fact, only Methods A and C are really needed.)

The paper is organized as follows: After some preliminaries in Section 2, we will generalize Methods A and C in Section 3 deducing the first part of the main theorem.

In Section 4, we will elaborate on Method B and finally in Section 5 we will prove that the exceptional cases are true exceptions, concluding the main theorem. In Section 6, we will give a stronger and more general version of the main theorem, stating that it holds for S -arithmetic groups and not merely for arithmetic groups. Moreover, one can get any (finite) number of non-isomorphic subgroups with isomorphic profinite completions (not just pairs).

2. Preliminaries

Throughout we assume that k is a number field. The set of places of k is denoted by $V(k)$; it is the union of the set of archimedean places $V_\infty(k)$ and the set of finite places $V_f(k)$. The completion of k at $v \in V(k)$ is denoted by k_v . Let \mathcal{O}_k denote the ring of integers of k and for a finite place $v \in V_f(k)$, denote by $\mathcal{O}_{k,v}$ the ring of integers of k_v . The ring of finite adeles $\mathbb{A}_k^f = \prod_{v \in V_f(k)}^* k_v := \{(x_v)_v \in \prod_{v \in V_f(k)} k_v : x_v \in \mathcal{O}_{k,v} \text{ for all but finitely many places}\}$ is the restricted product over all the finite completions of k . If k is clear from the context, we will omit the letter k from all the above.

Let $\tilde{\mathbf{G}}$ be a connected, simply connected, absolutely almost simple k -linear algebraic group, with a fixed faithful k -representation $\rho : \tilde{\mathbf{G}} \rightarrow GL(n_\rho)$. A subgroup $\Gamma \subseteq \tilde{\mathbf{G}}(k)$ is called arithmetic if it is commensurable with $\tilde{\mathbf{G}}(\mathcal{O})$ (see [10, 13] for more details about arithmetic groups). We will usually write \mathbf{G} for the adjoint form of $\tilde{\mathbf{G}}$ (which is the universal form), and by $\pi : \tilde{\mathbf{G}} \rightarrow \mathbf{G}$ the universal covering map, it is a central isogeny, and $\mathbf{Z} := \ker \pi = \mathbf{Z}(\tilde{\mathbf{G}})$ is a finite group. The $V_\infty(k)$ -rank of $\tilde{\mathbf{G}}$ is $\text{rank}_{V_\infty(k)} \tilde{\mathbf{G}} := \sum_{v \in V_\infty(k)} \text{rank}_{k_v} \tilde{\mathbf{G}}$, where $\text{rank}_{k_v} \tilde{\mathbf{G}}$ is the dimension of a maximal k_v -split torus; $\tilde{\mathbf{G}}$ is said to have high ∞ -rank if its $V_\infty(k)$ -rank is ≥ 2 .

We will use Margulis’ superrigidity in a rather delicate manner. The particular version we use is the following.

Theorem 2.1 (Margulis’ superrigidity). *Assume $\tilde{\mathbf{G}}$ is of high ∞ -rank, and let $\Gamma_1, \Gamma_2 \subseteq \tilde{\mathbf{G}}(k)$ be arithmetic subgroups. Assume further that $\Gamma_i \cap \mathbf{C}(k) = 1$. If $\varphi : \Gamma_1 \rightarrow \Gamma_2$ is an isomorphism, then there exists a unique k -automorphism Φ of $\tilde{\mathbf{G}}$ and a unique automorphism σ of k such that $\varphi(\gamma) = \Phi(\sigma^0(\gamma))$ for every $\gamma \in \Gamma_1$, where σ^0 is the automorphism of $\tilde{\mathbf{G}}(k)$ induced by σ .*

Proof. Identify Γ_1 and Γ_2 as arithmetic subgroups of the adjoint group $\mathbf{G}(k)$ via the universal covering map $\pi : \tilde{\mathbf{G}} \rightarrow \mathbf{G}$. Moving to the adjoint form enables us to use Margulis’ superrigidity [9, Theorem VIII.3.6. (ii)] extending the isomorphism φ to an automorphism of $\text{Res}_{k/\mathbb{Q}} \mathbf{G}$. By the properties of the restriction of scalars functor, such an automorphism must be of the form $\Phi \circ \sigma^0$ for a k -automorphism Φ of \mathbf{G} and an automorphism σ of the field k [5, Proposition A.5.14].

The k -automorphism Φ of \mathbf{G} is of the form $C_g \circ \omega$ where C_g is conjugation by some $g \in \mathbf{G}(\bar{k})$ (preserving $\mathbf{G}(k)$) and ω is a Dynkin automorphism. Both can be interpreted

as automorphisms of $\tilde{\mathbf{G}}$ and these interpretations act the same way upon Γ_1 . Hence, the assertion of the theorem. ■

If σ is an automorphism of k , it induces a permutation of the (finite) places of k , and thus an automorphism of the adelic group $\tilde{\mathbf{G}}(\mathbb{A}_k^f)$ by permuting its factors according to σ , call this automorphism $\sigma_{\mathbb{A}}^0$. If Φ is a k -automorphism of $\tilde{\mathbf{G}}$, it induces a unique k_v -automorphism $\Phi_v : \tilde{\mathbf{G}}(k_v) \rightarrow \tilde{\mathbf{G}}(k_v)$ for every finite place v of k and the product $(\Phi_v)_v : \prod_v \tilde{\mathbf{G}}(k_v) \rightarrow \prod_v \tilde{\mathbf{G}}(k_v)$ restricts to an automorphism $\Phi_{\mathbb{A}} : \tilde{\mathbf{G}}(\mathbb{A}_k^f) \rightarrow \tilde{\mathbf{G}}(\mathbb{A}_k^f)$ [13, §5]. Clearly $\Phi(\sigma^0(\gamma)) = \Phi_{\mathbb{A}}(\sigma_{\mathbb{A}}^0(\gamma))$ for every $\gamma \in \tilde{\mathbf{G}}(k)$ (we identify the group of rational points $\tilde{\mathbf{G}}(k)$ with its diagonal embedding in the group of adelic points $\tilde{\mathbf{G}}(\mathbb{A}_k^f)$) and is unique with this property. We thus get the following corollary.

Corollary 2.2. *Under the assumptions of the previous theorem. If $\varphi : \Gamma_1 \rightarrow \Gamma_2$ is an isomorphism. Then there exist unique automorphisms $\Phi_{\mathbb{A}}$ and $\sigma_{\mathbb{A}}^0$ of the adelic group $\tilde{\mathbf{G}}(\mathbb{A}_k^f)$, such that $\sigma_{\mathbb{A}}^0$ is induced from an automorphism of k , and $\Phi_{\mathbb{A}}$ is induced from a k -automorphism of $\tilde{\mathbf{G}}$ with $\varphi(\gamma) = \Phi_{\mathbb{A}}(\sigma_{\mathbb{A}}^0(\gamma))$ for every $\gamma \in \Gamma_1$.*

We will also need an adelic version of Margulis’ superrigidity stated and proven by Kammeyer and Kionke [6, Theorem 3.2].

Theorem 2.3. *Let \mathbf{G} be a connected, absolutely almost simple \mathbb{Q} -linear algebraic group of high ∞ -rank and $\Gamma \subseteq \mathbf{G}(\mathbb{Q})$ an arithmetic subgroup. If $\varphi : \Gamma \rightarrow \mathbf{G}(\mathbb{A}_{\mathbb{Q}}^f)$ is a homomorphism such that $\overline{\varphi(\Gamma)}$ has non-empty interior, then there exist a homomorphism of adelic groups $\eta : \mathbf{G}(\mathbb{A}_{\mathbb{Q}}^f) \rightarrow \mathbf{G}(\mathbb{A}_{\mathbb{Q}}^f)$ and a group homomorphism $\nu : \Gamma \rightarrow \mathbf{Z}(\mathbf{G})(\mathbb{A}_{\mathbb{Q}}^f)$ with finite image such that $\varphi(\gamma) = \nu(\gamma)\eta(\gamma)$ for all $\gamma \in \Gamma$. Moreover, η and ν are uniquely determined by this condition.*

2.1. Profinite groups and the congruence subgroup property

A family $(\{G_i\}_{i \in I}, \{\phi_{i,j}\}_{i \geq j \in I})$ is an inverse system of finite groups over the directed set I if the G_i s are finite groups, $\phi_{i,j} : G_i \rightarrow G_j$ are homomorphisms of groups whenever $i \geq j$ such that $\phi_{i,k} = \phi_{j,k}\phi_{i,j}$ for every $i \geq j \geq k$ and $\phi_{ii} = id_{G_i} \forall i$. A group G is called profinite if it is the inverse limit of an inverse system of finite groups over some directed set. A profinite group is a compact, Hausdorff, totally disconnected topological group, and a map of profinite groups is a continuous homomorphism of groups.

Example 2.4 (Profinite completion). Let Γ be a finitely generated group, and let \mathcal{N} be the set of finite index normal subgroups of Γ ; for $M, N \in \mathcal{N}$, declare that $M \leq N$ whenever $N \subseteq M$; it is a directed set. Consider the natural quotient homomorphisms $\phi_{N,M} : \Gamma/N \rightarrow \Gamma/M$, then the profinite group $\hat{\Gamma} = \lim_{\leftarrow N \in \mathcal{N}} \Gamma/N$ is called the *profinite completion* of Γ .

The profinite completion $\hat{\Gamma}$ and the set $\mathcal{C}(\Gamma)$ of isomorphism classes of the finite quotients of Γ hold the same information in the following manner.

Theorem 2.5 ([16, Theorems 3.2.2 and 3.2.7]). *If Γ and Λ are two finitely generated residually finite groups, then $\mathcal{C}(\Gamma) = \mathcal{C}(\Lambda)$ if and only if $\widehat{\Gamma} \cong \widehat{\Lambda}$*

There is a natural map $\iota : \Gamma \rightarrow \widehat{\Gamma}$ given by $\gamma \mapsto (\gamma N)_N$, and this map is injective if and only if Γ is residually finite; in this case, we identify Γ with its image $\iota(\Gamma)$. The pair $(\widehat{\Gamma}, \iota)$ satisfies a universal property: $\iota(\Gamma)$ is dense in $\widehat{\Gamma}$, and for every profinite group P , and every homomorphism $\varphi : \Gamma \rightarrow P$, there exists a unique homomorphism of profinite groups $\widehat{\varphi} : \widehat{\Gamma} \rightarrow P$ such that $\widehat{\varphi} \circ \iota = \varphi$.

There is a strong connection between the finite index subgroups of $\widehat{\Gamma}$ and those of Γ .

Proposition 2.6 ([16, Proposition 3.2.2]). *Let Γ be a finitely generated residually finite group, then there is a one-to-one correspondence between the set \mathcal{X} of all finite index subgroups of Γ and the set \mathcal{Y} of all finite index subgroups of $\widehat{\Gamma}$, given by*

$$\begin{aligned} X &\mapsto \overline{X}, & X &\in \mathcal{X} \\ Y &\mapsto Y \cap \Gamma, & Y &\in \mathcal{Y}, \end{aligned}$$

where \overline{X} denote the closure of X in $\widehat{\Gamma}$. Moreover, this bijection preserves normality, index, and quotients.

Note that we have implemented in the above the assertion of [12] that every finite index subgroup of $\widehat{\Gamma}$ is open.

Example 2.7 (Congruence completion). Let Γ be an arithmetic subgroup of $\widetilde{\mathbf{G}}(k)$. Consider the set \mathcal{C} of all congruence subgroups, that is, subgroups that contain $\Gamma[\mathcal{I}] := \Gamma \cap \ker(\phi_{\mathcal{I}} : \widetilde{\mathbf{G}}(\mathcal{O}_k) \rightarrow \widetilde{\mathbf{G}}(\mathcal{O}_k/\mathcal{I}))$ for some ideal $\mathcal{I} \triangleleft \mathcal{O}_k$, where $\phi_{\mathcal{I}}$ is the reduction map mod \mathcal{I} . As in the profinite completion, \mathcal{C} is a directed set by the inverse of inclusion, and one can form the *congruence completion* $\overline{\Gamma}$ of Γ with respect to this inverse system.

Thus, there is a surjective map $\widehat{\Gamma} \rightarrow \overline{\Gamma}$ between the profinite completion and the congruence completion. Call $C(\Gamma)$, the kernel of this map, the *congruence kernel*. The group Γ is said to have the *congruence subgroup property* if the congruence kernel $C(\Gamma)$ is a finite group. It is not difficult to see that the congruence subgroup property is actually a property of the ambient group $\widetilde{\mathbf{G}}$ and the field k .

It was conjectured by Serre [17] that if $\text{rank}_{V_{\infty}(k)}(\widetilde{\mathbf{G}}) \geq 2$ and $\Gamma \subseteq \widetilde{\mathbf{G}}(k)$ is an arithmetic subgroup then $C(\Gamma)$ is trivial or isomorphic to a subgroup of the roots of unity of k if k is totally imaginary. The conjecture has been proven in many instances, including for example, all the isotropic cases [15] and all anisotropic groups of type B_n, C_n, D_n (except for some triality forms of D_4), E_7, E_8, F_4 , and G_2 ([13, Chapter 9] and [14]).

2.2. A number theoretic lemma

Lemma 2.8. *Let $\widetilde{\mathbf{G}}$ be a connected, simply connected, absolutely almost simple k -linear algebraic group. There exist infinitely many finite places $v \in V_f(k)$ such that $\widetilde{\mathbf{G}}$ splits over k_v . Moreover, one can assume that for these places, $\mathbf{Z}(\mathcal{O}_v) = \mathbf{Z}(\mathbb{C})$.*

Proof. There exists a finite Galois field extension k'/k such that $\tilde{\mathbf{G}}$ splits over k' and $\mathbf{Z}(k') = \mathbf{Z}(\mathbb{C})$. By Chebotarev’s density theorem [11, Corollary 13.6], there exist infinitely many primes $\mathfrak{p} \subseteq \mathcal{O}_k$ that split completely in k' . In particular, if such a prime lies under a prime $\mathfrak{p}' \subseteq \mathcal{O}_{k'}$ then $k_v \cong k'_{v'}$, where v and v' are the places corresponding to the primes \mathfrak{p} and \mathfrak{p}' , respectively. Thus, for such a place v , $\tilde{\mathbf{G}}$ splits over k_v .

Moreover, as the center $\mathbf{Z}(k')$ is finite, for all but finitely many places, $\mathbf{Z}(k') = \mathbf{Z}(\mathcal{O}_{k',v'})$. So there exist infinitely many places $v \in V_f(k)$ with $\mathbf{Z}(\mathcal{O}_{k,v}) = \mathbf{Z}(\mathbb{C})$ and $\tilde{\mathbf{G}}$ splits over k_v . ■

3. First part of the main theorem: Existence

In this section we will prove the first part of Theorem 1.1, the existence part. It will follow from the two theorems below.

Theorem 3.1. *Let $\Gamma \subseteq \tilde{\mathbf{G}}(k)$ be an arithmetic subgroup. Assume that $\tilde{\mathbf{G}}$ has the congruence subgroup property and type different than E_8, F_4 , or G_2 . Then there exist two non-isomorphic finite index subgroups $\Gamma_1, \Gamma_2 \subseteq \Gamma$ with isomorphic profinite completions.*

Proof (Following Method A). Moving to a finite index subgroup, one can assume that there exists a finite set of primes S and a compact open subgroup $\Lambda \subseteq \prod_{v \in S} \tilde{\mathbf{G}}(k_v)$ commensurable with $\prod_{v \in S} \tilde{\mathbf{G}}(\mathcal{O}_v)$ such that

$$\hat{\Gamma} = \bar{\Gamma} \cong \Lambda \times \prod_{v \notin S} \tilde{\mathbf{G}}(\mathcal{O}_v). \tag{1}$$

Indeed, the congruence kernel $C(\Gamma) \subseteq \hat{\Gamma}$ is finite, so one can find a finite index subgroup $\Gamma'^P \subseteq \hat{\Gamma}$ of the above form; by Proposition 2.6, there exists a finite index subgroup $\Gamma' \subseteq \Gamma$ such that $\hat{\Gamma}' = \Gamma'^P$. Let $\pi : \tilde{\mathbf{G}} \rightarrow \mathbf{G}$ be the canonical central isogeny to the adjoint form \mathbf{G} , by moving again to a finite index subgroup one can assume that $\Gamma \cap C(k) = \{1\}$, and $\hat{\Gamma}$ is still of the same form as (1).

Let $S_{\mathbb{Q}}$ be the set of all rational primes lying under some valuation in S , and $S_{\text{full}} = \{v : v \mid p \text{ for some } p \in S_{\mathbb{Q}}\}$. By Lemma 2.8, one can find two valuations $\mathfrak{p}, \mathfrak{q} \notin S_{\text{full}}$ lying over different rational primes p and q , respectively, and such that $\mathbf{Z}(\mathcal{O}_{\mathfrak{p}}) = \mathbf{Z}(\mathcal{O}_{\mathfrak{q}}) = \mathbf{Z}(\mathbb{C})$, and thus also elements $1 \neq \rho_{\mathfrak{p}} \in \mathbf{Z}(\mathcal{O}_{\mathfrak{p}})$ and $1 \neq \rho_{\mathfrak{q}} \in \mathbf{Z}(\mathcal{O}_{\mathfrak{q}})$ of the same order. For every $v \mid p, q$, let Δ_v be a finite index subgroup of $\tilde{\mathbf{G}}(\mathcal{O}_v)$ with $\mathbf{Z}(\mathcal{O}_v) \cap \Delta_v = \{1\}$. Define $\varepsilon_{\mathfrak{p}} = (\varepsilon_{\mathfrak{p},w})_w, \varepsilon_{\mathfrak{q}} = (\varepsilon_{\mathfrak{q},w})_w \in \hat{\Gamma}$, where

$$\varepsilon_{\mathfrak{p},w} := \begin{cases} 1 & \text{if } w \neq \mathfrak{p} \\ \rho_{\mathfrak{p}} & \text{if } w = \mathfrak{p} \end{cases}, \quad \varepsilon_{\mathfrak{q},w} := \begin{cases} 1 & \text{if } w \neq \mathfrak{q} \\ \rho_{\mathfrak{q}} & \text{if } w = \mathfrak{q} \end{cases}.$$

Now, we define the following subgroups of $\widehat{\Gamma}$:

$$\begin{aligned} \Delta &= \Lambda \times \prod_{v \notin S, v \nmid p, v \nmid q} \widetilde{\mathbf{G}}(\mathcal{O}_v) \times \prod_{v|p \text{ or } v|q} \Delta_v \\ \Delta_1 &= \langle \Delta, \varepsilon_p \rangle \cong \Delta \times \langle \varepsilon_p \rangle \\ \Delta_2 &= \langle \Delta, \varepsilon_q \rangle \cong \Delta \times \langle \varepsilon_q \rangle. \end{aligned}$$

Clearly Δ_1 and Δ_2 are isomorphic finite index subgroups of $\widehat{\Gamma}$. By Proposition 2.6, there exist finite index subgroups $\Gamma_i \subseteq \Gamma$ with $\widehat{\Gamma}_i = \Delta_i$ for $i = 1, 2$. We will finish the proof by showing that Γ_1 and Γ_2 cannot be isomorphic.

Assume to the contrary that there exists an isomorphism $\varphi : \Gamma_1 \rightarrow \Gamma_2$. By Corollary 2.2, there exist unique adelic automorphisms $\Phi_{\mathbb{A}}$ and $\sigma_{\mathbb{A}}^0$ of $\widetilde{\mathbf{G}}(\mathbb{A}_k^f)$ such that $\sigma_{\mathbb{A}}^0$ is induced from an automorphism of k and $\Phi_{\mathbb{A}}$ is induced from a k -automorphism of $\widetilde{\mathbf{G}}$ such that $\varphi(\gamma) = \Phi_{\mathbb{A}}(\sigma_{\mathbb{A}}^0(\gamma))$ for every $\gamma \in \Gamma_1$, taking closures one deduces that $(\widetilde{\Phi}_{\mathbb{A}} \circ \sigma_{\mathbb{A}}^0)(\widehat{\Gamma}_1) = \widehat{\Gamma}_2$. On the other hand, the induced map between the profinite completions $\widehat{\varphi} : \widehat{\Gamma}_1 \rightarrow \widehat{\Gamma}_2$ is unique with $\widehat{\varphi}(\gamma) = \varphi(\gamma)$ for every $\gamma \in \Gamma_1$. Thus, it must be that $\widehat{\varphi} \equiv (\widetilde{\Phi}_{\mathbb{A}} \circ \sigma_{\mathbb{A}}^0)|_{\widehat{\Gamma}_1}$. In particular, it implies that the p th place of $\widehat{\Gamma}_1$ is mapped isomorphically onto the $\sigma(p)$ th place of $\widehat{\Gamma}_2$. This is a contradiction since $\sigma(p) \nmid p$, so the $\sigma(p)$ th place of $\widehat{\Gamma}_2$ is centerless, but the p th place of $\widehat{\Gamma}_1$ has a non-trivial center. ■

Theorem 3.2. *Let $\Gamma \subseteq \widetilde{\mathbf{G}}(k)$ be an arithmetic subgroup. Assume further that $\widetilde{\mathbf{G}}(k)$ has the congruence subgroup property and that k is a number field of degree $d \geq 2$ over \mathbb{Q} . Then there exist two non-isomorphic finite index subgroups $\Gamma_1, \Gamma_2 \subseteq \Gamma$ with isomorphic profinite completions.*

Proof (Following Method C). As before, by moving to a finite index subgroup we can assume that $\Gamma \cap \mathbf{C}(k) = \{1\}$ and that $\widehat{\Gamma} = \overline{\Gamma} = \Lambda \times \prod_{v \notin S} \widetilde{\mathbf{G}}(\mathcal{O}_v)$, for some finite set of places S and $\Lambda \subseteq \prod_{v \in S} \widetilde{\mathbf{G}}(k_v)$ commensurable with $\prod_{v \in S} \widetilde{\mathbf{G}}(\mathcal{O}_v)$. Let $S_{\mathbb{Q}}$ and S_{full} be as before. By Chebotarev’s density theorem, there exist infinitely many rational primes that split completely in k [11, Corollary 13.6]; pick two such different primes $p, q \notin S_{\mathbb{Q}}$. Say $\mathfrak{p}_1, \dots, \mathfrak{p}_d | p$ and $\mathfrak{q}_1, \dots, \mathfrak{q}_d | q$. For $i = 1, 2$, let $\Gamma_i := \Gamma(\mathfrak{p}_i \mathfrak{q}_1)$ be the principle congruence subgroups modulo $\mathfrak{p}_i \mathfrak{q}_1$, then obviously $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$.

By Corollary 2.2, if Γ_1 and Γ_2 are isomorphic. As above, there exist unique adelic automorphisms $\Phi_{\mathbb{A}}$ and $\sigma_{\mathbb{A}}^0$ of $\widetilde{\mathbf{G}}(\mathbb{A}_k^f)$ such that $\sigma_{\mathbb{A}}^0$ is induced from an automorphism of k and $\Phi_{\mathbb{A}}$ is induced from a k -automorphism of $\widetilde{\mathbf{G}}$ such that $\varphi(\gamma) = \Phi_{\mathbb{A}}(\sigma_{\mathbb{A}}^0(\gamma))$ for every $\gamma \in \Gamma_1$, taking closures one deduces that $(\widetilde{\Phi}_{\mathbb{A}} \circ \sigma_{\mathbb{A}}^0)(\widehat{\Gamma}_1) = \widehat{\Gamma}_2$. As $\sigma_{\mathbb{A}}^0$ acts by permuting the places and $\widetilde{\Phi}_{\mathbb{A}}$ acts place-wise, it must be that $\sigma_{\mathbb{A}} : \mathfrak{q}_1 \mapsto \mathfrak{q}_1$. We claim that such σ must be trivial, which will finish the proof.

Indeed, let N be the Galois closure of k , $G := \text{Gal}(N/\mathbb{Q})$, $H := \text{Gal}(N/k)$, let r be a prime lying over \mathfrak{p}_1 and G_r be its decomposition group. The correspondence $H \backslash G / G_r \rightarrow \{\mathfrak{p}_1, \dots, \mathfrak{p}_d\}$ given by $H\sigma G_r \mapsto \sigma \mathfrak{p}_1$ is a one-to-one correspondence (of G -sets), hence G_r must be trivial (see [11, p. 55]). Thus, $\text{Aut}(k)$ acts freely on the set of primes $\{\mathfrak{p}_1, \dots, \mathfrak{p}_d\}$, as needed. ■

4. Another set of examples

The following theorem is not needed for the proof of the main theorem, but following [Method B](#), it gives many more examples of non-profinitely rigid arithmetic groups.

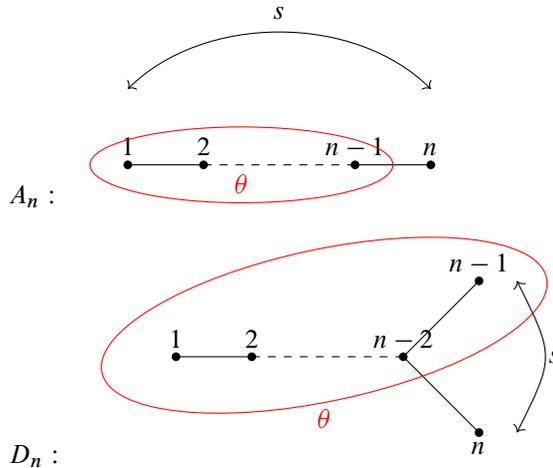
Theorem 4.1. *Let $\Gamma \subseteq \tilde{\mathbf{G}}(k)$ be an arithmetic subgroup. Assume further that $\tilde{\mathbf{G}}$ has (1) the congruence subgroup property; (2) type $A_n (n \geq 2)$, $D_n (n \geq 4)$, or E_6 ; and (3) there exists an archimedean place k_v such that $\text{rank}_{k_v} \tilde{\mathbf{G}} \geq 2$. Then there exist two non-isomorphic finite index subgroups $\Gamma_1, \Gamma_2 \subseteq \Gamma$ with isomorphic profinite completions.*

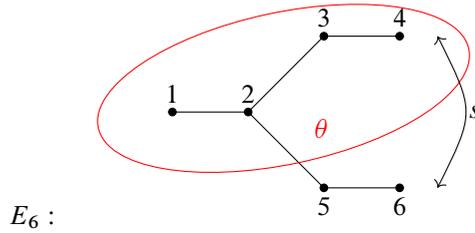
Note that A_n, D_n, E_6 are exactly the types of Dynkin diagrams with a non-trivial symmetry.

Proof (Following [Method B](#)). As before, by moving to a finite index subgroup we can assume that $\Gamma \cap \mathbf{C}(k) = \{1\}$ and that $\hat{\Gamma} = \bar{\Gamma} = \Lambda \times \prod_{v \notin S} \tilde{\mathbf{G}}(\mathcal{O}_v)$, for some finite set of places S and $\Lambda \subseteq \prod_{v \in S} \tilde{\mathbf{G}}(k_v)$ commensurable with $\prod_{v \in S} \tilde{\mathbf{G}}(\mathcal{O}_v)$.

By Lemma 2.8, one can find two different primes $p, q \notin S_{\text{full}}$ (where $S_{\text{full}} \supseteq S$ is as in the previous section) lying over different rational primes, such that $\tilde{\mathbf{G}}$ splits over both k_p and k_q . Fix root systems for $\tilde{\mathbf{G}}(k_p)$ and $\tilde{\mathbf{G}}(k_q)$, and let Σ be the set of simple roots, identified for both groups. For $v = p, q$, the reduction maps $\pi_v : \tilde{\mathbf{G}}(\mathcal{O}_v) \rightarrow \tilde{\mathbf{G}}(\mathcal{O}_v/v)$ are onto, and $G_v := \tilde{\mathbf{G}}(\mathcal{O}_v/v)$ is the split universal Chevalley group of the same type as $\tilde{\mathbf{G}}$ over the residue field. The root systems we fixed define corresponding root systems for G_v .

For each of the types involved, there exists a non-trivial symmetry s of the Dynkin diagram, pick a subset $\theta \subseteq \Sigma$ which is non-invariant under s , for example,





The symmetry s of the Dynkin diagram induces an isomorphism φ_v of G_v [18, Corollary to Theorem 29]. For a subset $R \subseteq \Sigma$ of simple roots, let $P_{v,R}$ be the parabolic subgroup of G_v corresponding to R , then $P_{v,\theta}$ and $P_{v,s\theta}$ are non-conjugate in G_v , but isomorphic via φ_v .

Now, let $\pi : \Gamma \rightarrow G_p \times G_q$ be the canonical reduction map modulo $\mathfrak{p}\mathfrak{q}$. Consider the following two congruence subgroups:

$$\Gamma_1 := \pi^{-1}(P_{\mathfrak{p},\theta} \times P_{\mathfrak{q},\theta});$$

$$\Gamma_2 := \pi^{-1}(P_{\mathfrak{p},s\theta} \times P_{\mathfrak{q},\theta}).$$

Then $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_2$ are isomorphic via $\Phi := (\Phi_v)_v$ where $\Phi_v \equiv id$ for $v \neq \mathfrak{p}$ and $\Phi_{\mathfrak{p}}$ is the isomorphism of $\widetilde{\mathbf{G}}(k_{\mathfrak{p}})$ induced by the non-trivial symmetry s of the Dynkin diagram. It remains to show that Γ_1 and Γ_2 cannot be isomorphic.

Assume to the contrary that there exists an isomorphism $\varphi : \Gamma_1 \rightarrow \Gamma_2$. By Corollary 2.2, there exist unique adelic automorphisms $\Phi_{\mathbb{A}}$ and $\sigma_{\mathbb{A}}^0$ of $\widetilde{\mathbf{G}}(\mathbb{A}_k^f)$ such that $\sigma_{\mathbb{A}}^0$ is induced from an automorphism σ of k and $\Phi_{\mathbb{A}}$ is induced from a k -automorphism of $\widetilde{\mathbf{G}}$ such that $\varphi(\gamma) = \Phi_{\mathbb{A}}(\sigma_{\mathbb{A}}^0(\gamma))$ for every $\gamma \in \Gamma_1$. Moreover, $\Phi_{\mathbb{A}}$ is of the form $C_g \cdot \omega_{\mathbb{A}}$ where C_g is conjugation by some $g \in \widetilde{\mathbf{G}}(\bar{k})$, ω is an outer automorphism which comes from a symmetry of the Dynkin diagram, and $\omega_{\mathbb{A}}$ just acts as ω at each place. In particular, $\pi_{\mathfrak{p}}^{-1}(P_{\mathfrak{p},\theta})$ is mapped onto $g\pi_{\mathfrak{p}}^{-1}(P_{\sigma(\mathfrak{p}),\omega\theta})g^{-1}$ and $\pi_{\mathfrak{q}}^{-1}(P_{\mathfrak{q},\theta})$ is mapped onto $g\pi_{\mathfrak{q}}^{-1}(P_{\sigma(\mathfrak{q}),\omega\theta})g^{-1}$. By our choice of \mathfrak{p} and \mathfrak{q} it must be that $\sigma(\mathfrak{p}) = \mathfrak{p}$ and $\sigma(\mathfrak{q}) = \mathfrak{q}$.

We have that $g^{-1}\pi_{\mathfrak{p}}^{-1}(P_{\mathfrak{p},\theta})g = \pi_{\mathfrak{p}}^{-1}(P_{\mathfrak{p},\omega\theta})$, multiplying g from both sides by elements of $\widetilde{\mathbf{G}}(k)$, we can assume that conjugation by g preserves the root system that was fixed in the beginning of the proof. If $\theta \neq \omega\theta$, let α be a simple root in $\omega\theta \setminus \theta$, then the action of g on the one-parameter unipotent subgroup U_{α} must be as scalar multiplication by some f_{α} with $\text{val}_{\mathfrak{p}}(f_{\alpha}) = 1$. Thus, the action on the opposite one-parameter unipotent subgroup $U_{-\alpha}$ is given as scalar multiplication by $1/f_{\alpha}$, but then $g^{-1}\pi_{\mathfrak{p}}^{-1}(P_{\mathfrak{p},\theta})g \not\subseteq \pi_{\mathfrak{p}}^{-1}(P_{\mathfrak{p},\omega\theta})$. Hence, θ must be equal to $\omega\theta$. The same argument implies that $\omega\theta = s\theta$, and so $\theta = \omega\theta = s\theta$. But θ was chosen to be non-invariant under s , a contradiction. ■

5. Second part of the main theorem: The exceptional cases

Lemma 5.1. *Let \mathbf{G} be a connected, simply connected, and absolutely almost simple \mathbb{Q} -linear algebraic group of type $E_8, F_4,$ or G_2 . Then, \mathbf{G} splits over \mathbb{Q}_p for every p .*

Proof. The \mathbb{Q}_p forms of the group \mathbf{G} are classified by the first Galois cohomology set $H^1(\mathbb{Q}_p, \text{Aut}(\mathbf{G}))$. For the groups considered, the universal and the adjoint forms coincide; moreover, there are no symmetries for their Dynkin diagram, hence $\text{Aut}(\mathbf{G}) \cong \mathbf{G}$. As the field \mathbb{Q}_p is local and non-archimedean, by [13, Theorem 6.4], the Galois cohomology group $H^1(\mathbb{Q}_p, \mathbf{G})$ is trivial. Thus, there is only one (up to an isomorphism) \mathbb{Q}_p form for \mathbf{G} ; in particular, this form must be the split form. ■

Remark. If \mathbf{G} is a k -split simple k -linear algebraic group, then the automorphism group, $\text{Aut}(\mathbf{G}(k))$, of $\mathbf{G}(k)$ is completely known. Precisely, following the notations of [18, Theorem 30], each automorphism can be written as the product of an inner, a diagonal, a graph, and a field automorphism. We have used graph automorphisms for **Method B** and field automorphisms for **Method C**. The group of diagonal automorphisms (modulo the inner ones) has a connection with the center of the universal form [18, Exercise following Theorem 30], which was used for **Method A**. Thus, restricting ourselves to the exceptional cases, where \mathbf{G} has type $E_8, F_4,$ or G_2 and $k = \mathbb{Q}$ (and also for \mathbb{Q}_p), all automorphisms are inner.

Theorem 5.2. *Let \mathbf{G} be a connected, simply connected, and absolutely almost simple high ∞ -rank \mathbb{Q} -linear algebraic group of type $E_8, F_4,$ or G_2 . If $\Gamma_1, \Gamma_2 \subseteq \mathbf{G}(\mathbb{Q})$ are two arithmetic subgroups with isomorphic profinite completions, then Γ_1 and Γ_2 are isomorphic.*

Proof. Let $\Gamma_1, \Gamma_2 \subseteq \mathbf{G}(\mathbb{Q})$ be two arithmetic subgroups. As noted in the preliminaries, the congruence kernel is trivial for these groups, so one can write $\overline{\Gamma}_i = \widehat{\Gamma}_i = \Lambda_i \times \prod_{p \notin S} \mathbf{G}(\mathbb{Z}_p)$ where S is a finite set and Λ_i are commensurable with $\prod_{p \in S} \mathbf{G}(\mathbb{Z}_p)$. Assume that $\Phi : \overline{\Gamma}_1 \cong \widehat{\Gamma}_1 \xrightarrow{\sim} \widehat{\Gamma}_2 \cong \overline{\Gamma}_2 \subseteq \mathbf{G}(\mathbb{A})$ is an isomorphism between the profinite completions of the two. By adelic superrigidity 2.3, there exists a unique homomorphism of adelic groups

$$\tilde{\Phi} : \mathbf{G}(\mathbb{A}^f) \rightarrow \mathbf{G}(\mathbb{A}^f)$$

such that $\tilde{\Phi}|_{\Gamma_1} \equiv \Phi \circ \iota|_{\Gamma_1}$, as $\mathbf{G}(\mathbb{A}^f)$ is centerless by our assumption on the type of \mathbf{G} . Moreover, using the uniqueness of the map, $\tilde{\Phi}$ must be an isomorphism, and $\tilde{\Phi}|_{\overline{\Gamma}_1} \equiv \Phi$.

Consider the homomorphisms $\tilde{\Phi}_{p,q} : \mathbf{G}(\mathbb{Q}_p) \rightarrow \mathbf{G}(\mathbb{Q}_q)$ which are the composite

$$\mathbf{G}(\mathbb{Q}_p) \xrightarrow{\iota_p} \mathbf{G}(\mathbb{A}^f) \xrightarrow{\tilde{\Phi}} \mathbf{G}(\mathbb{A}^f) \xrightarrow{\pi_q} \mathbf{G}(\mathbb{Q}_q)$$

of the inclusion in the p th place, $\tilde{\Phi}$, and the projection onto the q th place. This is a continuous homomorphism between a p -adic group and a q -adic group, so if $p \neq q$ it must

be a locally constant map. So its image is a normal countable subgroup of $\mathbf{G}(\mathbb{Q}_q)$; in particular, it is not of finite index, and hence must be trivial [13, Proposition 3.17]. Thus, $\tilde{\Phi} = (\tilde{\Phi}_{p,p})_p$ is given by an isomorphism at each place, and $\tilde{\Phi}_{p,p}$ must be conjugation by some $y'_p \in \mathbf{G}(\mathbb{Q}_p)$ (see the remark above).

We truncate $\tilde{\Phi}$ in the following manner:

$$\text{write } y = (y_p)_p, \quad \text{where } y_p = \begin{cases} y_p = y'_p & p \in S \\ y_p = 1 & p \notin S \end{cases}.$$

By our choice of S , conjugation by y is again an isomorphism between $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$. By the strong approximation theorem [13, Theorem 7.12], there exists some $g \in \mathbf{G}(\mathbb{Q})$ with $g \in y\overline{\Gamma}_1$. Thus, conjugation by $g \in \mathbf{G}(\mathbb{Q})$ is an isomorphism between $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$. We have that $\Gamma_i = \overline{\Gamma}_i \cap \mathbf{G}(\mathbb{Q})$, which imply that Γ_1 and Γ_2 can be conjugated by g , as needed. ■

6. Final remarks

It is possible to generalize our methods even further. For example, using [Method A](#), we can find finite index subgroups of $\Gamma := SL_2(\mathbb{Z}[1/p])$ which are not profinitely rigid. Explicitly (for $p \neq 2, 3, 5$), the following finite index subgroups of $SL_2(\mathbb{Z}[1/p])$ are non-isomorphic, but their profinite completions are

$$\Gamma_1 := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}[1/p]) : \begin{array}{l} b, c \equiv 0 \pmod{3, 5} \\ a, d \equiv \pm 1 \pmod{3} \\ a, d \equiv 1 \pmod{5} \end{array} \right\};$$

$$\Gamma_2 := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}[1/p]) : \begin{array}{l} b, c \equiv 0 \pmod{3, 5} \\ a, d \equiv 1 \pmod{3} \\ a, d \equiv \pm 1 \pmod{5} \end{array} \right\}.$$

Indeed, just as in Section 3, $\widehat{\Gamma}_1 \cong \widehat{\Gamma(15)} \times \mathbb{Z}/2\mathbb{Z} \cong \widehat{\Gamma}_2$, where $\Gamma(15)$ is the principle congruence subgroup of Γ of level 15.

Let us stress out that it is still unknown whether or not $SL_2(\mathbb{Z}[1/p])$ itself is profinitely rigid, and in fact, there are some reasons to believe it is profinitely rigid (see, for example, [4, §4]). On the other hand, increasing slightly the dimension, it has been shown that $SL_4(\mathbb{Z}[1/p])$ is not profinitely rigid [4].

We would like to state a stronger version of Theorem 1.1 which includes the above example. First, we need some further notations. Let $S \subseteq V(k)$ be a finite set of places containing all the archimedean places. The ring of S -integers of the number field k is

$$\mathcal{O}_{k,S} := \{x \in k : v(x) \geq 0 \forall v \notin S\}.$$

Let $\tilde{\mathbf{G}}$ be a connected, simply connected, absolutely almost simple k -linear algebraic group with a fixed faithful k -representation $\rho : \tilde{\mathbf{G}} \rightarrow GL(n_\rho)$. A subgroup $\Gamma \subseteq \tilde{\mathbf{G}}(k)$

is called an S -arithmetic subgroup if it is commensurable with $\tilde{\mathbf{G}}(\mathcal{O}_S)$. As in Section 2, there is a map from the profinite completion to the congruence completion, denote its kernel by $C(\Gamma, S)$. The group Γ is said to have the congruence subgroup property (with respect to S) if $C(\Gamma, S)$ is a finite group. Again, this is actually a property of the ambient group $\tilde{\mathbf{G}}$, the field k , and the set S . The proofs given throughout the paper carry over to establish.

Theorem 6.1. *Let n be a positive integer, k a number field, S a finite set of places of k containing all the archimedean places, and $\tilde{\mathbf{G}}$ a connected, simply connected, and absolutely almost simple k -linear algebraic group such that $\sum_{v \in S} \text{rank}_{k_v} \tilde{\mathbf{G}} \geq 2$ and such that $\tilde{\mathbf{G}}(k)$ satisfies the congruence subgroup property (with respect to S). Let $\Gamma \subseteq \tilde{\mathbf{G}}(k)$ be an S -arithmetic subgroup. Then, unless $\tilde{\mathbf{G}}$ has type G_2 , F_4 , or E_8 and $k = \mathbb{Q}$, Γ has infinitely many sequences of pairwise non-isomorphic finite index subgroups $\Gamma_1, \dots, \Gamma_n$ with isomorphic profinite completions.*

As in the main theorem, the exceptional cases are indeed exceptional. Moreover, in these cases, if Γ_1 is an S_1 -arithmetic subgroup and Γ_2 is an S_2 -arithmetic subgroup with $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$ then $S_1 = S_2$ and $\Gamma_1 \cong \Gamma_2$.

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