

Quasi-BNS invariants

Nicolaus Heuer and Dawid Kielak

Abstract. We introduce the notion of quasi-BNS invariants, where we replace homomorphisms to \mathbb{R} by homogenous quasimorphisms to \mathbb{R} in the theory of Bieri–Neumann–Strebel invariants. We prove that the quasi-BNS invariant $Q\Sigma(G)$ of a finitely generated group G is open; we connect it to approximate finite generation of almost kernels of homogenous quasimorphisms; finally, we prove a Sikorav-style theorem connecting $Q\Sigma(G)$ to the vanishing of the suitably defined Novikov homology.

1. Introduction

In 1987, Bieri–Neumann–Strebel [3] introduced the geometric invariant Σ for a group G , nowadays more often called the BNS invariant. The invariant is a set of non-trivial homomorphisms $G \rightarrow \mathbb{R}$ with the following fundamental properties (with the last statement proved by Sikorav [14]).

Theorem 1.1 ([3, 14]). *Let G be a finitely generated group, and let $\phi: G \rightarrow \mathbb{R}$ be a non-zero homomorphism.*

- (1) *The BNS invariant $\Sigma(G)$ is open.*
- (2) *If $\text{im } \phi = \mathbb{Z}$, then ϕ is an algebraic fibering (i.e., $\ker \phi$ is finitely generated) if and only if $\{\phi, -\phi\} \subseteq \Sigma(G)$.*
- (3) *We have $\phi \in \Sigma(G)$ if and only if the first Novikov homology of (G, ϕ) vanishes, that is, $H_1(G; \widehat{\mathbb{Z}G}^\phi) = 0$.*

Item (1) is used to construct new algebraically fibred characters out of an old one (when the abelianisation of G has rank at least 2). The combination of items (2) and (3) gives a practical way of finding algebraic fibrings. This approach was used by Friedl–Tillmann [10] in the setting of one-relator groups and by the second author in the setting of RFRS groups [11]. It is worth noting that the inspiration behind both papers lies in the realm of 3-manifolds – the former article was inspired by the work of Thurston [15] (which also plays a role in [3]), whereas the latter drew from the work of Agol [1].

Since their inception, BNS invariants have been generalised in two main directions. First, we might want to understand $\ker \phi$ better. In the context of Theorem 1.1 (2), Bieri–Renz [4] introduced the *higher BNS invariants* which are capable of computing higher finiteness properties of $\ker \phi$. Second, we might want to look beyond homomorphisms to \mathbb{R} . In [2], Bieri–Geoghegan noticed that a homomorphism (or *character*) $\phi: G \rightarrow \mathbb{R}$ can be identified with an action of G by translation on a CAT(0) space \mathbb{R} . They then developed a theory in which the role of ϕ is taken by an isometric action of G on a proper CAT(0) space.

This article seeks to generalise BNS invariants by allowing greater freedom in the choice of ϕ as well; the extra flexibility is found however not by looking at more general geometric actions, as was the case for Bieri–Geoghegan, but by leaving geometry entirely. We will replace homomorphisms $\phi: G \rightarrow \mathbb{R}$ by *quasimorphism*, which can be roughly described as homomorphisms up to a uniform error. The big advantage quasimorphisms have over homomorphisms to \mathbb{R} (and actions on CAT(0) spaces) is that they are ubiquitous. Take, for example, an infinite hyperbolic group G with property (T) – concretely, take a uniform lattice in $\mathrm{Sp}(n, 1)$; more examples were constructed by Žuk [16] (see also [12] for another proof) and Caprace [7]. The abelianisation of G is finite, and more generally its actions on CAT(0) spaces are severely restricted [13]. But the group G admits infinitely many non-homothetic quasimorphisms (some of them very explicit), as constructed by Epstein–Fujiwara [9]; see also [6].

Allowing quasimorphisms leads to the notion of *quasi-BNS invariants*, denoted $Q\Sigma(G)$. This invariant satisfies the following theorem.

Theorem 1.2. *Let G be finitely generated, and let $\phi: G \rightarrow \mathbb{R}$ be a homogenous quasimorphism with non-zero defect.*

- (1) *The set $Q\Sigma(G)$ is an open subset of \mathbb{R}^G endowed with the product topology.*
- (2) *The approximate kernel $\mathrm{Aker} \phi$ of ϕ is approximately finitely generated if and only if $\{\phi, -\phi\} \subseteq Q\Sigma(G)$.*
- (3) *We have $\phi \in Q\Sigma(G)$ if and only if the first Novikov homology of (G, ϕ) vanishes, that is, $H_1(G; \widehat{\mathbb{Z}G}^\phi) = 0$.*

There are various notions in the above statement that require introduction. The approximate kernel $\mathrm{Aker} \phi$ is a natural generalisation of a kernel and includes elements of G whose image under ϕ is very close to 0. Approximate finite generation is again a natural generalisation of finite generation and is defined using connectivity of the Rips complex. Novikov rings and Novikov homology are defined in the same way as in the classical setting.

The above theorem, as well as other results of this article, indicates that quasi-BNS invariants are probably the correct analogues of BNS invariants in the setting of approximate group theory, as developed, for example, by Cordes–Hartnick–Tonić [8]. This is an emerging topic with great promise.

2. Quasimorphisms and approximate groups

We recall the definition of an approximate group and define finiteness properties of approximate groups in Subsection 2.1. This paper studies approximate groups which arise as *approximate kernels* of quasimorphisms; see Subsection 2.3.

2.1. Approximate groups

Definition 2.1. Let G be a group. A subset $A \subseteq G$ is called an *approximate group* if there is a finite subset $X \subseteq G$ such that $A \cdot A \subseteq A \cdot X$. Here and in what follows, $A \cdot B = \{ab \mid a \in A, b \in B\}$ for $A, B \subseteq G$.

Let G be a group with a finite generating set S (which is not necessarily symmetric), and let d denote the word metric on G with respect to S . Let $A \subseteq G$ be a subset. The *Rips complex* $R_S(A, n)$ of A with parameter n is a simplicial complex defined as follows: its 0th skeleton consists of the elements of A , and a subset $\{x_1, \dots, x_{k+1}\} \subseteq A$ of pairwise distinct $k + 1$ elements spans a simplex if and only if its diameter is less than n . Here, the diameter of the set $\{x_1, \dots, x_{k+1}\}$ is defined as $\max\{d(x_i, x_j) \mid 1 \leq i, j \leq k + 1\}$.

Definition 2.2. We say that a subset A of G is *approximately connected* if and only if there exists a finite generating set S of G and $N \in \mathbb{N}$ such that the Rips complex $R_S(A, N)$ is connected.

It is easy to see that the Rips complex $R_S(A, n)$ is connected for every $n \geq N$, and hence the definition above is equivalent to saying: for every finite generating set S , there exists $N \in \mathbb{N}$ such that $R_S(A, N)$ is connected.

Definition 2.3 ([8, Definition 3.28]). We say that an approximate group $A \subseteq G$ of a finitely generated group G is *approximately finitely generated* if and only if A is approximately connected.

In [8], this notion is referred to as *geometric approximate finite generation*, but we will use the shorter form here.

2.2. Paths

In this subsection, we set up some notation for paths in a group G endowed with a finite generating set S .

Definition 2.4. A *path* p in G is a sequence (g_0, g_1, \dots, g_k) of elements of G such that $g_i^{-1}g_{i+1} \in S^\pm = S \cup S^{-1}$ for every i . For such a path p , we will set $o(p) = g_0$ to be the origin and $t(p) = g_k$ to be the terminus of p . We will say that p *connects* g_0 to g_k . We set $p^{-1} = (g_k, \dots, g_0)$.

Given a set-theoretic function $\alpha: G \rightarrow \mathbb{R}$, we set

$$\max \alpha(p) = \max\{\alpha(g_i) \mid i = 1, \dots, k\}$$

and similarly $\min \alpha(p) = \min\{\alpha(g_i) \mid i = 1, \dots, k\}$.

Given another path $q = (h_0, \dots, h_n)$ in G , we define the *concatenation* of p and q to be

$$pq = (g_0, \dots, g_k, g_k h_0^{-1} h_1, \dots, g_k h_0^{-1} h_n).$$

Note that the last definition applies in particular to the situation in which $p = (g_0)$ is a single-vertex path, in which case $pq = (g_0)q$ is a translate of q starting at g_0 . Also, the last definition allows us to write p^n for every $n \in \mathbb{Z} \setminus \{0\}$.

We can reformulate the notion of approximate connectivity using paths (we omit the obvious proof).

Lemma 2.5. *Let A be a subset of a group G endowed with a finite generating set S . The subset A is approximately connected if and only if there exists a constant $K \geq 0$ such that for every $a, b \in A$ there exists a path p connecting a to b such that for every vertex v on the path we have $d(v, A) \leq K$.*

2.3. Quasimorphisms and approximate kernels

Definition 2.6. A map $\phi: G \rightarrow \mathbb{R}$ is called a *quasimorphism* if there is some $C > 0$ such that for all $g, h \in G$, $|\phi(g) + \phi(h) - \phi(gh)| \leq C$. We call the smallest such C the *defect* of ϕ and denote it by $D(\phi)$. We say that ϕ is *homogenous* if $\phi(g^n) = n\phi(g)$ for every $g \in G, n \in \mathbb{Z}$.

Every quasimorphism ϕ is in bounded pointwise distance to a unique homogenous quasimorphism: the function $\bar{\phi}$ defined pointwise via

$$\bar{\phi}(g) = \lim_{n \rightarrow \infty} \frac{\phi(g^n)}{n}$$

is a quasimorphism of defect at most $2D(\phi)$ and $|\phi(g) - \bar{\phi}(g)| \leq D(\phi)$ for every $g \in G$, cf. [5, Lemma 2.21] of Calegari. The set of homogenous quasimorphisms forms a vector space under pointwise addition and will be denoted by $Q^h(G)$. We will topologise $Q^h(G)$ in a non-standard way. We write it as a disjoint union

$$Q^h(G) = \bigsqcup_{x \in [0, \infty)} \{\phi \in Q^h(G) \mid D(\phi) = x\}$$

and topologise each subset $\{\phi \in Q^h(G) \mid D(\phi) = x\}$ by embedding it into \mathbb{R}^G with product topology and then using subspace topology. Note that this in particular separates homomorphisms from proper quasimorphisms.

Lemma 2.7 ([5, Lemma 2.24]). *For a homogenous quasimorphism ϕ on a group G , we have*

$$D(\phi) = \sup_{g, h \in G} \phi([g, h]).$$

Proposition 2.8. *For every $\phi \in Q^h(G)$, the set*

$$\text{Aker}(\phi) = \{g \in G \mid |\phi(g)| \leq 2D(\phi)\}$$

is an approximate subgroup. We will call this set the approximate kernel of ϕ .

Proof. Suppose that $D(\phi) = 0$. Then ϕ is a homomorphism and hence $\text{Aker}(\phi)$ is the honest kernel of ϕ .

Now suppose that $D(\phi) \neq 0$. By Lemma 2.7, there exists $c = [g, h]$ such that

$$\frac{4D(\phi)}{5} < \phi(c) \leq D(\phi).$$

We claim that $\text{Aker}(\phi) \cdot \text{Aker}(\phi) \subseteq \text{Aker}(\phi) \cdot X$, where

$$X = \{c^5, c^4, \dots, c^{-4}, c^{-5}\}.$$

Let $g, h \in \text{Aker}(\phi)$. The definitions tell us that $\phi(ghc^n) = \phi(gh) + n\phi(c) + d_1(n)$ and $\phi(gh) = \phi(g) + \phi(h) + d_2$, where $d_1(n)$ depends on n and where $|d_1(n)|, |d_2| \leq D(\phi)$ for every n . Hence

$$\begin{aligned} |\phi(ghc^n)| &= |\phi(g) + \phi(h) + n\phi(c) + d_1(n) + d_2| \\ &\leq |\phi(g) + \phi(h) + n\phi(c) + d_2| + D(\phi). \end{aligned}$$

Also, as $|\phi(g) + \phi(h) + d_1| \leq 5D(\phi)$, we may choose $n \in \{-5, \dots, 5\}$ such that

$$|\phi(g) + \phi(h) + n\phi(c) + d_2| \leq D(\phi),$$

and so $|\phi(ghc^n)| \leq 2D(\phi)$. Hence $ghc^n \in \text{Aker}(\phi)$ and thus $gh \in \text{Aker}(\phi) \cdot X$. ■

Interestingly, for our purposes, the size of X in the definition of approximate groups could have been bounded uniformly by 11.

Observe that the approximate kernel of ϕ is the same as the approximate kernel of $\lambda\phi$ for every $\lambda \in \mathbb{R} \setminus \{0\}$.

For later purposes, the following characterisation of approximate finite generation will be useful.

Proposition 2.9. *Let $\phi: G \rightarrow \mathbb{R}$ be a homogenous quasimorphism with non-zero defect $D = D(\phi)$, and suppose that G is equipped with a finite generating set S . Then $\text{Aker}(\phi)$ is approximately finitely generated if and only if there is a constant $K > 0$ such that for every two elements $g, h \in \text{Aker}(\phi)$ there is a path p in G from g to h such that $\min \phi(p) \geq -K$ and $\max \phi(p) \leq K$.*

Proof. As in the proof of Proposition 2.8, there exists an element $c \in G$ such that

$$\frac{4D}{5} < \phi(c) \leq D.$$

Let $C = d(1, c)$.

Suppose first that we have a constant K as in the statement. Let $g, h \in \text{Aker}(\phi)$ be given. The assumption gives us a path p connecting g to h in G such that $\min \phi(p) \geq -K$ and $\max \phi(p) \leq K$. Let k be any vertex of p . There exists $m \in \mathbb{Z}$ with $|m| \leq \frac{5K}{4D}$ such that

$$|\phi(kc^m)| \leq 2D,$$

and therefore $kc^m \in \text{Aker}(\phi)$. Note that $d(k, kc^m) \leq |m|C \leq \frac{5KC}{4D}$. This means that for every $k \in p$ we may find $k' \in \text{Aker}(\phi)$ such that $d(k, k') \leq \frac{5KC}{4D}$. If k_1 and k_2 are successive points on p , then $d(k'_1, k'_2) \leq \frac{5KC}{2D} + 1$, and so g and h lie in the same connected component of $R_S(\text{Aker}(\phi), n)$ for every $n \geq \frac{5KC}{2D} + 1$, as required.

Now suppose that $R_S(\text{Aker}(\phi), n)$ is connected for some n . Let

$$|\phi(S)| = \max_{s \in S} |\phi(s)|$$

and $K = n(|\phi(S)| + D) + 2D$. Take $g, h \in \text{Aker}(\phi)$. Since $R_S(\text{Aker}(\phi), n)$ is connected, there exists a path q in $R_S(\text{Aker}(\phi), n)$ connecting g to h . This path can easily be transformed to a path in G connecting g to h , where each edge in q is replaced by a path of length at most n in G . It is immediate that for every vertex k of p we have $|\phi(k)| \leq n(|\phi(S)| + D) + 2D = K$. ■

The above statement is also true for a homomorphism $\phi: G \rightarrow \mathbb{Z}$. Indeed, in the proof, we take $c \in G$ to be such that $\phi(c)$ generates \mathbb{Z} , and otherwise argue in a similar vein. Interestingly, though, the statement is false for homomorphisms with a bigger image, as we will now show.

Example 2.10. Let $G = F_2 \times \mathbb{Z}$, where $F_2 = \langle a, b \rangle$ and $\mathbb{Z} = \langle c \rangle$. Let $\phi: G \rightarrow \mathbb{R}$ be a homomorphism induced by $\phi(a) = 1, \phi(b) = 0, \phi(c) = \sqrt{2}$. Clearly, $\ker \phi = \langle a^{-i} b a^i : i \in \mathbb{Z} \rangle$ is not finitely generated. But given any two elements $g, h \in \ker \phi$ and a path p in $F_2 \times \{0\}$ with generators $\{a^{\pm 1}, b^{\pm 1}\}$ connecting g to h , we may modify p using the generator c so that the resulting path q satisfies $\min \phi(q) \geq -3$ and $\max \phi(q) \leq 3$.

3. Quasi-BNS invariants

3.1. Definition

For a homogenous quasimorphism $\phi \in Q^h(G)$, we let

$$G_\phi = \{g \in G \mid \phi(g) > 0\}.$$

Analogously to the classical definition, we set

$$Q\Sigma(G) = \{\phi \in Q(G) \mid G_\phi \text{ is approximately connected}\} \subseteq Q^h(G).$$

Note that this set is invariant under positive scalar multiplication, that is, for every $\lambda \in \mathbb{R}^+$, $\phi \in Q\Sigma(G)$ if and only if $\lambda\phi \in Q\Sigma(G)$. Note also that this definition extends the classical one, that is, a non-trivial homomorphism $\phi: G \rightarrow \mathbb{R}$ lies in $Q\Sigma(G)$ if and only if it lies in $\Sigma(G)$.

The following is immediate but useful.

Proposition 3.1. *Let $G = \langle S \rangle$ with S finite, and let $\phi: G \rightarrow \mathbb{R}$ be a homogenous quasimorphism. Then $\phi \in Q\Sigma(G)$ if and only if there is a constant $K > 0$ such that for every two points $g, h \in G_\phi$ there is a path p from g to h such that $\min \phi(p) \geq -K$. In this case, there is a constant $K' > 0$ such that for any two elements $g, h \in G$ there is a path p from g to h such that $\min \phi(p) \geq \min\{\phi(g), \phi(h)\} - K'$.*

3.2. Approximate finite generation of approximate kernels

We will relate approximate finite generation of the approximate kernel $\text{Aker}(\phi)$ to $Q\Sigma(G)$, in analogy with the situation for the classical BNS invariant.

Theorem 3.2. *Let ϕ be a homogenous quasimorphism with positive defect. Then $\phi, -\phi \in Q\Sigma(G)$ if and only if the approximate kernel $\text{Aker}(\phi)$ is approximately finitely generated.*

Proof. Suppose that ϕ is a homogenous quasimorphism with $D = D(\phi) > 0$. Let $c = [g, h]$ be a commutator such that $\frac{4}{5}D < \phi(c) \leq D$, as before.

Assume that $\text{Aker}(\phi)$ is approximately finitely generated, and let $g_1, g_2 \in G_\phi$. Analogously to the proof of Proposition 2.8, there are $n_1, n_2 \geq 0$ such that

$$g'_1, g'_2 \in \text{Aker}(\phi)$$

with $g'_i = g_i c^{-n_i}$. Let p_c be a path from 1 to c . By Proposition 2.9, there is a uniform $K > 0$ such that we can find a path p from g'_1 to g'_2 with $\min \phi(p) \geq -K$. Finally, the path $p' = (g_1) \cdot p_c^{-n_1} \cdot p \cdot p_c^{n_2}$ connects g_1 to g_2 , and we have $\min \phi(p')$ uniformly bounded. By Proposition 3.1, we have shown that $\phi \in Q\Sigma(G)$; by symmetry, $-\phi \in Q\Sigma(G)$ as well.

Now assume that both $\phi, -\phi \in Q\Sigma(G)$. To prove that $\text{Aker} \phi$ is approximately finitely generated, we will use a peak reduction argument.

We take a finite symmetric generating set S for G containing c . Given a path, we define its vertex v to be *inessential* if and only if it is preceded by $vc^{\pm 1}$ and followed by $vc^{\pm 1}$ where the signs can be different. A vertex that is not inessential will be called *essential*.

Since $\phi, -\phi \in Q\Sigma(G)$, let K' be the second constant from Proposition 3.1 which works for both ϕ and $-\phi$; for convenience we take $K' > 2D$. Let

$$n = \left\lfloor \frac{5}{4D} \left(K' + \max_{s,t \in S} \phi(st) + D \right) \right\rfloor + 3.$$

Since $-\phi \in Q\Sigma(G)$, for every pair $(s, t) \in S^2$, there exists a path $q'_{s,t}$ such that

$$q_{s,t} = (1, c^{-1}, \dots, c^{-n})q'_{s,t}(1, c, \dots, c^n)$$

connects 1 to st , and its essential vertices v satisfy

$$\begin{aligned} \phi(v) &\leq \max\{\phi(c^{-n}), \phi(stc^{-n})\} + K' \\ &\leq \max_{s,t \in S} \phi(st) - n\phi(c) + D + K' \\ &\leq \max_{s,t \in S} \phi(st) - \frac{4nD}{5} + D + K' \\ &\leq \max_{s,t \in S} \phi(st) - \frac{4D}{5} \left(\frac{5}{4D} (K' + \max_{s,t \in S} \phi(st) + D) + 2 \right) + K' \\ &< -\frac{8}{5}D \\ &< -D. \end{aligned}$$

Let N be such that $-N < \min \phi(q_{s,t})$ for every (s, t) and such that N is greater than $K' + 2D$.

Now take $x, y \in \text{Aker}(\phi)$. We claim that there exists a path p connecting x to y such that

$$-N < \min \phi(p) \leq \max \phi(p) \leq M + 2D,$$

where $M = 3D + \max_{s \in S} |\phi(s)|$.

To prove the claim, we need two definitions: given a path p' connecting x to y , the *height* of p' is defined to be $\max \lfloor \phi(v) \rfloor$, with the maximum ranging over essential vertices. An essential vertex v of p' is a *peak* if and only if $\lfloor \phi(v) \rfloor$ coincides with the height of p' .

Let p be a path from x to y with

$$-N < \min \phi(p)$$

and such that p has the smallest possible height, and then the smallest possible number of peaks among all paths satisfying the inequality above; the path p exists since $-N < -2D - K' \leq \min\{\phi(x), \phi(y)\} - K'$. Suppose that the height of p is greater than M . Let v_1 be the first peak of p , and note that v_1 is neither x nor y , since $|\phi(x)|$ and $|\phi(y)|$ are strictly smaller than $3D$. Let v_0 be the vertex preceding v_1 , and let v_2 be the vertex following v_1 , so we have $v_2 = v_1t$ and $v_1 = v_0s$ for some generators s and t . Now we replace the subpath (v_0, v_1, v_2) in p by $(v_0)q_{s,t}$ and form a new path p' this way. By construction, the essential vertices of $(v_0)q_{s,t}$ cannot be peaks, since we have already shown that an essential vertex w of $q_{s,t}$ satisfies $\phi(w) < -D$, and so the essential vertices of $(v_0)q_{s,t}$ have to have ϕ -values strictly lower than $\phi(v_0)$, and v_0 is not a peak. Also, every essential vertex of p' is either an essential vertex of p or an essential vertex of the subpath $(v_0)q_{s,t}$. Hence p' has fewer peaks or lower height than p . Also,

$$\min \phi(p') = \min\{\min \phi(p), \min \phi((v_0)q_{s,t})\},$$

and we have

$$\min \phi(p) > -N.$$

Let us now compute the second minimum:

$$\begin{aligned} \min \phi((v_0)q_{s,t}) &\geq \phi(v_0) + \min \phi(q_{s,t}) - D \\ &> \phi(v_1) - \max_{s \in S} |\phi(s)| - D - N - D \\ &> M - \max_{s \in S} |\phi(s)| - N - 2D \\ &> D - N \\ &> -N. \end{aligned}$$

Together these inequalities yield $\min \phi(p') > -N$, contradicting the minimality of p . We conclude that the height of p is at most M .

We now remove all inessential vertices from p that appear when the path p backtracks along an edge corresponding to c . Note that a vertex of the new path is essential if and only if at least one of the vertices it came from was essential. Let v be a vertex of p such that $p(v) > M$. Since v is inessential, it lies on a segment using only the generators c or c^{-1} connecting two essential vertices, say u and w . Without loss of generality, let us take $v = uc^\alpha$ and $w = uc^\beta$, with $0 \leq \alpha \leq \beta$. We have

$$M \geq \phi(w) \geq \phi(u) + \beta\phi(c) - D,$$

and so

$$M - \phi(u) + D \geq \beta\phi(c) \geq \alpha\phi(c) \geq \phi(v) - \phi(u) - D$$

yielding $\phi(v) \leq M + 2D$. This finishes the proof. \blacksquare

As a corollary, we obtain Theorem 1.2 (2).

4. Sikorav's theorem for quasi-BNS invariants

The aim of this section is to generalise the following theorem due to Sikorav [14] which provides a homological characterisation of elements $\phi \in \Sigma(G)$.

Theorem 4.1 (Sikorav [14]). *Let $\phi: G \rightarrow \mathbb{R}$ be a non-trivial homomorphism, where G is a finitely generated group. Then $\phi \in \Sigma(G)$ if and only if $H_1(G, \widehat{\mathbb{Z}G}^\phi) = 0$.*

Here, $\widehat{\mathbb{Z}G}^\phi$ denotes the Novikov ring with respect to ϕ defined as follows: the elements of $\widehat{\mathbb{Z}G}^\phi$ are formal sums $\sum_{g \in G} \lambda_g g$ with $\lambda_g \in \mathbb{Z}$ and such that for every $x \in \mathbb{R}$ the set

$$\{g \in G \mid \phi(g) < x, \lambda_g \neq 0\}$$

is finite. Addition is pointwise. Let us now define multiplication. Take $x = \sum_{g \in G} \lambda_g g$, $y = \sum_{g \in G} \mu_g g \in \widehat{\mathbb{Z}G}^\phi$. We define

$$xy = \sum_{g \in G} \nu_g g$$

by the formula

$$\nu_g = \sum_{h \in G} \lambda_h \mu_{h^{-1}g}.$$

The sum on the right-hand side is finite. Indeed, since ϕ restricted to those elements h for which $\mu_h \neq 0$ is bounded below by some constant, say κ , we are summing over the finite set of elements h of G with $\phi(h) \leq \phi(g) - \kappa$, and such a set is finite by the definition of $\widehat{\mathbb{Z}G}^\phi$. Moreover, given an $x \in \mathbb{R}$ and $g \in G$ such that $\nu_g \neq 0$ and $\phi(g) < x$, we see that the only elements h that can contribute to the sum above satisfy $\phi(h) < x - \kappa$, and hence there are only finitely many of them. A similar argument shows that there are only finitely many elements of the form $h^{-1}g$ with $\mu_{h^{-1}g}$ contributing to the sum, and hence there are only finitely many such elements g .

Our multiplication extends the group ring multiplication of $\mathbb{Z}G$.

Exactly the same definition gives a ring when ϕ is a quasimorphism. We need to check that the multiplication still makes sense. If D is the defect of ϕ , then in the inequalities above we need to add D on the right-hand side. This does not affect the finiteness of the subsets of G .

We will generalise Sikorav’s result to the case of quasi-BNS invariants.

Theorem 4.2. *Let $\phi: G \rightarrow \mathbb{R}$ be a non-trivial homogenous quasimorphism, where G is a finitely generated group. The following are equivalent:*

- (1) $\phi \in Q\Sigma(G)$.
- (2) *There exists an open neighbourhood U of ϕ such that for every $\psi \in U$ we have $H_1(G, \widehat{\mathbb{Z}G}^\psi) = 0$.*
- (3) $H_1(G, \widehat{\mathbb{Z}G}^\phi) = 0$.

Proof. We take a finite symmetric generating set S of G which contains an element c with $\phi(c) > 0$.

Let $X^{(1)}$ denote the Cayley graph of G with respect to S , modified slightly so that if $s \in S$ is an involution, then the edges corresponding to s and s^{-1} emanating from the same vertex are identified. We extend $X^{(1)}$ to a 2-dimensional acyclic G -CW-complex X by adding G -orbits of 2-cells. We identify the 0-skeleton $X^{(0)}$ of X with G , and this way we can view ϕ as a map $X^{(0)} \rightarrow \mathbb{R}$. We extend ϕ to X inductively by defining ϕ of a cell C to be the minimum of the value of ϕ on all the 0-cells contained in C .

This choice of X also allows us to continue using our notation for paths, since any two vertices in $X^{(0)} = G$ differing by a generator from S are connected in $X^{(1)}$ by a unique edge.

We now define a *Novikov n -chain* in X to be a formal linear combination of n -cells in X with coefficients in \mathbb{Z} such that for every $x \in \mathbb{R}$ there are only finitely many n -cells in the chain with non-zero coefficient that are mapped to $(-\infty, x)$ by ϕ . It is immediate that $H_1(G, \widehat{\mathbb{Z}G^\phi})$ coincides with the space of Novikov 1-cycles divided by Novikov 2-boundaries, with cycles and boundaries defined in the usual way. We are now ready to start the proof.

(1) \Rightarrow (2) Suppose that $\phi \in Q\Sigma(G)$. Let $K \geq 0$ be such that for every two elements of $a, b \in G_\phi$ there exists a path p connecting them such that

$$\min \phi(p) \geq \min\{\phi(a), \phi(b)\} - K.$$

Pick n such that $n\phi(c) > K + D(\phi) + 1$. Let U be an open neighbourhood of ϕ defined by

$$U = \{\psi \in Q^h(G) \mid D(\psi) = D(\phi), n\psi(c) > K + D(\psi) + 1\}.$$

For every $s \in S$, we construct a 1-cycle z_s by combining two paths connecting c^n to sc^n . The first path is $(c^n)(1, c^{-1})^n(1, s)(1, c)^n$; the second path p_s comes from the fact that $\phi \in Q\Sigma(G)$ and satisfies $\min \phi(p_s) \geq n\phi(c) - K > D(\phi) + 1$. We intersect U with further open sets and obtain $V \subseteq U$ such that for all $\psi \in V$ and $s \in S$ we have

$$\min \psi(p_s) > D(\psi) + 1.$$

Since z_s is a 1-cycle in X , and X has vanishing first homology, there exists a 2-chain y_s in X with $\partial y_s = z_s$.

Now every edge in p_s is of the form $g(1, s')$, $s' \in S$ and $g \in G$ with $\psi(g) > D(\psi) + 1$ for every $\psi \in V$. For every such edge, we add $-gy_{s'}$ to y_s and form a new 2-chain whose boundary is a 1-cycle made of the path $(c^{2n})(1, c^{-1})^{2n}(1, s)(1, c)^{2n}$ and a path whose vertices have ψ -values bounded below by $2(D(\psi) + 1) - D(\psi) = D(\psi) + 2$. Moreover, $\min \psi(gy_{s'}) \geq \psi(g) + \min \psi(y_{s'}) - D(\psi) > \min \psi(y_{s'}) + 1$. Proceeding inductively, we build a Novikov 2-chain y'_s whose boundary is the Novikov 1-cycle x'_s obtained from the edge $(1, s)$ and infinite rays emanating from 1 and s and using only the edges corresponding to c . The chains are Novikov with respect to every $\psi \in V$.

Now fix $\psi \in V$, and let z be any ψ -Novikov 1-cycle in X . Adding the boundaries of Novikov 2-chains y'_s with $s \neq c$ we just constructed, we easily arrange for z to consist solely of edges corresponding to c . But the only Novikov 1-cycle of this form is the 0 cycle.

(2) \Rightarrow (3) This is obvious.

(3) \Rightarrow (1) Now suppose that $H_1(G, \widehat{\mathbb{Z}G^\phi}) = 0$, that is, that every Novikov 1-cycle is the boundary of a Novikov 2-chain. Let $a, b \in G_\phi$ be two elements, and let q be any path connecting them in X . Consider the Novikov 1-cycle z formed by q and the two infinite rays emanating from a and b and using only the edges corresponding to c . By assumption, there exists a Novikov 2-chain y' such that $\partial y' = z$.

Let y be a 2-chain obtained from y' by keeping only the cells whose value under ϕ is strictly negative. Note that $\partial y - z$ is supported solely on cells with ϕ -value positive or

zero. Now the support of $\partial y - z$ must contain a path p' connecting ac^m to bc^n for some non-negative m and n . We construct the path p connecting a to b by first following the ray from a to ac^m , then by following p' , and finally by following the ray in the negative direction from bc^n to b . It is clear from the construction that $\min \phi(p) \geq -D(\phi)$. This finishes the proof. ■

As a corollary, we obtain Theorem 1.2 (1) and (3).

5. Examples

Example 5.1. Let F_n be the free group of rank $n \geq 2$. The quasi-BNS invariant $Q\Sigma(F_n)$ is empty. This can be seen as follows: let $\phi \in Q^h(F_n)$ be a non-trivial homogenous quasimorphism, and take $c \in F_n$ such that $\phi(c) > 0$. Using Lemma 2.7, we see that $\text{Aker}(\phi)$ contains the commutator subgroup of F_n . For every element $x \in \text{Aker}(\phi)$ not commuting with c we have that $c^{-n}xc^n$ lies in the $2D(\phi)$ -neighbourhood of $\text{Aker}(\phi)$. But there is no path from x to $c^{-n}xc^n$ which stays uniformly close to $\text{Aker}(\phi)$ for all $n \in \mathbb{N}$, since the natural Cayley graph of F_2 is a tree.

Example 5.2. For amenable groups, every homogenous quasimorphism is a homomorphism, hence in this case $Q\Sigma(G) = \Sigma(G)$.

Acknowledgements. The authors would like to thank Matt Zaremsky for comments on an earlier version of this article and the referee for pointing out a number of relevant articles.

Funding. This work has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme, grant agreement no. 850930.

References

- [1] I. Agol, [The virtual Haken conjecture](#) (with an appendix by I. Agol, D. Groves, and J. Manning). *Doc. Math.* **18** (2013), 1045–1087 Zbl [1286.57019](#) MR [3104553](#)
- [2] R. Bieri and R. Geoghegan, [Limit sets for modules over groups on CAT\(0\) spaces: from the Euclidean to the hyperbolic](#). *Proc. Lond. Math. Soc. (3)* **112** (2016), no. 6, 1059–1102 Zbl [1397.20051](#) MR [3537333](#)
- [3] R. Bieri, W. D. Neumann, and R. Strebel, [A geometric invariant of discrete groups](#). *Invent. Math.* **90** (1987), no. 3, 451–477 Zbl [0642.57002](#) MR [0914846](#)
- [4] R. Bieri and B. Renz, [Valuations on free resolutions and higher geometric invariants of groups](#). *Comment. Math. Helv.* **63** (1988), no. 1, 464–497 Zbl [0654.20029](#) MR [0960770](#)
- [5] D. Calegari, *sci. MSJ Mem. 20*, Mathematical Society of Japan, Tokyo, 2009 Zbl [1187.20035](#) MR [2527432](#)

- [6] D. Calegari and K. Fujiwara, [Stable commutator length in word-hyperbolic groups](#). *Groups Geom. Dyn.* **4** (2010), no. 1, 59–90 Zbl [1227.20040](#) MR [2566301](#)
- [7] P.-E. Caprace, [A sixteen-relator presentation of an infinite hyperbolic Kazhdan group](#). *Enseign. Math.* **64** (2018), no. 3/4, 265–282 Zbl [1505.20037](#) MR [3987144](#)
- [8] M. Cordes, T. Hartnick, and V. Tonic, [Foundations of geometric approximate group theory](#). [v1] 2020, [v3] 2024, arXiv:[2012.15303v3](#)
- [9] D. B. A. Epstein and K. Fujiwara, [The second bounded cohomology of word-hyperbolic groups](#). *Topology* **36** (1997), no. 6, 1275–1289 Zbl [0884.55005](#) MR [1452851](#)
- [10] S. Friedl and S. Tillmann, [Two-generator one-relator groups and marked polytopes](#). *Ann. Inst. Fourier (Grenoble)* **70** (2020), no. 2, 831–879 Zbl [1528.20095](#) MR [4105952](#)
- [11] D. Kielak, [Residually finite rationally solvable groups and virtual fibring](#). *J. Amer. Math. Soc.* **33** (2020), no. 2, 451–486 Zbl [1480.20102](#) MR [4073866](#)
- [12] M. Kotowski and M. Kotowski, [Random groups and property \(T\): Żuk’s theorem revisited](#). *J. Lond. Math. Soc. (2)* **88** (2013), no. 2, 396–416 Zbl [1309.20035](#) MR [3106728](#)
- [13] G. A. Niblo and L. Reeves, [Groups acting on CAT\(0\) cube complexes](#). *Geom. Topol.* **1** (1997), no. 1, 1–7 Zbl [0887.20016](#) MR [1432323](#)
- [14] J.-C. Sikorav, [Homologie de Novikov associée à une classe de cohomologie réelle de degré un](#). Ph.D. thesis, Thèse d’Etat, Université Paris-Sud, Orsay, 1987
- [15] W. P. Thurston, [A norm for the homology of 3-manifolds](#). *Mem. Amer. Math. Soc.* **59** (1986), no. 339, 99–130 Zbl [0585.57006](#) MR [0823443](#)
- [16] A. Żuk, [Property \(T\) and Kazhdan constants for discrete groups](#). *Geom. Funct. Anal.* **13** (2003), no. 3, 643–670 Zbl [1036.22004](#) MR [1995802](#)

Received 20 October 2022.

Nicolaus Heuer

London, UK; nicolaus.heuer.maths@gmail.com

Dawid Kielak

Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford OX2 6GG, UK; kielak@maths.ox.ac.uk