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Corrigendum to "On compact extensions of tracial W^*-dynamical systems"

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Abstract. In this corrigendum to Jamneshan and Spaas [Groups Geom. Dyn. 19, 913–955 (2025)], we report and address three issues in the original paper. These issues did not affect the results; the main results of Jamneshan and Spaas (2025) remain valid as stated. A fully revised version – incorporating these corrections, some notational adjustments, updated references, and fixed typographical errors – will be available on the arXiv under the same posting.

1. Overview of the corrections

In this introductory paragraph, we outline the issues and how they are addressed, referring to the detailed corrections in Section 2 below. We quote from [2] and adopt its notation without further comment.

The first issue concerns the proof of the implication (1) $\Rightarrow$ (2) in Theorem A of the original paper [2]. There is a mistake in the computation of $\|E_Q(\sigma_\gamma(f)^*\sigma_\gamma(f))\|_2$ at the end of that proof. Instead, one computes $\|\sigma_\gamma(f)\|_2^2$ as follows while keeping the remainder of the proof the same:

$$\begin{aligned}
\|\sigma_\gamma(f)\|_2^2 &= \tau(\sigma_\gamma(f)^*\sigma_\gamma(f)) \\
&= \tau(E_Q(\sigma_\gamma(f)^*\sigma_\gamma(f))) \\
&\leq \tau\left(E_Q(\sigma_\gamma(f)^*\sigma_\gamma(f)) + E_Q\left(\left(\sum_{i=1}^k \eta_i\kappa_i(\gamma)\right)^*\left(\sum_{i=1}^k \eta_i\kappa_i(\gamma)\right)\right)\right) \\
&= \tau\left(E_Q\left(\left(\sigma_\gamma(f) - \sum_{i=1}^k \eta_i\kappa_i(\gamma)\right)^*\left(\sigma_\gamma(f) - \sum_{i=1}^k \eta_i\kappa_i(\gamma)\right)\right)\right. \\
&\qquad \left. + E_Q\left(\sigma_\gamma(f)^*\left(\sum_{i=1}^k \eta_i\kappa_i(\gamma)\right)\right) + E_Q\left(\left(\sum_{i=1}^k \eta_i\kappa_i(\gamma)\right)^*\sigma_\gamma(f)\right)\right)
\end{aligned}$$

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$$\leq \left\| \sigma_\gamma(f) - \sum_{i=1}^{k} \eta_i \kappa_i(\gamma) \right\|_2^2 + 2 \sum_{i=1}^{k} \| E_Q(\sigma_\gamma(f)^* \eta_i) \|_2 \| \kappa_i(\gamma) \|$$
$$< 3\varepsilon.$$

The second issue involves the definition of conditional Hilbert–Schmidt operators. The original version (see [2, Example 2.9]) allows any element $K \in L^2(N) \otimes_Q L^2(N)$. However, such an element does not necessarily define a bounded operator. One must instead restrict to those K arising from the basic construction, as clarified in Definition 2.3 below. We thank Vadim Alekseev for pointing out this correction.

Relating to this correction, we also include Lemma 2.7 below, which is needed to verify that the operator K constructed in the proof of (1) $\Rightarrow$ (2) of [2, Theorem A] is indeed a conditional Hilbert–Schmidt operator. The other uses of conditional Hilbert–Schmidt operators in [2] present no further issues and only require notational updates to align with the revised definition.

The third and main issue concerns the proof of Lemma 2.11 in [2], and consequently Proposition 2.12 and the proof of (2) $\Rightarrow$ (3) in Theorem A, which rely on it. Specifically, the second inequality in the final displayed equation of the proof of Lemma 2.11 is incorrect and cannot be salvaged. We therefore replace these parts with a revised analysis of conditional Hilbert–Schmidt operators as follows:

- We add Proposition 2.1.
- We replace the part about conditional Hilbert–Schmidt operators in [2, Subsection 2.4] with Subsections 2.1, 2.2, and 2.3 below, which also incorporate the aforementioned changes regarding the definition of conditional Hilbert–Schmidt operators.
- We provide the modified proof of the implication (2) $\Rightarrow$ (3) of Theorem A in Subsection 2.4 below.

2. Conditional Hilbert–Schmidt operators

We record the following characterization of finite rank modules, which will be used in our further analysis below.

Proposition 2.1. *Let $\mathcal{H}$ be a right M-module and $(\xi_i)_{i\in I}$ be a conditionally orthonormal basis. Then $\mathcal{H}$ is of finite rank if and only if there exists a constant $C > 0$ such that $\sum_{i\in I} E_{\mathcal{Z}}(\langle \xi_i, \xi_i\rangle_M) \leq C 1_M$, where $\mathcal{Z}$ denotes the center of M and $E_{\mathcal{Z}}$ denotes its center-valued trace.*

Proof. This is exactly [1, Proposition 9.3.2 (i)] once one observes that the element $\widehat{E}_{\mathcal{Z}}(1)$ as defined in that proposition equals $\sum_{i\in I} E_{\mathcal{Z}}(\langle \xi_i, \xi_i\rangle_M)$. Indeed, by [1, Lemma 8.4.8], $\sum_{i\in I} L_{\xi_i} L_{\xi_i}^* = 1$, and as in the beginning of [1, Section 9.3], we have $\widehat{E}_{\mathcal{Z}}(TT^*) = E_{\mathcal{Z}}(T^*T)$ for any bounded, right M-linear operator $T : L^2(M) \to \mathcal{H}$. Since $\langle \xi_i, \xi_i\rangle_M = L_{\xi_i}^* L_{\xi_i}$, the claim thus follows immediately. ∎

The next three Subsections 2.1, 2.2, and 2.3 replace the original analysis of conditional Hilbert–Schmidt operators in [2, Subsection 2.4].

2.1. Conditional Hilbert–Schmidt operators

Our treatment of conditional Hilbert–Schmidt operators is motivated by the ones given in [5, Section 3.3.2] and [4, Appendix C], though some care has to be taken since the Q-valued inner product is only defined on the dense subspace of left-bounded vectors in general, and we can only make sense of conditional orthonormality on this dense subspace.

Before continuing, we recall the definition of the basic construction for inclusions of tracial von Neumann algebras.

Definition 2.2. Let (N, τ) be a tracial von Neumann algebra, and let $Q \subset N$ be a von Neumann subalgebra. Denote by $e_Q \in B(L^2(N))$ the orthogonal projection onto $L^2(Q)$. The von Neumann algebra $\langle N, e_Q\rangle$ generated by N and e_Q inside $B(L^2(N))$ is called the *basic construction* for the inclusion $Q \subset N$.

It is well known that $\langle N, e_Q\rangle$ consists exactly of the operators in $B(L^2(N))$ which commute with the right Q-action; see, for instance, [1, Proposition 9.4.2].

We can now define conditional Hilbert–Schmidt operators in the above framework. This definition appears, for instance, in [5, Definition 3.6] in the commutative framework.

Definition 2.3. Let (N, τ) be a tracial von Neumann algebra and $Q \subset N$ a von Neumann subalgebra. An operator $T \in B(L^2(N))$ is *Hilbert–Schmidt relative to* Q (or, *conditionally Hilbert–Schmidt* if Q is clear from context) if $T \in \langle N, e_Q\rangle$, that is, T is in the basic construction of $Q \subset N$, and $\sum_{\xi\in\Omega} \|T\xi\|_2^2 < \infty$ for every conditionally orthonormal set $\Omega \subset L^2(N)^0$, where we view $L^2(N)$ as a right Q-module.

We will denote by $\mathrm{HS}_{N,Q}$, or HS_Q, the set of conditional Hilbert–Schmidt operators, and we equip it with the Hilbert–Schmidt norm given by $\|T\|_{\mathrm{HS}}^2 := \sum_i \|Te_i\|_2^2$, where $\{e_i\}_i$ is any conditionally orthonormal basis of $L^2(N)$ as a right Q-module.

We note that it follows from Lemma 2.6 below that the Hilbert–Schmidt norm defined above is independent of the chosen basis.

Remark 2.4. One can define conditional Hilbert–Schmidt operators more generally between any two right Q-modules: Suppose $\mathcal{H}$ and $\mathcal{K}$ are right Q-modules. Then a map $T : \mathcal{H} \to \mathcal{K}$ is *conditionally Hilbert–Schmidt* if it is Q-linear and $\sum_{\xi\in\Omega} \|T\xi\|_2^2 < \infty$ for every conditionally orthonormal set $\Omega \subset \mathcal{H}^0$. However, we will mostly be interested in the specific case of Hilbert–Schmidt operators defined in Definition 2.3.

Remark 2.5. We point out that the Hilbert space norm $\|.\|_2$ is used for the sum in the above definition, whereas orthogonality of the involved elements is taken with respect to the conditional inner product $\langle . , . \rangle_Q$, whose associated norm is only defined on the dense set of bounded vectors. Also, note that in the unconditional situation where $Q = \mathbb{C}$, we recover the usual notion of a Hilbert–Schmidt operator.

In the setting of Definition 2.3, it turns out there is an easy way to describe the conditional Hilbert–Schmidt operators. First, recall that, given a von Neumann algebra inclusion $Q \subset N$, the basic construction $\langle N, e_Q \rangle$ is in general no longer a tracial von Neumann algebra, but it has a canonical semifinite trace $\hat{\tau}$, which on positive elements takes values in $[0, \infty]$, and which is defined on the dense *-subalgebra $\operatorname{span}\{x e_Q y \mid x, y \in N\}$ by

$$\hat{\tau}(x e_Q y) = \tau(xy)$$

(see, e.g., [1, Section 9]).

Lemma 2.6. *Let (N, τ) be a tracial von Neumann algebra and $Q \subset N$ a von Neumann subalgebra. An operator $T \in \langle N, e_Q \rangle$ is conditionally Hilbert–Schmidt if and only if $\hat{\tau}(T^*T) < \infty$.*

Proof. This is immediate from [1, Proposition 8.4.15], which implies that we have

$$\sum_i \|Te_i\|_2^2 = \hat{\tau}(T^*T),$$

for any operator $T \in \langle N, e_Q \rangle$, and any conditionally orthonormal basis $\{e_i\}_i$ of $L^2(N)$ as a Q-module. ∎

The previous lemma shows that we can view the space of conditional Hilbert–Schmidt operators HS_Q naturally as a subspace of $L^2(\langle N, e_Q \rangle, \hat{\tau})$, when we equip HS_Q with its Hilbert–Schmidt norm $\|T\|_{\mathrm{HS}}^2 = \sum_i \|Te_i\|_2^2 = \hat{\tau}(T^*T)$. In fact, $L^2(\langle N, e_Q \rangle, \hat{\tau})$ is the completion of HS_Q for the Hilbert–Schmidt norm. Note that HS_Q is not necessarily complete: for instance, when $Q = N$, we have $\langle N, e_N \rangle = N$ and $L^2(\langle N, e_N \rangle, \hat{\tau}) = L^2(N, \tau)$. Nevertheless, the next lemma shows that balls of finite operator-norm radius inside HS_Q are in fact closed, which is analogous to the fact that the unit ball of a tracial von Neumann algebra is complete for the trace norm (which on the unit ball coincides with the strong operator topology).

Lemma 2.7. *Let (N, τ) be a tracial von Neumann algebra and $Q \subset N$ a von Neumann subalgebra. Then for every $r > 0$, the convex set*

$$B_r := \{T \in \mathrm{HS}_Q \mid \|T\| \le r\}$$

is complete in the Hilbert–Schmidt norm. Here $\|T\|$ denotes the usual operator norm of $T \in B(L^2(N))$.

Proof. Assume $(T_n)_n$ is a Cauchy sequence in HS_Q for the Hilbert–Schmidt norm. For $x, y \in \mathrm{HS}_Q$, since $yy^* \le \|yy^*\|1$, we have $\|xy\|_{\mathrm{HS}}^2 = \hat{\tau}(xyy^*x^*) \le \|y\|^2 \|x\|_{\mathrm{HS}}^2$, and thus for any $S \in \mathrm{HS}_Q$, we get

$$\|T_n S - T_m S\|_{\mathrm{HS}} \le \|S\| \|T_n - T_m\|_{\mathrm{HS}}.$$

Since $(T_n)_n$ is Cauchy for the Hilbert–Schmidt norm, we get that $(T_n S)_n$ is Cauchy in $L^2(\langle N, e_Q\rangle, \hat{\tau})$ for every $S \in \mathrm{HS}_Q$. Since HS_Q is dense inside $L^2(\langle N, e_Q\rangle, \hat{\tau})$, we thus get an operator $T \in B(L^2(\langle N, e_Q\rangle, \hat{\tau}))$ determined by $T(S) = \lim_n T_n S$ for $S \in \mathrm{HS}_Q$. In particular, we get that T_n converges in the strong operator topology to T, and thus $\|T\| \leq r$. Since $\langle N, e_Q\rangle \subset B(L^2(\langle N, e_Q\rangle, \hat{\tau}))$ is a von Neumann algebra, and thus closed in the strong operator topology, we moreover conclude that $T \in \langle N, e_Q\rangle$. Finally, let $(p_i)_i$ be an increasing net of finite projections in $\langle N, e_Q\rangle$ converging to 1 in the strong operator topology. Then

$$\begin{aligned}\hat{\tau}(T^*T) &= \sup_i \hat{\tau}(T^* p_i T) = \sup_i \|p_i T\|_{\mathrm{HS}}^2 = \sup_i \lim_n \|p_i T_n\|_{\mathrm{HS}}^2 \\ &\leq \sup_i \lim_n \|p_i\|^2 \|T_n\|_{\mathrm{HS}}^2 < \infty.\end{aligned}$$

We conclude that $T \in \mathrm{HS}_Q$. Similarly, we get $\hat{\tau}((T_n - T)^*(T_n - T)) \to 0$, and hence T_n converges to T in the Hilbert–Schmidt norm. This finishes the proof of the lemma. ∎

2.2. Hilbert–Schmidt operators as convolution operators

In the commutative setting $L^\infty(Y) \subset L^\infty(X)$, given a Hilbert–Schmidt operator $K \in \mathrm{HS}_{L^\infty(Y)} \subset \langle L^\infty(X), e_{L^\infty(Y)}\rangle$, we can consider the conditional convolution operator $K *_Y \cdot : L^2(X) \to L^2(X)$ given by

$$(K *_Y f)(x) := \int_X K(x, x') f(x')\, d\mu_{\pi(x)}(x'). \tag{2.1}$$

Considering the inclusion $\iota_1 : L^2(X) \hookrightarrow L^2(X) \otimes_{L^\infty(Y)} L^2(X)$ in the first coordinate, we see that the corresponding orthogonal projection on this subspace is given by

$$\begin{aligned}p_1 = (\mathrm{id} \otimes E_{L^\infty(Y)}) : L^2(X) \otimes_{L^\infty(Y)} L^2(X) &\to L^2(X) \\ \xi \otimes_{L^\infty(Y)} \eta &\mapsto \xi E_{L^\infty(Y)}(\eta)\end{aligned}$$

for $\xi, \eta \in L^\infty(X)$. Similarly for ι_2 and p_2.

Using the canonical isomorphism $L^2(X \times_Y X) \cong L^2(X) \otimes_{L^\infty(Y)} L^2(X)$ (cf. [2, equation (2.1)]), we then get for $f \in L^2(X)$

$$K *_Y f = p_1(K\iota_2(f)) = (\mathrm{id} \otimes E_{L^\infty(Y)})(K(1 \otimes_{L^\infty(Y)} f)). \tag{2.2}$$

Indeed, if $K = \xi \otimes_{L^\infty(Y)} \eta$ for $\xi, \eta \in L^\infty(X)$ and $f \in L^\infty(X)$, then by definition we have

$$\begin{aligned}((\mathrm{id} \otimes E_{L^\infty(Y)})(K(1 \otimes_{L^\infty(Y)} f)))(x) &= ((\mathrm{id} \otimes E_{L^\infty(Y)})(\xi \otimes_{L^\infty(Y)} \eta f))(x) \\ &= (\xi E_{L^\infty(Y)}(\eta f))(x) \\ &= \xi(x) \int_X \eta(x') f(x')\, d\mu_{\pi(x)}(x') \\ &= \int_X \xi(x)\eta(x') f(x')\, d\mu_{\pi(x)}(x').\end{aligned}$$

By linearity and continuity, we then see from (2.1) that (2.2) indeed holds.

This formalism now readily extends to general von Neumann algebras. Let $Q \subset N$ be an inclusion of tracial von Neumann algebras. Then the Connes fusion tensor product $L^2(N) \otimes_Q L^2(N)$ plays the role of the relatively independent product, and it is well known that $L^2(\langle N, e_Q\rangle, \hat{\tau})$ is isomorphic to $L^2(N) \otimes_Q L^2(N)$ via the canonical map $xe_Q y \mapsto \hat{x} \otimes_Q \hat{y}$ (see, e.g., [1, Exercise 13.13]).

Now consider a conditional Hilbert–Schmidt operator $K \in \mathrm{HS}_Q \subset \langle N, e_Q\rangle$. Viewing K as an operator $K : L^2(N) \to L^2(N)$, we also write $K *_Q f$ instead of $K(f)$ for $f \in L^2(N)$ to emphasize the fact that we view K as a convolution operator, and by analogy with the commutative situation. We claim that

$$K *_Q f = (\mathrm{id} \otimes E_Q)(K(1 \otimes_Q f)) \tag{2.3}$$

for $f \in L^2(N)$. Indeed, since $\langle N, e_Q\rangle$ is generated as a von Neumann algebra by elements of the form $xe_Q x'$ with $x, x' \in N$, it suffices to check (2.3) for operators of that form. Fixing $x, x' \in N$, we then see that for any $y \in N$,

$$xe_Q x' *_Q \hat{y} = (xe_Q x')(\hat{y}) = \widehat{xE_Q(x'y)} \in L^2(N).$$

Similar to the commutative situation, we have that $xe_Q x'$ corresponds to $x \otimes_Q x'$ through the aforementioned isomorphism, and thus we see that (dropping the hats for notational convenience)

$$xe_Q x' *_Q y = xE_Q(x'y) = (\mathrm{id} \otimes E_Q)(x \otimes_Q x'y) = (\mathrm{id} \otimes E_Q)((x \otimes_Q x')(1 \otimes_Q y))$$

for all $y \in N$. Since N is dense in $L^2(N)$, this proves our claim (2.3).

We will most commonly use (2.3) when working with conditional Hilbert–Schmidt operators in the remainder of this paper, since it is in our situation often computationally easier to work with $L^2(N) \otimes_Q L^2(N)$ rather than with $L^2(\langle N, e_Q\rangle, \hat{\tau})$.

2.3. Some finiteness results for conditional Hilbert–Schmidt operators

In this section, we establish some finiteness results for conditional Hilbert–Schmidt operators, which will help us later in proving that certain associated modules are of finite rank.

We fix a tracial von Neumann algebra (N, τ), a von Neumann subalgebra $Q \subset N$, and we view $L^2(N)$ as a right Q-module. Recall that $L^2(N)^0$ denotes the dense subspace of left-bounded vectors equipped with the Q-valued inner product $\langle .\,, .\rangle_Q$. On $L^2(N)^0$, we associate with $\langle .\,, .\rangle_Q$ a Q-valued norm defined by

$$\|x\|_Q := \langle x, x\rangle_Q^{\frac{1}{2}}.$$

We note that in this situation,

$$\|x\|_Q^2 = \langle x, x\rangle_Q = E_Q(x^* x)$$

for $x \in N$. The usual Hilbert space norm on $L^2(N)$ will be denoted by $\|.\|_2$.

The main result of this section is Lemma 2.10 which shows that conditional Hilbert–Schmidt operators satisfy a stronger summability criterion than the one in their definition, where the 2-norm is replaced by the center-valued trace. For this, we first establish the following fact, which can be viewed as a conditional Bessel inequality. Surely this has been observed by experts before, but we include a proof for the reader's convenience.

Lemma 2.8. *If $\Omega \subset L^2(N)^0$ is a conditionally orthonormal set, then for every $K \in \mathrm{HS}_Q$,*

$$\sum_{f\in\Omega} (\mathrm{id} \otimes E_Q)(K(1 \otimes_Q f))(\mathrm{id} \otimes E_Q)(K(1 \otimes_Q f))^* \leq (\mathrm{id} \otimes E_Q)(KK^*). \tag{2.4}$$

Proof. In the computation below, we will write $1 \otimes f$ instead of $1 \otimes_Q f$ for notational convenience. If Ω is finite, we can directly compute

$$\begin{aligned}
0 &\leq (\mathrm{id} \otimes E_Q)\Big[\Big(K^* - \sum_{f\in\Omega}(1 \otimes f)(\mathrm{id} \otimes E_Q)(K(1 \otimes f))^*\Big)^* \\
&\qquad\qquad \cdot \Big(K^* - \sum_{f\in\Omega}(1 \otimes f)(\mathrm{id} \otimes E_Q)(K(1 \otimes f))^*\Big)\Big] \\
&= (\mathrm{id} \otimes E_Q)(KK^*) - 2\sum_{f\in\Omega}(\mathrm{id} \otimes E_Q)[K(1 \otimes f)(\mathrm{id} \otimes E_Q)(K(1 \otimes f))^*] \\
&\quad + \sum_{f,g\in\Omega}(\mathrm{id} \otimes E_Q)[(\mathrm{id} \otimes E_Q)(K(1 \otimes f))(1 \otimes f)^* \\
&\qquad\qquad \cdot (1 \otimes g)(\mathrm{id} \otimes E_Q)(K(1 \otimes g))^*] \\
&= (\mathrm{id} \otimes E_Q)(KK^*) - 2\sum_{f\in\Omega}(\mathrm{id} \otimes E_Q)(K(1 \otimes f))(\mathrm{id} \otimes E_Q)(K(1 \otimes f))^* \\
&\quad + \sum_{f,g\in\Omega}(\mathrm{id} \otimes E_Q)(K(1 \otimes f))E_Q(f^*g)(\mathrm{id} \otimes E_Q)(K(1 \otimes g))^* \\
&\leq (\mathrm{id} \otimes E_Q)(KK^*) - \sum_{f\in\Omega}(\mathrm{id} \otimes E_Q)(K(1 \otimes f))(\mathrm{id} \otimes E_Q)(K(1 \otimes f))^*,
\end{aligned}$$

where we used that Ω is conditionally orthonormal in the last inequality. If Ω is infinite, it follows from the above that the sum on the left-hand side of (2.4) converges, and then the same computation yields the desired result. ∎

Using this, we can now establish the following stronger summability criterion for conditional Hilbert–Schmidt operators using the Q-valued norm, but where for technical reasons we need to pass to adjoints. Below in Lemma 2.10, this technicality disappears thanks to the tracial property of the center-valued trace.

Lemma 2.9. *Let $K \in \mathrm{HS}_Q$ be a conditional Hilbert–Schmidt operator. Then there exists a constant $C > 0$ such that for any conditionally orthonormal subset $\Omega \subset L^2(N)^0$, we have*

$$\sum_{f \in \Omega} \|(K *_Q f)^*\|_Q^2 \leq C 1_Q.$$

Proof. Firstly, we note that if $K \in \mathrm{HS}_Q \subset \langle N, e_Q \rangle$, then $E_Q(KK^*) \leq \|KK^*\| 1_Q$. Furthermore, through the identification $K \mapsto \widehat{K} \in L^2(N) \otimes_Q L^2(N)$, we have $E_Q(KK^*) = (E_Q \otimes E_Q)(\widehat{K}\widehat{K}^*)$. We will drop the hat for notational convenience in what follows.

Set $C := \|KK^*\|$. We can now compute

$$\begin{aligned}\sum_{f \in \Omega} \|(K *_Q f)^*\|_Q^2 &= \sum_{f \in \Omega} \langle (K *_Q f)^*, (K *_Q f)^* \rangle_Q \\ &= \sum_{f \in \Omega} E_Q((\mathrm{id} \otimes E_Q)(K(1 \otimes_Q f))(\mathrm{id} \otimes E_Q)(K(1 \otimes_Q f))^*).\end{aligned}$$

By Lemma 2.8, we thus get

$$\sum_{f \in \Omega} \|(K *_Q f)^*\|_Q^2 \leq E_Q((\mathrm{id} \otimes E_Q)(KK^*)) = (E_Q \otimes E_Q)(KK^*) \leq C 1_Q.$$

This finishes the proof. ∎

For the next lemma, we denote by $\mathcal{Z} = \mathcal{Z}(Q)$ the center of Q, and by $E_{\mathcal{Z}} : Q \to \mathcal{Z}$ the corresponding center-valued trace.

Lemma 2.10. *Let $K \in \mathrm{HS}_Q$ be a conditional Hilbert–Schmidt operator. Then there exists a constant $C > 0$ such that for any conditionally orthonormal subset $\Omega \subset L^2(N)^0$, we have*

$$\sum_{f \in \Omega} E_{\mathcal{Z}}(\langle K *_Q f, K *_Q f \rangle_Q) \leq C 1_Q.$$

Proof. Using the tracial property of $E_{\mathcal{Z}}$, we compute

$$\begin{aligned}\sum_{f \in \Omega} E_{\mathcal{Z}}(\langle K *_Q f, K *_Q f \rangle_Q) &= \sum_{f \in \Omega} E_{\mathcal{Z}}(E_Q((K *_Q f)^*(K *_Q f))) \\ &= \sum_{f \in \Omega} E_{\mathcal{Z}}((K *_Q f)^*(K *_Q f)) \\ &= \sum_{f \in \Omega} E_{\mathcal{Z}}((K *_Q f)(K *_Q f)^*) \\ &= \sum_{f \in \Omega} E_{\mathcal{Z}}(E_Q((K *_Q f)(K *_Q f)^*)).\end{aligned}$$

By Lemma 2.9, $\sum_{f\in\Omega} E_Q((K *_Q f)(K *_Q f)^*) = \sum_{f\in\Omega} \|(K *_Q f)^*\|_Q^2 \le C1_Q$, and hence

$$\sum_{f\in\Omega} E_{\mathcal{Z}}(E_Q((K *_Q f)(K *_Q f)^*)) = E_{\mathcal{Z}}\Big(\sum_{f\in\Omega} \|(K *_Q f)^*\|_Q^2\Big) \le C1_Q.$$

This finishes the proof. ∎

2.4. Corrected proof

Finally, we give the corrected proof of the implication (2) ⇒ (3) of [2, Theorem A].

Proof. Let $K \in \mathrm{HS}_Q$ be Γ-equivariant and assume without loss of generality that $K \ge 0$. Write $p_\varepsilon := 1_{[\varepsilon,\infty)}(K)$, where we apply Borel functional calculus to K as an operator $K : L^2(N) \to L^2(N)$. Then p_ε is Γ-invariant, since it arises as a limit of polynomials in K, and thus $\mathcal{H}_\varepsilon := p_\varepsilon L^2(N)$ is a Γ-invariant Q-submodule of $L^2(N)$.

Moreover, by construction we have $p_\varepsilon K = K p_\varepsilon \ge \varepsilon p_\varepsilon$, or in other words $K \ge \varepsilon 1$ on the submodule $\mathcal{H}_\varepsilon$. In particular,

$$\langle K *_Q f, f\rangle_{L^2(N)} \ge \varepsilon \langle f, f\rangle_{L^2(N)} \tag{2.5}$$

for every $f \in \mathcal{H}_\varepsilon$. We claim that moreover

$$|E_{\mathcal{Z}}(\langle K *_Q f, f\rangle_Q)| \ge \varepsilon E_{\mathcal{Z}}(\langle f, f\rangle_Q). \tag{2.6}$$

Indeed, the latter inequality can be interpreted as an inequality of functions upon identifying $\mathcal{Z} \cong L^\infty(X,\mu)$ for some probability space (X,μ). Now if (2.6) fails, then we can find a positive measure subset $A \subset X$ such that, denoting $q = 1_A \in \mathcal{Z}$,

$$|E_{\mathcal{Z}}(\langle K *_Q fq, fq\rangle_Q)| = |E_{\mathcal{Z}}(\langle K *_Q f, f\rangle_Q)| q < \varepsilon E_{\mathcal{Z}}(\langle f, f\rangle_Q) q = \varepsilon E_{\mathcal{Z}}(\langle fq, fq\rangle_Q).$$

This implies $\langle K *_Q fq, fq\rangle_{L^2(N)} = \tau(\langle K *_Q fq, fq\rangle_Q) < \varepsilon\langle fq, fq\rangle_{L^2(N)}$, contradicting (2.5) since $fq \in \mathcal{H}_\varepsilon$.

Viewing $\mathcal{H}_\varepsilon$ as a right $\mathcal{Z}$-module, we note that

$$\langle f, g\rangle_{\mathcal{Z}} = E_{\mathcal{Z}}(f^*g) = E_{\mathcal{Z}}(E_Q(f^*g)) = E_{\mathcal{Z}}(\langle f, g\rangle_Q),$$

for $f, g \in \mathcal{H}_\varepsilon$. Since $\mathcal{Z}$ is moreover abelian, we have a conditional Cauchy–Schwarz inequality: for all $f, g \in \mathcal{H}_\varepsilon$,

$$|\langle f, g\rangle_{\mathcal{Z}}|^2 \le \langle f, f\rangle_{\mathcal{Z}} \langle g, g\rangle_{\mathcal{Z}}$$

(see [3]).

Using this together with (2.6), we thus get

$$\begin{aligned}
\varepsilon E_{\mathcal{Z}}(\langle f, f\rangle_Q) &\le |E_{\mathcal{Z}}(\langle K *_Q f, f\rangle_Q)| \\
&= |\langle K *_Q f, f\rangle_{\mathcal{Z}}| \\
&\le \langle K *_Q f, K *_Q f\rangle_{\mathcal{Z}}^{\frac12} \langle f, f\rangle_{\mathcal{Z}}^{\frac12} \\
&= E_{\mathcal{Z}}(\langle K *_Q f, K *_Q f\rangle_Q)^{\frac12} E_{\mathcal{Z}}(\langle f, f\rangle_Q)^{\frac12}.
\end{aligned}$$

Letting $\Omega \subset \mathcal{H}_\varepsilon$ be any conditionally orthonormal basis, which exists by [1, Proposition 8.4.11], we thus get that

$$\sum_{f \in \Omega} E_{\mathcal{Z}}(\langle f, f \rangle_Q) \leq \frac{1}{\varepsilon^2} \sum_{f \in \Omega} E_{\mathcal{Z}}(\langle K *_Q f, K *_Q f \rangle_Q).$$

Since $K \in \mathrm{HS}_Q$, we conclude from Lemma 2.10 that the right-hand side is bounded above by $\frac{C}{\varepsilon^2} 1_Q$ for some constant $C > 0$. Proposition 2.1 thus implies that $\mathcal{H}_\varepsilon$ is of finite rank.

Letting $\varepsilon \to 0$, we get that the union of the finitely generated Γ-invariant Q-submodules of the range of K is dense in the range of K. ■

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