Groups Geom. Dyn. 20 (2026), 1–20
DOI 10.4171/GGD/942

The space of actions, partition metric and combinatorial rigidity

Miklós Abért and Gábor Elek

Abstract. We introduce a natural pseudometric on the space of actions of d-generated groups. In this pseudometric, the zero classes correspond to the weak equivalence classes defined by Kechris, and the metric identification is compact. We achieve this by employing symbolic dynamics and an ultraproduct construction which also facilitates the extension of our results to unitary representations. As a byproduct, we show that the weak equivalence class of every free non-amenable action contains an action that satisfies the measurable von Neumann problem.

1. Introduction

Let $(X, \mathcal{B}, \mu)$ be a non-atomic standard Borel probability space. Unless otherwise specified, we assume $(X, \mathcal{B}, \mu)$ is the unit interval equipped with the usual Borel subsets and Lebesgue measure. An *automorphism* of $(X, \mathcal{B}, \mu)$ is defined as a measure-preserving Borel isomorphism of $(X, \mathcal{B}, \mu)$. We identify two isomorphisms if they act the same way up to a nullset. We define $\mathrm{Aut}(X, \mathcal{B}, \mu)$ as the group of equivalence classes of automorphisms of $(X, \mathcal{B}, \mu)$. Using a dense sequence $\{A_n\}_{n=1}^{\infty}$ in the measure algebra $\mathcal{M}(X, \mu)$ (see Section 2), one can define a metric on $\mathrm{Aut}(X, \mathcal{B}, \mu)$ as

$$\delta(\phi, \psi) = \sum_n 2^{-n} \mu(\phi(A_n) \bigtriangleup \psi(A_n)),$$

where $\bigtriangleup$ denotes symmetric difference. The metric δ induces the weak topology on $\mathrm{Aut}(X, \mathcal{B}, \mu)$. The metric space $(\mathrm{Aut}(X, \mathcal{B}, \mu), \delta)$ is separable and complete, making the group of automorphisms a Polish group.

For a finite alphabet S, let $\mathrm{A}(S) = \mathrm{Aut}(X, \mathcal{B}, \mu)^S$ denote the set of maps from S to $\mathrm{Aut}(X, \mathcal{B}, \mu)$. In other words, $\mathrm{A}(S)$ corresponds to the set of probability measure-preserving (p.m.p.) actions of the free group $\mathbb{F}_S$ on $(X, \mathcal{B}, \mu)$. The metric δ extends to $\mathrm{A}(S)$ and defines the weak topology on it. For more details, see the book of Kechris [11]. Note that the space $\mathrm{A}(S)$ contains all p.m.p. actions of groups generated by $|S|$ elements, where words in $\mathbb{F}_S$ that evaluate to 1 act trivially.

Inspired by a classical concept in representation theory, Kechris [11] introduced the weak containment relation on $\mathrm{A}(S)$ as follows.

Mathematics Subject Classification 2020: 37A15.
Keywords: space of actions, partition metric, weak equivalence.

Definition 1.1. Let $f, g \in \mathrm{A}(S)$ be two $\mathbb{F}_S$-actions. Then f *weakly contains* g ($f \succeq g$) if for any $n \geq 2$, Borel partition $C : X \to \{1, 2, \ldots, n\}$, a finite set $F \subset \mathbb{F}_S$ and $\varepsilon > 0$ there exists a Borel partition $D : X \to \{1, 2, \ldots, n\}$ such that

$$|\mu(f_\gamma D_i \cap D_j) - \mu(g_\gamma C_i \cap C_j)| \leq \varepsilon \quad (1 \leq i,\ j \leq n,\ \gamma \in F),$$

where $C_k = C^{-1}(k)$ and $D_k = D^{-1}(k)$.

This means that the way g acts on finite partitions of the standard Lebesgue space can be simulated by f with arbitrarily small error. The actions f and g are *weakly equivalent*, $f \sim g$, if they weakly contain each other. If Γ is a countable group, the weak equivalence of probability measure-preserving actions of Γ can be defined analogously to the weak equivalence of $\mathbb{F}_S$-actions.

One can immediately observe that if the actions f and g are conjugates, then they are weakly equivalent. However, the converse is far from being true, for example, all essentially free p.m.p. actions of the integers are pairwise weakly equivalent. Nevertheless, Kechris [11, Proposition 10.1] gave an equivalent definition for weak containment that involves the notion of conjugacy. Let $f, g \in \mathrm{A}(S)$. Then f weakly contains g ($f \succeq g$) if $C(g) \subseteq \overline{C(f)}$ where the closure is in the weak topology [11, Proposition 10.1]. Here, $C(g)$ denotes the conjugates of g in $\mathrm{A}(S)$.

The aim of this paper is to introduce and study the notion of *partition metric*, a natural pseudometric on $\mathrm{A}(S)$ such that the zero classes are exactly the weak equivalence classes and the metric identification of $\mathrm{A}(S)$ with respect to this pseudometric is compact. In the following, we give a condensed description of how to define the partition metric. For details, see Section 2.

One can extract the local structure of an action $f \in \mathrm{A}(S)$ by taking the stabilizer of a μ-random point in X. This gives a probability measure on the space of subgroups of $\mathbb{F}_S$ invariant under conjugation by $\mathbb{F}_S$. Such measures are called *invariant random subgroups (IRS)*. The term was coined in [2], where it is proved that every IRS arises as the stabilizer of a p.m.p. action. The IRS of an action encodes information about its freeness but omits many other details. For instance, all free actions have trivial IRSs.

Proposition 1. *Weakly equivalent actions have the same IRS.*

For an IRS λ, let $\mathrm{A}(S, \lambda)$ (the *fiber* of λ) be the set of actions in $\mathrm{A}(S)$ with IRS λ.

The notion of symbolic dynamics can be extended from free actions to an arbitrary element $f \in \mathrm{A}(S)$ as follows. A Borel partition $C : X \to \{1, \ldots, k\}$ defines a colored random rooted graph. This graph is constructed as the k-vertex-colored Schreier graph of the action on the orbit of a μ-random point of X. Applying the Hausdorff distance on the set of k-Borel partition of X leads to the *partition metric* $\mathrm{pd}(f, g)$.

This introduces a third equivalent definition of weak equivalence, complementing those given by Kechris. Specifically, two actions f and g are weakly equivalent if their partition distance equals zero.

Theorem 1. *The function* pd *is a pseudometric on* $\mathrm{A}(S)$. *The zero classes of* pd *correspond exactly to the weak equivalence classes. Also, the metric identification of* $\mathrm{A}(S)$ *with this pseudometric is compact. Moreover, the fiber* $\mathrm{A}(S, \lambda)$ *of any invariant random subgroup* λ *is compact as well.*

This metric identification is referred to as the space of actions modulo weak equivalence. The type of convergence within this space is called *local-global convergence*. The notions of partition metric and local-global convergence come from graph convergence theory. The partition metric has been introduced by Bollobás and Riordan in that setting [5]. Hatami, Lovász, and Szegedy [10] later developed the concept of local-global convergence for sequences of finite graphs with bounded degrees, demonstrating that graphings can serve as limit objects. The graph theoretic analogue of an IRS is a unimodular random rooted graph that has been introduced by Aldous and Lyons in [4].

An intriguing property of the partition metric is that the weak equivalence class of an action approximately satisfying a combinatorial property always contains an action satisfying it exactly. We demonstrate this phenomenon on the measurable von Neumann problem. A conjecture attributed to John von Neumann posits that any non-amenable group contains a free subgroup on two generators. However, this conjecture is not universally valid, as non-amenable torsion groups exist [12]. Nevertheless, the conjecture remains unresolved in the measurable setting, as stated below [9].

Problem 1 (Gaboriau–Lyons). Is it true that for any free p.m.p. action of a countable non-amenable group Γ there exists a free p.m.p. action of $\mathbb{F}_2$ on the same space such that for μ-almost all $x \in X$, the x-orbits satisfy $x^{\mathbb{F}_2} \subseteq x^{\Gamma}$?

The conjecture was settled in the affirmative for large enough Bernoulli actions by Gaboriau and Lyons [9]. Using their result, we prove the following theorem.

Theorem 2. *For any free p.m.p. action of a countable non-amenable group* Γ, *there exists a weakly equivalent action for which Problem 1 has an affirmative solution.*

The novel technique introduced in our paper is the use of ultraproducts. This allows us to extend our results to unitary representations.

Remark 1. The first version of this paper was uploaded to arXiv more than ten years ago. Since then, it has been cited and used in several research papers. Partly because of that, we decided not to change the mathematical content in this version. We hope that we made the exposition more clear.

The paper is built as follows. In Section 2, we define the basic notions and prove some lemmas. In Section 3, we introduce the ultraproduct technique needed. We prove compactness of the space of actions in Section 4. This will lead to the proof of Theorem 2 in Section 5. In Section 6, we prove Proposition 1 and establish the equivalence of

Kechris' notion of weak equivalence and our version using symbolic dynamics. Finally, in the short Section 7, we finish the proof of Theorem 1. Note that in a follow-up paper, Tucker-Drob [14] defined another pseudometric on the space of actions using Kechris' original notion and showed its compactness (using the ultraproduct technique).

2. Preliminaries

In this section, we provide the basic definitions and some relevant lemmas.

Invariant random subgroups and Schreier graphs. Let S be a finite set, and let the group $\mathbb{F}_S$ act transitively by permutations on the pointed countable set X. We define the *Schreier graph* of this action as follows: the vertex set is X, and for each $s \in S$ and vertex x, there is an s-labeled edge going from x to x^s. Let us root the Schreier graph at the distinguished point of X. Then the Schreier graph is a rooted, connected, edge-labeled graph. We identify Schreier graphs that are isomorphic as rooted, edge-labeled graphs. A particular case is when H is a subgroup of $\mathbb{F}_S$ and the action is the right coset action: we denote the corresponding Schreier graph by $\mathrm{Sch}(\mathbb{F}_S/H, S)$, rooted at H. It is easy to see that every Schreier graph can be obtained this way and that H can be obtained by evaluating all the returning walks in the graph using the edge labels.

For an abstract alphabet S, let $\mathrm{SC}(S)$ denote the set of isomorphism classes of Schreier graphs for the free group $\mathbb{F}_S$. An external way to get an element of $\mathrm{SC}(S)$ is to take any $2|S|$-regular graph and then label the directed edges by $S \cup S^{-1}$ such that the following hold:

(1) for every vertex x and every $s \in S \cup S^{-1}$, there is exactly one s-labeled edge leaving and arriving to x;

(2) for every directed edge, its label is the formal inverse of the label of the reverted edge.

For two rooted Schreier graphs G_1 and G_2, let the distance $d(G_1, G_2) = \frac{1}{r+1}$, where r is the maximal integer such that the r-balls around the root of G_1 and G_2 are isomorphic. The metric d turns $\mathrm{SC}(S)$ into a totally disconnected, compact space. The group $\mathbb{F}_S$ acts on $\mathrm{SC}(S)$ continuously by moving the root along the path that represents the acting word.

Let $\mathrm{SC}_k(S)$ denote the set of rooted Schreier graphs together with a k-vertex coloring. We can define the metric similarly to that for ordinary Schreier graphs, just that we consider vertex-colored isomorphisms of rooted balls in the definition of r. Again, this metric turns $\mathrm{SC}_k(S)$ into a totally disconnected, compact space, and $\mathbb{F}_S$ acts on $\mathrm{SC}_k(S)$ continuously by moving the root. Clearly, the color-forgetting map $\mathrm{SC}_k(S) \to \mathrm{SC}(S)$ is a continuous $\mathbb{F}_S$-equivariant surjection.

Let $\mathrm{Sub}(\mathbb{F}_S)$ denote the set of subgroups of $\mathbb{F}_S$. We can endow $\mathrm{Sub}(\mathbb{F}_S)$ with the topology inherited from the product topology on the set of subsets of $\mathbb{F}_S$: $\{0,1\}^{\mathbb{F}_S}$. This turns $\mathrm{Sub}(\mathbb{F}_S)$ into a compact space. The group $\mathbb{F}_S$ acts on $\mathrm{Sub}(\mathbb{F}_S)$ continuously by conjugation. A random subgroup of $\mathbb{F}_S$ is called an *invariant random subgroup* (IRS) if its distribution is a Borel measure invariant under the conjugation action. The name IRS was coined in [2].

For $f \in \mathrm{A}(S)$, let the *type of* f be $\mathrm{Stab}_{\mathbb{F}_S}(x)$ where x is a uniform μ-random point in X. It is easy to see that the type is an IRS of $\mathbb{F}_S$. In [2], it is proved that every IRS arises as the type of a p.m.p. action. For an IRS λ, let $\mathrm{A}(S,\lambda)$ the *fiber of* λ be the set of actions in $\mathrm{A}(S)$ with type λ. Another way to look at the type of an action $f \in \mathrm{A}(S)$ is to consider the Schreier graph of the action of $\mathbb{F}_S$ on the orbit of a uniform μ-random point in X, rooted at x. From this point of view, the type is a Borel probability distribution on $\mathrm{SC}(S)$ that is invariant under moving the root. This identification matches with the canonical bijection between $\mathrm{SC}(S)$ and $\mathrm{Sub}(\mathbb{F}_S)$, so there is no ambiguity.

The partition metric. Let $\mathrm{U}(S)$ and $\mathrm{U}_k(S)$ denote the set of $\mathbb{F}_S$-invariant Borel probability distributions on $\mathrm{SC}(S)$ and $\mathrm{SC}_k(S)$, respectively. Both sets are compact and metrizable under the weak topology. Let d_S and $d_{S,k}$ denote the metrics on $\mathrm{SC}(S)$ and $\mathrm{SC}_k(S)$, respectively, which define the weak topology on these sets. Let us endow the set of compact subsets of $\mathrm{U}_k(S)$, $\mathrm{Comp}(\mathrm{U}_k(S))$ with the Hausdorff metric $d_{\mathrm{Haus},S,k}$. It is important to note that both $\mathrm{U}_k(S)$ and $\mathrm{Comp}(\mathrm{U}_k(S))$ are compact metric spaces with respect to the metrics defined above. Note that the Hausdorff metric $d_{\mathrm{Haus},S,k}$ depends on the choice of the metric $d_{S,k}$. However, the topology of $\mathrm{Comp}(\mathrm{U}_k(S))$ does not.

For $f \in \mathrm{A}(S)$ and a Borel partition $C : X \to \{1,\ldots,k\}$, let $\Psi_f^C : X \to \mathrm{SC}_k(S)$ be the Borel map, where $\Psi_f^C(x)$ is the k-vertex-colored Schreier graph of the action of $\mathbb{F}_S$ on the orbit of x (with x as the root). Then the push-forward measure $(\Psi_f^C)_*(\mu)$ is an element of $\mathrm{U}_k(S)$.

Definition 2.1. By considering all possible k-Borel partitions C of X, the closure of $\bigcup_C (\Psi_f^C)_*(\mu)$ gives us a compact subset $H_S^k(f) \in \mathrm{Comp}(\mathrm{U}_k(S))$. This set is called the *global k-type of f*.

Definition 2.2. Let $f, g \in \mathrm{A}(S)$ be two actions. The k-partition pseudodistance $\mathrm{pd}_k(f,g)$ is defined as follows:

$$\mathrm{pd}_k(f,g) = d_{\mathrm{Haus},S,k}((\Psi_f^C)_*(\mu), (\Psi_g^C)_*(\mu)).$$

Definition 2.3. The *partition pseudometric*, denoted as $\mathrm{pd}(f,g)$, is given by

$$\mathrm{pd}(f,g) = \sum_{k=1}^{\infty} \frac{1}{2^k}\, \mathrm{pd}_k(f,g).$$

Thus, the actions f are encoded with the element $\prod_{k=1}^{\infty} H_S^k(f)$ of the compact "codespace" $\prod_{k=1}^{\infty} \mathrm{Comp}(\mathrm{U}_k(S))$. The main theorem entails that if for a sequence of actions $\{f\}_{n=1}^{\infty}$, the associated codes are convergent to an element s of the space $\prod_{k=1}^{\infty} \mathrm{Comp}(\mathrm{U}_k(S))$, then we have an action $f \in \mathrm{A}(S)$ such that s is the code of f.

Unitary representations. Let Γ be a countable group and $\alpha, \beta : \Gamma \to U(H)$ be unitary representations of Γ on a complex separable Hilbert space H. We say that β weakly contains (in the sense of Zimmer) [11] α if for any finite orthonormal system $v_1, v_2, \dots, v_n$ in H, a finite set $F \subset \Gamma$, and a real number $\varepsilon > 0$, there exists an orthonormal system $w_1, w_2, \dots, w_n$ such that for any $1 \leq i, j, \leq n$ and $\gamma \in F$

$$|\langle \alpha(\gamma) v_i, v_j \rangle - \langle \beta(\gamma) w_i, w_j \rangle| < \varepsilon.$$

We say that two representations are weakly equivalent if they weakly contain each other in the sense of Zimmer. Note that the original definition of weak containment and weak containment in the sense of Zimmer are slightly different (see [11, Appendix H]). In our paper, weak containment always means weak containment in the sense of Zimmer. We say that a unitary representation α *contains* β, if β is isomorphic to a subrepresentation of α. Now fix a unitary representation α. Let us consider the countable set of pairs (F, n), where $F \subset \Gamma$ is a finite set and $n \geq 1$ is a natural number. For any such pair, we have a product set $D^{n^2 \times |F|} = C_{F,n}$, where D is the unit disc of the complex plane. Let $K_{F,n}(\alpha)$ be the closure of the set

$$\left\{ \bigoplus_{\gamma \in F} \bigoplus_{1 \leq i,j \leq n} \langle \alpha(\gamma) v_i, v_j \rangle \mid (v_1, v_2, \dots, v_n) \text{ is an orth. syst.} \right\}$$

in $C_{F,n}$. Again, we associate a closed subset $Q(\alpha)$ of a compact product set to a representation by

$$Q(\alpha) = \prod_{F,n} K_{F,n}(\alpha) \subset \prod_{F,n} C_{F,n}.$$

The following lemma is straightforward.

Lemma 2.1. *The unitary representation α is weakly equivalent to β if and only if $Q(\alpha) = Q(\beta)$.*

We shall prove the following analogue of Theorem 1.

Theorem 3. *The image of Q is compact.*

Measure algebras. In [11], the author uses the measure algebra formalism instead of measure spaces. Similarly, we find the use of measure algebras advantageous in our proofs. Hence in this subsection, we list some well-known facts about measure algebras and group actions of measure algebras. A measure algebra $\mathcal{M}$ is a Boolean algebra with a finitely additive measure μ that is a complete metric space with respect to the distance $d(A, B) = \mu(A \triangle B)$. If (X, μ) is a Lebesgue probability space, then the equivalence classes of Borel sets (two sets are equivalent if their symmetric distance has measure zero) form a measure algebra, the Lebesgue algebra. Any separable atomless measure algebra is in fact isomorphic to the Lebesgue algebra. In general, if $(X, \mathcal{A}, \mu)$ is a measure space with a sigma-algebra, then $\mathcal{M}(X,\mu)$ denotes the associated measure algebra. Let $\alpha : \mathcal{M}(X,\mu) \to \mathcal{M}(Y,\nu)$ be an injective (measure-preserving) homomorphism between measure algebras. Then there exists a surjective Borel map $\Phi_\alpha : Y \to X$ such that for any Borel set $A \subseteq X$, $\overline{\Phi_\alpha^{-1}(A)} = \alpha(\overline{A})$, where $\overline{A}$ denotes the element of the measure algebra represented by the set A.

Definition 2.4. Let $\psi : \mathbb{F}_S \to \mathrm{Aut}(\mathcal{M}(X,\mu))$ be a representation of $\mathbb{F}_S$ by measure algebra automorphisms. Then there exists $f_\psi \in \mathrm{A}(S)$ such that for any $\gamma \in \mathbb{F}_S$ and Borel set A, $\psi(\gamma)(\overline{A}) = \overline{f_\psi(\gamma)(A)}$. The action f_ψ is called the action *associated* with ψ.

If $\psi : \mathbb{F}_S \to \mathrm{Aut}(\mathcal{M}(X,\mu))$ and $\phi : \mathbb{F}_S \to \mathrm{Aut}(\mathcal{M}(Y,\nu))$ are representations and $\alpha : \mathcal{M}(X,\mu) \to \mathcal{M}(Y,\nu)$ is an injective measure-preserving isomorphism commuting with the representations, then the associated map Φ_α commutes (up to null sets) with the associated actions $f_\psi, f_\phi \in \mathrm{A}(S)$. Then we say that f_ϕ *contains* f_ψ.

3. The ultraproduct technique

The main technical tools in our paper are the ultraproducts of actions and representations. In this section, we briefly recall the construction of ultrapowers of probability measure spaces from [7]. Let (X,μ) be a standard Borel probability measure space and ω be a nonprincipal ultrafilter. Let $\lim_\omega$ be the associated ultralimit $\lim_\omega : l^\infty \to \mathbb{R}$. The ultrapower of the set X is defined in the following way. Let $\widetilde{X} = \prod_{i=1}^\infty X_i$, where each X_i is a copy of our X, equipped with the atomless probability measure μ, that we denote by μ_i to avoid confusion. We say that $\widetilde{p} = \{p_i\}_{i=1}^\infty, \widetilde{q} = \{q_i\}_{i=1}^\infty \in \widetilde{X}$ are equivalent, $\widetilde{p} \sim \widetilde{q}$, if

$$\{i \in \mathbb{N} \mid p_i = q_i\} \in \omega.$$

Define $\mathbf{X} := \widetilde{X}/\sim$. Now let $\mathcal{P}(X_i)$ denote the Boolean algebra of Borel subsets of X_i, with measure μ_i. Then let $\widetilde{\mathcal{P}} = \prod_{i=1}^\infty \mathcal{P}(X_i)$ and $\mathcal{P} = \widetilde{\mathcal{P}}/I$, where I is the ideal of elements $\{A_i\}_{i=1}^\infty$ such that $\{i \in \mathbb{N} \mid A_i = \emptyset\} \in \omega$. Notice that elements of $\mathcal{P}$ can be identified with certain subsets of $\mathbf{X}$: if

$$\overline{p} = [\{p_i\}_{i=1}^\infty] \in \mathbf{X} \quad \text{and} \quad \overline{A} = [\{A_i\}_{i=1}^\infty] \in \mathcal{P},$$

then $\overline{p} \in \overline{A}$ if and only if $\{i \in \mathbb{N} \mid p_i \in A_i\} \in \omega$. Clearly, for $\overline{A} = [\{A_i\}_{i=1}^\infty]$ and $\overline{B} = [\{B_i\}_{i=1}^\infty]$, the following hold:

- $\overline{A}^c = [\{A_i^c\}_{i=1}^\infty]$,
- $\overline{A} \cup \overline{B} = [\{A_i \cup B_i\}_{i=1}^\infty]$,
- $\overline{A} \cap \overline{B} = [\{A_i \cap B_i\}_{i=1}^\infty]$.

Thus, $\mathcal{P}$ forms a Boolean algebra on $\mathbf{X}$. An important subalgebra of P, denoted P', is associated with sequences where, for a Borel set $A \in X$, $A_i = A$ for all i. Clearly, the

Boolean algebra $\mathcal{P}'$ is isomorphic to the Boolean algebra of Borel sets of X. Define $\overline{\mu}(\overline{A}) = \lim_\omega \mu_i(A_i)$. Then $\overline{\mu} : \mathcal{P} \to \mathbb{R}$ is a finitely additive probability measure. We will call $\overline{A} = [\{A_i\}_{i=1}^\infty]$ the *ultraproduct* of the sets $\{A_i\}_{i=1}^\infty$.

Definition 3.1. $N \subseteq \mathbf{X}$ is a *nullset* if for any $\varepsilon > 0$ there exists a set $\overline{A_\varepsilon} \in \mathcal{P}$ such that $N \subseteq \overline{A_\varepsilon}$ and $\overline{\mu}(\overline{A_\varepsilon}) \leq \varepsilon$. The set of nullsets is denoted by $\mathcal{N}$.

Definition 3.2. We call $B \subseteq \mathbf{X}$ a *measurable set* if there exists $\widetilde{B} \in \mathcal{P}$ such that $B \triangle \widetilde{B} \in \mathcal{N}$.

Proposition 2 ([7, Proposition 2.2]). *The measurable sets form a σ-algebra $\mathcal{B}_\omega$, and $\overline{\mu}(B) = \overline{\mu}(\widetilde{B})$ defines a probability measure on $\mathcal{B}_\omega$. We denote this measure space by $(\mathbf{X}, \overline{\mu})$.*

It is important to note that the measure algebra of this space is not separable.

Let $\{f^i\}_{i=1}^\infty \subset \mathrm{A}(S)$. The ultraproduct of these actions $\overline{f}$ is defined in the following way:

$$\overline{f}_\gamma([\{p_i\}_{i=1}^\infty]) = [\{f_\gamma^i(p_i)\}_{i=1}^\infty].$$

This way we defined a measure-preserving action of $\mathbb{F}_S$ on the ultraproduct space. If $f^i = f$ for all i, then $\overline{f} = f_\omega$ is called the ultrapower of f. A σ-algebra $\mathcal{B}'_\omega \subset \mathcal{B}_\omega$ is $\mathbb{F}_S$-invariant if, for any $\gamma \in \mathbb{F}_S$ and $U \in \mathcal{B}'_\omega$, $f_\omega(\gamma)(U) \in \mathcal{B}'_\omega$.

Proposition 3. *Let $f \in \mathrm{A}(S)$ and f_ω be its ultrapower. Let $\mathcal{B}'_\omega$ be an $\mathbb{F}_S$-invariant separable subalgebra of $\mathcal{B}_\omega$ on $\mathbf{X}$ containing the algebra $\mathcal{P}'$. Then the associated $\mathbb{F}_S$-action $g \in \mathrm{A}(S)$ (see Definition 2.4) is weakly equivalent to f.*

Proof. The measure algebra $\mathcal{M}(\mathbf{X}, \mathcal{P}')$ is isomorphic to the measure algebra $\mathcal{M}(X, \mu)$, hence g contains f.

Now let $A : \mathbf{X} \to \{1, 2, \ldots, k\}$ be a measurable partition of $\mathbf{X}$ such that $A^{-1}(i) \in \mathcal{B}'_\omega$ and let $V_1, V_2, \ldots, V_k$ be elements of $\mathcal{P}$ such that $\overline{\mu}(A^{-1}(i) \triangle V_i) = 0$ for any $1 \leq i \leq k$. Then we have a sequence of Borel partitions $\{X = V_1^i \cup V_2^i \cdots \cup V_k^i\}_{i=1}^\infty$ such that $[\{V_j^i\}_{i=1}^\infty] = V_j$ for any $1 \leq j \leq k$. By definition, for any $\varepsilon > 0$ and $\gamma \in \mathbb{F}_S$, the set

$$A_{j,l}^\gamma := \{i \mid |\mu(V_j^i \cap f_\gamma(V_l^i)) - \overline{\mu}(V_j \cap g_\gamma V_l)| < \varepsilon\}$$

is in the ultrafilter ω. Therefore, the action f weakly contains the action g. ∎

Proposition 4. *Let f and f_ω be as described earlier. Suppose $h \in \mathrm{A}(S)$ is weakly contained in f. Then there exists an $\mathbb{F}_S$-invariant separable subalgebra $\mathcal{B}'_\omega$ of $\mathcal{B}_\omega$ containing $\mathcal{P}'$ such that the associated $\mathbb{F}_S$-action g contains h.*

Proof. Identify X with the product space $\prod\{0, 1\}$, equipped with the usual product measure μ. If s is a $0-1$-string of length k, let A_s be the elements of X starting with the string s. Let $\mathrm{St}(n)$ denote the set of strings of length n. Let $\gamma_1, \gamma_2, \dots$ be an enumeration of the elements of $\mathbb{F}_S$. By our assumption, for any $n \geq 1$, there exists a Borel partition of X into 2^n pieces

$$\bigcup_{s_i \in \mathrm{St}(n)} B^n_{s_i} = X$$

such that for any $1 \leq i \leq k$ and strings s_a, s_b we have that

$$|\mu(B^n_{s_a} \cap f_{\gamma_i} B^n_{s_b}) - \mu(A_{s_a} \cap h_{\gamma_i} A_{s_b})| < \frac{1}{10^n}.$$

Notice that for $k \leq n$, B^n_s is well defined if s is a string of length k. Simply let

$$B^n_s = \bigcup_{s_i \text{ starts with } s} B^n_{s_i}.$$

Observe that

$$|\mu(B^n_s \cap f_{\gamma_i} B^n_{s'}) - \mu(A_s \cap h_{\gamma_i} A_{s'})| < \frac{1}{2^n}$$

if the strings s and s' have length not greater than n.

Let $\overline{B_s} = [\{B^n_s\}^\infty_{n=1}]$. Then clearly,

$$\overline{\mu}(\overline{B_s} \cap f_\omega(\gamma)\overline{B_{s'}}) = \mu(A_s \cap h_\gamma A_{s'})$$

for all strings s, s' and $\gamma \in \mathbb{F}_S$. Hence the subalgebra $\mathcal{C}_\omega$ generated by the sets $\overline{B_s}$ is $\mathbb{F}_s$ -invariant and $\mathbb{F}_s$-equivariantly isomorphic to the measure algebra of (X, μ). Therefore, if $\mathcal{B}'_\omega$ contains $\mathcal{C}_\omega$, then the associated action f_ω contains h. ∎

The following corollary was also proved in [6, Proposition 4.7]

Theorem 4. *If $f \in \mathrm{A}(S)$ weakly contains $h \in \mathrm{A}(S)$, then there exists $g \in \mathrm{A}(S)$ that is weakly equivalent to f that contains h.*

3.1. The ultraproduct of unitary representations

Let H be separable, complex Hilbert space and $\alpha_1, \alpha_2, \dots$ be unitary representations of the countable group Γ. We define the ultraproduct of the representations in the following way. First, we recall the notion of the ultrapower of H. Let $\prod^\infty_{n=1} H$ be the set of bounded sequences in H, and let $V \subset \prod^\infty_{n=1} H$ be the set of vectors $\{v_n\}^\infty_{n=1}$ such that $\lim_\omega \langle v_n, v_n \rangle = 0$. Clearly, V is a subspace of $\prod^\infty_{n=1} H$ with a well-defined inner product on $\prod^\infty_{n=1} H/V = \prod_\omega H$ by

$$\langle [\{v_n\}^\infty_{n=1}], [\{w_n\}^\infty_{n=1}] \rangle = \lim_\omega \langle v_n, w_n \rangle,$$

where $[\{v_n\}^\infty_{n=1}]$ denotes the element in $\prod_\omega H$ representing $\{v_n\}^\infty_{n=1} \in \prod^\infty_{n=1} H$. It is a standard result that $\prod_\omega H$ is a nonseparable Hilbert space. The ultraproduct action is defined by

$$\alpha_\omega(\gamma)(v) = [\{\alpha_n(\gamma)(v_n)\}^\infty_{n=1}].$$

Clearly, α_ω is a unitary representation of Γ. Again, we consider the special case, when $\alpha_n = \alpha$ for all $n \geq 1$. Let $\widehat{H} \subset \prod_\omega H$ be the subspace consisting of vectors in the form $[\{v_i\}_{i=1}^\infty]$, where $v_i = v_j$ for any $i, j \geq 1$. Then we have the following analogue of Proposition 3.

Proposition 5. *Let $\widehat{H} \subset K \subset \prod_\omega H$ be a separable Γ-invariant subspace. Then the restriction of α_ω on K is weakly equivalent to α.*

Proof. Clearly, α_ω weakly contains α. It is enough to show that α weakly contains α_ω. Let $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k \in K$, $F \subset \Gamma$ be a finite set and $\varepsilon > 0$, where $\underline{v}_i = [\{v_i^n\}_{n=1}^\infty]$. Let

$$S_{\gamma,i,j} = \{n \mid |\langle \alpha(\gamma) v_i^n, v_j^n \rangle - \langle \alpha_\omega(\gamma) \underline{v}_i, \underline{v}_j \rangle| < \varepsilon\}.$$

By the definition of the ultraproduct, $S_{\gamma,i,j} \in \omega$. Hence $\bigcap_{\gamma \in F} \bigcap_{1 \leq i,j \leq k} S_{\gamma,i,j} \in \omega$ as well. Thus the lemma follows. ∎

Now we prove the analogue of Proposition 4.

Proposition 6. *Let $\alpha : \Gamma \to U(H)$ be a representation that weakly contains the representation $\delta : \Gamma \to U(H)$. Then there exists $\beta : \Gamma \to U(H)$ weakly equivalent to α that contains δ.*

Proof. Let $\{v_n\}_{n=1}^\infty$ be an orthonormal basis for H. Enumerate the elements of Γ, $\{\gamma_i\}_{i=1}^\infty$. Since α weakly contains δ, there exists an orthonormal system $w_1^n, w_2^n, \dots, w_n^n$ such that for any $1 \leq i, j, k \leq n$,

$$|\langle \alpha(\gamma_i) w_j^n, w_k^n \rangle - \langle \delta(\gamma_i) v_j, v_k \rangle| < \frac{1}{2^k}.$$

Let $\underline{w}_j = [\{w_j^n\}_{n=1}^\infty] \in \prod_\omega H$, where $w_j^n = 0$ if $n < j$. Then for any $i, j, k \geq 1$,

$$\langle \alpha_\omega(\gamma_i) \underline{w}_j, \underline{w}_k \rangle = \langle \delta(\gamma_i) v_j, v_k \rangle.$$

Hence α_ω restricted to the Γ-invariant subspace generated by $\widehat{H}$ and the vectors $\{\underline{w}_j\}_{j=1}^\infty$ contains δ. ∎

4. Compactness

4.1. The combinatorics of finite balls

Define $U^{r,S}$ as the finite family of r-balls (up to rooted, labeled isomorphisms) around the roots of $\mathbb{F}_S$-Schreier graphs, that is, elements of $\mathrm{SC}(S)$. We apply the following convention. If $x, y \in V(\kappa)$, $\kappa \in U^{r,S}$, and

$$d(\mathrm{root}(\kappa), x) = d(\mathrm{root}(\kappa), y) = r,$$

then x and y are not adjacent in κ. That is, we set vertices on the boundary to be non-adjacent. Let $W^{r,S}$ be the set of reduced words of length at most r in $\mathbb{F}_S$. For $\kappa \in U^{r,S}$, we have a partition P_κ of $W^{r,S}$

$$w_1 \equiv_{P_\kappa} w_2$$

if $w_1(\mathrm{root}(\kappa)) = w_2(\mathrm{root}(\kappa))$. By our convention, $\kappa_1 = \kappa_2$ if and only if $P_{\kappa_1} = P_{\kappa_2}$. For $f \in \mathrm{A}(S)$, let $\Psi_f : X \to \mathrm{SC}(S)$ be the Borel map, where $\Psi_f(x)$ is the Schreier graph of the action of $\mathbb{F}_S$ on the orbit of x (with x as the root). For a point $x \in X$, we call the r-ball around $\Psi_f(x)$ the r-ball type of x with respect to f. If $\kappa \in U^{r,S}$, then $T(\kappa) \in \mathrm{SC}(S)$ denotes the set of Schreier graphs G such that $B_r(\mathrm{root}(G)) \simeq \kappa$. Clearly, $T(\kappa)$ is a clopen set. Define

$$T(\kappa, f) := \Psi_f^{-1}(T(\kappa))$$

as the measurable set of points $x \in X$ such that the r-ball type of x is κ.

We also consider labeled balls. Let $U^{r,S,l}$ be the finite set of all l vertex labelings of the elements of $U^{r,S}$, up to rooted labeled isomorphisms. Thus we have a map $U^{r,S,l} \to U^{r,S}$ mapping a vertex labeled graph to the underlying unlabeled graph. Again, for $\widetilde{\kappa} \in U^{r,S,l}$, $T(\widetilde{\kappa})$ denotes the set of elements $\alpha \in \mathrm{SC}_l(S)$ such that the r-ball around the root of α is isomorphic to $\widetilde{\kappa}$. If $f \in \mathrm{A}(S)$ and $D : X \to \{1, 2, \ldots, l\}$ is a Borel partition, then the r-ball around the root of $\Psi_f^D(x)$ is called the (r, l)-ball type of x with respect to f and D. For $\widetilde{\kappa} \in U^{r,S,l}$, we denote by $T(\widetilde{\kappa}, f, D)$ the set of vertices $x \in X$ with (r, l)-type $\widetilde{\kappa}$.

4.2. The compactness of the global types

We begin with a simple lemma.

Lemma 4.1. *Let $\{f_n\}_{n=1}^\infty \subset \mathrm{A}(S)$ be a sequence of actions. Also, let $\{A_n \mid X \to \{1, 2, \ldots, l\}\}_{n=1}^\infty$ be a sequence of partitions and $\overline{f}$, respectively, $\overline{A}$ be their ultraproducts. Then for any $\widetilde{\kappa} \in U^{r,S,l}$,*

$$\lim_\omega \mu(T(\widetilde{\kappa}, f_n, A_n)) = \overline{\mu}(T(\widetilde{\kappa}, \overline{f}, \overline{A})).$$

Proof. Let $x_n \in X_n$ and $\overline{x} := [\{x_n\}_{n=1}^\infty]$. Then, the (r, l)-ball type of x with respect to $\overline{f}$ is $\widetilde{\kappa}$ if and only if the set of numbers n for which the type of x_n with respect to f_n is $\widetilde{\kappa}$ is an element of the ultrafilter ω. Hence

$$[\{T(\widetilde{\kappa}, f_n, A_n)\}_{n=1}^\infty] = T(\widetilde{\kappa}, \overline{f}, \overline{A}),$$

and the lemma follows. ∎

Now let $\{f_n\}_{n=1}^\infty$ be as above such that $H_S^k(f_n)$ is a Cauchy sequence in the space of compact subsets of $\mathrm{U}_k(S)$ with the Hausdorff metric.

Lemma 4.2. *We have that*

$$\lim_{n\to\infty} H_S^k(f_n) = H_S^k(\overline{f}). \tag{4.1}$$

Proof. Let $p \in H^k_S(\overline{f})$, where $p = (\Psi^{\overline{A}}_{\overline{f}})_\star(\mu)$ with $\overline{A} = [\{A_n\}^\infty_{n=1}]$. By Lemma 4.1, it follows that

$$\lim_\omega (\Psi^{A_n}_{f_n})_\star(\mu) = p.$$

Recall that the ultralimit with respect to ω is well defined for any compact metric space. If $\lim_\omega(\Psi^{A_n}_{f_n})_\star(\mu) = p$, then $\lim_{k\to\infty}(\Psi^{A_{n_k}}_{f_{n_k}})_\star(\mu) = p$ for some subsequence. This proves that for all $k \geq 1$ we have that $\lim_{n\to\infty} H^k_S(f_n)$ contains $H^k_S(\overline{f})$. Now let $\{(\Psi^{A_n}_{f_n})_\star(\mu)\}^\infty_{n=1}$ be a sequence in $\mathrm{U}_k(S)$ converging to an element p. Then by Lemma 4.1, we have that $p \in H^S_k(\overline{f})$. Therefore, $H^k_S(\overline{f})$ contains $\lim_{n\to\infty} H^k_S(f_n)$. ■

Proposition 7. *The set* $\left\{\prod^\infty_{k=1} H^k_S(f)\right\}_{f\in \mathrm{A}(S)}$ *is compact in* $\prod^\infty_{k=1} \mathrm{Comp}(\mathrm{U}_k(S))$.

Proof. Let $\{f_n\}^\infty_{n=1}, \overline{f}$ be as in the previous lemmas. By Lemma 4.2, it is enough to prove that there exists $g \in \mathrm{A}(S)$ such that for all $k \geq 1$, $H^k_S(g) = H^k_S(\overline{f})$ holds.

Let $\mathcal{C} \subset \mathcal{B}_\omega$ be an $\mathbb{F}_S$-invariant separable subalgebra. Then,

$$\bigcup_{A'} (\Psi^{A'}_{f})_\star(\mu) \subseteq \bigcup_{A} (\Psi^{A}_{f})_\star(\mu),$$

where the left-hand side is taken over the set of all $\mathcal{C}$-partitions and the right-hand side is taken over the set of all $\mathcal{B}_\omega$-partitions. Therefore, if we pick $\mathcal{C}$ in such a way that $\bigcup_{A'}(\Psi^{A'}_{f})_\star(\mu)$ contains a dense subset of $H^k_S(\overline{f})$, then $H^k_S(g) = H^k_S(\overline{f})$, where g is the action associated to $\mathcal{C}$. ■

Proof of Theorem 3. We need to show that the space of representations modulo weak equivalence is compact. Let $\{\alpha_i\}^\infty_{i=1}$ be a sequence of unitary representations on the Hilbert space H such that $\{Q(\alpha_i)\}^\infty_{i=1}$ is convergent, that is, for any finite set $F \subset \Gamma$ and $n \geq 1$, $\{K_{F,n}(\alpha_i)\}^\infty_{i=1}$ converges to some compact set $L_{F,n}$ in the Hausdorff metric. We need to prove that there exists a representation α such that $K_{F,n}(\alpha) = L_{F,n}$ for any F and any n. Let α_ω be the ultraproduct of the α_i's on $\prod_\omega H$. For each $i \geq 1$ and $k \geq 1$, pick orthonormal systems $\{v^{i,k,s}_{F,n}\}^n_{s=1}$ such that

$$\bigoplus_{\gamma\in F} \bigoplus_{1\leq p,q\leq n} \langle \alpha_i(\gamma) v^{i,k,p}_{F,n}, v^{i,k,q}_{F,n}\rangle = z^k_{F,n}$$

and the set $\{z^k_{F,n}\}^\infty_{k=1}$ is dense in $L_{F,n}$. Let $\overline{v}^{k,s} = [\{v^{i,k,s}_{F,n}\}^\infty_{i=1}]$. Let K be the Γ-invariant subspace of $\prod_\omega H$ generated by $\widehat{H}$ and the vectors $\bigcup_{F,n}\bigcup^\infty_{k=1}\bigcup^k_{s=1} \overline{v}^{k,s}_{F,n}$. Let α be the restriction of the ultraproduct action α_ω onto K. By definition, $L_{F,n} \subseteq K_{F,n}(\alpha)$ for any F and n. Now we prove the converse. Let $x \in K_{F,n}(\alpha)$. Fix a real number $\varepsilon > 0$. Then there exists $\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_n \in \prod_\omega H$ such that for any $\gamma \in F$ and $1 \leq i, j \leq n$

$$|\langle \alpha(\gamma)\underline{w}_i, \underline{w}_j\rangle - x_{\gamma,i,j}| < \varepsilon,$$

where $x_{\gamma,i,j}$ is the coordinate of x associated to the triple (γ, i, j). By the definition of the ultraproduct, there exist orthonormal systems $\{t^k_1, t^k_2, \ldots, t^k_n\}^\infty_{k=1} \subset H$ such that for any $1 \leq p, q \leq n$

$$\lim_\omega \langle \alpha_k(\gamma) t^k_i, t^k_j\rangle = \langle \alpha(\gamma)\underline{w}_i, \underline{w}_j\rangle.$$

Hence we have a subsequence $\{n_k\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} \langle \alpha_{n_k}(\gamma) t_i^{n_k}, t_j^{n_k} \rangle = \langle \alpha(\gamma) \underline{w}_i, \underline{w}_j \rangle \tag{4.2}$$

for any triple (γ, i, j) as above. Therefore, there exists an element $y \in L_{F,n}$ such that each coordinate of y differs from the corresponding coordinate of x by at most ε. Consequently, $L_{F,n} = K_{F,n}(\alpha)$. ∎

Remark. In [6, Corollary 4.5], the authors prove an interesting compactness result: if $\{a_n\}_{n=1}^{\infty} \subset \mathrm{A}(S)$ is a sequence of actions, then there is a subsequence $n_0 < n_1 < n_2 \cdots$ and $\{b_{n_k}\}_{k=1}^{\infty} \subset \mathrm{A}(S)$ such that $a_{n_k} \sim b_{n_k}$ and $\{b_{n_k}\}_{k=1}^{\infty}$ converges in $\mathrm{A}(S)$ in the weak topology.

The reader may ask what is the relation of this result to our compactness theorem. In fact, the two theorems are independent as they use a different topology. The result stated in [6, Corollary 4.5] does not concern the compactness of the space of weak equivalence classes since it is quite possible that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to an action $a \in \mathrm{A}(S)$ and the sequence $\{b_{n_k}\}_{k=1}^{\infty}$ converges to an action $b \in \mathrm{A}(S)$ such that a and b are not weakly equivalent. Indeed, let $a_n = a$ for each $n \geq 1$, where a is a free action of the free group $\mathbb{F}_S$ that is not weakly equivalent to the Bernoulli action b. Such actions exist, for example, by [1]. By a result of Abért and Weiss [3], a weakly contains b. This implies that there exists a sequence of actions $\{b_n\}_{n=1}^{\infty}$ such that b_n is equivalent to a_n and $\{b_n\}_{n=1}^{\infty}$ converges to b (here we used the definition of weak containment given in the Introduction).

5. The proof of Theorem 2

Before starting the proof of the theorem, let us make some remarks. Weak equivalence of group actions shows some similarity to orbit equivalence of group actions. It is known that all free ergodic actions of a countably infinite amenable group are both weakly equivalent and orbit equivalent. By Epstein's theorem [8], for any non-amenable countable group Γ, there exist uncountably many pairwise orbit-inequivalent free actions of Γ. On the other hand, it is proved in [1] that for several non-amenable groups, there exist uncountably many pairwise weakly inequivalent actions. According to Popa's superrigidity theorem [13], there exist free actions α of Kazhdan groups Γ that are rigid in the sense that if another action is orbit equivalent to α, then the two actions are in fact isomorphic. In [1], it was shown that if two strongly ergodic profinite actions of a countable group are weakly equivalent, then they are isomorphic. This is however somewhat weaker than actual rigidity.

Question 5.1. Does there exist a countable group Γ with a weakly rigid action?

Remark. After this paper was out on arXiv, Robin Tucker-Drob proved that such weakly rigid actions do not exist [14]. We leave the question here for historical consistency.

Proof of Theorem 2. Let Γ be a countable group and $\alpha : \Gamma \curvearrowright (X, \mu)$ be a free action of Γ. According to a recent result of Abért and Weiss [3], α weakly contains all the Bernoulli actions of Γ. By [6, Proposition 4.7], there exists an action $\beta : \Gamma \curvearrowright (Y, \nu)$ that is weakly equivalent to α and contains the Bernoulli action $\delta : \Gamma \curvearrowright [0, 1]^\Gamma$. That is there exists a map π from Y to $[0, 1]^\Gamma$ commuting with the Γ-actions. By the theorem of Gaboriau and Lyons, there exists a free p.m.p. action γ of the free group of two generators on $[0, 1]^\Gamma$ such that for any $t \in \mathbb{F}_2$ and almost all $y \in [0, 1]^\Gamma$, $\gamma(t)(y) = g(y)$ for some $g \in \Gamma$, where $g(y)$ is the image of y under the Bernoulli action. Now we define the action γ' of $\mathbb{F}_2$ on (Y, ν) the following way. Let $x \in Y$, then $\gamma'(t)(x) = \beta(g)(x)$ if $\gamma(t)(\pi(x)) = g(\pi(x))$. Clearly, γ' is a free action of $\mathbb{F}_2$ satisfying the condition of Theorem 2. ■

6. The weak equivalence notions are equivalent

The goal of this section is to prove that our definition of weak equivalence and Kechris's original definition is, in fact, equivalent. That is, the following theorem holds.

Theorem 5. *The actions $f, g \in \mathrm{A}(S)$ are weakly equivalent if and only if for any $k \geq 1$, $H^k_S(f) = H^k_S(g)$.*

The "if" part is easy. Let $C : X \to \{1, 2, \ldots, k\}$ be a Borel partition. Let $\{C^n \mid X \to \{1, 2, \ldots, k\}\}_{n=1}^\infty$ be a sequence of Borel partitions such that

$$\lim_{n\to\infty} (\Psi_f^{C^n})_\star(\mu) = (\Psi_g^C)_\star(\mu)$$

in the weak topology. Then for any finite set $F \subset \mathbb{F}_S$ and $\varepsilon > 0$,

$$|\mu(f_\gamma C_i^n \cap C_j^n) - \mu(g_\gamma C_i \cap C_j)| \leq \varepsilon \quad (1 \leq i,\ j \leq n,\ \gamma \in F),$$

provided that n is large enough.

The "only if" part is more complex and will be demonstrated in the next two subsections.

6.1. The proof of Proposition 1

First, we have the following lemma.

Lemma 6.1. *Let $f, g \in \mathrm{A}(S)$ be $\mathbb{F}_S$-actions. Assume that for every $r > 0$ and $\kappa \in U^{r,S}$, it holds that $\mu(T(\kappa, f)) = \mu(T(\kappa, g))$. Then, the IRSs of f and g coincide.*

Proof. Invariant random subgroups are invariant measures on the compact totally disconnected space $\mathrm{Sub}(\mathbb{F}_S)$ which we identified with $\mathrm{SC}(S)$ in Section 2. Thus it is enough to show that for any clopen subset $T(\kappa) \in \mathrm{SC}(S)$, it holds that $\mu(\Psi_f^{-1}(T(\kappa))) = \mu(\Psi_g^{-1}(T(\kappa)))$ (see Section 4.1). However by definition, $\mu(\Psi_f^{-1}(T(\kappa))) = \mu(T(\kappa, f))$, $\mu(\Psi_g^{-1}(T(\kappa))) = \mu(T(\kappa, g))$, thus the lemma follows. ■

So, we need to prove that if $f, g \in \mathrm{A}(S)$ are weakly equivalent, then for every $r > 0$ and $\kappa \in U^{r,S}$, we have that $\mu(T(\kappa, f)) = \mu(T(\kappa, g))$. The following technical lemma will be crucial.

Lemma 6.2. *Let $f \in \mathrm{A}(S)$. Then for any $r > 0$ and $\delta > 0$, there exist a positive integer $n_{r,\delta,f}$ and a finite partition $X = \bigcup_{i=1}^{n_{r,\delta,f}} L_i \cup E_{r,\delta,f}$ with the following properties.*

(1) $\mu(E_{r,\delta,f}) < \delta$.

(2) *Each L_i is a subset of $T(\kappa, f)$ for some $\kappa \in U^{r,S}$. We denote this element κ by $\tau(L_i)$.*

(3) *If $w_1, w_2 \in W^{r,S}$ and $w_1 \not\equiv_{P_{\tau(L_i)}} w_2$, then*

$$f_{w_1}(L_i) \cap f_{w_2}(L_i) = \emptyset.$$

The equivalence relation $\equiv_{P_\kappa}$ has been defined in Section 4.1. Note that by our second condition, if $w_1 \equiv_{P_{\tau(L_i)}} w_2$, then $f_{w_1}(L_i) = f_{w_2}(L_i)$.

Proof. According to Luzin's theorem, there exists a compact set $C_\delta \subset X$ such that $\mu(X \backslash C_\delta) < \frac{\delta}{2}$, and all the coordinates of $f \in \mathrm{A}(S) = \mathrm{Aut}(X, \mu)^S$ are continuous on C_δ. Let $x \in C_\delta$. Define $\lambda(x)$ by

$$\lambda(x) := \inf_{w_1, w_2 \in W^{r,S}, f_{w_1}(x) \neq f_{w_2}(x)} d_X(f_{w_1}(x), f_{w_2}(x)),$$

where d_X is the standard metric on the unit interval X. Note that $\lambda(x) = 0$ if and only if x is a fixed point of the action f. Let $\chi > 0$ be a real number such that

$$\mu(x \mid 0 < \lambda(x) < \chi) < \frac{\delta}{2}.$$

By uniform continuity, there exists an $\varepsilon > 0$ such that if $x, y \in C_\delta$ and $d_X(x, y) < \varepsilon$, then $d_X(f_w(x), f_w(y)) < \chi$ for any $w \in W^{r,S}$. Now let $E_{r,\delta,f} := X \backslash C_\delta \cup \{x \mid 0 < \lambda(x) < \chi\}$. For $\kappa \in U^{r,S}$, choose an arbitrary finite partition of $T(\kappa, f) \backslash E_{r,\delta,f}$ by subsets of diameter less than ε. Let L be such a subset, $x, y \in L$ and $w_1 \not\equiv_{P_\kappa} w_2$. Then $d_X(f_{w_1}(x), f_{w_2}(x)) \geq \chi$ and $d_X(f_{w_2}(x), f_{w_2}(y)) < \chi$. Hence $f_{w_1}(x) \neq f_{w_2}(y)$, that is, $f_{w_1}(L)$ and $f_{w_2}(L)$ are disjoint subsets. ∎

We fix the integer $r > 0$ and $\delta > 0$ until the end of the proof of Theorem 5.

Now, let us introduce the notion of height for r-types. The set $U^{r,S}$ is an ordered set, where $\kappa \leq \lambda$ if P_κ is a refinement of P_λ. The height function $h_r : U^{r,S} \to \mathbb{N}$ is defined in the following way. If κ is a minimal element, then let $h_r(\kappa) = 1$, $\Sigma_r(1) = h_r^{-1}(1)$. If κ is a minimal element in $U^{r,S} \backslash \Sigma_r(1)$, then let $h_r(\kappa) = 2$, $\Sigma_r(2) = h_r^{-1}(2)$. If $\Sigma_r(1), \Sigma_r(2), \ldots, \Sigma_r(k)$ are already defined, then let $h_r(\kappa) = k + 1$ if κ is minimal in the set $U^{r,S} \backslash \bigcup_{i=1}^{k} \Sigma_r(i)$ and let $\Sigma_r(k+1) = h_r^{-1}(k+1)$.

Let $P := \bigcup_{i=1}^{n_{r,\delta,f}} L_i \cup E_{r,\delta,f}$ be a partition of X satisfying the conditions of Lemma 6.2. Let $\rho > 0$. We say that a partition $P^\rho := \bigcup_{i=1}^{n_{r,\delta,f}} L_i^\rho \cup E_{r,\delta,f}^\rho$ is a ρ-simulation of P if

(1) $|\mu(L_i) - \mu(L_i^\rho)| < \rho$ for any $1 \leq i \leq n_{r,\delta,f}$.

(2) $|\mu(f_{w_1}(L_i) \cap f_{w_2}(L_j)) - \mu(g_{w_1}(L_i^\rho) \cap g_{w_2}(L_j^\rho))| < \rho$ for any $1 \leq i, j \leq n_{r,\delta,f}$ and $w_1, w_2 \in W^{r,S}$.

Note that by weak equivalence, such ρ-simulations must exist.

Lemma 6.3. *Let $\kappa \in U^{r,S}$. Then*

$$\limsup_{\rho \to 0} \mu\left(\bigcup_{L_i \subset T(\kappa, f)} L_i^\rho \backslash T(\kappa, g) \right) \leq 3\delta \tag{6.1}$$

Proof. For $1 \leq i \leq n_{r,\delta,f}$, let

$$\widehat{L}_i^\rho = \{x \in L_i^\rho \mid g_{w_1}(x) \neq g_{w_2}(x) \text{ if } w_1 \not\equiv_{P_{\tau(L_i)}} w_2\}. \tag{6.2}$$

Observe that if $x \in \widehat{L}_i^\rho$, then the r-ball type of x with respect to g is less than or equal to the r-ball type of x with respect to f. Also, by the definition of a ρ-simulation

$$\lim_{\rho \to 0} \mu(\widehat{L}_i^\rho) = \mu(L_i). \tag{6.3}$$

Now we need another lemma to proceed. ■

Lemma 6.4. *For any $n \geq 1$,*

$$\left| \mu\left(\bigcup_{\lambda, h_r(\lambda) \leq n} T(\lambda, f) \right) - \mu\left(\bigcup_{\lambda, h_r(\lambda) \leq n} T(\lambda, g) \right) \right| \leq \delta. \tag{6.4}$$

Proof. By definition,

$$\mu\left(\bigcup_{\lambda, h_r(\lambda) \leq n} T(\lambda, f) \right) \leq \sum_{i, h_r(L_i) \leq n} \mu(L_i) + \delta,$$

where $h_r(L_i)$ is defined as $h_r(\lambda)$, if $L_i \subset T(\lambda, f)$. Also,

$$\mu\left(\bigcup_{\lambda, h_r(\lambda) \leq n} T(\lambda, g) \right) \geq \sum_{i, h_r(L_i) \leq n} \mu(\widehat{L}_i^\rho)$$

by the observation after (6.2). Hence by (6.3),

$$\mu\left(\bigcup_{\lambda, h_r(\lambda) \leq n} T(\lambda, g) \right) \geq \mu\left(\bigcup_{\lambda, h_r(\lambda) \leq n} T(\lambda, f) \right) - \delta.$$

Since f weakly contains g, the reverse inequality must hold:

$$\mu\left(\bigcup_{\lambda, h_r(\lambda) \leq n} T(\lambda, f) \right) \geq \mu\left(\bigcup_{\lambda, h_r(\lambda) \leq n} T(\lambda, g) \right) - \delta.$$

That is,

$$\left| \mu\left(\bigcup_{\lambda, h_r(\lambda) \leq n} T(\lambda, f) \right) - \mu\left(\bigcup_{\lambda, h_r(\lambda) \leq n} T(\lambda, g) \right) \right| \leq \delta.$$ ■

Now we finish the proof of Lemma 6.3. By the definition of the subset L_i's, it follows that

$$\left|\mu\left(\bigcup_{\lambda,h_r(\lambda)<h_r(\kappa)} T(\lambda,f)\right) - \sum_{L_i,h_r(L_i)<h_r(\kappa)} \mu(L_i)\right| < \delta. \tag{6.5}$$

By the observation after (6.2), if x is an element of $\bigcup_{i,L_i\subset T(\kappa,f)} \widehat{L}_i^{\rho}\backslash T(\kappa,g)$, then the r-ball type of x with respect to g is strictly smaller than $h_r(\kappa)$. Also, if $x\in\widehat{L}_j^{\rho}$ and $h(L_j)<h_r(\kappa)$, then the r-ball type of x with respect to g is strictly smaller than $h_r(\kappa)$, as well. Therefore, we have that

$$\bigcup_{\lambda,h_r(\lambda)<h_r(\kappa)} T(\lambda,g) \supseteq \bigcup_{j,h_r(L_j)<h_r(\kappa)} \widehat{L}_j^{\rho} \cup \bigcup_{L_i,L_i\subset T(\kappa,f)} \widehat{L}_i^{\rho}\backslash T(\kappa,g).$$

Thus by (6.3), if ρ is small enough, then

$$\mu\left(\bigcup_{\lambda,h_r(\lambda)<h_r(\kappa)} T(\lambda,g)\right) \geq \left(\sum_{j,h_r(L_j)<h_r(\kappa)} \mu(L_j)\right) - \delta + \mu\left(\bigcup_{L_i,L_i\subset T(\kappa,f)} L_i^{\rho}\backslash T(\kappa,g)\right). \tag{6.6}$$

Adding up the inequalities (6.4) (for $n=h_r(\kappa)+1$), (6.5), and (6.6), we get the statement of Lemma 6.3. ■

Now we finish the proof of Proposition 1. Since Lemma 6.3 holds for arbitrarily small positive δ and $\lim_{\rho\to 0}\mu(L_i^{\rho})=\mu(L_i)$ holds for all $1\leq i\leq n_{r,\delta,f}$, we get that $\mu(T(\kappa,g))\leq\mu(T(\kappa,f))$. As the reverse inequality must also hold, we have the equation

$$\mu(T(\kappa,g))=\mu(T(\kappa,f)).$$

This finishes the proof of Proposition 1. ■

6.2. The end of the proof of Theorem 5

The following lemma is a straightforward consequence of [11, Proposition 10.1], which was previously mentioned in the Introduction. We leave the details for the reader to verify.

Lemma 6.5. *The following statements are equivalent for $f,g\in\mathrm{A}(S)$:*

- *f weakly contains g.*
- *For any $m,n,k,l\geq 1$, $\delta>0$, finite set $F\subset\Gamma$, and partitions $A:X\to\{1,2,\dots,k\}$, $B:X\to\{1,2,\dots,l\}$, there exist partitions $C:X\to\{1,2,\dots,k\}$, $D:X\to\{1,2,\dots,l\}$ such that for all $1\leq r_1,r_2,\dots,r_m\leq k$, $1\leq q_1,q_2,\dots,q_n\leq l$, $\gamma_1,\gamma_2,\dots,\gamma_m\in F$ and $\delta_1,\delta_2,\dots,\delta_n\in F$*

$$\left|\mu\left(\bigcap_{i=1}^{m} f_{\gamma_i}(C_{r_i})\cap\bigcap_{j=1}^{n} f_{\delta_j}(D_{q_j})\right)-\mu\left(\bigcap_{i=1}^{m} g_{\gamma_i}(A_{r_i})\cap\bigcap_{j=1}^{n} g_{\delta_j}(B_{q_j})\right)\right|<\delta.$$

Now let $C:X\to\{1,2,\dots,l\}$ be a Borel partition of X. It is enough to prove that for any $\varepsilon>0$ and $r>0$, there exists a partition $C':X\to\{1,2,\dots,l\}$ of X such that

$$|(\Psi_f^C)_\star(\mu)(T(\widetilde{\kappa}))-(\Psi_g^{C'})_\star(\mu)(T(\widetilde{\kappa}))|<\varepsilon \tag{6.7}$$

holds for all $\widetilde{\kappa} \in U^{r,S,l}$. Indeed, it means that $H^l_S(f) \subseteq H^l_S(g)$.

For the rest of the proof, we fix an integer $r > 0$ and a real number $\delta > 0$. Let $T_\delta = \bigcup_{i=1}^{n_{r,\delta,f}} L_i \cup E_{r,\delta,f}$ be a Borel partition of X as in Lemma 6.2. We say that a pair of partitions of X, (C^ρ, T^ρ_δ) is a ρ-simulation of the pair (C, T_δ) if

- T^ρ_δ is a ρ-simulation of T_δ,
- for any $\widetilde{\kappa} \in U^{r,S,l}$ and $L_i \subset T([\widetilde{\kappa}], f)$, we have

$$\left|\mu\left(L^\rho_i \cap \bigcap_{w_j \in W^{r,S}} g_{w_j^{-1}}(C^\rho_{\widetilde{\kappa}(w_j)})\right) - \mu\left(L_i \cap \bigcap_{w_j \in W^{r,S}} f_{w_j^{-1}}(C_{\widetilde{\kappa}(w_j)})\right)\right| < \rho, \tag{6.8}$$

where $\widetilde{\kappa}(w_j)$ denotes the label of $w_j(\mathrm{root}(\widetilde{\kappa}))$ in $\widetilde{\kappa}$ and $[\widetilde{\kappa}]$ denotes the underlying r-ball type of $\widetilde{\kappa}$. By Lemma 6.5, such ρ-simulation exists.

Now by definition,

$$T([\widetilde{\kappa}], f) \cap \bigcap_{w_j \in W^{r,S}} f_{w_j^{-1}}(C_{\widetilde{\kappa}(w_j)}) = T(\widetilde{\kappa}, f, C). \tag{6.9}$$

Similarly,

$$T([\widetilde{\kappa}], g) \cap \bigcap_{w_j \in W^{r,S}} g_{w_j^{-1}}(C_{\widetilde{\kappa}(w_j)}) = T(\widetilde{\kappa}, g, C^\rho). \tag{6.10}$$

Hence by the definition of the L_i's, we have

$$\left|\mu\left(\bigcup_{L_i, L_i \subset T([\widetilde{\kappa}], f)} \left(L_i \cap \bigcap_{w_j \in W^{r,S}} f_{w_j^{-1}}(C_{\widetilde{\kappa}(w_j)})\right)\right) - (\Psi^C_f)_\star(\mu)(T(\widetilde{\kappa}))\right| < \delta. \tag{6.11}$$

Also, by (6.10), we have

$$\begin{aligned}&\left|\mu\left(\bigcup_{L_i, L_i \subset T([\widetilde{\kappa}], f)} \left(L^\rho_i \bigcap_{w_j \in W^{r,S}} g_{w_j^{-1}}(C_{\widetilde{\kappa}(w_j)})\right)\right) - (\Psi^C_g)_\star(\mu)(T(\widetilde{\kappa}))\right| \\ &\quad < \mu\left(\left(T([\widetilde{\kappa}], g) \setminus \bigcup_{L_i, L_i \subset T([\widetilde{\kappa}], f)} L^\rho_i\right) \cup \left(\bigcup_{L_i, L_i \subset T([\widetilde{\kappa}], f)} L^\rho_i \setminus T([\widetilde{\kappa}], g)\right)\right).\end{aligned} \tag{6.12}$$

Therefore, by Lemma 6.3, and the fact that $\mu(T([\widetilde{\kappa}], f)) = \mu(T([\widetilde{\kappa}], g))$, if both δ and ρ are sufficiently small, then

$$|(\Psi^C_f)_\star(\mu)(T(\widetilde{\kappa})) - (\Psi^{C^\rho}_g)_\star(\mu)(T(\widetilde{\kappa}))| < \varepsilon$$

holds. This finishes the proof of Theorem 5. ■

7. The proof of Theorem 1

Finally, we are able to prove Theorem 1. By Theorem 5, the weak equivalence classes correspond exactly to the zero classes of pd. Also, by Proposition 7, the metric space of the zero classes is compact. The compactness of the fiber $\mathrm{A}(S,\lambda)$ follows directly from the definition of the partition pseudometric pd. This finishes the proof of Theorem 1. ■

Funding. The first author was partially supported by the ERC Consolidator Grant Invgrogra 648017, OTKA Grant 67867, NK 78439, and TAMOP 4.2.1/B-09/1/KMR-2010-003.

References

[1] M. Abért and G. Elek, Dynamical properties of profinite actions. *Ergodic Theory Dynam. Systems* **32** (2012), no. 6, 1805–1835 Zbl 1297.37004 MR 2995875

[2] M. Abért, Y. Glasner, and B. Virág, The measurable Kesten theorem. *Ann. Probab.* **44** (2016), no. 3, 1601–1646 Zbl 1339.05365 MR 3502591

[3] M. Abért and B. Weiss, Bernoulli actions are weakly contained in any free action. *Ergodic Theory Dynam. Systems* **33** (2013), no. 2, 323–333 Zbl 1268.37006 MR 3035287

[4] D. Aldous and R. Lyons, Processes on unimodular random networks. *Electron. J. Probab.* **12** (2007), no. 54, 1454–1508 Zbl 1131.60003 MR 2354165

[5] B. Bollobás and O. Riordan, Sparse graphs: metrics and random models. *Random Structures Algorithms* **39** (2011), no. 1, 1–38 Zbl 1223.05271 MR 2839983

[6] C. T. Conley, A. S. Kechris, and R. D. Tucker-Drob, Ultraproducts of measure preserving actions and graph combinatorics. *Ergodic Theory Dynam. Systems* **33** (2013), no. 2, 334–374 Zbl 1268.05104 MR 3035288

[7] G. Elek and B. Szegedy, A measure-theoretic approach to the theory of dense hypergraphs. *Adv. Math.* **231** (2012), no. 3–4, 1731–1772 Zbl 1251.05115 MR 2964622

[8] I. Epstein, Orbit inequivalent actions of non-amenable groups. [v1] 2007, [v2] 2008, arXiv:0707.4215v2

[9] D. Gaboriau and R. Lyons, A measurable-group-theoretic solution to von Neumann's problem. *Invent. Math.* **177** (2009), no. 3, 533–540 Zbl 1182.43002 MR 2534099

[10] H. Hatami, L. Lovász, and B. Szegedy, Limits of locally-globally convergent graph sequences. *Geom. Funct. Anal.* **24** (2014), no. 1, 269–296 Zbl 1294.05109 MR 3177383

[11] A. S. Kechris, *Global aspects of ergodic group actions*. Math. Surveys Monogr. 160, American Mathematical Society, Providence, RI, 2010, 237 pp. Zbl 1189.37001 MR 2583950

[12] A. J. Ol'šanskiĭ, An infinite group with subgroups of prime orders (in Russian). *Izv. Akad. Nauk SSSR Ser. Mat.* **44** (1980), no. 2, 309–321 English translation: *Math. USSR Izv.* **16** (1981), no. 2, 279–289 Zbl 0475.20025 MR 0571100

[13] S. Popa, Cocycle and orbit equivalence superrigidity for malleable actions of w-rigid groups. *Invent. Math.* **170** (2007), no. 2, 243–295 Zbl 1131.46040 MR 2342637

[14] R. D. Tucker-Drob, Weak equivalence and non-classifiability of measure preserving actions. *Ergodic Theory Dynam. Systems* **35** (2015), no. 1, 293–336 Zbl 1351.37006 MR 3294302

Received 15 August 2011.

Miklós Abért
HUN-REN Alfréd Rényi Institute of Mathematics, Reáltanoda utca 13-15, 1053 Budapest, Hungary; miklos.abert@renyi.mta.hu

Gábor Elek
Lancaster University, Fylde Ave, Bailrigg, Lancaster LA1 4YR, UK; HUN-REN Alfréd Rényi Institute of Mathematics, Reáltanoda utca 13-15, 1053 Budapest, Hungary; g.elek@lancaster.ac.uk