

Translation-like actions by \mathbb{Z} , the subgroup membership problem, and Medvedev degrees of effective subshifts

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Abstract. We show that every infinite, locally finite, and connected graph admits a translation-like action by \mathbb{Z} , and that this action can be taken to be transitive exactly when the graph has either one or two ends. The actions constructed satisfy $d(v, v * 1) \leq 3$ for every vertex v . This strengthens a theorem by Brandon Seward. We also study the effective computability of translation-like actions on groups and graphs. We prove that every finitely generated infinite group with decidable word problem admits a translation-like action by \mathbb{Z} which is computable and satisfies an extra condition which we call decidable orbit membership problem. As a nontrivial application of our results, we prove that for every finitely generated infinite group with decidable word problem, effective subshifts attain all Π_1^0 Medvedev degrees. This extends a classification proved by Joseph Miller for \mathbb{Z}^d , $d \geq 1$.

1. Introduction

1.1. Translation-like actions by \mathbb{Z} on locally finite graphs

A right action $*$ of a group H on a metric space (X, d) is called a *translation-like action* if it is *free* (that is, $x * h = x$ implies $h = 1_H$, for $x \in X$, $h \in H$), and for each $h \in H$, the set $\{d(x, x * h) \mid x \in X\} \subset \mathbb{R}$ is bounded. If G is a finitely generated group endowed with the left-invariant word metric associated with some finite set of generators, then the action of any subgroup H on G by right translations $(g, h) \mapsto gh$ is a translation-like action. On the other hand, observe that despite the action $H \curvearrowright G$ by left multiplication is usually referred to as an action by translations, in general it is not translation-like for a left-invariant word metric.

Following this idea, Kevin Whyte proposed in [41] to consider translation-like actions as a generalization of subgroup containment and to replace subgroups by translation-like actions in different questions or conjectures about groups and subgroups. This was called a geometric reformulation. For example, the von Neumann conjecture asserted that a group is nonamenable if and only if it contains a nonabelian free subgroup. Its geometric reformulation asserts then that a group is nonamenable if and only if it admits a translation-like

action by a nonabelian free group. While the conjecture was proven to be false [35], Kevin Whyte proved that its geometric reformulation holds [41].

One problem left open in [41] was the geometric reformulation of Burnside's problem. This problem asked if every finitely generated infinite group contains \mathbb{Z} as a subgroup, and was answered negatively in [20]. Brandon Seward proved that the geometric reformulation of this problem also holds.

Theorem 1.1 (Geometric Burnside's problem [38]). *Every finitely generated infinite group admits a translation-like action by \mathbb{Z} .*

A finitely generated infinite group with two or more ends has a subgroup isomorphic to \mathbb{Z} , by Stallings's structure theorem. Thus, it is the one-ended case that makes necessary the use of translation-like actions. In order to prove Theorem 1.1, Brandon Seward proved a more general graph theoretic result.

Theorem 1.2 ([38, Theorem 1.6]). *Let Γ be a connected and infinite graph whose vertices have uniformly bounded degree. Then Γ admits a transitive translation-like action by \mathbb{Z} if and only if it is connected and has either one or two ends.*

This result proves Theorem 1.1 for groups with one or two ends, and indeed it says more, as the translation-like action obtained is transitive. The proof of this result relies strongly on the hypothesis of having uniformly bounded degree. Indeed, the uniform bound on $d_\Gamma(v, v * 1)$ depends linearly on a uniform bound for the degree of the vertices of the graph. Here, we strengthen Seward's result by weakening the hypothesis to the locally finite case and improving the bound on $d_\Gamma(v, v * 1)$ to 3.

Theorem 1.3. *Let Γ be an infinite, connected, and locally finite graph. Then Γ admits a transitive translation-like action by \mathbb{Z} if and only if it has either one or two ends. Moreover, the action can be taken with $d(v, v * 1) \leq 3$ for every vertex v .*

A problem left in [38, Problem 3.5] was to characterize which graphs admit a transitive translation-like action by \mathbb{Z} . Thus, we have solved the case of locally finite graphs, and it only remains the case of graphs with vertices of infinite degree.

We now mention an application of these results to the problem of Hamiltonicity of Cayley graphs. This is related to a special case of Lovász conjecture which asserts the following: If G is a finite group, then for every set of generators the associated Cayley graph admits a Hamiltonian path. Note that the existence of at least one such generating set is obvious ($S = G$), and the difficulty of the question, which is still open, is that it alludes every generating set. Now assume that G is an infinite group, S is a finite set of generators, and $\text{Cay}(G, S)$ admits a transitive translation-like action by \mathbb{Z} . This action becomes a bi-infinite Hamiltonian path after we enlarge the generating set, and thus it follows from Seward's theorem that every finitely generated group with one or two ends

admits a generating set for which the associated Cayley graph admits a bi-infinite Hamiltonian path [16, Theorem 1.8]. It is an open question whether this holds for every Cayley graph [16, Problem 4.8], but our result yields an improvement in this direction.

Corollary 1.4. *Let G be a finitely generated group with one or two ends, and let S be a finite set of generators. Then the Cayley graph of G with respect to the generating set $\{g \in G \mid d_S(g, 1_G) \leq 3\}$ admits a bi-infinite Hamiltonian path.*

This was known to hold for generating sets of the form $\{g \in G \mid d_S(g, 1_G) \leq J\}$, where $S \subset G$ is a finite generating set for G and J depends linearly on the vertex degrees in $\text{Cay}(G, S)$.

In the more general case where we impose no restrictions on ends, we obtain the following result for nontransitive translation-like actions. Observe that this readily implies Theorem 1.1.

Theorem 1.5. *Every infinite graph which is locally finite and connected admits a translation-like action by \mathbb{Z} . Moreover, the action can be taken with $d(v, v * 1) \leq 3$ for every vertex v .*

These statements about translation-like actions can also be stated in terms of powers of graphs. Given a graph Γ , its n th power Γ^n is defined as the graph with the same set of vertices, and where two vertices u, v are joined if their distance in Γ is at most n . It is well known that the cube of every finite and connected graph is Hamiltonian [11, 30, 37]. Our Theorem 1.3 generalizes this to infinite and locally finite graphs. That is, it shows that the cube of a locally finite and connected graph with one or two ends admits a bi-infinite Hamiltonian path.

We mention that Theorem 1.5 has been proved in [13, Section 4], using the same fact about cubes of finite graphs.

1.2. Computability of translation-like actions

Now we turn our attention to the problem of computing translation-like actions on groups or graphs. We recall that a graph is computable if there exists an algorithm which given two vertices, determines whether they are adjacent or not. If, moreover, the graph is locally finite, and the function that maps a vertex to its degree is computable, then the graph is said to be highly computable. This extra condition is necessary to compute the neighborhood of a vertex.

An important example comes from group theory: If G is a finitely generated group with decidable word problem and S is a finite set of generators, then its Cayley graph with respect to S is highly computable.

There is a variety of problems in graph theory that have no computable solution for infinite graphs. A classical example is the problem of computing infinite paths. König's infinity lemma asserts that every infinite, connected, and locally finite graph admits an infinite path. However, there are highly computable graphs which admit paths, all of which

are uncomputable [29]. Another example comes from Hall's matching theorem. There are highly computable graphs satisfying the hypotheses in the theorem, but which admit no computable right perfect matching [33]. These two results are used in the proof of Seward's theorem, so the translation-like actions from this proof are not clearly computable. We say that a translation-like action by \mathbb{Z} on a graph is computable when there is an algorithm which given a vertex v and $n \in \mathbb{Z}$ computes the vertex $v * n$.

Our interest in the computability of translation-like actions comes from symbolic dynamics, and the shift spaces associated with a group. We will need a computable translation-like action such that it is possible to distinguish in a computable manner between different orbits. We introduce here a general definition, though we will only treat the case where the acting group is \mathbb{Z} .

Definition 1.6. Let G be a group, and let $S \subset G$ be a finite set of generators. A group action of H on G is said to have *decidable orbit membership problem* if there exists an algorithm which given two words u and v in $(S \cup S^{-1})^*$ decides whether the corresponding group elements u_G, v_G lie in the same orbit under the action.

Note that if H is a subgroup of G , then the action $H \curvearrowright G$ by right translations has decidable orbit membership problem if and only if H has decidable subgroup membership problem (Proposition 4.9). Thus, this property can be regarded as the geometric reformulation, in the sense of Whyte [41], of the subgroup property of having decidable membership problem. The orbit membership problem has been studied for some actions by conjugacy and by group automorphisms (see [6, 8, 40] and references therein).

Our main result associated with computable translation-like actions on groups is the following.

Theorem 1.7. *Let G be a finitely generated infinite group with decidable word problem. Then G admits a translation-like action by \mathbb{Z} that is computable and has decidable orbit membership problem.*

The proof of Theorem 1.7 proceeds as follows. For groups with one or two ends, we will show the existence of a computable and transitive translation-like action by \mathbb{Z} , that is, a computable version of Theorem 1.3. This action has decidable orbit membership problem for the trivial reason that it has only one orbit. For groups with at least two ends we obtain the action from a subgroup. It follows from Stallings's structure theorem on ends of groups that a finitely generated groups with two or more ends have a subgroup isomorphic to \mathbb{Z} . We will show that, if the group has solvable word problem, then this subgroup has decidable membership problem. This proof is based on the computability of normal forms associated with Stallings's structure theorem (Proposition 4.7).

1.3. Medvedev degrees of effective subshifts

We now turn our attention to Medvedev degrees, a complexity measure which is defined using computable functions. Precise definitions of this and the following concepts are

given in Section 5. Intuitively, the Medvedev degree of a set $P \subset A^{\mathbb{N}}$ measures how hard is to compute a point in P . For example, a set has zero Medvedev degree if and only if it has a computable point. This complexity measure becomes meaningful when we regard P as the set of solutions to a problem. This notion can be applied to a variety of objects, such as graph colorings [36], paths on graphs, matchings from Hall's matching theorem, and others [1, Chapter 13]. In this article, we consider Medvedev degrees of subshifts.

Let G be a finitely generated group, and let A be a finite alphabet. A subshift is a subset of A^G which is closed in the prodiscrete topology and is invariant under translations. Dynamical properties of subshifts have been related to their computational properties in different ways. A remarkable example is the characterization of the entropies of two-dimensional subshifts of finite type as the class of nonnegative Π_1^0 real numbers [25].

Here, we address the problem of classifying what Medvedev degrees can be attained for a certain class of subshifts. For instance, this classification is known for subshifts of finite type in the groups \mathbb{Z}^d , $d \geq 1$. In the case $d = 1$, all subshifts of finite type have Medvedev degree zero, because all of them contain a periodic point, and then a computable point. In the case $d \geq 2$, subshifts of finite type can attain the class of Π_1^0 Medvedev degrees [39].

A larger class of subshifts is that of effective subshifts. A subshift over \mathbb{Z} is effective if the set of words which do not appear in its configurations is computably enumerable. This notion can be extended to a finitely generated group, despite some intricacies that arise in relation to the word problem of the group. We will only deal with groups with decidable word problem, and the notion of effective subshift is a straightforward generalization.

Answering a question left open in [39], Joseph Miller proved that effective subshifts over \mathbb{Z} can attain all Π_1^0 Medvedev degrees [34]. We generalize this result to the class of infinite, finitely generated groups with decidable word problem.

Theorem 1.8. *Let G be a finitely generated and infinite group with decidable word problem. The class of Medvedev degrees of effective subshift on G is the class of all Π_1^0 Medvedev degrees.*

The idea for the proof is the following. Given any subshift $Y \subset A^{\mathbb{Z}}$, we can construct a new subshift $X \subset B^G$ that simultaneously describes translation-like actions $\mathbb{Z} \curvearrowright G$, and elements in Y . Then Theorem 1.7 ensures that this construction preserves the Medvedev degree of Y , and the result follows from the known classification for \mathbb{Z} [34].

Despite the simplicity of the proof, we need to translate some computability notions from $A^{\mathbb{N}}$ to A^G ; this is done by taking a computable numbering of G . The notions obtained do not depend on the numbering and are compatible with notions already present in the literature that were defined by other means [2].

This construction using translation-like actions was introduced in [28] and has been used to prove different results in the context of symbolic dynamics. For example, to transfer results about the emptiness problem for subshifts of finite type from one group to

another [28], to produce aperiodic subshifts of finite type on new groups [13, 28], and to study the entropy of subshifts of finite type on some amenable groups [3].

Paper structure

In Section 2, we fix some notation and recall some basic facts on graph theory, group theory, and computability theory on countable sets. In Section 3, we show Theorems 1.3 and 1.5 about translation-like actions. In Section 4, we show some results about computable translation-like actions, including Theorem 1.7. This result is applied in Section 5 to prove Theorem 1.8 on Medvedev degrees.

2. Preliminaries

We denote by $f \circ g$ the function that applies g to the argument, and then f .

2.1. Graph theory

In this article all graphs are undirected and unlabeled. Loops and multiple edges are allowed. The vertex set of a graph Γ will be denoted by $V(\Gamma)$, and its edge set by $E(\Gamma)$. Each edge *joins* a pair of vertices, and is said to be *incident* to them. Two vertices joined by an edge are called *adjacent*. The *degree* $\deg_{\Gamma}(v)$ of the vertex v is the number of incident edges to v , where loops are counted twice. A graph is said to be *finite* when its edge set is finite, and *locally finite* when every vertex has finite degree.

In our constructions we will constantly consider induced subgraphs. Given a set of vertices $V \subset V(\Gamma)$, the *induced subgraph* $\Gamma[V]$ is the subgraph of Γ whose vertex set equals V , and whose edge set is that of all edges in $E(\Gamma)$ whose incident vertices lie in V . On the other hand, $\Gamma - V$ stands for the subgraph of Γ obtained by removing from Γ all vertices in V , and all edges incident to vertices in V . That is, $\Gamma - V$ equals the induced subgraph $\Gamma[V(\Gamma) - V]$. If Λ is a subgraph of Γ , we denote by $\Gamma - \Lambda$ the subgraph $\Gamma - V(\Lambda)$.

A *path* on Γ is an injective function $f: [a, b] \rightarrow V(\Gamma)$ that sends consecutive integers to adjacent vertices, where $[a, b] \subset \mathbb{Z}$. We introduce now some useful terminology for dealing with paths. We say that f *joins* $f(a)$ to $f(b)$ and define its *length* as $b - a$. We say that f *visits* the vertices in its image, and we denote this set by $V(f)$. We denote by $\Gamma - f$ the subgraph $\Gamma - V(f)$. The vertices $f(a)$ and $f(b)$ are called the *initial* and *final* vertices of f , respectively. When every pair of vertices in the graph Γ can be joined by a path, then we say that Γ is *connected*. In this case we define the *distance* between two vertices as the length of the shortest path joining them. This distance induces the *path-length* metric on $V(\Gamma)$, which we denote by d_{Γ} .

A *connected component* of Γ is a connected subgraph of Γ which is maximal for the subgraph relation. The *number of ends* of Γ is the supremum of the number of infinite connected components of $\Gamma - V$, where V ranges over all finite sets of vertices in Γ .

2.2. Words and finitely generated groups

We now review some terminology and notation on words, alphabets, and finitely generated groups. An *alphabet* is a finite set. The set of finite words on alphabet A is denoted by A^* . The empty word is denoted by ε . A word u of length n is a *prefix* of v when they coincide in the first n symbols.

Now let G be a group. The identity element of G is denoted by 1_G , or 1 if no confusion arises. Let $S \subset G$ be a finite set, and let S^{-1} be the set of formal inverses to elements in S . Given a word $w \in (S \cup S^{-1})^*$, we denote by w_G the group element obtained by multiplying in G the elements from S that constitute the word. We also write $u =_G v$ when the words u, v correspond to the same group element. A set $S \subset G$ is said to *generate* G if every group element can be written as a word in $(S \cup S^{-1})^*$, and G is *finitely generated* when it admits a finite generating set. A finite generating set $S \subset G$ induces the *left-invariant word metric* on G , denoted by d_S . The distance $d_S(g, h)$ is the length of the shortest word $w \in (S \cup S^{-1})^*$ such that $g(w)_G = h$.

If $S \subset G$ is a finite set of generators, we denote by $\text{Cay}(G, S)$ the (undirected, and right) *Cayley graph* of G relative to S . The vertex set of $\text{Cay}(G, S)$ is G , and the edge set of $\text{Cay}(G, S)$ is $\{(g, gs) \mid g \in G, s \in S \cup S^{-1}\}$. The edge (g, gs) joins the vertex g with the vertex gs . Note that the distance that this graph assigns to a pair of elements in G equals their distance in the word metric associated with the same generating set. The *number of ends* of a finitely generated group is the number of ends of its Cayley graph, for any generating set. This definition does not depend on the chosen generating set and can only be among the numbers $\{0, 1, 2, \infty\}$ [19, 27].

We now recall some algorithmic properties of groups and subgroups. The concept of decidable set of words is defined in the next subsection. Let $S \subset G$ be a finite set of generators, and let H be a subgroup of G . We say that H has *decidable subgroup membership problem* if $\{w \in (S \cup S^{-1})^* \mid w_G \in H\}$ is a decidable subset of $(S \cup S^{-1})^*$. This notion does not depend on the chosen generating set. In the particular case where $H = \{1_G\}$, the set defined above is called the *word problem* of G . The property of having decidable word problem is closely related to the property of being a computable group, which we discuss in more detail in the next subsection.

2.3. Computability theory on countable sets via numberings

We start by reviewing some classical notions from recursion theory or computability theory. All these facts are well known; the reader is referred to [12] for computability theory and to [22, Chapter 14] for a survey on numberings.

We will use the word *algorithm* to refer to the formal object of Turing machine. We will use other common synonyms such as “effective procedure.” A partial function $f: D \subset \mathbb{N} \rightarrow \mathbb{N}$ is *computable* if there is an algorithm satisfying the following. On input n , the algorithm halts if and only if $n \in D$, and in this case outputs $f(n)$. A subset $D \subset \mathbb{N}$ is *semi-decidable* when there is an algorithm that halts on input n if and only if $n \in D$. A set $D \subset \mathbb{N}$ is *decidable* when both D and $\mathbb{N} - D$ are semi-decidable.

All these notions extend directly to products \mathbb{N}^p , $p \geq 1$, and sets of words A^* , as these objects can be represented by natural numbers in a canonical way. In order to extend these notions to other objects such as graphs and countable groups, we take a unified approach via numberings.

Definition 2.1. A (bijective) *numbering* of a set X is a bijective map $\nu : N \rightarrow X$, where N is a decidable subset of \mathbb{N} . We call (X, ν) a *numbered set*. When $\nu(n) = x$, we say that n is a *name* for x , or that n represents x .

A numbering of X defines computability notions in X in the same manner that charts are used to define continuous or differentiable function on manifolds. For instance, a function $f : X \rightarrow X$ is computable on (X, ν) when the “function in charts” $\nu^{-1} \circ f \circ \nu$ is computable. There is a notion of equivalence for numberings: Two numberings ν, ν' of X are equivalent when the identity function $(X, \nu) \rightarrow (X, \nu')$ is computable. The Cartesian product $X \times X'$ of two numbered sets $(X, \nu), (X', \nu')$ admits a unique numbering—up to equivalence—for which the projection functions to $(X, \nu), (X', \nu')$ are computable. This provides definitions of computable functions and relations between different numbered sets, and we can freely speak about computable functions and relations between numbered sets. We will be interested in the following objects.

Definition 2.2. A graph Γ is computable if we can endow $V(\Gamma)$ and $E(\Gamma)$ with numberings, in such a manner that the relation of adjacency, and the relation $\{(e, u, v) \mid e \text{ joins } u \text{ and } v\}$ are decidable. We say that Γ is also *highly computable* when it is locally finite, and the vertex degree function $V(\Gamma) \rightarrow \mathbb{N}, v \mapsto \deg_{\Gamma}(v)$ is computable.

Definition 2.3. A numbering ν of a group G is said to be *computable* when it makes the group operation $G^2 \rightarrow G$ is computable. In this case, the pair (G, ν) is called a *computable group*.

These notions provide a formal and precise meaning to general statements about the computability of objects such as translation-like actions and bi-infinite paths on computable groups or graphs. For instance, a group action on a computable group G is computable when the function $(g, n) \rightarrow g * n$ from the numbered set $G \times \mathbb{Z}$ to the numbered set G is computable.

It is well known that algorithmic properties of finitely generated groups have a number of stability properties, such as being independent of the generating set. In terms of numberings, this is expressed as follows.

Proposition 2.4. *Let G be a finitely generated group. Then:*

- (1) *G admits a computable numbering if and only if it has decidable word problem.*
- (2) *If G admits a computable numbering, then all computable numberings of G are equivalent.*

- (3) If H is another finitely generated computable group, then any group homomorphism $f: G \rightarrow H$ is computable.

Proof sketch. Suppose that G has decidable word problem, let $S \subset G$ be a finite generating set, and let $\pi: (S \cup S^{-1})^* \rightarrow G$ be the function that sends a word to the corresponding group element. Using the decidability of the word problem, we can compute a set $N \subset (S \cup S^{-1})^*$ such that the restriction of π to N is a bijection. Being N a decidable subset of $(S \cup S^{-1})^*$, it admits a computable bijection with \mathbb{N} . The composition of these functions give a bijection $\nu: \mathbb{N} \rightarrow G$, and it is easy to verify that it is a computable numbering. The reverse implication is left to the reader. Items (2) and (3) are also left to the reader: The relevant functions are determined by the finite information of letter-to-word substitutions, and this allows to prove that they are computable. ■

We will also make use of the following well-known fact. The proof is straightforward, and left to the reader.

Proposition 2.5. *Let G be a finitely generated group with decidable word problem, and let S be a finite generating set. Then $\text{Cay}(G, S)$ is a highly computable graph.*

3. Translation-like actions by \mathbb{Z} on locally finite graphs

The goal of this section is to prove Theorems 1.3 and 1.5. That is, that every connected, locally finite, and infinite graph admits a translation by \mathbb{Z} , and that this action can be chosen transitive exactly when the graph has one or two ends. The actions that we construct satisfy that the distance between a vertex v and $v * 1$ is at most 3.

Our proof goes by constructing these actions locally, and in terms of 3-paths.

Definition 3.1. Let Γ be a graph. A 3-path on Γ is an injective function $f: [a, b] \rightarrow V(\Gamma)$ such that consecutive integers in $[a, b]$ are mapped to vertices whose distance is at most 3. A bi-infinite 3-path on Γ is an injective function $f: \mathbb{Z} \rightarrow V(\Gamma)$ satisfying the same condition on the vertices. A 3-path or bi-infinite 3-path is called *Hamiltonian* when it is also a surjective function.

It is well known that every finite and connected graph admits a Hamiltonian 3-path, where we can choose its initial and final vertex [11, 30, 37]. Here, we will need a slight refinement of this fact.

Lemma 3.2. *Let Γ be a graph that is connected and finite. For every pair of different vertices u and v , Γ admits a Hamiltonian 3-path f which starts at u , ends at v , and moreover satisfies the following two conditions:*

- (1) *The first and last “jump” have length at most 2. That is, if f visits w immediately after the initial vertex u , then $d_\Gamma(u, w) \leq 2$. Moreover, if f visits w immediately before the final vertex v , then $d_\Gamma(w, v) \leq 2$.*

- (2) *There are no consecutive “jumps” of length 3. That is, if f visits $w_1, w_2,$ and w_3 consecutively, then $d_\Gamma(w_1, w_2) \leq 2$ or $d_\Gamma(w_2, w_3) \leq 2$.*

Let us review some terminology on 3-paths before proving this result. When dealing with 3-paths, we will use the same terms introduced for paths in the preliminaries, such as initial vertex, final vertex, visited vertex, etc. Let f and g be 3-paths. We say that f extends g if its restriction to the domain of g equals g . We will extend 3-paths by concatenation, which we define as follows. Suppose that final vertex of f is at distance at most 3 from the initial vertex of g , and such that $V(f) \cap V(g) = \emptyset$. The *concatenation* of f, g is the 3-path that extends f , and after the final vertex of f visits all vertices visited by g in the same order. Finally, the *inverse* of the 3-path f , denoted by $-f$, is defined by $(-f)(n) = f(-n)$. Note that its domain is also determined by this expression.

Proof of Lemma 3.2. The proof is by induction of the cardinality of $V(\Gamma)$. The claim clearly holds if $|V(\Gamma)| = 2$. Now assume that Γ is a connected finite graph, and let u and v be two different vertices. We consider the connected components of the graph $\Gamma - \{v\}$ obtained by removing the vertex v from Γ . Let Γ_u be the finite connected component of $\Gamma - \{v\}$ that contains u , and let Γ_v be the subgraph of Γ induced by the set of vertices $V(\Gamma) - V(\Gamma_u)$. Thus, $u \in \Gamma_u, v \in \Gamma_v$, and both Γ_u and Γ_v are connected. Let us first assume that both Γ_u and Γ_v are graphs with at least two vertices. Then we can apply the inductive hypothesis on each one of them. Let f_u be a Hamiltonian 3-path on Γ_u as in the statement, whose initial vertex is u , and whose final vertex is adjacent to v . Let f_v be a Hamiltonian 3-path on Γ_v as in the statement, whose initial vertex is adjacent to v , and whose final vertex is v . Now if Γ_u only has the vertex u , then we define f_u as the 3-path that only visits that vertex. We follow the same procedure if Γ_v only has the vertex v . Observe that in any case, the distance from the final vertex of f_u to the initial vertex of f_v is at most 2. We claim that the 3-path f obtained by concatenating f_u, f_v verifies the required conditions. It is clear that f is a Hamiltonian 3-path on Γ whose initial vertex is u , and whose final vertex is v . Finally, that f satisfies the two numbered conditions in the statement follows from the fact that distance from the final vertex of f_u to the initial vertex of f_v is at most 2. ■

We will define bi-infinite 3-paths by extending finite ones iteratively. The following definition will be key for this purpose.

Definition 3.3. Let f be a 3-path on a graph Γ . We say that f is *bi-extensible* if the following conditions are satisfied:

- (1) $\Gamma - f$ has no finite connected component.
- (2) There is a vertex u in $\Gamma - f$ at distance at most 3 from the final vertex of f .
- (3) There is a vertex $v \neq u$ in $\Gamma - f$ at distance at most 3 from the initial vertex of f .

If only the first two conditions are satisfied, we say that f is *right-extensible*.

We will now prove some elementary facts about the existence of 3-paths that are bi-extensible and right-extensible. The proofs are elementary and are given by completeness.

Lemma 3.4. *Let Γ be a graph that is infinite, connected, and locally finite. Then for every pair of vertices u and v in Γ , there is a right-extensible 3-path whose initial vertex is u , and which visits v .*

Proof. As Γ is connected, there is a path f joining u to v . Now define Λ as the graph induced in Γ by the set of vertices that are visited by f , or that lie in a finite connected component of $\Gamma - f$. Notice that as Γ is locally finite, there are finitely many such connected components, and thus Λ is a finite and connected graph. By construction, $\Gamma - \Lambda$ has no finite connected component.

The desired 3-path will be obtained as a Hamiltonian 3-path on Λ . Indeed, as Γ is connected, there is a vertex w in Λ that is adjacent to some vertex in $\Gamma - \Lambda$. By Lemma 3.2 there is a 3-path f' which is Hamiltonian on Λ , starts at u , and ends in w . We claim that f' is right-extensible. Indeed, our choice of Λ ensures that $\Gamma - f'$ has no finite connected component, and our choice of w ensures that the final vertex of f' is adjacent to a vertex in $\Gamma - f'$. ■

Lemma 3.5. *Let Γ be a graph that is infinite, connected, and locally finite. Then for every vertex u in Γ , there is a bi-extensible 3-path in Γ that visits u .*

Proof. Let v be a vertex in Γ that is adjacent to u , with $v \neq u$. Let Λ be the subgraph of Γ induced by the set of vertices that lie in a finite connected component of $\Gamma - \{u, v\}$, or in $\{u, v\}$. As Γ is locally finite, there are finitely many such connected components, and thus Λ is a finite and connected subgraph of Γ . By construction, $\Gamma - \Lambda$ has no finite connected component.

The desired 3-path will be obtained as a Hamiltonian 3-path on Λ . Indeed, as Γ is connected there are two vertices $w \in V(\Gamma - \Lambda)$ and $w' \in V(\Lambda)$, with w adjacent to w' in Γ . As Λ has at least two vertices, we can invoke Lemma 3.2 to obtain a 3-path f that is Hamiltonian on Λ , whose initial vertex is w' , and whose final vertex is adjacent to w' . It is clear that then f is a bi-extensible 3-path in Γ . ■

Our main tool to construct bi-infinite 3-paths is the following result, which shows that bi-extensible 3-paths can be extended to larger bi-extensible 3-paths.

Lemma 3.6. *Let Γ be a graph that is infinite, connected, and locally finite. Let f be a bi-extensible 3-path on Γ , and let u and v be two different vertices in $\Gamma - f$ whose distance to the initial and final vertex of f is at most 3, respectively. Let w be a vertex in the same connected component of $\Gamma - f$ that some of u or v . Then there is a 3-path f' which extends f , is bi-extensible on Γ , and visits w . Moreover, we can assume that the domain of f' extends that of f in both directions.*

Proof. If u and v lie in different connected components of $\Gamma - f$, then the claim is easily obtained by applying Lemma 3.4 on each of these components. Indeed, by Lemma 3.4 there are two right-extensible 3-paths g and h in the corresponding connected components of $\Gamma - f$, such that the initial vertex of g is u , the initial vertex of h is v , and some of them visits w . Then the concatenation of $-g$, f , and h satisfies the desired conditions.

We now consider the case where u and v lie in the same connected component of $\Gamma - f$. This graph will be denoted Λ . Note that Λ is infinite because f is bi-extensible. We claim that there are two right-extensible 3-paths on Λ , g and h , satisfying the following list of conditions: The initial vertex of g is u , the initial vertex of h is v , some of them visits w , and $V(g) \cap V(h) = \emptyset$. In addition, $(\Lambda - g) - h$ has no finite connected component and has two different vertices u' and v' such that u' is at distance at most 3 from the last vertex of g , and v' is at distance at most 3 from the last vertex of h . Suppose that we have g, h as before. Then we can define a 3-path f' by concatenating $-g$, f , and then h . It is clear that then f' satisfies the conditions in the statement.

We now construct g and h . We start by taking a connected finite subgraph Λ_0 of Λ which contains u, v, w and such that $\Lambda - \Lambda_0$ has no finite connected component. The graph Λ_0 can be obtained, for instance, as follows. As Λ is connected, we can take a path f_u from u to w , and a path f_v from v to w . Then define Λ_0 as the graph induced by the vertices in $V(f_v), V(f_u)$, and all vertices in the finite connected components of $(\Lambda - f_v) - f_u$.

Let p be a Hamiltonian 3-path on Λ_0 from u to v , as in Lemma 3.2. The desired 3-paths f and g will be obtained by “splitting” p in two. As Λ is connected, there are two vertices $u_0 \in V(\Lambda_0), v' \in V(\Lambda - \Lambda_0)$ such that u_0 and v are adjacent in Λ . By the conditions in Lemma 3.2, there is a vertex v_0 in $V(\Lambda_0)$ whose distance from u_0 is at most 2, and such that p visits consecutively $\{u_0, v_0\}$. We will assume that p visits v_0 after visiting u_0 , the other case being symmetric. As $\Lambda - \Lambda_0$ has no finite connected component, there is a vertex u' in $\Lambda - \Lambda_0$ that is adjacent to v' . Thus, u_0 is at distance at most 2 from u' , and v_0 is at distance at most 3 from v' . Now we define g and h by splitting p at the vertex u_0 . More precisely, let $[a, c]$ be the domain of p , and let b be such that $p(b) = u_0$. Then h is defined as the restriction of p to $[a, b]$, and we define g by requiring $-g$ to be the restriction of p to $[b + 1, c]$. Thus, h is a 3-path from v to v_0 , and g is a 3-path from u to u_0 . By our choice of Λ_0 and p , the 3-paths h and g satisfy the mentioned list of conditions, and thus the proof is finished. ■

When the graph has one or two ends, the hypotheses of Lemma 3.6 on u, v , and w are trivially satisfied. We obtain a very simple and convenient statement: We can extend a bi-extensible 3-path so that it visits a vertex of our choice.

Corollary 3.7. *Let Γ be a graph that is infinite, connected, locally finite, and whose number of ends is either 1 or 2. Then for every bi-extensible 3-path f and vertex w , there is a bi-extensible 3-path on Γ that extends f and visits w . We can assume that the domain of the new 3-path extends that of f in both directions.*

We are now in position to prove some results about bi-infinite 3-paths. We start with the Hamiltonian case, which is obtained by iterating Corollary 3.7. When we deal with bi-infinite 3-paths, we use the same notation and abbreviations introduced before for 3-paths, as long as they are well defined.

Proposition 3.8. *Let Γ be a graph that is infinite, connected, locally finite, and whose number of ends is either 1 or 2. Then Γ admits a bi-infinite Hamiltonian 3-path.*

Proof. Let $(v_n)_{n \in \mathbb{N}}$ be a numbering of the vertex set of Γ . We define a sequence of bi-extensible 3-paths $(f_n)_{n \in \mathbb{N}}$ on Γ recursively. We define f_0 as a bi-extensible 3-path which visits v_0 . The existence of f_0 is guaranteed by Lemma 3.5. Now let $n \geq 0$ and assume that we have defined a 3-path f_n that visits v_n . We define f_{n+1} as a bi-extensible 3-path on Γ which extends f_n , visits v_{n+1} , and whose domain extends the domain of f_n in both directions. The existence of such a 3-path is guaranteed by Corollary 3.7. We have obtained a sequence $(f_n)_{n \in \mathbb{N}}$ such that for all n , f_n visits v_n , and f_{n+1} extends f_n . With this sequence, we define a bi-infinite 3-path $f: \mathbb{Z} \rightarrow V(\Gamma)$ by setting $f(k) = f_n(k)$, for n big enough. Note that f is well defined because f_{n+1} extends f_n as a function, and the domains of f_n exhaust \mathbb{Z} . By construction, f visits every vertex exactly once, and thus it is Hamiltonian. ■

We now proceed with the non-Hamiltonian case, where there are no restrictions on ends. We first prove that we can take a bi-infinite 3-path whose deletion leaves no finite connected component.

Lemma 3.9. *Let Γ be a graph that is infinite, connected, and locally finite. Then for every vertex v , there is a bi-infinite 3-path f that visits v , and such that $\Gamma - f$ has no finite connected component.*

Proof. By Lemmas 3.5 and 3.6, Γ admits a sequence $(f_n)_{n \in \mathbb{N}}$ of bi-extensible 3-paths such that f_0 visits v , f_{n+1} extends f_n for all $n \geq 0$, and such that their domains exhaust \mathbb{Z} . We define a bi-infinite 3-path $f: \mathbb{Z} \rightarrow \Gamma$ by setting $f(k) = f_n(k)$, for n big enough. We claim that f satisfies the condition in the statement, that is, that $\Gamma - f$ has no finite connected component. We argue by contradiction. Suppose that Γ_0 is a nonempty and finite connected component of $\Gamma - f$. Define V_1 as the set of vertices in Γ that are adjacent to some vertex in Γ_0 , but which are not in Γ_0 . Then V_1 is nonempty as otherwise Γ would not be connected, and it is finite because Γ is locally finite. Moreover, f visits all vertices in V_1 , for otherwise Γ_0 would not be a connected component of $\Gamma - f$. As V_1 is finite, there is a natural number n_1 such that f_{n_1} has visited all vertices in V_1 . By our choice of V_1 and n_1 , Γ_0 is a nonempty and finite connected component of $\Gamma - f_{n_1}$, and this contradicts the fact that f_{n_1} is bi-extensible. ■

Now the proof of the following result is by iteration of Lemma 3.9.

Proposition 3.10. *Let Γ be a graph that is infinite, connected, and locally finite. Then there is a collection of bi-infinite 3-paths $f_i: \mathbb{Z} \rightarrow \Gamma$, $i \in I$, such that $V(\Gamma)$ is the disjoint union of $V(f_i)$, $i \in I$.*

Proof. By Lemma 3.9, Γ admits a bi-infinite 3-path f_0 such that $\Gamma - f_0$ has no finite connected component. Each connected component of $\Gamma - f_0$ is infinite and satisfies the hypotheses of Lemma 3.9. Thus, we can apply Lemma 3.9 on each of these connected components. Iterating this process in a tree-like manner, we obtain a family of 3-paths $f_i: \mathbb{Z} \rightarrow \Gamma$, $i \in I$ whose vertex sets $V(f_i)$ are disjoint. As Lemma 3.9 allows us to choose a vertex to be visited by the bi-infinite 3-path, we can choose f_i ensuring that every vertex of Γ is visited by some f_i . In this manner, $V(\Gamma)$ is the disjoint union of $V(f_i)$, ranging $i \in I$. ■

Finally, we can obtain Theorems 1.3 and 1.5 from our statements in terms of bi-infinite 3-paths.

Proof of Theorem 1.3. Let Γ be a graph as in the statement. By Proposition 3.8, Γ admits a Hamiltonian bi-infinite 3-path f . We define a translation-like $*$: $V(\Gamma) \times \mathbb{Z} \rightarrow V(\Gamma)$ by the expression $v * n = f(f^{-1}(v) + n)$, $n \in \mathbb{Z}$. This translation-like action is transitive because f is Hamiltonian and satisfies $d_\Gamma(v, v * 1) \leq 3$ because f is a bi-infinite 3-path.

We now prove the remaining implication of the result. That is, that a connected and locally finite graph which admits a transitive translation-like action by \mathbb{Z} must have either one or two ends. This is stated in [38, Theorem 3.3] for graphs with uniformly bounded vertex degree, but the same proof can be applied to locally finite graphs. For completeness, we provide an alternative argument. Let Γ be a connected and locally finite graph which admits a transitive translation-like action by \mathbb{Z} , denoted $*$. As the action is free, $V(\Gamma)$ must be infinite, and thus Γ has at least one end. Suppose now that it has at least three ends to obtain a contradiction. Let $J = \max\{d_\Gamma(v, v * 1) \mid v \in V(\Gamma)\}$. As Γ has at least three ends, there is a finite set of vertices V_0 such that $\Gamma - V_0$ has at least three infinite connected components, which we denote by Γ_1, Γ_2 , and Γ_3 . By enlarging V_0 if necessary, we can assume that any pair of vertices u and v that lie in different connected components in $\Gamma - V_0$, are at distance d_Γ at least $J + 1$. Now as V_0 is finite, there are two integers $n \leq m$ such that V_0 is contained in $\{v * k \mid n \leq k \leq m\}$. By our choice of V_0 , it follows that the set $\{v * k \mid k \geq m + 1\}$ is completely contained in one of Γ_1, Γ_2 , or Γ_3 . The same holds for $\{v * k \mid k \leq n - 1\}$, and thus one of Γ_1, Γ_2 , or Γ_3 must be empty, a contradiction. ■

Proof of Theorem 1.5. Let Γ be a graph as in the statement, and let f_i , $i \in I$ as in Proposition 3.10. We define $*$: $V(\Gamma) \times \mathbb{Z} \rightarrow V(\Gamma)$ by the expression

$$v * n = f(f^{-1}(v) + n), \quad n \in \mathbb{Z},$$

where f is the only f_i such that v is visited by f_i . Observe that $v * 1$ is well defined because $V(\Gamma) = \bigsqcup_{i \in I} V(f_i)$. This defines a translation-like action by \mathbb{Z} , where the distance from v to $v * 1$, $v \in V(\Gamma)$ is uniformly bounded by 3. ■

Remark 3.11. The proof given in this section is closely related to the characterization of those infinite graphs that admit infinite Eulerian paths. This is a theorem of Erdős, Grünwald, and Weiszfeld [18]. In the recent work [9], the author of this article gave a different proof of the Erdős, Grünwald, and Weiszfeld theorem that complements the original result by also characterizing those finite paths that can be extended to infinite Eulerian ones. This characterization is very similar to the notion of bi-extensible defined here. Indeed, the proofs of Proposition 3.8 and the proof of the mentioned result about Eulerian paths follow the same iterative construction.

Remark 3.12. As we mentioned before, it is known that the cube of every finite and connected graph is Hamiltonian [11, 30, 37]. Proposition 3.8 can be considered as a generalization of this fact to locally finite graphs. That is, Proposition 3.8 shows that the cube of every locally finite and connected graph with either one or two ends admits a bi-infinite Hamiltonian path.

We end this section by rephrasing a problem left in [38, Problem 3.5].

Problem 3.13. Find necessary and sufficient conditions for a connected graph to admit a transitive translation-like action by \mathbb{Z} .

We have shown that for locally finite graphs, the answer to this problem is as simple as possible, involving only the number of ends of the graph. The problem is now open for graphs that are not locally finite. We observe that beyond locally finite graphs there are different and nonequivalent notions of ends [15], and thus answering the problem above also requires to determine which is the appropriate notion of ends.

4. Computable translation-like actions by \mathbb{Z}

The goal of this section is to prove Theorem 1.7. Namely, that every finitely generated infinite group with decidable word problem admits a translation-like action by \mathbb{Z} , with the additional property of being computable and with decidable orbit membership problem.

The proof of Theorem 1.7 is as follows. For groups with at most two ends, we prove the existence of a computable and transitive translation-like action. That is, we prove a computable version of Theorem 1.3. For groups with more than two ends, we prove the existence of a subgroup isomorphic to \mathbb{Z} and with decidable subgroup membership problem. Thus for groups with two ends, we provide two different proofs for Theorem 1.7. A group with two ends is virtually \mathbb{Z} , and it would be easy to give a direct proof, but the intermediate statements may have independent interest (Theorem 4.1 and Proposition 4.7).

4.1. Computable and transitive translation-like actions by \mathbb{Z}

The goal of this subsection is to prove that Theorem 1.3 is computable on highly computable graphs.

Theorem 4.1 (Computable Theorem 1.3). *Let Γ be a graph that is highly computable, connected, and has either one or two ends. Then Γ admits a computable and transitive translation-like action by \mathbb{Z} , where the distance between a vertex v and $v * 1$ is uniformly bounded by 3.*

We start by proving that the bi-extensible property (Definition 3.3) is algorithmically decidable on highly computable graphs with one end.

Proposition 4.2. *Let Γ be a graph that is highly computable, connected, and has one end. Then it is algorithmically decidable whether a 3-path f is bi-extensible.*

Proof. It is clear that the second and third conditions in the definition of bi-extensible are algorithmically decidable. For the first condition, note that as Γ has one end, we can equivalently check whether $\Gamma - f$ is connected. This is proved to be a decidable problem in [9, Lemma 5.6]. Note that the mentioned result concerns the removal of edges instead of vertices, but indeed this is stronger: Given f , we compute the set E of all edges incident to a vertex in $V(f)$, and then use [9, Lemma 5.6] with input E . ■

For graphs with two ends we prove a similar result, but we need an extra assumption.

Proposition 4.3. *Let Γ be a graph that is highly computable, connected, and has two ends. Let f_0 be a bi-extensible 3-path on Γ , such that $\Gamma - f_0$ has two infinite connected components. Then there is an algorithm that on input a 3-path f that extends f_0 decides whether f is bi-extensible.*

Proof. It is clear that the second and third conditions in the definition of bi-extensible are algorithmically decidable. We address the first condition. We prove the existence of a procedure that, given a 3-path f as in the statement, decides whether $\Gamma - f$ has no finite connected component. Given f , we start by computing the set E . In [9, Lemma 5.5], there is an effective procedure that halts if and only if $\Gamma - f$ has some finite connected component (the aforementioned result mentions the removal of edges instead of vertices, but indeed this is stronger: Given f , we compute the set E of all edges incident to a vertex in $V(f)$, and then use [9, Lemma 5.5] with input E).

Thus, we need an effective procedure that halts if and only if $\Gamma - f$ has no finite connected component. As f extends f_0 , this is equivalent to ask whether $\Gamma - f$ has at most two connected components. The procedure is as follows: Given f , we start by computing the set V_0 of vertices in $\Gamma - f$ that are adjacent to a vertex visited by f . Then for every pair of vertices in $u, v \in V_0$, we search exhaustively for a path that joins them, and that never visits vertices in $V(f)$. That is, a path in $\Gamma - f$. Such a path will be found if and only if the connected component of $\Gamma - f$ that contains u equals the one that contain v . We stop the procedure once we have found enough paths to write V_0 as the disjoint union $V_1 \sqcup V_2$, where every pair of vertices in V_1 (resp. V_2) is joined by a path as described. ■

We can now show an effective version of Proposition 3.8.

Proposition 4.4 (Computable Proposition 3.8). *Let Γ be a graph that is highly computable, connected, and has either one or two ends. Then Γ admits a computable bi-infinite Hamiltonian 3-path.*

Proof. Let $(v_i)_{i \in \mathbb{N}}$ be a numbering of the vertex set of the highly computable graph Γ . Now let f_0 be a 3-path which is bi-extensible and visits v_0 . If Γ has two ends, then we also require that $\Gamma - f_0$ has two infinite connected components. In this case we do not claim that the path f_0 can be computed from a description of the graph, but it exists and can be specified with finite information. After fixing f_0 , we just follow the proof of Proposition 3.8 and observe that a sequence of 3-paths $(f_n)_{n \in \mathbb{N}}$ as in this proof can be uniformly computed. That is, there is an algorithm which given n , computes f_n . The algorithm proceeds recursively: Assuming that $(f_i)_{i \leq n}$ have been computed, we can compute f_{n+1} by an exhaustive search. The search is guaranteed to stop, and the conditions that we impose on f_{n+1} are decidable thanks to Propositions 4.2 and 4.3. Finally, let $f: \mathbb{Z} \rightarrow V(\Gamma)$ be the Hamiltonian 3-path on Γ defined by $f(k) = f_n(k)$, for n big enough. Then it is clear that the computability of $(f_n)_{n \in \mathbb{N}}$ implies that f is computable. ■

Now, we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let Γ be as in the statement. By Proposition 4.4, Γ admits a bi-infinite Hamiltonian 3-path $f: \mathbb{Z} \rightarrow V(\Gamma)$ that is computable. Then it is clear that the translation-like action $*$: $V(\Gamma) \times \mathbb{Z} \rightarrow V(\Gamma)$ defined by $v * n = f(f^{-1}(v) + n)$, $n \in \mathbb{Z}$, is computable. ■

This readily implies Theorem 1.7 for groups with one or two ends.

Proof of Theorem 1.7 for groups with one or two ends. Let G be a finitely generated infinite group with one or two ends, and with decidable word problem. Let $S \subset G$ be a finite set of generators, and let $\Gamma = \text{Cay}(G, S)$ be the associated Cayley graph. As G has decidable word problem, this is a highly computable graph (Proposition 2.5). Then by Theorem 4.1, Γ admits a computable and transitive translation-like action by \mathbb{Z} . As the vertex set of Γ is G , this is also a computable and transitive translation-like action on G . This action has decidable orbit membership problem for the trivial reason that it has only one orbit. ■

Remark 4.5. As mentioned in the introduction, there are a number of results in the theory of infinite graphs that cannot have an effective counterpart for highly computable graphs. In contrast, we have the following consequences of Theorems 1.3 and 4.1:

- (1) A highly computable graph admits a transitive translation-like action by \mathbb{Z} if and only if it admits a computable one.
- (2) A group with decidable word problem has a Cayley graph with a bi-infinite Hamiltonian path if and only if it has a Cayley graph with a computable bi-infinite Hamiltonian path.

- (3) The cube of a highly computable graph admits a bi-infinite Hamiltonian path if and only if it admits a computable one.

The third item should be compared with the following result of Bean: There is a graph that is highly computable and admits infinite Hamiltonian paths, but only uncomputable ones [5]. Thus, the third item shows that for graphs that are cubes, it is algorithmically easier to compute infinite Hamiltonian paths.

It follows from our results that the problem of *deciding* whether a graph admits a bi-infinite Hamiltonian path is also algorithmically easier when we restrict ourselves to graphs that are cubes. Harel proved that the problem of Hamiltonicity is analytic-complete for highly computable graphs [23, Theorem 2]. On the other hand, it follows from Theorem 1.3 that for graphs that are cubes, it suffices to check that the graph is connected and has either one or two ends. These conditions are undecidable, but are easily seen to be arithmetical [31]. In view of these results, it is natural to ask if these problems are easier when we restrict ourselves to graphs that are squares.

Question 4.6. The problem of computing infinite Hamiltonian paths (resp. deciding whether an infinite graph is Hamiltonian) on highly computable graphs, is easier when we restrict to graphs that are squares?

4.2. Computable normal forms and Stallings’s theorem

In this subsection we prove Theorem 1.7 for groups with two or more ends. It follows from Stallings’s structure theorem on ends of groups that a group with two or more ends has a subgroup isomorphic to \mathbb{Z} . We will prove that, if the group has solvable word problem, then this subgroup has decidable membership problem. This will be obtained from normal forms associated with HNN extensions and amalgamated products. We now recall well-known facts about these constructions; the reader is referred to [32, Chapter IV].

HNN extensions are defined from a group $H = \langle S_H \mid R_H \rangle$, a symbol t not in S_H , and an isomorphism $\phi: A \rightarrow B$ between subgroups of H . The *HNN extension* relative to H and ϕ is the group with presentation $H *_{\phi} = \langle S_H, t \mid R_H, t a t^{-1} = \phi(a), \forall a \in A \rangle$. Now let $T_A \subset H$ and $T_B \subset H$ be sets of representatives for equivalence classes of H modulo A and B , respectively. The group $H *_{\phi}$ admits a normal form associated with the sets T_A and T_B . The sequence of group elements $h_0, t^{\varepsilon_1}, h_1, \dots, t^{\varepsilon_n}, h_n, \varepsilon_i \in \{1, -1\}$, is in *normal form* if (1) $h_0 \in H$, (2) if $\varepsilon_i = -1$, then $h_i \in T_A$, (3) if $\varepsilon_i = 1$, then $h_i \in T_B$, and (4) there is no subsequence of the form $t^{\varepsilon}, 1_H, t^{-\varepsilon}$. For every $g \in H *_{\phi}$, there exists a unique sequence in normal form whose product equals g in $H *_{\phi}$.

Amalgamated products are defined from two groups $H = \langle S_H \mid R_H \rangle$ and $K = \langle S_K \mid R_K \rangle$, and a group isomorphism $\phi: A \rightarrow B$, with $A \leq H$ and $B \leq K$. The *amalgamated product* of H and K relative to ϕ is the group with presentation $H *_{\phi} K = \langle S_H, S_K \mid R_H, R_K, a = \phi(a), \forall a \in A \rangle$. Now let $T_A \subset H$ be a set of representatives for H modulo A , and let $T_B \subset K$ be a set of representatives for K modulo B . The group $H *_{\phi} K$ admits a normal form associated with the sets T_A and T_B . A sequence of group elements

c_0, c_1, \dots, c_n is in *normal form* if (1) c_0 lies in A or B , (2) c_i is in T_A or T_B for $i \geq 1$, (3) $c_i \neq 1$ for $i \geq 1$, and (4) successive c_i alternate between T_A and T_B . For each element $g \in H *_\phi K$, there exists a unique sequence in normal form whose product equals g in $H *_\phi K$.

Stalling's structure theorem relates ends of groups with HNN extensions and amalgamated products [17]. This result asserts that every finitely generated group G with two or more ends is either isomorphic to an HNN extension $H *_\phi$, or isomorphic to an amalgamated product $H *_\phi K$. In both cases, the corresponding isomorphism ϕ is between finite and proper subgroups, and the groups H , or H and K , are finitely generated (see [14, pp. 34 and 43]). We will now prove that when G has decidable word problem, then the associated normal forms are computable. This means that there is an algorithm which given a word representing a group element g computes a sequence of words such that the corresponding sequence of group elements is a normal form for g . The proof is direct, but we were unable to find this statement in the literature.

Proposition 4.7. *Let G be a finitely generated group with two or more ends and decidable word problem. Then the normal form associated with the decomposition of G as HNN extension or amalgamated product is computable.*

Proof. Let us assume first that we are in the first case, so there is a finitely generated group $H = \langle S_H \mid R_H \rangle$, and an isomorphism $\phi: A \rightarrow B$ between finite subgroups of H , such that G is isomorphic to the HNN extension $H *_\phi = \langle S_H, t \mid R_H, tat^{-1} = \phi(a), a \in A \rangle$. A preliminary observation is that H has decidable word problem. Indeed, this property is inherited by finitely generated subgroups, and G has decidable word problem by hypothesis. The computability of the normal form will follow from two simple facts:

First, observe that the finite group $A = \{a_1, \dots, a_n\}$ has decidable membership problem in H . Indeed, given a word $w \in (S_H \cup S_H^{-1})^*$, we can decide if $w_H \in A$ by checking if $w =_H a_i$ for $i = 1, \dots, m$. This is an effective procedure as the word problem of H is decidable and is guaranteed to stop as A is a finite set. As a consequence of this, we can also decide if $u_H \in Av_H$ for any pair of words $u, v \in (S_H \cup S_H^{-1})^*$, as this is equivalent to decide if $(uv^{-1})_H$ lies in A . The same is true for B .

Second, there is a computably enumerable set $W_A \subset (S_H \cup S_H^{-1})^*$ such that the corresponding set T_A of group elements in H constitute a collection of representatives for H modulo A . We sketch an algorithm that computably enumerates W_A as a computable sequence of words. Set u_0 to be the empty word. Now assume that words u_0, \dots, u_n have been selected, and search for a word $u_{n+1} \in (S_H \cup S_H^{-1})^*$ such that $(u_{n+1})_H$ does not lie in $A(u_0)_H, \dots, A(u_n)_H$. The condition that we impose to u_{n+1} is decidable by the observation in the previous paragraph, and thus an exhaustive search is guaranteed to find a word as required. It is clear that the set W_A that we obtain is computably enumerable, and that the set T_A of the group elements of H corresponding to these words is a set of representatives for H modulo A . A set W_B corresponding to T_B can be enumerated analogously.

Finally, we note that we can computably enumerate sequences of words w_0, \dots, w_n that represent normal forms (with respect to T_A and T_B) for all group elements. Indeed, using the fact W_A and W_B are computably enumerable sets, we just have to enumerate sequences of words w_0, \dots, w_n such that w_0 as an arbitrary element of $(S_H \cup S_H^{-1})^*$, and the rest are words from W_A, W_B , or $\{t, t^{-1}\}$ that alternate as in the definition of normal form. In order to compute the normal form of a group element w_G given by a word w , we just enumerate these sequences w_1, \dots, w_n until we find one satisfying $w =_G w_1 \cdots w_n$, this is a decidable question as G has decidable word problem. We have proved the computability of normal forms as in the statement, in the case where G is a (isomorphic to) HNN extension.

If G is not isomorphic to an HNN extension, then it must be isomorphic to an amalgamated product. Then there are two finitely generated groups $H = \langle S_H \mid R_H \rangle$ and $K = \langle S_K \mid R_K \rangle$, and a group isomorphism $\phi: A \rightarrow B$, with $A \leq H$ and $B \leq K$ finite groups, such that G is isomorphic to $\langle S_H, S_K \mid R_H, R_K, a = \phi(a), \forall a \in A \rangle$. Now the argument is the same as the one given for HNN extensions. That is, A and B have decidable membership problem because they are finite, and there are two computably enumerable sets of words W_A and W_B corresponding to sets T_A and T_B as in the definition of normal form for amalgamated products. This, plus the decidability of the word problem, is sufficient to compute the normal form of a group element given as a word, by an exhaustive search. ■

We obtain the following result from the computability of these normal forms.

Proposition 4.8. *Let G be a finitely generated group with two or more ends and decidable word problem. Then it has a subgroup isomorphic to \mathbb{Z} with decidable subgroup membership problem.*

Proof. By Stallings’s structure theorem, either G is isomorphic to an HNN extensions, or G is isomorphic to an amalgamated product. We first suppose that G is isomorphic to an HNN extension $H *_\phi = \langle S_H, t \mid R_H, t a t^{-1} = \phi(a), a \in A \rangle$. Without loss of generality, we will assume that G is equal to this group instead of isomorphic, as the decidability of the membership problem of an infinite cyclic subgroup is preserved by group isomorphisms. We claim that the subgroup of G generated by t has decidable membership problem. Indeed, a group element g lies in this subgroup if and only if the normal form of g or g^{-1} is $1, t, 1, \dots, t, 1$. By Proposition 4.7, this normal form is computable, and thus we obtain a procedure to decide membership in the subgroup of G generated by t .

We now consider the case where G is isomorphic to an amalgamated product. Then there are two finitely generated groups $H = \langle S_H \mid R_H \rangle$ and $K = \langle S_K \mid R_K \rangle$, and a group isomorphism $\phi : A \rightarrow B$, with $A \leq H$ and $B \leq K$ finite groups, such that G is isomorphic to $\langle S_H, S_K \mid R_H, R_K, a = \phi(a), \forall a \in A \rangle$. As before, we will assume without loss of generality that G is indeed equal to this group. Now let T_A and T_B be the sets defined in Proposition 4.7 that are associated with the computable normal form, and let $u \in T_A, v \in T_B$ be both nontrivial elements. We claim that the subgroup of G generated by uv

is isomorphic to \mathbb{Z} and has decidable membership problem. Indeed, a group element g lies in this subgroup if and only if the normal form of g or g^{-1} is u, v, \dots, u, v . By Proposition 4.7, this normal form is computable, and thus we obtain a procedure to decide membership in the subgroup of G generated by t . ■

We now verify the fact that for translation-like actions coming from subgroups, the properties of decidable orbit membership problem and decidable subgroup membership problem are equivalent.

Proposition 4.9. *Let $H \leq G$ be finitely generated groups. Then H has decidable membership problem in G if and only if the action of H on G by right translations has decidable orbit membership problem.*

Proof. Let $*$ be the action defined by $G \times H \rightarrow G, (g, h) \mapsto gh$. The claim follows from the fact that two elements $g_1, g_2 \in G$ lie in the same $*$ orbit if and only if $g_1g_2^{-1} \in H$, and an element $g \in G$ lies in H if and only if it lies in the same $*$ orbit as 1_G .

It is clear how to rewrite this in terms of words, but we fill the details for completeness. For the forward implication, let $u, v \in (S \cup S^{-1})^*$ be two words, for which we want to decide whether u_G, v_G lie in the same orbit. We start by computing the formal inverse of v , denoted v^{-1} , and then check whether the word uv^{-1} lies in $\{w \in (S \cup S^{-1})^* \mid w_G \in H\}$. This set is decidable for hypothesis. For the reverse implication, assume that the action has decidable orbit membership problem. The set $\{w \in (S \cup S^{-1})^* \mid w_G \in H\}$ equals the set of words $w \in (S \cup S^{-1})^*$ such that w_G and 1_G lie in the same orbit, which is a decidable set by hypothesis. It follows that H has decidable subgroup membership problem in G . ■

We can now finish the proof of Theorem 1.7.

Proof of Theorem 1.7 for groups with two or more ends. Let G be a finitely generated infinite group with decidable word problem and at least two ends. By Proposition 4.8, there is an element $c \in G$ such that $\langle c \rangle$ is isomorphic to \mathbb{Z} and has decidable subgroup membership problem in G . The right action $\mathbb{Z} \curvearrowright G$ defined by $g * n = gc^n$ has decidable orbit membership problem by Proposition 4.9.

It only remains to verify that the function $G \times \mathbb{Z} \rightarrow G, (g, n) \mapsto g * n$ is computable in the sense of Subsection 2.3. This is clear, but we write the details for completeness. The group operation $f_1: G \times G \rightarrow G$ is computable by Proposition 2.4. Moreover, it is clear that the function $f_2: \mathbb{Z} \rightarrow G, n \mapsto c^n$ is computable. Then it follows that the function $f_3: G \times \mathbb{Z} \rightarrow G, (g, n) \mapsto f_1(g, f_2(n))$ is computable, being the composition of computable functions. But $f_3(g, n) = g * n$, and thus $*$ is a computable group action. ■

5. Medvedev degrees of effective subshifts

The goal of this section is to prove Theorem 1.8. That is, that on every infinite group with decidable word problem, the class of possible Medvedev degrees of effective subshifts is that of Π_1^0 degrees.

In summary, our proof is the application of a known construction that given a subshift $X \subset A^{\mathbb{Z}}$, outputs a subshift $Y \subset A^G$ whose configurations describe simultaneously translation-like actions $\mathbb{Z} \curvearrowright G$, and configurations in X . When we require this construction to preserve the Medvedev degree of the initial subshift X , then the existence of a computable translation-like action $\mathbb{Z} \curvearrowright G$ with decidable orbit membership problem arises as a natural condition.

Medvedev degrees of subshifts have only been discussed in the literature for $G = \mathbb{Z}^d$, and for this reason we will review computability aspects of the space A^G in detail. Given a group G with decidable word problem, we will translate computability notions from $A^{\mathbb{N}}$ to A^G using a computable numbering $\nu: \mathbb{N} \rightarrow G$. We verify that the computability notions in this space are independent of the chosen numbering, preserved by group isomorphisms, compatible with previous notions in the literature [2], and that an effective subshift is the same as a subshift that is effectively closed as a set. This equivalence is lost for groups whose word problem is algorithmically complex (see [2,4]).

5.1. Computability notions on the Cantor space

Here, we will review some standard concepts from the theory of computability on the Cantor space. A modern reference of computability theory on uncountable spaces is [7].

Let A be a finite alphabet. The set $A^{\mathbb{N}}$ is endowed with the prodiscrete topology, for which a sub-basis is the set of cylinders. A *cylinder* is a set of the form $[p] = \{x \in A^{\mathbb{N}} \mid x|_K = p\}$, where p is a *pattern*: a function from a finite set $K \subset \mathbb{N}$ to A . We identify a word $w = w_0 \cdots w_n \in A^*$ with the pattern $\{0, \dots, n\} \rightarrow A$, and thus $[w] = \{x \in A^{\mathbb{N}} \mid x_0 \cdots x_n = w_0 \cdots w_n\}$.

Definition 5.1. A set $X \subset A^{\mathbb{N}}$ is *effectively closed*, denoted Π_1^0 , if some of the following equivalent conditions hold:

- (1) The complement of X can be written as $\bigcup_{w \in L} [w]$, for a computably enumerable set of words $L \subset A^*$.
- (2) It is semi-decidable whether a word w satisfies $[w] \cap X = \emptyset$.
- (3) It is semi-decidable whether a pattern p satisfies $[p] \cap X = \emptyset$.

Definition 5.2. A partial function $F: D \subset A^{\mathbb{N}} \rightarrow B^{\mathbb{N}}$ is *computable* when there is a partial computable function on words $f: A^* \rightarrow B^*$ satisfying the following three conditions:

- (1) f is monotone for the prefix order on words.
- (2) For each x in the domain D , the length of $f(x|_{\{0, \dots, k\}})$ tends to infinity with k .

- (3) For every x in the domain D , and for every $k \in \mathbb{N}$, there is n big enough such that $F(x)(n)$ is the n th letter in the word $f(x|_{\{0, \dots, k\}})$.

It follows from the definition that a computable function must be continuous.

Example 5.3. The shift function $\sigma: A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$, $(\sigma x)(n) = x(n + 1)$ is computable. This is shown by the computable function $s: A^* \rightarrow A^*$, $s(w_0 w_1 \cdots w_n) = w_1 \cdots w_n$.

Definition 5.4. Let $X \subset A^{\mathbb{N}}$ and $Y \subset B^{\mathbb{N}}$. The sets X and Y are *computably homeomorphic* if there is a homeomorphism $\Phi: X \rightarrow Y$ such that both Φ and its inverse are computable functions.

Example 5.5. Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be a computable bijection, and let $F: A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$ be defined by $x \mapsto x \circ f$. Then F is a computable homeomorphism, and with computable inverse $x \mapsto x \circ f^{-1}$.

Example 5.6. If A and B are finite alphabets with cardinality at least 2, then the sets $A^{\mathbb{N}}$ and $B^{\mathbb{N}}$ are computably homeomorphic. Indeed, the usual homeomorphism between these sets is a computable function (see [26, Theorem 2-97]). A simple case is when $A = \{0, 1, 2, 3\}$ and $B = \{0, 1\}$. Then a computable homeomorphism is given by the letter-to-word substitutions $0 \mapsto 00$, $1 \mapsto 01$, $2 \mapsto 10$, $3 \mapsto 11$.

5.2. Medvedev degrees

Here, we review the lattice \mathfrak{M} of Medvedev degrees. A survey on this topic is [24].

Definition 5.7. Let $X \subset A^{\mathbb{N}}$ and $Y \subset B^{\mathbb{N}}$. We say that Y is *Medvedev reducible* to X , written $Y \leq_{\mathfrak{M}} X$, if there is a partial computable function Φ defined on all elements of X , and such that $\Phi(X) \subset Y$. We write $X \equiv_{\mathfrak{M}} Y$ when we have both reductions. A *Medvedev degree* is an equivalence class of $\equiv_{\mathfrak{M}}$, and we denote by \mathfrak{M} the set of Medvedev degrees. The pre-order $\leq_{\mathfrak{M}}$ becomes a partial order on \mathfrak{M} , and the degree of a set X is denoted by $\text{deg}_{\mathfrak{M}}(X)$.

The partially ordered set $(\mathfrak{M}, \leq_{\mathfrak{M}})$ is indeed a distributive lattice with a bottom element $0_{\mathfrak{M}}$, and a top element $1_{\mathfrak{M}}$. We remark that the Medvedev degree of a set X is meaningful when we regard X as the set of all solutions to a problem: It measures how hard is it to find a solution, where hard means hard to compute. For instance, $\text{deg}_{\mathfrak{M}}(X) = 0_{\mathfrak{M}}$ if and only if X has a computable point, while $\text{deg}_{\mathfrak{M}}(X) = 1_{\mathfrak{M}}$ if and only if X is empty. A prominent sublattice of \mathfrak{M} is that of Π_1^0 degrees.

Definition 5.8. A Medvedev degree is called Π_1^0 when it is the degree of a Π_1^0 nonempty subset of $\{0, 1\}^{\mathbb{N}}$.

5.3. Subshifts

Here, we review standard terminology for subshifts. The reader is referred to the book [10].

Let G be a finitely generated group, and let A be a finite alphabet. We endow A^G with the prodiscrete topology. A *subshift* is a subset $X \subset A^G$ which is closed and invariant under the group action $G \curvearrowright A^G$ by left translations $(gx)(h) \mapsto x(g^{-1}h)$. A *pattern* is a function p from a finite set $K \subset G$ to A , and it determines the *cylinder* $[p] = \{x \in A^G \mid x|_K = p\}$. If $gx \in [p]$ for some $g \in G$, we say that p *appears* on x . A set of *forbidden patterns* \mathcal{F} defines the subshift $X_{\mathcal{F}}$ of all elements $x \in A^G$ where no pattern of \mathcal{F} appears in x . Every subshift is determined by a maximal set of forbidden patterns, but it can have more than one defining set of forbidden patterns. A subshift is of *finite type* (SFT) if it can be defined with a finite set of forbidden patterns.

5.4. Computability on A^G

In this subsection we translate computability notions from $A^{\mathbb{N}}$ to A^G , where G is a finitely generated group with decidable word problem. Our goal is to provide a definition of Medvedev degree for subshifts. In simple words, we will take a computable bijection $\nu: \mathbb{N} \rightarrow G$, and use it to define a homeomorphism from $A^{\mathbb{N}}$ to A^G . We declare this homeomorphism to be computable, and in this manner we translate to A^G the concepts defined in $A^{\mathbb{N}}$. This process is well established in the theory of computability on uncountable spaces and is the subject of representation theory. A representation plays the same role as a numbering (Subsection 2.3), but for an uncountable set.

We recall now some definitions from [7, Chapter 9]. A *represented space* is a pair (X, δ) where X is a set and δ is a *representation* of X : a partial surjection $\delta: \text{dom}(\delta) \subset A^{\mathbb{N}} \rightarrow X$. In a represented space (X, δ) , a subset $Y \subset X$ is *effectively closed* when $\delta^{-1}(Y) \subset A^{\mathbb{N}}$ is an effectively closed set. Moreover, if $(X', \delta': A'^{\mathbb{N}} \rightarrow X')$ is another represented space, a function $F: X \rightarrow X'$ is *computable* when $\delta'^{-1} \circ F \circ \delta: A^{\mathbb{N}} \rightarrow A'^{\mathbb{N}}$ is a computable function. Finally, two representations of the same space X , $\delta: A^{\mathbb{N}} \rightarrow X$ and $\delta': A'^{\mathbb{N}} \rightarrow X$, are *equivalent* if the identity function from (X, δ) to (X, δ') is computable. Note that in this case, both representations induce the same computability notions on X .

In what follows we will focus on a specific representation of A^G , which is also a total function and a homeomorphism.

Definition 5.9. Let G be a finitely generated group with decidable word problem, and let ν a computable numbering of G . We define the representation δ by

$$\begin{aligned} \delta: A^{\mathbb{N}} &\rightarrow A^G \\ x &\mapsto x \circ \nu^{-1}. \end{aligned}$$

It follows from Proposition 2.4 that a group as in the statement admits a computable numbering, and that all these numberings are equivalent. In terms of representations, this is expressed as follows.

Proposition 5.10. *In Definition 5.9, any two computable numberings induce equivalent representations.*

Proof. Let ν' be another computable numbering of G , and let δ' be the associated representation of A^G . Let $F: A^G \rightarrow A^G$ be the identity function. Then $\delta'^{-1} \circ F \circ \delta: A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$ is given by $x \mapsto x \circ \nu^{-1} \circ \nu'$. We verify that this function is a computable homeomorphism. Indeed, as the numberings ν, ν' are equivalent (Proposition 2.4), the function $\nu^{-1} \circ \nu': \mathbb{N} \rightarrow \mathbb{N}$ is a computable bijection of \mathbb{N} , and this implies that $x \mapsto x \circ \nu^{-1} \circ \nu'$ is a computable homeomorphism (see Example 5.5). ■

We note that computability notions on A^G are also preserved by group isomorphisms.

Proposition 5.11. *Let G and G' be finitely generated groups with decidable word problem, and let $A^G, A^{G'}$ be endowed with the representation in Definition 5.9. If $f: G \rightarrow G'$ is a group isomorphism, then the associated function $F: A^{G'} \rightarrow A^G, x \mapsto x \circ f$ is a computable homeomorphism.*

The proof is similar to the proof of Proposition 5.10, but applying the third item in Proposition 2.4. This means that the computability notions on A^G are preserved if we rename group elements (e.g., by taking different presentations of the same group). We are ready to define the Medvedev degree of a subset of A^G .

Definition 5.12. Let G be a finitely generated group with decidable word problem. Given a subset $X \subset A^G$, we define $\text{deg}_{\mathfrak{M}}(X) = \text{deg}_{\mathfrak{M}}(\delta^{-1}X)$.

This definition does not depend on δ , as long as δ comes from a computable numbering of G . We now turn our attention to effectively closed subsets of A^G , and subshifts.

Proposition 5.13. *Let G be a finitely generated group with decidable word problem. Then a subset $X \subset A^G$ is effectively closed if and only if it is semi-decidable whether a pattern $p: K \subset G \rightarrow A$ satisfies $[p] \cap X = \emptyset$.*

Proof. We only prove the forward implication, the converse being similar. Given a pattern $p: K \subset G \rightarrow A$, we start by computing a pattern $p': K' \subset \mathbb{N} \rightarrow A$ such that $p = p' \circ \nu$. Then $[p] \cap X = \emptyset$ if and only if $[p'] \cap \delta^{-1}(X) = \emptyset$. But the latter relation is semi-decidable on p' as $\delta^{-1}(X)$ is effectively closed in $A^{\mathbb{N}}$. ■

In [2], the authors introduced a notion of effectiveness for subshift on general finitely generated groups. This notion is not explicitly associated with A^G as a represented space (or a computable metric space), but we shall verify now that for groups with decidable word problem, these approaches are equivalent.

The following definitions are taken from [2]. A *pattern coding* c is a finite set of tuples $\{(w_1, a_1), \dots, (w_k, a_k)\}$, where $w_i \in S^*$ and $a_i \in A$, and is *consistent* when $w_i =_G w_j$ implies $a_i = a_j$. A consistent pattern coding defines a pattern $p(c): K \subset G \rightarrow A$, where K

equals $\{(w_1)_G, \dots, (w_k)_G\}$, and $p((w_i)_G) = a_i$. A set of pattern codings \mathcal{C} defines the subshift $X_{\mathcal{C}}$ of all elements $x \in A^G$ such that no pattern of the form $p(c)$ appears in x , where c ranges over \mathcal{C} . A subshift X is *effective* if there is a computably enumerable set of pattern codings \mathcal{C} such that $X = X_{\mathcal{C}}$.

Proposition 5.14. *Let G be a finitely generated group with decidable word problem. Then a subshift $X \subset A^G$ is effective if and only if it is an effectively closed subset of A^G .*

Proof. If a subshift is an effectively closed subset of A^G , then by Proposition 5.13 the set of all patterns p with $[p] \cap X = \emptyset$ is computably enumerable. Let \mathcal{F} be this set of patterns, and let \mathcal{C} be the set of all pattern codings associated with patterns in \mathcal{F} . It is clear that \mathcal{C} is computably enumerable and $X = X_{\mathcal{C}}$, so X is an effective subshift as well.

We now consider the other direction. In [2, Lemma 2.3], it is shown that for a recursively presented group and in particular one with decidable word problem, an effective subshift has a maximal—for inclusion—computably enumerable set of pattern codings associated with forbidden patterns. Given an effective subshift X , we can write $X = X_{\mathcal{C}}$, where \mathcal{C} is a maximal—for inclusion—set of defining forbidden pattern codings. As G has decidable word problem, the set of consistent pattern codings is decidable, and thus we can computably discard those pattern codings that are not consistent. This, plus the previous fact, proves that the set of all patterns p with $[p] \cap X = \emptyset$ is computably enumerable. By Proposition 5.13, it follows that the set X is effectively closed. ■

Let us now make some comments about the computability of the action $G \curvearrowright A^G$ by translations. It follows from Proposition 2.4 and Example 5.5 that this action is computable.

Proposition 5.15. *Let G be a finitely generated group with decidable word problem. Then the group action $G \curvearrowright A^G$ is computable.*

It follows from Proposition 2.4 that all numberings of a group G as above that make the left (resp. right) action $G \curvearrowright G$, $(g, h) \mapsto gh$ computable are equivalent. In other words, the action of a group on itself characterizes those computable numberings of the group. This is a well-studied subject and leads to the notion of *computable dimension* of a group (see, for instance, [21]). It is natural then to ask whether something analogous happens for representations of the space A^G .

Question 5.16. Let G be a finitely generated group with decidable word problem. Are all representations of the space A^G that make the action $G \curvearrowright A^G$ computable equivalent?

5.5. The subshift of translation-like actions by \mathbb{Z} , and the proof of Theorem 1.8

In this subsection, we finally prove Theorem 1.8. Our standing assumption is that G is a finitely generated group, $S \subset G$ is a finite set of generators, and $J \in \mathbb{N}$. When we need G to have decidable word problem, we will specify it.

Definition 5.17. We define $T_J(\mathbb{Z}, G)$ as the set of all translation-like actions $*$: $G \times \mathbb{Z} \rightarrow G$, such that $\{d_S(g, g * 1) \mid g \in G\}$ is bounded by J .

Consider now the finite alphabet $B = B(1_G, J) \times B(1_G, J)$, where $B(1_G, J)$ is the ball $\{g \in G \mid d_S(g, 1_G) \leq J\}$. Every translation-like action $*$ $\in T_J(\mathbb{Z}, G)$ defines a configuration in B^G , denoted x_* , by the condition

$$\forall g \in G \quad x_*(g) = (l, r) \iff g * -1 = gl \text{ and } g * 1 = gr.$$

Definition 5.18. We define $X_J(\mathbb{Z}, G)$ as the set $\{x_* \in B^G \mid * \in T_J(\mathbb{Z}, G)\}$.

The informal idea is to interpret $x(g) = (l, r)$ as a pair of arrows: g has an outgoing arrow to gr , and an incoming arrow from gl . See Figure 1.

Proposition 5.19. *The set $X_J(\mathbb{Z}, G)$ is a subshift. If G has decidable word problem, then it is an effective subshift.*

Proof. We define for each element $x \in B^G$ a function $*_x: G \times \mathbb{Z} \rightarrow G$, which may not be a group action. L and R stand for the projections $B \rightarrow B(1_G, J)$ to the left and right coordinates, respectively. For $m \in \mathbb{Z}_{\geq 0}$ and $g \in G$, define $g *_x m$ by setting $g *_x 0 = g$, $g *_x 1 = gR(x(g))$, and $g *_x (m + 1) = (g *_x m) *_x 1$. For $m \in \mathbb{Z}_{\leq 0}$, define $g *_x m$ by $g *_x -1 = gL(x(g))$ and $g *_x (m - 1) = (g *_x m) *_x -1$.

If $p: K \subset G \rightarrow B$ is a pattern and $m \in \mathbb{Z}$, we give to $g *_p m$ the same meaning as before, as long as it is defined. Note that for arbitrary $x \in B^G$ and $n, m \in \mathbb{Z}$, the relation $(g *_x n) *_x m = g *_x (n + m)$ is not guaranteed to hold, but it does hold when n and m have the same sign.

Let \mathcal{J} be the set of all patterns $p: B(1_G, n) \rightarrow B$, $n \in \mathbb{N}$, such that some of the following conditions occur:

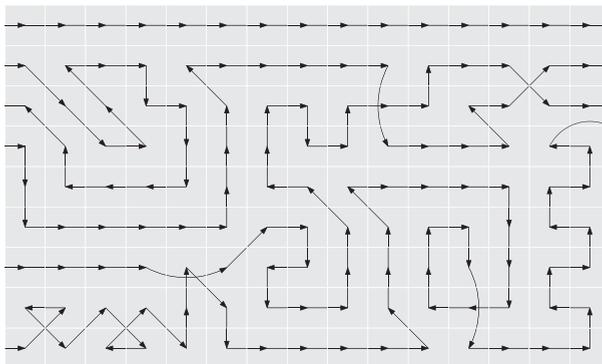


Figure 1. Representation of some orbits of a translation-like action in $T_2(\mathbb{Z}, \mathbb{Z}^2)$, or alternatively, a finite pattern in a configuration in $X_2(\mathbb{Z}, \mathbb{Z}^2)$. In this case, \mathbb{Z}^2 is endowed with the set of four generators $S = \{(\pm 1, 0), (0, \pm 1)\}$.

- (1) $(1_G *_{p} 1) *_{p} -1 \neq 1_G$.
- (2) $(1_G *_{p} -1) *_{p} 1 \neq 1_G$.
- (3) For some $m \in \mathbb{Z} - \{0\}$, $1_G *_{p} m = 1_G$.

We claim that $X_J(\mathbb{Z}, G) = X_{\mathcal{J}}$. The inclusion $X_J(\mathbb{Z}, G) \subset X_{\mathcal{J}}$ is straightforward. Indeed, given $* \in T_J(\mathbb{Z}, G)$, it is clear that no pattern of \mathcal{J} may appear on x_* by the definition of group action and translation-like action.

We prove now that $X_{\mathcal{J}} \subset X_J(\mathbb{Z}, G)$. Let $x \in X_{\mathcal{J}}$ be an arbitrary element. We first prove that $*_x$ is a translation-like action in $T_J(\mathbb{Z}, G)$. Indeed, it follows from the forbidden patterns in \mathcal{J} that for every $g \in G$, $(g *_x 1) *_x -1 = (g *_x -1) *_x 1 = g$. Then an easy induction on $\max\{|n|, |m|\}$ shows that $(g *_x n) *_x m = g *_x (n + m)$ for all $n, m \in \mathbb{Z}$. Thus $*_x$ is a group action. This action is free by the third condition on the set \mathcal{J} forbidden patterns, and the boundedness condition comes from the alphabet chosen. Thus, $*_x$ is a translation-like action in $T_J(\mathbb{Z}, G)$, and then $x_{(*_x)}$ lies in $X_J(\mathbb{Z}, G)$ by definition. But $x = x_{(*_x)}$, so it follows that x lies in $X_J(\mathbb{Z}, G)$. As x was an arbitrary element from $X_{\mathcal{J}}$, we obtain the desired inclusion $X_{\mathcal{J}} \subset X_J(\mathbb{Z}, G)$.

We now verify that, having G decidable word problem, the subshift $X_J(\mathbb{Z}, G)$ is effective. The definition of $*_p$ above is recursive: Given a pattern p on alphabet B and $m \in \mathbb{Z}$, we can decide if the group element $1_G *_{p} m$ is defined, and compute it. This shows that the conditions on patterns (1), (2), and (3) are decidable over patterns, and thus that \mathcal{J} is a decidable set of patterns. Thus, $X_{\mathcal{J}}$ is an effective subshift. ■

We now describe an effective subshift on G whose elements describe, simultaneously, translation-like actions, and configurations from a subshift over \mathbb{Z} (see Figure 2). Let A be an arbitrary finite alphabet, and let B be the alphabet already defined and which depends on the natural number J . Elements of $(A \times B)^G$ can be conveniently written as (y, x) for $y \in A^G$ and $x \in B^G$. We will write $\pi_A: A \times B \rightarrow A$ and $\pi_B: A \times B \rightarrow B$ for the projections to the first and second coordinates, respectively.

Definition 5.20. For a one-dimensional subshift $Y \subset A^{\mathbb{Z}}$, we define $Y[X_J(\mathbb{Z}, G)]$ as the set of all configurations $(y, x) \in (A \times B)^G$ such that the following two conditions are satisfied:

- (1) $x \in X_J(\mathbb{Z}, G)$.
- (2) For every $g \in G$, the element $m \mapsto y(g *_x m)$, $m \in \mathbb{Z}$, lies in Y .

Proposition 5.21. *The set $Y[X_J(\mathbb{Z}, G)]$ is a subshift. If G has decidable word problem and Y is an effective subshift, then $Y[X_J(\mathbb{Z}, G)]$ is an effective subshift.*

Proof. Let \mathcal{F} be the set of all patterns in \mathbb{Z} that do not appear in X , so that $X = X_{\mathcal{F}}$, and let \mathcal{J} be as in the proof of Proposition 5.19. That is, $X_{\mathcal{J}} = X_J(\mathbb{Z}, G)$. Define \mathcal{H} to be the set of all patterns $p: B(1_G, n) \rightarrow A \times B$, $n \in \mathbb{N}$, such that some of the following conditions hold:

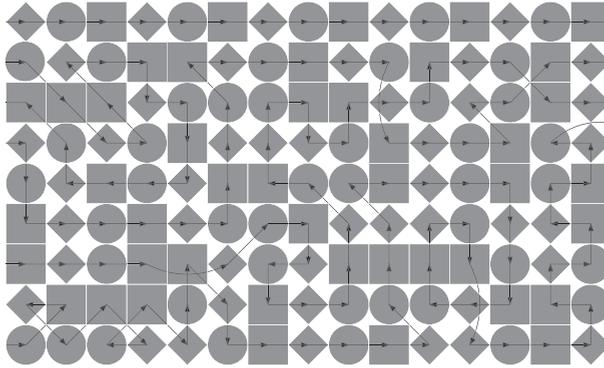


Figure 2. Representation of a finite pattern in a of configuration in $Y[X_2(\mathbb{Z}, \mathbb{Z}^2)]$. Here, A is the alphabet {circle, square, rhombus}, and $Y \subset A^{\mathbb{Z}}$ is the subshift of all sequences that alternate circle, square, and rhombus in that order.

- (1) The pattern $\pi_B \circ p: B(1_G, n) \rightarrow B$ lies in \mathcal{J} .
- (2) Let $q = \pi_B \circ p: B(1_G, n) \rightarrow B$. For some $m \in \mathbb{N}$, the elements $g *_q 1, \dots, g *_q m$ are all defined, lie in $B(1_G, n)$, and the pattern $r: \{1, \dots, m\} \subset \mathbb{Z} \rightarrow A, r(k) = \pi_A(g *_q k)$ lies in \mathcal{F} .

It is easily verified that $x \in Y[X_J(\mathbb{Z}, G)]$ if and only if $x \in X_{\mathcal{H}}$. This shows that $Y[X_J(\mathbb{Z}, G)]$ is a subshift.

Now assume that G has decidable word problem, and \mathcal{F} is a computably enumerable set. Then the first condition of \mathcal{H} is decidable on patterns: Given a pattern p , we can compute the pattern $q = \pi_B \circ p: B(1_G, n) \rightarrow B$, and we already proved that \mathcal{J} is a decidable set. The second condition of \mathcal{H} is semi-decidable: Given p and $m \in \mathbb{N}$, we can compute the pattern r and semi-decide whether it lies in \mathcal{F} . It follows that \mathcal{H} is a computably enumerable set. ■

These constructions were introduced in [28] in the more general case where there is a finitely generated group H instead of \mathbb{Z} . We will write $X_J(H, G)$ and $Y[X_J(\mathbb{Z}, G)]$ with the same meaning as before, but only for reference purposes. It is natural to ask what properties are preserved by the map $Y \mapsto Y[X_J(H, G)]$ that sends a subshift on H to a subshift on G . The following is known:

- (1) In [28], Jeandel proved that when H is a finitely presented group, this map preserves weak aperiodicity, and the property of being empty/nonempty. This was used to show the existence of weakly aperiodic subshifts on new groups, and the undecidability of the emptiness problem for subshifts of finite type on new groups.
- (2) In [3], Barbieri proved that when H and G are amenable groups, the topological entropy h satisfies the formula $h(Y[X_J(H, G)]) = h(Y) + h(X_J(H, G))$. This was used to classify the entropy of subshifts of finite type on some amenable groups.

In the present paper, we are interested in the algorithmic complexity of subshifts. We already verified that $Y \mapsto Y[X_J(\mathbb{Z}, G)]$ preserves the property of being an effective subshift, which is folklore. In the following result we use Theorem 1.7 to show that $Y \mapsto Y[X_J(\mathbb{Z}, G)]$ also preserves the Medvedev degree of a subshift when J is big enough.

Theorem 5.22. *Let G be a finitely generated infinite group with decidable word problem, and suppose that J is big enough so that $T_J(\mathbb{Z}, G)$ contains an element as in Theorem 1.7. Then for every subshift $Y \subset A^{\mathbb{Z}}$,*

$$Y \equiv_{\mathfrak{M}} Y[X_J(\mathbb{Z}, G)].$$

Proof. Recall that we have a Medvedev reduction $Y \geq_{\mathfrak{M}} X$ when there a computable function Φ defined on all elements of Y , and with $\Phi(Y) \subset X$. Intuitively, this means that there is an algorithm which from any element in Y is able to compute an element in X . In our case, we will consider computable functions between the represented spaces $A^{\mathbb{Z}}$ and $(A \times B)^G$, in the sense of Subsection 5.4.

Let $*$ be a translation-like action as in Theorem 1.7, and let J be big enough so that $*$ lies in $T_J(\mathbb{Z}, G)$. We first prove the inequality $Y \geq_{\mathfrak{M}} Y[X_J(\mathbb{Z}, G)]$. The intuitive idea is as follows. Given an element $y \in Y$, we define an element $(z, x) \in Y[X_J(\mathbb{Z}, G)]$ by setting $x = x_*$ (a computable point of B^G because $*$ is a computable function), and on each orbit described by x_* , we copy the sequence y . The fact that $*$ has decidable orbit membership problem is fundamental: When we compute the new element $z \in A^G$, we need to know if two arbitrary group elements g, h can be colored independently (when they lie in different orbits by $*$), or the color of one of them determines the color of the other (when they lie in the same orbit by $*$).

Let $(g_n)_{n \in \mathbb{N}}$ be a computable numbering of G . We compute a set of representatives for orbits of $*$ as follows. Define a decidable set $I \subset \mathbb{N}$ by the condition that $n \in I$ when g_n is the first element in its own orbit that appears in the numbering $(g_n)_{n \in \mathbb{N}}$. This condition is decidable because $*$ has decidable orbit membership problem. Thus, $\{g_i \mid i \in I\}$ contains exactly one representative for each orbit of $*$.

We now define a computable function $\Psi_A: A^{\mathbb{Z}} \rightarrow A^G$ as follows. On input y , we define $\Psi_A(y)$ by the expression

$$\Psi_A(y)(g_i * n) = y(n), \quad i \in I, n \in \mathbb{Z}.$$

The sets $\{g_i * n \mid n \in \mathbb{Z}\}$ partition G when we range $i \in I$, and thus we defined $\Psi_A(y)(g)$ for all $g \in G$. To see that Ψ_A is a computable function, we exhibit a procedure that given $y \in A^{\mathbb{Z}}$ and $g \in G$, outputs $\Psi_A(y)(g)$. First, compute $i \in I$ such that g lies in the same orbit as g_i . This is possible as I is a decidable set, and $*$ has decidable orbit membership problem. Then we use the fact that the action $*$ is computable to find $n \in \mathbb{Z}$ satisfying $g = g_i * n$. Finally, output $y(n)$. As mentioned, this proves that Ψ_A is a computable function.

Let $\Psi_B: A^{\mathbb{Z}} \rightarrow B^G$ be the function with constant value x_* , which is a computable because x_* is a computable point. We define now a function $\Psi: A^{\mathbb{Z}} \rightarrow (A \times B)^G$ by $z \mapsto \Psi(z) = (\Psi_A(z), \Psi_B(z))$. The function Ψ is clearly computable, and we have $\Psi(Y) \subset Y[X_J(\mathbb{Z}, G)]$ by construction. This proves the desired inequality $Y \geq_{\mathfrak{M}} Y[X_J(\mathbb{Z}, G)]$.

The remaining inequality $Y[X_J(\mathbb{Z}, G)] \geq_{\mathfrak{M}} Y$ is clear. From an arbitrary element in $Y[X_J(\mathbb{Z}, G)]$ we can compute an element in Y : On input (z, x) we just have to follow the arrows from 1_G , read the A component of the alphabet, and the sequence obtained lies in Y . More formally, we define the function $\Phi: Y[X_J(\mathbb{Z}, G)] \rightarrow Y$ by the expression

$$\Phi(z, x)(n) = z(1_G * n), \quad n \in \mathbb{Z}.$$

It is clear from the expression above that Φ is a computable function. This proves the desired inequality $Y \geq_{\mathfrak{M}} Y[X_J(\mathbb{Z}, G)]$. ■

We are now ready to prove Theorem 1.8.

Proof of Theorem 1.8. By Proposition 5.14, the Medvedev degree of every effective subshift on G is a Π_1^0 degree. It follows that the class of Medvedev degrees of effective subshifts on G is contained in the class of Π_1^0 Medvedev degrees.

We now prove that every Π_1^0 Medvedev degree is attained by a subshift on G . Let $P \subset \{0, 1\}^{\mathbb{N}}$ be an effectively closed set. By Miller’s theorem [34, Proposition 3.1], there is an effective subshift on Y on \mathbb{Z} , such that $P \equiv_{\mathfrak{M}} Y$. Suppose that J is big enough so that $T_J(\mathbb{Z}, G)$ contains an element as in Theorem 1.7. Then the subshift $Y[X_J(\mathbb{Z}, G)]$ is effective by Proposition 5.19, and it satisfies $P \equiv_{\mathfrak{M}} Y[X_J(\mathbb{Z}, G)]$ by Theorem 5.22. This finishes the proof. ■

Our proof of Theorem 1.8 has made extensive use of the hypothesis of decidable word problem, and it is unclear whether a similar method could work for recursively presented groups.

Question 5.23. Let G be a recursively presented infinite group. Is it true that effective subshifts on G attain all Π_1^0 Medvedev degrees?

Despite we have not considered recursively presented groups here, it can be proved that for recursively presented groups, the Medvedev degrees of an effective subshift must be a Π_1^0 degree [4, Section 3].

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