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Subfactors and Applications

Organized by
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Abstract. The theory of subfactors plays an important role in the discovery and analysis of quantum symmetries that seem to be ubiquitous in mathematics and physics. Subfactors profoundly interact with a wide range of different areas such as quantum topology, vertex operator algebras, quantum groups, free probability theory, quantum computing and quantum information theory, quantum field theory, conformal field theory, tensor categories, condensed matter physics, and, of course, operator algebras.

The aim of this workshop was to bring together an international group of researchers from these fields to disseminate recent results and to stimulate new collaborations. The focus was on operator algebraic, vertex operator algebraic and categorical aspects of quantum symmetries and their applications to open questions in mathematical physics. A substantial group of young mathematicians attended the workshop and were given the opportunity to present their work.

Mathematics Subject Classification (2020): 17B37, 17B69, 18M05, 18M15, 18M20, 18M30, 18N10, 46L37, 46L60, 47L90, 57K16, 81P45, 81P68, 81R15, 81T05, 81T40.

Introduction by the Organizers

The workshop *Subfactors and Applications*, organized by Dietmar Bisch (Nashville), Terry Gannon (Edmonton), Yasuyuki Kawahgashi (Tokyo) and Yoshiko Ogata (Kyoto) was very well attended with 48 in-person and two virtual participants. A large group attended for the first time an Oberwolfach workshop as many young mathematicians were invited. This created an exciting ambience with many intense interactions among the participants.

The aim of this workshop was to bring together experts working in areas that feature various aspects of quantum symmetries. These include the theory of subfactors, fusion and modular tensor categories (MTCs), quantum spin chains, algebraic quantum field theory (AQFT), mathematical conformal field theory (CFT), topological field theory (TFT), topological quantum computing and topological phases (TQC), and the theory of vertex operator algebras (VOA). The workshop brought these communities closer together and led to many fruitful interactions.

One important theme of the workshop was recent work by different groups on quantum spin systems and fusion category symmetries on lattices. This is intimately related to the investigation of exotic states of matter, such as topological phases or topological order. The workshop featured several talks related to this theme, displaying various approaches to this important topic, including a more fusion categorical approach and a more operator algebraic approach. Progress in this direction was presented by Stefan Hollands, Corey Jones, Pieter Naaijkens, David Penneys, Makoto Yamashita and others. We also had talks by several physicists (Jeongwang Haah and Frank Verstraete) that explained how the categorical tools and constructions are used for investigating entanglement and other exotic properties of quantum systems.

The connection between vertex operator algebras (VOAs) and tensor categories was another important subject of the workshop. The talks by Sebastiano Carpi, Bin Gui and Robert McRae touched on the intriguing connections between tensor categories and VOAs. James Tener explained recent work with André Henriques that shows how to construct a unitary VOA from a conformal net. Cain Edie-Michell presented work on conformal embedding categories and subfactors. Terry Gannon animated an informal Q&A session that helped to understand the connection between VOAs, modular tensor categories and subfactors. It was well attended with many good questions and many good answers.

The third main theme of the workshop was recent results on subfactors and planar algebras. This included the discovery of new planar algebras associated to quadrilaterals of intermediate subfactors (talk by Junhwi Lim), and the Delannoy planar algebra (Noah Snyder). Julio Caceres described a construction of new commuting squares and how a graph planar algebra embedding theorem can be used to determine that the resulting hyperfinite subfactors have Temperley-Lieb-Jones standard invariants. Zhengwei Liu's work on classifying exchange relation planar algebras, and hence new subfactors, shows the power of the quantum Fourier analysis tools that he and his group have developed. Masaki Izumi presented work on group-subgroup subfactors that constitutes a far-reaching generalization of Goldman's theorem for index 2 subfactors. Intriguing ideas involving a more categorial approach to von Neumann algebras were presented in the talks of André Henriques, Theo Johnson-Freyd and David Reutter.

We dedicated one afternoon session to the presentation of research by participants at the Ph.D. and postdoctoral level. The topics were wide ranging. For instance, Futaba Sato presented her work on heat semigroups on certain quantum automorphism groups, Roberto Hernández Palomares talked about interesting new quantum graphs, and Sergio Girón Pacheco explained classification of actions of tensor categories on Kirchberg algebras. In total, we had seven short talks during this session.

For perspective, the workshop also covered a variety of other topics, such as Katrin Wendland's work on symmetries of K3 surfaces within Mathieu groups, Catherine Meusburger's approach to Dijkgraaf-Witten TFT with defects and Julia Plavnik's bicrossed product construction for fusion categories that generalizes the one for Kac algebras. Ingo Runkel talked about gauging of non-invertible symmetries in 3-dimensional TFT and Christoph Schweigert showed the usefulness of tensor network states. There were several other talks on related subjects that are listed below.

The workshop *Subfactors and Applications* was exciting and very successful in stimulating new interactions between the subfactor, tensor categories and VOA communities.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-2230648, "US Junior Oberwolfach Fellows".

Workshop: Subfactors and Applications

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Abstracts

An operator algebraic approach to fusion category symmetry on the lattices

Corey Jones
(joint work with David E. Evans)

We propose a framework for fusion category symmetry on the (1+1)D lattice in the thermodynamic limit by giving a formal interpretation of SymTFT decompositions. Our approach is based on axiomatizing physical boundary subalgebra of quasi-local observables, and applying ideas from algebraic quantum field theory to derive the expected categorical structures. We show that given a physical boundary subalgebra B of a quasi-local algebra A, there is a canonical fusion category that acts on A by bimodules and whose fusion ring acts by locality preserving quantum channels on the quasi-local algebra such that B is recovered as the invariant operators. We give a formal definition of a topological symmetric state, and prove a Lieb-Schultz-Mattis type theorem forcing gaplessness. Using this, we show that for any fusion category with no fiber functor there always exists gapless pure symmetric states on an anyon chain.

References

[1] D.E. Evans and C. Jones, *An operator algebraic approach to fusion category symmetry on the Lattice*, arXiv:2507.05185.

The symmetries of Kummer-like K3 surfaces within the Mathieu groups

Katrin Wendland
(joint work with Kasia Budzik, Anne Taormina, Mara Ungureanu and Ida G. Zadeh)

We report on the recent preprint [2], which is motivated by Mathieu Moonshine [3, 4, 5], an observation in conformal field theory which states that the largest Mathieu group M_{24} governs the elliptic genus of K3 surfaces and which stands unexplained to the very day. In a series of papers [13, 14, 15, 16], Taormina&KW have proposed a conjectural explanation for Mathieu Moonshine, building on the idea that the action of M_{24} combines the action of all symmetry groups of K3 surfaces; this proposal is now known by the name of *symmetry surfing*. It has been successfully applied to Kummer surfaces [15, 6], where it was shown that the symmetry groups of these special K3 surfaces naturally generate the action of a maximal subgroup of M_{24} on the Niemeier lattice of type A_1^{24}. We therefore expect that a better understanding of symmetry groups of other families of K3 surfaces will add a relevant piece to the puzzle.

In the project presented here, we have chosen to investigate $\mathbb{Z}_3$-orbifold limits of K3. For this class of Kummer-like K3 surfaces, our results provide a counterpart to the extensive studies by Nikulin and others [10, 11, 1, 7, 9] of the geometry and symmetries of classical Kummer surfaces. As such, the project is purely in geometry and is hoped to be of interest independently of Mathieu Moonshine as well.

The classical Kummer construction yields a family of K3 surfaces by means of $\mathbb{Z}_2$-orbifolding any two-dimensional complex torus. For $\mathbb{Z}_3$-orbifold limits of K3, one similarly uses the product T of two identical $\mathbb{Z}_3$-symmetric elliptic curves and performs a $\mathbb{Z}_3$-orbifold construction on T. It is well-known [12, 17, 1] that this yields a K3 surface $X = \widetilde{T/\mathbb{Z}_3}$, and we show that all its holomorphic symplectic automorphisms are induced by holomorphic symplectic automorphisms of the underlying complex torus T. This allows us to determine the group of holomorphic symplectic automorphisms of $\mathbb{Z}_3$-orbifold limits of K3, a group which is isomorphic to $(\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_4$.

In determining the symmetry group, a detailed analysis of the lattice of integral homology $H_2(X, \mathbb{Z})$ for $\mathbb{Z}_3$-orbifold limits of K3 is key. It can be constructed by means of gluing techniques from the lattice $\pi_* H_2(T, \mathbb{Z})^{\mathbb{Z}_3}$ that is pushed forward under orbifolding from the underlying complex torus T, on the one hand, and on the other hand, the lattice generated by the irreducible components of the exceptional divisor that occurs when blowing up the quotient singularities in $T/\mathbb{Z}_3$. Both lattices are non-primitively embedded in the integral homology of the K3 surface. The latter lattice is a root lattice of type A_2^9, and the smallest primitive sublattice P of $H_2(X, \mathbb{Z})$ containing it is the analog of the Kummer lattice Π that has been central to Nikulin's investigations of Kummer surfaces [10, 11]. The lattice P had been determined before [1, 18], building on the results of [12], where the smallest primitive sublattice K of $H_2(X, \mathbb{Z})$ that contains $\pi_* H_2(T, \mathbb{Z})^{\mathbb{Z}_3}$ was determined. We give an independent, purely geometric derivation of the form of the lattice K and thereby of the Kummer-like lattice P, which ultimately allows us to track the symmetries of X as automorphisms of $H_2(X, \mathbb{Z})$.

In keeping with our motivation from Mathieu Moonshine, in addition we track the symmetry group of $\mathbb{Z}_3$-orbifold limits of K3 in terms of permutation groups on 12 resp. 24 elements within the Mathieu groups M_{12} and M_{24}.

To do so, as a key step, we show that every symmetry of X is already uniquely determined by its action on the lattice P. Since we also prove that the Niemeier lattice N of type A_2^{12} is the (unique) Niemeier lattice that has a primitive sublattice isomorphic to $P(-1)$, this allows us to express the symmetry group in terms of lattice automorphisms of N. The automorphism group of N possesses a known, natural projection to the Mathieu group M_{12}, given by those permutations of the twelve factors of A_2^{12} that lift to automorphisms of N. Although the lifting of the symmetries of X to automorphisms of N requires choices, we can show that the projection to M_{12} yields an injection of the symmetry group into M_{12} which is independent of these choices.

The idea to express symmetries of K3 surfaces in terms of liftings to automorphisms of Niemeier lattices goes back to Kondo [8], who instead of the lattice P uses a lattice whose orthogonal complement is acted on trivially by the symmetries. As such, our idea is a variation of Kondo's lattice techniques that Taormina&KW introduced earlier in their study of the symmetries of Kummer surfaces and the genesis of their symmetry surfing programme. Indeed, for Kummer surfaces and the Kummer lattice Π the analogous observations have been key for being able to symmetry surf the moduli space of Kummer K3s, since they imply that Π faithfully carries the action of every symmetry group of a Kummer K3, and that the combined action can be lifted to an action on the Niemeier lattice of type A_1^{24} and thereby to the action of a subgroup of M_{24}.

Since M_{12} is a subgroup of M_{24}, as a proof of concept, we are also able to construct an embedding that yields the largest Mathieu group M_{24} when the symmetry group of $\mathbb{Z}_3$-orbifold limits of K3 is combined with all symmetries of Kummer surfaces. This last step, so far, remains ad hoc. It requires a justification through geometry which we leave for future work.

References

[1] J. Bertin, *Réseaux de Kummer et surfaces K3*, Invent. Math. **93** (1988) 267

[2] K. Budzik, A. Taormina, M. Ungureanu, K. Wendland, I.G. Zadeh, *Tracking the symmetries of $\mathbb{Z}_3$-orbifold K3s within the Mathieu groups*, arXiv:2504.16248 [math.AG]

[3] T. Eguchi, H. Ooguri and Y. Tachikawa, *Notes on the K3 surface and the Mathieu group M_{24}*, Experimental Mathematics **20** (2011) 91, arXiv:1004.0956

[4] M. Gaberdiel, S. Hohenegger and R. Volpato, *Symmetries of K3 sigma models*, Commun. Number Theory Phys. **6** (2012) 1, arXiv:1106.4315

[5] T. Gannon, *Much ado about Mathieu*, Advances in Mathematics **301** (2016) 322, arXiv:1211.5531

[6] C.A. Keller and I.G. Zadeh, *Lifting 1/4-BPS states on K3 and Mathieu moonshine*, Communications in Mathematical Physics **377** (2020) 225, arXiv:1905.00035

[7] J.H. Keum, *Automorphisms of Jacobian Kummer surfaces*, Compositio Math. **107** (1997) 269

[8] S. Kondō, *Niemeier lattices, Mathieu groups and finite groups of symplectic automorphisms of K3 surfaces*, Duke Math. J. **92** (1998) 593

[9] S. Kondō, *The automorphism group of a generic Jacobian Kummer surface*, J. Algebraic Geom. **7** (1998) 589

[10] V. Nikulin, *On Kummer surfaces*, Math. USSR Isv. **9** (1975) 261

[11] V. Nikulin, *Finite automorphism groups of Kähler K3 surfaces*, Trans. Mosc. Math. Soc. **38** (1980) 71

[12] T. Shioda and H. Inose, *On singular K3 surfaces*, in Complex Analysis and Algebraic Geometry, W. Bailey and T. Shioda, eds., pp. 119–136, Cambridge Univ. Press (1977)

[13] A. Taormina and K. Wendland, *The overarching finite symmetry group of Kummer surfaces in the Mathieu group M_{24}*, JHEP **08** (2013) 125, arXiv:1107.3834

[14] A. Taormina and K. Wendland, *A twist in the M_{24} moonshine story*, Confluentes Mathematici **7** (2015) 83, arXiv:1303.3221

[15] A. Taormina and K. Wendland, *Symmetry-surfing the moduli space of Kummer K3s*, in Proc. of the Conference String-Math 2012, no. 90 in Proceedings of Symposia in Pure Mathematics, pp. 129–153, 2015, arXiv:1303.2931

[16] A. Taormina and K. Wendland, *Not doomed to fail*, Journal of High Energy Physics **2018** (2018) 1, arXiv:1708.01563
[17] M. Walton, *The Heterotic String on the Simplest Calabi-Yau Manifold and Its Orbifold Limits*, Phys. Rev. D **37** (1988) 377
[18] K. Wendland, *Consistency of orbifold conformal field theories on K3*, Adv. Theor. Math. Phys. **5** (2002) 429, arXiv:hep-th/0010281

Tensor network states, spherical Morita contexts and extruded graphs

Christoph Schweigert
(joint work with Julian Farnsteiner, Jürgen Fuchs, César Galindo, Jutho Haegeman, David Jaklitsch, Laurens Lootens, Frank Verstraete)

Projected entangled pair states (PEPS) and matrix product operators (MPO) are standard tools for the description of states in quantum information theory and quantum many-body physics. Following [3], we explained how a PEPS tensor and an MPO tensor can be associated to a pair consisting of a (spherical) fusion category and an appropriate module category over it. We then showed that spherical module categories provide the appropriate class of module categories. These notions find their natural home in the theory of spherical Morita contexts which was developed in [2] and applies beyond fusion categories to general finite categories.

Following [3], we then demonstrated that the contraction of PEPS and MPO tensors can be understood in terms of Turaev-Viro models on manifolds with boundary. Together with the calculus of extruded graphs developed in [1], this insight can be used to find generalizations of the standard MPO tensors.

References

[1] J. Farnsteiner, C. Schweigert, *The evaluation of graphs on surfaces for state-sum models with defects*, SIGMA 20 (2024) **102**, arXiv:2312.01946 [math.QA]
[2] J. Fuchs, C. Galindo, D. Jaklitsch and C. Schweigert, *Spherical Morita contexts and relative Serre functors*, Kyoto J. Math. 65(3), 537-594
[3] L. Lootens, J. Fuchs, J. Haegeman, C. Schweigert and F. Verstraete, *Matrix product operator symmetries and intertwiners in string-nets with domain walls*, SciPost Phys. **10**, 053 (2021), arXiv:2008.11187 [quant-ph]

Geometric formulation of Dijkgraaf-Witten TFT with defects

Catherine Meusburger

Dijkgraaf-Witten TQFT can be viewed as a gauge theory based on a finite group and is a special case of Turaev-Viro-Barrett-Westbury TQFT. We give a purely geometric formulation of Dijkgraaf-Witten TQFT with defects of all codimensions. It describes the stratification in terms of a graded graph and is formulated in terms of functors from the fundamental groupoids of different strata to groupoids constructed from the defect data. The strata of codimension 2 and 3 give rise to groupoid representations and intertwiners between them. This leads to an accessible description of Dijkgraaf-Witten TQFT with defects that allows for direct computations of invariants of stratified manifolds. We apply this to describe defects in Kitaev's quantum double model.

This is joint work with Joao Faria Martins [1].

References

[1] J. Faria Martins, C. Meusburger, A geometrical description of untwisted Dijkgraaf-Witten TQFT with defects. arXiv preprint arXiv:2410.18049.

Complete W*-categories, and a new notion of adjoint functor

André Henriques
(joint work with Dave Penneys, Nivedita)

W*-categories are the 'many object' versions of W*-algebras (a.k.a. von Neumann algebras). They were introduced by Ghez, Lima and Roberts [2], but have not been the subject of much in depth investigation since then [3] [1].

Our main observation is that complete W*-categories behave in many ways like Hilbert spaces. Most notably, every W*-category C admits a canonical sesquilinear functor

$$\langle\ ,\ \rangle_{\mathrm{Hilb}} : \overline{C} \times C \to \mathrm{Hilb},$$

first studied in [3], which we rename the 'Hilb-valued inner product'. Given complete W*-categories C and D, there is an antilinear equivalence of categories

$$\dagger : \mathrm{Func}(C, D) \xrightarrow{\cong} \mathrm{Func}(D, C)$$

called *adjoint*, characterised by the existence of unitary natural isomorphisms

$$\langle c, F^{\dagger}(d)\rangle_{\mathrm{Hilb}} \simeq \langle F(c), d\rangle_{\mathrm{Hilb}}.$$

This is genuinely distinct from the usual notion of adjoint functors. There is an extensive list of analogies between Hilbert spaces and complete W*-categories.

References

[1] Ulrich Bunke and Alexander Engel. Additive C*-categories and K-theory. arXiv:2010.14830, 2020.
[2] P. Ghez, R. Lima, and J. E. Roberts. W *-categories. Pacific J. Math., 120(1):79–109, 1985. MR808930.
[3] Shigeru Yamagami. Notes on operator categories. J. Math. Soc. Japan, 59(2):541–555, 2007. MR2325696.

von Neumann algebras are O(2) fixed points...somehow

Theo Johnson-Freyd, David Reutter

A (Wick-rotated) quantum field theory can answer questions like "what is the expectation value of ...?" and "what are the allowed local insertions at ...?" We are interested in QFTs with a choice of boundary condition. Some axioms, in the framed 2D TQFT case:

- "Questions" are expressed by pictures.
- "Pictures" are 2D manifolds some of whose boundary strata are *unglazed*, where you can glue the pictures together, whereas some boundary strata are *glazed*, where glue doesn't stick.
- Questions about expectation values are answered by (complex, say) numbers and questions about allowed local insertions are answered by (complex, say) vector spaces.
- All strata are framed. Near the glazed boundary, the framing is product. (Near unglazed boundaries the framing is unconstrained.

For example, point to a spot along the glazed boundary, and ask: "What are all the valid local insertions at this boundary point, with this (product) framing?" The answer will be automatically an associative unital algebra A. A different space: after removing the point of insertion, the framing could rotate as it goes around that point by $n \in \mathbb{Z}$ full rotations; the space A_n of local insertions with n units of framing vortex then becomes an A-A-bimodule. Moreover, $\bigoplus_{n\in\mathbb{Z}} A_n$ becomes a $\mathbb{Z}$-graded algebra, and it carries a degree-(-1) trace map $\mathrm{tr} : A_1 \to \mathbb{C}$.

Another vitally important axiom in physics is called *Remote Detectability.* In slogans, it says: "Anything that is consistent will other operator insertions is a valid operator." In the TQFT case, this implies that various pairings are perfect, and that various natural gluing maps are isomorphisms. In the case of a framed 2D TQFT with a boundary condition, it implies that the map $A_m \otimes_A A_n \to A_{m+n}$ is an isomorphism for all $m, n \in \mathbb{Z}$, so that in particular A_1 is invertible; and it implies that the pairing $A_1 \to A^\vee$ induced by tr is an isomorphism. In other words, A becomes a finite-dimensional separable algebra and A_1 becomes its Serre bimodule.

Actual quantum field theory is not framed. To unorient the story, one can work with pictures with no trangential structure (and whose "no tangential structure" is product near glazed boundaries). Compared to the framed story, this supplies extra isomorphisms, which upgrade A from a finite-dimensional separable algebra to a $*$-Frobenius algebra. Algebrotopologically, one can identify a certain "relative $\mathrm{O}(1) \hookrightarrow \mathrm{O}(2)$" action on the "relative space" {finite-dimensional separable associative algebras and isomorphisms} $\to$ {finite-dimensional separable associative algebras and Morita equivalences}, and match the axioms of $*$-Frobenius algebra exactly to the axioms of being a "relative $\mathrm{O}(1) \hookrightarrow \mathrm{O}(2)$" fixed point. Compare [SP09].

Actual quantum field theory also is not topological. The rule remains: near glazed boundaries, local structures are required to be product; for example, we allow only product metrics near the boundary. In favourable circumstances, one can hope to take a "deep UV limit" first in the space direction and then the time direction (and in the process breaking all manifest Lorentz invariance) and produce a "flat-space QFT with vanishing energy-momentum tensor"; from this, one can still hope to extract an associative algebra A of boundary insertions, but there is not enough room in the slide of your UV microscope to have framing changes. If one also allows the $\mathrm{O}(1)^2$-many reflections in space and in time, then one can produce a bit more: A becomes a $*$-algebra; there is a vector space H of "states" that can be inserted at the end of a long strip; H is an A-bimodule; H has a symmetric bilinear form with respect to which the action of A on H is a $*$-action.

Another vitally important axiom in physics is called *Reflection Positivity.* The boring easy part of this axiom is that reflections of questions act antiunitarily on answers. The interesting difficult part of this axiom says: "If a closed diagram admits a reflection symmetry, then its value is positive." Probably there should be also an axiom for open diagrams, but there is not yet consensus what a "positive vector space" really is. For example, this implies that the natural sesquilinear pairing on H built from a strip with insertions at both ends is necessarily positive-(semi)definite. On the other hand, Remote Detectability (and the memory of the topology on $\mathbb{R}$) then comes in and says that H with this pairing is necessarily Hilbert.

Build a strip with H's at both ends and some A's along the edges. These supply closed diagrams, and hence $\mathbb{C}$-valued maps on tensor products of A's and H's. Equivalently, they supply maps $A \to (\text{tensor product of } A\text{'s and } H\text{'s})^\vee$. These maps are not independent: they are related by fusion of operators. So there is some large diagram, and a map $A \to (\mathrm{colim}(\text{diagram}))^\vee = \lim((\text{diagram})^\vee)$. This limit ends up computing $\mathrm{End}_{\mathrm{RMod}(A)}(H_A)$, the algebra of endomorphisms of H thought of as a right A-module. Remote detectability then says: *This map* $A \to \mathrm{End}_{\mathrm{RMod}(A)}(H_A)$ *is an isomorphism!* But H is Hilbert. So A is von Neumann.

There is (probably) not a physical construction that takes a 2D QFT and produces a "deep UV theory" which is both Lorentz-invariant and has vanishing energy-momentum tensor: the deep UV, if it exists, is likely nontrivially conformal. But let's suppose that someone one does have a reflection-positive Lorentz-invariant (UV) theory on flat space with vanishing energy-momentum tensor. The Wick-rotated version of (reflection-positive) Lorentz invariance is $\mathrm{O}(2)$-symmetry. The rule remains: every stratum must be flat; near glazed boundary, the flat structure must be product. But there are now valid questions about spaces of operator insertions at corners of nontrivial angle θ. Remote detectability now says not that $A \to \mathrm{End}_{\mathrm{RMod}(A)}(H_A)$ is an iso, but rather that some map $A \to \lim((\text{much larger diagram})^\vee)$ is an iso.

On the other hand, we like von Neumann algebras. An idea going back to Segal [Seg04] suggests: given a von Neumann algebra A, assign to angle $\theta \in [0, \pi]$ the noncommutative L^p space $L^{2\pi/\theta}(A)$. Remarkably, this works: a theorem of Pavlov from [Pav17] is equivalent to the statement that *These assignments satisfy*

(Reflection Positivity and) Remote Detectability. In other words, von Neumann algebras have a secret O(2)-symmetry.

What is the "O(2)-action" on algebras that von Neumann algebras are the fixed points of? $L^p(A)$ is not invertible as an A-bimodule, meaning only "part" of O(2) acts; how to formulate "noninvertible O(2)-actions"? The L^p-space "$L^{2\pi/\theta}$" is only sensible when θ is convex (positive); what physics rules out concave angles? Are there other reflection-positive remote-detectable flat-space Lorentz-invariant 2D QFTs with boundary, or are von Neumann algebras the exact classification?

We have no new results or solutions to these puzzles. Our hope is that, by raising these puzzles, we will find a story that extends to higher dimensions. Perhaps, for example, there will be "von Neumann 2-algebras" (maybe bicommutant categories of [Hen17, HP23]?) that are selected by an $\mathrm{O}(1)^3$-symmetry but are secretly O(3)-symmetric.

References

[Hen17] A. Henriques. What Chern–Simons theory assigns to a point. *Proceedings of the National Academy of Sciences*, 114(51):13418–13423, 2017.

[HP23] A. Henriques and D. Penneys. Representations of fusion categories and their commutants. *Selecta Math. (N.S.)*, 29(3):Paper No. 38, 45, 2023.

[Pav17] D. Pavlov. Algebraic tensor products and internal homs of noncommutative L^p-spaces. *Journal of Mathematical Analysis and Applications*, 456(1):229–244, 2017.

[Seg04] G. Segal. The definition of conformal field theory. In *Topology, geometry and quantum field theory*, vol. 308 of *London Math. Soc. Lecture Note Ser.*, pp. 421–577. Cambridge Univ. Press, Cambridge, 2004.

[SP09] C. J. Schommer-Pries. *The classification of two-dimensional extended topological field theories.* ProQuest LLC, Ann Arbor, MI, 2009.

Teichmüller modular forms and holomorphic vertex operator algebras

Sebastiano Carpi

(joint work with Giulio Codogni)

Classical modular forms are holomorphic functions on the upper half plane $\mathbb{H}$ satisfying certain functional equations related to the action of the modular group $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}$ [9]. They are deeply related to the geometry of the moduli space of genus one compact complex curves. Vertex operator algebras (VOAs) give a mathematical description of chiral two-dimensional conformal field theories (chiral CFTs) [3]. One of the most remarkable and intriguing feature of VOAs is their relationship with modular forms.

VOAs with trivial representation theory are called holomorphic. They give a special important class of (modular invariant) two-dimensional CFTs. Their genus one partition functions give rise to classical modular forms. A very important example of holomorphic VOA is given by the moonshine VOA $V^\natural$ having central charge $c = 24$ and automorphism group isomorphic to the monster finite simple group $\mathbb{M}$, see [3, 5]. A central open problem in VOA theory is the conjecture on the uniqueness of the moonshine VOA [3] that can be stated in the following way: if $V = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} V_n$ is a holomorphic VOA with central charge $c = 24$ and weight-one subspace $V_1 = \{0\}$ then V is isomorphic to the moonshine $V^\natural$.

Teichmüller modular forms are higher genus generalizations of classical modular forms [7]. For for genus $g > 1$ the upper half plane is replaced by the Teichmüller space T_g and the higher genus modular group Γ_g is the (pure) mapping class group. The quotient T_g/Γ_g is the moduli stack $\mathcal{M}_g$ of genus g complex compact curves. Examples comes from the genus g theta series Siegel modular forms Θ_L associated to an unimodular even positive-definite lattice L.

In [1] we study the connection between holomorphic VOAs and Teichmüller modular forms and analyze various consequence of this connection. This is done by means of the concept of genus g partition functions. For a general two-dimensional CFT defined through functional integrals of fields living on Riemann surfaces, provided that the latter functional integrals make sense and have the right mathematical properties. In the influential paper [4] Friedan and Shenker argued that a two-dimensional CFT is completely determined by the collection $\mathcal{Z} = \{\mathcal{Z}_g : g \in \mathbb{Z}_{\geq 0}\}$ of all genus g partition functions and moreover, that this allows a description of the CFT in terms of the geometry of the moduli spaces $\mathcal{M}_g$.

In the case of a holomorphic VOA one can define all the genus g partition functions as formal power series without the need of functional integrals. A central result in [1] is the following: if V is a holomorphic VOA with central charge c then, for every non-negative integer g, the genus g partition function of V gives rise in a natural way to a non-zero Teichmüller modular form of weight $c/2$ unique up to a multiplication by a complex number. This can be seen as a generalization of the construction of Teichmüller modular forms from unimodular lattices, cf. [10]. Moreover, it gives strong constraints on the partition functions of holomorphic VOAs. As an example we prove that the validity of a weak form of the Harris-Morrison slope conjecture about the geometry of the moduli spaces of compact complex curves would imply that if V is a holomorphic VOA with central charge $c = 24$ and $V_1 = \{0\}$ then $\mathcal{Z}_V = \mathcal{Z}_{V^\natural}$.

Another result in [1] is the clarification of the relation between unitary holomorphic VOAs having the same genus g partition function for all g in connection with the conjecture in [4]. To this end, for any unitary holomorphic VOA V, we define unitary vertex operator subalgebra PV of V naturally associated with the collection $\mathcal{Z}_V = \{\mathcal{Z}_{V,g} : g \in \mathbb{Z}_{\geq 0}\}$ of all genus g partition functions of V. It turns out that $PV \subset V^{\mathrm{Aut}(V)}$.

If V and U are unitary we show that $\mathcal{Z}_V = \mathcal{Z}_U$ if and only if there is a unitary operator $\Phi : V \to U$ restricting to a vertex operator algebra isomorphism $\phi : PV \to PU$ and such that $\Phi Y^V(a, z) = Y^U(\phi(a), z)\Phi$ for all $a \in PV$. In particular, if the PV-module V has a unique VOA structure then U and V must be isomorphic. These relations can be also understood in terms of certain subfactors arising from inclusions of unitary VOAs through their correspondence with conformal nets [2]. This is because, if V is strongly local in the sense of [2] then the inclusion $PV \subset V$ gives rise to an inclusion of conformal nets $\mathcal{A}_{PV} \subset \mathcal{A}_V$ and to a corresponding irreducible subfactor $\mathcal{A}_{PV}(I) \subset \mathcal{A}_V(I)$ for any interval $I \subset S^1$.

These results open new perspectives on the conjectured uniqueness of the moonshine VOA and relate it to other important conjectures in different areas of mathematics. Assume for example that $PV^\natural$ coincide with the monster orbifold $V^{\natural^{\mathbb{M}}}$ and that the latter is strongly rational. Then, the uniqueness of $V^\natural$ would follow from the weak Harris-Morrison slope conjecture together with the results in [8].

References

[1] S. Carpi, G. Codogni *Vertex operator algebras, partition functions and Teichmüller modular forms*, In prepration.

[2] S. Carpi, Y. Kawahigashi, R. Longo, M. Weiner *From vertex operator algebras to conformal nets and back* Memoirs of the American Mathematical Society **254**(1213) (2018), vi+85 pp.

[3] I. Frenkel, J. Lepowsky, A. Meurman *Vertex operator algebras and the monster*, Academic Press, 1989.

[4] D. Friedan, S. Shenker *The analytic geometry of two-dimensional conformal field theory*, Nuclear Physics B **281** (1987), 509–545.

[5] T. Gannon *Moonshine beyond the monster. The bridge connecting algebra, modular forms and physics*, Cambridge University Press, Cambridge, 2006.

[6] J. Harris, I. Morrison *Slopes of effective divisors on the moduli space of stable curves*, Inventiones Mathematicae **99** (1990), 321–355.

[7] T. Ichikawa *On Teichmüller modular forms*, Mathematische Annalen **299** (1994), 731–740.

[8] A. Kirillov, V Ostrik *On a q-analogue of the McKay correspondence and the ADE classification of $\widehat{\mathfrak{sl}}_2$ conformal field theories*, Advances in Mathematics **171**(2) (2002), 183–227.

[9] J.-P. Serre *A course in arithmetic*, Springer, New York, 1973.

[10] M. P. Tuite, M. Welby. *Genus g Zhu recursion for vertex operator algebras and their modules*, Journal of Algebra and its Applications `https://doi.org/10.1142/S0219498826500763`

A 1d gapped boundary of a chiral theory over a 2d invertible subalgebra

Jeongwan Haah

We discuss a gapped quantum Hamiltonian in $2 + 1$d with a gapped boundary where the anyon theory of the bulk is chiral without any nontrivial boson. The example crucially uses a subalgebra of local operators that is not an infinite tensor product of matrix algebras [1].

References

[1] J. Haah, *Invertible subalgebras*, Commun. Math. Phys. **403** (2023), 661–698.

Holography for bulk-boundary local topological order

David Penneys
(joint work with Corey Jones, Pieter Naaijkens, and Daniel Wallick)

This talk was based on two articles:

- *Local topological order and boundary algebras* by Corey Jones, Pieter Naaijkens, David Penneys, and Daniel Wallick [JNPW23], and
- *Holography for bulk-boundary local topological order* by Corey Jones, Pieter Naaijkens, and David Penneys [JNP25]

Local topological order axioms. We begin with an *abstract quantum system* on a lattice $\mathcal{L}$, typically assumed to be $\mathbb{Z}^d$. We have a single unital C*-algebra A called the *quasi-local algebra*, and to each d-dimensional rectangle $\Lambda \subset \mathcal{L}$, we have a unital C*-algebra $A(\Lambda) \subset A$ satisfying the following axioms:

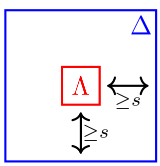
- $A(\emptyset) = \mathbb{C}1_A$,
- $\Lambda \subset \Delta$ implies $A(\Lambda) \subset A(\Delta)$,
- $\Lambda \cap \Delta = \emptyset$ implies $[A(\Lambda), A(\Delta)] = 0$ in A, and
- $\bigcup_\Lambda A(\Lambda)$ is norm-dense in A.

This net of algebras is equipped with a *net of projections* $p_\Lambda \in A(\Lambda)$ for each Λ satisfying $p_\Delta \leq p_\Lambda$ whenever $\Lambda \subset \Delta$; these projections are used instead of any local Hamiltonian.

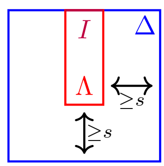
We give a rough sketch of the *local topological order* (LTO) axioms for (A, p); for the precise version, see [JNPW23].

- Whenever $\Lambda \ll \Delta$ (Δ sufficiently completely surrounds Λ), $p_\Delta A(\Lambda) p_\Delta = \mathbb{C}p_\Delta$, and
- Whenever $\Lambda \Subset \Delta$ (Δ sufficiently completely surrounds Λ on all but one side), there is an algebra $B(I)$ where $I = \partial\Lambda \cap \partial\Delta$ which is supported on sites near I such that $p_\Delta A(\Lambda) p_\Delta = B(I) p_\Delta$. This algebra $B(I)$ is independent of $\Lambda \subset \Delta$ beyond that $I = \partial\Lambda \cap \partial\Delta$.

$\Lambda \ll \Delta$:



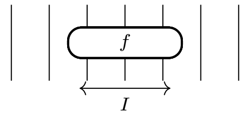
$\Lambda \Subset \Delta$:



One gets a canonical pure state on A by the formula $p_\Delta x p_\Delta = \psi(x) p_\Delta$ for $x \in A(\Lambda)$ and $\Lambda \ll \Delta$. Choosing a half-space $\mathbb{H} \subset \mathcal{L}$ and setting $A(\mathbb{H}) = \varinjlim_{\Lambda \subset \mathbb{H}} A(\Lambda)$, we get a net of boundary algebras $B = \varinjlim B(I)$ on $\partial\mathbb{H}$, together with a quantum channel $\mathbb{E} : A(\mathbb{H}) \to B$ defined by $p_\Delta x p_\Delta = \mathbb{E}(x) p_\Delta$ for $x \in A(\Lambda)$ and $\Lambda \Subset \Delta \subset \mathbb{H}$ with $\partial\Lambda \cap \partial\Delta = I$. One should think of this boundary algebra B as living on a *physical cut/boundary* of our abstract quantum spin system.

Examples of LTOs include almost all known topologically ordered commuting projector lattice models, including Kitaev's Toric Code and Quantum Double [Kit97, Kit03], the Levin-Wen model [LW05], and the Walker-Wang model [WW12]. In these cases, the boundary algebra can be described by a *fusion spin* *chain*, which is an abstract 1D spin chain built from a unitary fusion category (UFC) $\mathcal{C}$ and a strong tensor generator $X \in \mathcal{C}$. The local algebras are

$$B(I) \cong \mathrm{End}_{\mathcal{C}}(X^{\otimes I})$$

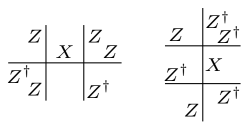


and inclusion is given by tensoring with copies of id_X as appropriate. One the recovers the bulk topological order by looking at C. Jones' *DHR bimodules* for this 1D fusion spin system [Jon24].

For the Walker-Wang model built from a unitary braided fusion category $\mathcal{B}$, the boundary algebra is a 2D net of algebras built from $\mathcal{B}$ and our strong tensor generator $X \in \mathcal{B}$ called a *braided fusion spin system.* Since $\mathcal{B}$ is braided, it makes sense to take a tensor product of objects at points in a 2D plane, so for each 2D rectangle I, we can define $\mathrm{End}_{\mathcal{B}}(X^{\otimes I})$. As an aside, we remark that this net of algebras has a canonical state corresponding to tensor powers of the map $1_{\mathcal{B}} \to X$, and in this state, the category of superselection sectors of this net of von Neumann algebras following [BBC+25] is equivalent to the completion (in the sense of [HNP24]) to $\mathcal{B}$ as a W*-category. We make the following conjectures, the second pointed out to us by C. Jones.

Conjecture 1. *The superselection sectors for this net is equivalent to the completion of $\mathcal{B}$ as a* W* *braided tensor category.*

Conjecture 2. *The braided fusion spin system for $SU(3)_1$ (or possibly its reverse) is bounded-spread isomorphic to a net of algebras from* [Haa23, Fig. 1] *discussed by Jeongwan Haah in his Oberwolfach talk given just prior to this talk, i.e., the net of algebras with local generators*

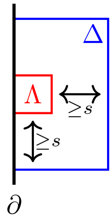


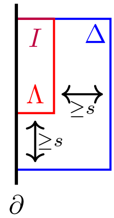
$$, \qquad X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}, \; \zeta = \exp(2\pi i/3)$$

on a 2D edge lattice with $\mathbb{C}^3$ spins on each edge.

Introducing a topological boundary. One can now introduce a *topological boundary* for our lattice; the term 'topological' here is used to distinguish this boundary from the 'physical' boundary cut obtained from choosing a half-space. The LTO axioms in the presence of a topological boundary are almost identical to the previous LTO axioms. Roughly speaking (for a precise version see [JNP25]),

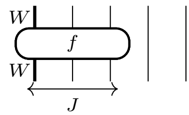
- Whenever $\Lambda \ll^{\partial} \Delta$, $p_\Delta A(\Lambda) p_\Delta = \mathbb{C} p_\Delta$, and
- Whenever $\Lambda \Subset^{\partial} \Delta$, there is an algebra $B^{\partial}(I)$ similar to $B(I)$ such that $p_\Delta A(\Lambda) p_\Delta = B^{\partial}(I) p_\Delta$.

$\Lambda \ll^{\partial} \Delta$: 

$\Lambda \Subset^{\partial} \Delta$: 

Again, one gets a net of algebras on the physical cut/boundary at the edge of a chosen half-space. Following [KK12], one gets topological boundaries for the 2D LTOs from our models above via module categories $\mathcal{M}$ for the corresponding UFCs $\mathcal{C}$, together with a choice of strong module generator $W \in \mathcal{M}$. For these topological boundaries, the boundary algebra is the a *fusion module spin chain*, where sites meeting the topological boundary give rise to the local algebras

$$B^{\partial}(J) \cong \mathrm{End}_{\mathcal{M}}(W \lhd X^{\otimes J-1})$$



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We then define a notion of boundary DHR-bimodule meant to capture the topological boundary excitations. We use subfactor theory to prove that that for these fusion module spin chains, the boundary DHR bimodules give exactly the dual category $\mathrm{End}(\mathcal{M}_{\mathcal{C}})$.

For our 3D Walker-Wang model from a UBFC $\mathcal{B}$, we get a 2D topological boundary from a untiary module tensor category [HPT16], i.e., a UFC $\mathcal{C}$ equipped with a unitary braided central functor $\mathcal{B} \to Z(\mathcal{C})$. Indeed, the half-braiding for $\mathcal{B}$ with $\mathcal{C}$ is exactly the data needed to attach the 3D $\mathcal{B}$-Walker-Wang model to the 2D $\mathcal{C}$-Levin-Wen model [HBJP23, GHK+24]. For such a 2D topological boundary, the category of boundary DHR bimodules has a canonical braiding as in [Jon24]. We use a folding trick and our 2D result for fusion module spin chains to prove that the 2D topological boundary excitations is equivalent to the *enriched center/Müger centralizer* $Z^{\mathcal{B}}(\mathcal{C}) = \mathcal{B}' \subset Z(\mathcal{C})$ [KZ18]. Using a 3D folding trick for our original 3D Walker-Wang model for $\mathcal{B}$, we get a 2D topological boundary labeled by $\mathcal{B}$ for the 3D bulk labelled by $\mathcal{B} \boxtimes \mathcal{B}^{\mathrm{rev}}$. Applying the above result, we see that the ordinary DHR bimodules for the 3D model is given by $Z^{\mathcal{B} \boxtimes \mathcal{B}^{\mathrm{rev}}}(\mathcal{B}) \cong Z_2(\mathcal{B})$, the Müger center of $\mathcal{B}$.

References

[BBC+25] Anupama Bhardwaj, Tristen Brisky, Chian Yeong Chuah, Kyle Kawagoe, Joseph Keslin, David Penneys, and Daniel Wallick. Superselection Sectors for Posets of von Neumann Algebras. *Comm. Math. Phys.*, 406(8):Paper No. 185, 2025. MR4927814 DOI:10.1007/s00220-025-05315-4 arXiv:2410.21454.

[GHK+24] David Green, Peter Huston, Kyle Kawagoe, David Penneys, Anup Poudel, , and Sean Sanford. Enriched string-net models and their excitations. *Quantum*, 8:1301, 2024. DOI:10.22331/q-2024-03-28-1301 arXiv:2305.14068.

[Haa23] Jeongwan Haah. Invertible subalgebras. *Comm. Math. Phys.*, 403(2):661–698, 2023. MR4645726 DOI:10.1007/s00220-023-04806-6 arXiv:2211.02086.

[HBJP23] Peter Huston, Fiona Burnell, Corey Jones, and David Penneys. Composing topological domain walls and anyon mobility. *SciPost Phys.*, 15(3):Paper No. 076, 85, 2023. MR4640433 doi10.21468/SciPostPhys.15.3.076 arXiv:2208.14018.

[HNP24] André Henriques, Nivedita, and David Penneys. Complete W*-categories. arXiv:2411.01678, 2024.

[HPT16] André Henriques, David Penneys, and James Tener. Categorified trace for module tensor categories over braided tensor categories. *Doc. Math.*, 21:1089–1149, 2016. MR3578212 DOI:10.4171/DM/553 arXiv:1509.02937.

[JNP25] Corey Jones, Pieter Naaijkens, and David Penneys. Holography for bulk-boundary local topological order. arXiv:2506.19969, 2025.

[JNPW23] Corey Jones, Pieter Naaijkens, David Penneys, and Daniel Wallick. Local topological order and boundary algebras. arXiv:2307.12552, 2023.

[Jon24] Corey Jones. DHR bimodules of quasi-local algebras and symmetric quantum cellular automata. *Quantum Topol.*, 15(3):633–686, 2024. MR4814692 DOI:10.4171/qt/216 arXiv:2304.00068.

[Kit97] A. Yu. Kitaev. Quantum computations: algorithms and error correction. *Uspekhi Mat. Nauk*, 52(6(318)):53–112, 1997. MR1611329.

[Kit03] A. Yu. Kitaev. Fault-tolerant quantum computation by anyons. *Ann. Physics*, 303(1):2–30, 2003. MR1951039 DOI:10.1016/S0003-4916(02)00018-0 arXiv:quant-ph/9707021.

[KK12] Alexei Kitaev and Liang Kong. Models for gapped boundaries and domain walls. *Comm. Math. Phys.*, 313(2):351–373, 2012. MR2942952 DOI:10.1007/s00220-012-1500-5 arXiv:1104.5047.

[KZ18] Liang Kong and Hao Zheng. Drinfeld center of enriched monoidal categories. *Adv. Math.*, 323:411–426, 2018. 1704.01447 DOI:10.1016/j.aim.2017.10.038 arXiv:1704.01447.

[LW05] Michael A. Levin and Xiao-Gang Wen. String-net condensation: A physical mechanism for topological phases. *Phys. Rev. B*, 71:045110, Jan 2005. doi10.1103/PhysRevB.71.045110 arXiv:cond-mat/0404617.

[WW12] Kevin Walker and Zhenghan Wang. (3+1)-tqfts and topological insulators. *Frontiers of Physics*, 7(2):150–159, 2012. DOI:10.1007/s11467-011-0194-z arXiv:1104.2632.

Geometry of 2d Finite Logarithmic CFT

Bin Gui

In recent years, I have been collaborating with Hao Zhang on the study of the vertex operator algebra (VOA) aspects of 2d logarithmic chiral conformal field theory. Specifically, in a series of papers [GZ23, GZ24, GZ25a], we have developed a systematic framework for understanding conformal blocks associated with C_2-cofinite VOAs that are not necessarily rational or self-dual.

The theory we have constructed—both in our finished work and ongoing research —currently stands as the only comprehensive approach capable of systematically addressing the following questions:

(1) Zhu's celebrated theorem on the modular invariance of characters for C_2-cofinite rational VOAs does not extend to irrational VOAs. In [Miy04], Miyamoto introduced pseudotraces of VOA modules, which do satisfy modular invariance. While the traditional characters—defined via trace—are well-aligned with Segal's geometric framework for conformal field theory, pseudotraces arise from intricate algebraic constructions. What is

the geometric significance of pseudotraces, and how can they be naturally incorporated into Segal's geometric formulation of CFT?

(2) Using pseudotraces, Miyamoto characterized in [Miy04] the space of vacuum torus conformal blocks in terms of higher Zhu algebras $A_n(V)$. However, these algebras are computationally challenging and their relationship to the representation category $\mathrm{Mod}(V)$ is unclear.

In [GR19], Gainutdinov and Runkel proposed a conjecture that more directly connects the space of vacuum torus conformal blocks to $\mathrm{Mod}(V)$. Specifically, they conjectured that this space is isomorphic to the space of symmetric linear functionals on $\mathrm{End}_V(G)$, where $G \in \mathrm{Mod}(V)$ is a fixed projective generator—with the isomorphism realized via the pseudotrace construction. How can one prove the Gainutdinov-Runkel conjecture?

(3) In TQFT, torus conformal blocks are studied via the Lyubashenko coend L [Lyu96]. Specifically, the TQFT perspective predicts that if V is strongly-finite (i.e. C_2-cofinite, self-dual, CFT-type, $\mathbb{N}$-graded), then the space of 1-pointed torus conformal block with insertion module $M \in \mathrm{Mod}(V)$ is isomorphic to $\mathrm{Hom}_V(M, L)$. How can this isomorphism be proved? And how can this picture be related to pseudotraces?

(4) From the TQFT perspective, modular functors/conformal blocks satisfy a certain **sewing-factorization** property [Lyu96, FS17]. In the C_2-cofinite rational case, this property is expressed in terms of direct sums over irreducible modules, and has been established recently, e.g., in [DGT24]. In the C_2-cofinite irrational case, the sewing-factorization property is expressed in terms of **coends**, for example, via horizontal composition of profunctors. How can this coend version of the sewing-factorization theorem be proved in the setting of C_2-cofinite VOAs?

(5) In the strongly-rational case, the equivalence between the categorical S-matrices (defined by Hopf links) and the modular S-matrices (defined by performing the modular transform $\tau \mapsto -1/\tau$) was proved in [Hua08, HK10]. In the strongly-finite case, the categorical S-transform is defined by the Hopf pairing of the Lyubashenko coend L. How can one prove the equivalence of the two S-transforms in the strongly-finite case?

In our series of papers [GZ23, GZ24, GZ25a], and especially in the last one, we completely solved Problem 4: we proved several equivalent versions of the sewing-factorization theorems, one of which is structurally equivalent to the formulation studied in the TQFT setting by [HR24], using horizontal composition of profunctors (defined by coends).

Problem 3 is also essentially solved in [GZ25a]; see the Introduction of [GZ25a]. In particular, the Lyubashenko coend L is given a VOA interpretation in [GZ25a].

A partial answer to Problem 1 is given in [GZ24]. Full answers to Problems 1 and 2 is presented in [GZ25b]. In this paper, we use the sewing-factorization theorem proved in [GZ25a] to show that for any C_2-cofinite VOA V, the end

$$E = \int_{M \in \mathrm{Mod}(V)} M \otimes_{\mathbb{C}} M' \qquad \in \mathrm{Mod}(V \otimes V)$$

(where M' is the contragredient module of M) has a canonical structure of an associative $\mathbb{C}$-algebra, that the category of certain left E-modules is linearly isomorphic to $\mathrm{Mod}(V)$, and that the space of vacuum torus conformal blocks of V is canonically isomorphic to the space of symmetric linear functionals on E. Using this result, the Gainutdinov-Runkel conjecture is proved in [GZ25b].

Problem 5 will be studied in a subsequent series of papers, all based on the sewing-factorization theorem proved in [GZ25a]. Note that our solution to Problem 3 already completes the first step towards solving Problem 5—namely, identifying the two vector spaces on which the two S-transformations we aim to compare act.

References

[DGT24] Chiara Damiolini, Angela Gibney, and Nicola Tarasca. On factorization and vector bundles of conformal blocks from vertex algebras. Ann. Sci. Éc. Norm. Supér. (4) 57(1):241–292, 2024.

[FS17] Jurgen Fuchs and Christoph Schweigert. Coends in conformal field theory. In Lie algebras, vertex operator algebras, and related topics, volume 695 of Contemp. Math., pages 65-81. Amer. Math. Soc., Providence, RI, 2017.

[GR19] Gainutdinov, A. M., & Runkel, I. (2019). The non-semisimple Verlinde formula and pseudo-trace functions. Journal of Pure and Applied Algebra, 223(2), 660-690.

[GZ23] Bin Gui and Hao Zhang. Analytic conformal blocks of C_2-cofinite vertex operator algebras I: Propagation and dual fusion products. arXiv:2305.10180, 2023.

[GZ24] Bin Gui and Hao Zhang. Analytic conformal blocks of $C2$-cofinite vertex operator algebras II: Convergence of sewing and higher genus pseudo-q-traces. arXiv:2411.07707, 2024.

[GZ25a] Bin Gui and Hao Zhang. Analytic conformal blocks of $C2$-cofinite vertex operator algebras III: The Sewing-Factorization Theorems. arXiv:2503.23995, 2025.

[GZ25b] Bin Gui and Hao Zhang. How are pseudo-q-traces related to (co)ends? arXiv:2508.04532, 2025

[HK10] Huang, Y. Z., & Kong, L. (2010). Modular invariance for conformal full field algebras. Transactions of the American Mathematical Society, 362(6), 3027–3067.

[HR24] Aaron Hofer and Ingo Runkel. Modular functors from non-semisimple 3d TFTs. Preprint, arXiv:2405.18038 [math.QA] (2024), 2024.

[Hua08] Huang, Y. Z., & Kong, L. (2010). Modular invariance for conformal full field algebras. Transactions of the American Mathematical Society, 362(6), 3027–3067.

[Lyu96] Lyubashenko, V. (1996). Ribbon abelian categories as modular categories. Journal of Knot Theory and Its Ramifications, 5(03), 311-403.

[Miy04] Masahiko Miyamoto. Modular invariance of vertex operator algebras satisfying $C2$-cofiniteness. Duke Math. J., 122(1):51-91, 2004.

[Zhu96] Yongchang Zhu. Modular invariance of characters of vertex operator algebras. J. Amer. Math. Soc., 9(1):237–302, 1996.

Planar algebras associated to cocommuting squares

Junhwi Lim

(joint work with Dietmar Bisch)

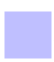
The 'generalized symmetries' of finite index II_1 subfactors are encoded by their planar algebras [6], or equivalently, by their strictly pivotal 2-categories. A natural foundational question is "What minimal structure can be universally expected from these symmetries?" For an arbitrary subfactor $N \subset M$, the planar algebra always contains the Temperley-Lieb-Jones algebra as a subalgebra [5]. When an intermediate subfactor $N \subset P \subset M$ is present, the planar algebra has Fuss-Catalan-Bisch-Jones algebra, introduced by Bisch and Jones [1]. However, the situation for a subfactor with two intermediate subfactors $N \subset P, Q \subset M$, remains an open problem. Such an inclusion is called a quadrilateral. One of the main difficulties of the problem is that the structure depend on the relative position of intermediate subfactors. Hence, when studying such planar algebras, a reasonable approach would be working with a subclass of quadrilaterals with some natural properties. The easiest class of quadrilaterals are the symmetric commuting square. Due to the highest degree of commutativity, they have a simple set of skein relations for their planar algebras [6, Lemma 2.11.2].

We studied quadrilaterals $N \subset P, Q \subset M$ with less degree of commutativity motivated by group-subgroup subfactor examples. They satisfy

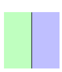
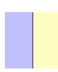
(1) $[P : N] = [Q : N] = [M : P] + 1 = [M : Q] + 1$;
(2) $p_2 q_2 = e_2$;
(3) $p_1 q_1 p_1 = \lambda p_1 + (1-\lambda) e_1$, $q_1 p_1 q_1 = \lambda q_1 + (1-\lambda) e_1$ for some $\lambda \in (0,1)$;
(4) $[e_{P_1 Q_1}, e_{Q_1 P_1}] = 0$;
(5) $[e_{M p_1 q_1 p_0 p_2 q_1 M}, e_{M q_1 p_0 p_2 q_1 p_1 M}] = 0$,

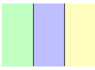
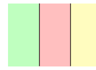
where e_i, p_i and q_i are the i-th Jones projection associated to N, P and Q, respectively, and the elements of the form e_X in the fourth and fifth condition are the projections from $L^2(M_1)$ and $L^2(M_2)$ onto $\overline{\text{span}} X$, respectively. The second condition says that $N \subset P, Q \subset M$ is the cocommuting square, a quadrilateral that has as half as much degree of commutativity of the symmetric commuting square. In addition, in the third condition, we have a weakened version of commuting square, where $p_1 q_1 p_1$ and $q_1 p_1 q_1$ are positive linear combinations of Jones projections. However, the second and the third conditions do not completely determine the structure of the planar algebras. Hence, we also consider the first condition, which is guaranteed to hold when $N \subset Q$ and $N \subset P$ are 3-supertransitive [7, Corollary 3.3], and the fourth and the last condition, which endow slightly more degree of commutativity. These conditions are realized by the quadrilaterals of fixed point algebras $M^{\mathbb{F} \rtimes \mathbb{F}^\times} \subset M^{\mathbb{F}^\times}, M^K \subset M$ where $\mathbb{F}$ is a finite field and K is a conjugate of $\mathbb{F}^\times$. We constructed more general class of quadrilaterals satisfying Conditions 1-5 with arbitrary integer index by using graph planar algebras.

There are diagrammatic expressions for Conditions 1-5. First, we assign the colors to the factors N, P, Q and M as follows:

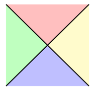
$N =$, $P =$ , $Q =$, $M =$.

Now, for the bimodules ${}_N L^2(P)_P$, ${}_P L^2(M)_M$, ${}_N L^2(Q)_Q$ and ${}_Q L^2(M)_M$, we assign the following diagrams:

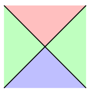
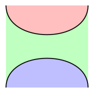
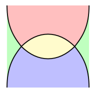
${}_N L^2(P)_P =$ , ${}_P L^2(M)_M =$ , ${}_N L^2(Q)_Q =$, ${}_Q L^2(M)_M =$, The dual bimodules of the above are expressed by horizontal reflection of the diagrams. Note that

 $= {}_N L^2(P) \otimes_P L^2(M)_M \cong {}_N L^2(M)_M \cong {}_N L^2(Q) \otimes_Q L^2(M)_M =$ .

Hence, there is a unitary intertwiner $u : {}_N L^2(Q) \otimes_Q L^2(M)_M \to {}_N L^2(P) \otimes_P L^2(M)_M$ that is expressed as

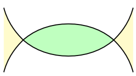
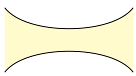
(1) $u =$ .

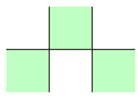
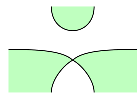
If we only use the diagram for u to express Conditions 1-5, the skein relations have up to eight terms. However, the number of terms can be reduced by employing another crossing given by

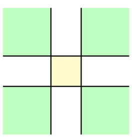
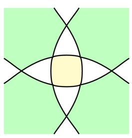
 $= \alpha$  $- \beta$ 

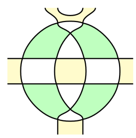
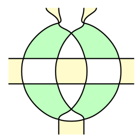
where $\alpha = [P : N]^{\frac{1}{2}}$ and $\beta = [M : P]^{\frac{1}{2}}$. Note that $\alpha^2 = \beta^2 + 1$ by Condition 1. The simplified skein relations are exhibited in the following proposition.

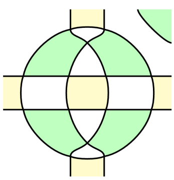
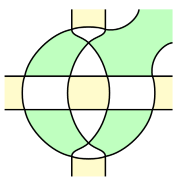
Proposition 1 ([2]). *Let $N \subset P, Q \subset M$ be an irreducible quadrilateral satisfying Condition 1.Then Conditions 2-5 hold iff its planar algebra satisfies the following:*

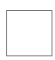
1)  $= \frac{\alpha}{\beta}$ 

2)  $=$ 

3)  $=$ 

4)  $= \beta$ 

5)  $= \alpha$ 

where  $=$  *or* .

Moreover, the unitary u together with the above skein relations form the presentation of planar algebras that are almost like the ones coming from subfactors. The precise description is given in the following theorem:

Theorem 2 ([2]). *In the planar algebra generated by the unitary u in (1), every closed diagram can be reduced to a scalar factor. Moreover, every vector space in the planar algebra is finite dimensional. In particular, the vector space with the boundary coloring $\underbrace{NPNPNP\cdots}_{2n \text{ letters}}$ has the Bell number $B(n)$as the dimension, and* *the one with* $\underbrace{NPMPNPMPNP\cdots}_{4n \text{ letters}}$ *the generalized Bell number* $B_{2,2}(n)$ *in the sense of Błasiak-Penson-Solomon* [3].

In the proof of the finite dimensionality, we show that there is a $1-1$ correspondence from the equivalence classes of diagrams to the set partitions and the result of [4]. We also proved the following theorem by showing that the vector space with the boundary coloring $\underbrace{NPNPNP\cdots}_{4k \text{ letters}}$ is isomorphic to the k-th partition algebra.

Theorem 3. *The quotient of the planar algebra generated by the unitary* u *in* (1) *by the elements of vanishing* 2*-norm is a subfactor planar algebra if and only if* $\alpha \in \mathbb{N}_{\geq 3}$. *If this is the case, the tensor category generated by* ${}_N L^2(M)_N$ *is* $\mathsf{Rep}(S_n)$ *with* $n = \alpha^2$.

References

[1] D. Bisch and V. Jones, *Algebras associated to intermediate subfactors*, Invent. Math. **128** (1997), no. 1, 89–157.

[2] D. Bisch, J. Lim *Planar algebras associated to cocommuting squares*, Manuscript in perparation (2025).

[3] P. Błasiak, K. Penson and A. Solomon, *The boson normal ordering problem and generalized Bell numbers*, Ann. Comb. **7** (2003), no. 2, 127–139.

[4] P. Codara, O. M. D'Antona and P. Hell, *A simple combinatorial interpretation of certain generalized Bell and Stirling numbers*, Discrete Math. **318** (2014), 53–57.

[5] V. Jones, *Index for subfactors*, Invent. Math. **72** (1983), no. 1, 1–25.

[6] V. Jones, *Planar algebras, I*, New Zealand J. Math. **52** (2021), 1–107.

[7] P. Grossman and M. Izumi, *Classification of noncommuting quadrilaterals of factors*, Internat. J. Math. **19** (2008), no. 5, 557–643.

Heat semigroups on quantum automorphism groups of finite dimensional C*-algebras

Futaba Sato

We investigate heat semigroups on a kind of compact quantum group called quantum automorphism groups of finite dimensional C*-algebras, denoted by $\mathrm{Aut}^+(B)$ defined for a pair of a finite dimensional C*-algebra B and an appropriate trace-state called Plancherel state on B introduced by Wang in 1998 [5]. These are important as a kind of quantum symmetries because quantum permutation groups S_n^+ and the "projective" versions of quantum orthogonal groups are included. For quantum permutation groups, it is shown that heat semigroups on those have ultracontractivity and hypercontractivity by Franz et al [2] and the concrete formula of heat semigroups are applied to show the sharp Sobolev embedding property of quantum permutation groups by Youn [6]. It is known that heat semigroups of $\mathrm{Aut}^+(B)$ have the same formula as those of S_n^+ with $\dim B = n$ [1]. However, properties of heat semigroups and their applications to noncommutative L^p-theoretical properties have not been appeared in lierature for general $\mathrm{Aut}^+(B)$.

In [4], we prove the properties of heat semigroups on $\mathrm{Aut}^+(B)$ such as hypercontractivity: for each p with $2 < p < \infty$, there exists $\tau_p > 0$ such that $||T_t x||_p \leq ||x||_2$ for any $t \geq \tau_p$. Furthermore, we obtain the sharpness of the Sobolev embedding property of $\mathrm{Aut}^+(B)$: for any $p \in (1, 2]$, we have the Hardy-Littlewood-Sobolev inequality

$$\left(\sum_{k \geq 0} \frac{n_k}{(1+k)^{s(\frac{2}{p}-1)}} ||\widehat{f}(k)||_{HS}^2 \right)^{\frac{1}{2}} \lesssim ||f||_p$$

if and only if $s \geq 3$. In the appendix of this paper, we give another proof for the concrete formula of heat semigroups on $\mathrm{Aut}^+(B)$ by considering the tube algebras of $\mathrm{Aut}^+(B)$ and S_n^+.

References

[1] M. Brannan, L. Gao, amd M. Junge, *Complete logarithmic Sobolev inequality via Ricci curvature bounded below II*, J. Topol. Anal. **15** (2023), no. 3, 741–794.

[2] U. Franz, G. Hong, F. Lemeux, M. Ulrich, and H. Zhang, *Hypercontractivity of heat semigroups on free quantum groups*, J. Operator Theory **77** (2017), no. 1, 61–76.

[3] S. Neshveyev and M. Yamashita, *A few remarks on the tube algebra of a monoidal category*, Proc. Edinb. Math. Soc. (2) **61** (2018), no. 3, 735–758.

[4] F. Sato, *Heat semigroups on quantum automorphism groups of finite dimensional C*-algebras*, arXiv:2503.03448 [math:QA], 2025.

[5] S. Wang, *Quantum symmetry groups of finite spaces*, Comm. Math. Phys. **195** (1998), no. 1, 195–211.

[6] S.-G. Youn, *On the Sobolev embedding properties for compact matrix quantum groups of Kac type*, Commun. Pure Appl. Anal. **19** (2020), no. 6, 3341–3366.

G-Crossed Extensions and Lattice Orbifolds

Sven Möller

(joint work with César Galindo, Simon Lentner)

G-crossed braided tensor categories (or their equivariantisations, which are often modular tensor categories) appear as categories of representations of fixed-point vertex algebras V^G under the action of some finite group G.

We develop techniques to determine these G-crossed categories in concrete examples, in particular when the original vertex algebra V has a pointed representation category (like a lattice or Heisenberg vertex algebra), i.e. when the fusion rules are characterised by an abelian group A equipped with a quadratic form.

Specifically, we generalise the $\mathbb{Z}/2\mathbb{Z}$-crossed Tambara–Yamagami categories, which have only one simple object in the twisted sector (and are only non-degenerate for A odd), to define a class of $\mathbb{Z}/2\mathbb{Z}$-crossed categories whose untwisted and twisted sector are parametrised by A and $A/(2A)$, respectively.

These give the representation categories of certain lattice $\mathbb{Z}/2\mathbb{Z}$-orbifold vertex algebras, but in contrast to earlier results, with all categorical data (and not just on the level of the fusion ring).

New hyperfinite subfactors with infinite depth

Julio Cáceres
(joint work with Dietmar Bisch)

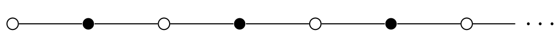
We will present new examples of irreducible, hyperfinite subfactors with trivial standard invariant and interesting Jones indices. These are obtained by constructing new finite dimensional commuting squares. We will use two graph planar algebra embedding theorems and the classification of small index subfactors to show that our commuting square subfactors cannot have finite depth. We also present one-parameter families of commuting squares that, by a classification result of Kawahigashi, will also yield irreducible infinite depth subfactors. This is joint work with Dietmar Bisch.

Subfactors are inclusions of certain algebras of operators on Hilbert spaces, that are closed under pointwise convergence on vectors. These are called *von Neumann algebras* and were introduced by Murray and von Neumann in the 1930's. Specifically, We are interested in subfactors of the hyperfinite II_1 factor $\mathcal{R}$, which is a tensor product of infinitely many copies of $M_2(\mathbb{C})$, and can be shown to be the unique approximately finite dimensional von Neumann algebra with a trace and trivial center. A hyperfinite subfactor is then a unital inclusion $N \subset M$ where both N and M are isomorphic to $\mathcal{R}$. In his seminal paper [13], Jones introduced the notion of index $[M : N]$ for a subfactor. This number behaves in a very similar way to the index for subgroups but need not be an integer. Jones also proved the surprising rigidity result

$$[M : N] \in \left\{4\cos^2\left(\frac{\pi}{n}\right),\ n = 3, 4, 5, \dots\right\} \cup [4, \infty]$$

Moreover, he showed that every index in this set is attained by a not necessarily *irreducible* hyperfinite subfactor. A subfactor is called irreducible if $\dim N' \cap M = 1$.

There is another invariant for subfactors, finer than the index, called the *principal graph*. It is a bipartite graph that describes the standard representation theory of the subfactor. Whenever this graph is finite we say the subfactor has *finite depth*, otherwise it has *infinite depth*. Subfactors whose principal graph is the Coxeter-Dynkin graph A_∞



play a special role, and we will refer to them as A_∞-subfactors. One of the main challenges in subfactor theory is classification. The most successful approach has been by classifying the *standard invariant* associated to the subfactor. Jones in [10] reinterpreted the standard invariant in terms of *planar algebras*. In particular, A_∞-subfactors have the Temperley-Lieb-Jones planar algebra $\mathrm{TLJ}(\delta)$ as their standard invariant, which is the "smallest" planar algebra there is. In [11] it is shown that hyperfinite finite depth subfactors are completely classified by their standard invariant. A complete classification of all planar algebras associated to subfactors with index less than 5.25 can be found in [2]. Thus all finite depth subfactors of the hyperfinite II_1 factor with index less than 5.25 are known. The same cannot be

Index	# of subfactors
$\frac{1}{2}(5+\sqrt{13})$	2
≈ 4.3772	2
$\frac{1}{2}(5+\sqrt{17})$	2
$3+\sqrt{3}$	2
$\frac{1}{2}(5+\sqrt{21})$	2
5	7
≈ 5.04892	2
$3+\sqrt{5}$	11

Table 1. Indices of hyperfinite finite depth subfactors < 5.25

said about *infinite depth* subfactors, in fact, any irreducible infinite depth subfactor with index less than 5 has the same TLJ(δ) standard invariant. In this situation, *new* invariants are needed to differentiate between infinite depth subfactors with the same index.

Given a II_1 factor M, Jones defined the invariant $\mathcal{C}(M)$ for M, where

$$\mathcal{C}(M) = \{[M : N] : \ N \subset M \text{ irreducible subfactor}\}.$$

$\mathcal{C}(M)$ is known for specific II_1 factors M by results of Popa, Shlyakhtenko and others, however the problem of computing $\mathcal{C}(\mathcal{R})$ is still wide open. The situation is slightly better if we only consider finite depth irreducible hyperfinite subfactors. Table 1 shows the list of all indices in $(4, 5.25]$ for irreducible hyperfinite finite depth subfactors.

Except for ≈ 5.04892, we realize these indices as the index of an irreducible hyperfinite infinite depth subfactor. Our work leads us to the following conjecture

Conjecture 1. *Every index of a finite depth irreducible hyperfinite subfactor is also the index of an A_∞-subfactor.*

We have shown the conjecture holds for $\frac{5+\sqrt{13}}{2}$, ≈ 4.3772, $\frac{5+\sqrt{17}}{2}$, $3+\sqrt{3}$, $\frac{5+\sqrt{21}}{2}$, 5 and $3+\sqrt{5}$. The novel idea in our work is to combine a classical construction of hyperfinite subfactors with planar algebras and fusion category techniques.

References

[1] J. K. Schou, *Commuting squares and index for subfactors*, Thesis (Ph.D.)–Odense Universitet Institut for Matematik og Datalogi (1990).

[2] N. Afzaly, S. Morrison, D. Penneys, *The classification of subfactors with index at most $5\frac{1}{4}$*, Mem. Amer. Math. Soc. **284** (2023), no. 1405, v+81.

[3] Y. Kawahigashi, *A characterization of a finite-dimensional commuting square producing a subfactor of finite depth*, Int. Math. Res. Not. IMRN (2023), no. 10, 8419–8433.

[4] P. Grossman, N. Snyder, *The Brauer-Picard group of the Asaeda-Haagerup fusion categories*, Trans. Amer. Math. Soc. **368** (2016), no. 4, 2289–2331.

[5] P. Grossman, M. Izumi, N. Snyder, *The Asaeda-Haagerup fusion categories*, J. Reine Angew. Math. **743** (2018), 261–305.

[6] V. Jones, D. Penneys, *The embedding theorem for finite depth subfactor planar algebras*, Quantum Topol. **2** (2011), no. 3, 301–337.

[7] V. Kodiyalam, V. S. Sunder, *On Jones' planar algebras*, J. Knot Theory Ramifications **13** (2004), no. 2, 219–247.
[8] E. Peters, *A planar algebra construction of the Haagerup subfactor*, Internat. J. Math. **21** (2010), no. 8, 987–1045.
[9] V. Jones, S. Morrison, N. Snyder, *The classification of subfactors of index at most 5*, Bull. Amer. Math. Soc. (N.S.) **51** (2014), no. 2, 277–327.
[10] V. Jones, *Planar algebras, I*, New Zealand J. Math. **52** (2021), 1–107.
[11] S. Popa, *Classification of amenable subfactors of type II*, Acta Mathematica **172** (1994), no. 2, 163–255.
[12] P. Grossman, S. Morrison, D. Penneys, E. Peters, N. Snyder, *The extended Haagerup fusion categories*, Ann. Sci. Éc. Norm. Supér. (4) **56** (2023), no. 2, 589–664.
[13] V. Jones, *Index for subfactors*, Invent. Math. **72** (1983), no. 1, 1–25.
[14] D. Bisch, V. Jones, *Algebras associated to intermediate subfactors*, Invent. Math. **128** (1997), no. 1, 89–157.

Quantum graphs and spin models

Roberto Hernández Palomares

Spin models for singly-generated Yang-Baxter planar algebras are known to be determined by certain highly-regular classical graphs such as the pentagon or the Higman-Sims graph. Examples of spin models include the Jones and Kauffman polynomials, as well as certain fiber functors. We will explore the notion of higher-regularity for quantum graphs as well as their potential to encode spin models. Time allowing, we will give examples of non-classical graphs enjoying these properties.

References

[1] N. Bravo Hernández, R. Hernández Palomares, F. Viales Solis. *Quantum graphs and spin models*, in preparation.

Classifying actions of tensor categories on Kirchberg algebras

Sergio Girón Pacheco
(joint work with Kan Kitamura, Robert Neagu)

The Kirchberg–Phillips theorem ([4, 5]) is a cornerstone of the classification theory for C*-algebras. In its original form, it is a KK rigidity result: two Kirchberg algebras are stably isomorphic if and only if they are KK-equivalent. Assuming the universal coefficient theorem (UCT), this yields a complete classification: two UCT Kirchberg algebras are stably isomorphic if and only if they have isomorphic K-groups, and any pair of countable abelian groups can be realised as K-groups of such an algebra.

Recently, Gabe and Szabó established a dynamical analogue, classifying pointwise outer actions of discrete amenable groups on Kirchberg algebras via group equivariant KK-theory ([2]). Combined with Meyer's result [7], this confirms Izumi's conjecture [8], that any two outer actions of countable torsion-free amenable groups on stable Kirchberg algebras are classified by the homotopy class of the induced maps on classifying spaces; a much simpler invariant than KK equivalence classes. For example, these homotopy groups can be computed for strongly self-absorbing C*-algebras via Dadarlat–Pennig theory [9].

In analogy with Popa's classification results for amenable subfactors ([3]) and it's subsequent reformulations, it is also natural to ask for a classification of *quantum symmetries* on Kirchberg algebras. The framework of unitary tensor category actions provides the right language to study these symmetries, their classification problem is also known to be closely related to the problem of classifying finite index inclusions of C*-algebras. Moreover, the recent work of Arano, Kitamura and Kubota [1] introduces the appropriate tensor category equivariant KK-theory.

In this talk I discuss the full analogue of Gabe and Szabó's theorem for actions of unitary tensor categories with countably many isomorphism classes of simple objects (hereinafter called countable UTC's). Specifically, if $\mathcal{C}$ is an amenable countable UTC and α, β are outer actions of $\mathcal{C}$–meaning that the associated functors into the bimodule categories of the underlying C*-algebra is full–on stable Kirchberg algebras, then α and β are equivalent if and only if they are $KK^{\mathcal{C}}$ equivalent. Similarly, in the unital case, α and β are (unitally) equivalent if and only if the isomorphism in KK preserves the class of the unit in K-theory.

This KK-rigidity lays the groundwork for future classification results. For instance, if $\mathcal{C}$ is as above and α, β are outer actions of $\mathcal{C}$ on the C*-algebra $\mathcal{O}_2$ that are equivariantly $\mathcal{O}_2$-stable in the sense of [6]–meaning that the action by bimodules is equivalent to its external tensor product with $\mathcal{O}_2$–then they are equivalent. In the group case, equivariant $\mathcal{O}_2$-stability is immediate for any outer action of a countable torsion free amenable group on $\mathcal{O}_2$. This raises a natural question: for which unitary tensor categories does equivariant $\mathcal{O}_2$ stability follow automatically from outerness?

References

[1] Y. Arano, K. Kitamura, Y. Kubota, *Tensor category equivariant KK-theory*, Adv. Math. **453** (2024).

[2] J. Gabe, G. Szabó, *The dynamical Kirchberg-Phillips theorem*, Acta Math. **232** (2024), 1–77.

[3] S. Popa, *Classification of amenable subfactors of type II*, Acta Math. **172(2)** (1994), 163–255.

[4] E. Kirchberg, *Exact C*-algebras, tensor products, and the classification of purely infinite C*-algebras*, Proceedings of the International Congress of Mathematicians, Vol. 1,2 (Z´urich, 1994), 943–954.

[5] N.C. Phillips, *A classification theorem for nuclear purely infinite simple C*-algebras*, Doc. Math. **5** (2000), 49–114, Birkh´auser, Basel, 1995.

[6] S. Evington, S. Girón Pacheco, C. Jones, *Equivariant $\mathcal{D}$-stability for actions of tensor categories*, to appear in J. Operator theory, arXiv:2401.14238.

[7] R. Meyer, *On the classification of group actions on C*-algebras up to equivariant KK-equivalence*, Ann. K-Theory **6(2)** (2021), 157–238.

[8] M. Izumi, *Group actions on operator algebras*, Proceedings of the International Congress of Mathematicians, Vol. 3, 1528–1548, Hindustan Book Agency, New Delhi, 2010.

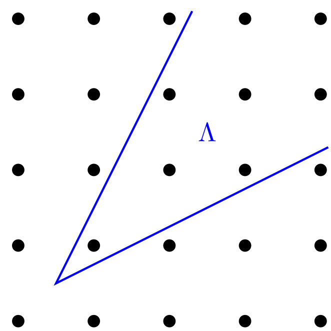
[9] M. Dadarlat, U. Pennig, *A Dixmier–Douady theory for storngly self-absorbing C*-algebras*, J. Reine Angew. Math., **718** (2016), 153–181.

Superselection sectors for a poset of von Neumann algebras

Daniel Wallick

(joint work with Anupama Bhardwaj, Tristen Brisky, Chian Yeong Chuah, Kyle Kawagoe, Joseph Keslin, David Penneys)

We consider spin systems on a $\mathbb{Z}^2$ lattice, where we associate $\mathbb{C}^{d_v}$ to each $v \in \mathbb{Z}^2$. These finite-dimensional Hilbert spaces represent quantum spin particles. We can describe the entire system of interacting particles by the UHF C*-algebra $\mathfrak{A} := \bigotimes_{v\in\mathbb{Z}^2} M_{d_v}(\mathbb{C})$. For a subset $S \subseteq \mathbb{Z}^2$, we associate the subalgebra $\mathfrak{A}_S := \bigotimes_{v\in S} M_{d_v}(\mathbb{C}) \subseteq \mathfrak{A}$. We will specifically consider the case where $S = \Lambda \cap \mathbb{Z}^2$ for a cone $\Lambda \subseteq \mathbb{R}^2$.



The physics of this system is given by a state $\omega_0 : \mathfrak{A} \to \mathbb{C}$. From this state, one can obtain the GNS representation $\pi_0 : \mathfrak{A} \to B(\mathcal{H}_0)$. Using this data, one can describe anyonic excitations using a DHR-inspired machinery. This approach was first introduced in [4] and was later expanded on in [5].

We reformulate these methods using the approach of [2] for conformal nets. We let $\mathcal{C}$ denote the poset of cones in $\mathbb{R}^2$. Note that $\mathcal{C}$ is closed under taking complements, i.e., if $\Lambda \in \mathcal{C}$, then $\Lambda^c \in \mathcal{C}$. To each $\Lambda \in \mathcal{C}$, we can associate the von Neumann algebra $\mathcal{R}_\Lambda := \pi_0(\mathfrak{A}_{\Lambda\cap\mathbb{Z}^2})'' \subseteq B(\mathcal{H}_0)$. Note that if $\Lambda \subseteq \Delta$, then $\mathcal{R}_\Lambda \subseteq \mathcal{R}_\Delta$. Furthermore, without loss of generality, we can assume that all of the algebras $\mathcal{R}_\Lambda$ are properly infinite [6]. Finally, we assume that the von Neumann algebras satisfy *Haag duality*, namely that $\mathcal{R}'_{\Lambda^c} = \mathcal{R}_\Lambda$. While Haag duality is hard to show in practice, it (or a generalization thereof) is a standard assumption in this area [5]. These assumptions further imply that for every $\Lambda \in \mathcal{C}$, ${}_{\mathcal{R}_\Lambda}\mathcal{H}_0$ is an *absorbing* $\mathcal{R}_\Lambda$-module [1, 3], that is, for every $\mathcal{R}_\Lambda$-module ${}_{\mathcal{R}_\Lambda}\mathcal{K}$, ${}_{\mathcal{R}_\Lambda}\mathcal{H}_0 \cong {}_{\mathcal{R}_\Lambda}\mathcal{H}_0 \oplus {}_{\mathcal{R}_\Lambda}\mathcal{K}$.

In [1], we define a *superselection sector* as a collection of normal unital $*$-representations $\pi_\Lambda : \mathcal{R}_\Lambda \to B(\mathcal{H})$ satisfying the following axioms:

- (isotony) if $\Lambda \subseteq \Delta$, then $\pi_\Delta|_{\mathcal{R}_\Lambda} = \pi_\Lambda$,
- (locality) for all $\Lambda \in \mathcal{C}$, $[\pi_\Lambda(\mathcal{R}_\Lambda), \pi_{\Lambda^c}(\mathcal{R}_{\Lambda^c})] = 0$, and
- (absorbing) for all $\Lambda \in \mathcal{C}$, π_Λ defines an absorbing $\mathcal{R}_\Lambda$-module.

This definition is equivalent to the usual definition of superselection sectors for spin systems [4, 5]. However, this definition makes sense even if the algebras $\mathcal{R}_\Lambda$ do not originate from a quantum spin system, and it works for more general posets than $\mathcal{C}$. We show that with general geometric axioms for the poset, the superselection sectors form a W^*-braided tensor category [1]. Furthermore, we show that the category we obtain is equivalent to the one defined for spin systems [5].

References

[1] A. Bhardwaj, T. Brisky, C.Y. Chuah, K. Kawagoe, J. Keslin, D. Penneys, and D. Wallick, *Superselection Sectors for Posets of von Neumann Algebras*, Comm. Math. Phys. **406** (2025), Article No. 185.

[2] F. Gabbiani and J. Fröhlich, *Operator algebras and conformal field theory*, Comm. Math. Phys. **155** (1993), 569–640.

[3] P. Ghez, R. Lima, and J.E. Roberts, *W^*-categories*, Pacific J. Math. **120** (1985), 79–109.

[4] P. Naaijkens, *Localized endomorphisms in Kitaev's toric code on the plane*, Rev. Math. Phys. **23** (2011), 347–373.

[5] Y. Ogata, *A derivation of braided C^*-tensor categories from gapped ground states satisfying the approximate Haag duality*, J. Math. Phys. **63** (2022), Article No. 011902.

[6] Y. Ogata, *Mixed state topological order: operator algebraic approach*, (2025), `https://arxiv.org/abs/2501.02398`.

Construct Subfactors by Classical and Quantum Computers

Zhengwei Liu

(joint work with Fan Lu)

We propose a new program to construct subfactors and planar algebras by classical and quantum computers. Brand new examples are discovered in this program. This is joint work with F. Lu, arXiv:2412.17790, and an updated version will appear soon.

Subfactors and planar algebras describe the quantum symmetries beyond the representation theory of groups, including quantum groups. These symmetries can be applied to classify 1+1 quantum spin chains and the boundary of unitary 2+1 topological quantum field theory, etc. People have tried various classifications of subfactors. However, new examples beyond the representation theory of groups are quite rare. The first example is the Haagerup subfactor with Jones index $\frac{3+\sqrt{13}}{2}$, which was discovered from the small index classification. It remains an open question that whether their is a Vertex operator algebra in conformal field theory whose representation category contains the Haagerup symmetry. A generic construction of new subfactors and planar algebras beyond group representation theory is highly demanded.

We investigated skein theoretical classification and construction of planar algebras. The planar algebras are presented by generators and parameterized relations. First, one needs to provide an evaluation algorithm based on enough relations, so that any closed diagram reduces to a polynomial of parameterized variables. Secondly, the value of different reductions process should be the same, which lead to a set of polynomial equations. Thirdly, if one wants a subfactor from the planar algebra, then ones need to verify the reflection positivity condition for certain variables.

We consider the n dimensional 2-box space of a planar algebra as a fusion bialgebra. We setup the generators as its minimal idemponents. The dual data, the quantum dimension d_i and the convolution coefficients N_{ij}^k are the $O(n^3)$ variables. They satisfy a set of degree 2 consistency equations, such as the associativity of the convolution. These equations are not enough to evaluate any closed diagram. Furthermore, we setup the exchange relation for the generators with $O(n^4)$ variables a_{ij}^{kl}, b_{ij}^{kl}. We derive the consistency equations as a set of $O(n^6)$ degree at most 3 polynomial equations.

In principal, one can solve a set of polynomial equations by the Gröberner basis. However, solving polynomial equations is NP hard problem. The computational complexity of the Gröberner basis algorithm to find algebraic solutions is double exponential w.r.t. the number of variables. The computational complexity to find numerical solutions is exponential. This exponential computation complexity is a common challenge in various classification program. Experts have introduced various methods to reduce the computational complexity, which are successful to construct small examples. However, these methods fail to construct large examples, due to the exponential computational growth of the computational complexity and the limited computational resource. A typical example is that the existence of generalized Haagerup subfactor for the group $\mathbb{Z}_p$ remains open, which is equivalent to solve Izumi's polynomial equations. We need to explore more mathematical structures behind the equations and find new algorithms to solve them.

Our first break is proving that the fusion bialgebra has an exchange relation if and only if the the fusion graph Γ_i is a forest for every i. Here $\Gamma_{i,j}^k$ is the support function of N_{ij}^k and Γ_i is the bipartite graph whose the vertices j,k are connected iff $\Gamma_{ij}^k = 1$ iff $N_{ij}^k \neq 0$. This is a very surprising correspondence between skein relations and fusion graphs. Given forests $\Gamma_\bullet$, we prove that

- The variables a_{ij}^{kl}, b_{ij}^{kl} are ± 1 or 0 determined by $\Gamma_\bullet$;
- N_{ij}^k is sum of the variables d_ℓ with integer coefficients determined by $\Gamma_\bullet$.
- All consistency equations reduces to linear equations, except the degree 2 associativity equation.

Therefore, $O(n)$ variables reduce to n variables of quantum dimensions d_i and they satisfy $O(n^6)$ linear equations. This is a highly over determined system. After solving linear equations, the remaining variables and the degree 2 associativity equations can be solved for a reasonably large n. We conjecture that the forest decomposition of the algebraic variety of the consistency equations is the irreducible decomposition. Each irreducible component is the solution of the associativity equation and linear equations.

We establish a computer algorithm to classify exchange relation fusion bialgebras:

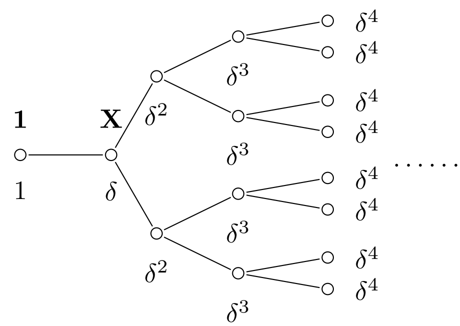
- Input: n, the dimension of 2-boxes;
- List all cases $\Gamma_{ij}^k \in \{0,1\}$, such that every Γ_i is a forest.
- List consistency equations, solve the linear ones and then solve the associativity equation by Gröbener basis.
- Output: solutions Γ_{ij}^k and variables d_i.

The running time of this algorithm on a personal computer is 1.2 seconds for $n \leq 5$ and a couple minutes for $n = 6$. The number of solutions is summarized in the following table:

n	types of forests	algebraic solutions	subfactors
3	16	7	6
4	1.024	24	20
5	1.048,576	88	61
6	68,719,476,736	275	198+

The table illustrates the rareness of planar algebras and subfactors. The column for subfactors is the number of forest types with algebraic solutions and the planar algebra has reflection positivity for certain variables, which produce subfactors. In general, the reflection positivity condition need to be checked case by case.

For $n = 3$, we obtain a brand new planar algebra with index $\delta^2 = -1$, whose principle graph is an infinite binary tree.



For $n \leq 5$, all solutions are from groups, Temperley-Lieb Jones, the binary tree planar algebras or their tensor/free product. We provide several criteria derived from Quantum Fourier analysis to test reflection positivity. The sieving efficiency is remarkablely 100%. All algebraic solutions passing the criteria produce subfactors.

For $n = 6$, there are 6 forest types which are neither groups, nor tensor/free products. They include

- a self dual planar algebra with index -2;
- a self dual one with index 10;
- a dual pair with index 8;
- a dual pair of one-parameter families.

The first unitary solution in the one-parameter family has index $6 + 2\sqrt{5}$ and it produces a new infinite depth subfactor.

The dimension $6 = 2 \times 3$ is the transition point that we start to see several non-trivial examples beyond (quantum) groups and tensor/free product. We expect to see more examples and new mathematical structures when n gets larger.

The remaining challenge is that the number of cases $\Gamma_{ij}^k \in \{0, 1\}$ is about $2^{n^3/6}$ modulo the permutation symmetry S_3, even though every single case is easy to solve. It exhausted the memory of a classical computer. How to efficiently find $\Gamma_{ij}^k \in \{0, 1\}$ with solutions?

This problem can be solved by our new quantum algorithm of solving binary polynomial equations. We can design a friendly Hamiltonian H, such that its time evolution e^{itH} on the state of one solution associated with Γ_{ij}^k is a superposition of other solutions up to an error rate $O(\varepsilon)$ and the running time is about $O(\varepsilon^{-2})$. Then the measurement of the output state will be another solution in a high probability. Repeating this sampling process, we will obtain most solutions considering the rareness of subfactors planar algebras. This quantum algorithm will not provide a mathematical classification. It provides a new paradigm and machine to produce new examples.

Anyonic Spin Chains and Subfactors

Stefan Hollands

Anyonic chains are a class of spin models in which the Hilbert space of the model is naturally defined not as a tensor product between the Hilbert space of each individual spin, but rather in terms of the "F-symbolds" or "quantum $6j$-symbols" of some given unitary fusion category. Such chains support a variety of operators corresponding, e.g., to local operators such as nearest neighboring spin interactions, as well as also "topological charges" which commute with all local operators.

In the work [1] reported in this talk, we investigated the setup of anyonic spin chains in the case that the fusion category in question is related to an inclusion $\mathcal{N} \subset \mathcal{M}$ of von Neumann factors. More precisely, we assume that $\mathcal{N}$ carries an action of a unitary modular tensor category ${}_{\mathcal{N}}X_{\mathcal{N}}$ by endomorphisms of $\mathcal{N}$, such that the extension $\mathcal{M}$ of $\mathcal{N}$ is characterized by a Q-system [2] ("canonical endomorphism"), θ, of the fusion category associated with $\mathcal{N}$. Then the braiding between the irreducible objects $\lambda, \mu, \dots$ in ${}_{\mathcal{N}}X_{\mathcal{N}}$ induces endomorphisms $\alpha_\lambda^\pm, \alpha_\mu^\pm$ of $\mathcal{M}$ called "alpha-induced" objects. These objects are in general not irreducible, a situation characterized by the matrix $Z_{\lambda,\mu} = \mathrm{dimHom}(\alpha_\lambda^+, \alpha_\mu^-)$ of non-negative integers.

In the setting of our work [1] the Hilbert space of the anyonic spin chain is given for a given length $2L$ by the set of intertwiners $H^{2L} = \mathrm{Hom}(\theta^L, \theta^L)$, which is also equal to the depth $2L$ relative commutant in the Jones tunnel associated with the inclusion $\mathcal{N} \subset \mathcal{M}$. Based on foundational work by Böckenhauer et al. [3], we first construct a set of mutually orthogonal projectors $Q_{\lambda,\mu}$ for any pair (λ, μ) such that $Z_{\lambda,\mu} \neq 0$. The construction is possible in a canonical manner for any L using alpha induction, and these projections give an orthogonal decomposition $H^{2L} = \oplus_{(\lambda,\mu)} H_{\lambda,\mu}^{2L}$. Each of the orthonormal subspaces $H_{\lambda,\mu}^{2L}$ is shown to be invariant under all local operators on the spin chain. We think of these subspaces as analogous to the "conformal block" in a putative $1+1$ dimensional CFT limit of the chain which is generated by acting with chiral operators together with a primary field of type (λ, μ) on the vacuum vector in the CFT Hilbert space.

The main result of our work [1] is a construction of defects in our setting. We characterize a specific defect of "type D" by a specific orthogonal projector P_D on the Hilbert space $H^{2L_1} \otimes H^{2L_2}$ of a bipartite chain. Our construction of P_D is aimed at being parallel to a construction of transparent defects in CFT [2]. It proceeds by first defining a set of operators $\Psi_{\lambda,\mu;w_1,w_2}$ on the Hilbert space of the bipartite chain, where w_1, w_2 run through an orthonormal basis of $\mathrm{Hom}(\alpha^+_\lambda, \alpha^-_\mu)$ for each given pair (λ, μ). These operators have the following key properties: (i) They mutually commute, (ii) they commute with all local operators on the bipartite chain away from the junction, (iii) their operator algebra is precisely equal to that of the "braided product of two full centers" in the category ${}_\mathcal{N}X_\mathcal{N}$, known [2] to classify defects in a CFT whose fusion category of any chiral half Virasoro sub-theory is ${}_\mathcal{N}X_\mathcal{N}$. Then we let $\mathcal{D}$ be the abelian operator algebra acting on $H^{2L_1} \otimes H^{2L_2}$ generated by the operators $\Psi_{\lambda,\mu;w_1,w_2}$. A specific defect of type D corresponds to a central projection P_D of this algebra $\mathcal{D}$. Finally, the subspace $P_D(H^{2L_1} \otimes H^{2L_2})$ is supposed to contain those physical states on the bipartite chain containing a defect of the type D. It can be seen from the algebraic relations in $\mathcal{D}$ that, in a sense, a defect of type D connects primary fields on both sides of the chain in a specific way. In the case of the anyonic spin chain based on the Ising category, the construction of the defects is consistent with the usual boundary conditions of the Ising primary fields across the defect.

References

[1] S. Hollands, *Anyonic chains – α-induction – defects – subfactors*, Commun. Math. Phys **399** 1549-1621 (2022),

[2] M. Bischoff et al., *Tensor categories of von Neumann algebras and applications to quantum field theory*, Springer Briefs Series in Mathematical Physics (2016)

[3] J. Bockenhauer and D. E. Evans, Y. Kawahigashi *On alpha induction, chiral generators and modular invariants for subfactors,* Commun. Math. Phys. **208** (1999), 429–487

Exact factorizations and bicrossed product of fusion categories

Julia Plavnik

(joint work with Monique Müller, and Héxtor M. Peña Pollastri)

Fusion categories are highly structured categories that appear naturally in many areas of mathematics and physics, such as low-dimensional topology, subfactors, mathematical physics, and representation theory, among others. More precisely, fusion categories are finite semisimple tensor categories. For more detailed defintions, we refer to [4]. We always consider our field to be algebraically closed of characteristic zero.

A central theme in the theory is the classification and construction of new examples. A lot of the inspiration for this comes from groups, since two important examples of fusion categories arise from finite groups: the so-called pointed fusion categories, which are the categories of finite-dimensional vector spaces graded by the group and associativity determined by a 3-cocycle, and finite-dimensional representations of the group. For example, there are notions of nilpotent and solvable fusion categories, which capture many of the features observed for groups but also have some differences.

When studying fusion categories, sometimes the focus is on understanding the fusion rules and the properties that only depend on them, that is, the Grothendieck structure. Something very valuable is to have the other piece of data pertaining to a fusion category: the associativity constraint, also described as $6j$-symbols or F-matrices.

For a group G, we say that it is an *exact factorization* of subgroups H and K if any element $g \in G$ can be written as $g = hk$, for unique $h \in H$ and $k \in K$. We will denote $G = H \bullet K$ an exact factorization of G by H and K [7], [8], [12]. The direct product and semidirect products of groups are examples of exact factorizations. There are examples not coming from semidirect products, such as the symmetric group $\mathbb{S}_n = \mathbb{S}_{n-1} \bullet C_n$, where C_n denotes the cyclic group of order n. An equivalent definition is that the intersection of H and K should be trivial and that the order $|G| = |H||K|$. With this definition, it is also easy to see that this notion is symmetric and does not depend on the order chosen for H and K. To construct new examples of exact factorizations, a useful notion is that of *matched pairs of groups*. A matched pair of groups is a collection $(H, K, \triangleright, \triangleleft)$ where H and K are groups, $\triangleright : K \times H \to H$ and $\triangleleft : K \times H \to K$ are left and right actions such that $(kt)\triangleleft h = (k \triangleleft (t \triangleright h))\,(t \triangleleft h)$, and k $\triangleright(hg) = (k \triangleright h)\,((k \triangleleft h) \triangleright g)$, for all $k, t \in K$, $h, g \in H$. Then, the *bicrossed product of groups* (or *Zappa -Zsép product*) is defined as a set as $H \bowtie K := H \times K$ with multiplication $(h, k)(g, t) = (h(k \triangleright g), (k \triangleleft g)t)$, $k, t \in K,\ h, g \in H$. The bicrossed product is an exact factorization of groups, and furthermore, any exact factorization of a finite group can be realized in this way.

These concepts extend beyond group theory to other algebraic contexts, such as Hopf algebras, Lie algebras [1], and C^*-algebras [3]. For fusion categories, Gelaki introduced the notion of exact factorizations [6], and later with Basak extended it to finite tensor categories [2].

A fusion category $\mathcal{B}$ is an exact factorization of fusion subcategories $\mathcal{A}$ and $\mathcal{C}$, and denoted $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$, if every simple object $B \in \mathcal{B}$ can be written uniquely as $A \otimes C$, with A a simple object in $\mathcal{A}$ and C a simple object in $\mathcal{C}$.

In our recent work [9], we explore exact factorizations of fusion categories from a structural and constructive viewpoint. We show that many fundamental invariants behave well under exact factorizations. This is true especially at the level of the Grothendieck ring. In particular, the universal grading group and the adjoint subcategory admit exact factorizations [9, Propositions 3.21 and 3.22]. The group of invertibles, and the pointed subcategory admit exact factorizations as well. More precisely, if $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$ is an exact factorization of fusion categories then the adjoint subcategory $\mathcal{B}_{\mathrm{ad}} = \mathcal{A}_{\mathrm{ad}} \bullet \mathcal{C}_{\mathrm{ad}}$ and the pointed $\mathcal{B}_{\mathrm{pt}} = \mathcal{A}_{\mathrm{pt}} \bullet \mathcal{C}_{\mathrm{pt}}$ subcategories are exact factorization of fusion categories, and the universal grading group $U(\mathcal{B}) =$

$U(\mathcal{A}) \bullet U(\mathcal{C})$ and the group of invertibles $G(\mathcal{B}) = G(\mathcal{A}) \bullet G(\mathcal{C})$. In particular, we show that $\mathcal{B}$ is nilpotent if and only if $\mathcal{A}$ and $\mathcal{C}$ are [9, Corollary 3.26]. Other properties, such as solvability, that in principle are not just determined by the fusion rules are not known to be preserved under exact factorizations.

We first defined the notion of exact factorization, matched pair, and bicrossed product for fusion rings [9, Subsections 3.3 and 3.4], and we proved that, as in the group case, any exact factorization of fusion rings can be realized in terms of bicrossed products [9, Theorem 3.14]. In this way, we give a complete answer to how the Grothendieck ring of an exact factorization of fusion categories looks like, but it is still important to find the possible categorifications, that is, the associativity constraints. In [9, Section 4], we also defined matched pairs and bicrossed products $\mathcal{A} \bowtie \mathcal{C}$ of fusion categories, which give rise to exact factorizations of fusion categories, but, at the moment, it is not known if all exact factorizations arise in this way. Already, the pointed case (see [6, Example 3.6]) hints that some cohomological data should play a role for the associativity constraint. The adjoint subcategory $(\mathcal{A} \bowtie \mathcal{C})_{\mathrm{ad}}$ of the bicrossed product is the Deligne product $\mathcal{A}_{\mathrm{ad}} \boxtimes \mathcal{C}_{\mathrm{ad}}$ of the adjoint subcategories [9, Corollary 3.24]. Then, for any exact factorization for which the universal grading groups involved are trivial, the problem of understanding exact factorizations reduces to finding all possible associativity constraints on the Deligne product of the fusion categories, that corresponds to a version of the Künneth formula for fusion categories [5]. In [10, Theorem 3.2], we show that when one of the subcategories of the exact factorization is the pointed fusion category vec_G of finite-dimensional G-graded vector spaces with trivial associativity, then the exact factorization can be realized as a bicrossed product. More precisely, if $\mathcal{B} = \mathrm{vec}_G \bullet \mathcal{C}$ then $\mathcal{B} \simeq \mathrm{vec}_G \bowtie \mathcal{C}$. It is still an open question if a similar statement is true for a general pointed fusion category vec_G^{ω}. To prove the aforementioned result, the strategy was to show that the bicrossed product with one of the categories being vec_G corresponds (under the Basak-Gelaki correspondance [2, Theorem 5.1]) to a construction introduced by Natale called *crossed extension*, which describe the fusion categories that fit in an abelian exact sequence and generalize equivariantization, see [11] for more details.

Another important question is about which structures are preserved under exact factorizations. Gelaki showed that if $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$ is braided, then it is the Deligne product $\mathcal{A} \boxtimes \mathcal{C}$ and the subcategories projectively centralize each other. In work in progress with S. Mondal, M. Müller, and H.M.Peña Pollastri, we are looking into exact factorizations of crossed braided categories.

References

[1] L. Agore and G. Militaru, *Classifying complements for Hopf algebras and Lie algebras*, Journal of Algebra **391** (2013), 193–208.

[2] T. Basak and S. Gelaki, *Exact factorizations and extensions of finite tensor categories*, Journal of Algebra **631** (2023), 46–71.

[3] N. Brownlowe, J. Ramagge, D. Robertson, and M. F, Whittaker, *Zappa–Szép products of semigroups and their C^*-algebras*, Journal of Functional Analysis **266** (6) (2014), 3937–3967.

[4] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, *Tensor Categories*, Mathematical surveys and monographs. American Mathematical Society (2015).
[5] P. Etingof, D. Nikshych, and V. Ostrik, private communication.
[6] S. Gelaki, *Exact factorizations and extensions of fusion categories*, Journal of Algebra **480** (2017), 505–518.
[7] G. Kac, *Extensions of groups to ring groups*, Mathematics of the USSR-Sbornik **5** (3) (1968), 451.
[8] G. Mackey, *Products of subgroups and projective multipliers*, Colloquia Math. Soc. Janos Bolyai **5**, Hilbert Space Operators, Tihany (Hungary) (1970), 401–413.
[9] M. Müller, H.M. Peña Pollastri, and J. Plavnik, *On bicrossed product of fusion categories and exact factorizations*, Preprint `arXiv:2405.10207` (2024).
[10] M. Müller, H.M. Peña Pollastri, and J. Plavnik, *Relation between bicrossed products and crossed extensions of fusion categories*, Preprint `arXiv:2507.08562` (2025).
[11] S. Natale, *Extensions of tensor categories by finite group fusion categories*, Mathematical Proceedings of the Cambridge Philosophical Society **170** (1) (2021), 161–189.
[12] M. Takeuchi, *Matched pairs of groups and bismash products of hopf algebras*, Communications in Algebra **9** (8) (1981), 841–882.

Rigidity for commutative algebras in braided tensor categories, with applications to vertex operator algebras

Robert McRae
(joint work with Thomas Creutzig, Kenichi Shimizu, Harshit Yadav and Jinwei Yang)

This talk was based on the paper [6], which was motivated by the problems of rigidity and semisimplicity for braided monoidal categories of modules for vertex operator algebras (VOAs). VOAs are an algebraic approach to the mathematical study of two-dimensional chiral conformal quantum field theories, as the chiral algebra of a conformal field theory is a VOA. The VOAs that appear in rational conformal field theory are called "strongly rational," and by a theorem of Huang [9], the representation category of a strongly rational VOA is a semisimple modular tensor category. Most VOAs are not strongly rational, but there is still a construction of braided monoidal structure on representation categories of more general VOAs due to Huang-Lepowsky-Zhang that works for many examples (see the review article [11]). However, it is in general difficult to say when such braided monoidal categories are rigid, since unlike for Hopf algebras, for example, rigidity is not built in to the monoidal category construction. Another question is when braided monoidal categories of modules for VOAs are semisimple.

One way to address these questions for a VOA V is to relate the representation theory of V to that of a known VOA, perhaps a subalgebra or extension of V. Thus we are led to consider the situation of a VOA inclusion $V \subseteq A$ where A is an object of a braided monoidal category $\mathcal{C}$ of V-modules. In this setting, A is a commutative algebra in $\mathcal{C}$ [10], where the unit morphism $\iota_A : V \to A$ is the inclusion and the multiplication $\mu_A : A \otimes A \to A$ is induced by the VOA structure of A. Let $\mathcal{C}_A$ be the monoidal category of left A-modules in $\mathcal{C}$ and $\mathcal{C}_A^{\mathrm{loc}}$ the braided monoidal category of local left A-modules. Then $\mathcal{C}_A^{\mathrm{loc}}$ is precisely the category of modules for A

considered as a VOA that are objects of $\mathcal{C}$ when considered as V-modules [10], and the braided monoidal structure on $\mathcal{C}_A^{\mathrm{loc}}$ is precisely the vertex algebraic structure of Huang-Lepowsky-Zhang [5]. Thus rigidity and semisimplicity questions for V-modules or for A-modules are reduced to category theory.

The first question we address in [6] is when rigidity or semisimplicity of $\mathcal{C}$ implies rigidity or semisimplicity of $\mathcal{C}_A$ and $\mathcal{C}_A^{\mathrm{loc}}$. If $\mathcal{C}$ is rigid and semisimple and A is a separable algebra in $\mathcal{C}$, then Kirillov and Ostrik showed quite some time ago that $\mathcal{C}_A$ and $\mathcal{C}_A^{\mathrm{loc}}$ are also rigid and semisimple [12], with the dual of an object M of $\mathcal{C}_A$ given by an A-module structure on the $\mathcal{C}$-dual of M. If $\mathcal{C}$ is ribbon, then the question of separability largely reduces to whether the categorical dimension of A in $\mathcal{C}$ is non-zero. For pseudo-unitary categories, this is no problem, but unfortunately most ribbon categories of VOA modules are not pseudo-unitary, so we need better criteria on A that guarantee rigidity of $\mathcal{C}_A$.

If $\mathcal{C}$ is a finite tensor category, then the solution is to consider exact rather than separable algebras in $\mathcal{C}$. A $\mathcal{C}$-algebra A is exact if for any object M in $\mathcal{C}_A$ and projective object P in $\mathcal{C}$, the module $M \otimes P$ is projective in $\mathcal{C}_A$. If A is an exact commutative haploid algebra in a finite braided tensor category $\mathcal{C}$, then it turns out $\mathcal{C}_A$ and $\mathcal{C}_A^{\mathrm{loc}}$ are both rigid, where the dual of an A-module M is no longer necessarily an A-module structure on the $\mathcal{C}$-dual of M [14]. The idea of the proof is to embed $\mathcal{C}_A$ and $\mathcal{C}_A^{\mathrm{loc}}$ into the monoidal category of A-bimodules in $\mathcal{C}$, which one then identifies with the category of exact $\mathcal{C}$-module endofunctors of $\mathcal{C}_A$, in which duals are given by adjoint functors.

Now we can use a very recent result of Coulembier-Stroiński-Zorman [4], which improves on a result of Etingof-Ostrik [8], that an algebra in a finite tensor category is exact (and thus $\mathcal{C}_A$ and $\mathcal{C}_A^{\mathrm{loc}}$ are rigid) if and only if it is a direct product of simple algebras. As a corollary, if A is a simple commutative algebra in a fusion category $\mathcal{C}$, then $\mathcal{C}_A$ and $\mathcal{C}_A^{\mathrm{loc}}$ are also fusion (because then $\mathbf{1}$ is projective in $\mathcal{C}$ and thus $M \otimes \mathbf{1} \cong M$ is projective in $\mathcal{C}_A$ for any M because A is exact). This corollary yields a very nice VOA application:

Theorem 1. *Any simple $\mathbb{N}$-graded VOA that contains a conformally-embedded strongly rational subalgebra is itself strongly rational and thus has a semisimple modular tensor category of representations.*

In particular, this theorem does not require any non-vanishing categorical dimension condition. Of course, we also have immediate rigidity applications for VOA extensions in non-semisimple module categories. For example, we say a VOA is strongly finite if it has all the properties of a strongly rational VOA except that its module category may be non-semisimple. Then:

Theorem 2. *If A is an $\mathbb{N}$-graded simple self-contragredient VOA extension of a strongly finite VOA V such that* $\mathrm{Rep}(V)$ *is rigid, then* $\mathrm{Rep}(A)$ *is rigid and is moreover a (not necessarily semisimple) modular tensor category.*

The second question we consider in [6] is the converse of the first one: If $\mathcal{C}_A^{\mathrm{loc}}$ is rigid, then when are $\mathcal{C}_A$ and $\mathcal{C}$ rigid? This question is more difficult than the first, and at the moment we have only some partial solutions, in particular [6, Theorem

3.21]. The first problem is that we need some way to get rigidity of $\mathcal{C}_A$ from its small subcategory $\mathcal{C}_A^{\text{loc}}$, since we want to exploit the induction (or free module) functor from $\mathcal{C}$ to $\mathcal{C}_A$. Second, we need at least some weak duality structure on $\mathcal{C}$ to get started with proving rigidity of $\mathcal{C}$.

The solution to the second problem is Grothendieck-Verdier category structure, as defined in [3], which for VOAs is given by contragredient modules [2]. Thus we show first that if A is a commutative algebra in a Grothendieck-Verdier category $\mathcal{C}$, then $\mathcal{C}_A$ is also a Grothendieck-Verdier category, and so is $\mathcal{C}_A^{\text{loc}}$ if $\mathcal{C}$ has a ribbon twist that squares to the identity on A. We then assume every simple object of $\mathcal{C}_A$ is local and use Grothendieck-Verdier duality and induction on length to show that if $\mathcal{C}$ is locally finite abelian and $\mathcal{C}_A^{\text{loc}}$ is rigid, then $\mathcal{C}_A$ is also rigid. The assumption that every simple object of $\mathcal{C}_A$ is local may seem strange, and it is definitely not true in general in semisimple settings, but it does seem to hold for free field realization-like VOA extensions in logarithmic conformal field theory that motivated our work in [6].

Finally, we prove in [6, Theorem 3.21] that if $\mathcal{C}$ is a suitable Grothendieck-Verdier category, A is a suitable commutative algebra in $\mathcal{C}$, and $\mathcal{C}_A^{\text{loc}}$ is rigid, then $\mathcal{C}$ is also rigid under some conditions. The first condition, as already mentioned, is that every simple object of $\mathcal{C}_A$ is local. The second condition amounts to the assumption that the induction functor from $\mathcal{C}$ to $\mathcal{C}_A$ commutes with Grothendieck-Verdier duals in $\mathcal{C}$ and $\mathcal{C}_A$. Third, we need a mild non-degeneracy condition on $\mathcal{C}$. In this setting, $\mathcal{C}_A$ is rigid and thus so is its Drinfeld center $\mathcal{Z}(\mathcal{C}_A)$. Then since induction lifts to a braided tensor functor $F : \mathcal{C} \to \mathcal{Z}(\mathcal{C}_A)$, rigidity of $\mathcal{C}$ follows from rigidity of $\mathcal{Z}(\mathcal{C}_A)$ if we can show that F is an embedding and commutes with Grothendieck-Verdier duals in $\mathcal{C}$ and rigid duals in $\mathcal{Z}(\mathcal{C}_A)$. These two properties of F follow from the second and third conditions above, and so indeed $\mathcal{C}$ is rigid.

As an application of [6, Theorem 3.21], we proved rigidity of the category of weight modules for the affine VOA of $\mathfrak{sl}_2$ at admissible levels in [7], where we take the algebra A to be Adamović's quantum Hamiltonian reduction [1] of the affine VOA of $\mathfrak{sl}_2$. The rigidity of this category was also proved independently and simultaneously using analytic methods in [13].

References

[1] D. Adamović, *Realizations of simple affine vertex algebras and their modules: the cases* $\widehat{sl(2)}$ *and* $\widehat{osp(1,2)}$, Comm. Math. Phys. **366** (2019), no. 3, 1025–1067.

[2] R. Allen, S. Lentner, C. Schweigert and S. Wood, *Duality structures for module categories of vertex operator algebras and the Feigin Fuchs boson*, Selecta Math. (N.S.) **31** (2025), no. 2, Paper No. 36, 57 pp.

[3] M. Boyarchenko and V. Drinfeld, *A duality formalism in the spirit of Grothendieck and Verdier*, Quantum Topol. **4** (2013), no. 4, 447–489.

[4] T. Coulembier, M. Stroiński and T. Zorman, *Simple algebras and exact module categories*, arXiv:2501.06629.

[5] T. Creutzig, S. Kanade and R. McRae, *Tensor categories for vertex operator superalgebra extensions*, Mem. Amer. Math. Soc. **295** (2024), no. 1472, vi+181 pp.

[6] T. Creutzig, R. McRae, K. Shimizu and H. Yadav, *Commutative algebras in Grothendieck-Verdier categories, rigidity, and vertex operator algebras*, arXiv:2409.14618.

[7] T. Creutzig, R. McRae and J. Yang, *Ribbon categories of weight modules for affine* $\mathfrak{sl}_2$ *at admissible levels*, arXiv:2411.11386.

[8] P. Etingof and V. Ostrik, *On the Frobenius functor for symmetric tensor categories in positive characteristic*, J. Reine Angew. Math. **773** (2021), 165–198.

[9] Y.-Z. Huang, *Rigidity and modularity of vertex tensor categories*, Commun. Contemp. Math. **10** (2008), suppl. 1, 871–911.

[10] Y.-Z. Huang, A. Kirillov, Jr. and J. Lepowsky, *Braided tensor categories and extensions of vertex operator algebras*, Comm. Math. Phys. **337** (2015), no. 3, 1143–1159.

[11] Y.-Z. Huang and J. Lepowsky, *Tensor categories and the mathematics of rational and logarithmic conformal field theory*, J. Phys. A **46** (2013), no. 49, Paper No. 494009, 21 pp.

[12] A. Kirillov, Jr. and V. Ostrik, *On a q-analogue of the McKay correspondence and the ADE classification of* $\mathfrak{sl}_2$ *conformal field theories*, Adv. Math. **171** (2002), no. 2, 183–227.

[13] H. Nakano, F. Orosz Hunziker, A. Ros Camacho and S. Wood, *Fusion rules and rigidity for weight modules over the simple admissible affine* $\mathfrak{sl}(2)$ *and* $\mathcal{N} = 2$ *superconformal vertex operator superalgebras*, arXiv:2411.11387.

[14] K. Shimizu and H. Yadav, *Commutative exact algebras and modular tensor categories*, arXiv:2408.06314.

Every conformal net has an associated unitary VOA

James Tener

(joint work with André Henriques)

Conformal nets and unitary vertex operator algebras (VOAs) are two prominent axiomatizations of two-dimensional chiral conformal field theories, and both of these notions have attracted substantial independent interest as mathematical areas of study. Since the two notions are supposed to axiomatize the same physical structure, it is widely believed that they are equivalent. That is, there are expected to be mutually inverse constructions of conformal nets from unitary VOAs, and vice versa. A precise version of this conjecture first appeared in the landmark article of Carpi-Kawahigashi-Longo-Weiner [CKLW18], which also provided a family of tools and ideas for studying the VOA-to-conformal net correspondence.

In recent joint work with Henriques [HT25a], we establish one direction of this correspondence: we show that there is a unitary VOA associated to every conformal net. We also show that the conformal net can be recovered from the unitary VOA from the following construction (see [CKLW18, RTT22]). Given a unitary VOA, we say that it is *AQFT-local* if smeared vertex operators $Y(v, f)$ and $Y(u, g)$ commute strongly when f and g have disjoint support. Here, "commute strongly" means that the von Neumann algebras generated by (the closures of) these unbounded operators commute. This is closely related to the notion of "strong locality" considered in [CKLW18], which is the property of AQFT-locality plus a technical "polynomial energy bounds" condition. When a unitary VOA is AQFT-local, there is an associated conformal net whose local algebras $\mathcal{A}(I)$ are generated by smeared fields $Y(v, f)$ with f supported in the interval I (shown in [CKLW18] for strongly local VOAs, and in [RTT22] under the weaker assumption of AQFT-locality). It is shown in [HT25a] that the unitary VOA associated to a conformal net is always AQFT-local, and the conformal net can be reconstructed from the construction outlined above. This yields an equivalence between conformal nets and AQFT-local unitary VOAs. Following [CKLW18], we conjecture that every unitary VOA is AQFT-local, but this problem remains open, and new ideas will be required.

The state-field correspondence of a unitary VOA can be encoded in a map which assigns to every configuration of distinct points in the open unit disk of $\mathbb{C}$, each labeled by a vector of the VOA, a vector in the Hilbert space completion of the VOA. The construction of a unitary VOA from a conformal net begins with an analogous construction of insertions along intervals, rather than points. More precisely, given a conformal net, the corresponding "worm insertions" are as follows: given a collection of disjoint intervals in the unit disk, each labeled by an element of the corresponding local algebra (in the sense of coordinate-free conformal nets [BDH15]), we construct a vector in the vacuum Hilbert space of the conformal net. This construction relies crucially on the fact that the vacuum Hilbert space carries a natural representation of the semigroup of (thin) annuli, which was studied in [HT24, HT25b]. The point insertions of the unitary VOA that corresponds to the conformal net are then constructed as finite linear combinations of worm insertions.

References

[BDH15] A. Bartels, C. L. Douglas, and A. Henriques. Conformal nets I: Coordinate-free nets. *Int. Math. Res. Not. IMRN*, (13):4975–5052, 2015.

[CKLW18] S. Carpi, Y. Kawahigashi, R. Longo, and M. Weiner. From vertex operator algebras to conformal nets and back. *Mem. Amer. Math. Soc.*, 254(1213), 2018.

[HT24] A. G. Henriques and J. E. Tener. The Segal-Neretin semigroup of annuli. *arXiv:2410.05929 [math.DG]*, 2024.

[HT25a] A. Henriques and J. Tener. Every conformal net has an associated unitary VOA. *arXiv:2507.20735 [math.OA]*, 2025.

[HT25b] A. G. Henriques and J. E. Tener. Integrating positive energy representations of the Virasoro algebra. *arXiv:2506.08684 [math.FA]*, 2025.

[RTT22] C. Raymond, Y. Tanimoto, and J. E. Tener. Unitary vertex algebras and Wightman conformal field theories. *Comm. Math. Phys.*, 395(1):299–330, 2022.

Entanglement and bimodule categories in quantum spin chains

Frank Verstraete

(joint work with Laurens Lootens, Clement Delcamp)

The fields of entanglement theory and tensor networks have recently emerged as central tools for characterising quantum phases of matter [1]. In this talk, I will discuss thse entanglement structure of ground states of gapped symmetric quantum lattice models, and use this to obtain the most efficient tensor network representation of those ground states [2]. We do this by showing that degeneracies in the entanglement spectrum arise through a duality transformation of the original model to the unique dual model where the entire dual (generalised) symmetry is spontaneously broken and subsequently no degeneracies are present. Physically, this duality transformation amounts to a (twisted) gauging of the unbroken symmetry in the original ground state. This result has strong implications for the complexity of simulating many-body systems using variational tensor network methods. For every phase in the phase diagram, the dual representation of the ground state that completely breaks the symmetry minimises both the entanglement entropy and the required number of variational parameters. We demonstrate the applicability of this idea by developing a generalised density matrix renormalisation group algorithm that works on (dual) constrained Hilbert spaces, and quantify the computational gains obtained over traditional tensor network methods in a perturbed Heisenberg model. Our work testifies to the usefulness of generalised non-invertible symmetries and their formal category theoretic description for the practical simulation of strongly correlated systems.

References

[1] L Lootens, J Fuchs, J Haegeman, C Schweigert, F Verstraete, *Matrix product operator symmetries and intertwiners in string-nets with domain walls*, SciPost Physics 10 (3), 053 (2021)

[2] L Lootens, C Delcamp, F Verstraete, *Entanglement and the density matrix renormalisation group in the generalised Landau paradigm* - preprint arXiv:2408.06334 (2024)

Standard Subspaces and Twisted Araki-Woods Subfactors

Ricardo Correa da Silva, Gandalf Lechner

The purpose of this talk was to describe a large family of interesting subfactors that are often of type III, without normal conditional expectation (infinite index), but despite these differences have various analogies with the more familiar finite index type II_1-subfactors. For instance, they require an underlying braiding and a modular theory version of the subfactor-theoretic Fourier transform. These subfactors go by the name of twisted Araki-Woods subfactors and have been introduced in [CdSL23]. They are based on two data: an inclusion of standard subspaces and a twist.

Inclusions of standard subspaces. A *standard subspace* H of a complex Hilbert space $\mathcal{H}$ is a closed real subspace $H \subset \mathcal{H}$ such that $H \cap iH = \{0\}$ and $\overline{H + iH} = \mathcal{H}$. Specific examples of standard subspaces arise from von Neumann algebras $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ with cyclic separating vector Ω as $H := \overline{\mathcal{M}_{\mathrm{sa}}\Omega}$. Although not all standard subspaces are of this form, the lattice $\mathrm{Std}(\mathcal{H})$ of all standard subspaces of $\mathcal{H}$ has interesting structural similarities to the lattice of von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$ and subfactors:

(i) Symplectic complementation $H \mapsto H' := \{v \in \mathcal{H} \,:\, 0 = \mathrm{Im}\langle v, h\rangle \,\forall h \in H\}$ is an order-reversing involution on $\mathrm{Std}(\mathcal{H})$, resembling the commutant of von Neumann algebras and the Bicommutant Theorem,

(ii) there is a natural notion of factor subspace, namely $H \in \mathrm{Std}(\mathcal{H})$ with $H \cap H' = \{0\}$,

(iii) proper irreducible inclusions $K \subsetneq H$ of factor subspaces $K, H \in \mathrm{Std}(\mathcal{H})$ exist (i.e. $K' \cap H = \{0\}$), resembling irreducible subfactors,

(iv) any inclusion $H_0 \subset H_1$ of standard subspaces $H_0, H_1 \in \mathrm{Std}(\mathcal{H})$ naturally extends to a tower and tunnel

$$\ldots \subset H_{-1} \subset H_0 \subset H_1 \subset H_2 \subset \ldots , \tag{1}$$

resembling iterations of Jones' basic construction.

While inclusions of standard subspaces do not come with an index, and are basically incompatible with (analogues of) conditional expectations, a good replacement for these missing tools is modular theory: Any $H \in \mathrm{Std}(\mathcal{H})$ defines a Tomita operator $S_H : H + iH \to H + iH$, given by $S_H(h_1 + ih_2) := h_1 - ih_2$, and the polar decomposition of this closed involution defines a one-parameter group of unitaries Δ_H^{it}, $t \in \mathbb{R}$, preserving H, and an antiunitary J_H mapping H onto H'. With this technique one for instance quickly checks that $H_2 := J_{H_1} J_{H_0} H$ and $H_{-1} := J_{H_0} J_{H_1} H_0$ are standard subspaces satisfying (1). One also checks that proper inclusions $K \subsetneq H$ can only exist for $\dim \mathcal{H} = \infty$ because $K \subset H$ is equivalent to an extension $S_K \subset S_H$ of Tomita operators.

Inclusions of standard subspaces can be seen as a spatial analogue of subfactors, and are of interest in their own right [CdSL23]. No canonical map from inclusions of standard subspaces to inclusions of von Neumann algebras exists, which is why we have to introduce more data to define twisted Araki-Woods subfactors.

Twisted Araki-Woods von Neumann algebras. Given a complex Hilbert space $\mathcal{H}$, an operator $T = T^* \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ with $\|T\| \leq 1$ is called a *twist* if the operators $P_{T,n} \in \mathcal{B}(\mathcal{H}^{\otimes n})$, $n \in \mathbb{N}$, iteratively defined by

$$P_{T,1} = 1, \quad P_{T,2} := 1 + T, \quad P_{T,n+1} = (1 \otimes P_{T,n})(1 + T_1 + T_1 T_2 + \ldots + T_1 \cdots T_n)$$

(in standard tensor leg notation), are all *positive*. In case T satisfies the Yang-Baxter equation, $P_{T,n}$ is the corresponding quantum symmetrizer.

Given a twist T, we consider the tensor algebra $\bigoplus_{n \geq 0} \mathcal{H}^{\otimes n}$ and the quotient by its left ideal $\bigoplus_{n \geq 0} \ker P_{T,n}$. Completed in the scalar product given by $\langle [\Psi], [\Phi] \rangle_T = \sum_{n \geq 0} \langle [\Psi]_n, P_{T,n} [\Phi]_n \rangle$, it becomes a Hilbert space (the T-twisted Fock space $\mathcal{F}_T(\mathcal{H})$), on which left tensor multiplication by $\xi \in \mathcal{H}$ defines an operator $a^*_{T,L}(\xi)$. With these definitions, the left twisted Araki-Woods von Neumann algebra with twist T and standard subspace $H \in \mathrm{Std}(\mathcal{H})$ is

$$\mathcal{L}_T(H) := \{ a^*_{T,L}(h) + a_{T,L}(h) \,:\, h \in H \}''. \tag{2}$$

Denoting by F the tensor flip, the von Neumann algebras $\mathcal{L}_{qF}(H)$ are second quantization factors for $q = 1$, generated by CAR algebras for $q = -1$, free group factors for $q = 0$ and H maximally abelian [Voi85], variations of free group factors for $q = 0$ and general standard subspace [Shl97]. So even in this very restricted class of examples one sees type I, II, and III von Neumann algebras, commutative and noncommutative ones, hyperfinite and non-hyperfinite ones, showing that $\mathcal{L}_T(H)$ depends crucially on H and T.

From the point of view of modular theory, it is most important to understand when the Fock vacuum $\Omega \in \mathcal{F}_T(\mathcal{H})$ is cyclic and separating for $\mathcal{L}_T(H)$.

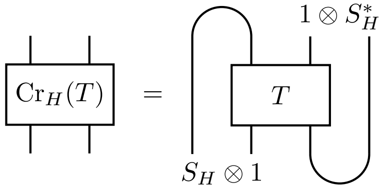


FIGURE 1. Graphical representation of the crossing map.

Theorem 1. *Let $H \in \mathrm{Std}(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ be a twist such that $[T, \Delta_H^{it} \otimes \Delta_H^{it}] = 0$ for all $t \in \mathbb{R}$. Then Ω is cyclic and separating for $\mathcal{L}_T(H)$ if and only if T satisfies the Yang-Baxter equation and is* crossing-symmetric *w.r.t. H (explained below).*

Crossing-Symmetry. In order to define crossing-symmetry, we begin by saying that an operator $T \in \mathcal{B}(\mathcal{H}^{\otimes 2})$ is crossable if the equation given in terms of matrix-coefficients

$$\langle \psi_1 \otimes \psi_2, \mathrm{Cr}_H(T)\psi_3 \otimes \psi_4 \rangle = \langle \psi_2 \otimes S_H^* \psi_4, T(S_H\psi_1 \otimes \psi_3)\rangle \tag{3}$$

defines a bounded operator $\mathrm{Cr}_H(T)$. In case the Hilbert space is infinite dimensional, one has to take into consideration that the vectors ψ_1 and ψ_4 must lie in the domain of S_H and S_H^*, respectively. A crossable operator is called crossing-symmetric if $\mathrm{Cr}_H(T) = T^*$.

The Yang-Baxter equation and crossing-symmetry, in the light of the theorem above, are equivalent to a KMS condition and, in particular, the crossing-symmetry carries all the analytic content of the KMS condition [CGL24]. Furthermore, one can immediately recognize the connection between the crossing map and the subfactor-theoretic Fourier transform when representing the map defined in (3) in graphical notation, where we highlight the dependence of the standard subspace through S_H and S_H^*, in contrast with the subfactor-theoretic Fourier transform.

The behaviour of two examples of twists under the crossing map is worth mentioning: The tensor flip F is always crossing-symmetric independent of H. The identity operator is crossable if and only if $\dim \mathcal{H} < \infty$, in which case $\mathrm{Cr}_H(1) = \mathrm{Tr}(\Delta_H)P_\xi$, where P_ξ is the orthogonal projection in the direction of the vector $\xi = \sum_{n=1}^{\dim \mathcal{H}} e_n \otimes S_H e_n$, which is a Temperley-Lieb projection, *i.e.* $(P_\xi \otimes 1)(1 \otimes P_\xi)(P_\xi \otimes 1) = \mathrm{Tr}(\Delta_H)^{-1} P_\xi \otimes 1$, as one could expected from the connection between the subfactor-theoretic Fourier transform and the Temperley–Lieb algebra.

Inclusions of Twisted Araki-Woods von Neumann algebras. It turns out that if T satisfies the Yang-Baxter equation, right tensor multiplication by $\xi \in \mathcal{H}$ also defines an operator denoted $a_{T,R}^*(\xi)$ and, similarly, the right twisted Araki-Woods von Neumann algebra $\mathcal{R}_T(H)$. Under the hypotheses of Theorem 1, namely, in case T is also crossing-symmetric w.r.t. H and satisfies $[T, \Delta_H^{it} \otimes \Delta_H^{it}] =$

0 for all $t \in \mathbb{R}$, we can also determine the commutant of the twisted Araki-Woods algebras as $\mathcal{L}_T(H)' = \mathcal{R}_T(H')$.

Given an inclusion of standard subspaces $K \subset H$, we have the corresponding inclusion of the von Neumann algebras $\mathcal{L}_T(K) \subset \mathcal{L}_T(H)$. We are interested in knowing when such inclusion is irreducible, *i.e.* when the relative commutant satisfies $\mathcal{C}(K,H) := \mathcal{L}_T(K)' \cap \mathcal{L}_T(H) = \mathbb{C} \cdot 1$, [CdSL23, CdSL25].

Theorem 2. *Let $H \in \mathrm{Std}(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ be a twist such that, $\|T\| < 1$, $[T, \Delta_H^{it} \otimes \Delta_H^{it}] = 0$ for all $t \in \mathbb{R}$, T satisfies the Yang-Baxter equation and is crossing-symmetric w.r.t. H. Then, if $\Delta_H^{\frac{1}{4}}|_K$ is non-compact, $\mathcal{C}(K,H) = \mathbb{C} \cdot 1$.*

It follows that, if we have a twist T satisfying the hypothesis of the theorem above for the standard subspace H and another standard subspace $K \subset H$ such that $\Delta_H^{\frac{1}{4}}|_K$ is non-compact, then both $\mathcal{L}_T(K)$ and $\mathcal{L}_T(H)$ are factors. It is known that in the particular case of $T = qF$, the algebras $\mathcal{L}_T(H)$ are always factors [KSW23], but for general T the question is still open.

We remark that, for applications in Algebraic Quantum Field Theory, one is often interested in having a large relative commutant. In that direction, we can say that, if in the situation of Theorem 2 it also holds that $\Delta_K^{-\frac{1}{4}}\Delta_H^{\frac{1}{4}}$ is trace class and its trace norm is less than 1, then the relative commutant $\mathcal{C}(K,H)$ is type III in case $\mathcal{L}_T(H)$ is a type III factor.

References

[CdSL23] R. Correa Da Silva and G. Lechner. Modular Structure and Inclusions of Twisted Araki-Woods Algebras. *Communications in Mathematical Physics*, 402 (3), pp. 2339–2386, 2023.

[CGL24] R. Correa da Silva, L. Giorgetti and G. Lechner. Crossing Symmetry and the Crossing Map. *Reviews in Mathematical Physics*, 2461005, 2024.

[CdSL25] R. Correa da Silva and G. Lechner. Inclusions of Standard Subspaces, arXiv:2506.16085, 2025.

[KSW23] M. Kumar, A. Skalski, M. Wasilewski. Full solution of the factoriality question for q-Araki-Woods von Neumann algebras via conjugate variables *Commun. Math. Phys.*, 2023.

[Shl97] D. Shlyakhtenko. Free quasi-free states. *Pacific J. Math.*, 177(2), pp. 329–368, 1997.

[Voi85] D. Voiculescu. Symmetries of some reduced free product C*-algebras. *Operator algebras and their connections with topology and ergodic theory*, pp. 556–588, 1985.

Group-subgroup subfactors revisited

Masaki Izumi

It is a famous story that the classical Goldman's theorem [1] inspired Vaughan Jones [10] to introduce the notion of an index for a subfactor. In modern terms, Goldman's theorem says that every subfactor of index 2 is given by the crossed product by the cyclic group $\mathbb{Z}_2$. In the case of index 3, a similar result holds in the sense that either the subfactor is given by the crossed product by $\mathbb{Z}_3$ if the principal graph is D_4, or it is given by the simultaneous crossed products of the symmetric groups $\mathfrak{S}_3 > \mathfrak{S}_2$ if the principal graph is A_5. In the talk, I present far-reaching generalizations of Goleman's type results obtained in [8].

Let G be a finite group and let H be a subgroup of G. Let α be an outer action of G on a factor R. Then we can get an irreducible subfator $R \rtimes_\alpha G \supset R \rtimes_\alpha H$, called a group-subgroup subfactor. We always assume that there is no non-trivial normal subgroup of G in H as such a normal subgroup is forgotten by the group-subgroup subfactor. Then the information of the pair $G > H$ is completely encoded in the transitive permutation group G acting on G/H, and (the conjugacy class of) H can be recovered from a point stabilizer. On the other hand, if G is a transitive permutation group acting on a finite set X, we can always get a pair of finite groups $G > H = G_x$ satisfying the above assumption, where G_x is the stabilizer of a point $x \in X$. Therefore when we discuss the group-subgroup subfactor arising from a pair of finite groups, we often refer to the corresponding transitive permutation group.

For a finite index subfactor $M \supset N$, we denote by $\mathcal{G}_{M\supset N}$ one of its principal graphs, more precisely, the induction-reduction graph between M-N and N-N bimodules generated by the bimodule ${}_M M_N$, that is, M as M-N bimodule. For $G > H$, we denote by $\Gamma_{G>H}$ the graph $\mathcal{G}_{M\supset N}$ with $M = R \rtimes G \supset N = R \rtimes H$. It is known that the tensor categorical structure of the inclusion $M \supset N$ is completely determined by $G > H$ (see [11]). The graph $\Gamma_{G>H}$ is given as follows. Let $G = \coprod_i Hg_iH$ be the double coset decomposition, and let $H_i = H \cap g_iHg_i^{-1}$. The odd vertices of $\Gamma_{G>H}$ are identified with the irreducible representations $\mathrm{Irr}(H)$ of H, and the even vertices are identified with $\coprod_i \mathrm{Irr}(H_i)$. The edges are given by induction and reduction between H and H_i.

In general we loose some information of $G > H$ by passing to $M = R \rtimes G \supset N = R \rtimes H$ (see [12], [7]).

Definition 1. We say that a Goldman's type theorem holds for $G > H$ if whenever a subfactor $P \supset Q$ satisfies $\mathcal{G}_{P\supset Q} = \Gamma_{G>H}$, there exists a subfactor $R \subset Q$ and an outer G-action α on R such that

$$P = R \rtimes_\alpha G \supset Q = R \rtimes_\alpha H.$$

Goldman's type theorems were first shown for some classes of Frobenius groups (see [5], [2], [3], [6]), and we recall its definition here. A transitive permutation group on a finite set is said to be regular if the action is free. A transitive permutation group G on a finite set X is said to be a Frobenius group if it is not regular and every $g \in G \setminus \{e\}$ has at most one fixed point. Let $H = G_{x_1}$ be a point stabilizer. Then G being Frobenius is equivalent to the condition that the H-action on $X \setminus \{x_1\}$ is free, and is further equivalent to the condition that $H \cap gHg^{-1} = \{e\}$ for all $g \in G \setminus H$. For a Frobenius group G,

$$K = G \setminus \bigcup_{x\in X} G_x$$

is a normal subgroup of G, called the Frobenius kernel, and G is a semi-direct product $K \rtimes H$ (see [13, 8.5.5]). The point stabilizer H is called a Frobenius complement.

Theorem 1. *Goldman's type theorem holds for every Frobenius group.*

Our original motivation for [6] is to apply the Goldman's type theorem to the classification of finite depth subfactors of index 5, which was established in [9]. For such an application, Frobenius groups are not enough and we needed a result applicable to the pairs $\mathfrak{A}_5 > \mathfrak{A}_4$ and $\mathfrak{S}_5 > \mathfrak{S}_4$.

Let k be a natural number greater than 1. A transitive permutation group G on a finite set X is said to be sharply k-transitive if the G-action on $X^{[k]}$ is regular, where $X^{[k]}$ is the set of distinct k-tuples of elements in X.

For $n \in \mathbb{N}$, let $X_n = \{1, 2, \ldots, n\}$. The defining action of the alternating group $\mathfrak{A}_n$ on X_n is sharply $n-2$-transitive. Since $X_n^{[n-1]}$ and $X_n^{[n]}$ are naturally identified, the defining action of $\mathfrak{S}_n$ on X_n is both sharply $n-1$ and n-transitive. Other than these two classes, we list all the sharply k-permutation groups below.

Every sharply 2-transitive permutation group G is known to be a Frobenius group, and hence of the form $G = \mathbb{Z}_p^k \rtimes H$ with a prime p and with a Frobenius complement H acting on $\mathbb{Z}_p^k \setminus \{0\}$ regularly. Let $q = p^k$, and let $\mathbb{F}_q$ be the finite field of of order q. Then $\mathbb{Z}_p^k$ is isomorphic to $\mathbb{F}_q$ as an additive group. Other than 7 exceptions, the Frobenius complecment H is isomorphic to a subgroup of $\mathbb{F}_q^\times \rtimes \mathrm{Aut}(\mathbb{F}_q)$ (see [4, Chapter XII, Section p]). The Affine group $H(q) = \mathbb{F}_q \rtimes \mathbb{F}_q^\times$ acting on $\mathbb{F}_q$ is a typical example of a sharply 2-permutation group. When p is an odd prime and $q = p^{2l}$, the field $\mathbb{F}_q$ has an involution $x^\sigma = x^{p^l}$. The group $S(q)$ has a Frobenius complement $\mathbb{F}_q^\times$ as a set, but its action on $\mathbb{F}_q$ is given as follows:

$$a \cdot x = \begin{cases} ax, & \text{if } a \text{ is a square in } \mathbb{F}_q^\times, \\ ax^\sigma, & \text{if } a \text{ is not a square in } \mathbb{F}_q^\times. \end{cases}$$

For example, the group $S(3^2)$ is isomorphic $\mathbb{Z}_3^2 \rtimes Q_8$.

There are exactly two families of sharply 3-transitive permutation groups $L(q)$ and $M(q)$, and they are transitive extensions of $H(q)$ and $S(q)$ respectively (see [4, Chapter XI, Section 2]). To describe their actions, it is convenient to identify the projective geometry $PG_1(q) = (\mathbb{F}_q^2 \setminus \{0\})/\mathbb{F}_q^\times$ with $\mathbb{F}_q \sqcup \{\infty\}$. The 3-transitive action of $L(q) = PGL_2(q)$ is given as follows:

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \cdot x = \frac{ax+b}{cx+d}.$$

The group $M(q)$ is $PGL_2(q)$ as a set, but its action on $PG_1(q)$ is given by

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \cdot x = \begin{cases} \frac{ax+b}{cx+d}, & \text{if } ad-bc \text{ is a square in } \mathbb{F}_q^\times, \\ \frac{ax^\sigma+b}{cx^\sigma+d}, & \text{if } ad-bc \text{ is not a square in } \mathbb{F}_q^\times. \end{cases}$$

Other than symmetric groups and alternating groups, the Mathieu groups M_{11} and M_{12} are the only sharply 4 and 5-transitive permutation groups, and their degrees are 11 and 12 respectively (see [4, Chapter XII, Section 3]).

Conjecture 2. *Golsman's type theorem holds for every sharply k-permutation group.*

Theorem 2. *The above conjecture is true for $k = 2, 3, 4$.*

Note that $PSL_2(q)$ is a 2-transitive permutation group on $PG_1(q)$, though it is not sharply 2-transitive.

Theorem 3. *Goldman's type theorem holds for*

$$PSL_2(q) > \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix};\ a \in \mathbb{F}_q^{\times},\ b \in \mathbb{F}_q \right\} / \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix};\ a^2 = 1 \right\}.$$

References

[1] Goldman, Malcolm. *On subfactors of factors of type II_1*. Michigan Math. J. **6** (1959), 167–172.

[2] Hong, Jeon Hee. *Subfactors with principal graph* $\mathbf{E}_6^{(1)}$, Acta Appl. Math. **40** (1995), no. 3, 255–264.

[3] Hong, Jeon Hee ; W. Szymański. *On finite subfactors with principal graph* $\mathbf{D}_{2n+1}/\mathbf{Z}_2$, J. Funct. Anal. **141** (1996), no. 2, 294–300.

[4] Huppert, Bertram; Blackburn. *Finite groups III.* Grundlehren der Mathematischen Wissenschaften, 243. Springer-Verlag, Berlin-New York, 1982.

[5] Izumi, Masaki. *Goldman's type theorem for index 3.* Publ. Res. Inst. Math. Sci. **28** (1992), no. 5, 833–843.

[6] Izumi, Masaki . *Goldman's type theorems in index theory.* Operator algebras and quantum field theory (Rome, 1996), 249–269, Int. Press, Cambridge, MA, 1997.

[7] Izumi, Masaki. *Characterization of isomorphic group-subgroup subfactors.* Int. Math. Res. Not. 2002, no. **34**, 1791–1803.

[8] M. Izumi, *Group-subgroup subfactors revisited*, Quantum Topol. **15** (2024), 721–778.

[9] Izumi, Masaki; Morrison, Scott; Penneys, David; Peters, Emily; Snyder, Noah. *Subfactors of index exactly 5.* Bull. Lond. Math. Soc. **47** (2015), no. 2, 257–269.

[10] Jones, Vaughan F. R. *Index for subfactors.* Invent. Math. **72** (1983), no. 1, 1–25.

[11] Kosaki, Hideki ; Yamagami, Shigeru. *Irreducible bimodules associated with crossed product algebras.* Internat. J. Math. **3** (1992), no. 5, 661–676.

[12] Kodiyalam, Vijay; Sunder, V. S. *The subgroup-subfactor.* Math. Scand. *86* (2000), no. 1, 45–74.

[13] Robinson, Derek J. S. *A course in the theory of groups.* Graduate Texts in Mathematics, 80. Springer-Verlag, New York, 1993.

Some recent results on superselection sector theory for topologically ordered models

Pieter Naaijkens

(joint work with Alex Bols, Mahdie Hamdan, Siddharth Vadnerkar)

Kitaev's quantum double model [5] is a prototypical example of a topologically ordered, long-range entangled quantum system. One interesting aspect is that such quantum systems can have anyonic excitations, whose algebraic properties are described by a braided (in our setting, typically even modular) tensor category. Our main goal is to obtain this tensor category from first principles for the quantum double model, using an operator-algebraic approach.

The quantum double model can be defined for any finite group [5]. We will consider the model for a finite group G, and defined on an infinite triangular lattice in 2D. More precisely, write Γ for the set of edges of the triangular lattice. At each edge we put a G-dimensional Hilbert space. For any finite set of edges $\Lambda_f \subset \Gamma$, we define the algebra of local observables $\mathfrak{A}(\Lambda_f) := \bigotimes_{x\in\Lambda_f} M_{|G|}(\mathbb{C})$. Inclusions of sets of edges define a (unital) inclusion of the corresponding observable algebras. Hence we get a net $\Lambda \mapsto \mathfrak{A}(\Lambda)$, and its direct limit in the category of C*-algebras is called the quasi-local algebra. The net is local in the sense that $[\mathfrak{A}(\Lambda_1), \mathfrak{A}(\Lambda_2)] = 0$ if $\Lambda_1 \cap \Lambda_2 = \emptyset$. Of interest are regions which are called *cones*, which are essentially obtained by intersecting a cone in $\mathbb{R}^2$ with Γ (see [7] for a more precise definition). For any subset $\Lambda \subset \Gamma$, we write Λ^c for its complement in Γ.

For the definition of the dynamics of the quantum double model we refer to [5]. For us it suffices to know that here there is a unique frustration free ground state, and we will write π_0 for the corresponding irreducible GNS representation. This representation satisfies:

- (Properly infiniteness) For each cone Λ, $\pi_0(\mathfrak{A}(\Lambda))''$ is an infinite factor [7].
- (Haag duality) For each cone Λ, $\pi_0(\mathfrak{A}(\Lambda))'' = \pi_0(\mathfrak{A}(\Lambda^c))'$.

At the time [2] was completed, we did not yet have a proof for Haag duality. However, Haag duality has been proven for the quantum double model for abelian groups [3]. We expect that the proof techniques can be generalised to non-abelian models. More importantly, a proof of Haag duality for cones for Levin–Wen string-net models has recently been announced [8]. This implies that at least the sufficient condition of approximate Haag duality holds.

From the assumptions above it follows (using the Doplicher–Haag–Roberts approach) that one can define a braided C*-category $\Delta(\pi_0)$ of superselection sectors (see [7] for the most general setting). This category is an invariant of the gapped ground state phase the initial frustration free ground state is in. It describes physical properties (such as braiding and fusion) of the anyons in the model.

The objects of $\Delta(\pi_0)$ can be identified with representations of $\mathfrak{A}$ satisfying the *superselection criterion*:

$$\pi \restriction \mathfrak{A}(\Lambda^c) \cong \pi_0 \restriction \mathfrak{A}(\Lambda^c) \qquad \text{for all cones } \Lambda. \tag{1}$$

That is, restricted to observables localised outside an *arbitrary* cone, the representation π is unitarily equivalent to π_0. An equivalence class of such representations is called a sector. We say that a sector is irreducible if a representative is an irreducible representation. These sectors correspond to the anyon types. The morphisms in the category are intertwiners between such representations.

For each vertex with an adjacent face (such a pair is called a *site*), we have a faithful unitary representation of $\mathcal{D}(G)$, the quantum double of the group algebra of G. Hence it is natural to expect that the representation theory of $\mathcal{D}(G)$ is related to the anyons. Our main result is that this is indeed the case [2]:

Theorem 1. *Let π_0 be the GNS representation of the frustration free ground state of the quantum double model for a finite group G. Then the categories $\Delta_f(\pi_0)$ and $\mathrm{Rep}_f\, \mathcal{D}(G)$ of finite dimensional unitary representations of $\mathcal{D}(G)$ are equivalent as braided* C*-*tensor categories.*

The subscript in $\Delta_f(\pi_0)$ means that we restrict to those representations π such that $\dim \mathrm{Hom}(\pi, \pi) < \infty$. We will comment on this later, but essentially it means we restrict to having finite direct sums only. The definition of the braiding and other structure on $\Delta_f(\pi_0)$ follows the standard DHR approach [3, 7]. The first step is showing that instead of representations satisfying (1), it is enough to consider localised and transportable *endomorphisms* of some C*-algebra $\mathfrak{B} \supset \pi_0(\mathfrak{A})$. In addition to the quasi-local observables, $\mathfrak{B}$ contains the cone von Neumann algebras (as long as the cone does not point in a "forbidden direction"). A monoidal product can then be defined via composition of endomorphisms, and a braiding using the localisation properties.

The construction of irreducible sectors is more complicated than in the abelian case [3]. To see why, it is helpful to recall the main idea. Pairs of excitations in the quantum double model can be created from the ground state using ribbon operators. The corresponding state only depends on the endpoints of the ribbon, and (in the abelian case) for each irrep of $\mathcal{D}(G)$ and choice of ribbon, there is a corresponding ribbon operators whose excitations are related to the irrep (and its conjugate). Then one can define an automorphism of $\mathfrak{A}$ via $\rho(A) := \lim_n F_{\xi_n} A F^*_{\xi_n}$, where ξ_n is a sequence of increasingly long ribbons obtained by moving one of its endpoints to infinity. It can be shown that $\pi_0 \circ \rho$ then satisfies (1).

For non-abelian irreps of $\mathcal{D}(G)$, however, one has to deal with *multiplets* of ribbon operators, which assemble into $k \times k$ unitary matrices $\mathbf{F} \in M_k(\mathfrak{A})$ of ribbon operators, where k is the dimension of the corresponding irrep. Hence, following [9], it is natural to consider *amplimorphisms*, i.e. $*$-homomorphisms $\mu : \mathfrak{A} \to M_k(\mathfrak{A})$. Similarly as before, we can then define $\mu(A) := \lim_{n\to\infty} \mathbf{F}_{\xi_n}(A \otimes I_k)\mathbf{F}^*_{\xi_n}$. It can be shown that $(\pi_0 \otimes \mathrm{id}_k) \circ \mu(A)$ again satisfies the selection criterion, motivating the study of the category $\mathrm{Amp}_f(\pi_0)$ of amplimorphism of $\mathfrak{B}$ such that all intertwiner spaces are finite dimensional.

The very explicit construction of amplimorphisms above enables one to explicitly write the so-called *charge transporters* as weak-operator limits of sequences of (matrices of) ribbon operators. The charge transporters are unitary intertwiners from an amplimorphism localised in one cone to another amplimorphism localised in a different cone. They are crucial in defining the braiding. Because the commutation properties of the ribbon operators are well understood in terms of the representation theory of $\mathcal{D}(\mathcal{G})$, this allows one to explicitly calculate the braiding. Similarly, fusion rules are readily obtained. This gives a braided equivalence between the category of (localised and transportable) amplimorphisms and $\mathrm{Rep}_f\, \mathcal{D}(G)$. The proof is then completed by the following two observations. First, using the properly infiniteness property of the cone algebras, the amplimorphims can be identified with the usual DHR *endo*morphisms of $\mathfrak{B}$. And secondly, the set of irreducible sectors constructed above is in fact a complete set of representatives of irreducible sectors [1].

From a theoretical point of view, it is somewhat unsatisfactory to have to restrict to objects with finite dimensional Hom-spaces, as this subcategory need not be closed under the monoidal product. (In our setting, this only follows once we have established that the irreducible sectors we construct are in fact *all* irreducible sectors, together with the fact that the fusion rules of these sectors are well understood.) Alternatively, one could restrict to the sectors which admit a conjugate, which *is* closed under the monoidal product [6].

Question. *Under what circumstances does a conjugate sector exist?*

In algebraic quantum field theory (AQFT) or for conformal nets this is automatically true in important examples, such as for massive particles (see e.g. [4] for an overview). It is also known that the existence of a conjugate for ρ is equivalent to the Jones index $[\rho(\mathfrak{B}) : \mathfrak{B}]$ being finite (cf. [6] and references therein). It would be interesting to have a physically relevant criterion for the case of superselection sectors of topologically ordered models that would guarantee this to be the case, similar to the results in AQFT or conformal nets.

References

[1] A. Bols, S. Vadnerkar, *Classification of the anyon sectors of Kitaev's quantum double model*, Commun. Math. Phys. **406**:188 (2025).

[2] A. Bols, M. Hamdan, P. Naaijkens, S. Vadnerkar, *The category of anyon sectors for non-abelian quantum double models.* Preprint, arXiv:2503.15611 (2025).

[3] L. Fiedler, P. Naaijkens, *Haag duality for Kitaev's quantum double model for abelian groups*, Rev. Math. Phys. **27**:1550021 (2015).

[4] D. Guido, R. Longo, *Relativistic invariance and charge conjugation in Quantum Field Theory*, Commun. Math. Phys. **148**:521–551 (1992).

[5] A. Kitaev, *Fault-tolerant quantum computation by anyons*, Ann. Phys. **303**:2–30 (2003).

[6] R. Longo, J.E. Roberts, *A theory of dimension*, K-Theory **11**:103–159 (1997).

[7] Y. Ogata, *A derivation of braided* C^**-tensor categories from gapped ground states satisfying the approximate Haag duality*, J. Math. Phys. **62**:011902 (2022).

[8] Y. Ogata, D. Pérez García, A. Ruíz de Alarcón, *Haag duality for 2D quantum spin systems.* Talk at "Anyons from Small to Large Scales", available at `https://www.mittag-leffler.se/activities/anyons-from-small-to-large-scales/` (2025).

[9] K. Szlachányi, P. Vecsernyés, *Quantum symmetry and braid group statistics in G-spin models*, Commun. Math. Phys. **156**:127–168 (1993).

Conformal Embedding Categories

Cain Edie-Michell

(joint work with Noah Snyder, Hans Wenzl)

Let

$$\mathcal{V}(\mathfrak{g}, k) \subseteq \mathcal{V}(\mathfrak{h}, 1)$$

be a conformal inclusion of Wess-Zumino-Witten vertex operator algebras [3]. It is well known that $\mathcal{V}(\mathfrak{h}, 1)$ has the structure of an etale algebra object in $\mathrm{Rep}(\mathcal{V}(\mathfrak{g}, k))$ [4]. This in turn gives a new (non-braided) tensor category $\mathrm{Rep}(\mathcal{V}(\mathfrak{g}, k))_{\mathcal{V}(\mathfrak{h},1)}$, the category of $\mathcal{V}(\mathfrak{h}, 1)$-modules internal to $\mathrm{Rep}(\mathcal{V}(\mathfrak{g}, k))$. Equivalently via Finkelberg's equivalence, these new categories are equivalent to $\overline{\mathrm{Rep}(U_q(\mathfrak{g}))}_A$, where A is the image of $\mathcal{V}(\mathfrak{h}, 1)$ under the Finkelberg equivalence.

While the combinatorial structure of the etale algebra A is known in all cases, the categorical structure of the multiplication map remains mysterious. This means that existence of the categories $\overline{\mathrm{Rep}(U_q(\mathfrak{g}))}_A$ is non-contructive, and their structure remains mysterious in general. A research program is hence to give presentations for the tensor categories $\overline{\mathrm{Rep}(U_q(\mathfrak{g}))}_A$ for all conformal embeddings. Some progress towards this program has already been made by Bigelow [1], who gave presentations for the categories coming from the conformal embeddings

$$\mathcal{V}(\mathfrak{sl}_2, 10) \subseteq \mathcal{V}(\mathfrak{so}_5, 1) \quad \text{and} \quad \mathcal{V}(\mathfrak{sl}_2, 28) \subseteq \mathcal{V}(\mathfrak{g}_2, 1),$$

and by Liu [5] who gave presentations the categories coming from the infinite family of conformal embeddings

$$\mathcal{V}(\mathfrak{sl}_N, N \pm 2) \subseteq \mathcal{V}(\mathfrak{sl}_{N(N\pm 1)/2}, 1).$$

In joint work with Noah Snyder we develop a new general method for giving presentations for the categories $\overline{\mathrm{Rep}(U_q(\mathfrak{g}))}_A$. We apply this new technique to give presentations for the infinite family of conformal embeddings

$$\mathcal{V}(\mathfrak{sl}_N, N) \subseteq \mathcal{V}(\mathfrak{so}_{N^2-1}, 1).$$

In addition we also construct a continuous familly of tensor categories which interpolates between this discrete infinite family. This interpolation category is similar in spirit to Delignes $\mathrm{Rep}(GL_t)$ categories [2]. We expect that our methods will work for all remaining infinite families of conformal embeddings, and for many of the sporadic examples as well.

While the methods developed by myself and Snyder give presentations whose Cauchy completions recover the conformal embedding categories, the techniques are non-constructive in the sense that the Cauchy completions are not deduced explicitly. In follow on work joint with Hans Wenzl, we explicitly deduce the structure of these Cauchy completions, and hence the combinatorial structure of the categories $\overline{\mathrm{Rep}(U_q(\mathfrak{sl}_N))}_A$ for the conformal embeddings $\mathcal{V}(\mathfrak{sl}_N, N) \subseteq \mathcal{V}(\mathfrak{so}_{N^2-1}, 1)$. We obtain a classification of the simple objects in terms of strict Young diagrams, along with certain sign choices. We also obtain explicit formulas for the fusion rules, and quantum dimensions. A key component of our proofs in this work is a surprising connection to the quantum isomeric Lie super algebra $U_q(\mathfrak{q}_N)$. This connection to quantum Lie super algebras appears to be special to type A, and does not seem to hold for conformal embeddings of other simple Lie algebras.

References

[1] S. Bigelow, *Skein theory for the ADE planar algebras*, J. Pure Appl. Algebra **214** (2010), 658–666.

[2] P. Deligne, *La catégorie des représentations du groupe symétrique S_t, lorsque t n'est pas un entier naturel*, Tata Inst. Fund. Res. Stud. Math. **19** (2007), 209–273.

[3] I. Frenkel, Y. Zhu, *Vertex operator algebras associated to representations of affine and Virasoro algebras*, Duke Math. J. **66** (1992), 123–168.

[4] A. Kirillov, V. Ostrik, *On a q-analogue of the McKay correspondence and the ADE classification of $\mathfrak{sl}_2$ conformal field theories*, Adv. Math. **32** (2002), 183–227.

[5] Z. Liu, *Yang-Baxter relation planar algebras*, Preprint.

The Delannoy planar algebra

Noah Snyder

The prototypical example of a planar algebra are the shaded Temperley-Lieb-Jones planar algebras with some loop parameter t [4, 5]. If you look at the even part of this planar algebra it can also be described as the universal monoidal category with a special Frobenius algebra of dimension t. The TLJ planar algebras have a close relationship to Deligne's interpolation category S_t [1], where you add symmetric crossings to the even part. This also corresponds to a universal property, namely that Deligne's S_t is the universal *symmetric* monoidal category with a *commutative* special Frobenius algebra.

Recently Harman and Snowden have developed a theory of measures on Oligomoprhic groups [2] which yields many new and interesting symmetric tensor categories. In their framework, you can recover Deligne's S_t as coming from the infinite symmetric group together with a measure which is determined by assigning t as the measure of the natural numbers as an S_∞-set. The simplest novel example coming from Harman-Snowden is called the Delannoy category [3] and comes from the group of order preserving automorphisms of the real line together with a measure coming from Euler characteristic.

In this talk, based on joint work in progress with Mikhail Khovanov, we reverse engineer back from the Delannoy category to a diagrammatically defined planar algebra. First, we give a diagrammatic description of the Delannoy category, using the universal property for the Delannoy category proved by Kriz [6]. Secondly, we see that by ignoring crossings we get a new planar algebra we call the Planar Delannoy Category. In the paper with Khovanov, though not in this talk, we will explain how to use the diagrams to completely describe the Delannoy category, recovering many of the main results of [3].

References

[1] P. Deligne, *La Catégorie des Représentations du Groupe Symétrique S_t, lorsque t n'est pas un Entier Naturel* Proceedings of the International Colloquium on Algebraic Groups and Homogeneous Spaces. January 2004, TIFR, Mumbai

[2] N. Harman and A. Snowden, *Oligomorphic groups and tensor categories*, arXiv:2204.04526

[3] N. Harman, A. Snowden, N. Snyder, *The Delannoy Category.* Duke Math. J. 173 (2024), no. 16, 3219-3291.

[4] V. Jones, *Index for Subfactors* Invent. Math. Volume 72, pages 1-25, (1983).

[5] V. Jones, *Planar algebras, I.* New Zealand J. Math. 52 (2021 [2021-2022]), 1-107.

[6] S. Kriz, *Quantum Delannoy Categories* `https://krizsophie.github.io/QuantumDelannoyCategory23111.pdf`

Witt-triviality of DHR bimodule categories

Makoto Yamashita
(joint work with Lucas Hataishi, David Jaklitsch, Corey Jones)

We look at nets of finite-dimensional C*-algebras $A_\bullet = (A_I)_{I \subset \mathbb{Z}}$ on a 1-dimensional lattice, and certain class of bimodules satisfying a localizability condition motivated by the Doplicher–Haag–Roberts theory of superselection sectors. This was introduced by Corey Jones [2], who showed that such bimodules form a unitary braided tensor category $\mathrm{DHR}(A_\bullet)$. Moreover, an analogue of the Longo–Roberts construction gives a realization of the Drinfeld center $\mathcal{Z}(\mathcal{C})$ for any unitary fusion category $\mathcal{C}$. In this joint work [1] with Lucas Hataishi, David Jaklitsch, and Corey Jones, we looked at the possibility of $\mathrm{DHR}(A_\bullet)$ in general, without knowing $\mathcal{C}$ beforehand. Under a suitable assumption on the subalgebra $C \subset A$ for $A = \bigvee_I A_I$ generated by "charge transporters" between sectors realized on the positive half line and the negative half line, we show that $\mathrm{DHR}(A_\bullet)$ admits nondegenerate braiding. Moreover, assuming that the Jones basic extension for the subalgebra $B_0 \subset A$, generated by the observables localized in either positive or negative half lines, is in the category $\mathrm{DHR}(A_\bullet)$, we show that $\mathrm{DHR}(A_\bullet)$ is in fact the Drinfeld center of the fusion category associated with the negative half line.

References

[1] Lucas Hataishi, David Jaklitsch, Corey Jones, and Makoto Yamashita, *On the structure of DHR bimodules of abstract spin chains*, preprint (arXiv:2504.06094), 2025.

[2] Corey Jones, *DHR bimodules of quasi-local algebras and symmetric quantum cellular automata*, Quantum Topol. **15** (2024), no. 3, 633–686.

Gauging non-invertible symmetries in 3d topological field theory

Ingo Runkel

1. Orbifold data

Let $\mathcal{C}$ be an additive idempotent-complete ribbon category.

Definition. [1] An *orbifold datum* $\mathbb{A}$ in $\mathcal{C}$ is a tuple consisting of

- $A \in \mathcal{C}$, a symmetric Frobenius algebra which is separable ($\mu \circ \Delta = \mathrm{id}_A$),
- $T \in \mathcal{C}$, an A-$A \otimes A$-bimodule,
- $\alpha \in \mathrm{End}(T \otimes T)$, $\psi \in \mathrm{End}_{AA}(A)$, $\phi \in \mathrm{End}(\mathbf{1})$,

subject to the conditions listed in [1].

The notion of a *module monoidal category* (Henriques, Penneys, Tener '15, Heinrich '23) gives a convenient way to encode some of the conditions satisfied by an orbifold datum. Namely, let $\mathcal{M} = A\text{-mod}_{\mathcal{C}}$ be the category of left A-modules in $\mathcal{C}$. Then T defines a functor $\otimes_{\mathcal{M}} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ by taking relative tensor products $\otimes_A$ (this is well-defined, as A is separable and $\mathcal{C}$ idempotent-complete). The conditions on α ensure that it defines an associator for $\otimes_{\mathcal{M}}$, so that $\mathcal{M}$ becomes a (possibly non-unital) monoidal category. There is a right $\mathcal{C}$-action turning $\mathcal{M}$ into a $\mathcal{C}$-module category in a compatible way. The additional conditions on $\mathbb{A}$ are dualisability properties of $\otimes_{\mathcal{M}}$.

Related notions are higher idempotents (Douglas, Reutter '18), condensation monads (Gaiotto, Johnson-Freyd '19), and 1-morphisms in BrTens (Brochier, Jordan, Snyder '18).

Theorem. [2] From $\mathcal{C}$ and $\mathbb{A}$ as above one obtains a new additive idempotent-complete ribbon category $\mathcal{C}_{\mathbb{A}}$.

For example, objects in $\mathcal{C}_{\mathbb{A}}$ are triples (M, τ_1, τ_2), where M is an A-A-bimodule in $\mathcal{C}$ and $\tau_i : T \otimes M \to M \otimes T$ are morphisms compatible with the A-actions in a certain way [2].

Theorem. [2, 3] Let $\mathcal{C}$, $\mathcal{D}$ be modular fusion categories. Then:

(1) If $\mathbb{A}$ in $\mathcal{C}$ is simple, then $\mathcal{C}_{\mathbb{A}}$ is a modular fusion category.
(2) $\mathcal{C}$ and $\mathcal{D}$ are Witt equivalent iff there is an orbifold datum $\mathbb{A}$ in $\mathcal{C}$ such that $\mathcal{C}_{\mathbb{A}} \cong \mathcal{D}$ as linear ribbon categories.

If $\mathcal{C}$ is a modular fusion category and $\mathcal{M} = A\text{-mod}_{\mathcal{C}}$ is unital, then there is a ribbon equivalence [3]

$$\mathcal{C} \boxtimes (\mathcal{C}_{\mathbb{A}})^{\mathrm{rev}} \cong \mathcal{Z}(\mathcal{M}) \ .$$

In this case one can equivalently describe $\mathcal{C}_{\mathbb{A}}$ as the reversed category of the commutant of $\mathcal{C}$ in the Drinfeld centre of $\mathcal{M}$.

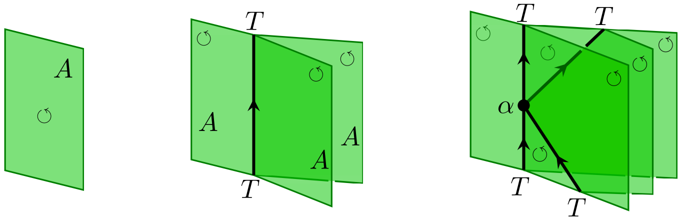
2. Examples

$\mathcal{C}$	$\mathbb{A}$	$\mathcal{C}_{\mathbb{A}}$	Ref.
any	for A commutative separable special Frobenius, $\mathbb{A} = (A, T = A, \alpha = \Delta \circ \mu, \dots)$	$\mathcal{C}_A^{\mathrm{loc}}$, the cat. of local A-modules.	[1]
$\mathcal{C} = \mathcal{B}_e$, with $\mathcal{B} = \bigoplus_{g \in G} B_g$ a G-crossed rib. cat.	$A = \bigoplus_{g \in G} A_g$ with $A_g = m_g^* \otimes m_g \in B_e$ for suitable $m_g \in B_g$.	$\mathcal{B}^G$, the equivariantisation of $\mathcal{B}$.	[4, 5]
$\mathrm{vec}_{\mathbb{C}}$	for $\mathcal{S}$ a spherical fusion category, $\mathbb{A} = (\bigoplus_{a \in \mathrm{Irr}\mathcal{S}} \mathbb{C}_a, \bigoplus_{a,b,c} \mathcal{S}(a \otimes b, c), \dots)$.	$\mathcal{Z}(\mathcal{S})$	[1]
$\mathrm{vec}_{\mathbb{C}}$	for H a finite-dim. semi-simple Hopf algebra over $\mathbb{C}$, $\mathbb{A} = (A = H, T = H \otimes H, \dots)$	$D(H)$-mod	[6]
Is, the Ising fusion category	$(A = \mathbf{1} \oplus \mathbf{1}, \dots)$	Galois-conjugate of $\mathcal{C}(sl_2, 10)$	[7]

In the last example, the module monoidal catgory is $\mathcal{M} = \text{Is} \oplus \text{Is}$. Denoting simple objects X in the first copy of Is by X_ι and in the second copy by X_τ, one has a Fibonacci-like product $\mathbf{1}_\tau \otimes_{\mathcal{M}} \mathbf{1}_\tau = \mathbf{1}_\iota \oplus \sigma_\tau$.

3. Relation to 3d TFT

Given a modular fusion category $\mathcal{C}$, one can define a Reshetikhin-Turaev type three-dimensional topological field theory $Z_\mathcal{C}$ for stratified bordisms. The 2-strata ("surface defects") are labelled by separable Frobenius algebras in $\mathcal{C}$, 1-strata ("line defects") by suitable modules over the algebras of adjacent surfaces, and 0-strata by morphisms. The entries A, T, α of an orbifold datum can then be represented as decorated 2, 1, and 0-strata as follows:



From this point of view, $\mathbb{A}$ becomes the data of a "gaugeable (possibly non-invertible) symmetry", and one can define a new 3d TFT $Z_\mathcal{C}/\mathbb{A}$, obtained from $Z_\mathcal{C}$ by "gauging $\mathbb{A}$". For example, the value of the gauged theory $Z_\mathcal{C}/\mathbb{A}$ on a closed 3-manifold M is

$$Z_\mathcal{C}/\mathbb{A}(M) := Z_\mathcal{C}(M \cup \{\text{defect foam}\}) .$$

Here, on the right hand side one stratifies M by the dual to a triangulation and decorates the 2, 1, 0-strata by A, T, α as shown above (plus additional decorations by ψ, ϕ, see [1] for details). One can then evaluate $Z_\mathcal{C}$ on this stratified version of M. By design, the conditions on $\mathbb{A}$ guarantee that the result is independent of the choice of dual triangulation.

It turns out that gauging $Z_\mathcal{C}$ by $\mathbb{A}$ and passing from $\mathcal{C}$ to $\mathcal{C}_\mathbb{A}$ are compatible:

Theorem [8] The topological field theories $Z_\mathcal{C}/\mathbb{A}$ and $Z_{\mathcal{C}_\mathbb{A}}$ are equivalent.

4. State spaces

Fix a modular fusion category $\mathcal{C}$ and an orbifold datum $\mathbb{A} = (A, T, \alpha, \psi, \phi)$ in $\mathcal{C}$. One way to obtain the vector space which the gauged TFT $Z_\mathcal{C}/\mathbb{A}$ assigns to a surface Σ is as the ground state space of an "internal Levin-Wen model" [6].

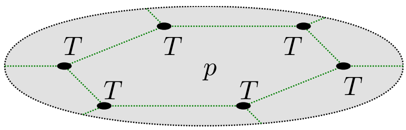
Namely, pick a dual triangulation of the surface Σ (i.e. 3-valent vertices, contractible 2-cells). Insert a copy of T (or T^*) at each vertex. Then the vector space

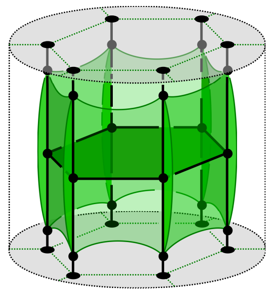
$$\mathcal{H} := Z_\mathcal{C}\big(\Sigma \cup \{T \text{ or } T^* \text{ at each vertex}\}\big)$$

is the state space of the internal Levin-Wen model. For each 2-cell p ("plaquette") of the surface, define the idempotent $P_p := Z_\mathcal{C}(M_p)$ as the linear endomorphism the TFT assigns to the bordism M_p shown below. The Hamiltonian of the internal Levin-Wen model is defined to be

$$H := \sum_p (\mathrm{id} - P_p) \ : \ \mathcal{H} \longrightarrow \mathcal{H} \ ,$$

where the sum runs over all plaquettes of Σ.



$M_p :=$ 

The left figure shows a patch of the surface Σ containing a plaquette p, and the right figure shows the corresponding patch of the bordims $M_p = \Sigma \times [0, 1]$ with a defect stratification inserted (only) above the plaquette p.

Theorem [6] The state space $Z_{\mathcal{C}}/\mathbb{A}(\Sigma)$ is canonically isomorphic to the ground state space of the Hamiltonian H.

References

[1] N. Carqueville, I. Runkel and G. Schaumann, *Orbifolds of Reshetikhin-Turaev TQFTs*, Theor. Appl. Categor. **35** (2020), 513–561.

[2] V. Mulevičius and I. Runkel, *Constructing modular categories from orbifold data*, Quantum Topol. **13** (2023), 459–523.

[3] V. Mulevičius, *Condensation inversion and Witt equivalence via generalised orbifolds*, arXiv:2206.02611 [math.QA].

[4] S. Heinrich, J. Plavnik, I. Runkel and A. Watkins, *Generalised orbifolds and G-equivariantisation*, arXiv:2506.08154 [math.QA].

[5] N. Carqueville and B. Haake, *2-Group symmetries of 3-dimensional defect TQFTs and their gauging*, arXiv:2506.08178 [math.QA].

[6] V. Mulevičius, I. Runkel and T. Voß, *Internal Levin-Wen models*, SciPost Phys. **17** (2024), 088.

[7] V. Mulevičius and I. Runkel, *Fibonacci-type orbifold data in Ising modular categories*, J. Pure Appl. Algebra **227** (2023), 107301.

[8] N. Carqueville, V. Mulevičius, I. Runkel, G. Schaumann and D. Scherl, *Reshetikhin–Turaev TQFTs close under generalised orbifolds*, Commun. Math. Phys. **405** (2024), 242.

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