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Dynamische Systeme

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6 July – 11 July 2025

Abstract. This workshop was part of the long-standing biannual series on Dynamical Systems at Oberwolfach, which began in 1981 with a meeting organized by Moser and Zehnder. It focused on recent advances and developments in dynamical systems, with particular attention to Hamiltonian systems and symplectic geometry. This year, special emphasis was placed on rigidity problems, periodic and connecting orbits.

Mathematics Subject Classification (2020): 35-XX, 37-XX, 53Dxx, 70Fxx, 70Hxx.

Introduction by the Organizers

The Dynamische Systeme workshop was organized by M.-C. Arnaud (Paris), S. Crovisier (Orsay), U. Hryniewicz (Aachen), M. Hutchings (Berkeley), with the help of T. Seara (Barcelona), who represented M.-C. Arnaud on site. It brought together 45 participants from 9 countries, representing a balanced mix of early-career, mid-career, and senior researchers. The event covered a broad spectrum of topics in dynamical systems, with a focus on classical Hamiltonian dynamics. Areas covered included symplectic dynamics and geometry, billiards, celestial mechanics, spectral rigidity, partial hyperbolicity, ergodic theory, dynamical convexity and Floer theory. Additional areas of dynamical systems were also represented.

Rafael Potrie has presented his classification of transitive partially hyperbolic diffeomorphisms on 3-manifolds obtained with S. Fenley. It answers conjectures due to Pujals and Hertz-Hertz-Ures. The proof is based on a general result about transverse foliations in dimension 3.

Vincent Colin presented his joint work with U. Hryniewicz and A. Rechtman on C^∞-generic properties of Reeb flows on closed 3-manifolds. The first result states that C^∞-generically every hyperbolic periodic orbit has homoclinic connections, thus improving on their previous result in collaboration with P. Dehornoy asserting that C^∞-generically the topological entropy is positive. The second result concerns existence of global surfaces of section with special properties, namely, containing prescribed sets of periodic orbits in their boundary or Legendrian links in their interior (up to C^0-small isotopy). The final result is about Legendrian knots having finitely many geometrically distinct Reeb chords: in the presence of a Birkhoff section, the presence of such a knot forces the Reeb flow to have exactly two periodic orbits and, due to results of D. Cristofaro-Gardiner, U. Hryniewicz, M. Hutchings and H. Liu, the ambient 3-manifold is a lens space and the dynamics has the exact same symplectic features of the quotient of an irrational ellipsoid.

Pedro Salomão, in joint work with Lei Liu, studies retrograde periodic orbits of the restricted circular three body problem. They consider the regularized (around collisions) energy surface slightly above the energy of L_1 and they prove that, for parameter mass close to $1/2$, it admits a $3-2-3$ foliation whose binding orbits are the retrograde orbits around the primaries and the Lyapunov orbit in the neck region. For energy values below the energy of L_1 they prove that Birkhoff's retrograde orbit conjecture holds.

Karen Butt showed that the entropy of the Liouville measure for the geodesic flow on surfaces with non-contant negative curvature is strictly increasing along the orbits of the Ricci flow, a joint work with A. Erchenko, T. Humbert and D. Mitsutani.

Oliver Edtmair reported on joint work with S. Seyfaddini where a positive answer to a question of V. I. Arnold about invariance of helicity for exact volume-preserving nowhere vanishing vector fields on homology 3-spheres is given. Their main result asserts that if the flows of two such vector fields are topologically conjugated then the vector fields have the same helicity.

Alessandra Nardi discussed the dynamical properties of a class of billards called symplectic billards and has presented recent rigidity results obtained with L. Baracco and O. Bernardi.

Agustin Moreno in a joint work with Arthur Limoge and Otto van Koert gave a talk about a A Poincaré–Birkhoff theorem for C^0-Hamiltonian maps. His motivation is the restricted three body problem where a previous result shows the existence of an open book decomposition. The symplectic form extends is also non-degenerate along the boundary, but the return map extends only continuously, his result gives the existence of periodic points of arbitrarily large minimal period.

Martin Leguil gave a talk about the rigidity of Anosov diffeomorphisms and notably presented his results with A. Gogolev on the deformation rigidity of diffeomorphisms near De la Llave's examples on $\mathbb{T}^4$.

Alberto Abbondandolo, in a joint work with Marco Mazzucchelli, studies the length spectrum rigidity and flexibility of spheres of revolution, where the geodesic flow is not hyperbolic. They consider S^1-invariant Riemannian metrics on S^2 that possess a unique equator, and, after defining type (p, q) geodesics, they give a definition of isoespectral metrics that allow them to give a rigidity result: roughly speaking if two metrics are isoespectral, then the geodesic flows are conjugated. They also present some extra properties.

Zhiyuan Zhang has exposed his recent advances with M. Tsujii on perturbations of the time-1 map of generic conservative Anosov flows in dimension 3, namely the proof of the topological mixing and of the existence of a physical measure.

Immaculada Baldoma, in a joint work with A. Florio, M. Leguil and T. M-Seara, proved that, among analytic strictly convex billiard, chaotic ones are open and dense, showing that given a rational rotation number, the set of analytic billiards having horseshoes associated to it are open and dense.

Basak Gürel reported on progress on the following important conjecture in Symplectic Dynamics: The Hamiltonian flow on a star-shaped energy level in a $2n$-dimensional symplectic vector space has either exactly n or infinitely many periodic orbits. In her recent joint work with E. Cineli and V. Ginzburg, one of the main statements is that the conjecture holds if the star-shaped energy level is nondegenerate, dynamically convex and centrally symmetric.

Mar Giralt proved the existence of Newhouse phenomena in the Restricted Planar Three Body Problem. In collaboration with I. Baldomá ans M. Guardia she studied the Lyapunov orbits around L3 and their stable and unstable manifold, and was able to prove the existence of transversal intersections for a family of Lyapunov orbits and, for a selected one, a tangential intersection which unfolds generically.

Sebastien Alvarez spoke about the space of surfaces of constant curvature inside a closed negatively curved 3-manifold, a 2-dimensional analogue of the geodesic flow. Together with B. Lowe and G. Smith he proves equidistribution properties of these surfaces.

Levin Maier's talk was about the interpretation of several PDEs of a Hamiltonian nature as magnetic geodesic flows on infinite dimensional Lie groups with a right-invariant metric. This extends celebrated work of V. I. Arnold on the interpretation of certain PDEs from fluid dynamics as geodesic flows on infinite-dimensional Lie groups with a right invariant metric. Several interesting results were presented, such as a Hopf-Rinow type theorem for energy levels above the so-called Mañé's critical value; it should be noted that the appropriate definition of Mañé's critical value was given in this new infinite-dimensional context.

Anna Florio's talk was devoted to the notion of h-flows which generalizes Anosov flows on non-compact manifolds. Her work with B. Schapira and A. Vaugon establishes the existence of measures maximizing the entropy under a SPR property.

Leonardo Masci reported on a Poincaré-Birkhoff type theorem for Hamiltonian systems in a symplectic vector space, that are "linear at infinity". A difficult and important conjecture due to A. Abbondandolo asserts that such systems have either exactly one or infinitely many periodic orbits. The main result of Masci's work provides infinitely many periodic orbits under the following assumptions: (i) the "linearized" dynamics at infinity is unitary and generic, and (ii) there exists a homologically visible fixed point whose linearized dynamics is nonresonant with the "linearized" dynamics at infinity.

Pierre Berger explained his construction of transitive analytic diffeomorphisms of the sphere, which proves an old conjecture by Birkhoff. His talk was illustrated by amazing animations and the proof, which follows Anosov-Katok approach, is based on deformations of complex structures.

The meeting was held in an informal and stimulating atmosphere. The weather was good and many participants attended the traditional walk to St. Roman on Wednesday.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-2230648, "US Junior Oberwolfach Fellows".

Workshop: Dynamische Systeme

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Abstracts

Transverse foliations and partially hyperbolic diffeomorphisms

Rafael Potrie
(joint work with Sergio Fenley)

Recently, using properties of foliations and transverse foliations in 3-manifolds, with Sergio Fenley we were able to obtain the following result:

Theorem A. *Let $f : M \to M$ be a (chain-)transitive partially hyperbolic diffeomorphism on a closed 3-manifold with fundamental group of exponential growth. Then, f is a collapsed Anosov flow.*

In a nutshell this means that M admits an Anosov flow φ_t and up to some continuous surjective map $h : M \to M$ the diffeomorphism f behaves as a self-orbit equivalence β of φ_t. That is, there is a homeomorphism $\beta : M \to M$ mapping oriented orbits of φ_t to oriented orbits of φ_t so that $f \circ h = h \circ \beta$. As a consequences of the techniques used in the proof and our result, we deduce that all definitions given in [BFP] of collapsed Anosov flows coincide and we refer the reader to that paper for more properties of such diffeomorphisms. This notion extends the notion of *discretized Anosov flows* appearing in the original Pujals' conjecture (see [BW, BFP, Mar]) to account for new examples (see [BGP, BGHP]). The classification is complete, since every possible (at least if orientable) collapsed Anosov flow can be realised as follows from a recent result of Bowden and Massoni. Manifolds with smaller fundamental group had already been dealt with [HP].

It is to be emphasized that a non-trivial consequence of our result is that a 3-manifold supporting a chain-transitive partially hyperbolic diffeomorphism must support a transitive Anosov flow. We refer the reader to [BW, CHHU, HP, BGP, BFP] for more context on this problem. The assumption of chain-transitivity is not crucial, it is used to ensure that leaves of some foliations appearing in the proof are by Gromov hyperbolic leaves, a generic assumption thanks to Candel's uniformization problem. In many cases, such as when f is absolutely partially hyperbolic, dynamically coherent, homotopic to the identity, it is not hard to see that the assumption is verified. It also holds unconditionally on hyperbolic and Seifert 3-manifolds [HP_2].

As a consequence of the previous result we also prove a conjecture due to Hertz-Hertz-Ures ([CHHU]), see [FP_2]:

Theorem B. *Let $f : M \to M$ be a conservative C^{1+} partially hyperbolic diffeomorphism in a closed 3-manifold whose fundamental group is not (virtually) solvable. Then, f is ergodic (and in fact a K-system).*

The proof of Theorem A builds on the existence of branching foliations [BI] and on a strategy first devised in the classification of partially hyperbolic diffeomorphisms of hyperbolic 3-manifolds ([$BFFP_1$, $BFFP_2$, FP]) that requires showing that center curves are quasi-geodesics in their corresponding weak-stable and weak-unstable (branching) leaves. While the proof in the hyperbolic manifold case achieves this through a detailed analysis of curves inside the leaves, with a crucial and continued used of the dynamics of f and properties of the strong stable and strong unstable foliations, the new strategy, which allows for a more general result, relies on a different approach initiated in [FP_3, FP_4, BaFP].

In our work, we try to understand the geometry of the flow defined by two transverse foliations and to obtain, assuming that the flowlines are not quasi-geodesic in their corresponding leaves, that there must be some structure incompatible with partial hyperbolicity. Our main result is then a completely general result about transverse foliations in 3-manifolds (related to some questions in [Th]):

Theorem C. *Let $\mathcal{F}_1, \mathcal{F}_2$ be two transverse foliations with Gromov hyperbolic leaves in a closed 3-manifold M. Then, if $\mathcal{G} = \mathcal{F}_1 \cap \mathcal{F}_2$ is the intersected foliation, it follows that either leaves of $\widetilde{\mathcal{G}}$ are quasi-geodesic in their corresponding $\widetilde{\mathcal{F}}_1$ and $\widetilde{\mathcal{F}}_2$ leaves, or, the foliation $\mathcal{G}$ contains a* generalized Reeb surface*.*

A generalized Reeb surface is a geometric object that can appear in a leaf of $\mathcal{F}_1$ or $\mathcal{F}_2$ which is foliated by $\mathcal{G}$ and has some particular geometric properties that are incompatible with the foliations coming from a partially hyperbolic diffeomorphism. Instead of defining properly the object, we close the summary with a corollary of the previous result:

Corollary D. *Let $\mathcal{F}_1, \mathcal{F}_2$ be two transverse foliations by Gromov hyperbolic leaves and let $\mathcal{G} = \mathcal{F}_1 \cap \mathcal{F}_2$ the intersected foliation. Then, $\mathcal{G}$ contains a closed leaf (a circle).*

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Birkhoff sections and Reeb chords in 3-dimensional contact manifolds

Vincent Colin

(joint work with Umberto Hryniewicz and Ana Rechtman)

Let M be a closed oriented 3-manifold, ξ a co-oriented *contact structure* and λ a positive contact form defining ξ, that is $\xi = \ker \lambda$ and $\lambda \wedge d\lambda > 0$. The *Reeb vector field* R_λ of λ is defined by the equations

$$\iota_{R_\lambda} d\lambda = 0 \qquad \text{and} \qquad \lambda(R_\lambda) = 1.$$

Its flow preserves the contact structure ξ and the volume form $\lambda \wedge d\lambda$. A special example of a Reeb vector field is given by the geodesic flow on the unit tangent bundle of a Riemannian surface (Σ, g). A Reeb vector field is *nondegenerate* when all its periodic orbits are nondegenerate: the transverse linearized first return map at a point of the orbit never has 1 in its spectrum.

An important tool for studying dynamics of flows on 3-manifolds was introduced by Poincaré and Birkhoff. A *Birkhoff section* for R_λ is an immersed compact surface S with boundary such that: ∂S is a collection of periodic orbits of R_λ, the interior of S is embedded and transverse to R_λ, and S intersects the orbit of R_λ starting at every point of M in bounded time. In particular, in the presence of a Birkhoff section the flow is entirely described by its first return map on S.

Beyond the classical study of periodic orbits and entropy of a Reeb flow, an important subject is that of *Reeb chords* along a *Legendrian* curve, that are a curve L everywhere tangent to ξ and the Reeb trajectories starting and ending on L.

Reeb vector fields are now known to have special strong properties.

- They always have at least one periodic orbit [14] (the *Weinstein conjecture*), and even two [6].
- A nondegenerate Reeb vector field always has either 2 or infinitely many periodic orbits [3]; if the first Chern class of ξ is torsion, the conclusion also holds in the degenerate case [7].

- Generically, a Reeb flow has > 0 topological entropy [2], that is, by Katok [11], equivalent to the existence of a homoclinic connection for a hyperbolic Reeb orbit.
- C^∞-generically, a Reeb vector field has a Birkoff section [2, 5].
- Every Legendrian curve has a Reeb chord [9, 10] (the *Arnold's chord conjecture*).

We present several improvements of these statements, see [4]:

Theorem A. *Let R be a Reeb vector field on a closed contact 3-manifold such that:*

(G1) *it is strongly nondegenerate;*
(G2) *its periodic orbits are equidistributed with respect to the Liouville measure;*
(G3) *it satisfies Zehnder's condition on elliptic periodic orbits.*

Then every hyperbolic periodic orbit has a transverse homoclinic orbit in each of the branches of its stable/unstable manifolds.

The hypothesis $(G1-3)$ are known to be C^∞ generic among Reeb flows. With this result at hands, we move forward to find nice Birkhoff sections.

Theorem B. *Let R be a Reeb vector field satisfying the hypotheses (G1), (G2) and (G3). Given a finite collection Γ of periodic orbits and a Legendrian link L, there exists a Legendrian link L' that is Legendrian isotopic to L by a C^0-small isotopy, and a Birkhoff section S for R such that*

(1) $\Gamma \subset \partial S$*;*
(2) *S is embedded;*
(3) *S contains L' (necessarily in its interior).*

Finally, we give a Reeb chord version of the "two or infinitely many Reeb periodic orbits" theorem proven in [3] for nondegenerate Reeb vector fields and in [8] in the general possibly degenerate case under the hypothesis that the first Chern class of the contact structure is torsion.

Theorem C. *If a Reeb vector field of a co-oriented contact structure ξ on a closed connected 3-manifold has a Birkhoff section, then every Legendrian knot L has infinitely many Reeb chords.*

If L has finitely many geometrically distinct Reeb chords, then the Birkhoff section is a disk or an annulus and the Reeb vector field has exactly two periodic orbits. Moreover, there are at least two geometrically distinct Reeb chords.

In the first part of Theorem C, the infinite number of chords could come from a periodic orbit intersecting the Legendrian knot in one point, that gives different, but not disjoint, chords obtained by covering the periodic orbit several times. The case of a Reeb vector field with exactly two periodic orbits was studied extensively in [7] where it is proved that the first return map on the Birkhoff section is an irrational pseudo-rotation, the two periodic orbits, together with their multiples, are elliptic non-degenerate, and their actions are related to the volume of the manifold as in the case of irrational ellipsoids. In this case, the ambient manifold M is either a lens space or the 3-sphere.

The proof of Theorem C relies on the existence of at least one Reeb chord provided by Hutchings and Taubes [9, 10], together with the fact that the first return map on a Birkhoff section has flux zero in the case of a Reeb flow.

Regarding the fact that, on one hand by a result of Alves and Mazzuchelli [1] every geodesic flow on the unit tangent bundle of a Riemannian surface has a Birkhoff section, and on the other hand infinitely many periodic orbits, we also get:

Corollary D. *If (Σ, g) is a closed Riemannian surface, then every Legendrian knot in $(UT\Sigma, \lambda_g)$ has infinitely many geometrically distinct Reeb (geodesic) chords.*

Since the unit sphere at every point defines a Legendrian knot we get, in particular, that for every $x \in \Sigma$ there are infinitely many distinct geodesic loops starting and ending at x, a classical result of Serre when Σ is the 2-sphere [13].

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Finite energy foliations in the restricted three-body problem

Pedro A. S. Salomão
(joint work with Lei Liu)

This report is about the main results of the preprint [4].

The circular planar restricted three-body problem studies the motion on the plane of a massless satellite attracted by two massive primaries that rotate along circular trajectories around their center of mass. In a rotating systems that fixes the primaries, this motion is governed by the Hamiltonian

$$H_\mu(p,q) = \frac{1}{2}|p+iq|^2 - \frac{\mu}{|q-(1-\mu)|} - \frac{1-\mu}{|q+\mu|} - \frac{1}{2}|q|^2.$$

Here, $q = q_1 + iq_2 \in \mathcal{C} \setminus \{-\mu, 1-\mu\}$ is the position of the satellite, and $p = p_1 + ip_2 \in \mathcal{C}$ is the conjugate momentum. The mass ratio $0 < \mu < 1$ is the unique parameter of the system after suitable normalizations. The primaries at $-\mu$ and $1-\mu$ are called the moon and the earth, respectively.

For energies $E < L_1(\mu)$ below the first Lagrange value, the energy surface $H^{-1}(E)$ contains two components $\mathcal{M}^e_{\mu,E}, \mathcal{M}^m_{\mu,E}$, projecting to punctured disk-like domains about the earth and the moon, respectively. If the energy E coincides with the first Lagrange value $L_1(\mu)$, then $\mathcal{M}^e_{\mu,L_1(\mu)}$ and $\mathcal{M}^m_{\mu,L_1(\mu)}$ touch each other at a common singularity, the first Lagrange point. For energies $L_1(\mu) < E < L_2(\mu)$ between the first and the second Lagrange values, the energy surface has a regular component $\mathcal{M}^{e\#m}_{\mu,E}$ corresponding to the connected sum of $\mathcal{M}^e_{\mu,E}$ and $\mathcal{M}^m_{\mu,E}$.

To regularize collisions with the primaries, we consider elliptic coordinates

$$q_1 = \frac{1}{2}\cosh x_1 \cos x_2, \quad q_2 = \frac{1}{2}\sinh x_1 \sin x_2,$$

and replace (p,q) with (y,x) satisfying $pdq = ydx$ to obtain the regularized Hamiltonian

$$\hat{H}(y,x) = \frac{1}{2}\left(\left(y_1 + \frac{\sin 2x_2}{8}\right)^2 + \left(y_2 + \frac{\sinh 2x_1}{8}\right)^2\right) + V(x) + (1-2\mu)\hat{V}(x),$$

where

$$V(x) = -\frac{h\cosh^2 x_1}{4} - \frac{\cosh x_1}{2} - \frac{\sinh^2(2x_1)}{128} + \frac{h\cos^2 x_2}{4} - \frac{\sin^2(2x_2)}{128},$$
$$\hat{V}(x) = \frac{\cos x_2}{2} - \frac{1}{16}(\frac{1}{2} - \mu + \cosh x_1 \cos x_2)(\cosh^2 x_1 - \cos^2 x_2).$$

The components of the regularized energy surface quotiented by the antipodal symmetry now become smooth compact submanifolds

$$\begin{aligned} &S^3 \overset{2:1}{\to} \mathcal{M}^e_{\mu,E}, \mathcal{M}^m_{\mu,E} \equiv \mathbb{R}P^3, && \forall E < L_1(\mu), \\ &S^1 \times S^2 \overset{2:1}{\to} \mathcal{M}^{e\#m}_{\mu,E} \equiv \mathbb{R}P^3 \# \mathbb{R}P^3, && \forall L_1(\mu) < E < L_2(\mu). \end{aligned} \tag{1}$$

In [1], Albers, Frauenfelder, van Koert, and Paternain observed that for energies up to slightly above $L_1(\mu)$, these regularized energy surfaces have contact type. In particular, the methods of pseudo-holomorphic curves apply. For energies below

$L_1(\mu)$, the flow is equivalent to a Reeb flow on $\mathbb{R}P^3$ equipped with the universally tight contact structure ξ_0. Birkhoff [2] used the shooting method to prove the existence of a retrograde orbit, i.e., a periodic orbit projecting to a simple closed curve around the primary moving opposite to the rotating system. He raised the question of whether the retrograde orbit bounds a disk-like global surface of section. In the neck region of $\mathbb{R}P^3\#\mathbb{R}P^3$, there exists an index-2 hyperbolic orbit $P_2 = P_{2,\mu,E}$, called the Lyapunov orbit. This orbit bounds a pair of closed disks whose interior is transverse to the flow. They form a regular two-sphere $\mathcal{S}$ which separates $\mathcal{M}^{e\#m}_{\mu,E}$ into two components whose closures, denoted by $\mathcal{M}^e_{\mu,E}$ and $\mathcal{M}^m_{\mu,E}$, are contactomorphic to $(\mathbb{R}P^3, \xi_0)$ with an open ball removed. One can prove using the same argument as Birkhoff that the interiors of $\mathcal{M}^e_{\mu,E}$ and $\mathcal{M}^m_{\mu,E}$ possess retrograde orbits P_3^e and P_e^m, respectively.

Definition 1. *Consider the following terminology from* [3] *for E slightly above $L_1(\mu)$:*

(i) *A $2-3$ foliation of $\mathcal{M}^e_{\mu,E}$ is a weakly convex foliation $\mathcal{F}^e$ of $\mathcal{M}^e_{\mu,E}$, i.e., the regular leaves consist of the hemispheres U_1, U_2 in $\mathcal{S} \setminus P_{2,E}$, a one-parameter family of planes asymptotic to $(P_3^e)^2$ and a rigid cylinder with a positive end at $(P_3^e)^2$ and a negative end at P_2. They are transverse to the flow and consist of projections to $\mathcal{M}^e_{\mu,E}$ of a finite energy foliation in the symplectization. A $2-3$ foliation $\mathcal{F}^m$ of $\mathcal{M}^m_{\mu,E}$ is defined similarly.*

(ii) *If $2-3$ foliations $\mathcal{F}^e$ and $\mathcal{F}^m$ of $\mathcal{M}^e_{\mu,E}$ and $\mathcal{M}^m_{\mu,E}$ exist, respectively, then $\mathcal{F}^e \cup \mathcal{F}^m$ is called a $3-2-3$ foliation of $\mathcal{M}^{e\#m}_{\mu,E}$.*

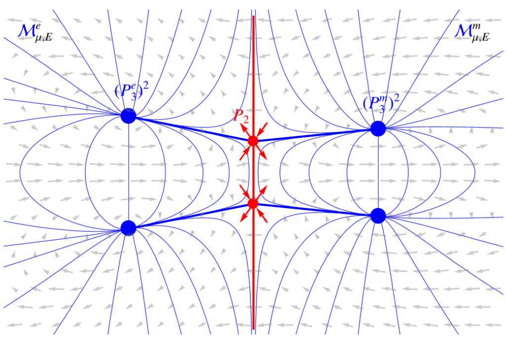

FIGURE 1. The $3-2-3$ foliation on the regularized component $\mathcal{M}^{e\#m}_{\mu,E} \equiv \mathbb{R}P^3\#\mathbb{R}P^3$.

The main result of this report states that for mass ratios sufficiently close to $1/2$ and energies slightly above the first Lagrange value, the regularized $\mathbb{R}P^3\#\mathbb{R}P^3$ component of the energy surface admits a $3-2-3$ foliation whose binding orbits are the retrograde orbits around the primaries and the Lyapunov orbit in the neck region.

Theorem A (Liu-S., 2025)**.** *The following statements hold for every* (μ, E) *sufficiently close to* $(1/2, -2)$*, with* $E > L_1(\mu)$*:*

(i) *The index-2 Lyapunov orbit* $P_2 \subset \mathcal{M}^{e\#m}_{\mu,E}$ *is the unique contractible periodic orbit with index* ≤ 2*. In particular,* $\mathcal{M}^{e\#m}_{\mu,E}$ *is weakly convex.*

(ii) *The regularized Hamiltonian flow on* $\mathcal{M}^{e\#m}_{\mu,E} \equiv \mathbb{R}P^3\#\mathbb{R}P^3$ *admits a* $3-2-3$ *foliation whose binding orbits are the retrograde orbits* P_3^e *and* P_3^m*, and the Lyapunov orbit* P_2 *around the first Lagrange point.*

(iii) *Each chamber* $\mathcal{M}^e_{\mu,E}$ *or* $\mathcal{M}^m_{\mu,E}$ *admits infinitely many periodic orbits and infinitely many homoclinic orbits to the Lyapunov orbit near* $l_1(\mu)$*. Moreover, if the stable and unstable manifolds of the Lyapunov orbit do not coincide, then the topological entropy of the flow on* $\mathcal{M}^{e\#m}_{\mu,E}$ *is positive.*

We also prove that Birkhoff's retrograde orbit conjecture holds for mass ratios sufficiently close to $1/2$ and every energy below the first Lagrange value.

Theorem B (Liu, S., 2025)**.** *There exists* $\epsilon_0 > 0$ *such that for every* $|\mu - 1/2| < \epsilon_0$ *and* $E < L_1(\mu)$*, the following statements hold:*

(i) *The* $\mathbb{R}P^3$*-components* $\mathcal{M}^e_{\mu,E}$ *and* $\mathcal{M}^m_{\mu,E}$ *are dynamically convex, i.e., every contractible periodic orbit has index at least* 3*.*

(ii) *Every retrograde orbit* $P_3^e \subset \mathcal{M}^e_{\mu,E} \equiv \mathbb{R}P^3$ *binds a rational open book decomposition whose pages are disk-like global surfaces of section. More generally, the same holds for every periodic orbit* $P \subset \mathcal{M}^e_{\mu,E}$ *which is transversely isotopic to a Hopf fiber. A similar statement holds for* $\mathcal{M}^m_{\mu,E}$*.*

(iii) *Let* $P' \subset \mathcal{M}^e_{\mu,E}$ *be the simple periodic orbit corresponding to a fixed point of the first return map associated to the global surface of section bounded by* P *as in (ii). Then the Hopf link* $P \cup P'$ *bounds an annulus-like global surface of section. A similar statement holds for* $\mathcal{M}^m_{\mu,E}$*.*

The proof of the theorem above strongly relies on the theory of Hofer-Wysocki-Zehnder on pseudo-holomorphic curves in symplectizations and the following convexity estimate that controls the indices of periodic orbits.

Theorem C (Liu, S., 2025)**.** *The regularized subsets* $\mathcal{M}^e_{1/2,E}$ *and* $\mathcal{M}^m_{1/2,E}$ *are strictly convex for every* $E \leq -2 = L_1(1/2)$*.*

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Monotonicity of the Liouville entropy along the Ricci flow

Karen Butt
(joint work with Alena Erchenko, Tristan Humbert, Daniel Mitsutani)

Let (M,g) be a closed negatively curved surface, and let $h_{\mathrm{Liou}}(g)$ denote its *Liouville entropy*, i.e., the measure-theoretic entropy of the geodesic flow on the unit tangent bundle S^gM with respect to the Liouville measure. In the case where g is a hyperbolic metric, i.e., a metric of constant negative curvature, $h_{\mathrm{Liou}}(g)$ coincides with the more familiar notion of *topological entropy* $h_{\mathrm{top}}(g)$; in other words, the Liouville measure is the measure of maximal entropy. By work of Bowen, the measure of maximal entropy is given by the equidistribution of periodic orbits (closed geodesics) [1], and one does not in general expect this to coincide with Liouville measure in the presence of geometric asymmetries.

Indeed, Katok proved that $h_{\mathrm{Liou}}(g) = h_{\mathrm{top}}(g)$ if and only if g is hyperbolic [5, Corollary 2.5]. More specifically, suppose M has Euler characteristic $\chi < 0$ and g is a negatively curved metric on M with total area A. Let $h_{\mathrm{hyp}} = \sqrt{-2\pi\chi/A}$, the common value of the Liouville and topological entropies for a hyperbolic metric on M, normalized as above. Katok proved that if g has *non-constant* negative curvature, then

$$h_{\mathrm{Liou}}(g) < h_{\mathrm{hyp}} < h_{\mathrm{top}}(g). \tag{1}$$

(In [5], it is also conjectured that equality of Liouville and topological entropies characterizes negatively curved locally symmetric metrics in higher dimensions, and this problem is not yet fully solved.)

Katok's proof uses that in dimension 2, every metric on a surface of negative Euler characteristic is conformally equivalent to a hyperbolic metric. This follows, for instance, from the uniformization theorem for Riemann surfaces; a more Riemannian-geometric way to see this is using Hamilton's *Ricci flow.* We recall that in dimension 2, the *normalized Ricci flow* is given by

$$\frac{\partial}{\partial\varepsilon} g_\varepsilon = -2(K_\varepsilon - \bar{K})g_\varepsilon, \tag{2}$$

where K_ε is the Gaussian curvature of g_ε and $\bar{K}$ is its average value. Hyperbolic metrics are fixed by the Ricci flow; for metrics of non-constant curvature, (2) defines a conformal family of negatively curved metrics $\varepsilon \mapsto g_\varepsilon$ of fixed area converging to a hyperbolic metric (of constant curvature $\overline{K}$) as $\varepsilon \to \infty$ [4, Theorem 3.3].

In [9], Manning considered the variation of the topological entropy along the normalized Ricci flow for closed negatively curved surfaces. Using the second inequality in (1), he proved the topological entropy strictly decreases along the normalized Ricci flow [9, Theorem 1] and asks if the corresponding monotonicity is true for the Liouville entropy [9, Question 3]. Our main result is that this is indeed the case:

Theorem A ([2], Theorem A)**.** *Let M be a smooth closed orientable surface of negative Euler characteristic. Let g_0 be a smooth Riemannian metric on M of* *non-constant negative Gaussian curvature. Let* $\varepsilon \mapsto g_\varepsilon$ *denote the normalized Ricci flow starting from* g_0*. Then*

$$\varepsilon \mapsto h_{\mathrm{Liou}}(g_\varepsilon) \text{ is strictly increasing for all } \varepsilon \geq 0.$$

Combining this monotonicity result with Manning's gives a new proof of Katok's aforementioned entropy rigidity theorem. Indeed, we have that for g non hyperbolic, the quantity $h_{\mathrm{top}}(g) - h_{\mathrm{Liou}}(g)$ strictly decreases along the normalized Ricci flow. On the other hand, the variational principle states $h_{\mathrm{top}}(g) - h_{\mathrm{Liou}}(g) \geq 0$. Hence, $h_{\mathrm{top}}(g) - h_{\mathrm{Liou}}(g) > 0$. Moreover, our proof of Theorem A gives a new proof of the first inequality in (1) (shown also in [8, Theorem 1] and [11, Corollary 1]).

The key ingredient in the proof of Theorem A is a new formula for the derivative of the Liouville entropy along an arbitrary area-preserving conformal perturbation of a negatively curved metric on a surface:

Theorem B ([2], Theorem D)**.** *Let* (M, g_0) *be a smooth closed negatively curved surface. Let* $g_\varepsilon = e^{2\rho_\varepsilon} g_0$ *be a* C^∞ *area-preserving conformal perturbation of* g_0 *and let* $\dot\rho_0 = \frac{d}{d\varepsilon}|_{\varepsilon=0}\, \rho_\varepsilon$*. Let* $h_{\mathrm{Liou}}(\varepsilon)$ *denote the Liouville entropy of* g_ε*. Then*

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} h_{\mathrm{Liou}}(\varepsilon) = -\frac{1}{2}\int_{SM} \dot\rho_0 w^s \, dm,$$

where m *is the Liouville measure for* g_0 *and* $-w^s(v)$ *is the mean curvature of the stable horosphere* (*or, strictly speaking, the geodesic curvature of the stable horocycle*) *determined by* v*.*

To prove Theorem B, we being with the well-known fact that, in negative curvature, the Liouville entropy can be expressed as the average, with respect to the Liouville measure, of the mean curvature of horospheres. (This was used by Knieper–Weiss to show the Liouville entropy varies smoothly with respect to the metric for negatively curved surfaces [6].) As in the work of Ledrappier–Shu [7], we use that this mean curvature is equal to the Laplacian of the corresponding Busemann function, and and can hence be expressed as the divergence of a vector field closely related to the geodesic spray.

A key tool, in both their work and ours, is a decomposition of the unit tangent bundle of the universal cover $\tilde M$ as the product of $\tilde M$ with $\partial\tilde M$, the visual boundary at infinity. As a consequence of this perspective, integrals of certain functions along half-infinite orbits of the geodesic flow appear naturally in the computations. We then use microlocal methods, more specifically, the formalism of Pollicott–Ruelle resonances, to express these integrals in terms of resolvents of the geodesic flow, as in the work of Faure–Guillarmou [3]. This key insight allows for dramatic simplification of our derivative formula.

To deduce Theorem A from Theorem B, we set $\dot\rho_0 = K_0 - \overline{K}$ in Theorem B, and we show positivity of the resulting derivative formula using a Jensen-type inequality. In fact, a simpler analogue of this positivity argument appears when considering the variation of the *mean root curvature* along the normalized Ricci flow. The mean root curvature is a geometric invariant introduced by Manning [8] which is defined for a negatively curved metric g on a closed surface M by

$$\kappa(g) := \frac{1}{A(g)} \int_M \sqrt{-K_g}\, dA_g, \tag{3}$$

where dA_g is the Riemannian area form of g, and $A(g)$ is the area defined by $A(g) = \int_M dA_g$.

The mean root curvature is small for metrics which concentrate curvature in regions of small area, and is maximized strictly at metrics of constant negative curvature, by Jensen's inequality and the Gauss–Bonnet theorem. In addition, it provides a lower bound for Liouville entropy: $\kappa(g) \leq h_m(g)$ with equality if and only if g is of constant negative Gaussian curvature [8, Theorem 2], [10]. Our last result is that this quantity is also monotonically increasing along the normalized Ricci flow:

Theorem C ([2], Theorem C). *Let M be a smooth closed orientable surface of negative Euler characteristic. Let g_0 be a Riemannian metric on M of non-constant negative Gaussian curvature. Let $\varepsilon \mapsto g_\varepsilon$ denote the normalized Ricci flow starting from g_0. Then*

$$\varepsilon \mapsto \kappa(g_\varepsilon) \text{ is strictly increasing for all } \varepsilon \geq 0.$$

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On the topological invariance of helicity

Oliver Edtmair
(joint work with Sobhan Seyfaddini)

Helicity is an invariant of exact volume preserving vector fields in dimension three. It was introduced in [12, 11, 10] in the context of (magneto)hydrodynamics, where it gives rise to conserved quantities. In my talk, I discussed joint work with Sobhan Seyfaddini [5], which addresses questions raised by Arnold [1] concerning topological properties of helicity.

Let (Y^3, μ) be a closed 3-manifold equipped with a volume form. A volume-preserving vector field X on Y is called *exact* if the closed 2-form $\iota_X \omega$ is exact. In this case, the *helicity* of X is defined as

$$\mathcal{H}(X) := \int_Y \alpha \wedge d\alpha,$$

where α is a primitive 1-form of $\iota_X \mu$. This integral turns out to be independent of the choice of α. Clearly, helicity is invariant under volume preserving-diffeomorphisms, i.e. $\mathcal{H}(f^* X) = \mathcal{H}(X)$ for every $f \in \mathrm{Diff}(Y, \mu)$.

We say that two volume-preserving vector fields X_1 and X_2 are *topologically conjugate* if there exists a volume- and orientation-preserving homeomorphism $f \in \mathrm{Homeo}^+(Y, \mu)$ which intertwines their flows, i.e. which satisfies $f \circ \varphi^t_{X_1} \circ f^{-1} = \varphi^t_{X_2}$ for all times t. In [1], Arnold posed the following questions; see also [6, 7, 2, 8].

Question 1 (Arnold). *Is helicity preserved under topological conjugacy, i.e. if X_1 and X_2 are two exact volume-preserving vector fields which are topologically conjugate, is it true that $\mathcal{H}(X_1) = \mathcal{H}(X_2)$? More generally, can helicity be extended to volume-preserving topological flows?*

Building on recent advances in C^0 symplectic geometry [3], we resolve these questions for flows without fixed points [5].

Theorem A. *Two nowhere vanishing, exact, volume-preserving, smooth vector fields which are topologically conjugate have the same helicity.*

Moreover, helicity admits an extension to fixed-point-free, exact, volume-preserving, topological flows whose flow lines have zero measure. This extension is invariant under conjugation by volume- and orientation-preserving homeomorphisms and compatible with the Calabi invariant in the sense of Eq. (1)*, and it is uniquely determined by these properties.*

In this theorem, compatibility with the Calabi invariant has the following meaning. Let (Σ, ω_Σ) denote an open surface equipped with an area form ω_Σ. Let $\mathrm{Ham}(\Sigma)$ be its group of compactly supported Hamiltonian diffeomorphisms. It was proven recently [4, 9] that the Calabi homomorphism $\mathrm{Cal}_\Sigma : \mathrm{Ham}(\Sigma) \to \mathbb{R}$ admits infinitely many extensions to the group of Hamiltonian homeomorphisms $\overline{\mathrm{Ham}}(\Sigma)$. Pick one such extension

$$\overline{\mathrm{Cal}}_\Sigma : \overline{\mathrm{Ham}}(\Sigma) \to \mathbb{R}.$$

We prove in [5] that the Calabi extensions for different surfaces Σ can be picked such that they are functorial with respect to area-preserving embeddings of surfaces.

Now, fix a topological volume-preserving flow ψ^t on (Y, μ) and suppose that we have a topological volume-preserving embedding

$$\alpha : ((0,1) \times \Sigma), dt \wedge \omega_\Sigma) \hookrightarrow (Y, \mu)$$

which intertwines the flow on $(0,1)\times\Sigma$ generated by the vector field ∂_t and the flow ψ^t. Consider a C^0 Hamiltonian isotopy $\varphi^t \in \overline{\mathrm{Ham}}(\Sigma)$ and note that its suspension to $(0,1) \times \Sigma$ is volume preserving. We refer to the tuple $\mathcal{P} := (\Sigma, \omega_\Sigma, \alpha, \varphi^t)$ as a *plug*. Given a plug $\mathcal{P}$, one can define a new volume-preserving flow $\psi^t \# \mathcal{P}$ on Y by replacing the flow ψ^t inside $\mathrm{im}(\alpha)$ with the suspension of φ^t. The compatibility condition between our helicity extension $\overline{\mathcal{H}}$ and the Calabi extension $\overline{\mathrm{Cal}}_\Sigma$ is given by

$$\overline{\mathcal{H}}(\psi^t \# \mathcal{P}) = \overline{\mathcal{H}}(\psi^t) + \overline{\mathrm{Cal}}_\Sigma(\varphi^1). \tag{1}$$

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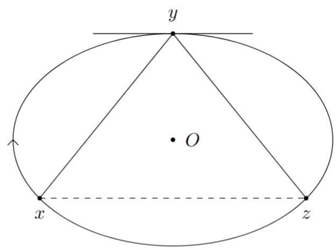
Symplectic billiards: integrability and spectral rigidity

Alessandra Nardi

(joint work with Luca Baracco and Olga Bernardi)

A mathematical billiard is a dynamical system describing the motion of a mass point (the billiard ball) inside a planar region (the billiard table) with, in general, piecewise smooth boundary. The ball moves with constant speed and without friction, following a rectilinear path. In the classical billiard, introduced by Birkhoff in [8], the planar region is a convex domain (often strictly convex) with a smooth boundary, and the law of motion is the reflection law.

In 2018, P. Albers and S. Tabachnikov introduced a new class of billiards called *symplectic billiards*, see [1]. As in the Birkhoff case, the billiard table is a convex planar region with smooth boundary, and the dynamics is described as follows. Three points x, y, z on the boundary are three consecutive points of a symplectic billiard orbit if and only if the tangent at the second point, y, is parallel to the line connecting x and z (see figure below).



Both Birkhoff and symplectic billiards' maps are monotone twist maps, preserving an area form. Moreover, in the Birkhoff case, the generating function is the length between two successive bounces; otherwise, in the symplectic case, the generating function is the area of the parallelogram given by two successive bounces, i.e.,

$$\omega(x, y) = \det(x, y).$$

Crucial questions for any billiard dynamics include integrability, which means the existence of a regular (i.e., at least C^0) foliation of the phase-space consisting of invariant, not null-homotopic curves. In particular, a billiard is called *totally integrable* if the foliation fills the whole phase-space. A celebrated result, proved by M. Bialy [6] in 1993, established that the only totally integrable Birkhoff billiards are circles. More recently, L. Baracco and O. Bernardi [2] proved that, in the case of symplectic billiards, ellipses are the only totally integrable billiard tables. It is consequently quite natural to try to apply the previous successful frameworks to search for other rigidity results by relaxing the total integrability assumption. In this direction, a fundamental contribution is due to M. Bialy and A.E. Mironov [7], proving the so-called Birkhoff–Poritsky conjecture for centrally symmetric C^2 strongly-convex (i.e., with positive curvature) domains Ω. The analog result for symplectic billiards was obtained in a joint work with L. Baracco and O. Bernardi.

Theorem A (Theorem 1.1 in [3]). *Let Ω be a centrally symmetric C^2 strongly-convex domain with boundary $\partial\Omega$. Assume that the symplectic billiard map $T : \mathcal{P} \to \mathcal{P}$ of $\partial\Omega$ has a (simple) continuous invariant curve $\delta \subset \mathcal{P}$ of rotation number* $1/4$ *(winding once around $\partial\Omega$) and consisting only of* 4*-periodic orbits. If one of the parts between δ and each boundary of the phase-space $\mathcal{P}$ is entirely foliated by continuous invariant closed (not null-homotopic) curves, then $\partial\Omega$ is an ellipse.*

The proof makes use of the consolidated integral approach coming from Hopf's method and of the affine equivariance of the symplectic billiard map in order to take back the problem to the isoperimetric inequality.

The results mentioned above are examples of *rigidity phenomena*: imposing the existence of the foliations described in the hypotheses of the previous theorems uniquely determines the domain to be an ellipse. Alternative approaches to exploring rigidity in symplectic billiards include the use of the β-Mather functions –whose expansion was computed in [4]– or the analysis of the *area spectrum.*

Let $\{x_j\}_{j=0}^{q}$ be a periodic trajectory for the symplectic billiard map, that is, $T(x_{j-1}, x_j) = (x_j, x_{j+1})$ for every $j = 1, \dots q-1$, and $x_0 = x_q$. Its *action* is defined as

$$\sum_{j=0}^{q-1} \omega(x_j, x_{j+1})$$

and if the orbit winds once around the boundary $\partial\Omega$, it is precisely twice the area of the polygon of vertices $\{x_j\}_{j=0}^{q-1}$. The area spectrum is defined as

$$\mathcal{A}(\Omega) = \mathbb{N}\{\text{action of all closed trajectories of } \Phi\} \cup \mathbb{N}\{A_\Omega\},$$

where A_Ω is the area of Ω. From the invariance of the symplectic billiards map under affine transformations of the plane, it is clear that given two strictly convex domains Ω and Ω' with the same area, if the corresponding symplectic billiard maps T_Ω and $T_{\Omega'}$ are conjugated by a unitary affine transformation of the plane, then $\mathcal{A}(\Omega) = \mathcal{A}(\Omega')$. Then a natural question arises: is it true that if $\mathcal{A}(\Omega) = \mathcal{A}(\Omega')$, then Ω and Ω' are necessarily equal up to a unitary affine transformation of the plane? A partial answer to this question was given in a joint work with L. Baracco and O. Bernardi for two different classes of domains.

Theorem B (Theorem 1 and 2 in [5]). (*a*) *Any finitely smooth axially symmetric strictly convex domain, with everywhere positive curvature and sufficiently close to an ellipse, and* (*b*) *any finitely smooth centrally symmetric strictly convex domain, even-rationally integrable, with everywhere positive curvature and sufficiently close to an ellipse, is* area spectrally rigid.

This means that any family $(\Omega_\tau)_{\tau\in[-1,1]}$ of domains in these classes such that $\mathcal{A}(\Omega_\tau) = \mathcal{A}(\Omega_{\tau'})$ for every $\tau, \tau' \in [-1, 1]$, is necessarily equi-affine, i.e. for every $\tau, \tau' \in [-1, 1]$, $\Omega_{\tau'} = B_{\tau,\tau'}\Omega_\tau$ with $B_{\tau,\tau'}$ unitary affine.

The analog result for Birkhoff billiards with the family of axially symmetric domains that are close to the circle is due to J. de Simoi, V. Kaloshin, and Q. Wei

[9]. The same result for symplectic billiards was obtained simultaneously and independently, for axially symmetric domains by C. Fierobe, A. Sorrentino, A. Vig in [10] with a slightly different technique.

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A Poincaré–Birkhoff theorem for C^0-Hamiltonian maps

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(joint work with Arthur Limoge, Otto van Koert)

In what follows, we study the well known circular, restricted three-body problem (CR3BP), review the existence of adapted open books, and state a fixed-point theorem inspired by the classical Poincaré–Birkhoff theorem.

Consider three bodies: Earth (E), Moon (M) and Satellite (S), with masses m_E, m_M, m_S. One has the following cases and assumptions.

- **(Restricted case)** $m_S = 0$.
- **(Circular assumption)** E, M move in circles.
- **(Planar case)** S moves in the ecliptic plane containing E, M;
- **(Spatial case)** S is allowed to move in $\mathbb{R}^3$.

The *mass ratio* is $\mu = \frac{m_M}{m_E+m_M} \in [0,1]$, and we normalize so that $m_E + m_M = 1$, and so $\mu = m_M$. In rotating coordinates, in which both primaries are at rest, the Hamiltonian describing the problem is actually autonomous:

$$H : \mathbb{R}^3 \backslash \{E, M\} \times \mathbb{R}^3 \to \mathbb{R}$$

$$H(q,p) = \frac{1}{2}\|p\|^2 - \frac{\mu}{\|q-M\|} - \frac{1-\mu}{\|q-E\|} + p_1 q_2 - p_2 q_1.$$

The planar problem is the subset $\{p_3 = q_3 = 0\}$ (invariant subproblem); the two relevant parameters are the *Jacobi constant* c (the energy value), and μ.

There are precisely five critical points of H, called the *Lagrangian points* $L_i = L_i(\mu)$, $i = 1, \ldots, 5$, ordered so that $H(L_1) < H(L_2) < H(L_3) < H(L_4) = H(L_5)$. The *low-energy* range corresponds to $c < H(L_1)$ (or slightly above). For $c \in \mathbb{R}$, let $\Sigma_c = H^{-1}(c)$. If

$$\pi : \mathbb{R}^3 \backslash \{E, M\} \times \mathbb{R}^3 \to \mathbb{R}^3 \backslash \{E, M\},\ \pi(q, p) = q,$$

the *Hill's region* of energy c is

$$\mathcal{K}_c = \pi(\Sigma_c) \in \mathbb{R}^3 \backslash \{E, M\}.$$

If $c < H(L_1)$, then $\mathcal{K}_c$ has three connected components: a bounded one around the Earth ($\mathcal{K}_c^E$), another bounded one around the Moon ($\mathcal{K}_c^M$), and an unbounded one. Let $\Sigma_c^E = \pi^{-1}(\mathcal{K}_c^E) \cap \Sigma_c$, $\Sigma_c^M = \pi^{-1}(\mathcal{K}_c^M) \cap \Sigma_c$. As c crosses $H(L_1)$, $\mathcal{K}_c^E$ and $\mathcal{K}_c^M$ get glued to each other into a new connected component $\mathcal{K}_c^{E,M}$, topologically their connected sum. These level sets are non-compact due to collisions, but can be compactified by Moser regularization.

Contact geometry in the CR3BP. It was only recently that the modern techniques from contact and symplectic geometry have been made to bear on the CR3BP:

Theorem A ([1, 2])**.** *If $c < H(L_1)$, the (regularized) hypersurfaces $\overline{\Sigma}_c^E, \overline{\Sigma}_c^M, \overline{\Sigma}_{P,c}^E$, $\overline{\Sigma}_{P,c}^M$ carry contact structures. The same holds for $\overline{\Sigma}_c^{E,M}, \overline{\Sigma}_{P,c}^{E,M}$, if $c \in (H(L_1), H(L_1) + \epsilon)$ for sufficiently small $\epsilon > 0$.*

Open book decompositions. We have the following fundamental notion from smooth topology.

Definition 1. *Let M be a closed manifold. A (concrete) open book decomposition on M is a fibration $\pi : M \backslash B \to S^1$, where $B \subset M$ is a closed, codimension-2 submanifold with trivial normal bundle. We further assume that $\pi(b, r, \theta) = \theta$ along some collar neighbourhood $B \times \mathbb{D}^2 \subset M$, where (r, θ) are polar coordinates on the disk factor. B is the* binding*, and the closure of the fibers $P = P_\theta = \overline{\pi^{-1}}(\theta)$ are the* pages*, which satisfy $\partial P_\theta = B$ for every θ.*

Definition 2 (Giroux)**.** *An open book on M is* adapted *to the dynamics of a (positive) contact form α if:*

(1) *$\alpha_B := \alpha|_B$ is a (positive) contact form for B;*
(2) *$d\alpha|_P$ is a (positive) symplectic form on the interior of every page P.*

Theorem B (Moreno–van Koert [4])**.** *For any $\mu \in [0, 1]$, c in the low-energy range, $\overline{\Sigma}_c$ admits a supporting open book decomposition for energies $c < H(L_1)$ that is adapted to the dynamics. Furthermore, if $\mu < 1$, then there is $\epsilon > 0$ such that the same holds for $c \in (H(L_1), H(L_1) + \epsilon)$. The binding is the planar problem.*

It follows that the dynamics is encoded by a Poincaré first return map, a symplectomorphism of any given page.

A trade-off. In the setup, we have the following trade-off:

- **(A)** Either the return map extends smoothly to the boundary, but the symplectic form degenerates; or
- **(B)** The symplectic form extends is also non-degenerate along the boundary, but the return map extends only continuously.

The two setups are equivalent, related by a conjugation which is smooth in the interior but only continuous at the boundary. In what follows, we choose **(B)**.

Definition 3. *Let $f : (W, \omega) \to (W, \omega)$ be a map on a Liouville domain, and α contact at $B = \partial W$. We say that f is a C^0-**Hamiltonian twist map** if:*

- ***(Hamiltonian)*** *$f|_{int(W)} = \phi^1_H$ is generated by a C^1 Hamiltonian $H_t : int(W) \to \mathbb{R}$;*
- ***(Extension)*** *Both f and the Hamiltonian H_t admit C^0 extensions to the boundary, but not necessarily C^1 extensions; and*
- ***(Weakened Twist Condition)*** *Near the boundary B, $h_t := \alpha(X_{H_t}) > 0$, and $h_t \to +\infty$ as we approach B.*

A Poincaré–Birkhoff theorem for C^0-Hamiltonian maps. The following is a higher-dimensional version of the classical Poincaré–Birkhoff theorem.

Theorem C (Limoge–Moreno [3], based on Moreno-van-Koert [5])**.** *Let $f : (W, \omega) \to (W, \omega)$ be a C^0-Hamiltonian twist map on a Liouville domain. Assume the following:*

- ***(fixed points)*** *All fixed points of f are isolated (i.e. finitely many);*
- ***(First Chern class)*** *$c_1(W) = 0$ if $\dim W \geq 4$;*
- ***(Symplectic cohomology)*** *$SH^\bullet(W)$ is non-zero in infinitely many degrees.*

Then f has simple interior periodic points of arbitrarily large minimal period.

There is also a version for Hamiltonian chords between Lagrangians, using wrapped Floer cohomology [3]. We expect that the third condition can be relaxed to $SH \neq 0$, using Ginzburg's approach to the Conley conjecture (SDMs), and so we should obtain the following:

Theorem D (Limoge–Moreno [3])**.** *Let Q be a closed, orientable manifild, and $f : (W, \omega) \to (W, \omega)$ a C^0-Hamiltonian twist map on a fiber-wise starshaped domain $W \subset T^*Q$. Then either f has infinitely many fixed points, or it has simple interior periodic points of arbitrarily large minimal period.*

Comparison to Moreno–van-Koert [5]**.** The following are the improvements on the fixed-point theorem:

- Twist condition is relaxed, now a C^1-**open** condition.
- The boundary degeneracy is addressed.
- Action growth in SH is used, as opposed to index growth, so that:
  1. Hamiltonian needs only be C^1 and not necessarily C^2.
  2. Index-definiteness is not needed.
  3. The contact structure at the boundary need not be globally trivial.

In both results, no assumption is made on non-degeneracy of orbits (needs local Floer homology and the mean index to deal with degenerate orbits).

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Deformational Rigidity of Anosov diffeomorphisms near De la Llave's example

Martin Leguil

(joint work with Andrey Gogolev, Federico Rodriguez Hertz)

We consider the following *rigidity problem*: given two smooth diffeomorphisms f, g which are topologically conjugated by some homeomorphism h, namely,

$$h \circ f = g \circ h, \tag{1}$$

when is it possible to show that f and g are actually *smoothly* conjugated?

One basic obstruction for this to hold comes from periodic points, more precisely, from their multipliers. Indeed, assuming the conjugacy map h can be chosen C^1, differentiating (1) at any periodic point $p = f^n(p)$, $n \geq 1$, yields

$$Dg^n(h(p)) = Dh(p)Df^n(p)Dh(p)^{-1},$$

hence $Df^n(p)$, $Dg^n(h(p))$ have the same multipliers.

For minimal systems, we may thus expect that a topological conjugacy can always be promoted to a smooth conjugacy. Yet, the example of circle diffeomorphisms shows that even "almost periodic" points can be problematic to keep the conjugacy as smooth as the systems themselves. Indeed, given two conjugated circle diffeomorphisms f and g whose orbits are "far from being periodic", namely, if their (common) rotation number ρ satisfies a *Diophantine condition*, results of Arnold, Herman, Yoccoz, Khanin-Sinai etc. [1, 10, 15, 11] show that if f, g are C^ω, resp. C^∞, then f and g are also C^ω, resp. C^∞-conjugated, while if ρ is Liouville, then the conjugacy may not be very regular.

For hyperbolic systems, the picture is quite different. Indeed, assume that $f\colon \mathbb{T}^k \to \mathbb{T}^k$ is an *Anosov diffeomorphism* of the torus $\mathbb{T}^k := \mathbb{R}^k/\mathbb{Z}^k$, $k \geq 2$, i.e., there exists a Df-invariant splitting $T\mathbb{T}^k = E^s \oplus E^u$, where the stable bundle E^s, resp. unstable bundle E^u is uniformly contracted, resp. expanded by Df. On the one hand, by *structural stability*, any diffeomorphism g sufficiently C^1-close to f is Anosov, and topologically conjugated to f. On the other hand, periodic points of f and g are dense in $\mathbb{T}^k$, and in fact, for any $p = f^n(p)$, it is easy to choose the perturbation g such that the multipliers of g at the continuation of p are distinct from those of f at p. In other words, for topologically conjugated Anosov diffeomorphisms, there are plenty of obstructions to lift for the conjugacy to be even differentiable. Yet, once it is differentiable, by some boostrap phenomenon, it can sometimes be improved to be more regular. Moreover, in the spirit of Livshits Theorem [12], the hope is that periodic points carry all the obstructions for the smoothness of the conjugacy between two such systems. This was indeed shown for Anosov diffeomorphisms on $\mathbb{T}^2$, and also for 3-dimensional transitive Anosov flows, by De la Llave-Marco-Moriyón and Pollicott in a series of papers (see e.g. [3, 13]).

Let us now consider an Anosov flow $X^t : M \to M$ on some closed Riemannian manifold M, with generator $X := \frac{d}{dt}|_{t=0} X^t$. If Y^t is generated by Y that is a C^1-small perturbation of X, then by Anosov's structural stability, these two flows are orbit-equivalent, that is, there exists a homeomorphism $\Phi : M \to M$ which sends X^t-orbits to Y^t-orbits. In general, such orbit equivalence usually cannot be improved to a conjugacy since the mismatch of periods of corresponding periodic orbits provide obstructions. It is a well-known corollary of the Livshits Theorem [12] that matching of all periods for a pair of transitive Anosov flows is a necessary and sufficient condition for the existence of a (time preserving) topological conjugacy.

Given two transitive Anosov flows X^t and Y^t on 3-manifolds which are topologically conjugated by a homeomorphism Φ,

$$\Phi \circ X^t = Y^t \circ \Phi, \tag{2}$$

we can wonder whether Φ can (most of the time) be promoted to a smooth conjugacy. In other words, does the existence of a C^0-conjugacy lift the obstructions coming from periodic multipliers? In a joint work with A. Gogolev and F.R. Hertz, we show that it is locally true in general, more precisely:

Theorem A (Gogolev-L.-Rodriguez Hertz [9]). *Let M be a 3-manifold such that the space $\mathcal{A}$ of C^∞ vector fields on M which generate transitive Anosov flows is non-empty. Then, there exists a C^1-open and C^∞-dense subset $\mathcal{U} \subset \mathcal{A}$ such that for any $X \in \mathcal{U}$, the Anosov flow X^t generated by X is locally rigid, i.e., if Y^t is an Anosov flow whose generator Y is sufficiently C^1-close to X, we have:*

$$X^t \text{ and } Y^t \text{ are } C^0\text{-conjugate} \quad \Leftrightarrow \quad X^t \text{ and } Y^t \text{ are } C^\infty\text{-conjugate.}$$

This result follows from the following result:

Theorem B (Gogolev-L.-Rodriguez Hertz [9]). *Let X^t, Y^t be two 3-dimensional transitive C^∞ Anosov flows which are C^0-conjugate by a homeomorphism Φ as in (2). Then, at least one of the following statements holds:*

1. *Φ swaps SRB measures of the two flows, i.e., $\Phi_* m_X^+ = m_Y^-$ and $\Phi_* m_X^- = m_Y^+$, where $m_\dagger^\pm$ is the positive/negative SRB measure of the flow $\dagger = X, Y$;*
2. *at least one of the stable/unstable distributions E_X^s, E_X^u, E_Y^s, E_Y^u is of class $C^{1+\alpha}$, $\alpha > 0$;*
3. *the flows X^t and Y^t are C^∞-conjugate.*

This result has a counterpart in finite regularity, although it requires a technical pinching condition. Moreover, the second case of the above list where one of the distributions is C^{1+} is highly non-generic. The previous statement generalizes a previous work by Gogolev-Rodriguez Hertz when the two flows are volume preserving, in which case, the conjugacy is smooth unless both flows are constant roof suspension flows.

The basic scheme of the proof is to show that periodic multipliers can be extracted from periodic expansions unless the stable/unstable distributions exhibit anomalous regularity. One key tool for that is the so-called *templates* introduced in the work of Tsujii-Zhang [14] on exponential mixing of 3-dimensional Anosov flows. Another important ingredient is the recent positive proportion Livshits Theorem of Dilsavor-Marshall Reber [4] which allows to globalize the argument, namely, it is sufficient to match multipliers for a positive set of periodic points to fall either in the first case or the third case of the above list.

Going back to Anosov diffeomorphisms, we can wonder whether a higher-dimensional version of the aforementioned rigidity results of De la Llave-Marco-Moriyón and Pollicott can be obtained. In other words, if f and g are two topologically conjugated Anosov diffeomorphisms with matchings multipliers, are they *smoothly* conjugated? For the 2-torus, the one-dimensionality of the stable and unstable foliations and the affine structures along their leaves plays an important role.

It turns out De la Llave [2] produced a family of counterexamples on the torus $\mathbb{T}^4$ as follows, see also [5, 6] for later expositions. Specifically, let A and B be automorphisms of $\mathbb{T}^2$ induced by hyperbolic matrices in $\mathrm{SL}(2,\mathbb{Z})$. We will assume that the smaller eigenvalues λ and μ of A and B, respectively, satisfy the following inequalities: $0 < \lambda < \mu < 1$. Define $\alpha = \log\mu/\log\lambda$ and notice that $\alpha \in (0,1)$. Let $\phi_0\colon \mathbb{T}^2 \to \mathbb{T}^2$ be a smooth function. A *De la Llave diffeomorphism* $L_{\phi_0}\colon \mathbb{T}^4 \to \mathbb{T}^4$ is defined as a skew-product

$$L_{\phi_0}\colon (x,y) \mapsto (Ax, By + \phi_0(x)), \quad (x,y) \in \mathbb{T}^2 \times \mathbb{T}^2.$$

Such diffeomorphism is Anosov and, if ϕ_0 is homotopic to a constant, is conjugate to the linear product automorphism L_0. More generally, if ϕ_1 is homotopic to ϕ_0 then the corresponding De la Llave diffeomorphism L_{ϕ_1} is conjugate to L_{ϕ_0} with conjugacy h given by

$$h\colon (x,y) \mapsto (x, y + \psi(x)),$$

where $\psi\colon \mathbb{T}^2 \to \mathbb{T}^2$ is null-homotopic. Then we can explicitly solve for ψ in terms of $\phi_1 - \phi_0$ and check that $\psi \in C^{\alpha}(\mathbb{T}^2)$. However, for some simple choices of $\phi_1 - \phi_0$ (one Fourier mode) the function ψ is not in the $C^{\alpha+}$ regularity class. Consequently, the conjugacy h is merely C^{α} regular despite the fact that all multipliers at corresponding periodic points of L_{ϕ_0} and L_{ϕ_1} are the same.

In an ongoing project with A. Gogolev, we show that this phenomenon is not typical near such examples. Before stating the result, let us give some definitions. Consider a one-parameter family of Anosov diffeomorphisms $\{F_s\}_{s\in[0,1]}$, associated with a continuous family of conjugacies $\{h_s\}_{s\in[0,1]}$, such that $h_s \circ F_0 = F_s \circ h_s$ and $h_0 = \mathrm{id}$. Such family is called *isospectral* if for every periodic point $p = F_0^n(p)$

the linearized return maps $DF_s^n(h_s(p))$ all have the same collection of eigenvalues for all $s \in [0,1]$. An Anosov diffeomorphism F_0 is called *deformation rigid* if for any isospectral one-parameter family based at F_0 the corresponding family of conjugacies h_s is, in fact, a family of C^1 diffeomorphisms. Our result is then:

Theorem C (Gogolev-L. [7]). *Let L_ϕ be a De la Llave diffeomorphism. Then there exists a C^1-small neighborhood U of L_ϕ and a C^2-open C^∞-dense subset $V \subset U$ such that each Anosov diffeomorphism $F \in V$ is deformation rigid.*

Note that L_ϕ and its perturbations have 2-dimensional stable and unstable distributions. We consider perturbations F with a partially hyperbolic splitting, namely the (un)stable bundle splits into the sum of a strong (un)stable and a weak (un)stable bundles. One key step in our proof is to show that for any isospectral family $\{F_s\}_s$ based at a typical perturbation F, the conjugacies $\{h_s\}_s$ have to preserve strong (un)stable foliations. For that, we consider leaves of these foliations at periodic points and use periodic expansions similar to those considered in [9].

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Length spectrum rigidity and flexibility of spheres of revolution

Alberto Abbondandolo
(joint work with Marco Mazzucchelli)

An important question in dynamics is to understand how much information is encoded in the periodic orbits of a system. In the case of geodesic flows on a negatively curved orientable closed Riemannian surface (M, g), this question is completely answered by the celebrated rigidity theorem of Otal [6] and Croke [4]: the map that assigns to each free homotopy class of closed loops in M the length of the unique closed geodesic in that class determines the metric g up to isometry. This metric rigidity phenomenon crucially relies on the hyperbolicity of the geodesic flow. It holds more generally for Anosov geodesic flows on orientable closed surfaces [5], but has no analogue on surfaces such as the two-sphere, where hyperbolicity cannot occur. Indeed, the two-sphere admits a rich family of pairwise non-isometric *Zoll metrics*, i.e., Riemannian metrics all of whose geodesics are closed and of equal length. However, it is worth noting that here dynamical rigidity still holds: the geodesic flows of any two Zoll metrics on the two-sphere with the same common geodesic length are smoothly conjugate [1].

Motivated by the example of Zoll metrics, we undertake a case study of a special class of metrics on the two-sphere.

We consider the unit sphere $S^2 \subset \mathbb{R}^3$ equipped with the smooth S^1-action given by rotations about the z-axis, and denote by $\mathcal{G}$ the set of S^1-invariant Riemannian metrics on S^2 that possess a unique equator, i.e., a single S^1-invariant unoriented closed geodesic.

Any unoriented closed geodesic γ that is neither the equator nor a meridian (i.e., a closed geodesic passing through the two fixed points of the S^1-action) has a non-zero winding number $p \in \mathbb{N}$ around the z-axis and intersects the equator transversely $2q$ times, for some $q \in \mathbb{N}$ with $\gcd(p, q) = 1$. Such a geodesic is said to be of *type* (p, q). For a fixed coprime pair $(p, q) \in \mathbb{N} \times \mathbb{N}$, we denote by $\mathcal{L}_g(p, q)$ the subset of $(0, +\infty)$ consisting of the lengths of all closed geodesics of type (p, q). This set may be empty, finite, infinite, or even uncountable.

We say that two metrics g_1 and g_2 in $\mathcal{G}$ are *isospectral* if their equators have the same length and the set-valued functions $\mathcal{L}_{g_1}$ and $\mathcal{L}_{g_2}$ coincide.

Given $g \in \mathcal{G}$, we denote by $T^1_g S^2$ the unit tangent bundle of (S^2, g) and by Γ_g the subset of $T^1_g S^2$ consisting of the two orbits of the geodesic flow corresponding to the equator. The S^1-action on S^2 lifts to an S^1-action on $T^1_g S^2$, under which Γ_g consists of two orbits. It is easy to verify that if the geodesic flows of two metrics in $\mathcal{G}$ are S^1-equivariantly conjugate, then the metrics are isospectral.

Conversely, we have the following dynamical rigidity result.

Theorem A. *Let $g_1, g_2 \in \mathcal{G}$ be isospectral smooth (resp. analytic) metrics. Then there exists a smooth (resp. analytic) S^1-equivariant diffeomorphism*

$$h : T^1_{g_1} S^2 \setminus \Gamma_{g_1} \to T^1_{g_2} S^2 \setminus \Gamma_{g_2}$$

which conjugates the geodesic flows of g_1 and g_2. Furthermore:

(a) *If the curvature of* g_1 *along the equator is positive, then the same holds for* g_2*, and* h *extends to a smooth (resp. analytic) conjugacy from* $T^1_{g_1}S^2$ *to* $T^1_{g_2}S^2$.

(b) *If the curvature of* g_1 *along the equator does not vanish to infinite order, then the same holds for* g_2*, and* h *extends to a continuous conjugacy from* $T^1_{g_1}S^2$ *to* $T^1_{g_2}S^2$.

Explicit examples show that if the assumptions of (a) and (b) are violated, the conjugacy may fail to extend smoothly or even continuously to the two orbits corresponding to the equator. The diffeomorphism h can be chosen to intertwine the contact forms whose Reeb flows are the geodesic flows of g_1 and g_2.

The proof of the above result relies on the integrability of the geodesic flow of S^1-invariant metrics on S^2, the Birkhoff section determined by the equator, and the following fact: a smooth real function on an interval is uniquely determined by the set of tangent lines to its graph. As noted during the workshop, the existence of the conjugacy outside Γ_{g_j} can also be deduced from a general theorem on integrable systems in Cieliebak's PhD thesis [3].

Any metric g in $\mathcal{G}$ can be uniquely written as

$$g = d\sigma^2 + r(\sigma)^2 d\theta^2,$$

where $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ denotes the longitude angle on S^2, and σ is the g-length parameter along the meridians starting from the south pole $(0, 0, -1)$. Here, r is a smooth function on $[0, m]$ which is positive on $(0, m)$, vanishes at 0 and m, and has a unique critical point (necessarily a maximum), corresponding to the equator. The quantity $m > 0$ denotes the length of the meridian arc from one pole to the other. The condition that g is smooth (resp. analytic) is equivalent to the fact that r extends to a smooth (resp. analytic) $2m$-periodic odd function on $\mathbb{R}$ such that $r'(0) = 1 = -r'(m)$. We refer to the function $r : [0, m] \to \mathbb{R}$ as the *profile function* of g. For instance, the profile function of the round sphere with curvature 1 is given by $r(\sigma) = \sin\sigma$ on the interval $[0, \pi]$.

Theorem B. *Let* $r_1 : [0, m_1] \to \mathbb{R}$ *and* $r_2 : [0, m_2] \to \mathbb{R}$ *be the profile functions of metrics* $g_1, g_2 \in \mathcal{G}$*. Then* g_1 *and* g_2 *are isospectral if and only if for every* $\rho \geq 0$ *we have*

$$\text{length}\left(\{\sigma \in [0, m_1] \mid r_1(\sigma) \geq \rho\}\right) = \text{length}\left(\{\sigma \in [0, m_2] \mid r_2(\sigma) \geq \rho\}\right).$$

In particular, for $\rho = 0$ *we obtain* $m_1 = m_2$.

The proof of the above theorem relies on the injectivity of the Abel transform. A metric $g \in \mathcal{G}$ is said to be $\mathbb{Z}_2$*-symmetric* if it is symmetric with respect to the reflection $(x, y, z) \mapsto (x, y, -z)$. An immediate consequence of Theorem 2 is the following metric rigidity result for $\mathbb{Z}_2$-symmetric metrics in $\mathcal{G}$.

Corollary C. *Let* $g_1, g_2 \in \mathcal{G}$ *be isospectral* $\mathbb{Z}_2$*-symmetric metrics. Then* g_1 *and* g_2 *are isometric.*

Another consequence is the following metric flexibility result, which shows that the space of metrics in $\mathcal{G}$ that are isospectral to a given one is extremely large

– essentially as large as the space of Zoll metrics in $\mathcal{G}$ – and carries a natural structure of an infinite-dimensional convex set.

Corollary D. *For any $g \in \mathcal{G}$ there exists a unique $\mathbb{Z}_2$-symmetric metric $S(g) \in \mathcal{G}$ that is isospectral to g. The fiber $S^{-1}(g_s)$ of any $\mathbb{Z}_2$-symmetric smooth (resp. analytic) metric g_s whose curvature at the equator does not vanish to infinite order can be described as follows: if g_s has profile function $r_s : [0,m] \to \mathbb{R}$, then the smooth (resp. analytic) metrics in $S^{-1}(g_s)$ are precisely those whose profile function r is given by*

$$r := r_s \circ \phi,$$

where $\phi : [0,m] \to [0,m]$ is the inverse of the diffeomorphism

$$[0,m] \to [0,m], \qquad \tau \mapsto \tau + \psi(\tau),$$

with $\psi : \mathbb{R} \to \mathbb{R}$ an arbitrary smooth (resp. analytic) odd function satisfying $\psi(m-\tau) = -\psi(\tau)$ for every $\tau \in \mathbb{R}$, $\psi'(0) = 0$, and $|\psi'| < 1$.

Using the parameter $\tau = \phi(\sigma) \in [0,m]$ and expressing the $2m$-periodic even function ψ' as a function of $\cos\left(\frac{\pi}{m}\cdot\right)$, we deduce that the metrics which are isospectral to the $\mathbb{Z}_2$-symmetric metric g_s as above are precisely those of the form

$$g = \left(1 + f\left(\cos\left(\tfrac{\pi}{m}\tau\right)\right)\right)^2 d\tau^2 + r_s(\tau)^2 d\theta^2,$$

where $f : [-1,1] \to (-1,1)$ is an odd function vanishing at 1. In the special case $m = \pi$ and $r_s(\sigma) = \sin\sigma$, we recover the classical formula for S^1-invariant Zoll metrics on S^2 with all geodesics of length 2π (see [2, Corollary 4.16]).

Remark. *The flexibility statement of Corollary 2 should be contrasted with the following rigidity result of Zelditch [7] for the class of analytic metrics in $\mathcal{G}$ of "simple type" (where the latter assumption includes the condition that for each positive number L, there exists at most one S^1-family of unoriented closed geodesics of length L, thereby excluding Zoll metrics): within this class, the eigenvalues of the Laplace–Beltrami operator determine the metric up to isometry.*

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Perturbation of the time-1 map of a generic volume-preserving 3-dimensional Anosov flow

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(joint work with Masato Tsujii)

A diffeomorphism is said to be *transitive* if it admits a dense orbit, and is said to be *stably transitive* (also known as persistently transitive, robustly transitive in the literatures) if this property holds for all of its small perturbations. The study of stably transitive diffeomorphisms can be traced back to the study of Anosov, Smale, etc. on structural stability. It is known that Anosov diffeomorphisms are C^1-stably transitive. The first C^1-stable, non-Anosov, transitive diffeomorphism is constructed by Shub [14] in 1968. Shub's example is a skew products of *derived-from-Anosov* (DA-diffeomorphsim) on $\mathbb{T}^4$. In 1978, Mañé [8] constructed a C^1-stably transitive DA-Anosov diffeomorphism on $\mathbb{T}^3$.

It has been an open question about whether there can be stably transitive diffeomorphism isotopic to the identity, until the solution by Bonatti-Diaz [1] in 1995, who proved that certain perturbation of the time-1 maps of Anosov flow is C^1-stably transitive. In their proof they introduce a geometric model called "blender", which is a special type of hyperbolic set. There are some other examples of C^1-stably transitive diffeomorphisms, for instance [10, 3, 4].

All of the above examples of stably transitive diffeomorphisms contain an abundance of periodic orbits, as hyperbolic set always appear by construction or by Katok's horseshoe theorem. The following natural question remains open:

Question 1. *Is there a stably transitive diffeomorphism without any periodic points ?*

A natural place to look for answer is among the time-1 maps of transitive Anosov flow, as studied in [1]. The following question raised in [2] remains open.

Question 2. *Let X be a transitive Anosov vector field that is not conjugate to a suspension, and let f be the time-one map of its flow. Is f stably transitive ? (The answer may depend on X)*

As stated in [2], *"A positive answer to this question would provide an initial example of a robustly transitive diffeomorphism without periodic points."* A more precise question about whether the time-one map of the geodesic flow on a negatively curved surface can be stably transitive already appeared explicitly in [6], [12, Problem 4] and [15, Problem 1]. Related question about whether there exists stably transitive non-Anosov affine diffeomorphism can be found in [7], [6], [12] and [11]. The following question of Palis and Pugh is also closely related to Question 2.

Question 3. [9, Problem 20] *Can the time-one map of an Anosov flow be approximated by an Axiom A diffeomorphism ?*

As already pointed out in [9], if the flow is a suspension of an Anosov diffeomorphism the answer to the above question is Yes. In [2], Bonatti and Guelman proved that such approximation can be done in the C^0-topology. Based on the strategy put forth in [2], Shi has shown in [13] that every partially hyperbolic automorphism on non-abelian $\mathbb{T}^3$ can be C^1 approximated by structurally stable Axiom A maps, providing a positive answer to Question 3 in the C^1-topology.

In a work-in-progress with Masato Tsujii, we give a positive answer to Question 1, and the first positive answer to Question 2 for a class of flows, albeit the topology considered in our theorem is the C^r-topology for some large r. In fact, our diffeomorphisms are stably topologically mixing.

Theorem A. *Let r be a large integer. Let g be a volume preserving 3-dimensional Anosov flow satisfying certain C^r-open and C^∞-dense condition. Then there exists an open neighborhood $\mathcal{U}$ of g^1 in $\mathrm{Diff}^r(M)$ such that the following is true for evey $f \in \mathcal{U}$:*

1. f is topologically mixing,

2. there is an f-invariant measure ν_f such that for a.e. p with respect to volume, the sequence of measures $\frac{1}{N}\sum_{n=0}^{N-1}\delta_{f^n(p)}$ converges to ν_f (in fact ν_f is the unique u-Gibbs state of f),

3. if f is volume preserving, then f is exponentially mixing for the volume.

We also obtain as a corollary the following partial answer to Question 3. As far as we know, this is the first negative answer to Question 3, in any topology.

Corollary 1. *Let r be given by Theorme A, and let g be a volume preserving Anosov flow given in Theorme A on a 3-dimensional manifold different from $\mathbb{T}^3$. Then g^1 cannot be approximated in the C^r topology by Axiom A maps.*

Theorem A follows quickly from our main result, Theorem B, which is about the speed of convergence to equilibrium, proved using ideas from [16, 17] and some newly developped analytical tools. It is worth mentioning that our proof is *not* based on the discussion of central Lyapunov exponent. An interesting result in a closely related setting was obtained by Dolgopyat [5].

Theorem B. *Let M be a compact 3-manifold. There exist integers $r, k > 0$, and a C^r-open and C^∞-dense subset $\mathcal{V}$ of volume preserving Anosov flows on M such that for any $g \in \mathcal{V}$, there exists $\kappa_g > 0$ and an open neighborhood $\mathcal{U}$ of g^1 in $\mathrm{Diff}^r(M)$ such that for any $f \in \mathcal{U}$, there is an f-invariant measure ν_f such that for any $u, v \in C^k(M)$, we have*

$$\left|\int u \cdot v \circ f^n dvol - \int udvol \int v d\nu_f\right| < C(f)e^{-n\kappa_g}\|u\|_{C^k}\|v\|_{C^k}, \quad n \geq 0.$$

In other words, we have shown that the push-forwards of a given probability measure on M with smooth density by iterates of f will converge exponentially fast to a (common) limit measure.

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Chaotic Dynamics in Generic Analytic Strictly Convex Billiards

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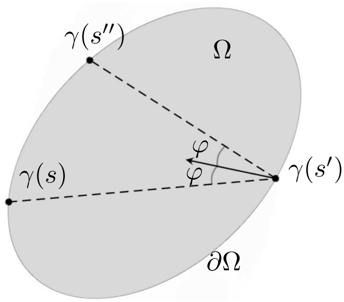
(joint work with Anna Florio, Martin Leguil, Tere Martínez-Seara)

The mathematical billiard maps were first introduced by Birkoff in [1]. They are dynamical systems which describe the motion of a massless particle, which moves among straight lines inside a closed billiard table Ω in such a way that, after the bounce, the angle of reflection equals the angle of incidence.

Let $\Omega = \Omega(\gamma)$ be a billiard table with $\partial\Omega = \gamma(\mathbb{T})$, for some closed curve γ, and consider f_γ the associated billiard map. We introduce

$$\mathbb{A} = \mathbb{T} \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Then f_γ is a map $f_\gamma : \mathbb{A} \to \mathbb{A}$ such that $f(s, \varphi) = (s', \varphi')$. The map f_γ inherits its regularity from γ and, in particular, if γ is an analytic curve, then f_γ is analytic in $\mathbb{T} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and extends continuously to $\mathbb{A}$. It is also well known that f_γ is a twist map and therefore Aubry-Mather theory can be applied.



This work concerns the abundance of analytic strictly convex Birkhoff billiards exhibiting chaotic motion through the conjugacy with the celebrated Smale horseshoe. Let us be more explicit. Let $r > 0$, we define the complexification of the torus as

$$\mathbb{T}_r = \{ s \in \mathbb{C} \,:\, \operatorname{Re} s \in \mathbb{T}, \, |\operatorname{Im} s| < r \}.$$

Then

$$\mathcal{C}^\omega_r(\mathbb{T}, \mathbb{R}^k) = \{\gamma : \overline{\mathbb{T}}_r \to \mathbb{R}^k, \text{ real analytic on } \mathbb{T}_r \text{ and continuous on } \overline{\mathbb{T}}_r \}$$

equipped with the norm

$$\|\gamma\|_r := \max_{s \in \overline{\mathbb{T}}_r} |\gamma(s)|$$

is a Baire Banach space. The space of analytic functions on $\mathbb{T}$ satisfy that

$$\mathcal{C}^\omega(\mathbb{T}, \mathbb{R}^k) = \bigcup_{r>0} \mathcal{C}^\omega_r(\mathbb{T}, \mathbb{R}^k).$$

A real analytic billiard table Ω strictly convex is characterized (non uniquely) as an element of the open set defined by

$$\mathcal{B}_r = \{\gamma \in \mathcal{C}^\omega_r(\mathbb{T}, \mathbb{R}^2), \, \gamma : \mathbb{T} \hookrightarrow \mathbb{R}^2, \, \gamma(\mathbb{T}) \text{ a strictly convex curve}\}$$

for some $r > 0$, simply by means of $\partial\Omega = \gamma(\mathbb{T})$. The main result is the following.

Theorem A. *Let $r > 0$. There exists a generic*[1] *subset $\mathcal{B}'_r \subset \mathcal{B}_r$ such that for all $\gamma \in \mathcal{B}'_r$ the following property hold.*

For any rational rotation number $p/q \in \mathbb{Q} \cap (0, 1)$, the billiard map associated with $\Omega(\gamma)$ has at least one hyperbolic periodic orbit of rotation number p/q having transverse homoclinic intersections between its stable and unstable invariant manifolds.

In particular $\mathcal{B}'_r$ is a dense subset of $\mathcal{B}_r$.

[1] A property is generic if it is shared by the elements of a residual set. In turn, a residual set is the countable intersection of open and dense sets.

Remark 1. *A consequence of Theorem A is the existence of Smale's horseshoe accumulating to the boundary of the billiard table.*

We observe that Theorem A is a trivial consequence of the next result.

Theorem B. *Let $r > 0$ and $p/q \in \mathbb{Q} \cap (0,1)$ be fixed. We denote by $\mathcal{V}_r^{p/q}$ the set of $\gamma \in \mathcal{B}_r$ such that the billiard map with billiard table $\Omega(\gamma)$ possesses a transverse homoclinic orbit associated with an hyperbolic periodic orbit of rotation number p/q.*

Then $\mathcal{V}_r^{p/q}$ is an open and dense set of $\mathcal{B}_r$ with the usual analytic topology, namely the one induced by the norm $\| \cdot \|_r$.

As a consequence the residual set

$$\mathcal{B}'_r := \bigcap_{p/1 \in \mathbb{Q} \cap (0,1)} \mathcal{V}_r^{p/q} \subset \mathcal{B}_r$$

satisfies Theorem A.

Remark 2. *In particular, Theorem B assures that the coexistence of chaotic dynamics with periodic orbits of any period is an open and dense phenomena for billiard maps in the analytic category.*

The openness property is clear. Indeed, on the one hand it is clear that transverse homoclinic points survive after $\mathcal{C}^1$-perturbations, and on the other hand, a $\mathcal{C}^2$-small perturbation of the curve γ defining the boundary of the billiard table induces a $\mathcal{C}^1$-small perturbation of the associated billiard map. Therefore, in order to prove Theorem B it only remains to show the density of the property of having transverse homoclinic orbit associated with hyperbolic periodic orbits of a given rotation number in $\mathbb{Q} \cap (0,1)$.

Theorem C. *Let $r > 0$, $\frac{p}{q} \in \mathbb{Q} \cap (0,1)$ and $\gamma \in \mathcal{B}_r$. We call $n(s)$ the unitary outward normal vector at $\gamma(s)$.*

Then, for any $\varepsilon > 0$, there exists an analytic function $\lambda_\varepsilon \in C_r^\omega(\mathbb{T}, \mathbb{R})$ – in fact, a trigonometric polynomial – with $\|\lambda_\varepsilon\|_r < \varepsilon$ such that, letting

$$\gamma_\varepsilon(s) = \gamma(s) + \lambda_\varepsilon(s)\, n(s),$$

the billiard map associated to the billiard table $\Omega(\gamma_\varepsilon)$ has a hyperbolic periodic orbit of rotation number $\frac{p}{q}$ with a transverse homoclinic point.

There has been several results about the genericity of the existence of transverse homoclinic orbits in several scenarios. For $\mathcal{C}^r$ diffeomorphisms, we mention [10, 9, 7, 8]. When restricting to convex billiards, in the differentiable case, we highlight the references [3, 11, 2, 4]. Finally, for analytic symplectic diffeomorphisms we mention the works [12] by Zehnder and [5] by Genecand where it is proven that the set of diffeomorphisms having an elliptic point and satisfying that there exist transverse homoclinic orbits at any neightborhood of the origin, is generic in some analytic topology.

All these works follow (more or less) a common strategy: if the dynamical system under consideration does not possess the desired property, a perturbation is constructed belonging to the same class (for instance when dealing with billiards maps, it has to be guaranteed that the perturbation is also a billiard map) and satisfying the condition. In the differentiable world, one can choose compactly supported perturbations that modify only a small neighbourhood of the phase space. In this way, the appropriate region is modified and moreover the perturbation might not destroy other properties of the map.

In the analytic world, this is not possible, but this technology can be leveraged in order to construct analytic perturbations from these compactly supported ones. Indeed, we combine the functional analysis tools developed in the pioneering work [12] of Zehnder on planar twist maps to provide a methodology for constructing analytic perturbations of maps in order to obtain transversality between the invariant manifolds of hyperbolic periodic orbits, with Aubry-Mather theory, in a similar way as in [5], which guarantees the existence of homoclinic points (not necessarily transverse) associated with hyperbolic periodic orbits of any rational rotation number.

Finally, besides dealing with analytic perturbation, another key difficulty in proving Theorem C lies in the fact that any modification in the billiard table induces a fibered perturbation of the phase space $\mathbb{A}$ of the billiard map f_γ (see Figure 1). In other words, even if the function $\varepsilon\lambda$ modifies only a small portion of $\partial\Omega$, the resulting effect propagates across a large region of the annulus $\mathbb{A}$. We overcome this difficulty by using Aubry-Mather theory.

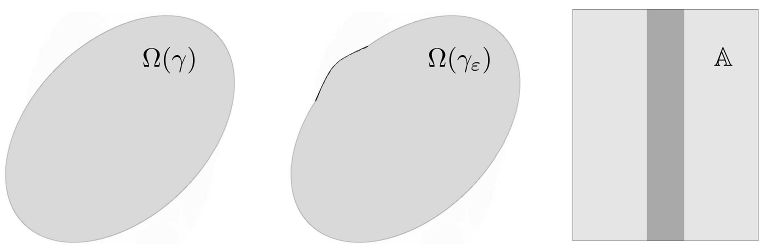


FIGURE 1. On the left, the initial billiard table. In the middle, a compactly supported perturbation of the billiard table, this would correspond to a compact support function λ in Theorem C. Even in this case, all the incidence angles are affected by the perturbation. On the right, the region in the phase space $\mathbb{A}$ affected by the perturbation.

Our approach is non-perturbative (we do not use Lazutkin coordinates, see [6]) and allows us to deal with invariant objects located away from the boundary of the billiard table.

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Chaotic phenomena to L_3 in the Restricted 3-Body Problem

Mar Giralt

(joint work with Inma Baldomá, Maciej J. Capiński, Marcel Guardia)

The Restricted Planar Circular 3-Body Problem (RPC3BP) models the motion of a body of negligible mass under the gravitational influence of two massive bodies, called the primaries, which perform circular motion and the massless body is coplanar with them. If one assumes that the ratio between the primaries masses μ is small, it models the dynamics of a Sun-Planet-Asteriod system.

Choosing a suitable rotating coordinate system, the position of the primaries can be fixed at $q_S = (\mu, 0)$ and $q_P = (\mu - 1, 0)$ and then, the position and momenta of the Asteroid, $(q, p) \in \mathbb{R}^2 \times \mathbb{R}^2$, are governed by the Hamiltonian system associated to the two degrees of freedom autonomous Hamiltonian $h = h_0 + \mu h_1$, where

$$h_0(q,p) = \frac{||p||^2}{2} - q^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} p - \frac{1}{||q||},$$

$$\mu h_1(q;\mu) = \frac{1}{||q||} - \frac{(1-\mu)}{||q-(\mu,0)||} - \frac{\mu}{||q-(\mu-1,0)||}.$$

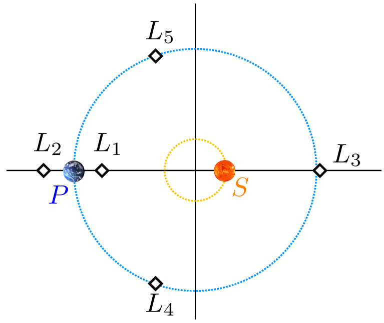

FIGURE 1. Projection onto the q-plane of the Lagrange equilibrium points for the RPC3BP on rotating coordinates.

One of the sources of instabilities in this setting are resonances where, typically, hyperbolic invariant objects with invariant manifolds appear. The goal of this work is to analyze instability phenomena at coorbital motions, that is when the Asteroid is at 1:1 mean motion resonance with the Planet (i.e. nearly equal periods) and performs close to circular motions. Several bodies in our Solar system belong to such regimes.

It is a classical result that h has five critical points, $L_1, .., L_5$, called the Lagrange points, see [8] and Figure 1. Notice that, on an inertial system of coordinates, these points correspond to circular periodic dynamics on 1:1 resonance with the primaries. In particular, we study the Lagrange point L_3, which is a saddle-center critical point. One can prove (see [10]) that the eigenvalues of the linearization around L_3, for $\mu \to 0$, satisfy that

$$\mathrm{Spec}(L_3) = \{\pm\sqrt{\mu}\,\nu(\mu), \pm i\,\omega(\mu)\}, \quad \text{with} \quad \begin{cases} \nu(\mu) = \sqrt{\frac{21}{8}} + \mathcal{O}(\mu), \\ \omega(\mu) = 1 + \mathcal{O}(\mu). \end{cases}$$

Since the ratio between the eigenvalues is $\mathcal{O}(\sqrt{\mu})$, the system possesses two time scales which translates to rapidly rotating dynamics coupled with a slow hyperbolic behavior around the critical point L_3.

The main result we present studies the existence of chaotic phenomena associated to L_3 and its invariant manifolds. The Lyapunov Center Theorem (see for instance [8]) ensures the existence of a family of periodic orbits emanating from the saddle-center L_3 which, close to the equilibrium point, are hyperbolic. In particular, that there exist $\mu_0, \varrho_0 > 0$ small enough such that, for $\mu \in (0, \mu_0)$, the system has a family of hyperbolic periodic orbits

$$\Pi_3 = \{P_{3,\varrho} \text{ periodic orbit } : h(P_{3,\varrho}) = \varrho^2 + h(L_3),\ \varrho \in (0, \varrho_0)\},$$

which depend regularly on $\varrho \in (0, \varrho_0)$ and satisfy that $\mathrm{dist}(P_{3,\varrho}, L_3) \to 0$ as $\varrho \to 0$ in the sense of Hausdorff distance. We denote by $W^u(P_{3,\varrho})$ and $W^s(P_{3,\varrho})$ its 2-dimensional unstable and stable manifolds. In [4], we obtain the following result.

Theorem A. *There exist $\mu_0 > 0$ and two functions $\varrho_{\min}, \varrho_{\max} : (0, \mu_0) \to [0, \varrho_0]$ of the form*

$$\varrho_{\min}(\mu) = \frac{\sqrt[6]{2}}{2} |\Theta| \mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}} \left[1 + \mathcal{O}\left(\frac{1}{|\log \mu|}\right)\right],$$
$$\varrho_{\max}(\mu) = \frac{\sqrt[6]{2}}{2} |\Theta| \mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}} \left[2 + \mathcal{O}\left(\frac{1}{|\log \mu|}\right)\right],$$

for certain $A > 0$ and $\Theta \neq 0$. Then, for a fixed $\mu \in (0, \mu_0)$, the following statements hold:

(1) *For $\varrho \in (\varrho_{\min}(\mu), \varrho_{\max}(\mu)]$: the invariant manifolds $W^u(P_{3,\varrho})$ and $W^s(P_{3,\varrho})$ intersect transversally at least twice.*
(2) *For ϱ close to $\varrho_{\min}(\mu)$, the flow f_ϱ unfolds generically at least one homoclinic quadratic tangency between $W^u(P_{3,\varrho_{\min}(\mu)})$ and $W^s(P_{3,\varrho_{\min}(\mu)})$.*

By the Smale-Birkhoff homoclinic Theorem (see [9, 7]), proving the existence of transverse intersections between $W^u(P_{3,\varrho})$ and $W^s(P_{3,\varrho})$ implies the existence of chaotic motions (Smale's horseshoes) exponentially close to L_3 and its invariant manifolds.

In addition, by [5], proving the generically unfolding of a quadratic homoclinic tangency implies the existence of (conservative) Newhouse domains. This leads to the existence of hyperbolic sets with Hausdorff dimension arbitrarily close to maximal and to the existence of an infinite number of elliptic islands (see [5, 6]).

To prove this result we require an asymptotic formula for the distance between the 1-dimensional stable and unstable manifolds of L_3. To present this formula, we introduce the classical symplectic polar coordinates where r is the radius, θ the argument of q, R is the radial linear momentum and G is the angular momentum. We consider as well the 3-dimensional section

$$\Sigma = \left[(r, \theta, R, G) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}^2 \,:\, r > 1,\, \theta = \frac{\pi}{2}\right]$$

and denote by $(r_*^u, \frac{\pi}{2}, R_*^u, G_*^u)$ and $(r_*^s, \frac{\pi}{2}, R_*^s, G_*^s)$ the first crossing of the invariant manifolds with this section. Then, we obtain the following result.

Theorem B. *There exists $\mu_0 > 0$ such that, for $\mu \in (0, \mu_0)$,*

$$||(r_*^u, R_*^u, G_*^u) - (r_*^s, R_*^s, G_*^s)|| = \sqrt[3]{4}\, \mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}} \left[|\Theta| + \mathcal{O}\left(\frac{1}{|\log \mu|}\right)\right],$$

where the constant $\Theta \in \mathbb{C}$ satisfies $\Theta \neq 0$ and the constant $A > 0$ is given by the real-valued integral

$$A = \int_0^{\frac{\sqrt{2}-1}{2}} \frac{2}{1-x} \sqrt{\frac{x}{3(x+1)(1-4x-4x^2)}} dx \approx 0.177744.$$

The asymptotic formula in the theorem is obtained in the papers [2, 3]. Then, in [1], by means of a computer assisted proof, we show that the constant Θ is not zero. The distance between the stable and unstable manifolds of L_3 is exponentially small with respect to $\sqrt{\mu}$. This is due to the rapidly rotating dynamics of the system and it is usually known as a beyond all orders phenomenon. As a result, classical perturbative methods (i.e the Melnikov-Poincaré method) can not be applied.

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Towards the Reeb Hofer–Zehnder and multiplicity conjectures

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(joint work with Erman Çineli, Viktor L. Ginzburg)

This report is based on the work [5] dedicated to the memory of Edi Zehnder who was a long-time co-organizer of the workshop Dynamische Systeme.

We consider the multiplicity question for prime closed orbits of Reeb flows on the boundary of a star-shaped domain in $\mathbb{R}^{2n}$ or, equivalently, on the standard contact sphere $S^{2n-1} \subset \mathbb{R}^{2n}$ in *higher* dimensions. This old and multifaceted question originates in classical mechanics and calculus of variations and goes back at least to Lyapunov's times if not earlier. It was studied by Ekeland, Hofer and Zehnder, Moser, Rabinowitz, Weinstein and many others. The following conjecture encompasses much of what we know or can expect to be true about prime closed orbits of Reeb flows on the boundary of $2n$-dimensional star-shaped domains.

Conjecture 1 (The n-or-∞ conjecture)**.** *Let M be the boundary of a star-shaped domain in $\mathbb{R}^{2n}$, which we assume to be smooth. Then*

- *Multiplicity conjecture: the Reeb flow on M has at least n prime closed orbits;*

- *Hofer–Zehnder conjecture: the Reeb flow on M has exactly n prime closed orbits whenever the flow is a Reeb pseudo-rotation, i.e., the number of prime closed orbits is finite.*

In dimensions $2n \geq 6$, the case where the novelty of [5] lies in, very little is known about either part of the conjecture in such a generality. For instance, without extra conditions on M, it is not even known that there must be more than one prime orbit or more than two if the flow is non-degenerate. The existence of at least one orbit was proved in [19] and served as the basis of the Weinstein conjecture, [22], later established in [21] for all contact type hypersurfaces in $\mathbb{R}^{2n}$. We expect the n-or-∞ conjecture to be true for a very broad class of Reeb flows on M, but possibly not all. Once some additional restrictions on M are imposed, usually along the lines of convexity or dynamical convexity or symmetry, much more is known; see, e.g., [1, 2, 6, 11, 12, 14, 18].

Our first main result from [5] establishes the multiplicity conjecture for dynamically convex Reeb flows and settles a conjecture usually attributed to Ekeland, [7, p. 198]. (Dynamical convexity, introduced in [16], is the lower bound $n+1$, the same as for ellipsoids in $\mathbb{R}^{2n}$, for the Conley–Zehnder index (or its lower semi-continuous extension) of closed orbits and, in general, less restrictive than convexity; see, e.g., [3, 13].)

Theorem A (Multiplicity; [5]). *Assume that the Reeb flow on the boundary $M^{2n-1} \subset \mathbb{R}^{2n}$ of a star-shaped domain is dynamically convex and has finitely many prime closed orbits, i.e., the flow is a Reeb pseudo-rotation.*

(i) *Then the flow has at least n prime closed orbits.*
(ii) *If, in addition, the flow is non-degenerate, it has exactly n non-alternating prime closed orbits, and all orbits have parity $n+1$.*

As a consequence, the Reeb flow on the boundary of a star-shaped domain has at least n prime closed orbits whenever the flow is dynamically convex.

The second main result of [5] is a higher-dimensional contact variant of Franks' celebrated 2-or-infinity theorem, [8, 9, 17], and, viewed from the symplectic dynamics perspective, settles a particular case of the Reeb Hofer-Zehnder conjecture; cf. [15, p. 263] and [20].

Theorem B (HZ-conjecture; [5]). *Assume that the Reeb flow on the boundary $M^{2n-1} \subset \mathbb{R}^{2n}$ of a centrally symmetric star-shaped domain has finitely many prime closed orbits and is dynamically convex and non-degenerate. Then the flow has exactly n prime closed orbits. These orbits are symmetric and non-alternating.*

Strictly speaking, Theorem B is the first result establishing the n-or-∞ dichotomy in higher dimensions. However, from a broader perspective, another closely related but logically independent result is [4, Thm. A] which asserts, in particular, that a non-degenerate dynamically convex Reeb flow with a hyperbolic closed orbit has infinitely many prime closed orbits; cf. [10].

The strategy of the proof of Theorems A and B is to compare the symplectic homology persistence module for a dynamically convex Reeb pseudo-rotation on

M with that of an irrational ellipsoid and show that they look the same. With this in mind, the proofs are based on several auxiliary results on the structure of the filtered symplectic homology and properties of closed orbits. In particular, the key to the argument is [5, Thm. C] asserting that the (non-equivariant) filtered symplectic homology is one-dimensional for every action threshold and any ground field whenever the number of prime closed orbits is finite and the flow is dynamically convex.

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Equidistribution and asymptotic counting of surfaces in negatively curved three manifolds

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(joint work with Ben Lowe, Graham Smith)

Closed geodesics and geometric rigidity. The abundance of closed geodesics in a closed, negatively curved 3-dimensional Riemannian manifold can be used to obtain many geometric rigidity results. For instance, knowing their exponential growth rate is sufficient to characterize the hyperbolic metric among all negatively curved metrics with a fixed volume. This is famously known as the Besson-Courtois-Gallot rigidity theorem [3].

In this talk, we focused on the dynamical study of higher-dimensional analogues of the geodesic flow, specifically the distribution and counting of closed surfaces in closed, negatively curved 3-manifolds.

In the sequel, (M, h_0) will denote a closed, connected 3-dimensional hyperbolic manifold. It is isometric to $\mathbb{H}^3/\Pi$, where Π is a cocompact lattice of $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^2)$. By Mostow's rigidity theorem the hyperbolic metric on M is unique up to isometry. Our main goal is to characterize the hyperbolic metric on M within the space of Riemannian metrics h in M that have negative sectional curvature everywhere ($\mathrm{sect}_h < 0$.)

Let $X = \widetilde{M}$ denote the universal cover of M, and let $\partial_\infty X$ denote its ideal boundary. This boundary is defined as the set of equivalence classes of geodesic rays under the relation of "staying at bounded distance". The group $\Pi \simeq \pi_1(M)$ (identified with a cocompact lattice of the group of direct isometries of X) acts on this ideal boundary.

Existence of closed and quasifuchsian surfaces. Let $C > 1$. a C-quasicircle of $\mathbb{CP}^1$ is a Jordan curve Λ that is the image of the real projective line $\Lambda = h(\mathbb{CP}^1)$ by a C-quasiconformal homeomorphism $h : \mathbb{CP}^1 \to \mathbb{CP}^1$.

A C-quasi-Fuchsian group is a discrete and torsion-free subgroup Γ of $\mathrm{PSL}_2(\mathbb{C})$ isomorphic to a cocompact lattice of $\mathrm{PSL}_2(\mathbb{R})$ whose limit set $\partial_\infty\Gamma$ is a C-quasi-circle. A 1-quasi-Fuchsian group is nothing but a Fuchsian surface subgroup of $\mathrm{PSL}_2(\mathbb{C})$.

Fuschsian subgroups of cocompact lattices of $\mathrm{PSL}_2(\mathbb{C})$ are quite rare. However, a breakthrough of Kahn-Marković showed that quasi-Fuchsian subgroups are abundant in such closed hyperbolic 3-manifolds groups.

Theorem A (Kahn-Marković's existence theorem [9])**.** *Let (M, h_0) be a closed hyperbolic 3-manifold. Then, there exists a hyperbolic surface S and an immersion $\iota : S \to M$ such that the induced map $\iota_* : \pi_1(S) \to \Pi$ is injective and the image is a quasi-Fuchsian group.*

They went further by providing a topological counting of conjugacy classes of such subgroups according to the genus. Given a cocompact lattice $\Pi \leq \mathrm{PSL}_2(\mathbb{C})$, we let QF denote the set of conjugacy classes $[\Gamma]$ of quasi-fuchsian subgroups of Π. Given $[\Gamma] \in \mathrm{QF}$ we let $\mathrm{g}(\Gamma)$ denote the genus of any representative Γ.

Theorem B (Kahn-Marković's topological counting [8])**.** *If* (M, h_0) *is a closed hyperbolic* 3*-manifold then*

$$\lim_{g\to\infty} \frac{1}{2g\log(g)} \log \#\{[\Gamma] \in \mathrm{QF};\ \mathrm{g}(\Gamma) \le g\} = 1. \tag{1}$$

We are interesting in a *geometric counting* of surfaces, analogous to the counting of geodesics. The first problem is to define natural geometric representatives of homotopy classes of surfaces.

Asymptotic Plateau problem and geometric representatives. There are two natural candidates for representing conjugacy classes of surface subgroups.

- *Minimal surfaces*, which are surfaces whose mean curvature H (the half trace of the shape operator) equals zero.
- *k-surfaces*, which are convex surfaces whose extrinsic curvature κ_{ext} (the determinant of the shape operator) equals a positive constant $k > 0$.

Given a quasi-circle $\Lambda \subset \partial_\infty X$ the asymptotic Plateau problem consists in finding an embedded disc $D \subset X$ that satisfies the required geometric condition (to be minimal or to be a k-surface) and whose ideal boundary $\partial_\infty D \subset \partial_\infty X$ coincides with Λ. There are always solutions (see [2] for minimal surfaces and [10] for k-surfaces). It is always unique for k-surfaces (see [10]). For minimal surfaces there is uniqueness provided the quasiconformality constant C of Λ is close enough to 1: see the recent [7] for results on uniqueness and non-uniqueness in that setting. When Λ is invariant by a surface subgroup of isometries Γ, the unique solution of the asymptotic Plateau problem is also invariant and the quotient gives the minimal surface, or the k-surface, representing the conjugacy class $[\Gamma]$.

In the present talk we are interested in k-surfaces since they are better behaved with respect to asymptotic Plateau problems.

Geometric counting and rigidity of the area spectra. If $[\Gamma] \in \mathrm{QF}$, we let $S_{k,h}([\Gamma])$ denote the unique closed k-surface in the conjugacy class of Γ for the metric h. We want to count them according to their area, and we expect the growth rate of this counting function to be superexponential. The next definition, and result are from [1]. A similar result was proven for minimal surfaces in groundbreaking work of Calegari-Marques-Neves [4]. The inspiration for the methods of the proofs can be found in Labourie's Bourbaki seminar on the latter work: [11].

Definition 1 (Area entropy of k-surfaces)**.** *Let h be a Riemannian metric with* $\mathrm{sect}_h \le -1$*. Fix* $0 < k < 1$ *and define the area entropy of k-surfaces as*

$$\mathrm{Ent}_k(M, h) = \liminf_{A\to\infty} \frac{1}{A\log(A)} \log \# \{[\Gamma] \in \mathrm{QF}; \mathrm{Area}_h(S_{k,h}([\Gamma]) \le A\}.$$

Theorem C (Geometric counting of k-surfaces [1])**.** *Let h be a Riemannian metric with* $\mathrm{sect}_h \le -1$ *and* $0 < k < 1$*. Then we have the following chain of inequalities*

$$\frac{H(M,h)^2}{2\pi} \ge \mathrm{Ent}_k(M,h) \ge \mathrm{Ent}_k(M,h_0) = \frac{1-k}{2\pi},$$

where $H(M,h)$ denotes the topological entropy of the geodesic flow.

Moreover the equality $\mathrm{Ent}_k(M,h) = \mathrm{Ent}_k(M,h_0)$ *holds if and only if h and h_0 are isometric.*

Let us define the *marked area spectrum* of quasi-Fuchsian k-surfaces.

$$\mathrm{MAS}_{k,h} : \mathrm{QF} \to \mathbb{R}^+, \qquad [\Gamma] \mapsto \mathrm{Area}_h(S_{k,h}([\Gamma])). \tag{2}$$

The following rigidity theorem was proven in [1]. It is an analogue of Hamenstädt's result of the rigidity of the hyperbolic marked length spectrum [6].

Theorem B. (Rigidity of the hyperbolic marked area spectrum. [1]) *Let (M,h_0) be a closed hyperbolic 3-manifold. Let h be a Riemannian metric on M with* $\mathrm{sect}_h \leq -1$ *and $k \in (0,1)$. Then* $\mathrm{MAS}_{k,h} = \mathrm{MAS}_{k,h_0}$ *if and only if h and h_0 are isometric.*

The proof uses an equidistribution result: if a quasi-Fuchsian k-surface in variable curvature has the same area than the corresponding surface in the hyperbolic metric then the sectional curvature must be constant equal to -1 in all tangent planes to the surface. We must prove these tangent planes equidistribute so that if marked area spectra coincide, the sectional curvature equals -1 in all tangent planes, so h must be isometric to h_0 by Mostow's rigidity.

The solution uses the resolution of a foliated Plateau problem in the spirit of Gromov [5], by proving that the unit tangent bundle of the ambient manifold is foliated by Gauss lifts of k-discs whose ideal boundaries are round circles at infinity. This foliation is proven to be conjugate to a homogeneous model which, by Ratner's theory [12], has very few whose ergodic invariant measures. Only one is fully supported, and the other ones, if they exist, are carried by a discrete set of discs. It is then enough to use Kahn-Marković's construction to produce a sequence of surfaces that becomes asymptotically Fuchsian and that equidistribute to a fully supported measure in the unit tangent space of the manifold.

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Geometric Hydrodynamics, Mañé Critical Value, and an Infinite-Dimensional Magnetic Hopf–Rinow Theorem

Levin Maier

Since V. Arnold's seminal discovery [2]–that the Euler equations of hydrodynamics, which govern the motion of an incompressible and inviscid fluid in a fixed domain (with or without boundary), can be interpreted as the geodesic equations on the group of volume-preserving diffeomorphisms of the domain, endowed with a right-invariant Riemannian metric (specifically, the L^2 metric)–many partial differential equations (PDEs) arising in mathematical physics have been reinterpreted within a similar geometric framework. These equations are formulated as geodesic equations on infinite-dimensional Lie groups equipped with a right-invariant Riemannian metric; see, for example, [3, 10] and the references therein.

In [10], it is further demonstrated that many PDEs in mathematical physics can be formulated as infinite-dimensional Newton equations. From a physical perspective, this provides a natural extension of the geodesic framework: while the geodesic equation describes the motion of a free particle, Newton's equation captures the dynamics of a particle under the influence of a potential force.

From this perspective, a physically natural next step is to study the motion of a charged particle in a magnetic field. Mathematically, this problem is framed within Hamiltonian dynamics, specifically through the theory of magnetic systems–pioneered by V. Arnold in [1]. The corresponding equations of motion, known as the *magnetic geodesic equations*, can be interpreted as geodesic equations modified by the *Lorentz force*, caused by the presence of an external magnetic field.

In [11], the author constructed *the first example of a partial differential equation (PDE) that admits a formulation as an infinite-dimensional magnetic geodesic equation*: the so-called *magnetic two-component Hunter–Saxton system.* In the present paper [12], we show that this example fits into a broader and more general framework. By combining the ideas of V. Arnold [1, 2], we introduce the notion of the *magnetic Euler–Arnold equation.*

This framework allows us to interpret several PDEs from fluid dynamics as magnetic Euler–Arnold equations. Examples include the Korteweg–de Vries equation, the generalized Camassa–Holm equation, the infinite conductivity equation, and the global quasi-geostrophic equations. That is, these equations describe the motion of a charged particle on an infinite-dimensional manifold under the influence of an external magnetic field. These interpretations are summarized in [12, Table 1].

All of this builds upon the *main theoretical advancement of* [12, Thm. 2.10], which extends Arnold's formulation of the geodesic equation on a Lie group with a right-invariant metric to so-called *right-invariant magnetic systems* on Lie groups; we refer to [12] for the precise definition.

This geometric formulation of the PDEs was used by the author in [11] to study the so-called *magnetic two-component Hunter–Saxton system* (M2HS). The following results about (M2HS) were obtained:

(1) *Infinitely many conserved quantities* [11, Cor. 5.4].
(2) *Geometric blow-up criteria* [11, Thm 6.1].
(3) *Existence of global weak solutions of low regularity* [11, Thm 6.10].
(4) *A Hopf–Rinow theorem for global weak solutions* [11, Thm 7.1].

For a geometric visualization of points (2) and (3), we refer the reader to [11, Fig. 2, Fig. 3]. After recalling some background on magnetic geodesics, we will explain what is meant in (4) by the Hopf–Rinow theorem in this context.

Before proceeding, we emphasize that energy is a conserved quantity in magnetic systems, reflecting their Hamiltonian structure. However, unlike standard geodesics, magnetic geodesics cannot, in general, be reparametrized to unit speed. As a result, it is natural to compare magnetic geodesics at different energy levels. The aim of this line of research is to develop a Hopf–Rinow-type theorem for magnetic geodesics with prescribed energy.

In the classical setting, Hopf and Rinow showed that any two points on a closed, finite-dimensional Riemannian manifold can be connected by a geodesic, regardless of its speed or energy. Here, we investigate whether an analogous result holds for magnetic geodesics: given two points and a fixed positive energy, does there exist a magnetic geodesic connecting them?

A central role in this question is played by the so-called Mañé critical values [14], which mark dynamical and geometric thresholds in the magnetic geodesic flow. In fact, it is known [7] that on closed, finite-dimensional manifolds, a magnetic version of the Hopf–Rinow theorem holds for energies above the Mañé critical value. For energies below this threshold, however, the situation is more subtle, and the validity of such a result remains unclear. For further background, we refer to [6, 8] and the references therein.

The remainder of this report is dedicated to extending this result to infinite-dimensional settings for certain classes of magnetic systems. Before discussing how subtle the Hopf–Rinow theorem becomes in infinite dimensions–even in the case of Riemannian Hilbert manifolds–we first motivate the problem with a positive example from recent work [11] by the author:

In [11], the author introduced the *Mañé critical value for exact infinite-dimensional magnetic systems* and demonstrated its relevance in [11, Thm. 7.1] through a magnetic version of the Hopf–Rinow theorem. Specifically, it was shown *that any two points can be connected by a magnetic geodesic of prescribed energy if* *and only if the energy exceeds the Mañé critical value.* This result provides a sharp threshold for the existence of magnetic geodesics and motivates the infinite-dimensional extension studied in the present work.

We conclude this report by outlining a *general framework–currently work in progress* [13] by the author in collaboration with F. Ruscelli–into which the previously discussed example naturally fits. To this end, we recall a recent landmark result of Bauer, Harms, and Michor [5], together with the necessary background and notation.

Half Lie groups arise exclusively in infinite dimensions. They are smooth manifolds and topological groups in which right translations are smooth, whereas left translations are only required to be continuous. The main examples are groups of diffeomorphisms of Sobolev class H^s or of class C^k.

If a half Lie group is equipped with a right-invariant strong Riemannian metric–that is, a Riemannian metric inducing an isomorphism between the tangent and cotangent bundles–then, quite unexpectedly, the full Hopf–Rinow theorem holds in this setting, as shown in [5, Thm. 7.7]. This is striking, since such a result does not generally hold even for Hilbert manifolds; see, for example, [4, 9].

In [13], Ruscelli and the author study half Lie groups equipped with right-invariant strong Riemannian metrics and right-invariant magnetic fields. For this class of spaces, they *introduce Mañé's critical values on the universal cover*, extending the ideas of [6] from finite to infinite dimensions. Moreover, the *magnetic version of the full Hopf–Rinow theorem holds in this setting* for energies above Mañé's critical value on the universal cover. In particular, in the case of a trivial magnetic field, one recovers [5, Thm. 7.7].

The *key step in the proof* is to show that, above this energy threshold, the lift of the magnetic geodesic flow to the universal cover is *conjugate to a Finsler geodesic flow.* This Finsler geodesic flow exhibits behavior analogous to that of the geodesic flows studied by Bauer, Harms, and Michor in [5].

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Entropies of H-flows on non-compact manifolds

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(joint work with Barbara Schapira, Anne Vaugon)

In collaboration with Barbara Schapira and Anne Vaugon, we introduce a class of chaotic flows on non-compact manifolds, which we call *H-flows*. These are defined by requiring that certain properties –classically verified by Anosov flows in the compact case– hold in our context. The primary interest in this new class lies in the fact that, under an additional dynamical condition known as *strongly positive recurrence*, we can prove the existence of an invariant measure of maximal entropy. More precisely, our main result is the following.

Theorem A (Florio–Schapira–Vaugon)**.** *Let $\phi\colon \mathbb{R}\times M \to M$ be a H-flow on a complete Riemmannian manifold (M,d). Suppose that ϕ is strongly positive recurrent. Then, there exists a ϕ-invariant probability measure $m_{\max}$ which maximizes the measure-theoretic entropy among all ϕ-invariant probability measures.*

In the compact case, a corresponding result was established by R. Bowen in [1], who proved that any transitive Anosov flow on a compact manifold admits a unique ergodic invariant probability measure maximizing entropy. For the non-compact case, similar results were obtained under the same strongly positive recurrence condition for transitive geodesic flows on non-compact manifolds with pinched negative curvature–by B. Schapira and S. Tapie in [5] and by S. Gouëzel, B. Schapira and S. Tapie in [3]. They demonstrated the existence and uniqueness of the measure of maximal entropy (or of a finite Gibbs measure, in the more general thermodynamical formalism).

This notion of strong positive recurrence also appears in other contexts, such as in [2], concerning non-uniformly hyperbolic diffeomorphisms on closed manifolds.

Let (M,d) be a complete Riemmannian manifold, with distance induced by the metric.

Definition 1. *A C^1 flow $\phi\colon \mathbb{R}\times M \to M$, generated by a vector field X, is a H-flow if it satisfies the following properties:*

(1) *For every $\tau\in[-1,1]$, the maps $\phi_\tau\colon M\to M$ are equi-Lipschitz.*

(2) *There exists constants* $a, b \in (0, +\infty)$ *such that* $a \leq \|X(x)\| \leq b$ *for all* $x \in M$.
(3) *The flow* ϕ *is transitive.*
(4) *The flow* ϕ *satisfies the closing lemma: for every compact set* K *and every* $\epsilon > 0$, *there exists* $\delta > 0$ *and* $T > 0$ *such that for every* $x \in K$ *and every* $t \geq T$ *such that* $d(x, \phi_t(x)) < \delta$, *there exists* $z \in M$ *and* $\tau \in]t - \epsilon, t + \epsilon[$ *so that* $\phi_\tau(z) = z$ *and* $d(\phi_s(x), \phi_s(z)) < \epsilon$ *for all* $s \in [0, t]$.
(5) *The flow* ϕ *is expansive: for every* $\nu > 0$, *there exists* $\epsilon > 0$ *such that if* $x, y \in M$ *and there exists* $s\colon \mathbb{R} \to \mathbb{R}$ *with* $s(0) = 0$ *and* $d(\phi_t(x), \phi_{s(t)}(y)) < \epsilon$ *for all* $t \in \mathbb{R}$, *then* $y = \phi_u(x)$ *for some* $u \in] - \nu, \nu[$.
(6) *The flow has the finite exact shadowing property: for every compact set* K *and every* $\delta > 0$, *there exist* $\eta > 0$ *and* $T > 0$ *such that if* $x_1, x_2 \in K$, $T_1 + T_2 > T$ *and* $d(\phi_{T_i}(x_i), x_j) < \eta$ *for* $i \neq j$, *then there exists* $y \in M$ *such that*
   (a) $d(\phi_s(y), \phi_s(x_1)) < \delta$ *for* $s \in [0, T_1]$, *and*
   (b) $d(\phi_{T_1+s}(y), \phi_s(x_2)) < \delta$ *for* $s \in [0, T_2]$.

This class of flows is particularly rich in periodic orbits. Moreover, any pseudo-orbit can be closely followed by a true orbit via "cut and paste" constructions. Given the abundance of periodic orbits, it's natural to study their exponential growth rate with respect to period, a quantity known as the *Gurevich entropy.*

Given a compact set K and a constant $C > 0$, we denote by $\mathcal{P}_K(T, C)$ the set of periodic orbits with period in $[T, T + C]$ that intersect K. The Gurevich entropy of a H-flow ϕ is then defined as:

$$h_{\mathrm{Gur}}(\phi) := \lim_{T\to\infty} \frac{1}{T} \log \#\mathcal{P}_K(T, C)\,.$$

Since we are dealing with non-compact manifolds, we also want to detect periodic orbits that may "escape" compact regions. For this, we define a refined notion.

For a compact set K, for $C > 0$ and $\epsilon > 0$, let $\mathcal{P}_K^\epsilon(T, C)$ denote the set of periodic orbits with period in $[T, T + C]$, intersecting K, but spending at most a fraction ϵ of their time inside K. The *Gurevich entropy at infinity* is defined by:

$$h^\infty_{\mathrm{Gur}}(\phi) := \inf_K \lim_{\epsilon\to 0} \limsup_{T\to\infty} \frac{1}{T} \log \#\mathcal{P}^\epsilon_K(T, C)\,,$$

where the infimum is taken over all compact sets K.

Definition 2. *A* H*-flow* ϕ *is strongly positive recurrent if* $h^\infty_{\mathrm{Gur}}(\phi) < h_{\mathrm{Gur}}(\phi)$.

This condition expresses that the complexity contributed by the behavior at infinity is exponentially smaller than the total complexity.

Having set the stage, we trun to the central objective of our work: to study the topological entropy $h_{\mathrm{top}}(\phi)$ of the flow. Thanks to the Variational Principle (which also holds in the non-compact case, by results of Handel and Kitchens [4]), we have:

$$h_{\mathrm{top}}(\phi) = \sup_\mu h_{\mathrm{KS}}(\mu)\,,$$

where the supremum is over all ϕ-invariant probability measures μ, and $h_{\mathrm{KS}}(\mu)$ denotes the Kolmogorov-Sinai entropy of μ. We construct an invariant measure $m_{\max}$ such that:

$$h_{\mathrm{KS}}(m_{\max}) = h_{\mathrm{top}}(\phi) = h_{\mathrm{Gur}}(\phi)\,.$$

A key step in our construction mirrors Bowen's method in the compact case [1]: we consider the ϕ-invariant probability measure obtained by normalizing the sum of Dirac measures supported on some periodic orbits of (approximately) fixed period, and then the sequence is built by letting the period goes to infinity. By taking a (subsequential) limit in the vague topology, we obtain a candidate measure. In non-compact settings, such sequences may "lose mass at infinity", but the strong positive recurrence condition ensures that the limiting measure in non-zero. Its normalization yields the desired measure of maximal entropy.

Beyond addressing the difficulties introduced by non-compactness, the proof also requires comparing various notions of entropy. In particular, the *chord entropy* –which measures the exponential growth rate of the number of separeted chords connecting two given open sets– plays a central role.

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Twisting and periodic orbits in asymptotically unitary Hamiltonian systems

Leonardo Masci

Introduction. In 1978, P. Rabinowitz [8] successfully used variational methods to show the existence of periodic solutions in a class of Hamiltonian systems on the standard symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$, defined by non-autonomous, time-periodic Hamiltonians having super-quadratic growth. This achievement surprised the Hamiltonian dynamicists of the time, as the Hamiltonian action functional was widely believed to be ill-suited for standard variational techniques.

A couple of years later, H. Amann and E. Zehnder [2, 3] initiated the study of the boundary case, not covered by Rabinowitz's techniques, where the Hamiltonians in question were assumed to grow quadratically. Under non-resonance hypotheses, one periodic solution was found to always exist for this class of systems. Joining forces with C. Conley [5], multiplicity results were obtained, and importantly, Morse-like inequalities between the indices of the orbits were shown to hold. This discovery led them to solve the Arnol'd conjecture on symplectic tori using similar techniques [4]. For an insider's view of this story, we recommend the beautiful article by Zehnder [9].

During and following these breakthroughs, much effort was expended to prove existence and multiplicity statements for periodic orbits in asymptotically linear Hamiltonian systems under weakened hypotheses. The culmination of this endeavor is the development, by A. Abbondandolo, of an infinite-dimensional relative Morse theory, suitable for the Hamiltonian action functional defined by these Hamiltonians with quadratic growth. We point to [1] and the references therein for more information on this line of research.

In the results mentioned above, the focus was on finding periodic solutions with period equal to the fundamental period of the coefficients of the system, which we can assume to be 1. We can think of these periodic solutions as *forced oscillations* in the Hamiltonian system. Much less is known about periodic solutions with higher period $k \in \mathbb{Z}$, which we will call *subharmonics*, as their period is a multiple of the fundamental period of the system. It seems natural to try to gather further knowledge on their existence, multiplicity and growth of period. Aiming toward this goal, in this talk the author formulated a natural twist condition, inspired by the classical Poincaré-Birkhoff theorem on area-preserving isotopies of the annulus, which applies to asymptotically linear Hamiltonian systems. Under further hypotheses, this twist condition was then used to show the existence of infinitely many non-trivial subharmonics with period growing to infinity.

Asymptotically linear Hamiltonian systems. Let us first define the class of Hamiltonian systems in study. An *asymptotically linear Hamiltonian system* on $(\mathbb{R}^{2n}, \omega_0)$ is defined by a smooth, non-autonomous, 1-periodic Hamiltonian $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times \mathbb{R}^{2n})$ which is *asymptotically quadratic*, in the following sense: there exists a smooth, 1-periodic path of symmetric matrices $A \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathrm{Sym}(2n))$ such that

$$|\nabla H(t,z) - A(t)| = o(|z|) \quad \text{as } |z| \to \infty.$$

We set $Q(t,z) = \frac{1}{2}\langle z, A(t)z\rangle$ and $h(t,z) = H(t,z) - Q(t,z)$. We call Q the *quadratic Hamiltonian at infinity* and h the *non-quadratic part.* Denote by φ_H^t the Hamiltonian flow of a Hamiltonian H. An *asymptotically linear Hamiltonian diffeomorphism (ALHD)* is a Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(\mathbb{R}^{2n})$ such that $\phi = \varphi_H^1$, where $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times \mathbb{R}^{2n})$ is an asymptotically quadratic Hamiltonian, which we call a *generating Hamiltonian.* Let ϕ be an ALHD and $H = Q + h$ be an asymptotically quadratic generating Hamiltonian for ϕ. The *linear map at infinity* for ϕ is $\phi_\infty = \varphi_Q^1$. The map ϕ_∞ depends only on ϕ, and not on the specific choice of generating Hamiltonian H [7].

Toy model. We first give an example of the kind of twisting phenomenon we want to capture, in a very simplified situation.

Let $\phi \in \mathrm{Ham}(\mathbb{R}^2)$ be a Hamiltonian diffeomorphism. Assume that there exists a compact subset $K \subset \mathbb{R}^2$ outside of which ϕ is a rotation of angle $\theta_\infty \notin 2\pi\mathbb{Z}$, and that ϕ admits a fixed point $z_0 \in K$ with rotation number $\theta_0 \neq \theta_\infty$. Such a ϕ is a special kind of ALHD, as it is equal to a linear symplectic map outside a compact set. We construct from ϕ an area- and orientation-preserving homeomorphism of the annulus $f\colon [0,1]\times S^1 \to [0,1]\times S^1$ by fixing a large invariant circle bounding a disc D which contains K, and blowing up the fixed point z_0. The assumption that $\theta_0 \neq \theta_\infty$ implies that f has different rotation numbers at the two boundaries of the annulus. Applying a version of the Poincaré-Birkhoff theoreom, e.g. Franks' [6, Corollary 2.4], we obtain that f, and consequently ϕ, has infinitely many non-trivial periodic points.

Index at infinity and twist condition. To formulate the twist condition, we will work with the Conley-Zehnder index. This is an integer $\mathrm{CZ}(M_t) \in \mathbb{Z}$ associated to a path of symplectic matrices $[0,1] \ni t \mapsto M_t \in \mathrm{Sp}(2n)$, which measures the number of twists that a symplectic basis undergoes under the action of the path. See [1, Chapter 1] for the definition and properties of this index.

As a relevant example, if $z_0 \in \mathbb{R}^{2n}$ is a fixed point of a Hamiltonian diffeomorphism ϕ and $H \in C^\infty\left(\mathbb{R}/\mathbb{Z} \times \mathbb{R}^{2n}\right)$ is a generating Hamiltonian, we can consider the Conley-Zehnder index of the linearized flow at the fixed point:

$$\mathrm{CZ}(z_0, H) := \mathrm{CZ}\left(D\phi_H^t\left(z_0\right)\right).$$

An interesting quantity, constructed in terms of the index, is the *mean Conley-Zehnder index*, defined by

$$\overline{\mathrm{CZ}}\left(z_0, H\right) := \lim_{k\to\infty} \frac{1}{k}\mathrm{CZ}\left(D\phi_H^{kt}\left(z_0\right)\right).$$

This quantity represents the rate of growth of the index under iteration, and can be seen as the generalization of the rotation number to fixed points of Hamiltonian diffeomorphisms.

With the toy model in mind, the idea is to use the mean Conley-Zehnder index to define a "rate of twisting at infinity" of an asymptotically linear Hamiltonian system. Let ϕ be an ALHD and $H = Q + h$ an asymptotically quadratic generating Hamiltonian. The ALHD ϕ is said to be *non-degenerate at infinity* if $\det\left(\phi_\infty - \mathbb{I}\right) \neq 0$. The *index at infinity* of the Hamiltonian H is

$$\mathrm{ind}_\infty H := \mathrm{CZ}\left(\varphi_Q^t\right).$$

The *mean index at infinity* of H is

$$\overline{\mathrm{ind}}_\infty H := \lim_{k\to\infty} \frac{1}{k}\mathrm{CZ}\left(\varphi_Q^{kt}\right).$$

In the toy model, the mean index at infinity is precisely $2\theta_\infty$, and the mean Conley-Zehnder index of the fixed point is $2\theta_0$. Therefore the twist condition there translates into a discrepancy of mean indices. With this in mind, we give the following

Definition 1. *Let ϕ be an ALHD. A fixed point $z_0 \in \mathrm{Fix}\,\phi$ is said to be* twist *if there exists an asymptotically quadratic generating Hamiltonian H for ϕ with*

$$\overline{\mathrm{CZ}}\left(z_0, H\right) \neq \overline{\mathrm{ind}}_\infty H.$$

The twist condition does not depend on the generating Hamiltonian chosen, even though the quantities involved do.

A Poincaré-Birkhoff theorem. Under further restrictions on the behaviour at infinity of the system, the twist condition has been used by the author to obtain the existence of infinitely many non-trivial subharmonics.

Let ϕ be an ALHD which is non-degenerate at infinity. We assume that $\phi = \varphi_H^1$ where $H = Q + h$ is asymptotically quadratic, and the non-quadratic part h is also bounded. Moreover, we assume that ϕ_∞ is a unitary map, i.e. both symplectic and orthogonal. Finally, we assume that ϕ has an isolated twist fixed point which is homologically visible. This last condition means that its local Floer homology is non-trivial. For example, if the fixed point is non-degenerate, i.e. $\det(D\phi(z_0) - \mathbb{I}) \neq 0$, then it is homologically visible.

Theorem A ([7]). *An ALHD ϕ as above has infinitely many fixed points or infinitely many non-trivial periodic points with increasing period.*

It would be interesting to understand the rate of growth of these subharmonics in terms of their period, namely, estimating the quantity

$$\underline{\mathrm{h}}(\phi) := \liminf_{k\to\infty} \frac{\log \# \operatorname{Fix} \phi^k}{\log k}.$$

Nothing is known about this quantity for asymptotically linear Hamiltonian systems. Taking inspiration from Franks' Poincaré-Birkhoff theorem [6, Corollary 2.4], we conjecture that if ϕ has infinitely many non-trivial periodic points, then $\underline{\mathrm{h}}(\phi) \geq 2$, i.e. the growth is at least quadratic.

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Analytic pseudo-rotations

Pierre Berger

We show the existence of transitive analytic symplectomorphisms of the sphere, the disk and the cylinder with finite numbers of periodic points.

This proves a conjecture of Birkhoff (1927) on the instability of elliptic points, and solves several other problems Birkhoff (1941), Herman (1998), Fayad-Katok (2004) and Fayad-Krikorian (2018) on the rigidity of pseudo rotations. Indeed this implies that an analytic symplectomorphism of the cylinder, the sphere or the disk with a finite number of periodic points is not necessarily topologically conjugated to a rotation, and can be even transitive.

To show this, we introduce a way to perform the approximation by conjugacy method of Anosov-Katok among surface analytic symplectomorphisms. To this end, we approximate smooth symplectic maps by analytic one, for a deformation of the complex structure of the surface induced by nearly holomorphic conjugacy.

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