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Real Analysis, Harmonic Analysis, and Applications

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Abstract. This workshop in real and harmonic analysis surveyed recent investigations in areas including geometric measure theory and restriction theory, multiple ergodic averages, local smoothing estimates, Schrödinger and Hörmander-type oscillatory integral operators, multilinear estimates and analysis on the Hamming cube. In particular, the workshop emphasized the recent solutions of two longstanding problems: the Kakeya phenomenon in dimension three, and almost everywhere convergence of long time averages associated to multiple commuting measure-preserving transformations. The methods presented during the workshop have yielded applications in ergodic theory, number theory, and computer science.

Mathematics Subject Classification (2020): 42-XX.

Introduction by the Organizers

The workshop focused on contemporary developments in real and harmonic analysis, with emphasis on recent solutions of two longstanding problems. The first of these was the Kakeya phenomenon in dimension three, while the second concerned almost everywhere convergence of long time averages associated to multiple commuting measure-preserving transformations of arbitrary probability spaces. Other topics included local smoothing inequalities, maximal functions, inequalities for solutions of divergence form elliptic partial differential equations, analysis on the Hamming cube, weighted norm inequalities, lattice point counting and exponential sums, and relations between curvature, concentration, and inequalities for various types of Fourier integrals.

Format. Sixteen participants presented 50 minute lectures on new developments in real and harmonic analysis and applications. Three themes were investigated in greater depth; each of these was the subject of a morning session comprised of paired talks by two researchers. In addition, in celebration of the recent resolution of the Kakeya problem in dimension 3, the workshop gathered for "Kakeya night," a multi-hour discussion session for digesting the proof methods presented earlier in a pair of lectures by H. Wang and J. Zahl.

Kakeya problem. The Kakeya conjecture in $\mathbb{R}^d$ asserts, *grosso modo*, that any subset of $\mathbb{R}^d$ containing unit line segments oriented in all directions must have dimension d. Examples of Lebesgue measure null sets with this property were constructed over a century ago. In the early 1970s, this was shown by Fefferman to have negative implications for the Lebesgue norm convergence of Fourier series in dimensions greater than 1. For dimension two, this circle of questions was resolved in strong formulations in the mid seventies by Carleson-Sjölin, Cordoba, and Fefferman. Corresponding results for higher dimensions have been regarded as a holy grail of Fourier analysis during the intervening half-century. Partial results have been obtained, and a wide array of ideas and techniques have been contributed, by numerous investigators in celebrated works.

In early 2025, H. Wang and J. Zahl posted a solution for the three-dimensional case, building on their own earlier work on the so-called "sticky" case. This timely MFO workshop became an ideal opportunity for the research community to assimilate and explore their ideas. With this end in mind, a midweek morning session was devoted to coordinated formal presentations by Wang and Zahl.

An informal discussion, geared towards all workshop participants and several hours in duration, was held that same evening in the lecture hall. It was structured around a blackboard recapitulation of central elements of the proof led by Guth, with frequent comments, questions, interruptions and pauses for further discussion. The authors were active participants, as was N. Katz, who had contributed some key ideas in earlier work.

Among points clarified in this discussion were the role of the class of convex sets in the proof's induction on scales scheme, obstacles to a solution in higher dimensions, and the relationship of the new work to earlier work on the sticky case; the new work relies on the earlier strong result for the sticky case, rather than being a more general superseding analysis.

Almost everywhere convergence of averages associated to commuting measure-preserving transformations. Another morning session was devoted to the solution by Kosz, Mirek, Peluse, and Wright of a 29-year-old conjecture of Bergelson concerned with almost everywhere convergence of averages in an ergodic-theoretic context. M. Mirek and J. Wright gave coordinated presentations.

This work brings together a wide variety of ideas and techniques from Fourier analysis, additive combinatorics, maximal functions, variational norms, exponential sums, and the Hardy-Littlewood circle method. A type of inverse theorem established by Peluse and Prendeville in connection with a Szemerédi-type problem is one key ingredient. Another notable technical element is a Fourier multiplier inequality, for $\mathbb{Z}^d$, which better facilitates localization to unions of major arcs than previously available tools.

Oscillatory integrals and Fourier restriction. Interest in the Kakeya problem stems in part from its connection with Fourier restriction and Bochner-Riesz multiplier operators. Both of the latter two problems can be regarded as instances of the far more general class known as Hörmander-type oscillatory integral operators. While it can be useful to study this more general construction, it is also true that many Hörmander-type operators behave very differently from Fourier restriction and Bochner-Riesz multiplier operators. In a dedicated morning session, S. Guo and R. Zhang reported on a new way to describe a sub-class of Hörmander-type operators that includes both Fourier restriction and Bochner-Riesz multiplier operators, but may rule out operators with more problematic behavior. The new class is defined by a property Guo and Zhang call "Bourgain's condition", and they described that methods that have worked in the settings of Fourier restriction and Bochner-Riesz multiplier operators also succeed for operators satisfying Bourgain's condition.

The polynomial partitioning method has been one useful tool in this context. In order to treat Hörmander-type operators that satisfy Bourgain's condition, one is led to consider curved variants of the Kakeya problem, in which line segments are replaced by appropriate families of curves. One of the issues discussed in the Kakeya context at this workshop was the question of which properties of families of line segments and of convex sets are essential. The negative results of Bourgain (which inspired the subclass of Hörmander-type operators defined by "Bourgain's condition") were based on a focusing phenomenon. What other obstructions arise for plausible inequalities remains an intriguing question.

Additional topics. Additional themes explored in the workshop included: operators with maximal-type behavior, both in the context of convex set testing conditions for positive operators (P. Gressman), and maximally-modulated singular Radon-type operators (D. Beltran); exponential sums as a tool for detecting equidistribution in arithmetic settings, both in the context of counting rational points (of bounded denominator) close to submanifolds (R. Srivastava) and using the new notion of L^p-set to characterize equidistribution of polynomial values associated to the set (T. Wooley); weighted norm inequalities, both in the context of a counterexample to the "A_2 conjecture" for matrix weights (A. Volberg) and the utility of sparse bounds for divergence form elliptic equations (O. Saari); local smoothing estimates, both in the context of lossless Strichartz and spectral projection estimates on convex cocompact hyperbolic surfaces (C. Sogge) and also seeking appropriate measures to study fractal versions of local smoothing (J. Roos); the analysis of Boolean functions, such as polynomial approximation problems on the n-dimensional hypercube as n goes to infinity (P. Ivanisvili); and finally square function estimates and Fourier integral operators defined by highly oscillatory Fourier multipliers concentrated near the truncated light cone in $\mathbb{R}^3$ (D. Müller).

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Workshop: Real Analysis, Harmonic Analysis, and Applications

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Abstracts

Hörmander-type oscillatory integral operators and Bourgain's condition

Ruixiang Zhang

For $a \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1})$, real $\phi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1})$ smooth in a neighborhood of supp$a$ and $\lambda > 1$, Hörmander considered the operator

$$T^\lambda f(x) = \int_{\mathbb{R}^{n-1}} e^{2\pi i \phi^\lambda(x;\xi)} a^\lambda(x;\xi) f(\xi) d\xi$$

where $\phi^\lambda(x;\xi) = \lambda\phi(\frac{x}{\lambda};\xi)$ and $a^\lambda(x;\xi) = a(\frac{x}{\lambda};\xi)$ and ϕ satisfies the following non-degeneracy condition:

- (H1) The rank of $\nabla_x \nabla_\xi \phi$ is $n-1$ throughout supp$a$.
- (H2) For the *Gauss map* $G(x;\xi)$ with $G = \frac{G_0(x;\xi)}{|G_0(x;\xi)|}$ and

  $$G_0(x;\xi) = \wedge_{j=1}^{n-1} \partial_{\xi_j} \nabla_x \phi(x;\xi),$$

  we have

  $$\det(\nabla_\xi)^2 \langle \nabla_x \phi(x;\xi), G(x;\xi_0)\rangle|_{\xi=\xi_0} \neq 0.$$

These are known as Hörmander type operators. An important such operator is the *Fourier restriction* operator with $\phi = \phi_{restr}(x;\xi) = x' \cdot \xi + x_n|\xi|^2$. Another important Hörmander type operator is the reduced Carleson-Sjölin operator that is important in the Bochner-Riesz conjecture.

Hörmander [8] asked whether all T^λ satisfy a similar L^p boundedness to that for the phase function ϕ_{restr}. This would unify the Fourier restriction Conjecture and the Bochner-Riesz Conjecture and is true in dimension 2 ([8], see also [3]), but unfortunately is known to be false in all dimensions $n > 2$, even with an additional positive definite condition for the phase [1, 2, 9, 7].

Bourgain proved more in [1]. To explain his result, note that diffeomorphisms in x and in ξ (separately) preserve the L^p-mapping properties of T^λ. Through these diffeomorphisms, it is elementary that one can change ϕ to a *normal form* around any point (taken to 0) in supp$a$:

$$\phi(x;\xi) = x_1\xi_1 + \cdots + x_{n-1}\xi_{n-1} + x_n\langle A\xi, \xi\rangle + O(|x_n||\xi|^3 + |x|^2|\xi|^2).$$

Bourgain [1] proved in dimension $n = 3$, T^λ fails to behave like the one with $\phi = \phi_{restr}$ unless around every point (in supp$a$), when the phase is expressed in the normal form, one has $\partial_{x_3}^2(\nabla_\xi)^2\phi|_{(0;0)}$ equal to a multiple of $\partial_{x_3}(\nabla_\xi)^2\phi|_{(0;0)}$.

In [4], we proved that a naive generalization of the above is true in every dimension: T^λ fails to behave like the one with $\phi = \phi_{restr}$ unless around every point (in supp$a$), when the phase is expressed in the normal form, one has $\partial_{x_n}^2(\nabla_\xi)^2\phi|_{(0;0)}$ equal to a multiple of $\partial_{x_n}(\nabla_\xi)^2\phi|_{(0;0)}$. (This is called *Bourgain's condition*)

Moreover, in [4] it was proved that the polynomial method (introduced to handle the T^λ with $\phi = \phi_{restr}$ by Guth [5, 6], and extended by e.g. [7]) works equally well to prove L^p mapping properties for a general T^λ with ϕ satisfying Bourgain's condition. It was conjectured in [4] that Bourgain's condition is also sufficient for a general ϕ to behave like the phase ϕ_{restr} in L^p mapping properties.

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Hörmander-type oscillatory integral operators and applications

Shaoming Guo

Consider Hörmander-type oscillatory integrals

$$T_N^{(\phi)} f(x,t) := \int_{\mathbb{R}^{n-1}} e^{iN\phi(x,t;y)} a(x,t;y) f(y) \mathrm{d}y. \tag{1}$$

Here $x \in \mathbb{R}^{n-1}, t \in \mathbb{R}, y \in \mathbb{R}^{n-1}$, and N is a large real number. Moreover, a is a smooth function supported in a small neighborhood of the origin. One example for the phase function $\phi(x,t;y)$ is given by

$$\phi(x,t;y) = x \cdot y + t|y|^2. \tag{2}$$

Hörmander [4] considered phase functions that are small perturbations of (2). More precisely, he considered phase functions of the form

$$\phi(x,t;y) = x \cdot y + t|y|^2 + O(|t||y|^3 + |(x,t)|^2|y|^2). \tag{3}$$

By applying elementary changes of coordinates, all non-degenerate phase functions can be written in the form (3). Hörmander [4] asked whether $T_N^{(\phi)}$ for a general ϕ in (3) satisfies a similar L^p bound to that for the phase function (2).

Bourgain [2] gave a surprising negative answer to Hörmander's question by showing that the phase function

$$\phi(x,t;\xi) = x \cdot y + ty_1y_2 + \frac{1}{2}t^2y_1^2 \tag{4}$$

satisfies a less favorable bound. To see why this is a "bad" phase function, we consider its associated Kakeya problem. The characteristic curves are given by

$$\nabla_\xi \phi(x,t;\xi) = w. \tag{5}$$

We solve (5) directly, and we can write a characteristic curve as

$$\big(w_1 - ty_2 - t^2 y_1, w_2 - ty_1, t\big). \tag{6}$$

For each (y_1, y_2), we pick the initial location $(w_1, w_2) = (0, -y_2)$, and we obtain a curved Kakeya set

$$\bigcup_{y_1,y_2} \{(-ty_2 - t^2 y_1, -y_2 - ty_1, t) : |t| \le 1\}. \tag{7}$$

It is elementary to see that the above set is contained in the hypersurface

$$\{(X,Y,Z) : X = YZ\}. \tag{8}$$

Therefore, this gives a two-dimensional Kakeya set instead of a three-dimensional one.

Bourgain [2] and Chen et al. [3] came up with a sufficient condition that eliminates the bad examples of the form (4). As an application, Beltran et al. [1] proved that certain Pierce-Yung operator is bounded.

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Lossless Strichartz and spectral projection estimates on convex cocompact hyperbolic surfaces

Christopher D Sogge

(joint work with Xiaoqi Huang, Zhongkai Tao and Zhexing Zhang)

In joint work with X. Huang, Z. Tao and Z. Zhang, we prove optimal lossless spectral projection and Strichartz estimates on convex cocompact hyperbolic surfaces which are analogs of classical Fourier-extension results in $\mathbb{R}^n$. We use ideas going back to Burq, Guillarmou and Hassell [5] who were able to prove Strichartz estimates for a more restrictive class of hyperbolic surfaces satisfying a pressure condition. Our results also strengthen recent spectral projection estimates of Anker, Germain and Léger [1], which involved λ^ε-losses.

Our of main results is the following

Theorem 1. *Let (M,g) be a convex cocompact hyperbolic surface and $N_0 \in \mathbb{N}$. Then for $\lambda \geq 1$ and every $p \in (2,\infty]$ we have*

$$\|\mathbf{1}_{[\lambda,\lambda+\delta]}(\sqrt{-\Delta_g})f\|_{L^p(M)} \leq C_p\,\delta^{1/2}\lambda^{\mu(p)}\|f\|_{L^2(M)},\ \delta \in [\lambda^{-N_0},1],$$

if $\mu(p) = \max\big(2(1/2-1/p)-1/2, 1/4-1/2p\big)$. *Also,*

$$\|e^{-it\Delta_g}f\|_{L^p_tL^q_x([0,1]\times M)} \leq C_p\,\|f\|_{L^2(M)},$$

if $1/2-1/q = 1/p$.

These results are optimal in terms of the exponents. Also, the bounds for the uniform spectral projection estimates need not hold for $\delta \in (0,1]$ and the Strichartz estimates for the Schrödinger propagator need not hold if $[0,1]\times M$ is replaced by $\mathbb{R}\times M$. They are the first sharp estimates which do not involve a pressure condition.

In order to prove these estimates we make use of known optimal L^2 local smoothing estimates for the Schrödinger propagator, which are due to Bourgain and Dyatlov [4] and others, as well as known estimates in the funnels of the convex cocompact hyperbolic surfaces from [5]. We can use an argument from Burq, Guillarmou and Hassell [5] along with our new log-scale estimates for the compact core.

The latter are a consequence of the following more general log-scale estimates for general manifolds of all dimensions which are of bounded geometry and have nonpositive sectional curvatures. It generalizes recent results for compact manifolds in joint work with Blair and Huang [2], [3], [6] and [7]. This is one of our main estimates and is the missing ingredient to allow us to prove Theorem 1 using the argument in [5].

Theorem 2. *Let (M,g) be an n-dimensional, $n \geq 2$, Riemannian manifold of uniformly bounded geometry all of whose sectional curvatures are non-positive. Then for $\lambda \ggg 1$ we have*

$$\|\mathbf{1}_{[\lambda,\lambda+\delta]}(\sqrt{-\Delta_g})f\|_{L^p(M)} \leq C_p\delta^{1/2}\lambda^{\mu(p)}\|f\|_{L^2(M)},\ \delta \in [(\log\lambda)^{-1},1],\ p \in (2,\infty],$$

if $\mu(p) = \max\big(n(\frac{1}{2}-\frac{1}{2})-\frac{1}{2}, \frac{n-1}{2}(\frac{1}{2}-\frac{1}{p})\big)$. *Also,*

$$\|e^{-it\Delta_g}f\|_{L^p_tL^q_x([0,1]\times M)} \leq C_p\,\|f\|_{L^2(M)},$$

if $n(1/2-1/q) = 1/p$ *and* $(p,q) \neq (2,\infty)$.

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Counting, Curvature and Convex Duality

Rajula Srivastava

Let $\mathcal{M}$ be a bounded immersed submanifold of $\mathbb{R}^M$ with boundary, of dimension n and codimension R. How many rational points with denominator of bounded size (height) are contained in a "small neighborhood" of $\mathcal{M}$? More precisely, for an integer $Q \geq 1$ and $\delta \in (0, 1/2)$, we define the counting function

$$N_{\mathcal{M}}(Q, \delta) := \#\{(\boldsymbol{p}, q) \in \mathbb{Z}^{M+1} : 1 \leq q \leq Q,\ \operatorname{dist}(\mathcal{M}, \boldsymbol{p}/q) \leq \delta/q\}.$$

Here dist denotes the distance with respect to the L^∞ norm on $\mathbb{R}^M$.

The study of rational points near manifolds has seen rapid development in the recent years. While the problem of obtaining precise asymptotics and upper bounds for $N_{\mathcal{M}}(Q, \delta)$ is interesting in its own right, it is also closely related to questions in Diophantine approximation and the dimension growth problem for submanifolds of $\mathbb{R}^M$.

The upper bound $N_{\mathcal{M}}(Q, \delta) \leq c_{\mathcal{M}} Q^{n+1}$ is trivial. Indeed, if $\mathcal{M}$ is a (compact piece) of a rational hyperplane in $\mathbb{R}^M$, then the above estimate is the best we can hope for. However, if $\mathcal{M}$ is curved in some sense, a probabilisitic heuristic suggests that

$$c'_{\mathcal{M}} \delta^R Q^{n+1} \leq N_{\mathcal{M}}(Q, \delta) \leq c_{\mathcal{M}} \delta^R Q^{n+1} \tag{1}$$

for δ above a critical threshold in terms of Q. A folklore conjecture, made precise by J.J. Huang in [5], asserts that the above estimates should be true for any smooth submanifold $\mathcal{M}$ under a "mild" nondegeneracy condition, in the range

$$\delta \geq Q^{-\frac{1}{R}+\epsilon} \tag{2}$$

for some $\epsilon > 0$ and $Q \to \infty$.

In [3], Beresnevich established the lower bound in (1) for *analytic*, manifolds satisfying this nondegeneracy condition, in the range $\delta > Q^{-\frac{1}{R}}$. In the recent work [2], Schindler, Technau and the author proved the indicated lower bound for *smooth* manifolds satisfying the same nondegeneracy condition in the range $\delta > Q^{-\frac{3}{2M-1}}$.

However, for upper bounds, very little is known, and only under very strong geometric conditions on the manifold $\mathcal{M}$. A breakthrough came in [4], where J.J. Huang proved an asymptotic for $N_{\mathcal{M}}(Q, \delta)$ when $\mathcal{M}$ is a sufficiently smooth hypersurface with *non-vanishing Gaussian curvature*, in the optimal range $\delta >$

$Q^{-1+\epsilon}$. Before this, such an asymptotic was only available in the case of planar curves with non-vanishing curvature, due to the infuential works of Huxley [8] and of Vaughan–Velani [7]. Moreover, in [6], Technau and the author showed that the conjecture is also true for certain hypersurfaces with Gaussian curvature vanishing at a single point, provided the "degree of flatness" is below a critical value depending only on the dimension of the hypersurface. Further, when the degree of flatness is large, [6] establishes a new asymptotic for $N_{\mathcal{M}}(Q,\delta)$ incorporating the contribution due to the "local flatness".

However, all of these results on upper bounds and asymptotics remain valid only within the conjectured range (2). It therefore came as a surprise when in [1], Schindler and Yamagishi established an asymptotic for $N_{\mathcal{M}}(Q,\delta)$ for manifolds $\mathcal{M}$ satisfying a *strong curvature condition*, in a range of δ which goes beyond (2) when the codimension is bigger than one. To describe this curvature condition, using the implicit function theorem, we may assume without loss of generality that $\mathcal{M}$ has the parametrization

$$\mathcal{M} := \{(\boldsymbol{x}, f_1(\boldsymbol{x}), \ldots, f_R(\boldsymbol{x})) \in \mathbb{R}^{n+R} : \boldsymbol{x} \in \overline{B_{\varepsilon_0}(\boldsymbol{x}_0)}\}.$$

Here $\boldsymbol{x}_0 \in \mathbb{R}^n$, $f_r : \mathbb{R}^n \to \mathbb{R}$ are C^∞ functions for $1 \le r \le R$ and $\overline{B_{\varepsilon_0}(\boldsymbol{x}_0)}$ denotes the closed ball in $\mathbb{R}^n$ centered at $\boldsymbol{x}_0$ and of small enough radius ε_0. Schindler-Yamagishi considered manifolds of the above form satisfying the following.

Curvature Condition: Given any $\boldsymbol{t} = (t_1, \ldots, t_R) \in \mathbb{R}^R \setminus \{\boldsymbol{0}\}$, there exists a constant $C_{\boldsymbol{t}} > 0$ such that

$$\text{(CC)} \qquad \min_{\boldsymbol{x} \in B_{2\varepsilon_0}(\boldsymbol{x}_0)} \left| \det H_{\sum_{i=1}^R t_i f_i}(\boldsymbol{x}) \right| > C_{\boldsymbol{t}}.$$

For manifolds satisfying the condition (CC), Schindler-Yamagishi established in [1] that (1) is true in the range

$$\text{(3)} \qquad \delta \ge Q^{-\frac{n}{n+2(R-1)}+\epsilon}.$$

Note that $Q^{-\frac{n}{n+2(R-1)}} < Q^{-\frac{1}{R}}$, whenever $R > 1$.

Our first result estabishes that for smooth manifolds satisfying condition (CC), the bounds in (1) hold true in an even bigger range of δ.

Theorem 1 ([9], Corollary 1.5)**.** *Suppose $\mathcal{M}$ satisfies condition* (CC)*. Then $N_{\mathcal{M}}(Q,\delta)$ satisfies* (1) *whenever*

$$\delta \ge \max\left(Q^{-\frac{n+2}{n+2R}+\epsilon}, Q^{-\frac{n}{n+2(R-1)-\frac{4}{n}}+\epsilon} \right)$$

for any sufficiently small $\epsilon > 0$ and $Q \to \infty$.

Our main estimate is an upper bound for the number of rational points with bounded denominators contained in a *non-isotropic* neighborhood of a smooth manifold $\mathcal{M}$ satisfying (CC). It specializes to the upper bound in Theorem 1 for isotropic neighborhoods of $\mathcal{M}$. For $Q \in \mathbb{Z}_{\ge 1}$ and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_R) \in (0, 1/2)^R$, we define the counting function

$$N_{\mathcal{M}}(Q, \boldsymbol{\delta}) := \#\left\{(\boldsymbol{a}, q) \in \mathbb{Z}^{n+1} : 1 \le q \le Q,\ \|q f_r(\boldsymbol{a}/q)\| \le \delta_r/q \text{ for } 1 \le r \le R\right\}.$$

Theorem 2 ([9], Corollary 1.10)**.** *Let* $\boldsymbol{\delta}^\times = \prod_{r=1}^R \delta_r$*. Suppose* $\mathcal{M}$ *satisfies condition* (CC)*. Then*

$$N_{\mathcal{M}}(Q, \boldsymbol{\delta}) \leq c_{\mathcal{M}} \boldsymbol{\delta}^\times Q^{n+1}$$

whenever

$$\min_{1 \leq r \leq R} \delta_r \geq \max\left(Q^{-\frac{n+2}{n+2R}+\epsilon}, Q^{-\frac{n}{n+2(R-1)-\frac{4}{n}}+\epsilon} \right),$$

for any sufficiently small $\epsilon > 0$ *and* $Q \to \infty$*.*

In [4], Huang used a novel combination of projective duality, stationary phase and induction on scales to develop a bootstrapping argument. The foundation of this argument was a self-improving estimate relying on the fact that the Legendre dual of a hypersurface with non-vanishing Gaussian curvature is also a hypersurface with the same property. Furthermore, the Legendre transform is an involution. Thus after every two steps of this iteration, one returns to the original counting problem.

In [1], a deep insight of Schindler-Yamagishi was generalizing the notion of Legendre duality to manifolds of arbitrary dimension n and codimension R, but satisfying the geometric condition (CC). However, after two steps of induction, their argument proceeds by using exactly one of the codimensions to project to a lower dimensional counting problem associated to a hypersurface in $\mathbb{R}^{n+1}$ and summing trivially in the remaining in $R-1$ codimension variables. This allows for the use of the sharp estimate for the rational point count close to hypersurfaces from [4] as a blackbox to deduce estimates for the rational point counting function associated to this family of projected hypersurfaces.

In our proof, instead of using the estimate for hypersurfaces as a blackbox, we use the involutive nature of the Legendre transform to return to the original counting problem associated to $\mathcal{M}$ after every two steps. Further, in a major departure from [4], our inductive argument develops a connection between counting functions associated with two entirely different geometric objects: the manifold $\mathcal{M}$ of codimension R on one hand, and a dual family of hypersurfaces in R^M on the other. In [1], this connection was only utilized in one direction. Finally, to exploit the information from all codimensions independently, our argument *necessarily requires* estimates for a non-isotropic counting function.

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Discrete approximation theory

Paata Ivanisvili
(joint work with Roman Vershynin, Xinyuan Xie)

Extended abstract

Problem. Let $f : \{0,1\}^n \to \mathbb{R}$ be an arbitrary function on the Hamming hypercube and let

$$E_d^n(f) \;:=\; \inf_{\deg(g)\le d} \|f-g\|_\infty$$

denote the best uniform approximation error of f by real polynomials of total degree at most d. Write

$$s(f) \;:=\; \max_{x\in\{0,1\}^n} \sum_{j=1}^{n} |f(x) - f(x^{(j)})|$$

for the *sensitivity* of f, where $x^{(j)}$ is obtained from x by flipping the j-th bit. Motivated by Huang's resolution of the sensitivity conjecture (which can be stated in a Jackson-type form for Boolean functions), we study the best constant

$$J(n,d) \;:=\; \inf\Big\{ J>0 :\; E_d^n(f) \;\le\; J\, s(f) \quad \text{for all } f:\{0,1\}^n\to\mathbb{R}\Big\}.$$

Our goal is to obtain sharp upper and lower bounds on $J(n,d)$ in several regimes and structural classes, as well as to understand the effect of dimensionality constraints on approximation by linear subspaces (Kolmogorov widths).

Background. A classical inequality of Pisier (in a form due to Wagner) gives the baseline

$$E_0^n(f) \;=\; \inf_{c\in\mathbb{R}} \|f-c\|_\infty \;\le\; s(f) \qquad \text{for all } f,$$

so $J(n,0) \le 1$ uniformly in n. For Boolean f one can combine Huang's theorem with standard reductions to obtain a *Jackson-type* bound

$$E_d^n(f) \;\le\; \frac{C}{d^c}\, s(f) \qquad (f:\{0,1\}^n \to \{0,1\})$$

for universal constants $C, c > 0$. Our results show that the situation for general real-valued f is subtler: in particular, such bounds cannot hold below a natural threshold degree, while strong positive results are available for high degrees and for symmetric functions.

Main results.

Proposition 1 (Symmetric functions: optimal upper and lower bounds)**.** *There exists a universal constant $C > 0$ such that for any symmetric $f : \{0,1\}^n \to \mathbb{R}$ and any $d \in (0, n]$,*

$$E_d^n(f) \;\leq\; \frac{C}{d}\, s(f).$$

Moreover, this rate is optimal: for every $n \in \mathbb{N}$ and $d \in (0, n]$ there exists a symmetric f with $s(f) > 0$ such that

$$E_d^n(f) \;\geq\; \frac{1}{8d}\, s(f).$$

Theorem 2 (Inapproximability below the half-degree threshold)**.** *For every $c_1 \in (0, 1/2)$ there exist $c_2 > 0$ and, for all sufficiently large n, a function $f : \{0,1\}^n \to \mathbb{R}$ with $s(f) > 0$ such that*

$$E_{c_1 n}^n(f) \;\geq\; c_2\, s(f).$$

In particular, $J(n, c_1 n)$ is bounded below by a positive constant independent of n.

As immediate consequences:

Corollary 3 (Approximate degree does not control sensitivity for real-valued functions)**.** *Let $h : \mathbb{R} \to \mathbb{R}$ be any function with $h(t) \to \infty$ as $t \to \infty$. There is no inequality of the form*

$$s(f) \;\geq\; h\big(\widetilde{\deg}(f)\big)$$

that holds for all $f : \{0,1\}^n \to [-1, 1]$ and all $n \geq 1$.

Corollary 4 (Failure of reverse Bernstein in the tail space)**.** *For any $c_1 \in (0, 1/2)$ there exists $c_2 > 0$ such that, for all sufficiently large n, there is a nonzero function f supported on Fourier levels $\geq c_1 n$ (i.e., $f \in L^1_{\geq c_1 n}$) with*

$$\|f\|_1 \;\geq\; c_2\, \|\Delta f\|_1,$$

where Δ is the (unnormalized) graph Laplacian on the cube.

Theorem 5 (Kolmogorov widths: negative result for low-dimensional models)**.** *Fix $c_1 \in (0, 1/2)$. There exist $c_2 > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and any linear subspace $E \subset \mathbb{R}^{\{0,1\}^n}$ with*

$$\dim E \;\leq\; \binom{n}{\leq c_1 n},$$

there is $f : \{0,1\}^n \to \mathbb{R}$ with $s(f) > 0$ for which

$$\inf_{g \in E} \|f - g\|_\infty \;\geq\; c_2\, s(f).$$

Theorem 6 (Kolmogorov widths: a universal half-dimension model)**.** *There exists a subspace $E \subset \mathbb{R}^{\{0,1\}^n}$ of dimension 2^{n-1} such that, for all $f : \{0,1\}^n \to \mathbb{R}$,*

$$\inf_{g \in E} \|f - g\|_\infty \;<\; \frac{s(f)}{n}.$$

High-degree approximation: a Jackson kernel on the cube. A key step is a "Jackson kernel" type estimate obtained by averaging f against a univariate polynomial in the Hamming weight. Write $x \oplus y$ for the coordinatewise XOR, let $X \sim \mathrm{Bin}(n, 1/2)$, and let $h : \mathbb{R} \to \mathbb{R}$ be any polynomial of degree $\leq d$; define $H(y) = h(y_1 + \cdots + y_n)$. Then:

Theorem 7 (Kernel upper bound)**.** *For all $f : \{0,1\}^n \to \mathbb{R}$ and all $d \in [0, n]$,*

$$E_d^n(f) \;\leq\; \max_{x \in \{0,1\}^n} \Big| f(x) - \mathbb{E}_y\big[f(y)\, H(y \oplus x)\big] \Big| \;\leq\; 3\, \frac{s(f)}{n}\, \mathbb{E}[\, X\, |h(X)|\,],$$

where the expectation is over $y \sim \mathrm{Unif}(\{0,1\}^n)$ and $X \sim \mathrm{Bin}(n, 1/2)$.

This estimate leads to several quantitative corollaries.

Corollary 8 (Krawtchouk-based bound)**.** *Let $k_{n,\ell}$ denote the smallest positive root of the Krawtchouk polynomial of degree ℓ (orthogonal with respect to $\mathrm{Bin}(n, 1/2)$). Then for all $d \in [0, n]$,*

$$E_d^n(f) \;\leq\; 3\, \frac{k_{n,\lfloor d/2 \rfloor + 1}}{n}\, s(f).$$

Corollary 9 (A simple linear bound)**.** *For all $d \in [0, n]$,*

$$E_d^n(f) \;\leq\; 3\Big(1 - \frac{d}{n}\Big)\, s(f).$$

Corollary 10 (High-degree regime: near-threshold behavior)**.** *Let $\delta = 1 - \frac{d}{n}$. There exists a universal constant $C > 0$ such that, for all $d \in [0, n]$,*

$$E_d^n(f) \;\leq\; C\, \min\Big\{ \delta,\; \max\{\delta^2,\; n^{-2/3}\} \Big\}\, s(f).$$

The last bound exhibits two natural scales as d approaches n: a quadratic decay $O(\delta^2)$ until the universal cutoff $n^{-2/3}$ arising from the location of the smallest positive Krawtchouk root, and a linear regime $O(\delta)$ for even higher degrees.

Proof ideas in brief.

- *Symmetric case.* When f depends only on the Hamming weight, the problem reduces to univariate polynomial approximation on $\{0, 1, \ldots, n\}$ with respect to $\mathrm{Bin}(n, 1/2)$. Classical extremal polynomials yield the $O(1/d)$ upper bound, and explicit functions show matching $\Omega(1/d)$ lower bounds.
- *Lower bounds / inapproximability.* We deploy a *sign-pattern/packing* argument on the cube. A quantitative version of a theorem of Lorenz (metric entropy versus Lipschitz approximation by finite-dimensional spaces) is combined with a construction of many well-separated functions of small sensitivity to force large uniform error for any degree $< (\frac{1}{2} - \varepsilon)n$.
- *Kolmogorov widths.* The negative subspace result follows by transferring the sign-pattern argument to arbitrary subspaces of dimension at most $\binom{n}{\leq c_1 n}$. The positive result constructs an explicit half-dimensional subspace that captures the "low-frequency" content of every f well enough to guarantee $\|f - g\|_\infty \lesssim s(f)/n$.

- *Kernel method.* The averaging operator $f \mapsto \mathbb{E}_y[f(y)H(y \oplus x)]$ is a polynomial of degree $\leq d$ in x whenever h has degree $\leq d$. Carefully choosing h (via positivity and normalization, or via quadrature/Krawtchouk theory) and bounding the sensitivity of the averaging map yields the kernel upper bound. Optimizing $\mathbb{E}[X|h(X)|]$ produces the bounds above.

Open directions. Our results pinpoint a qualitative threshold at degree $\frac{1}{2}n$: below it, no uniform Jackson-type inequality of the form $E_d^n(f) \leq o(1) \cdot s(f)$ can hold for all real-valued f; above it, one has nontrivial decay in $J(n,d)$, with precise rates near $d \approx n$. It remains of interest to determine whether $J(n, \alpha n) \to 0$ as $n \to \infty$ for fixed $\alpha > 1/2$, and to clarify the optimal transition profile between the linear, quadratic, and $n^{-2/3}$ regimes in the high-degree limit.

Notation. We write $\binom{n}{\leq m} = \sum_{k=0}^{m} \binom{n}{k}$ and use the standard discrete Laplacian $\Delta f(x) = \sum_{j=1}^{n} \big(f(x) - f(x^{(j)})\big)$. All implicit constants are universal unless explicitly stated.

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A counterexample to matrix weight A_2 conjecture

Alexander Volberg

(joint work with Komla Domelevo, Stefanie Petermichl, Sergei Treil)

We (Komla Domelevo, Stefanie Petermichl, Sergei Treil, Alexander Volberg) show that the famous matrix $\mathbf{A}_2$ conjecture is false: the norm of the Hilbert Transform in the space $L^2(W)$ with matrix weight W is estimated below by $C[W]_{\mathbf{A_2}}^{3/2}$.

Recall that a (d-dimensional) matrix weight on $\mathbb{R}$ is a locally integrable function on $\mathbb{R}$ with values in the set of positive-semidefinite $d \times d$ matrices[1]. The weighted space $L^2(W)$ is defined as the space of all measurable functions $f : \mathbb{R} \to \mathbb{F}^d$, (here $\mathbb{F} = \mathbb{R}$, or $\mathbb{F} = \mathbb{C}$) for which

$$\|f\|_{L^2(W)}^2 := \int (W(x)f(x), f(x))_{\mathbb{F}^d} \mathrm{d}x < \infty\,;$$

here $(\cdot,\cdot)_{\mathbb{F}^d}$ means the standard inner product in $\mathbb{F}^d$.

A matrix weight W is said to satisfy the matrix $\mathbf{A}_2$ condition (write $W \in \mathbf{A}_2$) if

$$[W]_{\mathbf{A_2}} := \sup_I \left\| \langle W \rangle_I^{1/2} \langle W^{-1} \rangle_I^{1/2} \right\|^2 < \infty\,, \tag{1}$$

where I runs over all intervals. The quantity $[W]_{\mathbf{A_2}}$ is called the $\mathbf{A}_2$ *characteristic of the weight* W. In the scalar case, when W is a scalar weight w, this coincides with the classical A_2 characteristic $[w]_{A_2}$.

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[1]There are of course similar definitions on the unit circle $\mathbb{T}$ or $\mathbb{R}^N$

Let $\mathcal{H}$ denote the Hilbert transform,

$$\mathcal{H}f(s) = \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{f(t)}{s-t}\mathrm{d}t.$$

In this paper, we disprove the famous *matrix* $\mathbf{A}_2$ *conjecture*, which stated that for any $\mathbf{A}_2$ weight W, $[W]_{\mathbf{A}_2} \le \boldsymbol{\mathcal{Q}}$

$$\|\mathcal{H}g\|_{L^2(W)} \le C\boldsymbol{\mathcal{Q}}\|g\|_{L^2(W)} \qquad \forall g \in L^2(W).$$

More precisely, our main result is:

Theorem 1. *There exists a constant $c > 0$ such that for all sufficiently large $\boldsymbol{\mathcal{Q}}$ there exist a 2×2 matrix weight $W = W_{\boldsymbol{\mathcal{Q}}}$ (with real entries), $[W]_{\mathbf{A}_2} \le \boldsymbol{\mathcal{Q}}$ and a function $g \in L^2(W)$, $g : \mathbb{R} \to \mathbb{R}^2$, $g \ne 0$ such that*

$$\|\mathcal{H}g\|_{L^2(W)} \ge c\boldsymbol{\mathcal{Q}}^{3/2}\|g\|_{L^2(W)}.$$

In fact, by picking a sufficiently small c we can state it for all $\boldsymbol{\mathcal{Q}} \ge 1$. By a simple reduction, we can state it for all dimensions $d \ge 2$ of matrices.

Theorem 1 also shows that the upper bound

$$\|T\|_{L^2(W)\to L^2(W)} \le C_T\boldsymbol{\mathcal{Q}}^{3/2}$$

obtained in [14] for general Calderón–Zygmund operators is sharp.

In the scalar case, the A_2 conjecture, i.e. the estimate $\|T\|_{L^2(w)\to L^2(w)} \le C_T[w]_{A_2}$ turned out to be true [15], [16], [7], so it is now known as the A_2 Theorem. It was a long standing open problem with a fascinating history that we describe briefly below.

Motivation. An important motivation for sharp estimates of the Hilbert transform in $L^2(W)$ with matrix weight W comes from probability theory, more precisely from the theory of stationary Gaussian processes. For Gaussian processes all information is encoded in the means and correlations, so the study of multivariate stationary Gaussian processes (say with discrete time) is reduced to the study of the subspaces $z^n\mathbb{C}^d$, $n \in \mathbb{Z}$, in the Hilbert space $L^2(\mathbf{W})$, where $\mathbf{W}$ is the spectral measure of the process. For a multivariate process of dimension d the spectral measure is a $d \times d$ matrix-valued measure.

The regularity properties of stationary stochastic processes in terms of their spectral measures $\mathbf{W}$ is a classical area that attracted the attention of many mathematicians, and a huge bibliography can be found in [9], [11], [18]. In the case of scalar processes many different types of regularities were studied and very detailed results were found in many papers, to name just a few [4], [5], [6].

One of the questions about regularity was the question when the angle between past and future of the process is positive, which reduces to the question when the Riesz Projection P_+, or, equivalently the Hilbert Transform $\mathcal{H}$ is bounded in the weighted space $L^2(\mathbf{W})$. This question goes back to N. Wiener and P. Masani, and was the main motivation behind the famous Helson–Szegö Theorem, which solved the problem in the one-dimensional case.

Another motivation comes from the theory of Toeplitz operators (Riemann Hilbert Problem). A Toeplitz operator T_F in the vector-valued Hardy space $H^2(\mathbb{C}^d)$ is defined as

$$T_F f = P_+(Ff), \qquad f \in H^2(\mathbb{C}^d),$$

where P_+ is the Riesz Projection, i.e. the orthogonal projection from L^2 onto the Hardy space H^2. The $d \times d$ matrix-valued function $F \in L^\infty(M_{d\times d})$ is called the symbol of the Toeplitz operator. It is well known that the operator T_F is invertible if and only if it can be factorized as

$$F = G_1^* G_2, \qquad G_{1,2}^{\pm 1} \in H^2(M_{d\times d}) \tag{2}$$

(the functions $G_{1,2}$ and their inverses are in the matrix-valued Hardy class H^2), and the formal inverse

$$f \mapsto G_2^{-1} P_+((G_1^{-1})^* f) \tag{3}$$

of T_F is bounded. It is not hard to see that the operator (3) is bounded if and only if the weighted estimate

$$\|P_+ f\|_{L^2(V)} \le C \|f\|_{L^2(W)} \qquad \forall f \in L^2(W) \tag{4}$$

with weights $W = G_1 G_1^*$, $V = (G_2^{-1})^* G_2^{-1}$ holds, and the norm of the operator is the best constant C in (4). This looks like a two weight inequality, but in reality it reduces to the one weight case. Namely, the invertibility of T_F implies that F is invertible in L^∞, and therefore it is not hard to check that the weights V and W are equivalent, i.e.

$$A^{-1} W \le V \le AW, \qquad \text{for some } A \in (0, \infty);$$

the inequality is understood as a matrix inequality. Therefore at the cost of the constant A, (4) is equivalent to the one weight estimate

$$\|P_+ f\|_{L^2(W)} \le C \|f\|_{L^2(W)} \qquad \forall f \in L^2(W), \tag{5}$$

i.e. to the boundedness of P_+ in the weighted space $L^2(W)$. On the real line the Riesz Projection P_+ and the Hilbert Transform are related as $\mathcal{H} = -iP_+ + i(\mathbf{I} - P_+)$, so P_+ is bounded in $L^2(W)$ if and only if $\mathcal{H}$ is, and the norms are equivalent in the sense of two-sided estimates.

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The Kakeya set conjecture in $\mathbb{R}^3$

Hong Wang, Joshua Zahl

A Besicovitch set is a compact set $K \subset \mathbb{R}^n$ that contains a unit line segment pointing in every direction. Besicovitch[1] constructed examples of such sets in $\mathbb{R}^2$ (and by extension $\mathbb{R}^n$ for $n \geq 2$) that have Lebesgue measure zero. The Kakeya set conjecture asserts that every Besicovitch set in $\mathbb{R}^n$ has Minkowski and Hausdorff dimension n. For $n = 2$ the conjecture was proved by Davies [2] in 1971. In this pair of joint talks, we discuss the proof of the Kakeya set conjecture for $n = 3$; this was proved in the sequence of papers [4, 5, 6], and aspects of the proof were simplified by Guth in [3]; our discussion follows this latter work.

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The multilinear circle method

Mariusz Mirek, James Wright

In a series of two talks, we described a new method in number theory which has implications in ergodic theory, additive combinatorics and harmonic analysis.

Exponential sums are ubiquitous in number theory and the Hardy-Littlewood Circle Method is a robust method to study these sums. Outside of number theory, especially in Ergodic Theory and Harmonic Analysis, exponential sums arise as the Fourier multipliers of translation-invariant operators. When the operators are *linear*, pointwise information about the multiplier/exponential sum gives *explicit* information about the operator via the Fourier transform and Plancherel's theorem. The classical circle method gives us precise pointwise information about exponential sums. However, when the operators are *multilinear*, then the Fourier multiplier/exponential sum is an *implicit* object and a variant of the circle method, the so-called *multilinear circle method*, is needed.

There are a multitude of new ideas, tools and techniques in the multilinear circle method. For example, inverse theorems from additive combinatorics (notoriously difficult to establish) play a central role, the Ionescu-Wainger multiplier theorem and its enhancements are important as well as L^p and Sobolev smoothing estimates from harmonic analysis, structural statements using the Hahn-Banach theorem, and the full strength of the classical circle method all come together in a decisive way. To keep things manageable for this brief note, we will concentrate only on the *minor arc* contribution in the multilinear circle method.

The linear circle method

An example of the *linear circle method* comes from ergodic theory. In the mid to late 1980s, Bourgain [3], [4], [5] established pointwise almost everywhere convergence for the *linear* ergodic averages

$$A_N f(x) \;=\; \frac{1}{N}\sum_{n=1}^{N} f(T^{P(n)}x).$$

Here $T : X \to X$ is a measure-preserving transformation and $P \in \mathbb{Z}[\mathrm{x}]$ is a general polynomial with integer coefficients. This generalises Birkhoff's famous pointwise ergodic theorem [2] when $P(n) = n$. Pointwise almost everywhere convergence lies much deeper than L^2 convegence. In fact L^2 convergence of the above polynomial ergodic averages A_N follows easily from the spectral theorem or the so-called *van der Corput trick*.

The key estimates to establish are $L^p(X)$ bounds for the maximal function $Mf(x) = \sup_N |A_N f(x)|$ and by transference, it suffices to prove $\ell^p(\mathbb{Z})$ bounds for

$$\mathcal{M}f(k) \;=\; \sup_N \Big| \frac{1}{N} \sum_{n=1}^{N} f(k - P(n)) \Big|.$$

The averages $\mathcal{A}_N f(x) = 1/N \sum_{n=1}^N f(k - P(n))$ are translation-invariant with the exponential sum $m_N(\theta) = 1/N \sum_{n=1}^N e^{2\pi i P(n)\theta}$ as the underlying Fourier multiplier.

The maximal function $\mathcal{M}$ is a discrete analogue of a *Singular Radon Transform*, a theory Stein developed with his collaborators since the late 1960s. The discrete maximal function $\mathcal{M}$, its higher dimensional versions and its singular integral cousins form the central objects of an area of harmonic analysis called *Discrete Analogues in Harmonic Analysis*. At the heart of this area is a multiplier theorem of A. Ionescu and S. Wainger [10] which, among other things, efficiently handles approximate operators on the major arcs at *small scales*.

Inspired by the theory of Singular Radon Transforms, Bourgain realised that a Sobolev smoothing estimate for $\mathcal{A}_N$ plays a crucial role in establishing the ℓ^p bounds for $\mathcal{M}$. To set this up, consider the *major arcs*

$$\mathcal{M}_{N,d,\delta^{-C}} \;:=\; \bigcup_{\substack{a/q \in \mathbb{T} \\ q \le \delta^{-C}}} \Big[a/q - \delta^{-C} N^{-d},\ a/q + \delta^{-C} N^{-d} \Big] \tag{1}$$

from the classical circle method. Here $d = \deg(P)$. The important Weyl bound for exponential sums, the key ingredient to handle the minor arc contribution in the circle method, states that for every $C \ge 1$, there is a small $c \in (0,1]$ such that for all $N \ge 1$ and $\delta \in (0,1]$, we have

$$|m_N(\theta)| \;=\; \Big| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i P(n)\theta} \Big| \;\le\; c^{-1} \big[\delta^c \;+\; N^{-c} \big]$$

whenever $\theta \notin \mathcal{M}_{N,d,\delta^{-C}}$.

The smoothing estimate takes the following form. If the Fourier transform of $f \in \ell^2(\mathbb{Z})$ vanishes on the major arcs $\mathcal{M}_{N,d,\delta^{-C}}$, then

$$\|\mathcal{A}_N f\|_{\ell^2(\mathbb{Z})}^2 = \|m_N \widehat{f}\|_{L^2(\mathbb{T})}^2 = \int_{\mathbb{T}} |m_N(\theta)|^2 |\widehat{f}(\theta)|^2 \, d\theta \;\le\; c^{-1} \big[\delta^c + N^{-c} \big] \|f\|_{\ell^2}^2 \tag{2}$$

and so we get a power gain for the averaging operator $\mathcal{A}_N$ when f is Fourier supported outside the major arcs $\mathcal{M}_{N,d,\delta^{-C}}$. Here we see how the Fourier transform and Plancherel's theorem can be used in the *linear* theory. As (2) is equivalent to the Weyl bound for exponential sums, we also call the Sobolev smoothing bound (2), the *linear Weyl estimate*.

By smoothly projecting $f = \Pi_{N,\delta} f + (I - \Pi_{N,\delta}) f$ onto the major and minor arcs, we see that the smoothing estimate (2) (or linear Weyl estimate) immediately takes care of the minor arc contribution $(I - \Pi_{N,\delta})f$, at least in ℓ^2. For ℓ^p, the Ionescu-Wainger multiplier theorem is very useful to move from ℓ^2 to ℓ^p.

It is interesting to note that Bourgain observed in [6] that Sobolev smoothing bounds such as (2) can be used to give quantitative bounds for polynomial patterns in sets of positive density. For the averages $\mathcal{A}_N$ the corresponding progressions are $\{x, x + P(n)\}$, which is related to the polynomial Furstenberg-Sárközy theorem. See [9] for the best quantitative bounds in the original setting of the squares $(P(n) = n^2)$.

The multilinear circle method

In ergodic theory, there is considerable interest in *multilinear* ergodic averages ever since Furstenberg's ergodic theoretic proof [8] of Szemerédi's theorem [13]: there exists arbitrarily long arithmetic progressions in any set of integers of positive density. Combined with his correspondence principle, Furstenberg reduced matters to establishing his multiple recurrence theorem on a general probability space (X, Σ, μ): for every $k \geq 1$ and every $A \in \Sigma$ with $\mu(A) > 0$,

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-kn}A) \ > \ 0.$$

Here $T : X \to X$ is a measure-preserving transformation. This is closely associated to the L^2 convergence of the multilinear ergodic averages

$$A_N(f_1, \ldots, f_k)(x) \ = \ \frac{1}{N}\sum_{n=1}^{N} f_1(T^n x) f_2(T^{2n}x) \cdots f_k(T^{kn}x),$$

a problem we've seen is much easier than the pointwise almost everywhere convergence problem.

For the connection with (and applications to) additive combinatorics, it is important to keep in mind the relationship of L^2 convergence of ergodic (linear or multilinear) averages to the *existence* of integer patterns in sets of positive density. It is a much more recent development (initiated in [11]) that in a similar manner, pointwise almost everywhere convergence is related to *quantitative* bounds for integer patterns in sets of positive density.

In 1996, Bergelson and Liebman [1] extended Furstenberg's multiple recurrence theorem to general polynomial progressions and higher dimensions, proving

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu(A \cap T_1^{-P_1(n)}A \cap T_2^{-P_2(n)}A \cap \cdots \cap T_k^{-P_k(n)}A) \ > \ 0$$

where $\{P_1, \ldots, P_k\} \subset \mathbb{Z}[\mathrm{x}]$ and the measure-preserving trasformations $\{T_1, \ldots, T_k\}$ commute. Consequently, we can conclude the *existence* of arbitrarily long multi-dimensional polynomial progressions in sets of positive density. The corresponding multilinear ergodic averages are

$$A_N(f_1,\dots,f_k)(x) \;=\; \frac{1}{N}\sum_{n=1}^{N} f_1(T_1^{P_1(n)}x)f_2(T_2^{P_2(n)}x)\cdots f_k(T_k^{P_k(n)}x),$$

and L^2 convergence of these averages are now understood due to the work of Walsh [14]. The question of pointwise almost everywhere convergence (and the corresponding quantitative bounds for higher dimensional polynomial progressions) lies much deeper.

In joint work with D. Kosz and S. Peluse, we established the following Sobolev smoothing bound for multilinear averages when the polynomials $\mathcal{P} = \{P_1,\dots,P_k\}$ have distinct degrees. As in the linear case, by transference, the analysis of A_N above are reduced to studying the discrete analogue

$$\mathcal{A}_N(f_1,\dots,f_k)(x) = \frac{1}{N}\sum_{n=1}^{N} f_1(x-P_1(n)e_1)f_2(x-P_2(n)e_2)\cdots f_k(x-P_k(n)e_k)$$

on the integer lattice $\mathbb{Z}^k$. Here $\{e_j\}$ are the usual directional basis vectors on $\mathbb{Z}^k$. We denote by $\mathcal{M}^j_{N,d_j,\delta^{-C}}$ the major arcs $\mathcal{M}_{N,d_j,\delta^{-C}}$ from (1) but only in the jth coordinate. Here $d_j = \deg(P_j)$.

Theorem 1. *Let $1 < p_1,\dots,p_k < \infty$ be exponents such that $\frac{1}{p_1}+\cdots+\frac{1}{p_k} = \frac{1}{p} \le 1$. For all $C_1, C_2 \ge 1$ there exists a small $c \in (0,1)$, possibly depending on $k, \mathcal{P}, p_1,\dots,p_k, p, C_1, C_2$, such that the following holds: for all $N \ge 1$ and $\delta \in (0,1]$, let $f_i \in \ell^{p_i}(\mathbb{Z}^k)$, $1 \le i \le k$, and suppose for some j, the jth Fourier transform of f_j vanishes on the major arcs $\mathcal{M}^j_{N,d_j,\delta^{-C}}$. Then*

$$\|\mathcal{A}_N(f_1,\dots,f_k)\|_{\ell^p(\mathbb{Z}^k)} \le c^{-1}(\delta^c + N^{-c}) \prod_{1\le i\le k} \|f_i\|_{\ell^{p_i}(\mathbb{Z}^k)}. \tag{3}$$

We emphasise that the Fourier transform and Plancherel's theorem are no longer viable tools (even when each $p_i = 2$ for all i) as in the linear case. Even though the multilinear averages $\mathcal{A}_N$ are translation-invariant and has the exponential sum

$$m_N(\vec{\theta}) \;=\; \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i[\theta_1 P_1(n)+\cdots+\theta_k P_k(n)]}$$

as the underlying Fourier multiplier, we can not access this multiplier *explicitly*, using pointwise information of $m_N(\vec{\theta})$ from the classical circle method, to help us establish the smoothing estimate (3). As in the linear theory, we call the bound (3) the *multilinear Weyl estimate.*

Theorem 1 takes care of the minor arc contribution for $\mathcal{A}_N$ and reduces matters to the situation where all the functions f_j are supported on the major arcs $\mathcal{M}^j_{N,d_j,\delta^{-C}}$. The proof of Theorem 1 is very involved; we needed to establish inverse theorems for the averages $\mathcal{A}_N$ (previously unknown) and together with the Hahn Banach theorem, we derive structural information for each of the k adjoints $\mathcal{A}_N^{*,j}$ of $\mathcal{A}_N$.

Also new ℓ^p improving bounds for $\mathcal{A}_N$ are used, requiring the recent, complete resolution of the Vinogradov Mean Value Theorem.

We also establish Theorem 1 in the real setting where $\mathbb{Z}$ is replaced by $\mathbb{R}$. This is important in the multilinear circle method since on the major arcs, we approximate the discrete averages by their continuous variants. More precisely, suppose $\mathcal{P} = \{P_1, \ldots, P_k\} \subset \mathbb{R}[\mathrm{x}]$ have distinct degrees. Consider the continuous multilinear averages

$$\mathcal{A}_N^{\mathbb{R}}(f_1, \ldots, f_k)(x) = \frac{1}{N}\int_0^N f_1(x - P_1(t)e_1)f_2(x - P_2(t)e_2)\cdots f_k(x - P_k(t)e_k)\,dt$$

and the *real* major arc $\mathcal{M}^{\mathbb{R}}_{N,d,\delta^{-C}} = [-\delta^{-C}N^{-d}, \delta^{-C}N^{-d}]$, a single interval centred at the origin. We prove that (3) holds for $\mathcal{A}_N^{\mathbb{R}}$ with $\ell^p(\mathbb{Z}^k)$ replaced by $L^p(\mathbb{R}^k)$.

Recently the multilinear averages $\mathcal{A}_N^{\mathbb{R}}$ have attracted the attention of harmonic analysts. In [7], the bilinear avarages

$$B_N(f,g)(x,y) \;=\; \frac{1}{N}\int_0^N f(x-t,y)g(x,y-t^2)\,dt$$

were examined and (3) for B_N was established. The Sobolev smoothing bound has numerous applications as shown in [7] for B_N. In particular, using the argument by Bourgain in [6], they give the following quantitative bound: for all $N, \epsilon > 0$ and for all $A \subset [0,N] \times [0,N^2]$ with $|A| \geq \epsilon N^3$, we have

$$(4) \qquad \left|\left\{(t,x,y) \in [0,N]^2 \times [0,N^2] : (x,y),(x+t,y),(x,y+t^2) \in A\right\}\right| \geq \beta N^4$$

where $\beta = \exp(-e^{1/\epsilon^C})$. This is a continuous version of the recent quantitative bounds by Peluse-Prendiville-Shao [12] for integer *squorners* $\{(j,k),(j+n,k),(j,k+n^2)\}$.

From our Sobolev smoothing bound (3) for $\mathcal{A}_N^{\mathbb{R}}$, one can extend the quantitative bound (4) to general polynomial corner configurations associated to $\mathcal{P} = \{P_1, \ldots, P_k\} \subset \mathbb{R}[\mathrm{x}]$ with distinct degrees: for every $\epsilon \in (0,1)$, there is a $\beta(\epsilon, \mathcal{P}) \in (0,1)$ and $N_0(\epsilon, \mathcal{P}) \in \mathbb{Z}_+$ such that for any $N \geq N_0$ and $A \subset [0, N^{d_1}] \times \cdots \times [0, N^{d_k}] =: I_N$ with $|A| \geq \epsilon N^D$ (where $D = d_1 + \cdots + d_k$), we have

$$(5) \qquad \left|\left\{(t,x) \in [0,N] \times I_N : x, x + P_1(t)e_1, \ldots, x + P_k(t) \in A\right\}\right| \geq \beta N^{D+1}.$$

We can take $N_0 = \lfloor e^{C\epsilon^{-C}} \rfloor$ and $\beta = e^{-C\epsilon^{-C}}$; compare with (4). When each polynomial P_j is a monomial, the bound (5) scales and holds true for all $N > 0$, not just $N \geq N_0$.

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A Pierce–Yung operator in the plane

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(joint work with Shaoming Guo, Jonathan Hickman)

Let $d \geq 2$ be an integer and $K : \mathbb{R}^{d-1} \to \mathbb{R}$ be a Calderón–Zygmund kernel, that is, a tempered distribution agreeing with a C^1 function $K(x)$ for $x \neq 0$ and such that $\widehat{K} \in L^\infty$ and $|\partial_x^\alpha K(x)| \leq A|x|^{-n-|\alpha|}$ for $0 \leq |\alpha| \leq 1$. A slight generalisation of the Carleson–Hunt theorem was obtained by Sjölin [6], who proved that the maximally modulated singular integrals

$$Tf(x) := \sup_{v \in \mathbb{R}^{d-1}} \Big| \int_{\mathbb{R}^{d-1}} f(x-t) e^{iv\cdot x} K(t)\,\mathrm{d}t \Big|$$

are bounded on $L^p(\mathbb{R}^{d-1})$ for all $1 < p < \infty$. In 1995, Stein [7] raised the question about whether Sjölin's result still holds true if the modulations $v \cdot x$ are replaced by an arbitrary polynomial modulation of a fixed degree. An affirmative answer was given by Stein and Wainger [8] for polynomials missing linear terms, and by Lie [3] in the general case. We note that if one avoids linear terms, the resulting operator is non-modulation invariant and its behaviour differs from that of the Carleson operator; in particular, after a linearisation and a TT^* argument, it is amenable to oscillatory integral techniques such as van der Corput's lemma.

A few years ago, Pierce and Yung [5] incorporated Radon-transform behaviour to the Stein–Wainger framework. They considered the operators

$$T_{\mathrm{par}} f(x,y) := \sup_{v \in \mathbb{R}^N} \Big| \int_{\mathbb{R}^{d-1}} f(x-t, y-|t|^2) e^{iP_v(t)} K(t)\mathrm{d}t \Big|,$$

where $P_v(t) = \sum_{j=2}^{N} v_j p_j(t)$ and p_j is a homogeneous polynomial of degree j, with $p_2(t)$ not a multiple of $|t|^2$. They proved that, for $d \geq 3$, the operator T_{par} is bounded on $L^p(\mathbb{R}^d)$ for all $1 < p < \infty$. Note that, in the spirit of Stein–Wainger, the above setup is not modulation invariant, since it avoids linear modulations and quadratic modulations of the type $|t|^2$.

The proof method in [5] does not extend to $d = 2$. One of the reasons is that the kernel of the linearised version of $T_{\text{par}}T_{\text{par}}^*$ is no longer given by an oscillatory integral. Our main result is a 2-dimensional version of the result of Pierce and Yung in the case of cubic modulations.

Theorem 1. *For $x, y \in \mathbb{R}$, consider the operator*

$$\mathcal{C}f(x,y) := \sup_{v\in\mathbb{R}} \Big|\text{p.v.} \int_{\mathbb{R}} f(x-t, y-t^2) e^{ivt^3} \frac{\mathrm{d}t}{t}\Big|,$$

initially defined for Schwartz functions $f \in \mathcal{S}(\mathbb{R}^2)$. There exists $C_p > 0$ such that

$$\|\mathcal{C}f\|_{L^p(\mathbb{R}^2)} \leq C_p \|f\|_{L^p(\mathbb{R}^2)}$$

holds for all $1 < p < \infty$.

The same result has also been recently obtained by Hsu and Lie [2] with a different approach.

Our proof method is inspired by that of Mockenhoupt, Seeger and Sogge [4] for the $L^p(\mathbb{R}^2)$ boundedness of the circular maximal function for $p > 2$. We obtain, for $p > 2$, a *local smoothing estimate* on L^p of

$$\mathcal{H}_n^w f(x,y) := \frac{1}{(2\pi)^2} \int_{\widehat{\mathbb{R}}^2} e^{i(x\xi+y\eta)} \int_{\mathbb{R}} e^{-i2^n(t\xi+t^2\eta-t^3w)} \beta(t) \frac{\mathrm{d}t}{t} \widehat{f}(\xi,\eta) \mathrm{d}\xi \mathrm{d}\eta,$$

where $1 \leq w \leq 8$. Here the operator $\mathcal{H}_n^w$ is a suitably localised piece of the Fourier multiplier operator featuring in the definition of $\mathcal{C}$, and by local smoothing we mean that we take an L^p integration in the w variable, and expect a better estimate than for a fixed w. Traditional L^2 methods allow to deal with the *most singular* parts of the Fourier multiplier, given by $(\phi_{\xi,\eta}^w)'(t) = (\phi_{\xi,\eta}^w)''(t) = 0$ where

$$\phi_{\xi,\eta}^w(t) := \langle (t, t^2, t^3), (\xi, \eta, -w) \rangle.$$

The vanishing of the first two order derivatives can only happen in a small w-interval, which gives a gain when integrating with respect to w. Thus, it is the *less singular* region of the multiplier, given by $(\phi_{\xi,\eta}^w)'(t) = 0$, the one that requires a more delicate analysis and where $p > 2$ becomes crucial in our arguments. By stationary phase, this gives raise to a variable-coefficient propagator with phase function

$$\Phi_{\pm}(x,y,w;\xi,\eta) := x\xi + y\eta + \phi_{\xi,\eta}^w \circ t_{\pm}(w;\xi,\eta)$$

where $t_{\pm}(w;\xi,\eta) := \frac{\eta \pm \sqrt{\eta^2+3w\xi}}{3w}$ satisfy $(\phi_{\xi,\eta}^w)'(t_{\pm}(w;\xi,\eta)) = 0$. These phase functions satisfy the *Nikodym non-compression hypotheses* recently introduced in [1], although they are not globally valid. One can then decompose the operator into two pieces. The main piece is that for which this condition holds, and one can obtain favorable L^p estimates thanks to (a quantified version of) the main result in [1]. The remaining piece, which can be seen as an error, can be handled on L^p via decoupling inequalities, a typical tool to obtain local smoothing estimates for propagators.

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Sparse Calderón-Zygmund estimates for divergence form elliptic equations

Olli Saari

(joint work with Hua-Yang Wang, Yuanhong Wei)

Background. Let $n \geq 2$ and consider a bounded, open, and connected set $\Omega \subset \mathbb{R}^n$. For simplicity, we assume that the boundary of Ω is C^2-smooth. Let $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be measurable in the first n variables and $C^1(\Omega)$ in the last n variables. We assume the standard uniform ellipticity conditions: there exist $\lambda > 0$ and $\Lambda < \infty$ such that for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$,

$$\xi \cdot (\nabla_\xi a)(x,\xi)\xi \geq \lambda|\xi|^2, \quad |a(x,\xi)| \leq \Lambda.$$

The most important special case is when the mapping is linear in the ξ-variable, that is, when there exists a matrix-valued function $A : \Omega \to \mathbb{R}^{n\times n}$ such that $a(x,\xi) = A(x)\xi$. We study the equation

$$-\operatorname{div} a(x, \nabla u(x)) = \operatorname{div} F(x)$$

posed in Ω with zero boundary values in the Sobolev sense, and we consider weak solutions.

The classical Calderón–Zygmund theory (or L^p–Schauder theory) builds on the following results:

- There exist constants $C, \varepsilon > 0$, depending only on n, λ, Λ, and Ω, such that if $F \in L^q(\Omega)$ with $|q-2| < \varepsilon$, then

$$\|\nabla u\|_{L^q(\Omega)} \leq C \|F\|_{L^q(\Omega)} .$$

  This is known as Meyers' estimate [10].
- If F is of vanishing mean oscillation (VMO), then ε can be taken arbitrarily close to 1 (see [2]).

The literature on this type of estimates is now extensive. In our work, we prove Calderón–Zygmund estimates within the framework of sparse bounds, rather than using $L^p(\Omega)$ norms. Sparse bounds emerged from simplifications in the resolution of the A_2 conjecture [4]. In particular, we highlight [6], [8], and [7] as key references from the point of view of our approach.

Results. Let $\theta \in (0,1]$. Recall that a family of cubes $\mathcal{F}$ is said to be θ-sparse if for each $Q \in \mathcal{F}$, there exists a measurable subset $E_Q \subset Q$ such that

$$\sum_{Q \in \mathcal{F}} 1_{E_Q} \leq 1, \quad \inf_{Q \in \mathcal{F}} \frac{|E_Q|}{|Q|} \geq \theta.$$

We recall that a bound

$$\left| \int \nabla u(x) \cdot g(x)\, dx \right| \\ \leq C \sup_{\mathcal{F}\ \theta\text{-sparse}} \sum_{Q \in \mathcal{F}} |Q| \left(\frac{1}{|Q|} \int_{3Q} |F(x)|^r\, dx \right)^{1/r} \left(\frac{1}{|Q|} \int_{3Q} |g(x)|^s\, dx \right)^{1/s}$$

for all test functions g, with $\theta, s, r \in [1, \infty)$ fixed and $C = C(n, r, s, \theta)$, implies $L^p(\Omega) \to L^p(\Omega)$ bounds for all $r < p < s'$. We refer to such a bound as an (r, s) sparse bound.

Our main results in [11] are the following:

- $(2, q)$-sparse bounds for q sufficiently close to 2, analogous to Meyers' estimate under the same assumptions;
- (q, q)-sparse bounds for all $q \in (1, 2)$ when the coefficient function is linear in the gradient variable and of vanishing mean oscillation in the spatial variable;
- A $(1, 1)$-sparse bound when the coefficient function is linear in the gradient variable and Dini continuous in the spatial variable.

The proofs rely on the method of approximation by solutions to equations with zero right-hand side, following [1], and the sparse iteration technique from [9]. Moreover, we take advantage of the regularity results for equations with VMO coefficients [2, 5] and Dini continuous coefficients [3].

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Subconvex L^p-sets, Weyl's inequality, and equidistribution

Trevor D. Wooley

We investigate sets of natural numbers $\mathcal{A}$ whose associated exponential sums satisfy a certain subconvexity property. Throughout, when $\mathcal{A} \subseteq \mathbb{N}$ and $N \geq 1$, we write $\mathcal{A}(N) = \mathcal{A} \cap [1, N]$ and $A(N) = \mathrm{card}(\mathcal{A}(N))$. As usual, we abbreviate $e^{2\pi i z}$ to $e(z)$. Then, when p is a positive real number, we define the mean value

$$I_p(N; \mathcal{A}) = \int_0^1 \Big| \sum_{n \in \mathcal{A}(N)} e(n\alpha) \Big|^p \, \mathrm{d}\alpha.$$

Definition 1. *Suppose that $\mathcal{A}$ is a non-empty subset of $\mathbb{N}$.*

(a) *We say that $\mathcal{A}$ is a weakly subconvex L^p-set if $0 < p < 2$ and, for all $\varepsilon > 0$ and all real numbers N sufficiently large in terms of p and ε, one has*

$$I_p(N; \mathcal{A}) \ll N^{\varepsilon - 1} A(N)^p.$$

(b) *We say that $\mathcal{A}$ is a strongly subconvex L^p-set if $0 < p < 2$ and, for all real numbers N sufficiently large in terms of p, one has*

$$I_p(N; \mathcal{A}) \ll N^{-1} A(N)^p.$$

It transpires that strongly subconvex L^p-sets $\mathcal{A}$ have positive lower density, and thus the associated mean value $I_p(N; \mathcal{A})$ exhibits better than square-root cancellation. It might seem that subconvex L^p-sets are unusual subsets of the integers that should be extraordinarily difficult to find. However, we provide examples illustrating that subconvex L^p-sets are abundant. The set of all natural numbers $\mathbb{N}$ is a trivial example of a weakly subconvex L^1-set which is also a strongly subconvex L^p-set whenever $p > 1$. Consider next, when $r \geq 2$, the set $\mathcal{N}_r$ of r-free numbers, defined by $\mathcal{N}_r = \{n \in \mathbb{N} : \pi^r | n \text{ for no prime } \pi\}$. It transpires that $\mathcal{N}_r$ is a weakly subconvex $L^{1+1/r}$-set, and when $p > 1 + 1/r$ it is a strongly subconvex L^p-set, as follows from the work of Keil [1, Theorem 1.2]. Further, when $\alpha > 0$ and β are real numbers with $1/\alpha$ of finite Diophantine type, then the Beatty set

$$\mathcal{B}(\alpha, \beta) = \{n \in \mathbb{N} : n = \lfloor \alpha m + \beta \rfloor \text{ for some } m \in \mathbb{N}\}$$

is a weakly subconvex L^1-set, and when $p > 1$ it is a strongly subconvex L^p-set. We show how new examples of subconvex L^p-sets may be obtained from old ones.

Our main focus lies on Weyl sums. When $k \geq 2$ and $\alpha_i \in \mathbb{R}$ $(0 \leq i \leq k)$, consider the polynomial $\psi(x; \boldsymbol{\alpha}) = \alpha_k x^k + \ldots + \alpha_1 x + \alpha_0$, and define the exponential sum

$$\Psi_k(\boldsymbol{\alpha}; N) = \sum_{1 \leq n \leq N} e(\psi(n; \boldsymbol{\alpha})).$$

By Dirichlet's approximation theorem, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$ and $|\alpha_k - a/q| \leq 1/q^2$. In these circumstances, it follows from Weyl's inequality (see [2, Lemma 2.4]) that for each $\varepsilon > 0$ and each large real number N, one has

$$\Psi_k(\boldsymbol{\alpha}; N) \ll N^{1+\varepsilon}(q^{-1} + N^{-1} + qN^{-k})^{2^{1-k}}.$$

Analogous estimates, in which the variables n defining this exponential sum are restricted to a subset of the natural numbers, are typically far weaker. Our first result on Weyl's inequality shows that estimates are not inferior for weakly subconvex L^p-sets, at least when $k \geq 3$ and $1 \leq p \leq 4/3$.

Theorem 2. *Suppose that $\mathcal{A}$ is a weakly subconvex L^p-set for some real number p with $1 \leq p \leq 4/3$, and $k \geq 3$. Let $(\alpha_0, \alpha_1, \ldots, \alpha_k) \in \mathbb{R}^{k+1}$, and suppose that $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a, q) = 1$ and $|\alpha_k - a/q| \leq 1/q^2$. Then, for each $\varepsilon > 0$ and each large real number N, one has*

$$\sum_{n \in \mathcal{A}(N)} e(\alpha_k n^k + \ldots + \alpha_1 n + \alpha_0) \ll N^{1+\varepsilon}(q^{-1} + N^{-1} + qN^{-k})^{2^{1-k}}.$$

This theorem is a corollary of a more general conclusion applicable whenever $k \geq 2$ and $\mathcal{A}$ is any subconvex L^p-set with $1 \leq p < 2$. Define

$$\sigma_p(k) = \begin{cases} \frac{1}{p} - \frac{1}{2}, & \text{when } k = 2, \\ 2^{3-k}\left(\frac{1}{p} - \frac{1}{2}\right), & \text{when } k \geq 3 \text{ and } 4/3 < p < 2, \\ 2^{1-k}, & \text{when } k \geq 3 \text{ and } 1 \leq p \leq 4/3, \end{cases}$$

and

$$\tau_p(k) = \begin{cases} \left(\frac{2}{p} - 1\right)\frac{1}{k^2-k-2}, & \text{when } k \geq 3 \text{ and } \frac{k^2-k}{k^2-k-1} < p < 2, \\ \frac{1}{k(k-1)}, & \text{when } k \geq 3 \text{ and } 1 \leq p \leq \frac{k^2-k}{k^2-k-1}, \\ 0, & \text{when } k = 2. \end{cases}$$

Theorem 3. *Suppose that $\mathcal{A}$ is a weakly subconvex L^p-set for some real number p with $1 \leq p < 2$, and $k \geq 2$. Let $(\alpha_0, \alpha_1, \ldots, \alpha_k) \in \mathbb{R}^{k+1}$, and suppose that $a \in \mathbb{Z}$*

and $q \in \mathbb{N}$ *satisfy* $(a,q)=1$ *and* $|\alpha_k - a/q| \le 1/q^2$*. Then, for each* $\varepsilon > 0$ *and each large real number* N*, one has*

$$\sum_{n\in\mathcal{A}(N)} e(\alpha_k n^k + \ldots + \alpha_1 n + \alpha_0) \ll N^{1+\varepsilon}(q^{-1} + N^{-1} + qN^{-k})^{\omega_p(k)},$$

where

$$\omega_p(k) = \max\{\sigma_p(k), \tau_p(k)\}.$$

Finally, we mention some consequences for the equidistribution modulo 1 of polynomial sequences. Consider a real sequence $(s_n)_{n=1}^\infty$ and the associated fractional parts $\{s_n\} = s_n - \lfloor s_n \rfloor$. This sequence is said to be *equidistributed modulo* 1 when, for each pair of real numbers a and b with $0 \le a < b \le 1$, one has

$$\lim_{N\to\infty} N^{-1}\mathrm{card}\{1 \le n \le N : a \le \{s_n\} \le b\} = b - a.$$

Theorem 4. *Suppose that* $\mathcal{A} = \{a_1, a_2, \ldots\}$*, with* $a_1 < a_2 < \ldots$*, is a strongly subconvex* L^p*-set with* $1 \le p < 2$*. Let* $k \ge 2$*, suppose that* $(\alpha_0, \alpha_1, \ldots, \alpha_k) \in \mathbb{R}^{k+1}$*, and define the polynomial* $\psi(x;\boldsymbol{\alpha}) = \alpha_k x^k + \ldots + \alpha_1 x + \alpha_0$*. Then, provided that one at least of the coefficients* $\alpha_2, \ldots, \alpha_k$ *is irrational, the sequence* $(\psi(a_n;\boldsymbol{\alpha}))_{n=1}^\infty$ *is equidistributed modulo* 1*.*

We remark that strongly subconvex L^p-sets exist having the property that equidistribution modulo 1 may fail for the sequence $(\psi(a_n;\boldsymbol{\alpha}))_{n=1}^\infty$ when α_1 is irrational. Thus, the hypothesis on the coefficients in this theorem cannot be relaxed to assert only that one of $\alpha_1, \ldots, \alpha_k$ is irrational.

The proofs of our new results rest on exploiting the basic property of subconvex L^p-sets in combination with orthogonality and appropriate applications of Hölder's inequality. Suppose then that $\mathcal{A}$ is a strongly subconvex L^p-set for some real number p with $1 \le p < 2$, and write

$$g(\alpha) = \sum_{n\in\mathcal{A}(N)} e(n\alpha).$$

Consider a unimodular arithmetic function $c : \mathbb{N} \to \mathbb{C}$, and put

$$h(\alpha) = \sum_{1\le n\le N} c(n)e(n\alpha).$$

Then it follows by orthogonality that

$$\sum_{n\in\mathcal{A}(N)} c(n) = \int_0^1 g(\alpha)h(-\alpha)\,\mathrm{d}\alpha.$$

Thus, by applying Hölder's inequality, we find that

$$\Big|\sum_{n\in\mathcal{A}(N)} c(n)\Big| \le \left(\int_0^1 |g(\alpha)|^p\,\mathrm{d}\alpha\right)^{1/p}\left(\int_0^1 |h(\alpha)|^{p/(p-1)}\,\mathrm{d}\alpha\right)^{1-1/p}.$$

The first mean value on the right hand side here may be estimated optimally by using the definition of a strongly subconvex L^p-set. Since $p/(p-1) > 2$, the second mean value is amenable to methods combining orthogonality and Weyl's inequality. When $p/(p-1) \geq 4$, the second mean value may even be interpreted in terms of a second order Weyl differencing process. In many circumstances, this approach suffices to obtain non-trivial upper bounds for the average of $c(n)$ over the set $\mathcal{A}(N)$.

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L^p-estimates for FIO-cone multipliers

Detlef Müller

(joint work with Stefan Buschenhenke, Spyridon Dendrinos, and Isroil Ikromov)

Let $\Gamma := \{\xi_1^2 + \xi_2^2 = \xi_3^2,\, 1/2 \leq \xi_3 \leq 1\}$ be the truncated light cone in $\mathbb{R}^3$, denote for $R \gg 1$ by

$$\Gamma_R = \Big\{ \frac{1}{2R} \leq \frac{\xi_1^2 + \xi_2^2}{\xi_3^2} - 1 \leq \frac{1}{R},\, 1/2 \leq \xi_3 \leq 1 \Big\}$$

the set of points whose horizontal distance to Γ is comparable to $1/R$, and let m_R be the corresponding cone multiplier, i.e., a smooth version of the characteristic function of Γ_R. Thanks to work by Mockenhaupt [5] and the more recent breakthrough on the associated reverse square estimate by Guth, Wang and Zhang [1], we know that the corresponding convolution operator T_{m_R} satisfies bounds $\|T_{m_R}\|_{p\to p} \leq C_\epsilon R^{\epsilon|\frac{1}{2}-\frac{1}{p}|}$ for every $\epsilon > 0$ in the range $4/3 \leq p \leq 4$. The work by Guth, Wang and Zhang also leads to a proof of Sogge's local smoothing conjecture (compare [6]) for solutions to the wave equation on $\mathbb{R}^2$.

By an FIO-cone multiplier we mean a Fourier multiplier of the form $e^{i\lambda\phi} m_R$, where ϕ is a real phase function homogenous of degree 1, which is smooth away from the light cone (but may exhibit singularities at the cone), and where $\lambda \gg 1$.

In our theory of FIO-cone multipliers, we devise suitable classes of phase functions ϕ by requiring certain estimates on their derivatives which are closely related to the geometry of the light cone, and prove L^p-estimates in the range above for the corresponding convolution operators. Our estimates are stronger by a factor $R^{-|\frac{1}{2}-\frac{1}{p}|}$ than the estimates that one would get from a direct application of the method of Seeger, Sogge and Stein [7] for estimating FIOs. In a similar way as for the cone multiplier, decoupling estimates would not allow to prove these results either.

Our proofs build on the work by Guth, Wang and Zhang, by making use of a deep characterization in [1] of the L^4-norm of the afore-mentioned square function by means of certain L^2-based semi-norms.

An important application of our results is to maximal averages along smooth hypersurfaces in $\mathbb{R}^3$: If S is a smooth hypersurface in $\mathbb{R}^3$, consider the associated averaging operators $A_t, t > 0$, given by

$$A_t f(x) := \int_S f(x - ty)\, d\sigma(y),$$

where $d\sigma$ denotes the surface measure on S. The associated maximal operator is given by

$$\mathcal{M}_S f(x) := \sup_{t>0} |A_t f(x)|, \quad (x \in \mathbb{R}^3).$$

The study of such maximal operators had been initiated by E.M. Stein's work on the spherical maximal function [9]. In a series of papers [4], [3], [2], for almost all real analytic hypersurfaces S (and even larger classes of finite type surfaces) in $\mathbb{R}^3$ satisfying a natural transversality assumption, the range of Lebesgue spaces $L^p(\mathbb{R}^3)$ on which the maximal operator $\mathcal{M}_S$ is bounded has been determined explicitly in terms of Newton diagrams associated to S, in some cases up to the critical exponent $p_c = p_c(S)$. The latter is determined by the property that M_S is L^p bounded for $p > p_c$, and unbounded for $p < p_c$. In [2], also a "geometric conjecture" had been stated and proved for the same class of surfaces, which roughly claims that p_c can be determined by testing $\mathcal{M}_S$ on characteristic functions of symmetric convex bodies.

This conjecture, and a related conjecture about a description of p_c in terms of Newton diagrams, has remained open only for a small class of "exceptional" surfaces exhibiting singularities of type A in the sense of Arno'ld's classification. A prototypical surfaces from this class is the graph S_n of

$$1 + \frac{1}{1 - x_1} x_2^2 + x_1^n, \qquad n \geq 5$$

over a sufficiently small neighborhood of the origin. Our conjectures claim for this surface that $p_c = 2\frac{n+1}{n+3}$. However, classical methods based on interpolation between L^1 and L^2 estimates only allow to cover the smaller range $p > 2\frac{n+2}{n+4}$ (if n is even, the surface S_n is even convex, so that the results from [8] apply to it, but these again only yield the latter range).

Our estimates for FIO-cone multipliers allow to prove our conjectures also for these surfaces S_n, i.e., that $p_c = 2\frac{n+1}{n+3}$, by interpolating here between $L^{4/3}$ and L^2 estimates. The proof requires essentially the full thrust of our FIO-cone multiplier estimates.

We expect that a slight extension of our theory of FIO-cone multipliers will eventually allow for a full proof of this geometric conjecture.

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Convex Set Testing Conditions for Positive Operators

Philip T. Gressman

In many distinct contexts within harmonic analysis (see [1, 2, 3]), it has been fruitful to study multilinear objects whose kernel is the absolute value of a determinant whose columns are vector-valued functions with geometric significance. A canonical example of such an object is given by

$$I(f_1, \ldots, f_d) := \int |\det(\gamma(u_1), \ldots, \gamma(u_d))|^{-s} \prod_{j=1}^{d} f_j(u_j)\, du_1 \cdots du_n,$$

where the parameters $u_1, \ldots, u_d$ typically range within an open subset U of parameters in $\mathbb{R}^p$ and $\gamma : U \to \mathbb{R}^d$ typically parametrizes some smooth p-dimensional submanifold of $\mathbb{R}^d$. Generally speaking, the goal is to establish boundedness of this functional on products of Lebesgue spaces $L^p(U)$. If μ denotes Lebesgue measure on the image of γ, the above functional can also be written as

$$I(f_1, \ldots, f_d) = \int |\det(x_1, \ldots, x_d)|^{-s} \prod_{j=1}^{d} f_j(x_j)\, d\mu(x_1), \ldots, d\mu(x_d),$$

and studied on products of Lebesgue spaces $L^p(\mu)$, where each vector $x_1, \ldots, x_d$ now belongs to $\mathbb{R}^d$. While this change of perspective is purely formal, it was demonstrated in [4] that this new perspective yields productive insights, and in particular allows one to understand $I(f_1, \ldots, f_d)$ geometrically in terms of the Oberlin condition (see D. Oberlin [6, 7] and Gressman [5]), which is satisfied by measures μ admitting some $\alpha > 0$ and $C < \infty$ such that $\mu(K) \leq C(\mathrm{vol}(K))^\alpha$ for all compact, convex sets $K \subset \mathbb{R}^d$, where $\mathrm{vol}(K)$ is the standard Euclidean d-dimensional volume of K. In the case of the given functional $I(f_1, \ldots, f_d)$, it also happens that one need only test the measure μ on compact, convex sets K which are symmetric with respect to $0 \in \mathbb{R}^d$.

The deep connection between multilinear functionals with determinant-like kernels and the Oberlin condition raises the question as to whether other natural families of multilinear functionals are governed in the same way by natural generalizations of the Oberlin condition. One key to answering this question is to recognize the role of determinants in computing Euclidean volumes of parallelepipeds. For various reasons, it also happens to be simpler to decompose kernels dyadically according to size and to look not at Lebesgue spaces L^p but the Lorentz spaces $L^{p,1}$. A central new result of this kind is captured by the following theorem.

Theorem 1. *Suppose $X_1, \dots, X_k$ are finite dimensional real vector spaces, and let $s_1, \dots, s_k$ be nonnegative exponents such that $s_k > 0$. Let*

$$\frac{1}{\theta_0} := \sum_{j=1}^{k} s_j \prod_{\ell < j} d_\ell.$$

There exist constants $C, c > 0$ such that the following is true. Suppose that $L : X_1 \times \cdots \times X_k \to \mathbb{R}$ is multilinear, that $\mu_1, \dots, \mu_k$ are nonnegative Borel measures which are finite on all compact sets, and $A, \delta > 0$ are real numbers such that

$$\prod_{j=1}^{k} (\mu_j(K_j))^{s_j} \leq cA$$

for all compact, symmetric convex sets $K_1 \subset X_1, \dots, K_k \subset X_k$ for which

$$\sup_{x_1 \in K_1, \dots, x_k \in K_k} |L(x_1, \dots, x_k)| \leq C\delta.$$

Then for all $\theta \in [0, \theta_0]$ and all Borel sets $F_1 \subset X_1, \dots, F_k \subset X_k$ such that $\mu_j(F_j) < \infty$ for each $j \in \{1, \dots, k\}$,

$$\int \chi_{|L(x_1, \dots, x_k)| \leq \delta} \prod_{j=1}^{k} \chi_{F_j}(x_j) d\mu_1(x_1) \cdots d\mu_k(x_k) \leq A^\theta \prod_{j=1}^{k} (\mu_j(F_j))^{1 - \theta s_j}.$$

By exploiting a lifting technique similar to that which appears in the case of multilinear determinant functionals, the above theorem can be applied in a broad array of contexts which includes any multilinear kernel which is the absolute value of a polynomial. An important limitation of the theorem, however, is that in some cases, the value of θ_0 could be improved. As such, the value of θ_0 specified above should be understood as limited by the method of proof rather than an intrinsic endpoint for the associated multilinear functional.

When pushing beyond the range of inequalities implied by Theorem 1, it becomes important to develop qualitative and quantitative understandings of functions $F_\delta(x)$ on $\mathbb{R}^d$ which have the form

$$F_\delta(x) := \int \chi_{|x \cdot y| \leq \delta} d\nu(y)$$

for some probability measure ν on $\mathbb{R}^d$. The first new result in this direction demonstrates that the superlevel sets

$$E_\alpha := \left\{ x \in \mathbb{R}^d \ : \ F_\delta(x) \geq \alpha \right\}$$

of F_δ exhibit increasing complexity as the parameter α tends to zero.

Theorem 2. *For any $0 < \alpha_1 < \alpha_2 < 1$, there is a collection of symmetric convex sets $C_1, \ldots, C_N$ such that*

$$E_{\alpha_2} \subset \bigcup_{i=1}^N C_i \subset E_{\alpha_1}.$$

Suppose that D is any positive integer such that $(\alpha_2 - \alpha_1)^{-1}(1 - \alpha_2) < D$. Then for any line $\ell \subset \mathbb{R}^d$,

$$\ell \cap \bigcup_{i=1}^N C_i$$

is a union of finitely many closed intervals, at most D of which contain a point in E_{α_2}.

Closely related to this result is the following theorem, which shows that weak-L^q norms of F_δ can be estimated simply by testing the function F_δ against indicator functions of convex sets, provided that q is sufficiently large.

Theorem 3. *Suppose μ is a σ-finite Borel measure on $\mathbb{R}^d$. Suppose there is some $A \geq 0$, $p \in [1, \infty)$ such that every closed, symmetric convex set $K \subset \mathbb{R}^d$ has*

$$\int_K F_{d\delta}(x) d\mu(x) \leq A[\mu(K)]^{1-\frac{1}{p}}.$$

Then every measurable set $E \subset \mathbb{R}^d$ satisfies

$$\int_E F_\delta(x) d\mu(x) \leq C \big[A \mu(E)^{1-\frac{1}{p}} \big]^\theta \big[\mu(E) ||\nu|| \big]^{1-\theta}$$

where $\theta := 1/(dp^{-1} + 1)$ and the constant C depends only on d and p.

The proofs of these results rely substantially on the following elementary lemma.

Lemma 1 (Convex multilinearization)**.** *Given $x_1, \ldots, x_d \in \mathbb{R}^d$, let*

$$\mathcal{C}_x := \left\{ x \in \mathbb{R}^d \ : \ x = \sum_{j=1}^d \theta_j x_j \text{ for some } \theta_1, \ldots, \theta_d \in [-1, 1] \right\}.$$

Suppose μ is a σ-finite nonnegative Borel measure on a real finite-dimensional vector space X of dimension d which is finite on all compact sets. For each $\rho \geq 0$, there is a constant $C_{d,\rho}$ such that

$$\int f d\mu \leq \left[C_{d,\rho} \int \left(\int_{\mathcal{C}_x} f d\mu \right)^\rho \prod_{j=1}^d f(x_j) d\mu(x_1) \cdots d\mu(x_d) \right]^{\frac{1}{d+\rho}} \tag{1}$$

for all nonnegative Borel-measurable functions f on X.

Beyond basic integration facts like Fubini's Theorem and Hölder's inequality, the proof of this lemma uses only elementary linear algebra and a basic symmetry observation.

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A fractal local smoothing problem

Joris Roos

(joint work with David Beltran, Alex Rutar, Andreas Seeger)

It is well-known (Miyachi [10], Peral [11]) that the half-wave propagator satisfies the fixed-time estimate

$$\|e^{it\sqrt{-\Delta}}f\|_{L^p(\mathbb{R}^d)} \le C_t \|f\|_{L^p_{s_p}(\mathbb{R}^d)} \tag{1}$$

for $p \in [2,\infty)$ and for every fixed $t > 0$, where $s_p = (d-1)(\frac{1}{2} - \frac{1}{p})$. The local smoothing phenomenon (Sogge [14]) states that the required smoothness improves with an additional time average in L^p, i.e.

$$\Big(\int_1^2 \|e^{it\sqrt{-\Delta}}f\|^p_{L^p(\mathbb{R}^d)}\,dt\Big)^{\frac{1}{p}} \le C\|f\|_{L^p_s(\mathbb{R}^d)} \tag{2}$$

holds for some $s < s_p$ when $p > 2$. The classical local smoothing conjecture [14] postulates that this holds for all $s > \sigma_p = \max(0, s_p - \frac{1}{p})$ when $p > 2$. Wolff [15] showed this for large enough p using an early decoupling inequality. The decoupling approach has been pushed to its limits by Bourgain and Demeter [4] who proved that the conjecture holds for $p \ge \frac{2(d+1)}{d-1}$. Guth, Wang and Zhang [8] finally proved the conjecture for all $p > 2$ in the case $d = 2$. We now propose a fractal version of the conjecture. By localizing and discretizing, the estimate can be reformulated as

$$\Big(\sum_t \|e^{it\sqrt{-\Delta}}P_j f\|^p_{L^p(\mathbb{R}^d)}\Big)^{1/p} \le C\,2^{js}\|f\|_{L^p(\mathbb{R}^d)}, \tag{3}$$

for $s > \max(\frac{1}{p}, s_p)$ and $p > 2$, where the sum over t runs over $1 + 2^{-j}n$ for $n = 0, \ldots, 2^j$ and P_j denotes Littlewood–Paley projection to frequencies $|\xi| \approx 2^j$ for $j \geq 1$. In the fractal problem we consider an arbitrary set $E \subset [1,2]$ and a maximal 2^{-j}-separated subset E_j and let the sum over t run only over E_j. Given E, we define the *Legendre–Assouad function* $\nu_E^\sharp$ as

$$\nu_E^\sharp(\alpha) = \overline{\lim}_{\delta\to 0} \frac{\log(\sup_{\delta \leq |J| \leq 1} |J|^{-a} N(E \cap J, \delta))}{\log(\frac{1}{\delta})}, \tag{4}$$

where $\alpha \in \mathbb{R}$ and $N(E, \delta)$ denotes the minimum number of intervals of length δ required to cover E and the supremum runs over intervals $J \subset [1,2]$ with $|J| \geq \delta$. This function originates in the study of sharp L^p improving bounds for spherical maximal functions with fractal sets of dilations [3], [12], [1]. We conjecture that (3) holds when the sum runs over $t \in E_j$ and whenever $p > 2$ and

$$s > \tfrac{1}{p}\nu_E^\sharp(ps_p).$$

In [2] it was shown that this holds for radial functions and that it fails for $s < \frac{1}{p}\nu_E^\sharp(ps_p)$. Setting $E = [1,2]$ recovers the classical local smoothing conjecture (2). The Legendre–Assouad function is closely related to the *Assouad spectrum* of E, which is the function $\theta \mapsto \dim_{\mathrm{A},\theta} E$ defined as the infimum over all exponents $a > 0$ for which there exists a constant C such that

$$N(E \cap J, \delta) \leq C(|J|/\delta)^a$$

for all intervals J with $|J| = \delta^\theta$ and $\delta \in (0,1)$ (Fraser–Yu [7]). The *upper Assouad spectrum* arises when the requirement $|J| = \delta^\theta$ is replaced by $|J| \geq \delta^\theta$ (Fraser–Hare–Hare–Troscheit–Yu [6]). The Assouad spectrum is a continuous function on $[0,1)$ and the limit $\lim_{\theta\to 1-} \dim_{\mathrm{A},\theta} E$ exists and is called the *quasi-Assouad dimension* $\gamma = \dim_{\mathrm{qA}} E$ (Lü–Xi [9], Fraser [5]). It was shown in [2] that $\nu_E^\sharp$ is the Legendre transform of the function $\nu_E(\theta) = -(1-\theta)\dim_{\mathrm{A},\theta} E$ on $[0,1]$. Note that the Legendre transform of a function only depends on its convex hull, which implies that the Legendre transform of ν_E stays unchanged when the Assouad spectrum is replaced by the upper Assouad spectrum. Since it is known which functions may occur as the (upper) Assouad spectrum of a set (Rutar [13]), one may determine which functions occur as Legendre–Assouad functions: a function $\tau : [0,\infty) \to [0,\infty)$ satisfies $\nu_E^\sharp|_{[0,\infty)} = \tau$ for some $E \subset [1,2]$ if and only if τ is nondecreasing, convex, and satisfies $\tau(\alpha) = \alpha$ for $\alpha \geq 1$. In particular, we witness a striking contrast to the classical local smoothing problem: in the classical problem, it suffices to establish the estimate at the critical exponent $p = \frac{2d}{d-1}$ giving the remaining range by interpolation. The characterization of possible Legendre–Assouad functions shows that in the interesting range $p \in [2, p_\gamma]$ (with $p_\gamma = \frac{2(d-1+\gamma)}{d-1}$), interpolation between $p = 2$ and $p = p_\gamma$ generally does not recover the sharp estimate (3). Also note that (3) always follows from (2) for $p \geq \frac{2d}{d-1}$.

There is also an $L^p \to L^q$ variant of this problem, where for every $1 < p \leq q < \infty$, $q > p'$, we conjecture that

$$\Big(\sum_{t \in E_j} \| e^{it\sqrt{-\Delta}} P_j f \|_{L^q(\mathbb{R}^d)}^q \Big)^{\frac{1}{q}} \leq C\, 2^{js} \|f\|_{L^p(\mathbb{R}^d)}$$

holds for all $j \geq 1$ when

$$s > \tfrac{d+1}{2}\big(\tfrac{1}{p} - \tfrac{1}{q}\big) + \tfrac{1}{q} \nu_E^\sharp\big(\tfrac{q(d-1)}{2}\big(1 - \tfrac{1}{p} - \tfrac{1}{q}\big)\big).$$

This matches lower bounds proved in [2], where the conjecture is also verified for the easy case $p = 2$. A solution would also immediately imply a conjecture about closures of $L^p \to L^q$ type sets of spherical maximal functions with fractal dilation sets.

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