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Large Scale Stochastic Dynamics

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Abstract. The goal of this workshop was to explore the recent advances in the mathematical understanding of the macroscopic properties which emerge on large space-time scales from interacting microscopic particle systems. The talks addressed the following topics: stochastic homogeneization, hydrodynamic limits, Markov chain mixing times, superdiffusivity in out-of-equilibrium 2-dimensional systems, random walks in random environments.

Mathematics Subject Classification (2020): 82C05, 82C20, 82C24, 82C40, 60J10.

Introduction by the Organizers

The workshop *Large scale stochastic dynamics* is the continuation of the highly successful series of Oberwolfach workshops with the same title, whose organising team included along the years T. Bodineau, C. Landim, S. Olla, H. Spohn and the present organisers. This new edition, organised by P. Caputo (Roma Tre), F. Toninelli (TU Wien) and B. Tóth (University of Bristol and Alfréd Rényi Institute of Mathematics), was well attended with 57 participants (46 in person and 11 online) with broad geographic representation, including postdocs and graduate students, working in diverse intertwining areas of probability and statistical mechanics.

The workshop was devoted to the wide mathematical problem of understanding emergent structures on large space-time scales in the evolution of physical systems. These are modelled by particle systems, namely high-dimensional Markov processes and/or by systems of particles with deterministic (Hamiltonian) dynamics where randomness comes only with the initial conditions. With respect to the previous editions of this series of workshops, there was a larger focus on the presently very active topic of stochastic homogeneization. Very interesting links with singular SPDES/SDEs have been emphasized by several of the talks.

During the meeting, 12 talks of 50 minutes, and 15 talks of 30-35 minutes were scheduled. Thursday's talks were especially intended to celebrate Erwin Bolthausen's 80th birthday, and his many fundamental contributions to this field. In addition, an evening "open problem session" was organised with 9 more short informal presentations of 10 minutes, many of which by younger participants: Daniel Keliger, Nikolaos Zygouras, Jonas Köppl, Fabio Toninelli, Frederike Lemming, Balázs Maga, Federico Sau, Márton Balázs and Assaf Shapira. In our choice of 30 talks, we tried to illuminate major recent advances in the field and to expose and address at least some aspects of the works for each of the participants. The chosen schedule format (with a long afternoon break until 4:30 pm, intended to favour discussions and interactions) was unanimously appreciated by the participants. The evening session was the occasion to learn both about intriguing open problems in this area, and about the recent results of early career participants. Both the talks and the evening session triggered further discussions afterwards.

A summary account of the 50- and 35-minute presentations is given below.

- In two-dimensional *random field Ising models* with weak disorder, the addition of independent Gaussian fields leads to a unique local minimizer of the energy, in contrast to the degenerate minimizers of the pure Ising model. [Otto]
- The fluctuations of the weakly asymmetric facilitated exclusion process converge to the *stochastic heat equation* with Dirichlet boundary condition. A major difficulty is handling the singularity at time 0 of the heat kernel. [Blondel]
- The open asymmetric simple exclusion process exhibits high-density, low-density, and maximal current phases, depending on boundary parameters. At the triple point where all three phases meet, the mixing time scales as $N^{3/2}$. [Schmid]
- New criteria for gelation are proposed for the Marcus-Lushnikov *coagulation process*. The analysis includes the emergence of a giant particle, and a large-deviation framework for particle trajectories. [Andreis]
- A variant of the totally asymmetric exclusion process is connected to *true self-avoiding walks* and the *true self-repelling motion*, with new relations providing insights into super-diffusive behavior. [Massoulié]
- A critical drift-diffusion driven by a *divergence-free Gaussian field*, is related to diffusions on $\mathrm{SL}(n)$, explaining the intermittency observed in the original system. [Wagner]
- At the critical point of the Glauber/Kawasaki reaction-diffusion dynamics, *magnetisation fluctuations are non-Gaussian* and described by a nonlinear SDE. The analysis requires separation of slow and fast modes. [Landim]

- Several examples of *self-interacting random walks*, including the Lorentz gas and reinforced walks can be analyzed in high dimensions by new techniques. The results imply *diffusive behavior*. [Elboim]
- A unified framework of *localization schemes* reduces mixing time analysis of Markov chains to simpler problems, yielding new proofs and sharp bounds, including $O(n \log n)$ mixing for Glauber dynamics in the hardcore model. [Chen]
- A *superdiffusive central limit theorem* for the stochastic Burgers equation in dimension $d = 2$ is the first such scaling limit for a critical, singular SPDE. [Moulard]
- The *Simple Exclusion with Traps* exhibits a frozen and an ergodic phase. The transience time to either the frozen or ergodic component exhibits a *sharp cut-off phenomenon*. [Erignoux]
- A Glauber dynamic of the two-dimensional dimer model is shown to satisfy the *gradient condition* and to be diffusive on large scales. Its diffusion matrix is explicitly computed. [Giles]
- A new approach to bootstrap percolation is based on proving that the model is equivalent in a strong sense to its local version. This allows one to understand the so-called *bootstrap percolation paradox*. [Hartarsky]
- Large deviation theory for the zero-range process is connected to fluctuating hydrodynamics and macroscopic fluctuation theory. Mathematically, it leads to critical parabolic-hyperbolic PDEs. [Fehrman]
- Particles in the symmetric exclusion process on $\mathbb{Z}^d$ are highly correlated. Despite this, *extremal particles follow asymptotically a Gumbel distribution*, as would be the case for independen random variables. [Sethuraman]
- While classical results on random walks in random environment are given for almost-every realization of the enviromnent ω, a *deterministic* condition on ω is provided that guarantees the validity of LLN and CLT. [Biskup]
- The Glauber dynamic of the Sherrington-Kirkpatrick mean-field spin glass model satisfies a uniform (modified) log-Sobolev inequality for small but volume-independent inverse temperature. [Bodineau]
- The *stochastic heat flow (SHF)* is a non-trivial (and non-Gaussian) scaling limit of the directed polymer in random environment. The SHF turns out to be a so-called *black noise*. [Caravenna]
- Maximal displacement in *branching random walks* and branching Markov chains grows linearly with logarithmic corrections. [Gantert]
- The probability of anomalous deviations of the directed landscape can be estimated via an explicit large-deviation functional. [Virág]
- *Einstein relations* and scaling limits for reversible diffusions are derived for systems in various types of random environments. [Mathieu]
- Random walks in *Dirichlet environments* are connected to *edge-reinforced* and *vertex-reinforced* processes, allowing the analysis of invariant measures from the particle's point of view, particularly in dimension two. [Sabot]

- Random walks conditioned to stay above a *concave obstacle* exhibit fluctuations of order $n^{1/3}$ around the obstacle, generalizing known results for Ferrari-Spohn diffusions in the quadratic case. [Velenik]
- *Lace expansion*, traditionally used for high-dimensional critical models like weakly self-avoiding walk, are being extended to the discrete *Heisenberg group*. [Kozma]
- *Log-Sobolev inequalities* in mean-field particle systems are derived by leveraging the convexity of the free energy projected onto the mean. [Dagallier]
- The Brownian web distance on coalescing random walks is related to the *Brownian web*, and the *directed landscape*. Weighted versions yield height distributions interpolating between Gaussian and Tracy-Widom. [Vető]
- Refinement of a well known argument by Fröhlich-Spencer shows *logarithmic growth* of height variance in p-SOS models. [Ott]

Workshop: Large Scale Stochastic Dynamics

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Abstracts

Some recent progress in coagulation processes

Luisa Andreis
(joint work with Tejas Iyer, Wolfgang König, Heide Langhammer, Elena Magnanini, Robert I.A. Patterson)

Since Smoluchowski introduced his well-known coagulation equation in 1917, there has been an active line of research focused on understanding the properties of the solutions to this equation and related models for coagulation. In particular, in 2000, Norris introduced a generalised version of the model, which he named the cluster coagulation model [3]. This model was intended to extend the framework established by Smoluchowski, allowing particles to have additional properties beyond their mass, such as shape or spatial location.

In this talk we focus on some recent progress in the study of the particle system that converges to such limiting (spatial) coagulation equation, often called the Marcus-Lushnikov process. Here particles have a mass and a spatial location (in a general Polish space $\mathcal{S}$) and after independent exponential random times, pairs of particles merge into a single one, with their masses being summed. The location of the resulting new particle in $\mathcal{S}$ is chosen according to a certain kernel.

In particular, we present a recent sufficient criterion on the coagulation rate for the appearance of a giant particle (i.e. a particle whose mass is non-negligible with respect to the total mass of the system) in the spatial setting [1]. This improves existing criteria for the occurrence of the so called *gelation* phase transition in the spatially homogeneous framework as well, proving in particular the longstanding conjecture that homogeneous kernels with degree $\gamma > 1$ are indeed gelling (as long as they do not vanish on the diagonal).

Additionally, we present a first approach to study large deviations of the trajectory of such a Markov process in the large volume limit [2]. Borrowing techniques from statistical mechanics, we express the distribution of particles in terms of a reference Poisson point process and a pairwise interaction term. Based on this formula, we derive a (conditional) large-deviation principle for the joint distribution of the particles, with an explicit identification of the rate function. We characterize its minimizer(s) through a variational problem. Finally, we prove that, in certain cases (specifically in the absence of *gelation*), these minimizers indeed solve the spatial version of the Smoluchowski coagulation equation.

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De-randomized conductance models

Marek Biskup

A conductance model on $\mathbb{Z}^d$ is an assignment of a number $\mathsf{c}(e) \in (0,\infty)$ to each nearest-neighbor edge e of $\mathbb{Z}^d$. Given such an assignment, which we will label as ω, we define a discrete-time Markov chain $X = \{X_n\}_{n\geq 0}$ on $\mathbb{Z}^d$ by prescribing its transition probabilities as

$$\mathsf{P}_\omega(x,y) := \frac{\mathsf{c}_\omega(x,y)}{\pi_\omega(x)} \quad \text{for} \quad \pi_\omega(x) := \sum_{z\colon (x,z)\in \mathsf{E}(\mathbb{Z}^d)} \mathsf{c}_\omega(x,z),$$

where $\mathsf{c}_\omega(x,y)$ stands for the conductance of edge (x,y) in assignment ω. The symmetry condition $\mathsf{c}_\omega(x,y) = \mathsf{c}_\omega(y,x)$ implies that π_ω is a reversible measure.

The Markov chain X generalizes the ordinary simple symmetric random walk (which corresponds to ω with $\mathsf{c}_\omega(\cdot) = 1$). The prime question of interest is for what ω the chain X exhibits the "usual" behavior; i.e., obeys the Law of Large Numbers, the CLT, an Invariance Principle, the Law of the Iterated Logarithm.

The stated question is answered affirmatively for all periodic ω by representing X as a function of a finite-state Markov chain. Another class of conductance models for which an affirmative answer exists are ω's sampled from translation-invariant ergodic laws. This relies on technical tools from *stochastic homogenization*: first, the corrector method brought to the subject area by Kipnis and Varadhan [3] and, second, heat-kernel estimates and/or elliptic regularity theory brought in by Sidoravicius and Sznitman [4] for uniformly elliptic ω and by Andres, Deuschel and Slowik [1] under suitable moment conditions. However, the effective output is a full-measure set of conductances for which the above holds, with the null set implicit and impossible to control. In particular, barring the periodic cases, the method offers no way to decide whether an Invariance Principle holds for any particular ω of interest.

The work [2] reported here resolves this by identifying a large deterministic set of conductances for which the desired conclusions can be proved without reliance on the Spatial Ergodic Theorem and/or L^2-limits under environment law that are the main sources of implicit null sets in the stochastic-homogenization approach. We will present it under the simplifying assumption of uniform ellipticity; i.e., when ω belongs to the set $\Omega := [a,b]^{\mathsf{E}(\mathbb{Z}^d)}$ for some $0 < a \leq b < \infty$. Note that Ω, endowed with product topology, is a compact metrizable space. We will write $C(\Omega)$ for the space of continuous functions on Ω, use $\tau_x\colon \Omega \to \Omega$ to denote the shift by x and set $\Lambda_r := [-r,r]^d \cap \mathbb{Z}^d$. Two definitions are needed:

Definition 1. *We say that $\omega \in \Omega$ is* averaging *if for all $f \in C(\Omega)$,*

$$\ell_\omega(f) := \lim_{r\to\infty} \frac{1}{|\Lambda_r|} \sum_{x\in\Lambda_r} f \circ \tau_x(\omega) \quad \text{exists.}$$

For ω avearging, $f \mapsto \ell_\omega(f)$ is a positivity-preserving continuous linear functional on $C(\Omega)$ with $\ell_\omega(1) = 1$. Since Ω is compact, the Riesz Representation Theorem yields existence of a unique probability measure $\mathbb{P}_\omega$ on Ω such that

$\ell_\omega(f) = \int f(\omega')\mathbb{P}_\omega(\mathrm{d}\omega')$ for all $f \in C(\Omega)$. As is also checked from the definition, $\mathbb{P}_\omega$ is translation invariant, i.e., $\mathbb{P}_\omega \circ \tau_x^{-1} = \mathbb{P}_\omega$ for each $x \in \mathbb{Z}^d$.

Definition 2. *An averaging $\omega \in \Omega$ is said to be* ergodic *if $\mathbb{P}_\omega$ is (jointly) ergodic with respect to the translations of $\mathbb{Z}^d$.*

This notion is introduced to exclude the situations when samples from $\mathbb{P}_\omega$ are not representative of ω as happens for instance when ω takes value 1 in one half-space and value 2 in the other half-space. Denote

$$\Omega^\star := \bigl\{\omega \in \Omega\colon\ \text{averaging} \ \wedge\ \text{ergodic}\bigr\}.$$

The main conclusion of [2] is then summarized in:

Theorem 3. *Let $\omega \in \Omega^\star$. Then*

(1) *the Weak Law of Large Numbers holds, i.e.,*

$$\frac{X_n}{n} \underset{n\to\infty}{\longrightarrow} 0$$

in probability, and

(2) *an Invariance Principle holds, i.e.,*

$$\Bigl\{\frac{1}{\sqrt{n}}X_{\lfloor tn\rfloor}\colon t \ge 0\Bigr\} \underset{n\to\infty}{\overset{\text{law}}{\longrightarrow}} \text{Brownian motion}$$

Here, in both cases, X is started from the origin in $\mathbb{Z}^d$.

A short version of the statement is that X behaves as "usual" whenever ω satisfies the conclusion of the Spatial Ergodic Theorem. Here we note that $\Omega^\star$ is translation invariant and full-measure under any ergodic law on Ω. The above result thus subsumes the stochastic-homogenization approach and is even stronger as it is unaffected by zero-density perturbations of ω.

The proofs of Theorem 3 run parallel to the stochastic setting except that all stochastic averaging must be built out of spatial averaging. A starting point is the formulation of the "point of view of the particle" which allows representing averages of functions along the sequence of environments seen by X by way of a stochastic averaging with respect to $\mathbb{P}_\omega$. A key input here are the heat-kernel estimates that apply throughout Ω. The corrector is not introduced as it likely does not exist in the desired generality; instead, we work with suitable approximations.

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Fluctuations in the weakly asymmetric facilitated exclusion process

Oriane Blondel

(joint work with Guillaume Barraquand, Marielle Simon)

We are interested in the facilitated exclusion process on $\mathbb{Z}$ with asymmetry rates (p, q). This process has transitions $\bullet\bullet\circ \to \bullet\circ\bullet$ at rate p and $\circ\bullet\bullet \to \bullet\circ\bullet$ at rate q. It is known that [4], in finite volume, depending on the initial number of particles, the system ends up in one of the two following situations after a random transition time:

(1) a state where all particles are isolated, which is frozen under the dynamics;
(2) a state dubbed "ergodic" where all empty sites are surrounded by particles.

In the latter case the term ergodic comes from the fact that (except for limit cases) all such states with fixed number of particles are connected by the dynamics. Also in the ergodic situation, it is possible to view the dynamics as a simple exclusion process by pairing each empty site with the particle on its right. The pairs $\circ\bullet$ and non-paired particles then follow an simple exclusion process dynamics.

The hydrodynamic limit of this process is known [5, 7] to be given by

(1) $\partial_t \rho = \Delta\left(\frac{2\rho-1}{\rho}\mathbf{1}_{\rho>1/2}\right)$ in the diffusive time scale if $p = q$;
(2) $\partial_t \rho + (2p-1)\partial_x\left(\frac{(1-\rho)(2\rho-1)}{\rho}\mathbf{1}_{\rho>1/2}\right) = 0$ in the hyperbolic time scale if $p \neq q$.

We are interested in fluctuations around the rightmost particle in the case of a step initial configuration $\cdots\bullet\bullet\bullet\circ\circ\circ\cdots$. As in [1], we map the corresponding process to an open ASEP, with injections at rate p and no removal at the origin. In the weakly asymmetric case $p = \frac{1}{2}e^{\varepsilon}, q = \frac{1}{2}e^{-\varepsilon}$, this can be studied through the microscopic Cole-Hopf transform $(\mathcal{Z}_t(x))_{t,x}$ and the techniques of Bertini-Giacomin [3]. Previous works with a similar strategy include

(1) [3] considers the WASEP on the line and shows convergence of the rescaled transform $\mathcal{Z}^{\varepsilon}_t(u) := Z_{\varepsilon^{-4}t}(\varepsilon^{-2}u)$ to the multiplicative stochastic heat equation (SHE)
$$\partial_t \mathcal{Z} = \frac{1}{2}\Delta \mathcal{Z} + \mathcal{Z}\xi.$$
(2) [6], with a special choice of reservoir dynamics that puts the system close to the triple point of the phase diagram for the ASEP current; the limit equation is the SHE with Neumann boundary condition.
(3) [8], with a similar choice of reservoir dynamics as [6], but – contrary to previous works – allowing for empty initial condition, shows convergence of $\varepsilon^{-1}\mathcal{Z}^{\varepsilon}$ to the SHE with Neumann boundary condition and δ_0 initial condition.

The main result in [2] is the following. For the weakly asymmetric open ASEP on $\mathbb{Z}_+$ with $(\frac{1}{2}e^{\varepsilon}, 0)$ reservoir dynamics at the origin, $\varepsilon^{-2}\mathcal{Z}^{\varepsilon}$ converges to the solution of the SHE with Dirichlet boundary condition and initial condition $-2\delta_0'$.

A major difficulty is handling the singularity at time 0 of the heat kernel on $\mathbb{R}_+$ with Dirichlet boundary condition..

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Functional inequalities for microscopic dynamics with random interactions

Thierry Bodineau

(joint work with Roland Bauerschmidt, Benoit Dagallier)

In this talk, we review a method to derive functional inequalities by decomposing a Gibbs measure into simpler measures. This strategy doesn't rely on detailed features of the interaction, but only on the spectral structure of the two-body interaction matrix. Thus it is well suited to study the Glauber dynamics of models with random interactions. We illustrate this approach in the case of the Sherrington-Kirkpatrick (SK) model, but it has been also applied to different dynamics including the Kawasaki dynamics on regular random graphs [3] and mean field models with diluted interactions [5]. We refer to [4] for a survey of the general method and its links with the renormalisation group.

Let Λ be a finite set and $(M_{xy})_{x,y\in\Lambda}$ be a symmetric matrix. We consider a Gibbs measure of the form :

$$\nu(d\sigma) = \frac{1}{Z} e^{-\frac{1}{2}(\sigma, M\sigma)} \prod_{x\in\Lambda} \mu(d\sigma_x), \quad (\sigma, M\sigma) = \sum_{x,y\in\Lambda} M_{xy}\, \sigma_x \cdot \sigma_y, \tag{1}$$

For different choices of the interaction matrix, we can recover :

- The Curie-Weiss model : $M_{xy} = -\frac{\beta}{N}$ for all $x, y \leq N$
- The SK model : $M = \beta H$ with $\beta > 0$ and H a $N \times N$ GOE matrix consisting of independent Gaussian entries with variance $1/N$ above the diagonal.
- The Ising model on a graph : for M the adjacency matrix of the graph

We consider the Glauber dynamics with generator

$$LF(\sigma) = \sum_{x\in\Lambda} \frac{1}{2} c(\sigma, \sigma^x)\big(F(\sigma^x) - F(\sigma)\big)$$

and jump rates given by the heat bath dynamics

$$c(x,\sigma) = \frac{\nu(\sigma^x)}{\nu(\sigma^x) + \nu(\sigma)}. \tag{2}$$

The corresponding Dirichlet form reads

$$D_\nu(F,G) = \sum_{\substack{x\in\Lambda\\ \sigma}} \frac{\nu(\sigma)}{2} c(\sigma,\sigma^x)\big(F(\sigma^x) - F(\sigma)\big)\big(G(\sigma^x) - G(\sigma)\big).$$

We say that the measure ν satisfies a *modified Log-Sobolev inequality* with constant $\lambda > 0$ if for any test function $F \geq 0$

$$\mathrm{Ent}_\nu F = \mathbb{E}_\nu\big(\Phi(F)\big) - \Phi(\mathbb{E}_\nu F) \leq \frac{2}{\lambda} D_\nu(F, \log F) \quad \text{with} \quad \Phi(x) = x\log x. \tag{3}$$

Theorem 1. *Let $\langle M\rangle$ be the difference between the largest and the smallest eigenvalue of M and assume that $\langle M\rangle < 1$. $\exists\gamma$, such that ν satisfies a modified Log-Sobolev inequality uniformly with respect to the set Λ:*

$$\mathrm{Ent}_\nu(F) \leq \frac{2}{\gamma}\left(\frac{1+\langle M\rangle}{1-\langle M\rangle}\right) D_\nu(F,\log F). \tag{4}$$

As a consequence of this theorem one can show :

Corollary 2. *Let $\Gamma_N(M)$ be the modified Log-Sobolev constant associated with the quenched SK measure on N sites with coupling matrix $M = \beta H$. Then the SK model with $\beta < 1/4$ satisfies a uniform LSI in the following sense: there is $c_\beta < \infty$ such that*

$$\lim_{N\to\infty} \mathbb{P}_N\big(\Gamma_N(M) > c_\beta\big) = 1, \tag{5}$$

where $\mathbb{P}_N$ stands for the GOE distribution of the coupling matrix.

Theorem 1 was derived in [2] for dynamics with jump rates of the form

$$c(x,\sigma) = \frac{1}{2}\left(1 + \frac{\nu(\sigma)}{\nu(\sigma^x)}\right).$$

For these rates, a Log-Sobolev inequality was established in [2]. It was noticed in [7] that the heat bath jump rates (2) were more relevant when considering the SK model and a spectral gap inequality was obtained in [7] under the same condition $\beta < 1/4$. The approach of [7] follows the stochastic localisation scheme and we refer to [6] for a survey of this method. Recently, the threshold 1/4 in Corollary 2 has been improved to 0.295 in [1].

As explained in [4, Inequality (6.86)], a minor modification of the proof of [2] leads to the derivation of the modified Log-Sobolev inequality as stated in Theorem 1 and Corollary 2 for the Glauber dynamics with the heat bath rates (2).

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2D directed polymers and stochastic heat flow

Francesco Caravenna

(joint work with Quentin Berger, Anna Donadini, Rongfeng Sun, Nicola Turchi, Nikos Zygouras)

We consider *directed polymers in random environment*, a key model in disordered systems which describes a random walk on $\mathbb{Z}^d$ interacting via a Gibbs measure with a space-time random environment composed by i.i.d. random variables. We focus on the partition function Z_N^β of the model, where N denotes the system size and $\beta > 0$ is the inverse temperature (or coupling constant).

It was shown in the seminal paper [4] by Erwin Bolthausen that Z_N^β is a positive martingale, hence it converges a.s. to a limit Z_∞^β. A *phase transition* is observed, namely there exists $\beta_c = \beta_c^{(d)} \geq 0$ such that the following holds: for $\beta \leq \beta_c$ one has $Z_\infty^\beta > 0$ a.s. (*weak disorder*) and the behavior of the polymer is diffusive, similar to the unperturbed random walk; for $\beta > \beta_c$ one has $Z_\infty^\beta = 0$ a.s. (*strong disorder*) and the behavior of the polymer is localised and conjecturally super-diffusive, very different from the unperturbed random walk.

The critical value $\beta_c = \beta_c^{(d)}$ is strictly positive in high space dimensions $d \geq 3$, while in low dimensions $d = 1, 2$ one has $\beta_c = 0$, that is any $\beta > 0$ radically changes the behavior of the random walk. Since the partition function vanishes $Z_N^\beta \to 0$, while $Z_N^{\beta=0} \equiv 1$, it is natural to look for an *intermediate disorder regime*: can one rescale $\beta = \beta_N \to 0$ as $N \to \infty$ in such a way that $Z_N^\beta \to \mathscr{Z} > 0$ where $\mathscr{Z}$ is a non-trivial *random* limit? For $d = 1$ a positive answer was given in [1] with $\beta_N = \hat{\beta}/N^{1/4}$, where $\mathscr{Z}$ is the solution of the *stochastic heat equation*. For $d = 2$ it was shown in [6] that the correct rescaling is $\beta = \hat{\beta}/\sqrt{\log N}$ and a *phase transition* emerges on this scale with critical value $\hat{\beta}_c = \sqrt{\pi}$: for $\hat{\beta} < \hat{\beta}_c$ the limit $\mathscr{Z}$ is a log-normal random variable, while for $\hat{\beta} \geq \hat{\beta}_c$ one has $\mathscr{Z} = 0$.

We focus henceforth on the critical dimension $d = 2$. Understanding what happens at the critical value $\hat{\beta} = \hat{\beta}_c = \sqrt{\pi}$ requires to consider the *space-time* *random field of partition functions* $Z_N^\beta(k,z)$ for random walks starting at time k from position $z \in \mathbb{Z}^2$. The diffusively rescaled field $u_N(t,x) := Z_N^\beta(N(1-t), \sqrt{N}x)$ solves a discretised version of the *stochastic heat equation*: however, such a SPDE is ill-defined in space dimension $d = 2$, so there was no known candidate process to which u_N could converge. It was shown in [8] that, in a whole critical window $\hat{\beta} = \hat{\beta}_c(1+\theta/\log N)$ with $\theta \in \mathbb{R}$, the diffusively rescaled partition functions $u_N(t,x)\,\mathrm{d}x$, converges to a universal process of random measures on $\mathbb{R}^2$, called the *critical 2D Stochastic Heat Flow (SHF)*.

Several properties of the SHF have been investigated. An axiomatic characterisation based on moments and independence was given in [16], building on a Chapman-Kolmogorov property from [10]. Convergence to the SHF of the regularised solution of the stochastic heat equation was also obtained in [16]. The second moment of the solution had been shown to converge in the seminal paper [3], while third and higher moments were determined in [7, 13]. We refer to [15, 12] for recent results about moment asymptotics. Singularity and regularity of the SHF as a random measure was investigated in [9].

Recent progress on the behavior of the SHF in the strong disorder limit $\theta \to \infty$ was established in [2]. It was shown that in this limit the SHF locally vanishes as a random measure on $\mathbb{R}^2$, with sharp explicit estimates on the mass of large balls. This was obtained by bounding truncated and fractional moments, exploiting refined change of measure arguments and coarse-graining techniques. A proof that the SHF locally vanishes as $\theta \to \infty$ (with no quantitative bounds) was independently obtained in [11], as a corollary of a *conditional GMC structure* enjoyed by the SHF on path space, established in the same paper.

Finally, we mention a noise sensitivity property for directed polymer partition functions recently proved in [5], which yields the *independence between SHF and white noise*. This also follows by the independent approach in [14], where the SHF was shown to be a so-called black noise.

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Localization schemes for mixing time analysis of Markov chains

Yuansi Chen
(joint work with Ronen Eldan)

We introduce the localization schemes framework [4] for analyzing the mixing time of Markov chains. At the heart of the framework is the concept of a localization scheme which, to every probability measure , assigns a martingale of probability measures which localize in space as time evolves. The use of localization schemes allows us to reduce the mixing time analysis on the original target distribution to that on many simpler transformed distributions. We demonstrate this framework via three examples. First, we show how coordinate-by-coordinate localization scheme gives an alternative interpretation of the spectral independence framework for mixing time analysis. Second, we apply the framework to derive the mixing time of a proximal sampling algorithm for sampling log-concave distributions, as well as the state-of-the-art mixing time analysis of the hit-and-run algorithm for sampling isotropic convex bodies. Finally, we discuss negative fields localization to obtain the first $O(n\log(n))$ mixing time bound of the Glauber dynamics for sampling the hardcore model in the tree-uniqueness regime.

Problem introduction. Suppose that we would like to sample from a measure ν on a set Ω. For the sake of discussion, suppose that either $\Omega = \{-1,1\}^n$ is the Boolean hypercube or $\Omega = \mathbb{R}^n$. A common type of sampling algorithms is to introduce a Markov chain whose stationary distribution is ν and which mixes rapidly. For example, on $\Omega = \{-1,1\}^n$, a widely used Markov chain for sampling from ν is the Glauber dynamics. At any state $x \in \mathrm{supp}(\nu)$, the Glauber dynamics chooses a uniformly random coordinate $i \in [n]$ and transitions to the next state according to the law ν conditioned on the event that all coordinates other than i are fixed. We say a Markov chain $(X_t)_{t\geq 0}$ with stationary measure ν mixes rapidly if for any error tolerance $\epsilon > 0$, there is a reasonably-small time $t(\epsilon)$ such that the total variation distance between the law of X_t and ν is smaller than ϵ for $t \geq t(\epsilon)$.

In recent years, two seemingly-unrelated new techniques were introduced to study Markov chain mixing through functional inequalities:

- The *spectral independence* notion was first introduced by Anari, Liu and Oveis Gharan [1] to develop a theoretical framework which reduces mixing time analysis to establishing spectral independence for measures on $\{-1, 1\}^n$. The proof of the main results there has close connections with the field of high-dimensional expanders. The *stochastic localization* technique, first introduced by the second author in [2], is the key ingredient used in the proofs of several functional inequalities, both in the continuous setting where $\Omega = \mathbb{R}^n$ and ν is a *log-concave* measure and in the setting of the Boolean hypercube. In particular, the technique gives the first sub-polynomial bounds, due to the first author ([3]), for the so-called Kannan-Lovász-Simonovits conjecture ([5]) and Bourgain's slicing problem (see [7]).

In this work, we unify and extend these two techniques towards a new framework which can be used to establish mixing time analysis in various settings. By walking through several examples of sampling problems via the proposed framework, we also point out the common principles underlying the aforementioned two techniques.

One main principle underlying both techniques is that concentration bounds on a measure can be deduced from bounds on the covariance structure of a certain family of measures which are transformations of the original measure. In the spectral independence framework, a sufficient condition for a spectral gap is the boundedness of the *pairwise influence matrices* [1] of restrictions of the measure. It is not hard to see that the pairwise influence matrices are related to covariance matrices. In the stochastic localization framework, a spectral gap is implied by the boundedness of the covariance matrix along a certain stochastic process which is associated with the measure.

Starting with a new point of view which shows that the above two reductions follow from the exact same argument, our work has three main contributions. First, we generalize the stochastic localization framework to introduce localization schemes, which also gives rise to a natural associated family of Markov chains. Second, using the localization schemes, we simplify the mixing time proofs that were introduced in the work related to the spectral and entropic independence, bypassing the need on the theory of high-dimensional expanders. Third, we not only provide self-contained and (arguably) simpler proofs for many previous mixing time analyses, but also establish new mixing time results.

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A criterion on the free energy for log-Sobolev inequalities in mean-field models

Benoit Dagallier
(joint work with Roland Bauerschmidt, Thierry Bodineau)

We consider N particles in $\mathbb{R}^d$ with mean field interaction as described by the following probability measure:

$$m_T^N(dx) = \frac{1}{Z_T^N} e^{-H_T^N(x)}\, dx,$$

where for $x \in (\mathbb{R}^d)^N$ the energy $H_T^N(x)$ is given by:

$$H_T^N(x) = \frac{1}{2NT} \sum_{i,j=1}^N W(x_i, x_j) + \sum_{i=1}^N V(x_i).$$

Above, $V : \mathbb{R}^d \to \mathbb{R}$ is a confinement potential, $W : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is an interaction term, with strength parametrised by the temperature $T > 0$. The constant Z_T^N is a normalisation making m_T^N a probability measure. The typical example we have in mind is the so-called Curie-Weiss model where $W(x_1, x_2) = -(x_1, x_2)$ $(x_1, x_2 \in \mathbb{R}^d)$, and $V(x_1) = \lambda|x_1|^4/4 - |x_1|^2/2$ $(x_1 \in \mathbb{R}^d, \lambda > 0)$.

The aim of the talk is to bound the speed of convergence of the following Langevin dynamics, known to converge to m_T^N in long time:

$$\mathrm{d}X_t^N = -\nabla H_T^N(X_t^N)\,\mathrm{dt} + \sqrt{2}\,\mathrm{d}B_t^N,$$

with $B_\cdot^N$ a standard Brownian motion in $(\mathbb{R}^d)^N$. This is done by bounding the log-Sobolev constant of the dynamics, that is the best constant $\gamma > 0$ such that, for all smooth compactly supported test functions $F : (\mathbb{R}^d)^N \to \mathbb{R}$:

$$\mathrm{Ent}_{m_T^N}(F^2) \le \frac{2}{\gamma} \mathbb{E}_{m_T^N}\big[|\nabla F|^2\big],$$

where $\mathrm{Ent}_{m_T^N}(F^2) = \mathbb{E}_{m_T^N}[F^2 \log(F^2)] - \mathbb{E}_{m_T^N}[F^2] \log \mathbb{E}_{m_T^N}[F^2]$.

In cases such as the Curie-Weiss model there is a phase transition: there is a temperature $T_c > 0$ above which γ is supposed to be bounded below uniformly in

N (and vanishing with N if $T < T_c$). T_c is defined in terms of the free energy/large deviation rate function of the model:

$$T_c = \inf \Big\{ T > 0 : \mathscr{F}_T \text{ has a unique minimiser} \Big\},$$

with $\mathscr{F}_T$ the real-valued functional acting on probability measures $\rho = \rho(x)\,dx$ on $\mathbb{R}^d$ according to:

$$\mathscr{F}_T(\rho) = \int \rho(x) \log \rho(x)\,dx + \int V(x)\,\rho(dx) - \frac{1}{2T}\Big(\int x\,\rho(dx) \Big)^2,$$

and $\mathscr{F}_T(\rho) = \infty$ if ρ is not absolutely continuous. Previous results such as [2] only prove uniformity in N of γ for temperatures $T \gg T_c$ (but have the advantage of applying in much more general settings than quadratic interactions).

We prove in [1] that the log-Sobolev constant γ of m_T^N is indeed bounded below uniformly in N for any $T > 0$ such that the free energy has the following convexity property:

$$\exists \lambda_T > 0, \forall m \in \mathbb{R}^d, \qquad \nabla^2 \hat{\mathscr{F}}_T(m) \geq \lambda_T\, \mathrm{id},$$

where $\hat{\mathscr{F}}_T : \mathbb{R}^d \to \mathbb{R}$ is the following projection of the free energy on the mean:

$$\hat{\mathscr{F}}_T(m) = \inf \Big\{ \mathscr{F}_T(\rho) : \int x\,\rho(dx) = m \Big\}, \qquad m \in \mathbb{R}^d.$$

The proof relies on a one-step renormalisation argument, which concretely corresponds to using the following Gaussian identity:

$$\exp\Big[\frac{1}{2NT} \Big| \sum_{i=1}^N x_i \Big|^2 \Big] \propto \int_{\mathbb{R}^d} \exp\Big[-\frac{N|\varphi|^2}{2T} + \frac{1}{T}\Big(\varphi, \sum_{i=1}^N x_i \Big) \Big]\,d\varphi.$$

This splits the measure m_T^N in two probability measures: an infinite-temperature part (i.e. a product measure) driven by an external field $\varphi \in \mathbb{R}^d$, and a low-dimensional part $\nu_r(d\varphi) \propto e^{-NV_T(\varphi)}\,d\varphi$, where the so-called renormalised potential V_T reads:

$$\begin{aligned} V_T(\varphi) &= \frac{|\varphi|^2}{2T} - \log \int_{\mathbb{R}^d} e^{-V(x_1) + (\varphi, x_1)/T}\,dx_1 \\ &= \frac{|\varphi|^2}{2T} + \inf_\rho \Big\{ \int \rho(x) \log \rho(x)\,dx + \int V(x)\,\rho(dx) - \frac{1}{T}\Big(\varphi, \int x\,\rho(dx) \Big) \Big\}. \end{aligned}$$

All information on phase transitions is encoded in V_T, in particular on its convexity. This convexity is equivalent to convexity of $\hat{\mathscr{F}}_T$ as the two can be shown to more or less be Legendre transforms of one another.

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Self-interacting walks in high dimensions

Dor Elboim
(joint work with Antoine Gloria, Felipe Hernandez, Gady Kozma, Allan Sly)

A self-interacting random walk is a random process evolving in an environment which depends on its history. In this talk, we will discuss a few examples of these walks including the Lorentz gas, the mirror walk, the once-reinforced walk and the cyclic walk in the interchange process. I will present a method to analyze these walks in high dimensions and prove that they behave diffusively.

Transience and mixing time for the FEP and the SSEP with traps

Clément Erignoux
(joint work with Brune Massoulié)

The Facilitated Exclusion Process (FEP) has attracted a lot of attention in recent years as a prototypical kynetically constrained lattice gas with hard constraint, which is both gradient and non-reversible w.r.t. Bernoulli product measures. It exhibits two distinct macroscopic phases, one ergodic and the other one frozen, depending on whether the density is super or subcritical, which are both reached after a transience time. In the supercritical regime, once the ergodic phase is reached, the FEP roughly behaves as the classical SSEP, and its mixing time in particular can be estimated. Starting from a general transient state, however, the transience time needs to be sharply estimated to retain access to the mixing time, which is not by any means straightforward because the FEP is not attractive.

For this reason, many results on the FEP so far have relied to various extent on mapping it to other processes [2, 3], in particular to the SSEP and a facilitated Zero-Range Process, which are both attractive. In that spirit, we introduced a new model, called SSEP with traps (SWT), whose transience time can be estimated as well as its mixing time, as a parent model of the SSEP. The SWT is defined on the ring of size K, where K represents the number of particles in the original FEP. Each site k of the SWT is either occupied by a particle ($\xi_k = 1$), empty ($\xi_k = 0$), or a trap of depth $a > 0$ ($\xi_k = -a$). Like in the SSEP, particles jump at rate 1 to neighboring sites not already containing a particle. Whenever a particle jumps towards a trap, it gets stuck there : the particle is destroyed, and the trap depth decreases by 1.

Total trap depth in the system and number of particles can both only decrease, and decrease by the same amount. In particular, the SWT remains is a transient state as long as both particles and traps are present in the system. Afterwards, either all particles have disappeared (subcritical case) and the system freezes, or all traps have disappeared (supercritical case) and we are left with a classical -ergodic- SSEP. Transient , ergodic and frozen SSEP are in direct correspondance with those of the FEP, so that to estimate the FEP's transience time it is enough to estimate the SWT's.

The big upside is that the SWT is attractive. In particular any configuration can be associated with a critical configuration that takes longer to leave the transient component. To estimate the SWT's worst transience time, we therefore estimate its critical transience time starting with configurations with a single trap, which can be coupled with boundary-driven SSEP. The general case can also be treated by delicate coupling arguments, and proves that the transience time for the SWT undergoes cutoff in a region of size $o(t^\star_K)$ around $t^\star_K := K^2 \log K/\pi^2$.

Estimating the transience time allows to estimate the SWT's mixing time as well, which is in most cases is not of the same order as the transience time inducing cutoff. These results can then be transfered to the FEP, both for the transience time and the mixing time [4], both significantly improving on previous results [2, 1].

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Large deviations of the zero range process and conservative SPDE

Benjamin Fehrman

(joint work with Benjamin Gess, Daniel Heydecker)

We will discuss the derivation of a full large deviations principle for the zero range particle process in finite and infinite volume, and its connections to macroscopic fluctuation theory and fluctuating hydrodynamics. In particular, we will explain how such questions lead to the analysis of certain parabolic-hyperbolic PDE in energy critical spaces, whose well-posedness is based on concepts of renormalized solutions and the equation's kinetic form. We will then introduce the complimentary theory of fluctuating hydrodynamics in the context of the zero range process, which is based on an approximating sequence of conservative stochastic PDEs. We will study their stochastic dynamics, including through the construction a random dynamical system and invariant measure, and make rigorous in this context the informal connection between fluctuating hydrodynamics and macroscopic fluctuation theory.

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Some questions on branching Markov chains

Nina Gantert

(joint work with Viktor Bezborodov, Alice Callegaro, Carlo Scali)

Branching random walks are systems of particles which produce offspring and also move in space. A simple model is the following: particles produce offspring according to a fixed offspring law μ, the offspring particles take i.i.d. steps in space according to a fixed probability measure ν on $\mathbb{R}^d$. Assume that the mean m of μ measure satisfies $1 < m < \infty$. This is a classical model, see [5, 4, 7, 10, 8], with a lot of recent interest, see [13] for a monograph on the topic. Replacing the branching random walk with a branching Brownian motion, there is considerable recent progress about the maximal distance $M_t^{(d)}$ of the particle cloud to the origin at time t. In the one-dimensional case, it is known, under additional assumptions, that $M_t^{(1)}$ grows linearly and the second term is logarithmic. There are refined results about the point process of particles seen from the right-most particle. In the multidimensional case, it was proved for branching Brownian motion in $\mathbb{R}^d$ that the maximal distance $M_t^{(d)}$ to the origin satisfies the following:

$$(M_t^{(d)} - m_t^{(d)})_{t\geq 0} \text{ is tight, where } m_t^{(d)} = \sqrt{2}t + \frac{d-4}{2\sqrt{2}} \ln t \,. \tag{1}$$

see [12]. Later, it was proved that $M_t^{(d)} - m_t^{(d)}$ converges in law to a random shift of a Gumbel law, see [9]. Building on [9, 14, 2] investigate the corresponding extremal point process and prove convergence to a randomly shifted decorated Poisson point process.

In a joint work with Viktor Bezborodov, see [3], we have generalized (1) to the case of radially symmetric step distributions ν. Our main results is the following. Define for $u \in \mathbb{R}$

$$\Phi(u) = m \cdot \mathbb{E}\Big[e^{u\langle X,\theta\rangle}\Big]$$

where X has law ν and $\theta \in \mathbb{S}^{d-1}$, where $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$. Let

$$\Psi(u) = \ln \Phi(u), \qquad \Psi'(u) = \frac{\Phi'(u)}{\Phi(u)}. \tag{2}$$

Note that since ν is radially symmetric, the functions $\Phi(u)$, $\Phi'(u)$, $\Psi(u)$, and $\Psi'(u)$ do not depend on the direction $\theta \in \mathbb{S}^{d-1}$. The following equation plays an important role:

$$u\Psi'(u) - \Psi(u) = 0. \tag{3}$$

We assume that there exists $\lambda \in (0, \infty)$ with $\Phi(\lambda) < \infty$ solving equation (3). Let

$$r_t := \frac{\Psi(\lambda)}{\lambda} t + \frac{d-4}{2\lambda} \ln t \,. \tag{4}$$

Then, under additional assumptions for which we refer to [3],

$$\sup_{t\geq 0} \mathbb{P}\big\{\big|M_t^{(d)} - r_t\big| \geq y \big| \mathcal{S}\big\} \to 0 \quad \text{as} \quad y \to \infty\,. \tag{5}$$

Here, $\mathcal{S}$ is the event that the branching process survives. In other words, conditioned on survival, the laws of $(M_t^{(d)} - r_t)$ are tight.

We also discuss branching Markov chains: here the offspring particles take independent steps according to a Markov chain. This includes the case where the Markov chain is a random walk in random environment. We mention some work in progress with Alice Callegaro and Carlo Scali where the Markov chain is a one-dimensional random walk in random environment. The results are similar to the results in [6] and [11], where the fixed offspring distribution is replaced with a random sequence. It turns out that the fluctuations of the environment dominate and the second term of $M_t^{(1)}$ is of order $\sqrt{t}$.

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Diffusivity of Glauber dynamics for dimers

Harry Giles
(joint work with Giuseppe Cannizzaro, Fabio Toninelli)

We consider Glauber dynamics on dimer configurations of the hexagonal lattice. To each configuration, there is a canonical association with a discrete stepwise height function $\hbar : \mathbb{Z}^2 \to \mathbb{Z}$. Therefore, the Glauber dynamics can also be seen as a model of discrete surface evolution in $2+1$ dimensions.

The dynamics that we consider were previously studied in [1, 2] and convergence was proven under diffusive scaling $N^{-1}\hbar(N^2t, Nx)$ to a hydrodynamic limit, given by the solution of the following non-linear PDE:

$$\partial_t h = \mu(\nabla h) \sum_{i,j=1}^{2} a_{ij}(\nabla h)\partial^2_{ij} h$$

with explicit coefficients $a \in \mathbb{R}^{2\times 2}, \mu \in \mathbb{R}$ that are given in terms of trigonometric functions of ∇h.

Naturally, one would like to determine the order and nature of fluctuations around the hydrodynamic limit. Linearising the equation leads one to conjecture that the fluctuations ought to be described by the additive stochastic heat equation. In detail, if the dimer configurations are initialised according to the Gibbs stationary state π_ρ, under which $\mathbb{E}[\nabla\hbar] = \rho$, then the lower order fluctuations $\hbar(N^2t, Nx) - \mathbb{E}[\hbar(N^2t, Nx)]$ should converge to the solution of

$$\partial_t h = \mu(\rho)\nabla \cdot a(\rho)\nabla\psi + \sqrt{2\mu}\xi$$

in which $\xi : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ is a space-time white noise. In particular, the model lies in the Edwards–Wilkinson (EW) class.

We present a Green-Kubo formula for the diffusivity of the model, defined according to

$$D_{ij}(t) = \frac{1}{t}\sum_{x\in\mathbb{Z}^2} x_i x_j S_t(x)$$

in which $S_t(x)$ is the two-point kernel $S_t(x) \sim \mathbb{E}[\nabla\hbar_t(x) \cdot a(\rho)\nabla\hbar_0(0)]$ after suitable recentering and renormalisation. The Green-Kubo formula yields an explicit expression for the diffusivity, $D(t) = 2\mu a$, which confirms that the fluctuations are indeed in the EW class.

One of the many challenges of the model are the long range correlations present in the Gibbs state π_ρ, in which two point correlations decay like $|x|^{-2}$. In particular, they are not summable, and the definition of $D(t)$ as above is put into question. We overcome this problem, and others, by utilising the congested nature of dimers under the stationary state. The explicit formulas are a consequence of the fact that the dynamics satisfy a type of gradient condition.

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Locality in bootstrap percolation

Ivailo Hartarsky
(joint work with Christian Maura, Augusto Teixeira)

Bootstrap percolation is a class of cellular automata which may be viewed as models of nucleation and metastability. They also present deep connections to the the low-temperature stochastic Ising model, kinetically constrained models and others. In Froböse bootstrap percolation, iteratively, any vertex of the square lattice $\mathbb{Z}^2$ that is the only healthy vertex of 4-cycle becomes infected and infections never heal.

In [1], we prove that if vertices are initially infected independently with probability $p \to 0$, then with high probability the infection time of the origin τ is

$$\exp\left(\frac{\pi^2}{6p} - \frac{\pi\sqrt{2+\sqrt{2}}}{\sqrt{p}} + \frac{O(\log^2(1/p))}{\sqrt[3]{p}}\right).$$

We achieve this by proposing a new paradigmatic view on bootstrap percolation based on locality. Namely, we show that studying the Froböse model is equivalent in an extremely strong sense to studying its local version. That is, the infection time τ_{loc} of the local Froböse model satisfies

$$1 \le \frac{\tau_{\mathrm{loc}}}{\tau} \le \exp\left(\log^{19}(1/p)\right)$$

with high probability as $p \to 0$. In the local model, growth occurs starting from a single location, rather than everywhere in parallel. This greatly simplifies its study, since it reduces to an explicit finite range Markov chain on $\mathbb{N}^2 \times S$ for a suitable finite space S encoding the size and frame state of a growing rectangle.

The locality viewpoint is also useful for understanding the so-called bootstrap percolation paradox regarding the systematic discrepancies between rigorous results and numerical predictions on the asymptotics of τ and related quantities. Indeed, in [2], we propose and implement an exact (deterministic) algorithm which exponentially outperforms previous Monte Carlo approaches. It computes the probability that the framed rectangle Markov chain reaches states corresponding to a critical droplet (suitably large rectangle fully infected only using infections contained in it). Using the resulting data for extremely large systems (much larger than any physically meaningful size) allows us to clearly showcase and quantify the slow convergence proved rigorously, as well as to explain the previous disagreements between theory and numerics. The treatment applies identically to the more classical two-neighbour model on the plane.

We also present work in progress on the modified two-neighbour model. In this model, a vertex becomes infected when it belongs to a 4-cycle of $\mathbb{Z}^2$ with two non-adjacent infections. This model presents several additional challenges, particularly because its locality ratio τ_{loc}/τ is of order $1/\sqrt{p}$, much larger than the Froböse case. This much weaker locality is visible even in direct Monte Carlo simulations. Thanks to the development of robust a priori bounds, we are able to prove the above quantitative locality and then, thanks to the local model, we obtain

$$\tau = \exp\left(\frac{\pi^2}{6p} - \frac{\sqrt{2+\sqrt{2}}}{2\sqrt{p}} - \frac{\Theta(1)}{\sqrt{p}}\right).$$

This model is the first step towards generalising the locality approach to other bootstrap percolation models in the critical universality class.

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Lace expansion on the Heisenberg group

Gady Kozma

Lace expansion is a technique for understanding critical models in high enough dimension. Started with the seminal paper [3] handling weakly self-avoiding walk and with important contributions from [4] and [5], it had evolved to cover many more models and give much more precise results.

In the talk we described an approach to lace expansion developped jointly with Erwin Bolthausen and Remco van der Hofstad (and based, originally, on [1]). We focused on the analytic core of lace expansion, which is the new part in our approach. This approach was used in [2] to give a new proof for the classic result of weakly self-avoiding walk in 5 or more dimensions. Finally, we mentioned ongoing work to use this approach for the Heisenberg group (or, to be more precise, versions of the discrete Heisenberg group with at least 5 dimensional volume growth).

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Critical dynamical fluctuations in reaction-diffusion processes

Claudio Landim
(joint work with Benoit Dagallier)

We consider a one-dimensional microscopic reaction-diffusion process obtained as a superposition of a Glauber and a Kawasaki dynamics. The reaction term is tuned so that a dynamical phase transition occurs in the model as a suitable parameter is varied. We study dynamical fluctuations of the density field at the critical point.

We characterise the slowdown of the dynamics at criticality, and prove that this slowdown is induced by a single observable, the global density (or magnetisation). We show that magnetisation fluctuations are non-Gaussian and characterise their limit as the solution of a non-linear SDE. We prove, furthermore, that other observables remain fast: the density field acting on the fast modes (i.e. on mean-0 test functions) and with Gaussian scaling converges, in the sense of finite dimensional distributions, to a Gaussian field with space-time covariance that we compute explicitly.

The proof relies on a decoupling of slow and fast modes relying in particular on a relative entropy argument. Major technical difficulties include the fact that local equilibrium does not hold due to the non-linearity, and proving replacement estimates on diverging time intervals due to critical slowdown.

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From the lifted TASEP to true self-avoiding walks

Brune Massoulié
(joint work with Clement Erignoux, Werner Krauth, Francois Simenhaus, Cristina Toninelli)

The lifted TASEP is a variant of the totally asymmetric exclusion process where at each time-step, instead of trying to move forward a uniformly chosen particle, we try to move forward a particle marked by a pointer, which then may pass the pointer to another particle. We establish connections from this system to true self-avoiding walks (TSAW) and deduce some timescales of the dynamics [7, 4].

This model was introduced by physicists [5] as a toy model for non-reversible event chain Monte Carlo algorithms, which are expected to reach stationarity faster than reversible dynamics. The lifted TASEP, as well as many non-reversible Markov processes used in Monte Carlo [3], are liftings of Markov chains. There are theoretical guarantees over how much a non-reversible lifting can reduce the mixing time [1] (time to be close to stationarity) or the relaxation time [3] compared to the initial Markov chain. It is therefore important to understand if the lifted TASEP accelerates compared to simple exclusion and how far this is optimal.

Furthermore, it was observed numerically by [6] that at stationarity, the pointer's motion seems to have the same super-diffusive scaling and the same density as the true self-repelling motion (TSRM), the latter having been computed in [2].

We first give some heuristics on the system's behaviour, based on the observation that the pointer's motion depends on the particle density around it [7].

We then study a slight modification of the model on $\mathbb{Z}$ and make two connections with true self-avoiding walks:

- Started from a "step" initial configuration, meaning all particles in the left half and all empty sites in the right half, the pointer performs a zero temperature version of the TSAW with directed edges (introduced in [8]).
- In general, the particle system coincides exactly with the toy model for the TSRM that was introduced in [9].

The first observation allows to understand well the system started from the step: the explored zone grows in a diffusive way, and the law of the configuration in the explored zone is very close to stationarity.

With the second observation, we can use the "maze" representation of [9] to study the model. This explains the super-diffusive scaling of the pointer's motion at stationarity as well as the TSRM density observed in the limit by [6].

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Einstein relations and scaling limits for reversible diffusions in a random environment

Pierre Mathieu

(joint work with Alessandra Faggionato, Nina Gantert, Quentin Ghibaudo, Andrei Piatnitski)

The derivation of the (badly) so-called *Einstein* relation expressing the identity of the effective diffusivity and the (derivative of the) effective velocity under a small forcing for processes in a random environment is a nice, largely open, question that both illustrates the *fluctuation-dissipation theorem* [6] and extends the *linear response theory* of Ruelle [1] to highly non-hyperbolic models. It, so-far, resisted different attempts based on functional analytic tools e.g. [5] until we solved it with a proof based on a combination of PDE estimates and probabilistic tools, in the case of environments with a finite range of dependence [3]. Indeed our proof uses the decomposition of the trajectory of a reversible diffusion in a random environment with finite range of dependence into i.i.d. pieces along a sequence of regeneration times [9], [10]. A by-product of [3] is a complete description of the, diffusive and ballistic, scaling limits of the position of the diffusion in all the regimes where time goes to infinity and the forcing vanishes. (The diffusive regimes had already been derived from homogenization results in [7] for quite general models with a stationary ergodic environment.) A similar strategy may also be applied to analyse other Fourier modes when the forcing is periodic in time, thus yielding so-called *Nyquist relations* [4], [2]. It can also be extended to other observables than the position of the diffusion, to get a full fluctuation-dissipation theorem [8].

In the talk, I gave an introduction to this topic and insisted on the need to formulate alternative proofs to get a better understanding of the exact scope of the Einstein relation.

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The Critical Stochastic Burgers Equation

Quentin Moulard
(joint work with Giuseppe Cannizzaro, Fabio Toninelli)

The Stochastic Burgers Equation (SBE) was introduced by van Beijeren, Kutner, and Spohn [1] in the 1980s as a mesoscopic model for driven diffusive systems, such as ASEP. In the subcritical dimension $d = 1$, it coincides with the derivative of the KPZ equation whose large-scale behaviour is polynomially superdiffusive and governed by the KPZ Fixed Point. By contrast, in the supercritical dimensions $d \geq 3$, it was recently shown to be diffusive and to rescale to an anisotropic stochastic heat equation. At the critical dimension $d = 2$, the SBE was conjectured to be logarithmically superdiffusive with a precise exponent, but this had only been established up to lower-order corrections. In recent joint work with G. Cannizzaro and F. Toninelli [2], we derive the precise large-time asymptotics of the diffusivity in $d = 2$ and show that, after rescaling with the logarithmic corrections, the fluctuations satisfy a central limit theorem. This establishes the first superdiffusive scaling limit result for a critical SPDE, beyond the weak-coupling regime.

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Mixing times for the open ASEP

Dominik Schmid
(joint work with Patrik Ferrari)

The open asymmetric simple exclusion process (open ASEP) is among the best studied examples of an interacting particle system. It can intuitively be described as follows. Consider a segment of length $N \in \mathbb{N}$ and bias parameter $q \in [0, 1)$. Each site of the segment is either occupied by a particle, or left vacant. The particles perform independent random walks with jumps to the right at rate 1, and to the left at rate q. However, a jump is performed if and only if the target is a vacant site. This exclusion rule ensures that each site is occupied by at most one particle at a time. In addition, for some fixed $\alpha, \beta, \gamma, \delta \geq 0$, we let particles enter at site 1 at rate α, exit at site N at rate β, exit at site 1 at rate γ, and enter at site N at rate δ, respectively, subject to the exclusion constraint.

Depending on the choice of boundary parameters, it is well-known that the open ASEP can be partitioned into three phases: The high density phase, the low density phase, and the maximal current phase. In joint work with Ferrari, we study the speed of convergence of the open ASEP towards the stationary measure in terms of total-variation mixing times. We focus on the triple point, where all three phases meet and the stationary distribution is uniform on the state space, and show that the mixing time is of order $N^{3/2}$. This extends earlier work [4, 5] for mixing times of the open TASEP, where $q = \gamma = \delta = 0$, which crucially relied on an alternative representation of the open TASEP as a corner growth model.

For the proof of mixing times at the triple point, we follow the overall strategy from Gantert et al. in [2] for mixing times of the open ASEP in the high and low density phase. We investigate the basic coupling between two open ASEPs, started from the extremal configurations, where all sites are either fully occupied by particles or are left empty, respectively. In particular, the corresponding disagreement process starts from all sites on $\{1, \dots, N\}$ being initially occupied by second class particles. We now obtain an upper bound on the mixing by providing estimates on the time it takes for all second class particles to exit the segment. To this end, we use arguments from [2] to bound the exit time via effective bounds on the expected current and the current fluctuations for the open ASEP. In the work [1], we achieve effective bounds on the expected current by a detailed analysis of a classical formula by Uchiyama, Sasamoto, Wadati from [6]. For the most delicate part of the argument, i.e., controlling the fluctuations of the current of the open ASEP, we introduce a novel multi-species coupling argument. This allows us to compare current fluctuations between the open ASEP and the ASEP on the integers. For the latter, precise moderate deviations were recently established by Landon and Sosoe [3].

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Some new results about the roughening transition

Sébastien Ott

(joint work with Hugo Duminil-Copin, Gady Kozma, Florian Schweiger)

We revisit the famous 1981 paper of Fröhlich and Spencer, [4], on the roughening transition in a multiple-steps project. The roughening transition is a conjectured change of behaviour of the interfaces in the 3D Ising model: going from $O(1)$ fluctuations for $T < T_R$ to unbounded variance of height difference for $T \in [T_R, T_c]$, where T_c denotes the critical temperature of the system. The bound $T_R > 0$ was proven in the seminal work of Dobrushin, [3]. The bound $T_R < T_c$ is only proven to occur in effective models of interfaces: integer-valued height functions on $\mathbb{Z}^2$. Whilst new approaches to the roughening transition (for height functions) arose in the past few years, see for example [1, 2], or [5], they impose severe limitations on the models they apply to.

The project is to revisit and extend the FS argument, initially designed for nearest-neighbour height function models on $\mathbb{Z}^2$ to a larger class of models (long range, many-body interactions), allowing eventually to go beyond effective models. Our main motivation is to be able to handle the following modified Ising model:

$$H(\sigma) = -\sum_x (\sigma_x\sigma_{x+e_1} + \sigma_x\sigma_{x+e_2} + J\sigma_x\sigma_{x+e_3}).$$

As $J \to \infty$, the interface of this model under Dobrushin boundary conditions converges to the SOS model, whilst for $J = 1$, it is the standard Ising model.

The first step of our program was recently posted on the arXiv, [6]. There, we clean up and optimize the presentation of the FS argument in the case of the nearest-neighbour p-SOS model with $p \in (0, 2]$, obtaining that, at sufficiently low temperature, the variance of the spin in the middle of a box of size n scales at least like $\ln(n)$, whatever the boundary conditions. In a second step, we will remove the condition of being nearest-neighbour/finite range, and allow for interactions with polynomially decaying couplings. In a third step, we will extend the method to handle many-body interactions, allowing in particular to prove the roughening transition in the modification of the Ising model mentioned above with J large enough.

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On minimizing curves in a Brownian potential

Felix Otto
(joint work with Matteo Palmieri, Christian Wagner)

Given independent Gaussians $\{\xi(z)\}_z$ on the lattice $\mathbb{Z}^d$, the random field Ising model assigns to a configuration $\sigma : \mathbb{Z}^d \to \{0,1\}$ the energy

$$E(\sigma) = P(\sigma) - \epsilon F(\sigma) := \sum_{\text{edges}(z,z')} |\sigma(z) - \sigma(z')| - \epsilon \sum_{\text{vertices } z} \xi(z)\sigma(z),$$

where the parameter ϵ amounts to the disorder strength. Although the Ising model, which is recovered at $\epsilon = 0$, admits the two local minimizers[1] $\sigma_* \equiv 0$ and $\sigma_* \equiv 1$, the following uniqueness result proven by Aizenman and Wehr holds at any non-trivial disorder.

Theorem 1 (see [1]). *If $\epsilon \neq 0$, then almost surely there exists a unique local minimizer if and only if $d \leq 2$.*

The critical role of dimension $d = 2$ can be seen by the following heuristic: When changing the spins inside a box Λ, the changes in perimeter P and field energy F are proportional to $\#\partial\Lambda$ and $(\#\Lambda)^{\frac{1}{2}}$, which exhibit the same scaling behavior on large scales precisely in dimension $d = 2$.

Assuming now $d = 2$, Ding and Wirth recently gave upper and lower bounds on the correlation length in the regime $0 < \epsilon \ll 1$.

Theorem 2 (see [3]). *Let σ_* be the minimizer of E with the constraint $\sigma(z) = 0$ for $z \notin [-L, L]^2$. Then $\mathbb{E}\sigma_*(0) \geq \frac{1}{3}$ if and only if $\ln L \gtrsim \epsilon^{-\frac{4}{3}}$.*

Prior to that Leighton and Shor studied a variant of the problem in form of an isoperimetric problem: maximize

$$\frac{F(\Sigma)}{P(\Sigma)} := \frac{\#\{Z : Z \in \Sigma\} - \mathcal{L}^2(\Sigma)}{\mathcal{H}^1(\partial\Sigma)} \qquad \text{for a Poisson Point Process } Z$$

among all polygons Σ with side-length ≥ 1 contained in $[-L, L]^2$. This problem is the dual, in the sense of convex analysis, to the semi-discrete Wasserstein W_∞ matching between Z and the Lebesgue measure $\mathcal{L}^2$ on $[-L, L]^2$.

Theorem 3 (see [4]).

$$\mathbb{E} \sup_{\Sigma} \frac{F(\Sigma)}{P(\Sigma)} \sim \ln^{\frac{3}{4}} L.$$

The two field terms, $F(\sigma)$ and $F(\Sigma)$, amount to different approaches to introducing an ultra violet-cut off (at scale 1) to a two-dimensional white noise. Consequently, they are expected to exhibit the same large-scale behavior.

[1] meaning that $E(\sigma_*) - E(\sigma) \leq 0$ whenever σ_* and σ differ in finitely many points

Our interest lies in a reduced model: Starting from E, we zoom in on a portion of the boundary $\partial\{\sigma_* = 1\}$, which we assume to be the graph of a height function h_* that can be modeled as a minimizer of

$$E(h) = P(h) - \epsilon F(h) := \frac{1}{L}\int_0^L dx \frac{1}{2}(\frac{dh}{dx})^2 - \epsilon \frac{1}{L}\sum_{x=1}^{L-1} W(x, h(x))$$

among all piecewise linear functions on intervals of size 1 with $h(0) = h(L) = 0$. Replacing $E(\sigma)$ by $E(h)$ we make two approximations: a geometric linearization of the perimeter P and an approximation of discrete random walks by continuous Brownian motions $\{W(x,\cdot)\}_x$ in the field F.

Up to the rescaling $h = \epsilon^{\frac{2}{3}}\hat{h}$, we can assume $\epsilon = 1$ in the linearized model. We established the following homogenization result.

Theorem 4. *There exists a deterministic constant $E_* \in (0,\infty)$ such that almost surely*

$$\lim_{L\to\infty} \frac{\min_h (P - F)}{\ln L} = -E_* \qquad \text{and} \qquad \lim_{L\to\infty} \frac{P(h_*)}{\ln L} = \frac{1}{3}E_*.$$

However, the laws of $L^{-1}h_*(L\cdot)$ are tight in the Hölder space $C^{1-}([0,1])$.

Our proof relies on the existence of a net of paths $\mathcal{H} \subset \{h : [0,L] \to [0,L]\}$ such that, on the one hand, h_* in well approximated (in the energy topology) by an element $\bar{h}_* \in \mathcal{H}$, and on the other hand, the cardinality of $\mathcal{H}$ is sufficiently small to control random errors (which appear in form of a supremum over $\mathcal{H}$) associated with replacing h_* by $\bar{h}_*$. Thereby we rely on the following second result.

Theorem 5. *Given the Littlewood-Paley decomposition $h_* = \sum_l h_{*l}$ over dyadic scales $1 \le l < L$, we have that*

$$\text{for every } l \qquad P(h_{*l}) \lesssim 1 \text{ in terms of exponential moments.}$$

As a corollary of Theorem 4, we obtain a refinement of Talagrand's chaining bound for suprema of Gaussians with an alternative proof particular to our setting.

Corollary 6. *With the constant from Theorem 4*

$$E_* = 3\Big(\frac{1}{4}\lim_{L\uparrow\infty} \frac{\sup_{P(h)\le 1} F(h)}{\ln^{\frac{3}{4}} L}\Big)^{\frac{4}{3}}.$$

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About the point of view of the particle for random walks in Dirichlet environment

Christophe Sabot
(joint work with Adrien Perrel)

Random Walks in random Dirichlet Environment (RWDE) is the model of random walks in random environment with i.i.d. Dirichlet distributed transition probabilities at each site. With this specific choice of environment the annealead law is the directed Edge Reinforced Random Walk. Besides, under some condition on the weights the time-reversed walk is again a RWDE. This was used e.g. in dimension $d \geq 3$, to prove the existence of an invariant measure for the process viewed from the particle, absolutely continuous with respect to the static law. We will present a new identity for RWDE, inspired by the Vertex Reinforced Jump Process (VRJP), more precisely by its non-reversible generalization, the $\star$-VRJP. From this identity we will deduce some sufficient conditions in dimension $d = 2$ for the existence or non-existence of an absolutely continuous invariant measure for the process viewed from the particle.

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Extremes in symmetric exclusion on $\mathbb{Z}^d$

Sunder Sethuraman
(joint work with Adrian Gonzalez Casanova, Michael Conroy)

We consider the scaling limits of extremes $X_t = \max\{x_1 : \eta(x) = 1\}$ in symmetric exclusion processes, with nearest-neighbor jump rates, starting from types of 'step' profiles. In $d = 1$, such profiles are versions of $\eta(x) = 1(x \leq 0)$, and in $d > 1$ the profiles have support on parts of half spaces. Although the rates are symmetric, since large spaces are occupied, the extremes have an effective drift. Distances traveled in time t of the extremes are of order $\sqrt{t \log t}$, beyond the diffusive hydrodynamic scale.

We show under proper scaling that the distributional limits are of Gumbel type, as it would be when the particles are independent. We consider also in $d = 1$ how much of a 'step', that is the size of $L = L(t)$ when the initial condition is $\eta(x) = 1(-L < x \leq 0)$, is needed for the same limits to arise. It turns out when $L/\sqrt{t/\log t} \to \infty$ the limits (in this array indexed by t) are the same as if $L = \infty$. When otherwise and $L \uparrow \infty$, the extremes are of order $\sqrt{t \log L}$ and the properly scaled limits are different Gumbel distributions. If $L < \infty$ is bounded, the behavior is diffusive. We also give associated Poisson point process limits for these $d = 1$ scaled extreme statistics, which in particular captures the joint behaviors of the extremal particles.

We discuss two ways of proving such limits: one by use of the Strong Rayleigh property, and another by computing moments of certain counts. This talk is based on the paper/preprints [2, 3, 1].

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Random walk above a concave obstacle

Yvan Velenik

(joint work with Sébastien Ott)

The random walk. Let $(X_k)_{k\geq 1}$ be i.i.d. $\mathbb{Z}$-valued random variables satisfying

$$\mathbb{E}[X_1] = 0 \quad \text{and} \quad \mathbb{E}[e^{\delta|X_1|}] < \infty,$$

for some $\delta > 0$. Let $H(t) = \log \mathbb{E}[e^{tX_1}]$ and $I(x) = \sup_{t\in\mathbb{R}}(tx - H(t))$. Introduce $a_* = \inf\{k \in \mathbb{Z} : \mathbb{P}(X_1 = k) > 0\}$ and $b_* = \sup\{k \in \mathbb{Z} : \mathbb{P}(X_1 = k) > 0\}$. Denote by $S_n = X_1 + \cdots + X_n$ the corresponding random walk and by $\mathbb{P}$ its law.

The obstacle. Fix $a, b \in \mathbb{R}$ such that $a_* < a < b < b_*$. Let $h \in C^3([0, 1])$ be such that $h(0) = 0$ and

$$\forall x \in [0, 1], \qquad h'(x) \in [a, b] \text{ and } h''(x) < 0.$$

Denote by $h_n(k) = nh(k/n)$ for all $k \in \{0, \ldots, n\}$.

Main results. Let $\mathbb{P}_n = \mathbb{P}(\cdot \mid S_n = \lceil h_n(n)\rceil)$. Our first result concerns the probability of the event $\mathscr{A} = \{\forall k \in \{0, \ldots, n\}, S_k \geq h_n(k)\}$ under $\mathbb{P}_n$: there exist $c_+ \geq c_- > 0$ and $n_0 \geq 1$ such that for any $n \geq n_0$,

$$e^{-c_+ n^{1/3}} \leq e^{n\int_0^1 I(h'(s))\,\mathrm{d}s}\, \mathbb{P}_n(\mathscr{A}) \leq e^{-c_- n^{1/3}}.$$

To state the remaining results, let us introduce the measure $\mathbb{P}_n^h = \mathbb{P}_n(\cdot \mid \mathscr{A})$. Then, there exist $C > 0$ and $c_+ \geq c_- > 0$ such that

$$\begin{gathered} C^{-1}e^{-c_- t^{3/2}} \leq \mathbb{P}_n^h(S_k \geq h_n(k) + tn^{1/3}) \leq Ce^{-c_+ t^{3/2}}, \\ h_n(k) + c_- n^{1/3} \leq \mathbb{E}_n^h[S_k] \leq h_n(k) + c_+ n^{1/3}, \\ C\,e^{-c_-(l-k)/n^{2/3}} \geq \mathrm{cov}_n^h[S_k, S_l] \geq C^{-1}e^{-c_+(l-k)/n^{2/3}}, \end{gathered}$$

provided that $n - Cn^{2/3} \geq l \geq k \geq Cn^{2/3}$ and $t = \mathsf{o}(n^{1/9})$.

Relaxing the assumptions on the obstacle. We also investigate what happens when the assumptions on the obstacle are relaxed. We only do that in a restricted framework: Gaussian random walks $(S_k)_{-n\leq k\leq n}$ with i.i.d. increments of law $\mathcal{N}(0,\beta)$ for some $\beta > 0$, and an obstacle described by the function $h_p : [-1,1] \to [0,1]$, $h_p(x) = 1 - |x|^p$ for some $p \geq 1$. Note that h_p is not twice continuously differentiable at 0 when $p \in [1,2)$, and that $h_p''(0) = 0$ when $p > 2$. Let $\mathbb{Q}_n^h$ be the law of the random walk conditioned on $\{S_{-n} = S_n = 0\} \cap \{\forall k \in \{0,\dots,n\},\ S_k \geq h_n(k)\}$, and $\alpha_p = (p-1)/(2p-1)$. Then, there exist $c_+ \geq c_- > 0$ such that

$$e^{-c_+ t^{1/(1-\alpha_p)}} \leq \mathbb{Q}_n^h(S_0 \geq n + tn^{\alpha_p}) \leq e^{-c_- t^{1/(1-\alpha_p)}}.$$

In particular, there exist $c_+ \geq c_- > 0$ such that, for all $n \geq 1$,

$$n + c_- n^{\alpha_p} \leq \mathbb{E}_n^h[S_0] \leq n + c_+ n^{\alpha_p}.$$

The Brownian web distance

Bálint Vető

(joint work with Martin Hairer, Bálint Virág)

We define the random walk web distance in [7] as a natural directed distance on the trajectories of coalescing simple random walks on the even integer lattice $\mathbb{Z}_\mathrm{e}^2 = \{(i,n) \in \mathbb{Z}^2 : i+n \text{ is even}\}$ so that from each $(i,n) \in \mathbb{Z}_\mathrm{e}^2$ there are two outgoing directed edges: to $(i+1,n-1)$ and to $(i+1,n+1)$. We assign independent random variables $(\xi_{(i,n)})$ with $\mathbf{P}(\xi_{(i,n)} = 1) = \mathbf{P}(\xi_{(i,n)} = -1) = 1/2$ to the vertices in $\mathbb{Z}_\mathrm{e}^2$. The random walk web Y is a family of coalescing random walks starting at each point of $\mathbb{Z}_\mathrm{e}^2$. For all $(i,n) \in \mathbb{Z}_\mathrm{e}^2$, we let $Y_{(i,n)}$ be the random walk starting at (i,n) with the first step to $(i+1, n+\xi_{(i,n)})$.

For any $(i,n;j,m) \in \mathbb{Z}^4$ we define the random walk web distance $D^{\mathrm{RW}}(i,n;j,m)$ to be the smallest integer k such that (j,m) can be reached from (i,n) by following a directed path in the graph $\mathbb{Z}_\mathrm{e}^2$ with k jumps between different random walk trajectories in Y. The value of $D^{\mathrm{RW}}(i,n;j,m)$ is infinite if there is no path from (i,n) to (j,m).

The Brownian web was constructed in [5] and further studied in [4]. In the Brownian web B almost surely a unique Brownian motion $B_{(t,x)}$ starts from almost every point (t,x) of the space-time $\mathbb{R}^2$. There is however a dense set of special points of type $(1,2)$ which have an incoming and two outgoing Brownian paths. The Brownian web distance $D^{\mathrm{Br}}(t,x;s,y)$ denotes the smallest $k \geq 0$ for which there exist points $(t,x) = (t_0,x_0),(t_1,x_1),\dots,(t_{k+1},x_{k+1}) = (s,y) \in \mathbb{R}^2$ with $t_0 \leq t_1 < t_2 < \dots < t_{k+1}$ and a continuous path $\pi : [t,s] \to \mathbb{R}$ with $\pi(t_i) = x_i$ for $i = 0,\dots,k+1$ so that for each $i = 0,\dots,k$ there is a path γ_i in the Brownian web with $\pi(r) = \gamma_i(r)$ for all $r \in [t_i,t_{i+1}]$. We set $D^{\mathrm{Br}}(t,x;s,y) = +\infty$ if there is no such $k \geq 0$. The definition slightly differs if the starting point (t,x) is a special point in the Brownian web and depending on the convention, we define the Brownian web distance D^{Br} and its lower semicontinuous verion $D^{\mathrm{Br,LSC}}$.

The Brownian web distance is integer-valued and has scaling exponents 0:1:2 as compared to 1:2:3 in the KPZ world. That is, for any $\alpha > 0$, we have the equality in distribution

$$(1)\quad \left(D^{\mathrm{Br}}(\alpha^2 t, \alpha x; \alpha^2 s, \alpha y), (t,x;s,y) \in \mathbb{R}^4\right) \stackrel{\mathrm{d}}{=} \left(D^{\mathrm{Br}}(t,x;s,y), (t,x;s,y) \in \mathbb{R}^4\right).$$

Furthermore, the Brownian web distance is the scale-invariant limit of the random walk web distance in the following sense. There is a coupling of the underlying random walk webs and Brownian web such that

$$(2)\quad D^{\mathrm{RW}}(nt, n^{1/2}x; ns, n^{1/2}y) \to D^{\mathrm{Br,LSC}}(t,x;s,y)$$

as $n \to \infty$ almost surely in the epigraph sense.

The shear limit of the Brownian web distance is still given by the Airy process. As $m \to \infty$, we have that

$$\frac{tm + 2zm^{2/3} - D^{\mathrm{Br}}(-tm, 2tm + 2zm^{2/3}; 0, \mathbb{R}_-)}{m^{1/3}} \stackrel{\mathrm{d}}{\Longrightarrow} \mathcal{L}(0,0;z,t)$$

where $\mathcal{L}$ is the directed landscape defined in [2]. We conjectured the convergence of the rescaled Brownian web distance to the directed landscape in all the variables in [7]. The conjecture was resolved in [3] and it was proved that

(3)

$$\frac{m + 2(z-y)m^{2/3} - D^{\mathrm{Br}}(-m, 2m + 2zm^{2/3}; 0, (-\infty, 2ym^{2/3}])}{m^{1/3}} \stackrel{\mathrm{d}}{\Longrightarrow} \mathcal{L}(y,0;z,1)$$

as $m \to \infty$.

Under logarithmic scaling, we prove the following law of large numbers and central limit theorem in the horizontal direction. There is $\mu \in \mathbb{R}$ and $\sigma \in (0,\infty)$ such that as $n \to \infty$

$$(4)\quad \frac{D^{\mathrm{RW}}(-2n, 0; 0, 0)}{\log(2n)} \longrightarrow \mu$$

almost surely and

$$(5)\quad \frac{D^{\mathrm{RW}}(-2n, 0; 0, 0) - \mu \log(2n)}{\sigma\sqrt{\log n}} \stackrel{\mathrm{d}}{\Longrightarrow} \chi$$

where χ has standard normal distribution.

The Bernoulli-Exponential first passage percolation was introduced in [1]. We studied this model in [6] as a weighted version of the random walk web distance. It is defined on $\mathbb{Z}^2_{\mathrm{e}}$ using the same directed edges. The first passage value $T(i,n;j,m)$ from (i,n) to (j,m) in $\mathbb{Z}^2_{\mathrm{e}}$ is given similarly as the minimal weight of directed paths. The weight of paths is defined using independent standard exponential random variables $(\eta_{(k,l)})$ assigned to every vertex $(k,l) \in \mathbb{Z}^2_{\mathrm{e}}$. The weight of a path is given by the sum of the $\eta_{(k,l)}$ variables at the jumps of the path. The corresponding height function $H(n,r) = \max\{k \in \mathbb{Z} : T(0,0;n,k) \le r\}$ is the highest position a path can reach at time n with weight at most r.

The rescaled height converges to a new explicit distribution. For any $s > 0$, as $n \to \infty$,

$$\frac{1}{\sqrt{n}} H\left(n, \frac{s}{\sqrt{n}}\right) \stackrel{\mathrm{d}}{\Longrightarrow} H_s \tag{6}$$

where the distribution of the random variable H_s is given by

$$\mathbf{P}(H_s < h) = \det(I - K_s)_{L^2((h,\infty))} \tag{7}$$

with the kernel

$$K_s(x,y) = \frac{1}{(2\pi i)^2} \int_{1+i\mathbb{R}} \mathrm{d}u \int_{\mathcal{C}_0} \mathrm{d}v \, \frac{e^{u^2/2 - yu - s/u}}{e^{v^2/2 - xv - s/v}} \frac{u}{v} \frac{1}{v-u} \tag{8}$$

where the integration contour $\mathcal{C}_0$ is a small circle around 0 with positive orientation such that it does not intersect $1 + i\mathbb{R}$.

The distribution of the limiting height interpolates between the Gaussian and the GUE Tracy–Widom distribution. The formal substitution $s = 0$ gives back the Gaussian distribution. As $s \to \infty$, we have that

$$2^{4/9} 3^{-1/3} s^{1/9} \left(H_s - 2^{-2/3} 3 s^{1/3}\right) \stackrel{\mathrm{d}}{\Longrightarrow} \mathrm{TW}. \tag{9}$$

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A critical drift-diffusion equation: connection to a diffusion on SL(n)

Christian Wagner

(joint work with Şefika Kuzgun, Peter S. Morfe, Felix Otto)

We discuss a connect of two seemingly unrelated objects, a drift diffusion equation in n-dimensional Euclidian space and a natural diffusion on the Lie group SL(n). This connection reveals an intermittency inherent in the former object that becomes aparent in the latter.

More specifically, on the one hand is the so-called diffusion in the curl of the two dimensional Gaussian free field and its generalization to higher dimensions $n \geq 2$. More precisely, we consider a process X solving the drift-diffusion equation

$$dX_t = b(X_t)dt + \sqrt{2}\, dW_t \tag{1}$$

in the n-dimensional Euclidean plane, where W denotes a standard Brownian motion and b is an isotropic, stationarity Gaussian field that is divergence free and independent of W. We impose that the Fourier transform of the covariance tensor $c(x-y) = \mathbb{E}b(x) \otimes b(y)$ of b is an $(2-n)$-homogeneous function on wave vectors k with $|k| \leq 1$ and vanishes provided $|k| > 1$. The cut-off (without loss of generality at wave length one) is necessary for the well-posedness of the SDE (1) but breaks the scale invariance of b. The latter would amount to $b(\lambda x) =_{\text{law}} \lambda^{-1} b(x)$, in which case X (formally) inherits the scaling $W_{\lambda^2 t} =_{\text{law}} \lambda W_t$ from Brownian motion. The aforementioned assumptions determine b up to a multiplicative constant, which we describe in terms of the Péclet number ε via $\mathbb{E}|b|^2 = \frac{n}{2}\varepsilon^2$. The process X obtained from (1) has recently received growing attention by the mathematical community. Starting from [9] precise $\sqrt{\log}$-super diffusive behaviour has been established in [2] and [3], which has been extended to a super diffusive central limit theorem in [1].

On the other hand is a natural diffusion on the special linear group $\mathrm{SL}(n)$, whose evolution is governed by the Stratonovich SDE

$$dF_\tau = F_\tau \circ dB_\tau, \tag{2}$$

where B denotes a Brownian motion on the Lie algebra $\mathfrak{sl}(n)$. To connect to equation (1) we require the Brownian motion B, and hence also F, to be invariant in law under the action of the orthogonal group $\mathrm{O}(n)$, and additionally choose B such that the Itô- and Stratonovich-interpretation of the SDE (2) yield the same process. These two axioms define F up to a multiplicative constant, which we fix by a normalization of its second motions via $\mathbb{E}|F_\tau|^2 = |\mathrm{id}|^2 \exp(\tau)$ provided F is started at the identity.

We connect these two objects on the level of the expected particle position that is given by

$$u(t,x) = \mathbb{E}_W[X_t], \quad \text{where } X \text{ solves (1) with initial condition } X_{t=0} = x.$$

The exception $\mathbb{E}_W$ is taken w.r.t. the Brownian motion W so that u still depends on the drift b, i.e. is a quenched quantity. Our main result states that increments of u are well-approximated by the diffusion F.

Theorem 1 (see [6] and [7]). *There exists a coupling of b and B such that for (x,T) we have*

$$\mathbb{E}\frac{1}{T}\int_0^T \Big|\frac{1}{|x|}(u(t,x) - u(t,0)) - F^t_{\tau(|x|^2),\tau(T)}\frac{x}{|x|}\Big|^2 dt \lesssim \varepsilon^2 \mathbb{E}|F_{0,\tau}|^2,$$

where $\tau(t) = \log(1+\frac{\varepsilon^2}{2}\log(1+t))$ and $F_{\tau_*,\cdot}$ denotes a solution of (2) with $F_{\tau_*,\tau} = \mathrm{id}$ provided $\tau \leq \tau_*$.

Our main tool to prove Theorem 1 is a scale-by-scale homogenization technique originally developed in [3] in its SDE based form first presented in [5]. Since F, given by (2), generalizes geometric Brownian motion to a matrix-valued process, it naturally displays non-Gaussian behaviour; more specifically

$$\mathbb{E}|F_{0,\tau}|^{2p} \sim_p \big(\mathbb{E}|F_{0,\tau}|^2\big)^{1+\frac{n+4}{n+2}(p-1)+\frac{4}{n+2}(p-1)^2},$$

see [4]. As a corollary of our main theorem we can transfer this non-Gaussian behaviour to the increments of u, and obtain a strengthening and an explanation of the intermittent behaviour first worked out in [8].

Corollary 2 (see [6]). *In dimension $n = 2$, in the regime*

$$\varepsilon^2\big(1+\frac{\varepsilon^2}{2}\log|x|^2\big)^{\frac{1}{2}} \ll 1$$

we have for any (x, T) and $1 < p < \infty$

$$\mathbb{E}\Big(\frac{1}{T}\int_0^T \big|\frac{1}{|x|}(u(t,x)-u(t,0))\big|^2\Big)^p \gtrsim_p \Big(\mathbb{E}\frac{1}{T}\int_0^T \big|\frac{1}{|x|}(u(t,x)-u(t,0))\big|^2\Big)^{1+\frac{3}{2}(p-1)}$$

with explicit asymptotic $\approx \big(\frac{1+\frac{\varepsilon^2}{2}\log T}{1+\frac{\varepsilon^2}{2}\log|x|^2}\big)^{\frac{1}{2}}$ of the second moment.

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