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# Algebraic K-theory

Organized by
Hélène Esnault, Berlin
Thomas Geisser, Tokyo
Lars Hesselholt, Copenhagen
Moritz Kerz, Regensburg

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Abstract. Algebraic $K$-theory is a generalization of linear algebra to rings and to geometric objects, leading to numerous applications across various mathematical fields. It plays a crucial role in number theory, algebraic and geometric topology, algebraic geometry, and analysis. Moreover, algebraic $K$-theory is intimately connected with motivic cohomology and motivic homotopy theory, which provide a deeper structural understanding of algebraic $K$-groups.

Recent advancements in the field have been significantly driven by the algebraic calculus of $\infty$-categories. This abstract framework has yielded substantial benefits, including applications to $p$-adic Hodge theory through the computation of $p$-adic $K$-theory and the utilization of trace methods, which effectively linearize algebraic $K$-theory. The perspective of stable $\infty$-categories has also enhanced our understanding of algebraic $K$-theory with respect to various localization constructions.

Furthermore, algebraic $K$-theory has made remarkable contributions to stable homotopy theory, notably in relation to the telescope conjecture. These advances and their implications will be thoroughly explored at the upcoming workshop. The event will feature several presentations on the applications of algebraic $K$-theory to neighboring areas of mathematics, showcasing the field's broad impact and ongoing developments.

*Mathematics Subject Classification (2020):* 14-XX, 19-XX.

## Introduction by the Organizers

The workshop *Algebraic $K$-theory* was attended by roughly fifty participants from various backgrounds with particular attention given to early career researchers. Most participants attended in person, but a small group attended online. The workshop covered a wide range of topics in algebraic $K$-theory and its applications with two areas of emphasis being Efimov's extension of $K$-theory to large categories and motivic homotopy theory. Regrettably, Efimov was unable to obtain a visa in time to attend the workshop.

**Results related to categorical methods in $K$-theory**
Aoki, Land, Levy, Ramzi, Scholbach, Sosnilo, Tamme, and Wagner reported on results concerning the categorical methods in $K$-theory and their applications.

Aoki spoke on applications of $(\infty, n)$-categories. He gave a characterization of the $\mathbb{Z}$-linearly symmetric monoidal $(\infty, 2)$-category of kernels of Berkovich motives in terms of generators and relations, and he expressed hope that an $(\infty, n)$-category of noncommutative $n$-motives can be defined for all $n \geq 1$, extending the $n = 1$ case due to Kontsevich. Land reported on work with Bayındır and Tamme to determine the $K$-groups of the ring $\mathbb{Z}_p[x]/(px)$ of functions on the coordinate axes in the arithmetic plane. Levy spoke on work with Sosnilo, unifying and vastly generalizing the categorifications by Burklund–Levy of Quillen's devissage theorem and by Elmanto–Sosnilo of the Dundas–Goodwillie–McCarthy theorem. The central new ingredient is the notion of a $c$-category of width $n$. Ramzi spoke on work with Volpe and Wolf producing an in-families version of the categorification by Bartels–Efimov–Nikolaus of the assembly map in Waldhausen $K$-theory. As a corollary, Chapman's theorem on the vanishing of the Whitehead torsion of a homeomorphism of a compact topological manifold follows without recourse to Hilbert cube manifolds. Scholbach spoke on joint work with Richarz on methods to address the categorical Künneth question: An abstract six-functor formalism in the sense of Liu–Zheng and Mann is a lax symmetric monoidal functor

$$D_S\colon\ \mathrm{Corr}(C_{/S}) \to \mathrm{Mod}_{D(S)}(\mathrm{Pr}^L).$$

Under what conditions is it symmetric monoidal? Sosnilo spoke on joint work with Ramzi and Winges categorifying the delooping of localizing invariants. The proof proceeds by geometrizing Grayson's binary complexes using the quotient stack of the coordinate axes in the affine plane by the multiplicative group. As a corollary, a map of spectrum-valued localizing invariants is an equivalence if and only if it induces an isomorphism on $\pi_0$. Tamme spoke on joint work with Kelly and Saito proving that if valuative dimension is substituted for Krull dimension, then Weibel's conjecture on the vanishing and regularity of negative $K$-groups holds for all qcqs schemes. The argument follows the proof by Kerz–Strunk–Tamme in the case of noetherian schemes, but uses derived schemes in an essential way. Finally, Wagner spoke on Habiro cohomology: Given a number field $F$ and an integer $\Delta$ divisible by 6 and by the discriminant of $F$, Garoufalidis–Scholze–Wheeler–Zagier have constructed a regulator map $K_3(F) \to \mathrm{Pic}(\mathcal{H}_{\mathcal{O}_F[1/\Delta]})$ with $\mathcal{H}_{\mathcal{O}[1/\Delta]}$ a formally étale algebra over the Habiro ring $\mathcal{H} = \lim_n \mathbb{Z}[q]^\wedge_{(q^n-1)}$. As a commutative algebra in spectra, the ring $\mathcal{O}_F[1/\Delta]$ descends uniquely along the unique map $\mathbb{S} \to \mathbb{Z}$ to $\mathcal{O}_F[1/\Delta]^+$, and Wagner shows that

$$\mathcal{H}_{\mathcal{O}_F[1/\Delta]} \simeq \lim_n \pi_0((\mathrm{THH}(\mathrm{KU} \otimes \mathcal{O}_F[1/\Delta]^+ / \mathrm{KU})^{C_n})^{hU(1)/C_n}).$$

He hopes to use Efimov rigidity to develop a refined version of the right-hand side and to relate it to Scholze's analytic Habiro cohomology.

**Results related to algebraic cycles and motivic homotopy theory**

Binda, Elmanto, Gazaki, Hoyois, and Wickelgren reported on results concerning algebraic cycles and motivic homotopy theory.

Binda spoke on joint work with Lundemo, Merici, and Park extending the Bloch–Esnault–Kerz fiber square to formal schemes $\mathfrak{X}$ proper and of semi-stable reduction over $\mathrm{Spf}(\mathcal{O}_K)$ with $K$ a local number field, the infinite root stack being a key ingredient. Elmanto spoke on joint work with Morrow to use their extension of the motivic filtration of $K$-theory to all schemes qcqs over a field $k$ to reprove a theorem of Levine, Krishna–Srinivas, and Krishna on zero cycles by translating this statement to a statement in syntomic cohomology. Gazaki spoke on joint work with Rathore proving that a conjecture of Colliot-Thélène stating that for $X$ a scheme smooth, projective, and geometrically connected over a local number field, the quotient $F^2(X)/F^2(X)_{\mathrm{div}}$ of the Albanese kernel by its maximal divisible subgroup is finite holds for surfaces geometrically dominated by a product of curves. A key ingredient in the proof is the motivic Borel–Moore homology introduced by Suslin. Hoyois reported on joint work with Annala and Iwasa proving that if $S$ is a regular $\mathbb{Q}$-scheme, then the tensor-unit $\mathbb{S}_S^{\mathrm{mot}}$ of their stably symmetric monoidal $\infty$-category of $\mathbb{P}^1$-motivic spectra $\mathrm{MS}(S)$ belongs to the full subcategory

$$\mathrm{SH}(S) \subset \mathrm{MS}(S)$$

spanned by the Morel–Voevodsky $\mathbb{A}^1$-motivic spectra. Finally, Wickelgren spoke on joint work with Brugallé and Rau that determines the value for (certain) del Pezzo surfaces over a field $k$ of characteristic zero of the $GW(k)$-valued refinement of Gromov–Witten invariants introduced by Kass–Levine–Solomon–Wickelgren.

**Miscellaneous results**

The workshop also featured a number of results outside the two groupings above. Artusa spoke on duality for condensed cohomology of the Weil group of a $p$-adic field. The question was raised during the workshop of whether this is an instance of suave duality in the Heyer–Mann six-functor formalism on condensed anima. Burklund introduced notions of 1-affine schemes and affine maps between them and showed that these encompass both animated commutative algebras in abelian groups and commutative algebras in animated abelian groups, as well as Bachmann–Hoyois normed rings. Flach spoke on joint work with Krause and Morin on a syntomic logarithm, and Krause spoke on joint work with Antieau on effective spectra. Finally, Mathew spoke on joint work with Bhatt, Vologodsky, and Zhang on sheared Witt vectors, a stacky decompletion ${}^sW(R) \to W(R)$ of the ring of $p$-typical Witt vectors considered by Drinfeld and Lau. It holds the promise of extending Zink's theory of displays to give a description of the moduli stack of $p$-divisible groups for derived $p$-complete rings.

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## Workshop: Algebraic K-theory

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# Abstracts

## Homotopy invariance of the motivic sphere

Marc Hoyois
(joint work with Toni Annala, Ryomei Iwasa)

Let $S$ be a derived scheme. A *motivic spectrum* over $S$ is a cohomology theory with Tate twists on $\mathrm{Sm}_S$ satisfying Nisnevich descent and elementary blowup excision [1, 2]. We denote by $\mathrm{MS}(S)$ the symmetric monoidal category of motivic spectra over $S$ and by $\mathbb{S}_S^{\mathrm{mot}} \in \mathrm{MS}(S)$ its unit. The Morel–Voevodsky stable $\mathbb{A}^1$-homotopy category $\mathrm{SH}(S)$ is the subcategory of $\mathbb{A}^1$-invariant theories in $\mathrm{MS}(S)$.

**Main Theorem.** *Let $S$ be a regular $\mathbb{Q}$-scheme. Then $\mathbb{S}_S^{\mathrm{mot}}$ is $\mathbb{A}^1$-invariant.*

This result is an analogue of the fact that algebraic K-theory, which is the unit in noncommutative motives, is $\mathbb{A}^1$-invariant on regular schemes. We hope that it is also true in positive characteristic, but our proof uses resolution of singularities in an essential way. Let us mention two important precursors of this theorem:

**Theorem 1** (Annala–Iwasa [1])**.** *For any derived scheme $S$, the cohomology theory $\mathbb{S}_S^{\mathrm{mot}}[\mathrm{Pic}][\beta^{-1}]$ is algebraic K-theory, where $\beta \in (\mathbb{S}_S^{\mathrm{mot}}[\mathrm{Pic}])(\mathbb{P}^1)$ is the Bott element $\mathcal{O}_{\mathbb{P}^1} - \mathcal{O}_{\mathbb{P}^1}(-1)$. In particular, it is $\mathbb{A}^1$-invariant if $S$ is regular.*

In the next statement, $\mathbb{N}_\varepsilon \subset \mathbb{S}^{\mathrm{mot}}(\mathrm{Spec}\,\mathbb{Z})$ is the multiplicative set of $\varepsilon$-integers $n_\varepsilon = \sum_{i=0}^{n-1}\langle(-1)^i\rangle$ for $n > 0$.

**Theorem 2** (H [3])**.** *For any derived scheme $S$, the cohomology theory $\mathbb{S}_S^{\mathrm{mot}}[\mathbb{N}_\varepsilon^{-1}]$ is rational motivic cohomology $\mathbb{Q}_S^{\mathrm{mot}}$, as defined by Beilinson. In particular, it is $\mathbb{A}^1$-invariant if $S$ is regular.*

Because of this result, it suffices to prove the main theorem after $\mathbb{N}_\varepsilon$-completion. The proof uses log geometry. A *log scheme* (in the sense of Deligne and Faltings) is a pair $(X, \partial X)$, where $X$ is a derived scheme and $\partial X \colon X \to (\mathbb{A}^1/\mathbb{G}_m)^I$ is a finite collection of generalized Cartier divisors on $X$. A morphism of log schemes $(f, \varphi)\colon (X, \partial X) \to (Y, \partial Y)$ is a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \partial X}\downarrow & & \downarrow{\scriptstyle \partial Y} \\ (\mathbb{A}^1/\mathbb{G}_m)^I & \xrightarrow[\varphi]{} & (\mathbb{A}^1/\mathbb{G}_m)^J, \end{array}$$

where $\varphi$ is a monoid map. A log scheme $(X, \partial X)$ over $S$ is called *smooth* if, for every subset $J \subset I$, the intersection $\partial_J X = \bigcap_{i\in J} \partial_i X$ is smooth over $S$. We denote by $\mathrm{Sch}^{\log}$ the category of log schemes and by $\mathrm{Sm}_S^{\log} \subset \mathrm{Sch}_S^{\log}$ the subcategory of smooth log schemes over $S$.

Given a log scheme $(X, \partial X) \in \mathrm{Sch}^{\log}$ and a suitable cohomology theory $E$, we define the *logarithmic $E$-cohomology* of $(X, \partial X)$ by

$$E(X, \partial X) = \text{total cofiber of the } I\text{-cube } J \mapsto E^{-\mathcal{N}_J}(\partial_J X),$$

where the cube uses the *covariant* functoriality of $E$ with respect to quasi-smooth closed immersions and $\mathcal{N}_J$ is the conormal sheaf of $\partial_J X \to X$.

Let us give some context for this definition of logarithmic cohomology. The purity theorem of Morel and Voevodsky implies:

**Theorem 3** (Morel–Voevodsky [5]). *If $E \in \mathrm{SH}(S)$ is an $\mathbb{A}^1$-invariant motivic spectrum and if $(X, \partial X) \in \mathrm{Sm}_S^{\log}$, then $E(X, \partial X) = E(X - \partial X)$.*

For general motivic spectra, we have the following fundamental result of Tang:

**Theorem 4** (Tang [6]). *If the cohomology theory $E$ satisfies* quasi-smooth blowup excision *and* $\mathbb{P}^1$-homotopy invariance*, then it is covariantly functorial with respect to quasi-smooth closed immersions, so that $E(X, \partial X)$ is defined. This is also true if $E \in \mathrm{MS}(S)$ and $(X, \partial X) \in \mathrm{Sm}_S^{\log}$.*

This was a key ingredient in the proof of Atiyah duality for motivic spectra [4], which in turn implies the following result:

**Theorem 5** (AHI [4]). *If $E \in \mathrm{MS}(S)$ is a module over the $\mathbb{A}^1$-invariant motivic sphere $\mathbb{S}_S^{\mathbb{A}^1}$ and if $(X, \partial X) \in \mathrm{SmProj}_S^{\log}$, then $E(X, \partial X)$ depends only on $X - \partial X$.*

By the main theorem, the assumption on $E$ is now vacuous if $S$ is a regular scheme of characteristic zero. But even in positive and mixed characteristic, this assumption is satisfied by many examples, such as prismatic, syntomic, de Rham, and Hodge cohomology. In these cases, the theorem says that the logarithmic cohomology of a relative sncd compactification of a smooth scheme is independent of the choice of compactification.

Coming back to the main theorem, the idea is to compare motivic spectra with the *logarithmic motivic spectra* introduced by Binda, Park, and Østvær [7]. For a derived scheme $S$, let $\mathrm{logSH}(S)$ be the category of cohomology theories with Tate twists on $\mathrm{Sm}_S^{\log}$ satisfying dividing Nisnevich descent and $\mathbb{A}^1_{\log}$-invariance. Here, $V_{\log}$ is the projective bundle compactification of a vector bundle $V$. The functors

$$\begin{array}{cc} \mathrm{Sm}_S \to \mathrm{Sm}_S^{\log} & \mathrm{Sm}_S^{\log} \to \mathrm{Sm}_S \\ X \mapsto (X, \emptyset) & (X, \partial X) \mapsto X - \partial X \end{array}$$

induce symmetric monoidal left adjoint functors $\mathrm{MS}(S) \to \mathrm{logSH}(S) \to \mathrm{SH}(S)$.

In the logarithmic context, the $\mathbb{A}^1$-invariance of the sphere was proved by Park:

**Theorem 6** (Park [8]). *Let $k$ be a field with resolution of singularities. Then $\mathbb{S}_k^{\log} \in \mathrm{logSH}(k)$ is $\mathbb{A}^1$-invariant.*

The key point is the fact that the interior functor $\mathrm{Sm}_k^{\log}[\mathrm{adm}^{-1}] \to \mathrm{Sm}_k$, where "adm" is the class of admissible blowups, admits a symmetric monoidal right adjoint sending $U$ to an sncd compactification $(\bar{U}, \partial\bar{U})$. This implies that the lax symmetric monoidal inclusion $\mathrm{SH}(k) \subset \mathrm{logSH}(k)$ is in fact strict.

**Conjecture.** *For any derived scheme $S$, $\mathrm{MS}(S) = \mathrm{logSH}(S)$.*

Clearly, this conjecture and Park's theorem imply the main theorem. To prove this conjecture, it would suffice to know that logarithmic $E$-cohomology for $E \in$

$\mathrm{MS}(S)$ is functorial for arbitrary morphisms in $\mathrm{Sm}_S^{\log}$ (a priori, it is only functorial for morphisms $(f,\varphi)$ as above where $I=J$ and $\varphi$ is a product of power maps). The functor $\mathrm{Sm}_S^{\log} \to \mathrm{MS}(S)$ sending $(X,\partial X)$ to the total fiber of the cube $J \mapsto \mathrm{Th}_{\partial_J X}(\mathcal{N}_J)$ would then induce an inverse to the functor $\mathrm{MS}(S) \to \mathrm{logSH}(S)$.

We bypass this conjecture by using the *infinite root stack*. Given a log scheme $(X,\partial X) \in \mathrm{Sch}^{\log}$, the $n$th root stack of $(X,\partial X)$ is defined by the pullback square

$$\begin{array}{ccc} \sqrt[n]{(X,\partial X)} & \longrightarrow & (\mathbb{A}^1/\mathbb{G}_m)^I \\ \downarrow & & \downarrow{\scriptstyle z^n} \\ X & \xrightarrow[\partial X]{} & (\mathbb{A}^1/\mathbb{G}_m)^I, \end{array}$$

and we set $\sqrt[\infty]{(X,\partial X)} = \lim_n \sqrt[n]{(X,\partial X)} \in \mathrm{Pro}(\mathrm{Stk}_X)$. For profinitely complete theories, Bhatt, Clausen, and Mathew observed that the logarithmic cohomology of $(X,\partial X)$ often coincides with the cohomology of $\sqrt[\infty]{(X,\partial X)}$. We contend that this should be true more generally for all $\mathbb{N}_\varepsilon$-complete motivic spectra.

Let $n_\pi \in \mathbb{S}^{\mathrm{mot}}(\mathrm{Spec}\,\mathbb{Z})$ be the element induced by the $n$th power map $\mathbb{P}^1 \to \mathbb{P}^1$, and let $\mathbb{N}_\pi = \{n_\pi \mid n > 0\}$.

**Theorem 7** (AHI)**.** *For any derived scheme $S$, the infinite root stack induces a functor* $\mathrm{logSH}(S) \to \mathrm{MS}(S)^\wedge_{\mathbb{N}_\pi}$. *Hence,* $\mathrm{MS}(S)^\wedge_{\mathbb{N}_\pi}$ *is a retract of* $\mathrm{logSH}(S)^\wedge_{\mathbb{N}_\pi}$.

Using an alternative presentation of $\mathrm{logSH}(S)$ proved in [7], it suffices to show that $\sqrt[\infty]{\mathbb{A}^n_{\log}}$ becomes contractible in $\mathrm{MS}(S)^\wedge_{\mathbb{N}_\pi}$, which uses several results from [2]. It then follows from Park's theorem that $(\mathbb{S}^{\mathrm{mot}}_k)^\wedge_{\mathbb{N}_\pi}$ is $\mathbb{A}^1$-invariant when $k$ has resolution of singularities. The next result concludes the proof of the main theorem:

**Theorem 8** (AHI)**.** *If $S$ is a derived $\mathbb{Q}$-scheme, then* $\mathrm{MS}(S)^\wedge_{\mathbb{N}_\varepsilon} = \mathrm{MS}(S)^\wedge_{\mathbb{N}_\pi}$.

To prove this, we use an extension to $\mathrm{MS}(S)$ of the Lefschetz–Hopf trace formula from [9], which is a consequence of Atiyah duality [4]; this approach only works in characteristic zero as we need $\mu_n$ to be étale. However, we expect that $n_\varepsilon = n_\pi$ in $\mathrm{MS}(\mathbb{Z})$; this would follow from the relation $\rho = \delta$ in $\mathrm{Ext}^1(\mathbb{S}^{\mathrm{mot}}, \mathbb{S}^{\mathrm{mot}}(1))$, where $\rho$ is induced by $-1 \in \mathbb{G}_m$ and $\delta$ by the diagonal of $\mathbb{G}_m$, which is known in $\mathrm{SH}(\mathbb{Z})$.

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## Berkovich 2-motives and normed ring stacks

Ko Aoki

This report summarizes my talk, which was based on my thesis in progress. The central object of study is the **Z**-linearly symmetric monoidal $(\infty, 2)$-category of Berkovich 2-motives.

In number theory, we study algebraic varieties over **Z**. To study its geometry, we use *cohomology*, which basically associates vector spaces (with additional structures) to varieties. In arithmetic geometry, there are several different cohomology theories, such as étale and de Rham cohomology, together with comparison results between them.

These constructions suggest the need for a unifying framework. Grothendieck advocated the notion of a *motive* $[X]$, intended as the "cohomological essence" of a variety $X$. Instead of being a single vector space, a motive lives in a category that looks like the category of vector spaces. One way to define such a category is through "systems of realizations", consisting of compatible families of vector spaces across different theories. Another approach is to isolate the axioms satisfied by any cohomology theory and then construct a universal presentably symmetric monoidal $(\infty, 1)$-category from varieties. This is the origin of categories such as SH(**Z**) or DA(**Z**). We focus on the latter approach here.

Cohomology theories often arise from categories of sheaves: étale cohomology from étale sheaves, de Rham cohomology from D-modules, and so on. Thus every reasonable cohomology theory comes with a *category of coefficients*, whose objects are those to which the cohomology functor applies. Each such category has *six operations*, as developed by Grothendieck. This motivates the passage from looking at cohomology theories to looking at coefficient theories. Hence we can imagine the theory of 2-*motives*, which provides a universal coefficient theory. It should form a category that looks like the $(\infty, 2)$-category of linear categories. Formulating this involves delicate foundational considerations.

Large categories such as the category of groups are not strictly sets, yet they can be treated through the theory of presentability, which studies categories described by small data. Presentable categories enjoy useful properties such as the adjoint functor theorem. Extending this framework to $(\infty, n)$-categories is delicate. Stefanich [8] first defined presentable $(\infty, n)$-categories via universe enlargements. In my own work [2], I gave a definition within standard set theory, avoiding universes. I also showed that the adjoint functor theorem holds in the 2-categorical case, though it fails in the 3-categorical one.

Within this framework I defined the notion of $n$-rigidity for **S**-linearly symmetric monoidal $(\infty, n)$-categories. This generalizes Gaitsgory's 1-rigidity, isolating a class of categories that are fully dualizable and self-dual in a strong sense. Also, I developed a general theory of descendability. The point is to do this, we cannot use limits freely as in the 1-categorical case, but still we can write down a workable definition without them.

**Example.** *There is a natural* $\mathbf{S}$*-linearly symmetric monoidal* $(\infty, 2)$*-category* $2\mathrm{SH}(\mathbf{Z})$ *that is generated by varieties over* $\mathbf{Z}$ *and satisfying* $\mathrm{Hom}([X], [Y]) = \mathrm{SH}(X \times Y)$*. The compositions are defined via six operations on* SH.

I proved that this admits a universality characterization:

**Theorem** (A. (see below))**.** *It is freely generated by affine varieties over* $\mathbf{Z}$ *with the product symmetric monoidal structure satisfying smooth base change, excision,* $\mathbb{A}^1$*-invariance, and Tate-stability.*

Here are some comments on this theorem: A similar set of axioms was used Drew–Gallauer [4] to characterize SH is a universal six-functor formalism in their sense. I do not know how to directly connect this with their statement. I also need to use a different formulation of excision here. Scholze [6] formulated this by also imposing the functoriality in the shriek direction, but here we can only consider the star direction and automatically get the shriek (uniquely and canonically) functoriality.

We then move on to ring stacks. So to construct the category of D-modules, Simpson invented *de Rham stack*. Namely, for a variety $X$ over $\mathbf{Q}$, we can associate a stack $X^{\mathrm{dR}}$ such that $\mathrm{QCoh}(X^{\mathrm{dR}})$ coincides with the derived category of D-modules on $X$. Drinfeld [5] and Bhatt–Lurie [3], during their work on prismatization, emphasized the aspect of *transmutation*: To get the assignment $X \mapsto X^{\mathrm{dR}}$, we need to specify what $(\mathbb{A}^1_{\mathbf{Q}})^{\mathrm{dR}}$ as a *ring stack* (i.e., a ring object in the category of stacks).

The main theme of Scholze's talk [6] connects these two ideas: Since $2\mathrm{SH}(\mathbf{Z})$ is generated by $[\mathbb{A}^1_{\mathbf{Z}}]$, this category should be the universal one generated by a ring stack satisfying a certain condition. Of course, once we get $[\mathbb{A}^1_{\mathbf{Z}}]$, we get a class of affine varieties, so we can just say that we could characterize this. I proved a certain easy-to-verify version:

**Theorem** (A.)**.** *It is freely generated by a ring stack that is suave, excisive (with respect to* $0$ *and* $\mathbb{G}_m$*),* $\mathbb{A}^1$*-invariant, and Tate-stable.*

Of course, this is practically useful, but in practice, we can also concretely construct realizations. In [7], Scholze developed the theory of *Berkovich motives*. In that case, constructing realizations is difficult, since we need to construct classes for finitary arc sheaves. For that, I proved the following:

**Theorem** (A.)**.** *The* $\mathbf{Z}$*-linearly symmetric monoidal* $(\infty, 2)$*-category of kernels of Berkovich motives (restricted to affinoids of topologically of finite type) is freely generated by a ring stack with absolute value that is suave, excisive,* $\mathbb{D}^1$*-invariant, Tate-stable, and satisfies Kummer and Artin–Schreier descent.*

As an application, we can construct the Habiro realization: We construct the 2-motivic realization first to obtain the 1-motivic realization functors. Note that this theorem is subtler than the previous ones. For example, we have hypercompleteness issues so that the same proof does not work with the sphere coefficients. Moreover, we cannot use arc descent, because it fails even in the totally disconnected situation by [1]. Still, the proof depends on perfectoid-like technique, i.e., passing to something not of finite type.

Finally, I expect a theory of noncommutative $n$-motives. For $n = 1$, the category of noncommutative (or localizing) motives is already known, and has recently been shown to be rigid by Efimov. My goal is to construct, for general $n$, an $\mathbf{S}$-linearly symmetric monoidal $(\infty, n)$-category of noncommutative $n$-motives that is $n$-rigid. Note that for Berkovich 2-motives, higher steps of categorification do not yield new information (in technical terms, this is a certain "affineness" phenomenon).

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## The syntomic logarithm

Matthias Flach
(joint work with Achim Krause, Baptiste Morin)

The syntomic logarithm is a construction in prismatic cohomology which was discovered by Achim Krause and which we use to give a new proof of the Beilinson fibre square and a new construction of the inverse of the Bloch-Kato exponential map.

### 1. The Bloch-Kato exponential map

Let $p$ be a prime number, $K/\mathbb{Q}_p$ be a finite extension and $V$ a $\mathbb{Q}_p$-representation of the absolute Galois group $G_K$ of $K$. The Bloch-Kato exponential map

$$\exp_V : D_{dR}(V)/D^0_{dR}(V) \to H^1(K, V)$$

is the connecting homomorphism in the long exact Galois cohomology sequence induced by the fundamental exact sequence of $p$-adic Hodge theory

$$0 \to \mathbb{Q}_p \to (B_{cris})^{\varphi=1} \to B_{dR}/B^+_{dR} \to 0$$

tensored with $V$. For example, for $n \geq 2$ the map $\exp_{\mathbb{Q}_p(n)} : K \simeq H^1(K, \mathbb{Q}_p(n))$ is an isomorphism. The following is our main result, generalizing [4][Thm. 4.2] from unramified to arbitrary $K/\mathbb{Q}_p$.

**Theorem 1.1.** *(Conjecture $C_{EP}(\mathbb{Q}_p(n))$ of* [5]*[App. C2]) Let $K/\mathbb{Q}_p$ be a finite extension of discriminant $D_K$ and $n \geq 2$. Then*

$$\det_{\mathbb{Z}_p}\left(\exp_{\mathbb{Q}_p(n)} \mathcal{O}_K\right) \cdot (n-1)!^{[K:\mathbb{Q}_p]} \cdot D_K^{1-n} = \det^{-1}_{\mathbb{Z}_p} R\Gamma(K, \mathbb{Z}_p(n))$$

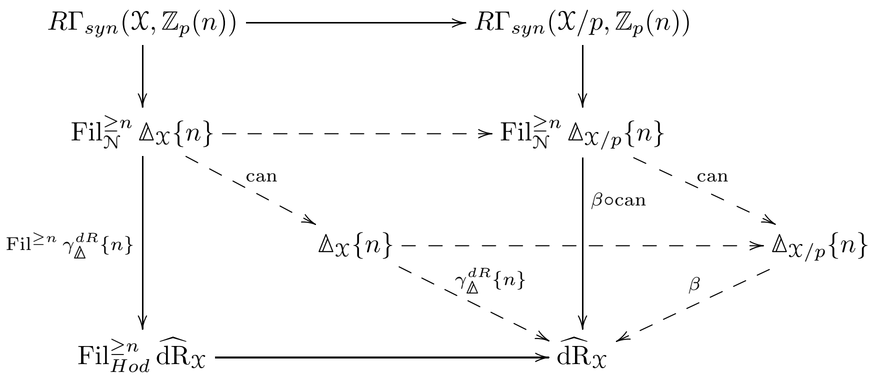
*inside* $\det^{-1}_{\mathbb{Q}_p} R\Gamma(K, \mathbb{Q}_p(n)) = \det_{\mathbb{Q}_p} H^1(K, \mathbb{Q}_p(n))$.

## 2. The Beilinson fibre square

The following proposition is immediate from the results of [3].

**Proposition 2.1.** *(The Beilinson square) For a derived $p$-adic formal scheme $\mathcal{X}$ one has a commutative diagram*

(1)




*where* $\mathrm{Fil}^{\geq \bullet}\, \gamma^{dR}_{\Delta}\{n\}$ *was defined in* [3]*[Construction 5.5.3] and the isomorphism $\beta$ in* [3]*[Thm. 5.4.2]. We call the square of solid arrows the Beilinson square. Taking horizontal fibres in (1) one obtains a map*

$$\log_{\mathcal{X}} : R\Gamma_{syn}(\mathcal{X}, \mathbb{Z}_p(n))^{\mathrm{rel}} \to \widehat{\mathrm{dR}}^{<n}_{\mathcal{X}}[-1]$$

*where* $F(\mathcal{X})^{\mathrm{rel}} := \mathrm{fibre}\,(F(\mathcal{X}) \to F(\mathcal{X}/p))$ *and* $\mathcal{X}/p := \mathcal{X} \otimes^L \mathbb{F}_p$.

**Theorem 2.1.** *Assume $\mathcal{X}$ is a quasi-compact, quasi-separated derived formal scheme such that $(L_{\mathcal{X}/\mathbb{Z}_p})^\wedge_p$ is perfect.*

a) *(The Beilinson fibre square). The map*

$$\log_{\mathcal{X}} : R\Gamma_{syn}(\mathcal{X}, \mathbb{Z}_p(n))^{\mathrm{rel}} \to \widehat{\mathrm{dR}}^{<n}_{\mathcal{X}}[-1]$$

*is a rational isomorphism, i.e. the Beilinson square is rationally Cartesian.*

b) *(Local volume computation) If $\mathcal{X}/\mathbb{Z}_p$ is proper then the source and target of $\log_{\mathcal{X}}$ are perfect complexes of $\mathbb{Z}_p$-modules. Moreover, there is an identity of $\mathbb{Z}_p$-lines in* $\det^{-1}_{\mathbb{Q}_p} \widehat{\mathrm{dR}}^{<n}_{\mathcal{X},\mathbb{Q}}$

$$\det_{\mathbb{Q}_p}(\log_{\mathcal{X},\mathbb{Q}})\left(\det_{\mathbb{Z}_p} R\Gamma_{syn}(\mathcal{X}, \mathbb{Z}_p(n))^{\mathrm{rel}}\right) = \det_{\mathbb{Q}_p}(\iota_{\mathbb{Q}})\left(\det^{-1}_{\mathbb{Z}_p} \widehat{\mathrm{dR}}^{<n,\mathrm{rel}}_{\mathcal{X}}\right) \cdot C_\infty(\mathcal{X}, n)^{-1}$$

*where*

$$C_\infty(\mathcal{X}, n) := \prod_{i \leq n-1;\, j} (n-1-i)!^{(-1)^{i+j} \dim_{\mathbb{Q}_p} H^j(\mathcal{X}_{\mathbb{Q}_p}, L\Omega^i)}.$$

We briefly sketch the deduction of Theorem 1.1. The key observation, already made in [1][Thm. 7.7], is that the Beilinson fibre square for $\mathcal{X} = \mathrm{Spf}(\mathcal{O}_{\mathbb{C}_p})$ recovers the fundamental exact sequence of $p$-adic Hodge theory. More precisely, there are $G_K$-isomorphisms

$$\begin{aligned} c_{et,\mathcal{O}_{\mathbb{C}_p}} : R\Gamma_{syn}(\mathrm{Spf}(\mathcal{O}_{\mathbb{C}_p}), \mathbb{Z}_p(n)) &\simeq R\Gamma(\mathbb{C}_p, \mathbb{Z}_p(n)) = \mathbb{Z}_p(n) \\ R\Gamma_{syn}(\mathrm{Spf}(\mathcal{O}_{\mathbb{C}_p}/p), \mathbb{Z}_p(n))_{\mathbb{Q}} &\simeq (B^+_{cris})^{p^{-n}\varphi=1} \\ \log_{\mathcal{O}_{\mathbb{C}_p},\mathbb{Q}}[1] : R\Gamma_{syn}(\mathrm{Spf}(\mathcal{O}_{\mathbb{C}_p}), \mathbb{Z}_p(n))^{\mathrm{rel}}_{\mathbb{Q}}[1] &\simeq \big(\widehat{\mathrm{dR}}^{<n}_{\mathcal{O}_{\mathbb{C}_p}}\big)_{\mathbb{Q}} \simeq \big(A_{cris}/F^n\big)_{\mathbb{Q}} \simeq B^+_{dR}/F^n \end{aligned}$$

and the top row in (1) becomes the (twisted) fundamental exact sequence

$$0 \to \mathbb{Q}_p(n) \to (B^+_{cris})^{p^{-n}\varphi=1} \to B^+_{dR}/F^n \to 0.$$

Functoriality of syntomic cohomology for $\mathrm{Spf}(\mathcal{O}_{\mathbb{C}_p}) \to \mathrm{Spf}(\mathcal{O}_K)$ gives a commutative diagram (of isomorphisms if $n \geq 2$)

$$\begin{array}{ccc} K & \xrightarrow{\exp_{\mathbb{Q}_p(n)}} & H^1(K, \mathbb{Q}_p(n)) \\ {\scriptstyle \log_{\mathcal{O}_K,\mathbb{Q}}}\uparrow{\scriptstyle \sim} & & \uparrow{\scriptstyle c_{et,\mathcal{O}_K,\mathbb{Q}}} \\ H^1_{syn}(\mathrm{Spf}(\mathcal{O}_K), \mathbb{Z}_p(n))^{\mathrm{rel}}_{\mathbb{Q}} & \xrightarrow{\sim} & H^1_{syn}(\mathrm{Spf}(\mathcal{O}_K), \mathbb{Z}_p(n))_{\mathbb{Q}} \end{array}$$

By [2][Thm. 1.8] the map $c_{et,\mathcal{O}_K}$ is an isomorphism if $n \geq 2$. Theorem 1.1 readily follows from this fact together with Theorem 2.1 b) for $\mathcal{X} = \mathrm{Spf}(\mathcal{O}_K)$.

## 3. The syntomic logarithm

We define *additive syntomic cohomology* by the fibre sequence

$$R\Gamma_{add}(\mathcal{X}, \mathbb{Z}_p(n)) \to \mathcal{N}^{\geq n}\Delta_{\mathcal{X}}\{n\} \xrightarrow{\mathrm{can}} \Delta_{\mathcal{X}}\{n\}.$$

Replacing the top row in (1) by additive syntomic cohomology we obtain a map

$$\gamma_{\mathcal{X}} : R\Gamma_{add}(\mathcal{X}, \mathbb{Z}_p(n))^{\mathrm{rel}} \to \widehat{\mathrm{dR}}^{<n}_{\mathcal{X}}[-1]$$

analogous to $\log_{\mathcal{X}}$.

**Theorem 3.1.** *Assume $\mathcal{X}$ is a quasi-compact, quasi-separated derived formal scheme such that $(L_{\mathcal{X}/\mathbb{Z}_p})^\wedge_p$ is perfect. There are filtrations $G^{\geq\star}R\Gamma_{syn}(\mathcal{X}, \mathbb{Z}_p(n))^{\mathrm{rel}}$ and $G^{\geq\star}R\Gamma_{add}(\mathcal{X}, \mathbb{Z}_p(n))^{\mathrm{rel}}$ and a commutative diagram for large enough $k$*

$$\begin{array}{ccccc} G^{\geq k}R\Gamma_{syn}(\mathcal{X}, \mathbb{Z}_p(n))^{\mathrm{rel}} & & \xrightarrow[\sim]{G^{\geq k}\mathrm{slog}^{\mathrm{rel}}_{\mathcal{X}}} & & G^{\geq k}R\Gamma_{add}(\mathcal{X}, \mathbb{Z}_p(n))^{\mathrm{rel}} \\ \downarrow{\scriptstyle \iota_{syn}} & & & & \downarrow{\scriptstyle \iota_{add}} \\ R\Gamma_{syn}(\mathcal{X}, \mathbb{Z}_p(n))^{\mathrm{rel}} & \xrightarrow{\log_{\mathcal{X}}} & \widehat{\mathrm{dR}}^{<n}_{\mathcal{X}}[-1] & \xleftarrow{\gamma_{\mathcal{X}}} & R\Gamma_{add}(\mathcal{X}, \mathbb{Z}_p(n))^{\mathrm{rel}} \end{array}$$

*such that the following hold.*

a) *The map $G^{\geq k}\mathrm{slog}^{\mathrm{rel}}_{\mathcal{X}}$, which we call the* syntomic logarithm *is an isomorphism.*

b) *There are finite filtrations* $\Phi^{\geq\star}\mathrm{Cone}(\iota_{syn})$ *and* $\Phi^{\geq\star}\mathrm{Cone}(\iota_{add})$ *and isomorphisms*

$$\mathrm{gr}^i_\Phi\mathrm{Cone}(\iota_{syn}) \simeq \mathrm{gr}^i_\Phi\mathrm{Cone}(\iota_{add})$$

*on associated graded. The associated graded are* $p^N$*-torsion for some* $N$*.*

c) *The map* $\gamma_{\mathcal{X}}$ *is a rational isomorphism. If* $\mathcal{X}/\mathbb{Z}_p$ *is proper then*

$$\det_{\mathbb{Q}_p}(\gamma_{\mathcal{X},\mathbb{Q}})\left(\det_{\mathbb{Z}_p} R\Gamma_{add}(\mathcal{X},\mathbb{Z}_p(n))^{\mathrm{rel}}\right) = \det_{\mathbb{Q}_p}(\iota_{\mathbb{Q}})\left(\det^{-1}_{\mathbb{Z}_p}\widehat{\mathrm{dR}}_{\mathcal{X}}^{<n,\mathrm{rel}}\right)\cdot C_\infty(\mathcal{X},n)^{-1}.$$

Part b) implies that $\iota_{syn}$ and $\iota_{add}$ are rational isomorphisms. Together with a) and c) this gives Thm. 2.1 a). If $\mathcal{X}/\mathbb{Z}_p$ is proper part b) implies

$$\det_{\mathbb{Q}_p}(\log'_{\mathbb{Q}_p})\left(\det_{\mathbb{Z}_p} R\Gamma_{syn}(\mathcal{X},\mathbb{Z}_p(n))^{\mathrm{rel}}\right) = \det_{\mathbb{Z}_p} R\Gamma_{add}(\mathcal{X},\mathbb{Z}_p(n))^{\mathrm{rel}}$$

where $\log'_{\mathbb{Q}_p} = \iota_{add,\mathbb{Q}_p}\circ G^{\geq k}\mathrm{slog}^{\mathrm{rel}}_{\mathcal{X},\mathbb{Q}_p}\circ\iota^{-1}_{syn,\mathbb{Q}_p}$. Together with c) this gives Thm. 2.1 b).

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# Zero-cycles on smooth projective surfaces over $p$-adic fields


Evangelia Gazaki

(joint work with Jitendra Rathore)


For a smooth projective geometrically connected variety $X$ over a field $k$ the Chow group of zero-cycles $CH_0(X)$ has a filtration

$$CH_0(X) \supset F^1(X) \supset F^2(X) \supset 0,$$

where $F^1(X)$ is the subgroup of zero-cycles of degree 0, and $F^2(X)$ is the kernel of the Albanese map $\mathrm{alb}_X : F^1(X) \to Alb_X(k)$. The latter is a generalization to higher dimensions of the Abel-Jacobi map of smooth projective curves. A famous conjecture of Colliot-Thélène ([2]) predicts that if $X$ is defined over a finite extension $k$ of the $p$-adic field $\mathbb{Q}_p$, then the Albanese kernel $F^2(X)$ has a decomposition

$$F^2(X) \simeq D \oplus F,$$

where $D$ is a divisible group and $F$ a finite group. In a celebrated paper S. Saito and K. Sato ([5]) proved that the (generally larger) group $F^1(X)$ has a decomposition $F^1(X) \simeq D\oplus F$, where $F$ is a finite group and $D$ is a group divisible by any integer $m$ coprime to the residue characteristic $p$. The full conjecture (in particular the "$p$-part") is only known in very limited cases. We note that proving the conjecture is equivalent to proving that the quotient $F^2(X)/F^2(X)_{\text{div}}$ is finite, where $F^2(X)_{\text{div}}$ is the maximal divisible subgroup of $F^2(X)$.

In this talk I reported on recent joint work with J. Rathore, where we proved the full conjecture for many new classes of surfaces. Our first theorem is the following.

**Theorem 1.** *Suppose that $\pi : X \dashrightarrow Y$ is a generically finite rational map between smooth projective surfaces and the conjecture is true for the base change $X_L := X \otimes_k L$ for every finite extension $L/k$. Then it is true for $Y$.*

Using work of Raskind and Spiess ([3]), our result proves the conjecture for surfaces that are *geometrically dominated* by a product of curves. That is, for surfaces $X$ that over the algebraic closure $\overline{k}$ admit a dominant rational map $\pi : C_1 \times C_2 \dashrightarrow X$, under the assumption that the Jacobians of $C_1, C_2$ have a mixture of good ordinary and multiplicative reduction. Some important new classes of surfaces for which we obtain evidence are: isotrivial fibrations $X \xrightarrow{\pi} C$ (that is, fibrations for which all smooth fibers are isomorphic), symmetric squares $\mathrm{Sym}^2(\mathrm{C})$ of smooth projective curves, geometrically simple abelian surfaces with a mixture of good ordinary and multiplicative reduction, Fermat diagonal surfaces $\{a_0x_0^m + a_1x_1^m + a_2x_2^m + a_3x_3^m = 0\} \subset \mathbb{P}^3_k$. More generally, we give evidence for a surface $X$ which is a resolution of singularities of a quotient $(C_1 \times C_2)/G$, where $G$ is a finite group acting faithfully on $C_1 \times C_2$.

The key new method we introduce is to work and pose similar questions for open subvarieties by replacing the Chow group $CH_0$ with a (generally larger) class group of zero-cycles, namely *Suslin's singular homology* $H_0^{\text{sus}}$ defined by Suslin ([7]). In the talk I used a more recent definition due to Wiesend ([8]), which turns out to be isomorphic with Suslin singular homology when the base field $k$ is perfect. Let $X$ be a smooth projective variety over a $p$-adic field $k$ and $j : U \hookrightarrow X$ a dense open immersion. We recall that the group $H_0^{\text{sus}}(U)$ is the quotient of the free abelian group $Z_0(U) := \bigoplus_{x \in U_0} \mathbb{Z}(x)$ on all closed points of $U$ modulo the following subgroup. For every $C \hookrightarrow U$ closed integral curve, let $\tilde{C}$ be its normalization and $\overline{C}$ be its smooth completion. Let $\pi : \overline{C} \to X$ be the induced morphism. Then we divide $Z_0(U)$ by the subgroup generated by $\mathrm{div}(f)$, where $f \in k(C)^\times$ is such that $f = 1$ on $\overline{C} \setminus \tilde{C}$.

Similarly to the projective case, the group $H_0^{\text{sus}}(U)$ admits a degree map $\deg : H_0^{\text{sus}}(U) \to \mathbb{Z}$. We denote by $F^1(U)$ its kernel. Moreover, there is a generalized Albanese map

$$alb_U : F^1(U) \to Alb_U(k),$$

where $Alb_U$ is a semi-abelian variety constructed by Serre in one of his exposés ([6]). It is an extension

$$0 \to T \to Alb_U \to Alb_X \to 0$$

of $Alb_X$ by a torus $T$. When $U(k) \neq \emptyset$, then $Alb_U$ is universal for morphisms $U \to G$ to semiabelian varieties $G$. We denote by $F^2(U)$ the kernel of $\mathrm{alb_U}$. The main technical theorem I discussed in this talk, from where Theorem 1 follows, is the following.

**Theorem 2.** *Let $X$ be a smooth projective geometrically connected surface over a $p$-adic field $k$. Let $j : U \hookrightarrow X$ be a dense open immersion such that the complement $X \backslash U$ is either of dimension $0$, or a simple normal crossing divisor. Then the group $F^2(X_L)/F^2(X_L)_{\text{div}}$ is finite for every finite extension $L/k$ if and only if the group $F^2(U_L)/F^2(U_L)_{\text{div}}$ is finite for every finite extension $L/k$.*

During the talk I gave an overview of the proof of the above theorem, focusing on the harder case when $D = X\backslash U$ is a simple normal crossing divisor. First, using the interpretation of the groups $CH_0(X)$ and $H_0^{\text{sus}}(U)$ as motivic cohomology we obtain a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & & & CH^2(X,1) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle g} & & \downarrow \\
 & & & & SK_1(D) & \xrightarrow{\varepsilon} & T(k) \\
 & & & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \delta} \\
0 & \longrightarrow & F^2(U) & \longrightarrow & F^1(U) & \xrightarrow{\text{alb}_{\text{U}}} & Alb_U(k) \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\
0 & \longrightarrow & F^2(X) & \longrightarrow & F^1(X) & \xrightarrow{\text{alb}_{\text{X}}} & Alb_X(k) \\
 & & & & \downarrow & & \\
 & & & & 0. & &
\end{array}$$

Here $CH^2(X,1)$ is Bloch's higher Chow group of $(2,1)$-cycles. Moreover, the group $SK_1(D)$ is a group we defined, which is a generalization of the $K$-group $SK_1(C)$ for a smooth projective connected curve $C$. Using the known structure of the $K$-group $K_2(k)$, as well as the Class Field Theory of curves over $p$-adic fields of Bloch ([1]) and S. Saito ([4]) which describes the structure of $SK_1(C)$, we showed that the quotient $SK_1(D)/SK_1(D)_{\text{div}}$ fits into a short exact sequence

$$0 \to H_1 \to SK_1(D)/SK_1(D)_{\text{div}} \to H_2 \oplus T'(k) \to 0,$$

where $H_1, H_2$ are finite groups and $T'$ is a torus of larger dimension than the maximal subtorus $T$ of $Alb_U$. The theorem follows after showing that the toric part $T'(k)$ does not survive in the group $F^2(U)$. To achieve this, we first showed that the cokernel of the map $SK_1(D) \xrightarrow{\varepsilon} T(k)$ is finite. The last step was to annihilate the error term $\dim(T') - \dim(T)$ by using special $(2,1)$-cycles arising from the Néron-Severi group of $X$.

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## $K$-theory and geometric topology


Maxime Ramzi

(joint work with Marco Volpe, Sebastian Wolf)


### 1. Introduction

Given a pointed, connected finitely dominated CW-complex $X$, the Wall obstruction $w_X \in K_0(\mathbb{Z}[\pi_1(X)])$ is a classical invariant defined by Wall, such that $X$ is actually finite up to homotopy if and only if $w_X$ is in the image of

$$\mathbb{Z} \cong K_0(\mathbb{Z}) \to K_0(\mathbb{Z}[\pi_1(X)]).$$

In fact, letting $B := \Pi_\infty(X)$ denote the underlying homotopy type/anima of $X$, there is an isomorphism $\pi_0 K((\mathrm{Sp}^B)^\omega) \cong K_0(\mathbb{Z}[\pi_1(X)])$, where $\mathrm{Sp}^B$ denotes the $\infty$-category of ($\infty$-)local systems on $B$. Recall that the condition that $X$ be finitely dominated corresponds to $B$ being *compact* as an anima, or homotopy type. Along this isomorphism, the class $w_X$ is the connected component of a canonical point $w_B = [\mathbb{S}_B]$ in the $K$-theory space $A(B) := K((\mathrm{Sp}^B)^\omega)$, which we abusively denote similarly. Any given finite cell decomposition of $X$ provides an actual lift of that point along the *assembly map* $A(\mathrm{pt}) \otimes B \to A(B)$. The data of such a lift $w_X^{\mathrm{loc}} \in (A(\mathrm{pt}) \otimes B) \times_{A(B)} \{w_B\}$ therefore comes with any realization of $X$ as a CW-complex. Whitehead showed that a homotopy equivalence $f : X \to Y$ between finite CW-complexes is compatible with this lift if and only if it is homotopic to a *simple homotopy equivalence*, which is a composite of particularly explicit and simply describable homotopy equivalences.

Thus, one can think of the pair $(B, w_X^{\mathrm{loc}})$ as remembering the homotopy type $B$ of a CW-complex $X$ together with some shadow of its cell structure. The following theorem of Chapman's is thus a big surprise:

**Theorem 1** (Chapman)**.** *Any homeomorphism between finite CW-complexes is compatible with the lifts $w^{\mathrm{loc}}$, and in particular is homotopic to a simple homotopy equivalence.*

In this talk, I gave a brief account of Bartels' and Nikolaus's revisiting of this and related theorems (such as West's theorem on compact topological manifolds) using Efimov's continuous $K$-theory. Their approach helps one understand these results in a clearer way, and allows for extensions of their results to other settings. In the remainder of the talk, I also reported on joint work with Volpe and Wolf where we extend their approach to the parametrized setting, and use this to obtain versions of these results in families. Among other things, we use this to reprove and generalize the topological Dwyer–Weiss–Williams index theorem.

## 2. Continuous $K$-theory and sheaf categories

The scope of algebraic $K$-theory has evolved over time, from rings to exact categories to stable ($\infty$-)categories, which were until recently the natural generality of inputs for algebraic $K$-theory. Up to idempotent-completion, the category of such gadgets is equivalent to $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$, the category of compactly generated presentable stable categories and compact-preserving, colimit-preserving functors between them. Therefore, we may as well define $K$-theory (and related invariants) on these instead, as long as we are careful about the morphisms[1].

One key observation in this direction is that kernels of compact localizations between compactly-generated categories need not be themselves compactly-generated, but they are nonetheless *dualizable*, or equivalently, compactly assembled. That is, they still have *some* form of finiteness to them. Since structural properties of $K$-theory are closely related to these localization sequences, the following extension theorem of Efimov is quite natural, though it allows for remarkably more flexibility:

**Theorem 2** (Efimov, [2, Theorem 0.1])**.** *The $K$-theory functor $K : \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega} \to \mathrm{Sp}$ admits a unique extension to a* localizing invariant[2] *on $\mathrm{Pr}^{\mathrm{dual}}_{\mathrm{st}}$, the category of dualizable presentable stable categories, and* strongly continuous *functors*[3] *between them.*

**Remark 1.** *Efimov's theorem is actually much more general, as it applies to any localizing invariant, and proves not only uniqueness of extensions but also of maps between these extensions, namely, it establishes an equivalence of categories of localizing invariants on $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$ and $\mathrm{Pr}^{\mathrm{dual}}_{\mathrm{st}}$.*

For our purposes, the following is the key example:

**Example 1.** *Let $X$ be a locally compact Hausdorff space. In this case, the category of sheaves of spectra on $X$, $\mathrm{Sh}(X, \mathrm{Sp})$ is dualizable. Furthermore, its $K$-theory is equivalent to the spectrum of compactly supported global sections on $X$, $K(\mathrm{Sh}(X)) \simeq \Gamma_c(X, K(\mathrm{Sp}))$. Note that it is rarely compactly generated, cf.* [3], [2, Section 6.4].

[1] If we do not restrict our attention to functors that preserve compact objects, then functors such as $\bigoplus_{\mathbb{N}}$ can be used to implement the Eilenberg swindle.

[2] This means sending localization sequences to co/fiber sequences.

[3] This restriction is the appropriate generalization of "compact-preserving functors" to the dualizable setting.

This means that geometrically defined objects and functors on the sheaf side provide canonical maps or points in sheaf cohomology spaces, which often only depend on the underlying homotopy type of $X$ – these points and maps, themselves, typically depend on the homeomorphism type of $X$ rather than only its homotopy type.

Using a categorification of the assembly map arising from a geometric co-pairing $\delta_X : \mathrm{Sp} \to \mathrm{Sp}^{\Pi_\infty X} \otimes \mathrm{Sh}(X, \mathrm{Sp})$, Bartels and Nikolaus reprove Chapman's theorem, as well as West's theorem, but also provide a general blueprint for these types of questions.

**Theorem 3** (Bartels–Nikolaus)**.** *For $X$ a compact Hausdorff space which is furthermore locally contractible, the co-pairing $\delta_X$ induces, after applying $K$-theory and suitably dualizing, the assembly map*

$$K(\mathrm{Sp}) \otimes \Pi_\infty(X) \to K(\mathrm{Sp}^{\Pi_\infty X})$$

*The left adjoint to pullback $p_\# : \mathrm{Sh}(X, \mathrm{Sp}) \to \mathrm{Sp}$ is furthermore strongly continuous, and defines a canonical point in the source of this map which lifts the Wall class $w_{\Pi_\infty(X)} \in A(\Pi_\infty(X))$.*

## 3. Parametrized categories and motives

With Volpe and Wolf, we study some basic properties of $\mathrm{Pr}^{\mathrm{L}}_X$, for $X$ a topological space (though some of them only under additional assumptions on $X$). This is a category of parametrized-presentable categories, following Martini–Wolf's work [5]. With the foundations set up, we can describe parametrized analogues of Bartels' and Nikolaus' constructions. Most of the geometry is similar, though some verifications take more work.

One extra difficulty lies in the imprecise theorem stated above, where the "suitably dualizing" procedure is partly subtle in the parametrized context. We salvage this by studying a parametrized version of Blumberg, Gepner and Tabuada's localizing motives $\mathrm{Mot}^{\mathrm{loc}}$ [1] (see also Hoyois–Scherotzke–Sibilla [4, Section 5] for another version of a relative theory of motives, though they stick to the small or compactly generated setting since their work predates Efimov's breakthrough), which we denote by $\mathrm{Mot}^{\mathrm{loc}}_X$. One precise theorem we prove in this context is the following comparison:

**Theorem 4** (R.–Volpe–Wolf)**.** *Let $X$ be a locally compact Hausdorff space.*

*The canonical symmetric monoidal,* $\mathrm{Mot}^{\mathrm{loc}}$*-linear colimit-preserving functor*

$$\mathrm{Sh}(X; \mathrm{Mot}^{\mathrm{loc}}) \to \mathrm{Mot}^{\mathrm{loc}}_X$$

*is fully faithful, and has a strong symmetric monoidal right adjoint. Furthermore, its image contains the localizing motive of* $\mathrm{Sh}(Y, \mathrm{Sp})$ *for any map $Y \to X$ of locally compact Hausdorff spaces.*

This theorem is sufficient for our applications, though let us also mention that conditional on some unpublished (and hard) results of Efimov, we can also prove that this functor is in fact an equivalence. Using the foundations of parametrized presentability mentioned above together with the above theorem, we prove, among other things, the following generalization of the Dwyer–Weiss–Williams topological index theorem (for simplicity, we do not state it here in full generality):

**Theorem 5** (R.–Volpe–Wolf)**.** *Let $f : Y \to X$ be a fibration between finite CW-complexes with fibers $F_x$. In this case, the parametrized Wall class*

$$w_{\Pi_\infty(F_x)} \in \ A(\Pi_\infty(F_x)),$$

*a global section of the local system of A-theory spectra $x \mapsto A(\Pi_\infty(F_x))$, lifts along the parametrized assembly $A(\mathrm{pt}) \otimes \Pi_\infty(F_x) \to A(\Pi_\infty(F_x))$ to a parametrized local Wall class $w^{\mathrm{loc}}_{F_x} \in A(\mathrm{pt}) \otimes \Pi_\infty(F_x)$, i.e. there is a canonical global section $w^{\mathrm{loc}}_{F_x}$ of the local system $A(\mathrm{pt}) \otimes \Pi_\infty(F_x)$ lifting $w_{\Pi_\infty(F_x)}$.*

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## K-theory of mixed characteristic coordinate axes


Markus Land

(joint work with Georg Tamme, Özgür Bayındır)


The talk was about computing algebraic $K$-groups of certain rings such as $\mathbb{Z}[x]/px$, for a prime number $p$. This ring is isomorphic to $\mathbb{Z} \times_{\mathbb{F}_p} \mathbb{F}_p[x]$, and can hence be thought of as the ring of functions on the union of the 1-dimensional affine space over $\mathbb{F}_p$ and $\mathrm{spec}(\mathbb{Z})$ intersecting in $\mathrm{spec}(\mathbb{F}_p)$. Thinking of $\mathrm{spec}(\mathbb{Z})$ as an arithmetic line over $\mathrm{spec}(\mathbb{F}_p)$, we call $\mathbb{Z}[x]/px$ the ring of functions on a mixed characteristic coordinate axes and our result reads as follows:

**Theorem 1.** *For $n \geq 0$, there is a canonical isomorphism*

$$K_n(\mathbb{Z}[x]/px) \cong K_n(\mathbb{Z}) \oplus \begin{cases} \mathbb{W}_{\frac{n}{2}}(\mathbb{F}_p) & \textit{if } n \textit{ is even} \\ 0 & \textit{if } n \textit{ is odd.} \end{cases}$$

I then first recalled a theorem of Hesselholt's about the equal characteristic (i.e. geometric) coordinate axes which states that for all $n \geq 0$ we similarly have

$$K_n(\mathbb{F}_p[x,y]/xy) \cong K_n(\mathbb{F}_p) \oplus \begin{cases} \mathbb{W}_{\frac{n}{2}}(\mathbb{F}_p) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

and that our proof will proceed by showing that the two relative terms appearing in the mixed and the equal characteristic cases are equivalent.

To do so, I recalled the following theorem, joint with Georg Tamme, obtained by combining [3, 4]. Fix a base ring $k \in \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{sp})$, e.g. $k = \mathbb{S}$ or $k = \mathbb{Z}$. In what follows, the term *ring spectrum* may always be interpreted as $k$-algebra spectrum.

**Theorem 2.** *To any pullback square of ring spectra as on the left side, one can associated a ring spectrum* $\odot$ *and a refined diagram as on the right side*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array} \qquad\qquad \begin{array}{cccc} A & \longrightarrow & B & \\ \downarrow & & \downarrow & \searrow \\ A' & \longrightarrow & \odot & \\ & \searrow & & \searrow \\ & & & B' \end{array}$$

*and the inner square of the right hand side is sent to a pullback square by any localizing invariant, in particular by $K$-theory.*

*In addition, if one is given a map $A_0 \to A$ such that the induced map $B \otimes_{A_0} A' \to B'$ is an equivalence, then so is the canonical map $A' \amalg_{A_0} B \to \odot$.*

Here, $A' \amalg_{A_0} B$ refers to the pushout in $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{sp})$ or $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}(k))$ if one works over the base $k$. In addition, the ring spectrum $\odot$ is compatible with base change in various forms. Next, I explained the following example.

**Example 1.** *Consider the pullback square for the functions on the geometric coordinate axes over* $\mathrm{spec}(\mathbb{Z})$ *given as*

$$\begin{array}{ccc} \mathbb{Z}[x,y]/xy & \longrightarrow & \mathbb{Z}[x] \\ \downarrow & & \downarrow \\ \mathbb{Z}[y] & \longrightarrow & \mathbb{Z} \end{array}$$

*Then the associated ring $\odot$ is given by $\mathbb{Z}[t]$ with $|t| = 2$, that is, the free $\mathbb{E}_1$-$\mathbb{Z}$-algebra on a generator in degree 2.*

Indeed, the map $\mathbb{Z}[x,y] \to \mathbb{Z}[x,y]/(xy)$ is a choice for a map $A_0 \to A$ as above; hence $\odot \simeq \mathbb{Z}[x] \amalg_{\mathbb{Z}[x,y]} \mathbb{Z}[y]$ and I explained how to compute this pushout to be $\mathbb{Z}[t]$. Via this description, $\mathbb{Z}[t]$ comes equipped with the structure of a $\mathbb{Z}[x,y]$-algebra. As a consequence, I explained the following result from [4].

**Corollary 1.** *Let $R$ be a commutative ring equipped with elements $x, y \in R$. Then $R$ is naturally a $\mathbb{Z}[x,y]$-algebra and the ring $\odot$ associated to the pullback square*

$$\begin{array}{ccc} R/xy & \longrightarrow & R/x \\ \downarrow & & \downarrow \\ R/y & \longrightarrow & R/(x,y) \end{array}$$

*is equivalent to $R/x \amalg_R R/y$ by Theorem 2 above, as well as to $\mathbb{Z}[t] \otimes_{\mathbb{Z}[x,y]} R$ by Example 1 and compatibility with base change. If $(x,y)$ is a regular sequence in $R$,* *the Tor spectral sequence gives an isomorphism* $\pi_*(\odot) \cong R/(x,y)[t]$*, where again* $|t| = 2$*.*

If one knew in the above situation that the $R$-algebra $\odot$ admits an $R/(x,y)$-algebra structure, then it would follow that $\odot$ is equivalent to the free algebra over $R/(x,y)$ on a single generator in degree 2, that is, to the graded ring $R/(x,y)[t]$, thought of as an $R/(x,y)$-algebra spectrum. However, our results do not give that (or when) such an algebra structure exists. And indeed, in general, it is not true that $\odot \simeq R/(x,y)[t]$:

**Example 2.** *Consider the ring* $R = \mathbb{Z}[x]$ *with elements* $x$ *and* $x - p$*. Then one obtains* $\odot \simeq \mathbb{Z}/\!\!/p = \mathbb{Z} \amalg_{\mathbb{Z}[x]} \mathbb{Z}$*, where the left hand map in the pushout is given by* $x \mapsto 0$ *and the right hand map by* $x \mapsto p$*. But this implies that* $\odot \otimes_{\mathbb{Z}} \mathbb{F}_p \simeq \mathbb{F}_p/\!\!/p \simeq \mathbb{F}_p/\!\!/0 \simeq \mathbb{F}_p[u]$ *where* $|u| = 1$*. On the other hand* $\mathbb{F}_p[t] \otimes_{\mathbb{Z}} \mathbb{F}_p \simeq (\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p)[t] \simeq \mathbb{F}_p[\mathbb{E}psilon, t]/\mathbb{E}psilon^2$ *where* $|t| = 2$ *and* $\mathbb{E}psilon = 2$*. This shows that* $\odot$ *and* $\mathbb{F}_p[t]$ *are* not *equivalent as* $\mathbb{Z}$*-algebras. In fact, it follows from computations of Davis–Frank–Patchkoria* [1] *that for p odd,* $\odot$ *and* $\mathbb{F}_p[t]$ *are not even equivalent as ring spectra, i.e. as* $\mathbb{S}$*-algebras; see* [4, Ex. 4.33].

In the case of the mixed characteristic coordinate axes, which is the example $(\mathbb{Z}[x], x, p)$, our main result follows as soon as we can show that $\odot \simeq \mathbb{F}_p[t]$ in this case. I then briefly explained that the classification of $\mathbb{Z}$-algebra spectra $A$ with $\pi_*(A) \cong \mathbb{F}_p[s]$ is, for $|s| \geq 1$, by Koszul duality equivalent to the classification of $\mathbb{Z}$-algebra spectra $B$ with $\pi_*(B) \cong \Lambda_{\mathbb{F}_p}[e]$, with $|e| = -|s| - 1$ and that this classification has been studied by Dwyer–Greenlees–Iyengar [2] in the case $|e| = -1$. The case $|e| < -1$ is, to the best of my knowledge, open at this point.

In joint work currently in preparation with Bayındır, we have the following results in this direction to offer.

**Theorem 3.** *Let* $A$ *be* $\mathbb{Z}$*-algebra spectrum with* $\pi_*(A) = \mathbb{F}_p[t_{2k}]$ *where* $|t_{2k}| = 2k$ *and* $k > 0$*. Then we have:*

(1) *If there exists a map* $\mathbb{F}_p \to A$ *over* $\mathbb{S}$ *or* $\mathbb{Z}$*, then there exists an equivalence* $\mathbb{F}_p[x_{2k}] \to A$ *(over* $\mathbb{S}$ *or* $\mathbb{Z}$*, respectively).*
(2) *If* $k \geq p - 1$*, then there exists a map* $\mathbb{F}_p \to A$ *over* $\mathbb{S}$*.*

Now in the mixed characteristic coordinate axes we have a diagram of $\mathbb{Z}$-algebras

$$\begin{array}{ccc} \mathbb{Z}[x]/px & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{F}_p[x] & \longrightarrow & \odot \end{array}$$

showing that there exists a map $\mathbb{F}_p \to \odot$ and hence an equivalence $\mathbb{F}_p[x_2] \to \odot$. Then we can import Hesselholt's computation to our situation as claimed and obtain Theorem 1. As another application, I discussed the following:

**Corollary 2.** *For $\ell \geq p-1$, there is a pullback square*

$$\begin{array}{ccc} K(\mathbb{Z}[x]/x(p-x^\ell)) & \longrightarrow & K(\mathbb{Z}) \\ \downarrow & & \downarrow \\ K(\mathbb{Z}[x]/(p-x^\ell)) & \longrightarrow & K(\mathbb{F}_p[t]) \end{array}$$

*Proof sketch.* More generally, for any polynomial $f$ in $X^\ell$ and with constant term $p$, the pullback square

$$\begin{array}{ccc} \mathbb{Z}[x]/xf & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Z}[x]/f & \longrightarrow & \mathbb{F}_p \end{array}$$

can be made $\mathbb{Z}/\ell\mathbb{Z}$-graded in a way such that $\odot$ is also $\mathbb{Z}/\ell\mathbb{Z}$-graded and the weight $\omega(t)$ of $t \in \pi_*(\odot) \cong \mathbb{F}_p[t]$ is 1. Denoting by $\mathrm{gr}_0(\odot)$ the weight 0-part of the $\mathbb{Z}/\ell\mathbb{Z}$-graded ring $\odot$, we have a map $\mathrm{gr}_0(\odot) \to \odot$ and $\pi_*(\mathrm{gr}_0(\odot)) \cong \mathbb{F}_p[t^\ell]$. Hence, part (2) of Theorem 3 gives a map $\mathbb{F}_p \to \mathrm{gr}_0(\odot) \to \odot$, and part (1) then implies that $\mathbb{F}_p[t] \simeq \odot$ as claimed. □

Going more towards the classification problem of Dwyer–Greenlees–Iyengar, Bayındır and I then prove:

**Theorem 4.** *Let $k > 0$. Then the following hold true.*

(1) *There exist infinitely many pairwise distinct $\mathbb{Z}$-algebra spectra with $\pi_* \cong \mathbb{F}_p[t_{2k}]$. All but finitely many of these are equivalent over $\mathbb{S}$ to $\mathbb{F}_p[t_{2k}]$.*
(2) *There exist infinitely many pairwise distinct dg enhancements on the triangulated category $\mathrm{Ho}(\mathrm{Mod}(\mathbb{F}_p[t_{2k}^{\pm 1}]))$, in fact, infinitely many pairwise distinct $\mathbb{Z}$-linear structures on the stable $\infty$-category $\mathrm{Mod}(\mathbb{F}_p[t_{2k}^{\pm 1}])$.*

The remaining time in the talk was spent on indicating the most important aspects of the proofs of Theorems 3 and 4.

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## Sheared Witt vectors, after V. Drinfeld, E. Lau, and T. Zink

Akhil Mathew
(joint work with Bhargav Bhatt, Vadim Vologodsky, Mingjia Zhang)

Let $R$ be any ring. We have the associated ring $W(R)$ of $p$-typical Witt vectors of $R$. This is equipped with a Frobenius $F : W(R) \to W(R)$ and Verschiebung $V : W(R) \to W(R)$ such that $FV = p$. Moreover, $F$ is a ring homomorphism which refines to a natural $\delta$-structure. There are several characterizations of the functor $W$. For instance, Joyal proved that $W(R)$ is the cofree $\delta$-ring on $R$.

It was observed by V. Drinfeld and E. Lau, based on the construction in certain cases due to T. Zink [5] (see also [3]), that for derived $p$-complete rings $R$ such that $(R/p)_{\mathrm{red}}$ is perfect, $W(R)$ has a natural "decompletion" with respect to the Verschiebung. The purpose of this project is to study and give some characterizations of this decompletion.

As usual, let $\hat{W}(R) \subset W(R)$ consist of those elements $x = \sum_{i\geq 0} V^i([x_i])$ where $x_i \in R$ is nilpotent for all $i$ and zero for $i \gg 0$. It is well known that $\hat{W}(R) \subset W(R)$ is an ideal, stable under Frobenius, Verschiebung, and the $\delta$-operation.

**Definition 1.** *Let $R$ be a $p$-nilpotent ring with $R_{\mathrm{red}}$ perfect. Then*

$$ {}^sW(R) = W(R) \times_{W(R)/\hat{W}(R)} \varprojlim_F W(R)/\hat{W}(R). $$

*If $R$ is a $p$-complete ring with bounded $p$-power torsion and with $(R/p)_{\mathrm{red}}$ perfect, we define ${}^sW(R) = \varprojlim_n {}^sW(R/p^n)$.*

Moreover, ${}^sW(R)$ can be defined for any derived $p$-complete ring $R$ such that $(R/p)_{\mathrm{red}}$ is perfect (for example by left Kan extending from those $R$ whose $p$-power torsion is bounded).

The construction ${}^sW(R)$ is naturally a $\delta$-ring, equipped with a $\delta$-ring Frobenius $F : {}^sW(R) \to {}^sW(R)$. When $p > 2$, ${}^sW(R)$ carries a natural Verschiebung operator lifting the Verschiebung on $W(R)$.[1] The interactions between these operations can be axiomatized as follows.

**Definition 2** (Magidson [4])**.** *A* $\delta$-Cartier ring *is a $\delta$-ring $A$ (with $\delta$-ring Frobenius $F : A \to A$) that is additionally equipped with an additive map $V : A \to A$ such that:*

1. *$FV = p$.*
2. *For all $a, b \in A$, $V(aF(b)) = V(a)b$.*
3. *(Drinfeld) For all $a \in A$, $\delta(V(a)) = a - p^{p-2}V(a^p)$.*

These conditions have many further consequences: for instance, $V : A \to A$ is automatically injective.

If $R$ is any ring, then $W(R)$ with its usual structure is a $\delta$-Cartier ring. If $p > 2$ and $R$ is derived $p$-complete with $(R/p)_{\mathrm{red}}$ perfect, then ${}^sW(R)$ is a $\delta$-Cartier

[1] When $p = 2$, the operator $y \mapsto V([-1]y)$ on $W(R)$ lifts to ${}^sW(R)$. One can develop an analog of the theory for $p = 2$, which we omit here.

ring. The map ${}^sW(R) \to W(R)$ induces an isomorphism modulo $V$, and thus after $V$-completion.

**Theorem 1** (Magidson [4])**.** *If $A$ is any $\delta$-Cartier ring and $R$ is any ring, then maps of $\delta$-Cartier rings $A \to W(R)$ are in bijection with maps of rings $A/V \to R$.*

As a consequence, the category of $V$-complete $\delta$-Cartier rings is equivalent to the category of rings, via the functor $R \mapsto W(R)$. If we relax $V$-completeness (but retain derived $p$-completeness), we can characterize ${}^sW(R)$ in a dual manner.

**Theorem 2.** *Suppose $p > 2$. Let $R$ be a derived $p$-complete ring such that $(R/p)_{\mathrm{red}}$ is perfect. Then for any $\delta$-Cartier ring $B$ which is derived $p$-complete, maps of $\delta$-Cartier rings ${}^sW(R) \to B$ are in bijection with maps of rings $R \to B/V$.*

In particular, the category of derived $p$-complete rings $R$ with $(R/p)_{\mathrm{red}}$ perfect embeds as a full subcategory of the category of derived $\delta$-Cartier rings via ${}^sW$. The essential image is those derived $p$-complete $\delta$-Cartier rings $C$ such that for any derived $p$-complete $\delta$-Cartier ring $C'$, maps from $C \to C'$ are the same as maps $C \to (\hat{C'})_V$.

**Example 1.** *Let $P$ be a perfect $\mathbb{F}_p$-algebra. Then the map ${}^sW(P) \to W(P)$ is an isomorphism.*

**Example 2.** *Any element of $W(\mathbb{Z}_p)$ has a unique expression of the form*

$$\sum_{i\geq 0} V^i(x_i),$$

*for $x_i \in \mathbb{Z}_p$. For $p > 2$, ${}^sW(\mathbb{Z}_p) \subset W(\mathbb{Z}_p)$ consists of those expressions such that $x_i \to 0$ in the $p$-adic topology.*

**Example 3.** *Let $R = P/I$ be the quotient of a perfect $\mathbb{F}_p$-algebra $P$ by an ideal $I \subset P$. Then ${}^sW(R)$ is the quotient of ${}^sW(P) = W(P)$ by the $p$-complete ideal generated by $V^n([i])$ for all $n \geq 0, i \in I$.*

When $R = \mathbb{F}_p[x^{1/p^\infty}]/(x)$, then ${}^sW(R)$ is the derived $p$-completion of the direct sum of the multiples of $[x^i]$ for all $i \in \mathbb{Z}[1/p]_{\geq 0}$. In particular, it behaves like a sort of graded decompletion of $W(R)$. For instance, the element $\sum_{n\geq 0} p^n[x^{1/p^n}] \in W(P)$ maps to a nonzero class in ${}^sW(R)$, but to zero in $W(R)$. Note in particular that ${}^sW(R) \to W(R)$ is surjective in this case.

There is also a characterization of ${}^sW$, at least locally in the flat topology, purely in terms of $\delta$-rings. This is based on the following observation: given a square-zero extension of $\delta$-rings, $\delta$ is an additive map on the kernel.

**Theorem 3.** *Let $R$ be a derived $p$-complete ring such that $F : W(R) \to W(R)$ is surjective (e.g., a semiperfect $\mathbb{F}_p$-algebra). Then the map ${}^sW(R) \to W(R)$ is a square-zero extension of $\delta$-rings such that $\delta$ acts as an isomorphism on the kernel, which is derived $p$-complete and $p$-torsionfree. Moreover, it is the universal square-zero extension of $W(R)$ in $\delta$-rings which has this property.*

Let $\mathbb{G}_a^\sharp = \mathrm{Spec}(\mathbb{Z}[x, x^i/i!])$ be the divided power additive group; this is a group scheme with a natural map $\mathbb{G}_a^\sharp \to \mathbb{G}_a$. In [2], the identity (over $\mathrm{Spec}\mathbb{Z}_{(p)}$)

$$W/^{\mathbb{L}}p \simeq \mathrm{cone}(\mathbb{G}_a^\sharp \to \mathbb{G}_a)$$

is proved; it plays an essential approach in the theory of prismatization of [2, 1].

**Proposition 1.** *For $p > 2$, there is a natural equivalence of stacks over* $\mathrm{Spf}\mathbb{Z}_p$*,*

$${}^sW/^{\mathbb{L}}p \simeq \mathrm{cone}(\mathbb{G}_a^{\hat{\sharp}} \to \mathbb{G}_a),$$

*where $\mathbb{G}_a^{\hat{\sharp}} \subset \mathbb{G}_a^\sharp$ is the ind-subscheme cut out by the condition that all sufficiently high degree elements vanish.*

As observed by Drinfeld, this can be used to formulate a "sheared" analog of prismatization (ongoing joint work of the authors and A. Kanaev), where the role of the Witt vectors is replaced by ${}^sW$. In sheared prismatization, the category of quasi-coherent sheaves yields a deformation of the category of $\mathcal{D}$-modules (rather than $\mathcal{D}$-modules with locally nilpotent $p$-curvature as in usual prismatization).

**Corollary 1.** *The functor $R \mapsto ({}^sW(R)/^{\mathbb{L}}p^n)$ commutes with filtered colimits, for any $n$.*

This gives another sense in which ${}^sW$ is a decompletion of $W$, since $W/^{\mathbb{L}}p$ does not commute with filtered colimits.

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## Gromov–Witten invariants in Hermitian K-theory: k-rational del Pezzo surfaces


Kirsten Wickelgren

(joint work with Erwan Brugallé, Johannes Rau)


Gromov–Witten invariants in Hermitian K-theory allow one to obtain an arithmetically meaningful count of curves satisfying constraints over a field $k$ without assuming that $k$ is the field of complex or real numbers. They were developed in joint work with Kass, Levine, and Solomon [3, Theorem 1 and 2]. Let $\mathrm{GW}(k)$

denote the Grothendieck–Witt ring of $k$. For del Pezzo surfaces $X$ under suitable hypotheses, there is a Gromov–Witten invariant

$$N^{\mathbb{A}^1}_{X,\beta}(A) \in \mathrm{GW}(k)$$

enumerating genus 0 degree $\beta$ curves on $X$ passing through general point constraints $\operatorname{Spec} A \to X$ where $k \to A$ is a finite étale algebra of the appropriate degree. The rank of $N^{\mathbb{A}^1}_{X,\beta}(A)$ recovers the integer valued Gromov–Witten invariants $N_{X,\beta}$. For a map $k \to \mathbb{R}$ the signature recovers the beautiful Welschinger invariants $W_X(\beta, s)$ where $s$ is the number of copies of $\mathbb{C}$ in $A \otimes_k \mathbb{R}$

In joint work with Erwan Brugallé and Johannes Rau [1], we give a complete calculation of these invariants $N^{\mathbb{A}^1}_{X,\beta}(A)$ for k-rational del Pezzo surfaces $X$ of degree greater than 5. Moreover, we give these invariants structure. Let $\mathrm{Et}_{n,k} : \mathrm{Fields}_k \to \mathrm{Set}$ be the functor from the category of field extensions of $k$ to the category of sets which takes $L$ to the set of finite étale $L$-algebras of degree $n$. Define $W : \mathrm{Fields}_k \to \mathrm{Set}$ to be the functor which takes $L$ to the set underlying the Witt group $W(L)$ of $L$. Serre defines a Witt invariant to be a natural transformation $\mathrm{Et}_{n,k} \to W$. We show that for a fixed surface $X$ and degree $\beta$, the association

$$A \mapsto N^{\mathbb{A}^1}_{X,\beta}(A)$$

is an unramified Witt invariant.

We then construct a multivariable unramified Witt invariant which conjecturally contains all of these invariants for k-rational surfaces. Serre classified all Witt invariants as the free $W(k)$-module with basis given by the wedge powers of the trace form, $\lambda_0, \ldots, \lambda_m$ for $m = [n/2]$. Serre proves that a Witt invariant which is 0 on multiquadratic extensions is identically 0. A useful basis of Witt invariants for our purposes is $\beta_0, \ldots, \beta_m$ characterized by

$$\beta_i(\prod_{j=1}^{m} k[x]/(x^2 - a_1)(\times k)) = P_i(\langle 2, 2a_1\rangle, \ldots, \langle 2, 2a_m\rangle)$$

where $P_i$ denotes the $i$th elementary symmetric polynomial. We say that a Witt invariant is $\beta$-integral if it is of the form $\sum_{i=0}^{m} b_i\beta_i$ with $b_i \in \mathbb{Z}$.

Let $\overline{n} = (n_1, \ldots, n_r)$ and $\overline{d} = (d_1 \ldots, d_r)$ be tuples of positive integers. For $s < n = \sum n_i$, consider the blowups $X_{n,s}$ of $\mathbb{P}^2$ at $n$ points in general position, such that $s$ pairs are complex conjugate and the remainder are real. Let $L$ denote the class of a line in $\operatorname{Pic} \mathbb{P}^2$. Let $D \in \operatorname{Pic} X_{n,s}$ be given by $d_0 L$ minus $d_1$ times the first $n_1$ exceptional divisors minus $d_2$ times the next $n_2$ exceptional divisors and so on

$$\begin{aligned} D = d_0 L - d_1(E_1 + \ldots + E_{n_1}) - d_2(E_{n_1+1} + \ldots E_{n_1+n_2}) - \ldots \\ - d_r(E_{n_1+\ldots+n_{r-1}+1} + \ldots + E_{n_1+\ldots+n_r}). \end{aligned}$$

We show there is a unique $\beta$-integral invariant

$$WW_{\overline{n},\overline{d}} : \mathrm{Et}_{n_0} \times \mathrm{Et}_{n_r} \to W$$

satisfying

$$WW_{D,\mathbb{R}}(\mathcal{E}_{-1}^{s_0} \times \mathbb{R}^{n_0-2s_0}, \ldots, \mathcal{E}_{-1}^{s_r} \times \mathbb{R}^{n_r-2s_r}) = \mathrm{Wel}_{X_{n_1+\ldots+n_r, s_1+\ldots+s_r}}(d; s_0)$$

in $\mathrm{W}(\mathbb{R}) \cong \mathbb{Z}$.

We conjecture the following.

**Conjecture 1.** *Let $k$ be a perfect field of characteristic not 2 or 3.*

*Fix $(n_1, \ldots, n_r) \in \mathbb{N}^r$ and $(A_1, \ldots, A_r) \in \mathrm{Et}_{\overline{n}}(k)$ and suppose that $X$ is a rational del Pezzo surface of degree at least 4 constructed as the blow up of $\mathbb{P}^2_k$ along the zero-dimensional subschemes $\mathbf{p}_1, \ldots, \mathbf{p}_r \subset \mathbb{P}^2_k$ such that $\mathbf{p}_i = \operatorname{Spec} A_i$. Then*

$$WW_D(A_0, A_1, \ldots, A_r) = Q_{X,D}(A_0).$$

We prove this conjecture for del Pezzo surfaces of degree greater than 5 and obtain the calculation. To do this, we study the behavior of these Gromov–Witten invariants during an algebraic analogue of surgery on del Pezzo surfaces [2]. We obtain a surprisingly simple formula when uncomputable terms cancel out with an identity in (twisted) binomial coefficients in the Grothendieck–Witt group.

Over an algebraically closed field, del Pezzo surfaces are isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$ or to a blow-up of $\mathbb{P}^2$ at fewer than 9 points in general position. When the points being blown-up acquire certain special configurations such as 3 points lying on a line or 5 points lying on a conic, the blow-up contains a curve of self-intersection $-2$ and the map to projective space associated to the canonical divisor crushes the $-2$-curve to a node. We start with a family over $k[[t]]$ where the general fiber is a del Pezzo surface and the special fiber has a single rational node with completed local ring the quotient of $k[[x, y, z, t]]$ by $Q(x, y, z) + t$. In a compactification of a moduli space of del Pezzo surfaces, this corresponds to a point on a wall or boundary component. Pulling back by $t \mapsto d't^2$ and passing to the general fiber produces many different directions in which to leave the wall into the moduli space of del Pezzo surfaces. We parametrize these directions by the discriminant $d$ of $Q(x, y, z) + d't^2$, and let $\Sigma_d$ denote the corresponding general fiber. While the curve counts on the central fiber are uncomputable, they cancel using a quadratically enriched combinatorics, giving a “quadratic Abramovich–Bertram formula:”

**Theorem 1.** *([2]) Let $\mathcal{X} \to \operatorname{Spec} k[[t]]$ be a 1-nodal Lefschetz fibration of del Pezzo surfaces of degree at least 4. Suppose $k$ is a characteristic 0 field, and $\Sigma_1$ is $k((t))$-rational and $\mathcal{X}_0$ is $k$-rational. Then for all $D$ in $\operatorname{Pic}\Sigma(d)$, and all finite étale extensions $k \to A$ of degree $-K_{\Sigma(d)} \cdot D - 1$ we have*

$$N^{\mathbb{A}^1}_{\Sigma(d),D}(A((t))) = N^{\mathbb{A}^1}_{\Sigma(1),D}(A((t))) + (\langle 2\rangle - \langle 2d\rangle)\sum_{j\geq 1}(-1)^j N^{\mathbb{A}^1}_{\Sigma(1),D-j\gamma}(A((t))).$$

We prove the toric case of Conjecture 1 with the powerful methods of [4]. We use Theorem 1 to reduce to the toric case in degree $> 5$.

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## Vanishing of negative K-groups of quasi-compact, quasi-separated schemes


Georg Tamme

(joint work with Shane Kelly, Shuji Saito)


In this talk, I explained a proof of the following theorem.

**Theorem 1.** *Let $X$ be a quasi-compact and quasi-separated scheme of valuative dimension $d$. Then*

$$K_{-n}(X) = 0 \qquad \textit{for all } n > d,$$

$$K_{-n}(X) \xrightarrow{\cong} K_{-n}(\mathbb{A}^r_X) \qquad \textit{for all } r \geq 0, n \geq d.$$

The notion of valuative dimension vdim was introduced by Jaffard in [5]. One always has $\dim(X) \leq \mathrm{vdim}(X)$, where $\dim(X)$ denotes the Krull dimension, and equality holds if $X$ is Noetherian. However, the valuative dimension is better behaved for non-Noetherian schemes.

Theorem 1 has a long history. For Noetherian schemes $X$, the statement of the Theorem was known as "Weibel's conjecture" [12]. The cases $\dim(X) \leq 1$ are classical, and Weibel proved the case of surfaces [13]. For varieties over characteristic 0 fields, the conjecture was proven in [2]. Assuming resolution of singularities, there were also results for varieties over fields of positive characteristic [3, 10]. The conjecture was fully resolved in [8]. Finally, the generalisation to non-Noetherian schemes, i.e. Theorem 1, is proved in [6].

The strategy of proof is that sketched by Kerz in the Noetherian setting in [9]. The proof works by induction on $d$. Using Zariski hyperdescent [1], one reduces to the case that $X$ is affine and local. Using nilinvariance of non-positive $K$-theory on affine schemes, one may then further assume that $X$ is reduced. Using excision and Zorn's lemma, we further reduce to the case that $X$ is integral.

For the inductive step, a key input is the following result.

**Proposition 1** (Kerz-Strunk [7])**.** *Let $X$ be an integral affine scheme, $i > 0$, and $\gamma \in K_{-i}(X)$. Then there exists a finitely presented, proper closed subscheme $Z$ of $X$ such that $p^*(\gamma) = 0$ in $K_{-i}(\mathrm{Bl}_Z(X))$ where $p\colon \mathrm{Bl}_Z(X) \to X$ denotes the blowup.*

To finish the inductive step, it then suffices to show that the map of relative $K$-theory spectra $K(X,Z) \to K(\mathrm{Bl}_Z(X),E)$, where $E$ denotes the exceptional divisor, is $(-d+1)$-connective.

To achieve this, we prove that, for $X = \mathrm{Spec}(R)$ local, integral, any blowup as in Proposition 1, admits a finite morphism to a blowup $\mathrm{Bl}_Y(X)$ in a closed subscheme $Y$ cut out by at most $d$ equations. This follows from the theory of reductions of ideals (see, e.g., [4]) as follows.

**Proposition 2.** *Let $I \subset R$ be a finitely generated ideal, and let $J \subseteq I$ be a reduction, i.e. there exists an $n$ such that $JI^n = I^{n+1}$. Then there is a natural finite morphism $\mathrm{Bl}_I(R) \to \mathrm{Bl}_J(R)$.*

For $R$ local with residue field $\kappa$, $I \subset R$ a finitely generated ideal, the analytic spread $\ell(I)$ is defined as the Krull dimension of the "fibre cone" $(\bigoplus_{n\geq 0} I^n) \otimes_R \kappa$.

**Proposition 3.** *For $R$ and $I$ as above, we have $\ell(I) \leq \mathrm{vdim}(R)$.*

**Proposition 4.** *For $R$ and $I$ as above, there exists a reduction $J$ of some power $I^n$ generated by $\ell(I)$ elements.*

In the Noetherian setting, these results are proven in [4], but the proofs adapt to the non-Noetherian setting.

Given a closed subscheme $V(I) = Z \hookrightarrow X = \mathrm{Spec}(R)$ as in Proposition 1, we thus find a reduction $J$ of $I^n$ generated by $\ell$ elements $f_1,\dots,f_\ell$ with $\ell \leq d$. We now consider the derived blowup $\widetilde{X}$ of $X$ in $f_1,\dots,f_\ell$. The map $\mathrm{Bl}_Z(X) = \mathrm{Bl}_I(R) \to X$ then factors as the composite

$$\mathrm{Bl}_I(R) = \mathrm{Bl}_{I^n}(R) \to \mathrm{Bl}_J(R) \hookrightarrow \widetilde{X} \to X,$$

where the first two maps are finite. If $\widetilde{E} \to \widetilde{X}$ denotes the (derived) exceptional divisor, the map $K(X,Z) \to K(\widetilde{X},\widetilde{E})$ is an equivalence by a generalisation [8] of a result of Thomason. Using that $\widetilde{X}$ has a covering by $\ell$ affine, open subschemes, the desired connectivity then follows by an induction on $\ell$ from derived nilinvariance of non-positive $K$-theory on connective ring spectra and the following.

**Proposition 5.** *Let $p\colon Y \to X$ be a finite morphism of affine scheme which is an isomorphism outside a finitely presented closed subscheme $Z \hookrightarrow X$. Then the map of relative $K$-theory spectra*

$$p^*\colon K(X,Z) \to K(Y,p^{-1}(Z))$$

*is 0-connective.*

As $K$-theory commutes with filtered colimits, in the latter proposition we may assume moreover that $p$ is finitely presented. Under that assumption, the proposition follows from pro-excision results of [11].

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## Duality for condensed cohomology of the Weil group of a $p$-adic field

Marco Artusa

Duality theorems are central statements in arithmetic geometry. For $p$-adic fields, the foundations are due to Tate, who proves duality results for their Galois cohomology. Let $F$ be a $p$-adic field, we fix a separable closure $\overline{F}$ and we set $G_F \coloneqq \mathrm{Gal}(\overline{F}/F)$. The local Tate duality can be stated as follows

**Theorem 1** (Local Tate Duality, [10, Theorem 2.1])**.** *Let $M$ be a finite Galois module. Then we have a perfect cup-product pairing*

$$\mathrm{H}^q(G_F, M) \times \mathrm{H}^{2-q}(G_F, \underline{\mathrm{Hom}}(M, \overline{F}^{\times})) \to \mathrm{H}^2(G_F, \overline{F}^{\times}) = \mathbb{Q}/\mathbb{Z} \tag{1}$$

*between finite abelian groups.*

The goal of this talk is twofold: to replace Galois cohomology with a new cohomology, and to extend Tate's result from finite coefficients to more general and *topological* ones.

Why should we replace Galois cohomology? First, if we try to generalise Theorem 1 within the framework of Galois cohomology, we don't go very far. For $M = \mathbb{Z}$ and $q = 0$, (1) becomes

$$\mathbb{Z} \times \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z},$$

which is not perfect: to make it perfect, one should replace $\mathbb{Z}$ with its profinite completion (this holds in more generality by Tate-Nakayama duality). The second problem with classical Galois cohomology is that topology is not taken into account. For example, $\overline{F}^{\times}$ and $\mathrm{H}^q(G_F, \overline{F}^{\times})$ should be endowed with a topology, while the classical framework only considers them as discrete objects. These two issues are related. Indeed, (1) is expressed in terms of the functor $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z}) : \mathrm{Ab}^{\mathrm{op}} \to \mathrm{Ab}$, which is *not* an equivalence of categories. When we take topology into account, the situation improves. If LCA denotes the category of *locally compact* abelian groups, we have an a equivalence of categories

$$(-)^{\vee} := \underline{\mathrm{Hom}}(-, \mathbb{R}/\mathbb{Z}) : \mathrm{LCA}^{\mathrm{op}} \to \mathrm{LCA},$$

Our improved duality is expressed in terms of this functor (the *Pontryagin duality*).

## 1. A new cohomology

The new cohomology replacing $\mathrm{H}^q(G_F, -)$ fits in Geisser-Morin's conjectural picture (see [3, Section 6]). To define it, we follow intuitions by Lichtenbaum (see [8]) and we use the recent theory of Condensed Mathematics (developed by Clausen-Scholze [9] and Barwick-Haine [2]). We mimick the topos-theoretic definition of Galois cohomology, replacing $G_F$ with the *Weil group* $W_F$ and the topos of sets with the topos of *condensed sets* Cond(Set): from the categorical point of view, this topos is similar to Set, but it contains "nice enough" topological spaces as a full subcategory stable by all limits. We obtain a fixed point functor

$$(-)^{W_F} : W_F - \mathrm{Cond}(\mathrm{Mod}) \to \mathrm{Cond}(\mathrm{Ab})$$

and we define the *condensed cohomology of the Weil group* of $F$ as

$$\mathbf{H}^q(W_F, -) := R^q(-)^{W_F}.$$

**Remark 2.** *The cohomology groups $\mathbf{H}^q(W_F, M)$ are condensed abelian groups, hence they can be naturally topologised. Having a topology for such objects is just a property (being in a certain subcategory of* Cond(Ab)*), not a structure.*

**Remark 3.** *$W_F = \lim_U W_F/U$ is a prodiscrete topological group, where $U$ runs among open normal subgroups of $I_F$. We are* not *considering it as an object of* Cond(Grp) *but as a pro-object. Thus its classifying topos is not $B_{\lim_U W_F/U}$ but it is $B_{\hat{W}_F} := \lim_U B_{W_F/U}$. The first (more intuitive) choice would give a cohomology which is not well-behaved for our purposes (see* [1, Proposition 3.4]*).*

## 2. Duality

**The new coefficients.** Let FLCA be the quasi-abelian category of locally compact abelian groups of finite ranks (see [4, Definition 2.6]). We define $\mathrm{D}^{\mathrm{perf}}_{\mathbb{Z},\mathbb{R}}$ as the smallest stable $\infty$-subcategory of $\mathrm{D}^b(\mathrm{FLCA})$ containing $\mathbb{Z}$ and $\mathbb{R}$. Examples of objects of this category are: finitely generated abelian groups (with their discrete topology), finite-dimensional real vector spaces (with their Euclidean topology), $(\mathbb{R}/\mathbb{Z})^n$ (with its compact Hausdorff topology). Another example is $\theta := [\mathbb{Z} \hookrightarrow \mathbb{R}/\mathbb{Z}]$, where the map sends $1 \in \mathbb{Z}$ to an irrational $\alpha \in \mathbb{R}/\mathbb{Z}$. This map has a dense image, hence $\theta$, contrarily to the other examples, does not belong to the full subcategory $\mathrm{LCA} \subset \mathrm{Cond(Ab)}$.

**The dualising object.** Let $L$ be the completion of the maximal unramified extension $F^{\mathrm{ur}}$ and let $\overline{L}$ be a separable closure of $L$ which contains $\overline{F}$. We define

$$\mathbb{R}/\mathbb{Z}(1) \coloneqq \mathrm{cofib}(\overline{L}^{\times}[-1] \overset{\mathrm{val}}{\to} \mathbb{R}[-1]) =: \mathrm{cofib}(\mathbb{Z}(1) \to \mathbb{R}(1)) \in \mathrm{D}^b(W_F - \mathrm{Cond(Mod)}).$$

One can show that we have $\mathbf{H}^2(W_F, \mathbb{R}/\mathbb{Z}(1)) = \mathbb{R}/\mathbb{Z}$ and $\mathbf{H}^q(W_F, \mathbb{R}/\mathbb{Z}(1)) = 0$ for all $q \geq 3$.

**The new duality.** For all $M \in \mathrm{D}^b(W_F - \mathrm{Cond(Mod)})$ we set

$$M^{\mathrm{D}} \coloneqq R\underline{\mathrm{Hom}}(M, \mathbb{R}/\mathbb{Z}(1)).$$

The main theorem is the following (see [1, Theorem 4.27])

**Theorem 4.** *Let $M \in \mathrm{D}^{\mathrm{perf}}_{\mathbb{Z},\mathbb{R}}$ with a continuous action of $G$, finite quotient of $G_F$. Then we have a perfect cup-product pairing*

$$R\mathbf{\Gamma}(W_F, M) \otimes^L R\mathbf{\Gamma}(W_F, M^{\mathrm{D}}) \to \mathbf{H}^2(W_F, \mathbb{R}/\mathbb{Z}(1))[-2] = \mathbb{R}/\mathbb{Z}[-2]$$

*in* $\mathrm{D}^b(\mathrm{FLCA})$. *If moreover we have* $\mathrm{H}^q(M) \in \mathrm{FLCA}$ *for all $q$, we obtain an induced perfect cup-product pairing in* FLCA *for all $q$*

$$\mathbf{H}^q(W_F, M) \otimes \mathbf{H}^{2-q}(W_F, M^{\mathrm{D}}) \to \mathbf{H}^2(W_F, \mathbb{R}/\mathbb{Z}(1)) = \mathbb{R}/\mathbb{Z}. \tag{2}$$

**Remark 5.** *The condition* $\mathrm{H}^q(M) \in \mathrm{FLCA}$ *excludes examples like $\theta$, for which* $\mathbf{H}^q(W_F, M) \notin \mathrm{FLCA}$ *in general. For these objects,* $\mathbf{H}^q(W_F, M)$ *has a maximal* non-separated *subgroup* $\mathbf{H}^q(W_F, M)^{\mathrm{ns}}$ *and maximal* locally compact *quotient* $\mathbf{H}^q(W_F, M)^{\mathrm{lc}} \coloneqq \mathbf{H}^q(W_F, M)/\mathbf{H}^q(W_F, M)^{\mathrm{ns}}$. *The general duality is given by two different perfect pairings*

$$\mathbf{H}^q(W_F, M)^{\mathrm{lc}} \otimes \mathbf{H}^{2-q}(W_F, M^{\mathrm{D}})^{\mathrm{lc}} \to \mathbb{R}/\mathbb{Z},$$
$$\mathbf{H}^q(W_F, M)^{\mathrm{ns}} \otimes^L \mathbf{H}^{1-q}(W_F, M^{\mathrm{D}})^{\mathrm{ns}} \to \mathbb{R}/\mathbb{Z}[1].$$

This result is an improvement of Theorem 1. If $M$ is finite, (2) canonically identifies with (1), and this pairing is perfect by the local Tate duality. If $M$ is finitely generated, this theorem is an improved version of Tate-Nakayama duality, which does not need profinite completion to hold; if $M$ is a finite-dimensional real vector space, (2) defines a new "exotic" duality between finite-dimensional real vector spaces. If $M = \mathbb{R}/\mathbb{Z}$ and $q = 1$, (2) becomes $(W_F^{\mathrm{ab}})^{\vee} \otimes F^{\times} \to \mathbb{R}/\mathbb{Z}$, which yields the reciprocity isomorphism of local class field theory "à la Weil"

$$F^{\times} \overset{\sim}{\to} W_F^{\mathrm{ab}}.$$

## 3. Perspectives: higher local fields and solid Milnor $K$-theory

We present some future directions, which aim to extend previous results to higher local fields. Everything in this section is conjectural.

0-local fields are finite fields, 1-local fields are what we call local fields, and inductively $d$-local fields are complete discrete valuation fields with a $(d-1)$-local field as a residue field. Kato (see [5],[6],[7]) extends local class field theory to higher local fields. If $F$ is a $d$-local field, the role of $F^\times$ is taken by $K_d^{\mathrm{M}}(F)$, the $d$th Milnor $K$-theory of $F$. Kato shows that we have a reciprocity morphism

$$\psi_F : K_d^{\mathrm{M}}(F) \to G_F^{\mathrm{ab}}$$

such that $\psi_F/n$ is an isomorphism for all $n \in \mathbb{N}$. The reasons why $\psi_F$ is not an isomorphism without passing to a finite quotient are essentially two: first, the Galois group should be replaced by the Weil group; secondly, $K_d^{\mathrm{M}}(F)$ contains huge divisible subgroups, as it is built using a tensor product which is *algebraic* and does not take the "topology" of the field into account. For a $d$-local field $F$, we aim to define its Weil group $W_F$ and its *solid* Milnor $K$-theory $\mathbf{K}_d^{\mathrm{M}\blacksquare}(F)$, where the "topology" of $F$ is taken into account and where the algebraic tensor product is replaced by the solid tensor product $- \otimes^{\blacksquare} -$ (see [9, Lecture V]). Moreover, we aim to show that we have an isomorphism of solid abelian groups

$$\mathbf{K}_d^{\mathrm{M}\blacksquare}(F) \xrightarrow{\sim} W_F^{\mathrm{ab}}. \tag{3}$$

This result should come from a generalisation of Theorem 4 to higher local fields. This involves endowing higher local fields with condensed structures, defining complexes $\mathbb{Z}(d), \mathbb{R}(d), \mathbb{R}/\mathbb{Z}(d) \in \mathrm{D}^b(W_F - \mathrm{Cond}(\mathrm{Mod}))$ and showing that we have $\mathbf{H}^d(W_F, \mathbb{Z}(d)) = \mathbf{K}_d^{\mathrm{M}\blacksquare}(F)$. Finally, $\mathbf{K}_d^{\mathrm{M}\blacksquare}(F)/n$ should canonically coincide with $K_d^{\mathrm{M}}(F)/n$ and (3)$/n$ should canonically coincide with $\psi_F/n$. Hence, obtaining (3) would be an improvement of Kato's higher local class field theory.

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## Trace methods beyond connective rings

Ishan Levy
(joint work with Vova Sosnilo)

I explained work joint with Vova Sosnilo on extending trace methods beyond connective rings. The fundamental result of trace methods is the Dundas–Goodwillie–McCarthy theorem, which says that the fiber of the cyclotomic trace map is a truncating invariant, i.e is a localizing invariant that is an isomorphism on maps of connective ring spectra that are $\pi_0$-surjections with nilpotent kernel. Previously, work of Elmanto–Sosnilo [1] extended this to a categorical setting, showing that nilpotent extensions of bounded weighted categories are an isomorphism on truncating invariants, and previous work of mine [2] showed that this is true for 1-connective maps of $-1$-connective rings.

In our work, we introduce the notion of a $c$-category, which is a stable category with extra structure that in particular allows us to extend trace methods to it. To give the definition, we first recall some preliminaries about weight structures. Given a stable category $\mathcal{C}$, and a full subcategory $\mathcal{C}_{\leq 0}$, there is a unique weight structure on $\mathrm{Ind}(\mathcal{C})$ with the property that an object $y \in \mathrm{Ind}(\mathcal{C})$ is connective in the weight structure if and only if the mapping spectra $\mathrm{map}(x, y)$ is connective for each $x \in \mathcal{C}_{\leq 0}$. We call such a weight structure on Ind(C) *compactly generated.*

A *$c$-category of width $n$* is a stable category $\mathcal{C}$ with a compactly generated weight structure on $\mathrm{Ind}(\mathcal{C})$ such that each object of $\mathcal{C}$ is bounded in the weight structure, and such that any object $x \in \mathrm{Ind}(\mathcal{C})$ that is connective in the weight structure can be written as filtered colimit of compact objects $y_\alpha$ along morphisms that factor through $-n$-connective objects in the weight structure.

We show that a $c$-category admits a finite resolution (in the sense of chain complexes of stable categories) by module categories of connective rings. In particular, for a $c$-category $\mathcal{C}$ of width $n$, there is a connective ring spectrum $R$ such that for a localizing invariant $E$, there is an isomorphism $E(R) \cong \Sigma^n E(\mathcal{C})$. We define a notion of nilpotent extension of $c$-categories, and show that nilpotent extensions of $c$-categories are isomorphisms on all truncating invariants.

Many examples of $c$-categories naturally arise. For example, quasi-coherent sheaves on a qcqs scheme, and many quasi-geometric stacks, have natural $c$-structures. Moreover, the theory of $c$-categories is robust enough that checking that a map of such stacks induces a nilpotent extension is one that can be checked fppf-locally. Another family of examples comes from taking fixed points of continuous fcd pro-$p$-group actions on connective rings. For example, the map $(\mathbb{Z}/p^n\mathbb{Z})^{hG} \to \mathbb{F}_p^{hG}$ is a map of rings that induces a nilpotent extension of $c$-categories on module categories, where $G$ is a fcd pro-$p$-group, and $(-)^{hG}$ denotes the continuous homotopy fixed points by the trivial action. In particular, our results allow one to computationally access $K$-theory of rings such as $(\mathbb{Z}/p^n\mathbb{Z})^{hG}$ in terms of TC.

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## A cohomology theory in mixed chromatic characteristic

Robert Burklund

In this talk I presented an extension of the category of affine schemes containing both affine derived schemes and affine spectral schemes together with a cohomology theory in this context recovering in its various aspects de Rham cohomology, syntomic and prismatic cohomology, $TC$ and $THH$.

## Habiro cohomology & THH($-$/ku)

Ferdinand Wagner

*Habiro cohomology* is supposed to be a (positive) answer to the following question of Peter Scholze:

**Question.** Is there a version of $q$-de Rham cohomology with coefficients not in the power series ring $\mathbb{Z}[[q-1]]$, but in the *Habiro ring*

$$\mathcal{H} \overset{\text{def}}{=} \lim_{m\in\mathbb{N}} \mathbb{Z}[q]^{\wedge}_{(q^m-1)}\,?$$

$q$-de Rham cohomology itself, constructed by Bhatt–Scholze [2], is already a very interesting cohomology theory: It is defined globally, i.e. for smooth schemes over $\mathbb{Z}$, and after completion at any prime $p$ it recovers prismatic cohomology relative to the $p$-adic $q$-de Rham prism $(\mathbb{Z}_p[[q-1]],[p]_q)$. This makes it the only known non-trivial case in which prismatic cohomology for various primes combines into a global object. One reason to look for a Habiro-refinement of $q$-de Rham cohomology is the following recent result:

**Theorem/Construction 1** (Garoufalidis–Scholze–Wheeler–Zagier, [4])**.** *Let $F$ be a number field and let $\Delta$ be divisible by $6\operatorname{disc} F$. Then there exists a formally étale $\mathcal{H}$-algebra $\mathcal{H}_{\mathcal{O}_F[1/\Delta]}$ and a regulator map*

$$K_3(F) \longrightarrow \operatorname{Pic}\big(\mathcal{H}_{\mathcal{O}_F[1/\Delta]}\big)$$

We remark that the line bundles in the image of the regulator map above will become trivial after completion at $(q-1)$, so there's no way to see this regulator using only $q$-de Rham cohomology.

It turns out that the best possible version of Habiro cohomology doesn't exist, but by now we do have several non-optimal constructions: Scholze constructed *analytic Habiro cohomology* [7], which takes values in modules over an analytic version of the Habiro ring. In my thesis I construct *algebraic Habiro cohomology*, whose values are often finitely generated modules over the completion of $\mathcal{H}[1/N]$.

In this talk we focussed on the algebraic version, which can be summarised in the following theorem:

**Theorem/Construction 2.** *Define* $\mathrm{AniRing}^{q\text{-Hdg}}$ *to be the category of pairs* $(R, \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_R)$*, where* $R$ *is an animated ring and* $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_R$ *is filtration on the derived* $q$*-de Rham complex of* $R$*, which* $q$*-deforms the Hodge filtration on* $\mathrm{dR}_R$ *and satisfies a few natural compatibilities. Given such a pair, define the* $q$*-Hodge complex*

$$q\text{-Hdg}_R \overset{\mathrm{def}}{=} \operatorname{colim}\left(\mathrm{fil}^0_{q\text{-Hdg}}\, q\text{-dR}_R \xrightarrow{(q-1)} \mathrm{fil}^1_{q\text{-Hdg}}\, q\text{-dR}_R \xrightarrow{(q-1)} \cdots\right)^\wedge_{(q-1)}.$$

- ($a$) *The forgetful functor* $\mathrm{AniRing}^{q\text{-Hdg}} \to \mathrm{AniRing}$ *is not essentially surjective. However, on the category of smooth* $\mathbb{Z}$*-algebras* $R$ *in which all primes* $p \leq \dim(R/\mathbb{Z})$ *are invertible, the forgetful functor does have a section.*
- ($b$) *The functor* $q\text{-Hdg}_{(-)}\colon \mathrm{Sm}_{\mathbb{Z}}^{q\text{-Hdg}} \to \mathcal{D}(\mathbb{Z}[[q-1]])$ *factors canonically and non-trivially through a functor*

$$q\text{-}\mathcal{H}\mathrm{dg}_{(-)}\colon \mathrm{Sm}_{\mathbb{Z}}^{q\text{-Hdg}} \longrightarrow \mathcal{D}(\mathcal{H})$$

  *valued in the derived* $\infty$*-category of the Habiro ring.*
- ($c$) *In the situation of Theorem/Construction* 1, $q\text{-}\mathcal{H}\mathrm{dg}_{\mathcal{O}_F[1/\Delta]} \simeq \mathcal{H}_{\mathcal{O}_F[1/\Delta]}$.

The general non-existence of a section of $\mathrm{AniRing}^{q\text{-Hdg}} \to \mathrm{AniRing}$ is the price we have to pay for an *algebraic* (as opposed to *analytic*) theory. Besides smooth $\mathbb{Z}$-algebras with the property from Theorem/Construction 2($a$), a rich source of examples of objects in $\mathrm{AniRing}^{q\text{-Hdg}}$ can be constructed using topological Hochschild/negative cyclic homology over ku.

**Theorem/Construction 3.** *Let* $R$ *be quasisyntomic and* $2 \in R^\times$*. Suppose that* $R$ *admits a lift to an* $\mathbb{E}_2$*-ring spectrum* $\mathbb{S}_R$ *satisfying* $R \simeq \mathbb{S}_R \otimes \mathbb{Z}$*. Then the derived* $q$*-de Rham complex of* $R$ *can be equipped with a filtration* $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_R$ *in such a way that*

$$(R, \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_R) \in \mathrm{AniRing}^{q\text{-Hdg}},$$

*and such that the associated graded of the* $S^1$*-equivariant even filtration on* $\mathrm{TC}^-(\mathrm{ku} \otimes \mathbb{S}_R/\mathrm{ku})$ *is the completion of the filtration* $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_R$ *(up to shift):*

$$\mathrm{gr}^\star_{\mathrm{ev},hS^1} \mathrm{TC}^-(\mathrm{ku} \otimes \mathbb{S}_R/\mathrm{ku}) \simeq \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_R[2\star].$$

In the case where $R = \mathbb{Z}[x]$ with spherical lift $\mathbb{S}[x]$, this theorem was first shown in unpublished work of Raksit. The theorem can be regarded as a $q$-de Rham/ku-analogue of Antieau's result [1] that

$$\mathrm{gr}^\star_{\mathrm{ev},hS^1} \mathrm{HC}^-(R) \simeq \mathrm{fil}^\star_{\mathrm{Hdg}}\, \widehat{\mathrm{dR}}_R[2\star].$$

In the situation from Theorem/Construction 3, one can also describe the $q$-Hodge complex $q\text{-Hdg}_R$ and its descent to the Habiro ring $q\text{-}\mathcal{H}\mathrm{dg}_R$ in terms of $\mathrm{THH}(-/\mathrm{ku})$. In the talk we sketched a construction of a suitable even filtration on the genuine fixed points $\mathrm{THH}(\mathrm{KU} \otimes \mathbb{S}_R/\mathrm{KU})^{C_m}$ for every finite cyclic subgroup $C_m \subseteq S^1$. This leads to the following result:

**Theorem 4.** *In the situation of Theorem/Construction* 3*,*

$$q\text{-Hdg}_R \simeq \mathrm{gr}^0_{\mathrm{ev},hS^1}\,\mathrm{TC}^-(\mathrm{KU}\otimes\mathbb{S}_R/\mathrm{KU})\,,$$

$$q\text{-}\mathcal{H}\mathrm{dg}_R \simeq \lim_{m\in\mathbb{N}} \mathrm{gr}^0_{\mathrm{ev},S^1}\left(\mathrm{THH}(\mathrm{KU}\otimes\mathbb{S}_R/\mathrm{KU})^{C_m}\right)^{h(S^1/C_m)}\,.$$

In the situation of Theorem/Construction 1, we obtain the following special case: If $\mathbb{S}_{\mathcal{O}_F[1/\Delta]}$ denotes the unique spherical $\mathbb{E}_\infty$-lift of the étale $\mathbb{Z}$-algebra $\mathcal{O}_F[1/\Delta]$, then

$$\mathcal{H}_{\mathcal{O}_F[1/\Delta]} \cong \lim_{m\in\mathbb{N}} \pi_0\left(\mathrm{THH}(\mathrm{KU}\otimes\mathbb{S}_{\mathcal{O}_F[1/\Delta]}/\mathrm{KU})^{C_m}\right)^{h(S^1/C_m)}\,.$$

We hope to use this in future work to obtain a description of the regulator from Theorem/Construction 1 in terms of a trace map from $K$-Theory to THH(−/ku).

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## The stacky view on the Grayson construction


Vladimir Sosnilo

(joint work with Mazime Ramzi, Vladimir Sosnilo, Christoph Winges)


Given a stable $\infty$-category a very elementary and naive definition of its $\mathrm{K}_0$ does not immediately generalize to the higher groups. Typically, one has to define the connective K-theory spectrum via Waldhausen's $S_\bullet$-construction (or, equivalently, Quillen's Q-construction) for the connective part, and the nonconnective deloopings are built using a construction of Thomason [7]. This procedure appears very ad hoc, but it gives a reasonable invariant because it satisfies the two main properties we expect K-theory to have:

(1) it generalizes $\mathrm{K}_0$: $\pi_0\,\mathrm{K}(\mathcal{C})\simeq \mathrm{K}_0(\mathcal{C})$ for $\mathcal{C}\in\mathrm{Cat}^{\mathrm{perf}}$;

(2) it is a localizing invariant, i.e. it sends a localization sequence $\mathcal{D}\to\mathcal{C}\to(\mathcal{C}/\mathcal{D})^{\mathrm{ic}}\in\mathrm{Cat}^{\mathrm{perf}}$ to a fiber sequence.

This does not fully justify the definition, because it might still be possible that there is not a unique such functor. A result of Blumberg–Gepner–Tabuada shows that K-theory is at least universal among them:

**Theorem 1** ([1, Theorem 9.8], [2])**.** *For any (acessible) localizing invariant* $F : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Sp}$ *with a natural transformation* $\mathrm{Core}(-) \to \Omega^\infty F(-)$ *there exists a unique extension to a map of localizing invariants* $\mathrm{K} \to F$.

This, in particular, means that any accessible functor $E$ satisfying (1) and (2) admits a map $\mathrm{K} \to E$, which is an isomorphism on $\pi_0$.

In [4] Grayson proposed an alternative construction of K-theory based on the following result:

**Theorem 2.** *There exists a functor* $\Gamma : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Cat}^{\mathrm{st}}$ *such that* $\mathrm{K}\,\Gamma(\mathcal{C}) \simeq \Sigma\,\mathrm{K}(\mathcal{C})$.

The mere existence of such a functor already implies that at least for positive $i$ the values of $\mathrm{K}_i$ are determined by the values of $\mathrm{K}_0$ on all objects of $\mathrm{Cat}^{\mathrm{perf}}$. Note that the Calkin construction gives a similar equivalence

$$\mathrm{K}(\mathrm{Calk}(\mathcal{C})) \simeq \Omega\,\mathrm{K}(\mathcal{C}),$$

also shows that it is true for negative $i$. Together with Ramzi and Winges, in [6, Appendix A] we prove the following generalization of Grayson's result:

**Theorem 3** (Ramzi–Sosnilo–Winges, Efimov)**.** *There exists a functor* $\Gamma : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Cat}^{\mathrm{st}}$ *induces an equivalence* $E\Gamma(\mathcal{C}) \simeq \Sigma E(\mathcal{C})$ *for any (not necessarily filtered colimit preserving) localizing invariant.*

**Corollary 1.** *Any morphism of localizing invariants valued in spectra* $F \to G$ *is an equivalence as long as it is an equivalence on* $\pi_0$*.*

This, together with Theorem 1, shows that there is a unique localizing invariant $E$ satisfying (1) and (2).

The functor $\Gamma$ in Theorem 2 is very explicit but infinitary, so it is still combinatorially hard to understand in practice, especially when we cannot control filtered colimits.

A proof of Theorem 3 was also given in [3] using dualizable categories. There Efimov showed that sheaves on the real line valued in a given dualizable category form a dualizable analogue of the functor $\Gamma$. The goal of this talk is to give a different direct proof by providing a geometric interpretation of $\Gamma$ in the context of small stable $\infty$-categories.

We introduce the notion of weighted $\mathrm{A}^1$-invariance and consider the spectral stacks $[\mathbb{A}^1/\mathbb{G}_m]$ and $[X/\mathbb{G}_m]$, where $X$ is the union of axes on the affine plane and all the actions have the same weight. The summation map $[X/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ is then shown to be a weighted $\mathbb{A}^1$-equivalence. We then prove that any localizing invariant (in fact, any functor on stacks satisfying projective bundle formula), satisfies weighted $\mathbb{A}^1$-invariance. Now in the diagram

$$\begin{array}{ccccc}
E(\mathrm{Perf}_{0,[\mathbb{A}^1/\mathbb{G}_m]} \otimes \mathcal{C}) & \longrightarrow & E(\mathrm{Perf}_{[\mathbb{A}^1/\mathbb{G}_m]} \otimes \mathcal{C}) & \longrightarrow & E(\mathcal{C}) \\
\downarrow & & \downarrow & & \downarrow \\
E(\mathrm{Perf}_{0,[X/\mathbb{G}_m]} \otimes \mathcal{C}) & \longrightarrow & E(\mathrm{Perf}_{[X/\mathbb{G}_m]} \otimes \mathcal{C}) & \longrightarrow & E(\mathcal{C}) \times E(\mathcal{C}).
\end{array}$$

the horizontal maps are fiber sequences, the middle vertical map is an equivalence, and the right vertical map has $E(\mathcal{C})$ as the cofiber. This means that the cofiber of the left vertical map is $\Omega E(\mathcal{C})$. This allows us to define $\Gamma\mathcal{C}$ as the categorical cone construction applied a functor $\mathrm{Perf}_{0,[\mathbb{A}^1/\mathbb{G}_m]} \otimes \mathcal{C} \to \mathrm{Perf}_{0,[X/\mathbb{G}_m]} \otimes \mathcal{C}$.

**Corollary 2.** *K-theory commutes with arbitrary products, i.e. for any family* $\mathcal{C}_i \in \mathrm{Cat}^{\mathrm{perf}}$ *the map*

$$\mathrm{K}(\prod_i \mathcal{C}_i) \to \prod_i \mathrm{K}(\mathcal{C}_i)$$

*is an equivalence.*

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# Infinite root stacks and the Bloch–Esnault–Kerz fiber square


Federico Binda

(joint work with Tommy Lundemo, Alberto Merici, Doosung Park)


## 1. $p$-adic deformation of $K$-theory classes

1.1. **The Bloch–Esnault–Kerz fiber square.** Let $K$ be a local field of mixed characteristic $(0,p)$, let $\mathcal{O}_K$ be its ring of integers, and $k$ its residue field. The starting point of our investigation is the following result, originally proved by Bloch–Esnault–Kerz for $K_0$ classes (and additional restrictions on $p$) and motivated by Fontaine–Messing's $p$-adic variational Hodge conjecture:

**Theorem 1** (Bloch–Esnault–Kerz [3], Beilinson [2], Antieau–Mathew–Morrow–Nikolaus [1]). *Let $\mathfrak{X}$ be a proper smooth formal scheme over* $\mathrm{Spf}(\mathcal{O}_K)$ *and let* $\mathfrak{X}_0$

*be its special fiber. Then the square*

$$\begin{array}{ccc} K^{\mathrm{cts}}(\mathfrak{X};\mathbb{Q}_p) & \longrightarrow & K(\mathfrak{X}_0;\mathbb{Q}_p) \\ \downarrow & & \downarrow \\ \prod_i \mathrm{Fil}^{\geq i} R\Gamma_{dR}(\mathfrak{X}_K)[2i] & \longrightarrow & \prod_i R\Gamma_{dR}(\mathfrak{X}_K)[2i] \end{array}$$

*is cartesian, where the map $K(\mathfrak{X}_0;\mathbb{Q}_p) \to \prod_i R\Gamma_{dR}(\mathfrak{X}_K)[2i]$ is induced by the crystalline Chern character and Berthelot's crystalline-to-de Rham comparison isomorphism $R\Gamma_{dR}(\mathfrak{X}_K) \simeq R\Gamma_{crys}(\mathfrak{X}_0) \otimes K$.*

The above theorem gives a cohomological criterion for lifting classes to continuous $K$-theory [2, 11], reducing effectively the original conjecture to a (hard) algebraization problem. When $\mathfrak{X}$ is not smooth nor proper, a refined obstruction to lifting classes has been constructed by Beilinson and Antieau–Mathew–Morrow–Nikolaus: for $x \in K_i(\mathfrak{X}_0;\mathbb{Q}_p)$, there exists a class

$$o(x) \in \bigoplus_{r\geq 0} H^{2r-i}(L\Omega_{\mathfrak{X}}/L\Omega_{\mathfrak{X}}^{\geq r})_{\mathbb{Q}_p}$$

such that $x$ lifts to $K_i^{\mathrm{cts}}(\mathfrak{X};\mathbb{Q}_p)$ if and only if $o(x) = 0$. Here, $L\Omega_{\mathfrak{X}}$ denotes the $p$-adic derived de Rham cohomology of $\mathfrak{X}$. See [1, Theorem E].

However, the obstruction class $o(x)$ is rather implicit and lacks a crystalline interpretation, in contrast with the smooth case.

1.2. **Hyodo-Kato cohomology and the log BEK square.** To tackle this problem, we recall the following fact: when $\mathfrak{X}$ is proper and has semistable reduction over $\mathrm{Spf}(\mathcal{O}_K)$, the de Rham cohomology of the (smooth, rigid analytic) generic fiber of $\mathfrak{X}$ can be recovered by means of the Hyodo–Kato cohomology

$$R\Gamma_{dR}(\mathfrak{X}_K) \simeq R\Gamma_{HK}(\mathfrak{X}_0/k^0) \otimes_{W(k)} K \tag{2}$$

of the *logarithmic* special fiber, seen as a smooth log scheme over the *logarithmic point* $k^0 = (\mathbb{N} \to k, 1 \mapsto 0)$. The Hyodo–Kato cohomology is a $(\varphi, N)$-module over $W(k)$, and can be defined using an appropriate version of the crystalline site following work of Kato and Fontaine–Illusie (note however that functoriality is tricky: see [6] for a motivic approach working directly on the rigid analytic generic fiber).

**Theorem 3** (See [8]). *For $\mathfrak{X}$ proper and semistable, a class $x \in K_i(\mathfrak{X}_0;\mathbb{Q}_p)$ lifts to $K_i^{\mathrm{cts}}(\mathfrak{X};\mathbb{Q}_p)$ if and only if $ch_{HK}(x) \in \bigoplus_{r\geq 0} H_{HK}^{2r-i}(\mathfrak{X}_0/k^0)[1/p]$ belongs to $\bigoplus_{r\geq 0} \mathrm{Fil}^{\geq r} H_{dR}^{2r-i}(\mathfrak{X}_K/K)$ under the Hyodo–Kato isomorphism* (2)*.*

This solves the problem in the semistable case of identifying the obstruction class introduced by Beilinson.

Following the strategy of [1], in order to prove Theorem 3 it is necessary to identify the cofiber of the natural map $K^{\mathrm{cts}}(\mathfrak{X};\mathbb{Q}_p) \to K(\mathfrak{X}_0;\mathbb{Q}_p)$ with an appropriate variant of $TC(-;\mathbb{Q}_p)$, taking into account the additional information coming from the log structure. We do this by means of Rognes' logarithmic (topological) cyclic homology [18], and its cousins logarithmic (topological) Hochschild, periodic and negative cyclic homology. The key ingredient is the following

**Theorem 4** (See [7, 8])**.** *Let $(R, Q)$ be a pre-log $\mathbb{Z}_p$-algebra.*

(a) *There exists a commutative square*

$$\begin{array}{ccc} TC((R,Q);\mathbb{Z}_p) & \longrightarrow & TC((R\otimes_{\mathbb{S}}\mathbb{F}_p,Q);\mathbb{Z}_p) \\ \downarrow & & \downarrow \\ HC^{-}((R,Q);\mathbb{Z}_p) & \longrightarrow & HP((R,Q);\mathbb{Z}_p) \end{array}$$

*which becomes cartesian after inverting $p$.*

(b) *if $(R, Q)$ is a regular, log regular* [16, Theorem III.1.11.1.]*, and vertical* [16, Definition I.4.3.1.] *log ring over $\mathcal{O}_K^\sharp = (\mathcal{O}_K, (\pi))$, then there is a cartesian square of the form*

$$\begin{array}{ccc} K(R;\mathbb{Q}_p) & \longrightarrow & K(R/p;\mathbb{Q}_p) \\ \downarrow & & \downarrow \\ HC^{-}((R,Q)/\mathcal{O}_K^\sharp;\mathbb{Z}_p) & \longrightarrow & HP((R,Q)/\mathcal{O}_K^\sharp;\mathbb{Z}_p). \end{array}$$

*Proof.* Part (a) can be deduced by adapting the arguments in [1], or directly using the descent results discussed below. Part (b) is a consequence of [1, Theorem A], part (a), the vanishing of $HC((R/p, Q); \mathbb{Q}_p)$ using the fact that $R/p$ is an $\mathbb{F}_p$-algebra, and the existence of Gysin/residue sequences for logarithmic $TC$ and its variants (see [10, Proposition 8.6.7] and [17, Theorem 1.2]). □

**Remark 5.** Using the motivic filtration on logarithmic $TC$ [9] (analogous to the one in [4]), part (a) of Theorem 4 induces on the graded quotients the analogue of [1, Theorem 6.17], involving the log syntomic cohomology sheaves $\mathbb{Z}_p(i)(R, Q)$ and $p$-adic derived log de Rham cohomology.

## 2. Saturated descent and infinite root stacks

There are several advantages in introducing a logarithmic structure in the previous discussion. First, log invariants naturally arise when considering Gysin sequences for non $\mathbb{A}^1$-homotopy invariant cohomology theories (such as prismatic or syntomic cohomology), even for classical (i.e., non logarithmic) schemes. Second, we can effectively use the "magic" of log geometry (in Kato's words) and treat semistable degenerations as if they were smooth morphisms (leading for example to Theorem 3, in complete analogy with Theorem 1). It is however convenient to reduce statements involving log rings to the non-log counterparts. This can be done by means of the following two principles:

(1) Saturated Kummer descent, after Kato and Nizioł (a $p$-adic variant of Abhyankar's lemma);
(2) The infinite root stack approach, after Talpo and Vistoli.

It turns out that 1) and 2) in fact coincide with profinite coefficients, as we now explain.

2.1. **Saturated descent.** Let $\varphi\colon P \to Q$ be an injective map of saturated monoids [16, §I.1.3]. We say that $\varphi$ is Kummer if it is $\mathbb{Q}$-surjective, i.e., for every $y \in Q$ there is $n \in \mathbb{N}$ such that $ny \in \mathrm{Im}(\varphi)$. A standard example for a fine and saturated (fs) monoid $M$ is the map to the colimit perfection $M_{\mathrm{perf}} = \mathrm{colim}_{\times p} M$.

For $(A, M)$ a prelog ring with $A$ a complete $\mathbb{Z}_p$-algebra, we can form the cosimplicial diagram of $p$-complete rings:

$$C_{M_{\mathrm{perf}}}(A, M)^{\bullet} = (A\widehat{\otimes}_{\mathbb{Z}_p\langle M\rangle}\mathbb{Z}_p\langle M_{\mathrm{perf}}\rangle \rightrightarrows A\widehat{\otimes}_{\mathbb{Z}_p\langle M\rangle}\mathbb{Z}_p\langle M_{\mathrm{perf}} \oplus^{\mathrm{sat}}_M M_{\mathrm{perf}}\rangle \ldots)$$

where the coproduct on the right takes place in the category of saturated monoids (this is crucial), and the angular brackets denote the $p$-adic completion of the monoid algebras. The descent property mentioned above takes the following form:

**Theorem 6.** *Let $(A, M)$ be a "special" quasi-syntomic pre-log $\mathbb{Z}_p$-algebra (see [7] for details, for example, the p-adic completion of a free log algebra). Then*

$$\widehat{\mathbb{\Delta}}_{(A,M)} \simeq \lim_{\Delta} \widehat{\mathbb{\Delta}}_{(C_{M_{\mathrm{perf}}}(A,M)^{\bullet}, M^{\bullet})} \simeq \lim_{\Delta} \widehat{\mathbb{\Delta}}_{C_{M_{\mathrm{perf}}}(A,M)^{\bullet}}$$

In particular, logarithmic, Nygaard completed, prismatic cohomology of a large class of quasi-syntomic rings can be computed in non-logarithmic terms. This is a key ingredient in the proof that the same descent property holds for $THH(-;\mathbb{Z}_p)$, $TC(-;\mathbb{Z}_p)$ and variants.

2.2. **Infinite root stack.** The infinite root stack $\sqrt[\infty]{X}$ of an fs log scheme $X = (\underline{X}, \mathcal{M}_X)$ was introduced by Talpo and Vistoli in [19] as a geometric incarnation of the Kato-Nakayama space [12]. It is a pro algebraic stack over $\underline{X}$, with the property that the map $\sqrt[\infty]{X} \to \underline{X}$ is an isomorphism at every point $x \in X$ where the log structure is trivial. For a divisorial log structure given by a (derived) Cartier divisor $\underline{X} \to \mathbb{A}^1/\mathbb{G}_m$, it can be defined as the limit over $n$ of the pullback to $\underline{X}$ of the multiplication by $n$ map $\mathbb{A}^1/\mathbb{G}_m \to \mathbb{A}^1/\mathbb{G}_m$.

It was suggested by Bhatt, Clausen and Mathew [15] that $p$-adic cohomological invariants of log schemes should be extracted from the infinite root stack. This is possible thanks to the following simple observation due to Nizioł and Kato: for $\varphi\colon P \to Q$ a Kummer map as above, there is a simplicial homotopy equivalence in the category of saturated monoids:

$$Q^{\bigoplus^{\mathrm{sat}}_P(\bullet+1)} \simeq Q \oplus (Q^{\mathrm{gp}}/P^{\mathrm{gp}})^{\bigoplus \bullet}$$

This leads to the following fact. Let $(A, M)$ be a "special" quasi-syntomic pre-log $\mathbb{Z}_p$-algebra. Then $C_{M_{\mathrm{perf}}}(A, M)^{\bullet}$ is eqiuvalent to the Cech nerve of the atlas $\mathrm{Spf}(A\widehat{\otimes}_{\mathbb{Z}_p\langle M\rangle}\mathbb{Z}_p\langle M_{\mathrm{perf}}\rangle) \to \sqrt[p^{\infty}]{\mathrm{Spf}(A, M)}$. We deduce the following meta-corollary:

**Corollary 7.** *All the invariants for log rings derived from Rognes' log topological Hochschild homology coincide, after p-completion, with the invariants defined using the infinite-root stack.*

As a sample application, it is possible to show that the Nygaard completion of the site-theoretic log prismatic cohomology of Koshikawa and Koshikawa-Yao [13, 14] coincide with the definition introduced in [9]. This is the log analogue of the comparison theorem in Bhatt-Scholze [5, Theorem 13.1].

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# Zero cycles on singular varieties

Elden Elmanto

(joint work with Matthew Morrow)

The goal of this talk is to explain a short cohomological approach of most cases of the following result. More details will appear in an upcoming paper, joint with Matthew Morrow.

**Theorem 1** (Levine, Krishna–Srinivas, Krishna)**.** *Let $k$ be an algebraically closed field and $A$ a reduced, finite type $k$-scheme of dimension $d \geq 2$. Then the subgroup $F^d\mathrm{K}_0(\mathrm{A}) \subset \mathrm{K}_0(\mathrm{A})$ is torsionfree.*

Here, $F^d\mathrm{K}_0(\mathrm{A})$ is the subgroup of $\mathrm{K}_0(\mathrm{A})$ generated by classes of smooth points of codimension $d$ points of Spec(A). Classically, it is considered as replacement for the Chow group of zero cycles for schemes which are not necessarily smooth over a field.

**Remark 2.** *For some context, Theorem 1 is an answer to a question posed by Murthy in* [9, Open Question (2.12)], *in the context of splitting problems for vector bundles over affine varieties. In particular, Theorem 1 shows that the top chern class in $F^d\mathrm{K}_0(\mathrm{A})$ is a complete obstruction to splitting off a trivial rank one summand in a rank $d$ vector bundle over* Spec(A)*. For torsion prime to the characteristic, Theorem 1 was proved by Levine in an unpublished manuscript* [8]*. A major breakthrough was made for normal varieties in* [7] *and a proof of the full conjecture can be found in* [6]*. Our approach here is different and uses vanishing results for étale/syntomic cohomology of affine varieties which are algebro-geometric analogs of the Andreotti-Frankel theorem in topology.*

Our approach to this result relies on the extension of motivic cohomology to all equicharacteristic quasicompact, quasiseparated schemes introduced in [3]. Let us review parts of this theory that we will use in the proof.

**Theorem 3.** *Let $k$ be a field. There is a functorial, multiplicative, $\mathbb{N}$-indexed and complete filtration on the $K$-theory of an algebraic variety $X$ over $k$:*

$$\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K(X)} \to \mathrm{K(X)}.$$

*Writing its graded pieces and associated cohomology as:*

$$\mathbb{Z}(j)^{\mathrm{mot}}(X) := \mathrm{gr}^j_{\mathrm{mot}}\mathrm{K(X)[-2j]} \qquad \mathrm{H}^{\mathrm{i}}_{\mathrm{mot}}(\mathrm{X}, \mathbb{Z}(\mathrm{j})) := \mathrm{H}^{\mathrm{i}}(\mathbb{Z}(\mathrm{j})^{\mathrm{mot}}(\mathrm{X})),$$

*there is a functorial* motivic-to-syntomic *comparison map*[1]

$$\mathbb{Z}(j)^{\mathrm{mot}}(X)/p \to \mathbb{F}_p(j)^{\mathrm{syn}}(X)$$

*satisfying the following properties*

(1) *if $p$ is invertible in $k$, then we have a functorial equivalence under the motivic-to-syntomic comparison map:*

$$\mathbb{Z}(j)^{\mathrm{mot}}(X)/p \simeq L_{\mathrm{cdh}}\tau^{\leq j}R\Gamma_{\acute{e}t}(-, \mu_p^{\otimes j})(X).$$

(2) *If $p = 0$ in $k$ then we have a cartesian square*

$$\begin{array}{ccc} \mathbb{Z}(j)^{\mathrm{mot}}(X)/p & \longrightarrow & \mathbb{F}_p(j)^{\mathrm{syn}}(X) \\ \downarrow & & \downarrow \\ R\Gamma_{\mathrm{cdh}}(X, \Omega^j_{\log})[-j] & \longrightarrow & R\Gamma_{\acute{e}t}(X, \Omega^j_{\log})[-j]. \end{array}$$

[1]Let us recall that if $p$ is invertible in $k$ then syntomic cohomology of $X$ is its étale cohomology, with an appropriate Tate twist. On the other hand, if $p = 0$ in $k$ then weight-$j$ mod-$p$ syntomic cohomology is described as the left Kan extension of the functor $X \mapsto R\Gamma_{\text{ét}}(X; \Omega^j_{\log})[-j]$.

(3) *Weibel vanishing: for $j \geq 0$ we have*
$$H^{>j+\dim(X)}_{\mathrm{mot}}(X, \mathbb{Z}(j)) = 0.$$
(4) *Nisnevich-locally, the presheaf $H^j_{\mathrm{mot}}(-, \mathbb{Z}(j))$ identifies with Gabber-Kerz's improved Milnor $K$-theory of a local ring.*
(5) *After rationalisation, the filtration functorially splits. In particular, the resulting spectral sequence rationally degenerates.*

Key to a cohomological approach to Theorem 1 is the following result, which can be regraded as a cohomological version of the vanishing theorems of Suslin [10] and Geisser [4].

**Theorem 4.** *Let $X$ be an scheme of finite type over $k$, an algebraically closed field. Then for $j \geq \dim(X)$ the motivic-to-syntomic comparison map*
$$\mathbb{Z}(j)^{\mathrm{mot}}(X)/p \to \mathbb{F}_p(j)^{\mathrm{syn}}(X)$$
*is an equivalence.*

*Proof.* First, let us assume that $p$ is invertible in $k$. Then, by Theorem 3(1), the cofiber of the motivic-to-syntomic comparison map in weight $j$ is equivalent to the presheaf $L_{\mathrm{cdh}}\tau^{>j}R\Gamma_{\mathrm{ét}}(-, \mathbb{F}_p)$. To check that this latter presheaf is zero in the stated range, it suffices to prove that for proper cdh cover $Y \to X$ where $\mathrm{dim(Y)} \leq \mathrm{dim(X)}$, the presheaf $\tau^{>j}R\Gamma_{\mathrm{ét}}(-, \mathbb{F}_p)$ is Nisnevich-locally trivial. However, for any affine scheme Spec(A), étale over $Y$ we have that $R\Gamma_{\mathrm{ét}}(\mathrm{Spec(A)}, \mathrm{\mathbb{F}_p})$ is concentrated in degrees $\leq \mathrm{dim(A)} \leq \mathrm{dim(X)}$ by Artin's vanishing theorem [1, Corollaire XIV.3.2].

Next, assume that $p = 0$ in $k$. This time, by Theorem 3(2), the cofiber of the motivic-to-syntomic comparison map in weight $j$ is equvialent to the presheaf $R\Gamma_{\mathrm{cdh}}(-, \widetilde{\nu}(j))$ where $\widetilde{\nu}(j)$ is the $j$-th Artin–Schreier obstruction [3, Definition 4.29]. In the notation of the previous paragraph, it then suffices to prove that $\widetilde{\nu}(j)$ is Nisnevich-locally trivial on $Y$ for $j \geq \dim(X)$. Let $y \in Y$, we claim that $\widetilde{\nu}(j)(\mathcal{O}^h_{Y,y}) = 0$. We now appeal to the rigidity theorem of Antieau–Mathew–Morrow–Nikolaus [2, Theorem 5.2] which shows that $\widetilde{\nu}(j)(\mathcal{O}^h_{Y,y}) \cong \widetilde{\nu}(j)(\kappa(y))$ where $\kappa(y)$ is the residue field of $y \in Y$. Now, $\kappa(y)$ is transcendence degree $\leq \mathrm{dim(Y)} \leq \mathrm{dim(X)}$ over $k$; that the the Artin–Schreier obstruction vanishes is a consequence of the definition of Kato's $p$-dimension and the inequality of Kato–Kuzumaki [5, Corollary 2].

□

**Remark 5.** *Let $X$ be a quasiprojective variety over $k$ and $z^j(X, \bullet)$ be Bloch's cycle complex so that its* higher Chow groups *(with coefficients) are defined as the homology groups*
$$\mathrm{CH}^j(X, i; \mathbb{Z}/m) =: H_i(z^j(X, \bullet) \otimes^{\mathbb{L}} \mathbb{Z}/m).$$
*Then Suslin (in characteristic zero* [10]*, $j \geq \dim(X)$) and Geisser (in arbitrary characteristic* [4] *for $j = \dim(X)$) proved that there is an isomorphism*
$$\mathrm{CH}^j(X, i; \mathbb{Z}/m) \cong H^{2(\dim(X)-i)+j}_{\mathrm{syn}}(X; \mathbb{Z}/m(d-j))^{\sharp},$$

*where $X$ is an equidimensional quasiprojective variety over an algebraically closed field and $\sharp$ denote Pontrjagin dual (take maps into $\mathbb{Q}/\mathbb{Z}$).*

*It is expected that Bloch's cycle complex is a model for Borel–Moore motivic homology so that, in particular, it admits a functorial action of motivic cohomology which refines the action of $K$-theory on $G$-theory. Under this expected relationship, the above isomorphism is a Borel–Moore counterpart to Theorem 4.*

Equipped with Theorem 4, we proceed to reduce Theorem 1 to a problem in syntomic cohomology. First, we only assume that $X$ is a $k$-variety (not necessarily affine). Thanks to Theorem 3(3), the motivic spectral sequence for $X$ gives rise to an edge homomorphism

$$\mathrm{edge} : H^{2d}_{\mathrm{mot}}(X, \mathbb{Z}(d)) \to \mathrm{K}_0(X).$$

A key property of this edge homomorphism is that it covers $F^d\mathrm{K}_0(X)$ in the sense that $F^d\mathrm{K}_0(X) \subset \mathrm{Image(edge)}$. The proof of this involves the construction of a *cycle class map*; this is a functorial map $Z_0(X_{\mathrm{sm}}) \to H^{2d}_{\mathrm{mot}}(X, \mathbb{Z}(d))$ from the abelian group of zero cycles on the smooth locus of a $k$-variety $X$. It has the property that after post-composing with the edge map, it sends the class of a point $[x] \in Z_0(X_{\mathrm{sm}})$ to the same-named element in $\mathrm{K}_0(\mathrm{X})$.

Combined with Theorem 3(5) which implies that the edge map is a rational injection, Theorem 1 will follow once we know that $H^{2d}_{\mathrm{mot}}(X, \mathbb{Z}(d))$ is torsionfree. This we may check one prime at a time. Let $p$ be a prime then plugging in Theorem 4, the Bockstein sequence looks like

$$H^{2d-1}_{\mathrm{mot}}(X, \mathbb{Z}(d)) \to H^{2d-1}_{\mathrm{syn}}(X; \mathbb{F}_p(d)) \xrightarrow{\delta} H^{2d}_{\mathrm{mot}}(X, \mathbb{Z}(d)) \xrightarrow{\cdot p} H^{2d}_{\mathrm{mot}}(X, \mathbb{Z}(d)) \to \cdots$$

Now we use that $X$ is affine. If $p$ is invertible, then

$$H^{2d-1}_{\mathrm{syn}}(X; \mathbb{F}_p(d)) = H^{2d-1}_{\text{ét}}(X; \mathbb{F}_p) = 0$$

as soon as $2d-1 \geq d+1$ by Artin's vanishing theorem. In other words, $\cdot p$ is injective when $d \geq 2$. If $k$ is characteristic $p$, then we have that $H^{2d-1}_{\mathrm{syn}}(X; \mathbb{F}_p(d)) = 0$ whenever $2d-1 \geq d+2$ by [2, Theorem G]. In other words, $\cdot p$ is injective whenever $d \geq 3$.

When $p = 0$ in $k$ and $d = 2$ the result is equivalent to the surjectivity of the map $H^3_{\mathrm{syn}}(X, \mathbb{Z}_p(2)) \to H^3_{\mathrm{syn}}(X, \mathbb{F}_p(2))$ induced by mod-$p$ reduction. We are working on a direct proof of this surjectivity.

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## Towards a categorical Künneth formula for motives

Jakob Scholbach

Throughout, let $X, Y$ be two algebraic varieties over a field $k$ with finite Galois cohomological dimension. We consider the category $\mathrm{DM}(X)$ of étale motivic sheaves, as introduced and studied by Ayoub and Cisinski–Déglise, and the full stable presentable subcategories

$$\mathrm{DM}(X)'' \subset \mathrm{DM}(X)' \subset \mathrm{DM}(X)$$

that are generated, respectively, by $f_*\mathbf{Z}(n)$, for $f : T \to X$ smooth and proper, and by all dualizable objects. As a consequence of de Jong's resolution of singularities, these three categories coincide for $X = \operatorname{Spec} k$.

**Conjecture 1.** *If $Y$ is smooth and proper over $k$, the exterior product induces an equivalence (of stable $\infty$-categories)*

$$\mathrm{DM}(X)' \otimes_{\mathrm{DM}(k)} \mathrm{DM}(Y)' \overset{\cong}{\to} \mathrm{DM}(X \times_k Y)'.$$

We call this a *categorical Künneth formula.* In addition to its intrinsic beauty, we study this question because of potential applications in the Langlands program.

It is a consequence of a nontrivial theorem of Jin–Yang that the functor

$$\mathrm{DM}(X) \otimes_{\mathrm{DM}(k)} \mathrm{DM}(Y) \to \mathrm{DM}(X \times_k Y)$$

is fully faithful. The essence of the conjecture above is therefore to see how dualizable objects $P \in \mathrm{DM}(X \times Y)$ (i.e., "motivic local systems" on $X \times Y$) can be constructed out of such objects on the individual factors $X$ and $Y$.

**Étale torsion sheaves.** For a prime $\ell \neq \mathrm{char}(k)$, rigidity for etale motives asserts an equivalence $\mathrm{DM}(-) \otimes_{\mathrm{Mod}_{\mathbf{Z}}} \mathrm{Mod}_{\mathbf{Z}/\ell} = \mathrm{D}_{\mathrm{et}}(X)$, where the right hand side denotes the derived category of étale sheaves of $\mathbf{Z}/\ell$-vector spaces on $X$. A dualizable object is precisely a sheaf that is étale-locally constant and there given by a perfect complex of $\mathbf{Z}/\ell$-modules. For concreteness, we assume in this section that $k$ is separably closed, so that $\mathrm{D}_{\mathrm{et}}(\operatorname{Spec} k) = \mathrm{Mod}_{\mathbf{Z}/\ell}$, the derived category of $\mathbf{Z}/\ell$-vector spaces.

**Theorem 2.** [HRS24] *If $Y$ is smooth and a) $Y$ is proper or b) char $k = 0$, there is an equivalence*

$$\mathrm{D}_{\mathrm{et}}(X)' \otimes_{\mathrm{Mod}_{\mathbf{Z}/\ell}} \mathrm{D}_{\mathrm{et}}(Y)' = \mathrm{D}_{\mathrm{et}}(X \times_k Y)'.$$

This relates to, and uses as an input, the isomorphism

$$\pi_1(X\times_k Y) \overset{\cong}{\to} \pi_1(X)\times\pi_1(Y).$$

Such a formula does not hold over $\mathbf{F}_p$ in general, and Drinfeld's lemma provides a fix for this. This and further input leads to the following variant. For $X/\mathbf{F}_p$ and its base change $\overline{X}/\overline{\mathbf{F}}_p$, we let $\mathrm{D}_{\mathrm{et}}(X^W)$ be the category of *Weil sheaves*, i.e.

$$\mathrm{D}_{\mathrm{et}}(X^W) := \mathrm{Ind}\big(\{(F \in \mathrm{D}_{\mathrm{et,cstr}}(\overline{X}), \alpha : (\mathrm{Frob}_X \times \mathrm{id}_{\overline{\mathbf{F}}_p})^* F \cong F)\}\big).$$

A similar definition is done for more than one factor, as in the right hand side below. The result below applies also to not necessarily locally constant sheaves.

**Theorem 3.** [HRS24] *For $X, Y/\mathbf{F}_p$ (not necessarily smooth nor proper), there is an equivalence*

$$\mathrm{D}_{\mathrm{et}}(X^W) \otimes_{\mathrm{Mod}_{\mathbf{Z}/\ell}} \mathrm{D}_{\mathrm{et}}(Y^W) \cong \mathrm{D}_{\mathrm{et}}(X^W \times_{\overline{\mathbf{F}}_p} Y^W).$$

**A construction of a preimage.** Suppose $X$ admits a $k$-rational point $x$. The following construction is due to Gaitsgory–Rozenblyum–Kazhdan–Varshavsky. Let $e : X\times Y \to Y \overset{x\times\mathrm{id}}{\to} X\times Y$ and $p := X\times Y \to X$ be the projection. Fix a dualizable object $P \in \mathrm{DM}(X\times Y)$ and consider the evaluation map

$$\alpha_P : Q := p^* p_* \underline{\mathrm{Hom}}(e^*P, P) \otimes_{p^*p_*\underline{\mathrm{End}}(e^*P)} e^*P \to P.$$

It is easy to see that for $p$ smooth and proper, $Q \in \mathrm{DM}(X)' \otimes_{\mathrm{DM}(k)} \mathrm{DM}(Y)' \subset \mathrm{DM}(X\times Y)'$. Also, $(x\times \mathrm{id}_Y)^*\alpha_P$ is an isomorphism. I.e., this object $Q$ is a candidate for showing that $P$ lies in the full subcategory $\mathrm{DM}(X)' \otimes_{\mathrm{DM}(k)} \mathrm{DM}(Y)'$. We therefore aim to be in a situation where pullback to a closed point $x$ is conservative.

**Etale cohomological motives.** In the context of etale torsion or $\mathbf{Z}_\ell$-adic sheaves, it is known that $x^* : \mathrm{D}(X)' \to \mathrm{D}(\mathrm{Spec}\, k)$ is conservative (for $X$ connected). Taking our cue from this fact one can enforce an independent-of-$\ell$ six functor formalism that satisfies the above categorical Künneth formula.

**Definition 4.** *Let* $\widehat{\mathrm{DM}}(X) := \lim_{m\in\mathbf{N}^\times,\ \mathit{in}\ \mathrm{Pr}^{\mathrm{L}}_\omega} \mathrm{DM}(X) \otimes_{\mathrm{Mod}_{\mathbf{Z}}} \mathrm{Mod}_{\mathbf{Z}/m}$ *be the profinite completion of* DM. *(If we were to only consider $m = \ell^n$, this would be exactly $\mathrm{D}(X, \mathbf{Z}_\ell)$, by rigidity.) We consider the category of so-called* cohomological motives

$$\mathrm{DM}^\flat(X) := \mathrm{DM}(X)/\ker\left(\mathrm{DM}(X) \to \widehat{\mathrm{DM}}(X)\right).$$

Conjecturally, at least over fields of finite transcendence degree, the adic realization functor $\mathrm{DM}(X) \to \widehat{\mathrm{DM}}(X)$ is conservative, so that we expect $\mathrm{DM}(X) \cong \mathrm{DM}^\flat(X)$. Independently of this deep conjecture, we use the above construction to leverage the conservativity of $x^*$ from etale sheaves to cohomological motives.

**Theorem 5.** [RS25] *For $Y$ smooth and a) $Y$ proper or, b) $\mathrm{char}\, k = 0$, ind-dualizable objects in $\mathrm{DM}^\flat$ satisfy a categorical Künneth formula:*

$$\mathrm{DM}^\flat(X)' \otimes_{\mathrm{DM}^\flat(k)} \mathrm{DM}^\flat(Y)' \cong \mathrm{DM}^\flat(X\times_k Y)'.$$

A similar result also holds for cohomological Weil motives, along the lines of the second theorem above.

**Motives of smooth-proper families.** Current work in progress centers on the following question.

**Question 6.** *Is there an equivalence (for $Y$ smooth and proper)*

$$\mathrm{DM}(X)'' \otimes_{\mathrm{DM}(k)} \mathrm{DM}(Y)'' \overset{\cong}{\to} \mathrm{DM}(X \times_k Y)'' \text{ ?}$$

This would mean that the motive of any smooth proper family $T \to X \times Y$ is decomposable in terms of motives of smooth proper schemes over $X$ and over $Y$.

This question can be approached using Robalo's category of non-commutative motives. In a nutshell, for a variety $S/k$, there is a commutative diagram involving at the top right Kontsevich's category of smooth proper non-commutative schemes:

$$\begin{array}{ccccc}
\mathrm{SmPr}_S & & \xrightarrow{T/S \mapsto \mathrm{QC}(T)} & & \mathrm{SmPr}_S^{\mathrm{nc}} := (\mathrm{Mod}_{\mathrm{QC}(S)} \mathrm{Pr}^{\mathrm{L}}_{\omega})^{\mathrm{dbl,op}} \\
\downarrow & & & & \downarrow \\
\mathrm{Sm}_S & & & & \mathrm{Sm}_S^{\mathrm{nc}} := (\mathrm{Mod}_{\mathrm{QC}(S)} \mathrm{Pr}^{\mathrm{L}}_{\omega})^{\omega,\mathrm{op}} \\
\downarrow & & & & \downarrow \\
\mathrm{SH}(S) & \longrightarrow & \mathrm{Mod}_{\mathrm{KH}} \mathrm{SH}(S) & \longrightarrow & \mathrm{SH}^{\mathrm{nc}}(X) := \mathrm{P}(\mathrm{Sm}_S^{\mathrm{nc}}, \mathrm{Sp})[(\mathrm{Nis}, \mathbf{A}^1)^{-1}].
\end{array}$$

Homotopy K-theory has (non-commutative) Nisnevich descent and $\mathbf{A}^1$-invariance, so KH is an object in $\mathrm{SH}^{\mathrm{nc}}(X)$. The bottom right functor is fully faithful on ind-dualizable objects (Robalo).

In the sequel, let $P = \mathrm{QC}(T)$ be the non-commmutative motive of some smooth and proper $T \to X \times Y$. The map $\alpha_P : Q \to P$ can be defined in the presheaf category $\mathrm{P}(\mathrm{SmPr}^{\mathrm{nc}}_{X \times Y})$ (in a way compatible with the one in $\mathrm{SH}(X \times Y)$).

**Proposition 7.** *(Categorical Nakayama lemma) For a local ring $R$ with residue field $k$, the pullback $x^* : \mathrm{SmPr}^{\mathrm{nc}}_{\mathrm{Spec}\, R} \to \mathrm{SmPr}^{\mathrm{nc}}_{\mathrm{Spec}\, k}$ is conservative.*

Consequently, for $Y$ proper smooth, the following pullback is also conservative:

$$(x \times \mathrm{id}_Y)^* : \mathrm{SmPr}^{\mathrm{nc}}_{\mathrm{Spec}\, R \times_k Y} \to \mathrm{SmPr}^{\mathrm{nc}}_Y.$$

These results relate to the categorical Künneth formula for $\mathrm{DM}(-)''$ as follows: in view of the torsion results, it is enough to consider rational coefficients throughout. Descent allows to replace $k$ by any algebraic extension, guaranteeing the existence of $x \in X(k)$. Then $\mathrm{DM}_{\mathbf{Q}} = \mathrm{SH}_{\mathbf{Q}} \to \mathrm{Mod}_{\mathrm{KH}_{\mathbf{Q}}} \mathrm{SH}$ is conservative by Bott periodicity, and to show that $\alpha_P$ is an isomorphism in $\mathrm{DM}(X \times Y)_{\mathbf{Q}}$ it is enough to do this in $\mathrm{SH}^{\mathrm{nc}}(X \times Y)$. The results above show that the restriction of $\alpha_P$ to $\mathrm{Spec}\, \mathcal{O}_{X,x} \times Y$ is an isomorphism whenever the following is satisfied:

$$Q \in \mathrm{SmPr}^{\mathrm{nc}}_{X \times Y} (\subset \mathrm{P}(\mathrm{SmPr}^{\mathrm{nc}}_{X \times Y})). \tag{1}$$

A Morita-type argument shows that the object $Q' := \underline{\mathrm{Hom}}(e^* P, P) \otimes_{\underline{\mathrm{End}}(e^* P)} e^* P$ does lie in $\mathrm{SmPr}^{\mathrm{nc}}_{X \times Y}$ provided that $T \to X \times Y$ admits a section. Work in progress aims to identify criteria when $Q$ has the dualizability property (1). Provided this dualizability holds, one may conclude that $\alpha_P$ is an isomorphism in $\mathrm{DM}(X \times Y)$ by using a continuity argument and the fact that restriction to an open dense $U \subset X$ gives a conservative functor $\mathrm{DM}(X\times Y)^{\mathrm{dbl}}_{\mathbf{Q}} \to \mathrm{DM}(U\times Y)_{\mathbf{Q}}$.

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# Effective Spectra


Achim Krause

(joint work with Ben Antieau)


## 1. Effective...?

The word "effective" in the title of the talk refers to a notion we refer to as "computationally effective". This is not completely formal, so let us elaborate it in an example:

There are two ways to think about finite-dimensional vector spaces:

- A finite-dimensional $\mathbb{Q}$-vector space is a set $V$ with various maps $V \times V \to V$, $\mathbb{Q} \times V \to V$,... and some properties. A morphism between finite-dimensional $\mathbb{Q}$-vector spaces is a map of sets with some properties.
- A finite-dimensional $\mathbb{Q}$-vector space is specified by a natural number. A morphism between finite-dimensional $\mathbb{Q}$-vector spaces is specified by a matrix.

Of course, these perspectives are equivalent (for example, they describe equivalent categories), and so we can pass back and forth between them. The first one has many theoretical advantages, but the second one is how we actually make the objects of linear algebra amenable to computation. For example, in the second one we are able to represent any finite diagram in the category of finite-dimensional $\mathbb{Q}$-vector spaces in terms of a finite amount of data (some numbers and some matrices), and there exist algorithms for various questions we might want to ask about those diagrams (e.g. finding kernels and cokernels). We think of the second perspective as providing a "computationally effective model of finite-dimensional $\mathbb{Q}$-vector spaces".

Algebra and number theory are full of such effective models. We can describe number fields by minimal polynomials, varieties by Gröbner bases, and so on. This enables a rich body of computer-algebra based tabulation and classification work, such as the [1], a searchable database of number fields, elliptic curves and other objects.

This talk considers ongoing efforts to construct a similarly effective model for (finite) spectra. There are already perspectives that come close to this:

- Every finite spectrum is, up to a shift, a suspension spectrum of a finite CW complex. Their homotopy types can be modeled in terms of a finite amount of data, e.g. as finite simplicial complexes, and there are in fact not only theoretical results about algorithmic computability of various questions in unstable homotopy theory [2], there also exists a working implementation in [3]. However, the complexity of representing and working with finite simplicial complexes actually grows under suspension, even though the Freudenthal suspension theorem suggests that it should become easier (and stabilize). So representing spectra unstably seems indirect and impractical.
- There is a long history of practical and very successful applications of computer-algebraic methods to computations in stable homotopy theory, such as the recent sophisticated Adams spectral sequence computations of [4] leading to a resolution of the fate of the Kervaire invariant element in $\pi_{126}(\mathbb{S})$. While this project makes extensive use of automatically generating, analyzing and comparing Adams spectral sequences of many different finite spectra, the computer-algebraic part deals directly with the Adams spectral sequences, and there is no true representation of arbitrary finite spectra.

We do not expect an effective model for finite spectra to eclipse the capabilities of the subtle mixture of manual and automated methods which has been refined over the last decades to go very deep in the homotopy groups of $\mathbb{S}$ and selected other finite spectra, but instead open the possibility of a more "wide" systematic analysis of stable homotopy theory (which deals with many more finite spectra up to a more modest range of degrees, and a more exhaustive tabulation of things like Toda brackets of maps between them, etc.).

## 2. Comonadicity

Compact objects in $\mathcal{D}(\mathbb{Z})$ admit an effective description: They can all be represented by finite-length chain complexes of finitely-generated abelian groups, which can be written down by a sequence of matrices. There is an adjunction

$$\mathrm{Sp} \rightleftarrows \mathcal{D}(\mathbb{Z})$$

which is *monadic*: Objects of $\mathcal{D}(\mathbb{Z})$ can be viewed as spectra with an additional structure, namely the structure of a module over $\mathbb{Z}$.

If we pass to connective objects in the Postnikov t-structure, the adjunction

$$\mathrm{Sp}_{\geq 0} \rightleftarrows \mathcal{D}(\mathbb{Z})_{\geq 0}$$

is still monadic, but it is now also comonadic! This statement is essentially equivalent to strong convergence of the $\mathbb{Z}$-based Adams spectral sequence for $\mathbb{S}$, and follows from a connectivity argument. It can also be interpreted as some sort of descent for $\mathbb{S} \to \mathbb{Z}$.

Viewing $\mathcal{D}(\mathbb{Z})_{\geq 0}$ as $\mathbb{Z}$-module spectra, the comonad is given by the functor $\mathbb{Z} \otimes_{\mathbb{S}} -$, which forgets the $\mathbb{Z}$-module structure and induces a new one. Comonadicity tells us

$$\mathrm{Sp}_{\geq 0} \simeq \mathrm{CoAlg}_{\mathbb{Z}\otimes_{\mathbb{S}}-}(\mathcal{D}(\mathbb{Z})_{\geq 0}),$$

i.e. that we can view connective spectra as chain complexes with additional structure. It also is not hard to deduce from this a corresponding version for compact objects,

$$\mathrm{Sp}^{\omega} \simeq \mathrm{CoAlg}_{\mathbb{Z}\otimes_{\mathbb{S}}-}(\mathrm{Perf}(\mathbb{Z})),$$

which promises an effective model of spectra, provided we can give an algebraic model for the comonad $\mathbb{Z} \otimes_{\mathbb{S}} -$ and coalgebras over it.

## 3. The $Q$-construction

The functor $\mathbb{Z} \otimes_{\mathbb{S}} - : \mathcal{D}(\mathbb{Z}) \to \mathcal{D}(\mathbb{Z})$ is quite strange: It is exact, in fact preserves all colimits. In particular it is additive. But it cannot come from an additive functor $\mathrm{Ch}(\mathbb{Z}) \to \mathrm{Ch}(\mathbb{Z})$ of additive 1-categories, since then it would have to be $\mathbb{Z}$-linear!

Since $\mathcal{D}(\mathbb{Z})_{\geq 0}$ is the sifted-colimit completion of Latt, the category of finitely generated free abelian groups, the functor $\mathbb{Z} \otimes_{\mathbb{S}} - : \mathcal{D}(\mathbb{Z})_{\geq 0} \to \mathcal{D}(\mathbb{Z})_{\geq 0}$ is characterized by its restriction to Latt, an additive functor $\mathrm{Latt} \to \mathcal{D}(\mathbb{Z})_{\geq 0}$. This also cannot come from an additive functor $\mathrm{Latt} \to \mathrm{Ch}(\mathbb{Z})_{\geq 0}$ for analogous reasons.

**Theorem 1** (Eilenberg-MacLane [5])**.** *For $A \in \mathrm{Latt}$,*

$$\mathbb{Z} \otimes_{\mathbb{S}} A \simeq Q(A) = \big(\mathbb{Z}[A]/\sim \leftarrow \mathbb{Z}[A^2]/\sim \leftarrow \mathbb{Z}[A^4]/\sim \leftarrow \ldots\big)$$

(Of course, they prove a statement on the level of homology.)

Here the quotients in the chain complex on the right are obtained by quotienting out elements with a certain support condition, and the differentials are a kind of "cross effect map" taking the difference between addition on the inside and on the outside of $\mathbb{Z}[-]$. The combinatorics works out in such a way that the functor becomes additive up to chain homotopy, even though its constituent terms are far from being additive. The right hand side complex is written $Q(A)$ and referred to as "MacLane's $Q$-construction" (where it isn't clear to us why Eilenberg is typically not directly credited as well.)

The role of $Q(A)$ as prototypical example of a functor which is additive but not $\mathbb{Z}$-linear was further clarified by Johnson-McCarthy:

**Theorem 2** (Johnson-McCarthy [6])**.** *In fact, $\mathbb{Z} \otimes_{\mathbb{S}} -$ admits a description as Goodwillie derivative $\partial_1 L\mathbb{Z}[-]$, where $L\mathbb{Z}[-] : \mathcal{D}(\mathbb{Z})_{\geq 0} \to \mathcal{D}(\mathbb{Z})_{\geq 0}$ is the nonabelian derived functor ("animation") of $\mathbb{Z}[-] : \mathrm{Ab} \to \mathrm{Ab}$. More generally, for any functor $F : \mathrm{Ab} \to \mathrm{Ab}$, there is a description of $\partial_1 LF(-)$ (evaluated on $A \in \mathrm{Latt}$) as a complex of the form*

$$\big(F(A)/\sim \leftarrow F(A^2)/\sim \leftarrow F(A^4)/\sim \leftarrow \ldots\big)$$

It is worth noting that this is a genuinely different description of the first Goodwillie derivative than the one which is usually given (namely, as $\partial_1 F(X) = \operatorname{colim} \Omega^n F(\Sigma^n X)$).

As a side result, we develop a perspective on the first Goodwillie derivative which extends Johnson-McCarthy's result:

**Theorem 3** (A.-K.). *Let $\mathcal{C}, \mathcal{D}$ be stable $\infty$ categories and assume $\mathcal{D}$ has countable colimits. Write $\mathcal{F} = \mathrm{Fun}(\mathcal{A}, \mathcal{B})_*$ for the category of functors taking $0 \mapsto 0$, and consider the functor $q : \mathcal{F} \to \mathcal{F}$ given informally as*

$$qF(A) = \mathrm{colim}(F(A) \oplus F(A) \to F(A \oplus A) \rightrightarrows F(A)),$$

*where the first of the two maps on the right is $F$ applied to the codiagonal $A \oplus A \to A$, and the other is the sum $F(p_1) + F(p_2)$ with $p_i$ the two projections, and the colimit is a "restricted coequalizer", indexed over the nerve of the 1-category with objects $0, 1, 2$, a unique morphism $0 \to 1$, two morphisms $1 \to 2$ and a unique morphism $0 \to 2$.*

*Then $q$ with the natural transformation $\mathrm{id}_{\mathcal{F}} \to q$ coming from including the last $F(A)$ is a pointed endofunctor on $\mathcal{F}$, and:*

(1) *A functor $F \in \mathcal{F}$ admits a $q$-algebra structure (in the sense of a map $qF \to F$ with a homotopy beween the composite $F \to qF \to F$ and the identity) if and only if it is additive.*
(2) *If $F \in \mathcal{F}$ preserves sifted colimits, the Goodwillie derivative $\partial_1 F$ is the initial $q$-algebra under $F$.*

*Furthermore, an explicit description of the initial algebra over a pointed endofunctor applied to $q$ recovers the Johnson-McCarthy description of $\partial_1 F$.*

At the very least, this Goodwillie derivative interpretation of $\mathbb{Z} \otimes_{\mathbb{S}} -$ provides a very compelling perspective on spectra: There is an underived version of the adjunction between $\mathrm{Sp}_{\geq 0}$ and $\mathcal{D}(\mathbb{Z})_{\geq 0}$,

$$\mathrm{Set} \rightleftarrows \mathrm{Ab}$$

which is also not only monadic but also comonadic (although this is not a very useful perspective on sets). The comonad is $\mathbb{Z}[-] : \mathrm{Ab} \to \mathrm{Ab}$, and passing to nonabelian derived functors, we obtain a comonad $L\mathbb{Z}[-]$ on $\mathcal{D}(\mathbb{Z})_{\geq 0}$. Since the comonad $Q$ is obtained from this by passing to the first Goodwillie derivative, i.e. enforcing exactness, we can think of the passage to $\partial_1 L\mathbb{Z}[-]$-coalgebras as essentially "descending away the $\mathbb{Z}$-linearity in a universal way while preserving exactness", which gives a purely derived-algebraic incarnation of the theme that Sp is a sort of universal world in which exactness makes sense.

## 4. $\infty$-functors

The description of $\mathrm{Sp}_{\geq 0}$ as $Q$-coalgebras in $\mathcal{D}(\mathbb{Z})_{\geq 0}$, and $\mathrm{Sp}^\omega$ as $Q$-coalgebras in $\mathrm{Perf}(\mathbb{Z})$, in principle lets us describe a finite spectrum as a finite chain complex $C$ with a map $C \to Q(C)$, a homotopy between two maps $C \to Q^2(C)$, etc. (together with counitality data). This is a chain-complex style description of spectra, but not immediately useful (at least not to provide an "effective model" for spectra), since this is an infinite amount of data, and the complexes $Q(C)$ etc. are huge. In order for this to be made useful, we require a smaller description of $Q$.

To specify a sifted-colimit preserving functor $\mathcal{D}(\mathbb{Z})_{\geq 0} \to \mathcal{D}(\mathbb{Z})_{\geq 0}$, i.e. a functor $\mathrm{Latt} \to \mathcal{D}(\mathbb{Z})_{\geq 0}$, one has different options. The easiest (and classical) option is to simply write down an explicit functor $\mathrm{Latt} \to \mathrm{Ch}(\mathbb{Z})_{\geq 0}$, i.e. essentially producing a 1-categorical lift of the desired functor as in the following diagram.

$$\begin{array}{ccc} & & \mathrm{Ch}(\mathbb{Z})_{\geq 0} \\ & \nearrow & \downarrow \\ \mathrm{Latt} & \longrightarrow & \mathcal{D}(\mathbb{Z})_{\geq 0} \end{array}$$

For example, this is how Eilenberg-MacLane's description of the functor $Q$ is provided. Model category theory abstractly tells us that any functor can be described by such a 1-categorical lift, but that the resulting complexes might be big even if the homology of $F(A)$ for each $A \in \mathrm{Latt}$ is small. This is the case for the $Q$-construction, where each $Q(A)$ has homology groups given by the homology of the Eilenberg-MacLane spectrum corresponding to $A$, which in particular is finitely generated in each degree, but the complexes $Q(A)$ are countably infinitely generated in each degree.

There is a separate way of describing a functor $\mathrm{Latt} \to \mathcal{D}(\mathbb{Z})_{\geq 0}$, directly as functor of $\infty$-categories. If we view Latt as the nerve of the category with objects $\mathbb{Z}^n$ and integer matrices as morphisms, and $\mathcal{D}(\mathbb{Z})_{\geq 0}$ as dg nerve of levelwise projective chain complexes, such a functor is an assignment $F$ which takes every object $A = \mathbb{Z}^n \in \mathrm{Latt}$ to a chain complex $F(A)$, every morphism $f : A \to B$ to a chain map $F(f) : F(A) \to F(B)$, every pair of composable morphisms $f$ and $g$ to a chain homotopy between $F(g) \circ F(f)$ and $F(g \circ f)$, and more generally every $n$-tuple of composable morphisms $f_i : A_i \to A_{i+1}$ to a map

$$F(f_0, \ldots, f_{n-1}) : F(A_0) \to F(A_n)[1 - n],$$

whose boundary in $\mathrm{Hom}(F(A_0), F(A_n))$ needs to satisfy an identity similar to the dg description of $A_\infty$ structures. One can think of this type of data as a kind of cocycle description of functors into the derived category. To stress this perspective, we refer to such a gadget as an $\infty$-functor.

Every strict functor $\mathrm{Latt} \to \mathrm{Ch}(\mathbb{Z})_{\geq 0}$ does of course give rise to an $\infty$-functor, namely one where the higher coherences $F(f_0, \ldots, f_n)$ for $n > 1$ vanish. However, by allowing for nontrivial higher coherences, one gains additional flexibility in the form of the following "transport lemma":

**Lemma 1.** *Given an $\infty$-functor $F : \mathrm{Latt} \to \mathcal{D}(\mathbb{Z})_{\geq 0}$, and for each $A \in \mathrm{Latt}$ a chain homotopy equivalence $\eta_A : F(A) \simeq F'(A)$, there exists an $\infty$-functor extending the assignment $A \mapsto F'(A)$ on objects, and $F$ and $F'$ are equivalent in the $\infty$-category $\mathrm{Fun}(\mathrm{Latt}, \mathcal{D}(\mathbb{Z})_{\geq 0})$.*

This is not hard to prove, and in fact one can describe explicit formulas in terms of the homotopy equivalences $\eta_A$. But it tells us that any strict functor $\mathrm{Latt} \to \mathrm{Ch}(\mathbb{Z})_{\geq 0}$ which is pointwise of finite type, meaning that for each $A$, $F(A)$ has finitely generated homology groups, actually admits an equivalent $\infty$-functor $F'(A)$ which on objects really produces levelwise finitely generated chain complexes. Thus one can trade strictness (the vanishing of higher coherences) for finiteness conditions.

The finite-type conditions are satisfied by $Q$, since $Q(\mathbb{Z}^n) \simeq Q(\mathbb{Z})^n$ and $Q(\mathbb{Z}) \simeq \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z}$ is of finite type. In fact, if we pick a minimal resolution

$$Q^{\text{fin}}(\mathbb{Z}) := \mathbb{Z} \leftarrow 0 \leftarrow \mathbb{Z} \stackrel{2}{\leftarrow} \mathbb{Z} \leftarrow \dots$$

of the homology of the Eilenberg-MacLane spectrum $\mathbb{Z}$, then there exist chain homotopy equivalences $Q(\mathbb{Z}^n) \simeq Q^{\text{fin}}(\mathbb{Z})^n$, and one can use the above transport lemma to obtain an $\infty$-functor $Q^{\text{fin}}$ representing $\mathbb{Z} \otimes_{\mathbb{S}} -$, but taking finite-type values pointwise. Explicitly, the data required to describe this $\infty$-functor consists of maps

$$Q^{\text{fin}}(f_0, \dots, f_n) : Q^{\text{fin}}(\mathbb{Z})^{r_0} \to Q^{\text{fin}}(\mathbb{Z})^{r_n}[1-n]$$

for each sequence of composable integer matrices $f_i : \mathbb{Z}^{r_i} \to \mathbb{Z}^{r_{i+1}}$. Since the two chain complexes are levelwise finite-rank, if we restrict attention to a range of degrees, these $Q^{\text{fin}}(f_0, \dots, f_n)$ consist of finite-size matrices. Making this assignment explicit makes it possible to mechanically work with the functor $Q \simeq Q^{\text{fin}}$.

(We currently have SAGE code that makes it possible to compute $Q^{\text{fin}}(f_0, ..., f_n)$ through a range of degrees, and are noticing some very interesting polynomiality behaviour that suggests that this could be made more explicit, but this is very much work in progress.)

## 5. $Q$-Coalgebras

Analogously to how the $\infty$-functor perspective leads to a finite-type description of pointwise finite-type functors Latt $\to \mathcal{D}(\mathbb{Z})_{\geq 0}$, we are hoping to be able to similarly make explicit the data of a coherent comonad on $\mathcal{D}(\mathbb{Z})_{\geq 0}$. Provided such a description, the comonad structure on $Q$ transports over to a comonad structure on the equivalent functor $Q^{\text{fin}}$.

A *nonunital* $Q^{\text{fin}}$-coalgebra structure on a complex $C$ consists of maps $C \to Q^{\text{fin}}(C)$, a chain homotopy $C \to (Q^{\text{fin}})^2(C)[-1]$, and more generally maps $C \to (Q^{\text{fin}})^n(C)[1-n]$ with prescribed boundary. For a *unital* coalgebra structure, these maps additionally have to interact with the counit $Q^{\text{fin}} \to \text{id}$, and if we denote the fiber of this natural transformation by $\overline{Q^{\text{fin}}}$, the structure maps in the unital case land in torsors over maps $C \to \overline{Q^{\text{fin}}}^n(C)[1-n]$. Note that $\overline{Q^{\text{fin}}}$ increases connectivity by 2, which means that as soon as $n$ exceeds the amplitude of $C$ the higher coherences for a unital coalgebra structure become unique. This is related to the fact that the $\mathbb{Z}$-based Adams spectral sequence has a vanishing line of slope a priori at most 1. Since additionally all the structure maps are maps from $C$ into some finite-type complex, if $C$ is perfect we learn that $Q^{\text{fin}}$-coalgebra structures on $C$ can be described by a finite amount of data. (Which we hope to be able to make completely explicit.)

Obtaining a fully explicit finitary picture of $Q$ as a comonad and $Q$-coalgebras as outlined above is certainly ambitious, and this is very much work in progress. However, we want to note one potential application which is in reach even before carrying this out fully. We point to work of Baues and Baues-Jibladze [7], [8], where an algebraic model for the “secondary Steenrod algebra”, capturing essentially the homotopy 2-category of the full subcategory of spectra on shifts of $\mathbb{F}_p$, is studied and applied to give an algebraic description of the $E_3$ page of the Adams spectral sequence. This work is closely related to the study of the comonad structure on $Q$, and $Q$-coalgebra structures, up to a bicategorical level of coherences (more precisely, a $\mathbb{F}_p$-based version of $Q$). We expect that carrying out the program outlined above even to a finite level of coherences can be used analogously to determine later pages of the Adams spectral sequence.

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*Reporter: Adrien Morin*

# Participants

**Prof. Dr. Benjamin Antieau**
Department of Mathematics
Lunt Hall
Northwestern University
2033 Sheridan Road
Evanston, IL 60208-2730
UNITED STATES

**Ko Aoki**
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

**Marco Artusa**
Université de Strasbourg
67000 Strasbourg
FRANCE

**Prof. Dr. Joseph Ayoub**
Institut für Mathematik
Universität Zürich
Winterthurerstrasse 190
8057 Zürich
SWITZERLAND

**Prof. Dr. Tom Bachmann**
Institut für Mathematik
Johannes Gutenberg-Universität Mainz
Staudingerweg 9
55128 Mainz
GERMANY

**Prof. Dr. Federico Binda**
Dipartimento di Matematica
Università di Milano
Via C. Saldini, 50
20133 Milano
ITALY

**Tess Bouis**
University of Regensburg
Faculty of Mathematics
Regensburg 93040
GERMANY

**Prof. Dr. Robert Burklund**
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
2100 København
DENMARK

**Prof. Dr. Denis-Charles Cisinski**
Fakultät für Mathematik
Universität Regensburg
93040 Regensburg
GERMANY

**Prof. Dr. Guillermo Cortinas**
Dep. de Matemática / IMAS
Facultad de Ciencias
Exactas y Naturales
Universidad de Buenos Aires
Ciudad Universitaria Pab. 1
C 1428 EGA Buenos Aires
ARGENTINA

**Dr. Christian Dahlhausen**
Institut für Mathematik
Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

**Dr. Elden Elmanto**
University of Toronto
40 St. George St
Toronto M5S 2E4
CANADA

**Prof. Dr. Dr. h.c.mult. Hélène Esnault**
Fachbereich Mathematik und Informatik
Freie Universität Berlin
Arnimallee 7
14195 Berlin
GERMANY

**Prof. Dr. Matthias Flach**
Department of Mathematics
California Institute of Technology
Pasadena, CA 91125
UNITED STATES

**Dr. Evangelia Gazaki**
Department of Mathematics
University of Virginia
141 Cabell Drive
22902 Charlottesville VA 22902
UNITED STATES

**Prof. Dr. Thomas H. Geisser**
Department of Mathematics
Rikkyo University
3-34-1 Nishi-Ikebukuro, Toshima-ku
Tokyo 171-8501
JAPAN

**Dr. Fabian Hebestreit**
Fakultät für Mathematik
Universität Bielefeld
Bielefeld 33615
GERMANY

**Prof. Dr. Lars Hesselholt**
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
2100 København
DENMARK

**Prof. Dr. Jens Hornbostel**
Fakultät IV:
Mathematik und Naturwissenschaften
Bergische Universität Wuppertal
Gaußstraße 20
42119 Wuppertal
GERMANY

**Prof. Dr. Marc Hoyois**
Fakultät für Mathematik
Universität Regensburg
Universitätsstr. 31
93053 Regensburg
GERMANY

**Dr. Ryomei Iwasa**
Laboratoire de Rech. Informatique
CNRS, Université Paris Sud (Paris XI)
Centre d'Orsay, Bat. 490
91405 Orsay Cedex
FRANCE

**Prof. Dr. Shane Kelly**
Department of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

**Prof. Dr. Moritz Kerz**
Fakultät für Mathematik
Universität Regensburg
Universitätsstraße 31
93053 Regensburg
GERMANY

**Dr. Andrei Konovalov**
Fachbereich Mathematik
Universität Duisburg-Essen
45127 Essen
GERMANY

**Prof. Dr. Achim Krause**
Department of Mathematics
University of Oslo
P.O. Box 1053 Blindern
0316 Oslo
NORWAY

**Prof. Dr. Markus Land**
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstr. 39
80333 München
GERMANY

**Prof. Dr. Marc Levine**
Fakultät für Mathematik
Universität Duisburg-Essen
Universitätsstraße 2
45117 Essen
GERMANY

**Dr. Ishan Levy**
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
2100 København
DENMARK

**Lucas Mann**
Universität Münster
Einsteinstraße 62
48149 Münster
GERMANY

**Dr. Lorenzo Mantovani**
Institut für Mathematik
Johannes Gutenberg-Universität Mainz
Staudingerweg 9
Postfach 3980
55128 Mainz
GERMANY

**Dr. Akhil Mathew**
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago, IL 60637-1514
UNITED STATES

**Dr. Adrien Morin**
University of Copenhagen
Universitetsparken 5
2100 København
DENMARK

**Dr. Baptiste Morin**
Institut de Mathématiques de Bordeaux
CNRS
351 cours de la Libération
33 405 Talence Cedex 9
FRANCE

**Dr. Matthew Morrow**
Université Paris-Saclay, Laboratoire de
Mathématique d'Orsay
Bâtiment 307, rue Michel Magat
91405 Orsay
FRANCE

**Prof. Dr. Paul Arne Østvær**
Department of Mathematics
University of Milan
Via Cesare Saldini 50
20133 Milano
ITALY

**Vivien Picard**
Bergische Universität Wuppertal
Gaußstraße 20
42119 Wuppertal
GERMANY

**Dr. Maxime Ramzi**
Universität Münster
Einsteinstraße 62
48149 Münster
GERMANY

**Dr. Charanya Ravi**
Tata Institute of Fundamental Research, Mumbai
Homi Bhabha Road, Colaba
Mumbai 400005
INDIA

**Prof. Dr. Oliver Röndigs**
Fachbereich Mathematik/Informatik
Universität Osnabrück
49069 Osnabrück
GERMANY

**Dr. Kay Rülling**
Fakultät für Mathematik u. Naturwissenschaften
Bergische Universität Wuppertal
Gaußstr. 20
42119 Wuppertal
GERMANY

**Dr. Jakob Scholbach**
Università degli studi di Padova
Via Trieste 63
35121 Padova
ITALY

**Dr. Vladimir Sosnilo**
Am Vitusbach 13
93051 Regensburg
GERMANY

**Prof. Dr. Markus Spitzweck**
Fachbereich Mathematik/Informatik/Physik
Universität Osnabrück
49069 Osnabrück
GERMANY

**Prof. Dr. Georg Tamme**
Institut für Mathematik
Johannes Gutenberg-Universität Mainz
Staudingerweg 9
55128 Mainz
GERMANY

**Longke Tang**
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton NJ 08544-1000
UNITED STATES

**Luca Terenzi**
UMPA, ENS de Lyon
46, Allée d'Italie
69364 Lyon cedex 07
FRANCE

**Ferdinand Wagner**
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

**Dr. Guozhen Wang**
Shanghai Center for Mathematical Sciences
Fudan University
Jiangwan Campus
2005 Songhu Road
Shanghai Shi 200 438
CHINA

**Prof. Dr. Kirsten G. Wickelgren**
Department of Mathematics
Duke University
Room 117, Physics
120 Science Drive
Durham, NC 27708-0320
UNITED STATES

**Dr. Maria Yakerson**
IMJ-PRG
Sorbonne Université
4 place Jussieu
75005 Paris
FRANCE

**Dr. Inna Zakharevich**
Department of Mathematics
Cornell University
Malott Hall 587
Ithaca, NY 14850
UNITED STATES