Mathematisches Forschungsinstitut Oberwolfach

Report No. 33/2025

DOI: 10.4171/OWR/2025/33

Partial Differential Equations

Organized by
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20 July – 25 July 2025

Abstract. This workshop focused on nonlinear elliptic and parabolic partial differential equations, touching topics such as geometric flows, geometric variational problems and minimal surfaces, free boundaries, and geometric measure theory.

Mathematics Subject Classification (2020): 35Jxx, 35Kxx, 53C42, 49Q20, 49Rxx, 35R35, 35Pxx, 32W20.

Introduction by the Organizers

The workshop *Partial Differential Equations*, organized by Ailana Fraser (UBC), Xavier Ros-Oton (Barcelona) and Felix Schulze (Warwick) was held July 20–July 25, 2025. The meeting was attended by 48 participants with broad geographic representation. The program consisted of 22 talks and left sufficient time for discussions.

As in the tradition of the workshop, a main theme of the workshop was around PDE related to geometric and variational problems. New progress related to regularity theory for solutions of geometric PDE was announced in several talks. This included a talk in which generic regularity of area-minimizing hypersurfaces in ambient dimension 11 was presented. Another talk discussed work investigating the size of the singular set of stationary integral varifolds, showing that provided an ϵ-regularity property holds, that the branch set has codimension 2, and consequently that the whole singular set has codimension 1. A generic regularity result for stationary integral varifolds with only strongly isolated singularities inside Riemannian manifolds, in absence of any restriction on the dimension and codimension was discussed. Conditions that guarantee regularity of viscosity solutions to the special Lagrangian equation were presented, along with examples of singular solutions showing that these conditions are sharp. In the context of free boundary problems, it was presented that the Hausdorff dimension of the branching set at the boundary for minimizers of the one-phase Bernoulli free boundary problem is at most $n-2$. A classification of global solutions to the one-phase free boundary problem which are asymptotic to the De Silva-Jerison cone was presented.

Geometric properties of min-max geodesics were discussed, including upper bounds on their Morse index and number of self-intersections. The existence of local minimizers of a multiple bubble problem with one finite-volume and two infinite-volume chambers whose interface has a blowdown that is a singular minimizing cone was presented, demonstrating non-uniqueness of such local minimizers for a large number of dimensions.

Several presentations focused on recent developments in geometric flows. This included a talk where extensions of the Harnack inequality for curve shortening flow without convexity assumptions were presented. For the analogue in higher dimension, the mean curvature flow, one talk discussed the instability of the peanut solution. For more general ancient solutions with an asymptotically cylindrical profile, it was explained how to employ information about the dominant eigenmode to trace the finer asymptotic shape of the solution. A further contribution explained how to capitalise on a non-degeneracy condition for generic singularities to obtain a precise picture of the evolution locally prior and past the singularity. Another presentation discussed an extension to the classical avoidance principle and its implications. As an application of mean curvature flow to questions in geometry connectivity properties of the space of mean convex spheres and tori in three-manifolds was presented. Other geometric flows were the focus of further talks. One talk discussed uniqueness of smooth non-compact solutions to Ricci flow under natural scaling invariant curvature bounds. For general parabolic systems a talk about existence and uniqueness result for non-smooth initial data was given.

Other PDE aspects were covered as well, with several talks presenting recent progress on PDE topics related to physics or probability. This included a talk on a liquid drop model, which uses geometric and variational techniques. Another talk discusses work investigating travelling waves for a 2D equation arising in fluid dynamics, and its connection to free boundary problems. A further contribution gave a geometric view about regularity structures, arising in the study of stochastic PDEs. For kinetic equations a talk about the decay for large velocities for the Boltzmann equation was given. Finally, one talk discussed a new characterization of GBD, an energy space arising in linearized elasticity with cracks.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows".

Workshop: Partial Differential Equations

Table of Contents

Abstracts

The liquid drop model

Rupert L. Frank
(joint work with Mathieu Lewin, Robert Seiringer)

The following energy functional appears in physics to model both highly compressed nuclear matter found in the crust of neutron stars (Ravenhall–Pethick–Wilson, Hashimoto–Seki–Yamada) and microphase separation in diblock copolymers (Ohta–Kawasaki). Our main motivation comes from the first application.

Let $\rho \in (0,1)$. For $L > 0$ let $\Lambda_L := (-L/2, L/2)^3 \subset \mathbb{R}^3$ and for $\Omega \subset \Lambda_L$ measurable, let

$$\mathcal{E}_{\rho,L}[\Omega] := \operatorname{Per}\Omega + \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(\mathbb{1}_\Omega(x) - \rho\mathbb{1}_{\Lambda_L}(x))\,(\mathbb{1}_\Omega(y) - \rho\mathbb{1}_{\Lambda_L}(y))}{|x-y|}\,dx\,dy\,.$$

and consider the minimization problem

$$E(\rho, L) := \inf\{\mathcal{E}_{\rho,L}[\Omega] :\ \Omega \subset \Lambda_L\,,\ |\Omega| = \rho L^3\}\,.$$

We know that the thermodynamic limit exists:

$$\exists\ e(\rho) := \lim_{L\to\infty} \frac{E(\rho,L)}{L^3}\,.$$

This is subtle due to the long-range nature of the Coulomb potential. It can be derived from results by Alberti–Choksi–Otto (2009) (who prove much more). It can also be proved using techniques of Lebowitz, Lieb and Narnhofer.

Our main result [1] describes the behavior of $e(\rho)$ as $\rho \to 0$ and provides a justification of what is known as the *gnocchi phase* among the *nuclear pasta phases.*

Theorem. *There are positive constants μ_* and e_* such that, as $\rho \to 0$,*

$$e(\rho) = \mu_*\rho - e_*\rho^{4/3} + o(\rho^{4/3})\,.$$

Intuitively,

- Proving $O(\rho)$ corresponds to proving that there are droplets of size ~ 1.
- Proving $\mu_*\rho + o(\rho)$ corresponds to finding the shape of the droplet.
- Proving $\mu_*\rho + O(\rho^{4/3})$ corresponds to proving distance $\sim \rho^{-1/3}$ between droplets.
- Proving the theorem corresponds to undestanding the arrangement of the droplets.

We briefly summarize earlier work. The asymptotics $e(\rho) = \mu_*\rho + o(\rho)$ is due to Emmert–F.–König (2020). Similar leading order results are due to Knüpfer–Muratov–Novaga (2016) in the *ultra-dilute limit* $\rho \sim L^{-2}$. We stress that we first let $L \to \infty$, then $\rho \to 0$. Results reminiscent of the sub-leading order asymptotics are due to Choksi–Peletier (2010) in the *ultra-ultra-dilute limit* $\rho \sim L^{-3}$.

The constant μ_* is defined as follows. The liquid drop model without background is

$$\mathcal{E}[\Omega] := \operatorname{Per}\Omega + \frac{1}{2}\iint_{\Omega\times\Omega}\frac{dx\,dy}{|x-y|}\,, \qquad E(A) := \inf\left\{\mathcal{E}[\Omega]:\ |\Omega| = A\right\}\,.$$

As shown by F.–Lieb (2015) and Knüpfer–Muratov–Novaga (2016),

$$\mu_* := \inf_{A>0}\frac{E(A)}{A}$$

is attained. The minimizer for μ_* is widely believed, but not known to be a ball.

The constant e_* is defined as follows. The Jellium model (Wigner, 1934) is

$$\mathcal{E}^{\mathrm{J}}_L[X] := \sum_{n<m}\frac{1}{|X_n - X_m|} - \sum_{n=1}^{N}\int_{\Lambda_L}\frac{dx}{|X_n - x|} + \frac{1}{2}\iint_{\Lambda_L\times\Lambda_L}\frac{dx\,dy}{|x-y|}\,,\ X\in(\Lambda_L)^N,$$

$$E^{\mathrm{J}}_L := \inf\left\{\mathcal{E}^{\mathrm{J}}[X]:\ X\in(\Lambda_L)^N\,,\ N = L^3\right\}$$

We know that the thermodynamic limit exists (Lieb–Narnhofer, 1975):

$$\exists\ e^{\mathrm{J}} := \lim_{L\to\infty}\frac{E^{\mathrm{J}}_L}{L^3}\,.$$

Minimizers are widely believed, but not known to be arranged on a BCC lattice.

We define, with $A_{**} := \max\{A:\ E(A)/A = \mu_*\}$ (which is known to be finite),

$$e_* = -A_{**}^{2/3}\,e^{\mathrm{J}}\,.$$

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Generic regularity of minimizing hypersurfaces up to dimension 11

Christos Mantoulidis
(joint work with Otis Chodosh, Felix Schulze, Zhihan Wang)

Consider a smooth, closed, oriented $(n-2)$-dimensional submanifold $\Gamma\subset\mathbf{R}^n$. Among all smooth, compact hypersurfaces $M\subset\mathbf{R}^n$ with $\partial M = \Gamma$, we want to find one of least area ("minimizing"). Using geometric measure theory, one can prove the existence of a minimizer among a weaker class of objects, integral currents. A minimizer is now well-known to be a smooth hypersurface if $n\le 7$, or more generally smooth outside an $(n-8)$-dimensional singular set [1, 2, 3, 4, 5, 6]. In $\mathbf{R}^8$, the first dimension in which singularities appear [7], Hardt–Simon [8] showed that for a generic choice of Plateau boundary Γ, minimizing M are smooth. Smale [9] proved an analogous result for minimizers in 8-dimensional manifold codimension-1 integral homology classes, with generic Riemannian metrics.

These generic regularity results were recently extended with Chodosh and Schulze to cover $\mathbf{R}^9$ and $\mathbf{R}^{10}$ [10]. We also proved the analogous results in the context of 9- and 10-manifolds for minimization in codimension-1 integral homology classes [11]. In higher dimensions, we proved that a minimizing M will have a $\leq n - 10 - \epsilon_n$ dimensional singular set after perhaps a C^∞-perturbation of the Plateau boundary. See also [12].

We now extended this to $\mathbf{R}^{11}$, 11-manifolds, and a corresponding $\leq n - 11 - \epsilon_n'$ dimensional singular set after perhaps a C^∞-perturbation of the Plateau boundary or Riemannian metric [13]. Let us discuss what goes into the proof of this theorem for the Plateau problem in $\mathbf{R}^n$, focusing on the case $n \leq 11$. Let us denote

$$\mathcal{M}(\Gamma) = \{\text{minimizing integral currents in } \mathbf{R}^n \text{ with boundary } [\![\Gamma]\!]\}.$$

We agree to the following simplifying assumptions for the presentation:

- Γ is connected.
- $\mathcal{M}(\Gamma)$ is a singleton.

Set $\Gamma_0 := \Gamma$ and perturb Γ smoothly to $(\Gamma_s)_{s\in(-\delta,\delta)}$ by s times the unit normal to the unique minimizing $M \in \mathcal{M}(\Gamma)$ (by [6], $\operatorname{sing} M \cap \Gamma = \emptyset$) for some small $\delta > 0$. For each $s \in (-\delta, \delta)$, let $\mathcal{M}(\Gamma_s)$ be the set of all minimizers with boundary data Γ_s; each such is still of the form $[\![M_s]\!]$, with M_s enjoying similar a priori regularity as M. A cut-and-paste argument implies that

$$(\ddagger) \qquad [\![M_s]\!] \in \mathcal{M}(\Gamma_s),\ [\![M_{s'}]\!] \in \mathcal{M}(\Gamma_{s'}),\ s \neq s' \implies \bar{M}_s \cap \bar{M}_{s'} = \emptyset.$$

Define

$$\mathcal{L} = \cup_{s\in(-\delta,\delta)} \cup_{[\![M_s]\!]\in\mathcal{M}(\Gamma_s)} \bar{M}_s,$$
$$\mathcal{S} = \cup_{s\in(-\delta,\delta)} \cup_{[\![M_s]\!]\in\mathcal{M}(\Gamma_s)} \operatorname{sing} M_s.$$

In view of (‡), the following "timestamp" function $\mathbf{t} : \mathcal{L} \to (-\delta, \delta)$ is well-defined:

$$\mathbf{t}(x) = s \text{ for all } x \in \bar{M}_s,\ [\![M_s]\!] \in \mathcal{M}(\Gamma_s),\ s \in (-\delta, \delta).$$

We show, in a Sard-type manner, that $\mathbf{t}(\mathcal{S})$ has measure zero in $\mathbf{R}$. In particular, its complement is dense. For any such parameter s in the complement, minimizing $M \in \mathcal{M}(\Gamma_s)$ are all smooth.

In scales where $\mathcal{L}$ is suitably modeled by a minimizing hypercone $\mathbf{C}$, the following hold in a suitable (local, coarse) sense:

(i) $\mathcal{S}$ is $\leq (\dim \operatorname{spine} \mathbf{C})$-dimensional.

(ii) $\mathbf{t}$ is $\mathfrak{h}$-Hölder for all $\mathfrak{h} < 1 + \alpha(\mathbf{C})$, where $\alpha(\mathbf{C}) > 1$.

Varying $\mathbf{C}$ over all nonflat minimizing hypercones in $\mathbf{R}^n$, and using

$$\alpha_n = \inf_{\mathbf{C}} \alpha(\mathbf{C}) > 1 \text{ and } \max_{\mathbf{C}} \dim \operatorname{spine} \mathbf{C} = n - 8,$$

then a stratification and iteration shows, for $n \leq 10$, a Hausdorff dimension bound:

$$\dim \mathbf{t}(\mathcal{S}) \leq \sup_{\mathbf{C}} \tfrac{\dim \operatorname{spine} \mathbf{C}}{1+\alpha(\mathbf{C})} \leq \tfrac{n-8}{1+\alpha_n} \leq \tfrac{2}{1+\alpha_n} < 1.$$

Thus, $\mathbf{t}(\mathcal{S}) \subset \mathbf{R}$ has measure zero, and we are done.

When $n = 11$, this strategy becomes borderline. Indeed:

$$\mathbf{C} \in \operatorname{Rot}(\mathbf{C}_\circ \times \mathbf{R}^3),\ \mathbf{C}_\circ \subset \mathbf{R}^8 \text{ any minimizing quadratic hypercone}$$
$$\implies \tfrac{\dim \operatorname{spine} \mathbf{C}}{1+\alpha(\mathbf{C})} = \tfrac{3}{3} = 1.$$

At points of $\mathcal{S}$ suitably modeled by such a minimizing hypercone, our upper bound is "borderline" and thus insufficient. In fact, this is the only case that is borderline when $n = 11$. This suggests that we have the following cases to consider:

(a) Minimizing hypercones $\mathbf{C}$ with $\dim \operatorname{spine} \mathbf{C} \leq 2$. These satisfy:

$$\tfrac{\dim \operatorname{spine} \mathbf{C}}{1+\alpha(\mathbf{C})} \leq \tfrac{2}{1+\alpha_n} < 1.$$

(b) Minimizing hypercones $\mathbf{C}$ with $\dim \operatorname{spine} \mathbf{C} = 3$, i.e., $\mathbf{C} \in \operatorname{Rot}(\mathbf{C}_\circ \times \mathbf{R}^3)$, where $\mathbf{C}_\circ \subset \mathbf{R}^8$ is **not** a minimizing quadratic hypercone. One can show:

$$\alpha(\mathbf{C}) = \alpha(\mathbf{C}_\circ) \geq \alpha_8 + \Delta_8^{\text{non-qd}} \text{ where } \alpha_8 = 2 \text{ and } \Delta_8^{\text{non-qd}} > 0.$$

Then:

$$\tfrac{\dim \operatorname{spine} \mathbf{C}}{1+\alpha(\mathbf{C})} \leq \tfrac{3}{1+\alpha_8+\Delta_8^{\text{non-qd}}} = \tfrac{3}{3+\Delta_8^{\text{non-qd}}} < 1.$$

(c) All remaining hypercones, i.e., $\mathbf{C} \in \operatorname{Rot}(\mathbf{C}_\circ \times \mathbf{R}^3)$, where $\mathbf{C}_\circ \subset \mathbf{R}^8$ is some minimizing quadratic hypercone. Here, we engaged in fine analysis to get an upper bound on the dimension of $\mathbf{t}(\mathcal{S})$ in terms of a sharper right hand side. More generally, we treated this larger class of hypercones:

$$(\star_{k,n}) \qquad \mathbf{C} \in \operatorname{Rot}(\mathbf{C}_\circ \times \mathbf{R}^k),\ \mathbf{C}_\circ \subset \mathbf{R}^{n-k} \text{ regular, strongly integrable, strictly stable, strictly minimizing.}$$

Simon [14] pioneered the study of such hypercones. One can show that homogeneous Jacobi fields on $\mathbf{C}$ satisfying $(\star_{k,n})$ have a *gap* in their allowed degrees of homogeneity: either the degree equals 1, or it is $\geq 1 + \Delta_{\mathbf{C}}$, with $\Delta_{\mathbf{C}} > 0$. We proved that the local picture near points of $\mathcal{L}$ suitably modeled by $\mathbf{C}$ satisfying $(\star_{k,n})$ with Jacobi field degree d is as follows.

(i') If $d = 1$, we improved the (local, coarse) dimension bound in (i):

$$\mathcal{S} \text{ is } \leq (\dim \operatorname{spine} \mathbf{C} - 1)\text{-dimensional},$$

by more refined understanding of degree-1 Jacobi fields on such $\mathbf{C}$.

(ii') If $d \geq 1 + \Delta_{\mathbf{C}}$, we improved the (local, coarse) Hölder bound in (ii):

$$\mathbf{t} \text{ is } \mathfrak{h}\text{-Hölder for all } \mathfrak{h} < 1 + \alpha(\mathbf{C}) + \Delta_{\mathbf{C}},$$

by exploiting the faster-than-scaling closeness of the minimizer to $\mathbf{C}$.

Varying $\mathbf{C}$ over all hypercones satisfying $(\star_{k,n})$, one sees that:

$$\alpha(\mathbf{C}) = \alpha_{n-k},\ \Delta_n^{\text{qd}} = \inf_{\mathbf{C},(\star_{k,n})} \Delta_{\mathbf{C}} > 0.$$

Now $n = 11$, and we are in case (c). Then, $\alpha_{11-3} = \alpha_8 = 2$ implies:

$$\min\{\tfrac{\dim \operatorname{spine} \mathbf{C}-1}{1+\alpha(\mathbf{C})}, \tfrac{\dim \operatorname{spine} \mathbf{C}}{1+\alpha(\mathbf{C})+\Delta_{\mathbf{C}}}\} = \min\{\tfrac{2}{1+2}, \tfrac{3}{3+\Delta_{11}^{\text{qd}}}\} < 1.$$

At this point, a stratification and iteration more refined than in [10, 11], which distinguishes cases (a), (b), (c.i'), and (c.ii'), can be used to prove, once again

$$\dim \mathbf{t}(\mathcal{S}) < 1 \text{ in } \mathbf{R}^{11}.$$

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Generation of decay for solutions to the Boltzmann equation

Amélie Loher

The focus of this talk were recent developments on the Boltzmann equation, a PDE that describes dynamics of uncharged particles. We first explained how equations in kinetic theory happen on a mesoscopic scale, between the microscopic dynamics encoding the trajectories of all single particles, and the macroscopic scale of fluid equations. We mentioned the recent break through result by Deng-Hani-Ma [2] that rigouroulsy justifies the Boltzmann equation on arbitrary time scales (as long as the solution exists) as a Boltzmann-Grad limit from particle systems under the molecular chaos assumption.

We secondly discussed the different potentials that the particles can be subjected to: from hard, to moderately soft, up to very soft potentials. The well-posedness theory for the spatially homogenous equation started with the case of hard spheres through Carleman in 1933 [10], then continued with moderately soft potentials as can be found for instance in He [4], up to the recent result by Imbert-Silvestre-Villani in 2024 [8]. They showed that the Fisher information is monotone, from which they inferred the global existence of a unique solution to the Boltzmann equation.

We then turned to the spatially inhomogeneous case: the well-posedness is open. There are some perturbative results, some local well-posedness result, and some very weak notion of solution results (renormalised solution with defect measure) that are not known to be unique and too weak to be able to make sense of the equation even in a distributional sense. It is, however, known that there is underlying coercivity in the equation, which can for instance be exploited under further assumptions: Silvestre in 2016 [1] introduced a conditional regime for moderately soft and hard potentials, where a given solution is supposed to satisfy for each time and each point in space that the macroscopic mass is bounded above and below, and the kinetic energy and the entropy are bounded above. If this is satisfied, then the solution is in L^∞ in time, space, and velocity, where the bound depends only on the boundedness of the aforementioned quantities. This led to a series of papers by Imbert-Silvestre around 2020 [9, 7, 8, 6] with the final result which stated that any given classical solution to the spatially inhomogeneous Boltzmann equation in $(0,T) \times \mathbb{R}^d \times \mathbb{R}^d$ (with moderately soft and hard potentials) is smooth, provided that it satisfies the assumptions of the conditional regime. They assumed that the solution is periodic in space, and in case of moderately soft potentials that the initial condition is pointwisely decaying in velocity at some polynomial rate. The result was based on

a) local Hölder continuity,
b) local Schauder estimates,
c) change of variables to turn the local estimates into global ones,
d) bootstrap.

In order to make the bootstrap work, they also required

e) pointwse polynomial decay estimates in velocity.

This last point is the reason they had to impose a pointwise decay on the initial datum for soft potentials. It was then shown by Imbert-Mouhot-Silvestre in 2018 [8] that strong solutions to the Boltzmann equation which are periodic in space and subject to the conditional regime

(1) generate some polynomial pointwise decay in case of hard potentials,
(2) propagate decay in case of moderately soft and hard potentials.

For (1) the generation of decay was shown up to a rate that was bounded by $d+1$. Together with C. Imbert in 2025 [5], we studied a suitable notion of weak solutions to the Boltzmann equation in order to be able to deal with bounded spatial domains. We were able to treat the following boundary conditions: bounce back, inflow, specular reflection, diffuse reflection, and a linear combination of the last two which is known as Maxwell condition.

We can also treat the whole Euclidean space for the spatial directions. We worked in the conditional regime of Silvestre, and showed that the solutions generate any polynomial pointwise decay in velocity, not only for hard potentials, but also in case of soft potentials. In particular, this allows to get rid of the decay assumption on the initial datum in the result of Imbert-Silvestre [6]. The proof relies on a purely non-local effect: the fact that from a non-local equation one gets two, instead of one, regularity estimate out of an energy estimate. It also relies on the non-linearity of the equation: we first show the generation of some amount of decay, which we then use to get better estimates due to the non-linearity to show the generation of some more decay. This argument is then bootstrapped to get arbitrary polynomial decay.

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A geometric view upon regularity structures, charts and transition maps for the solution manifold

Felix Otto

(joint work with Lucas Broux, Markus Tempelmayr)

We consider nonlinear elliptic equations of the form

$$L_\lambda[u] := -\Delta u + \lambda u^m = f \tag{1}$$

in the whole d-dimensional space. Here, $m \geq 1$ is a positive integer and $\lambda \in \mathbb{R}$ is a 'coupling constant' modulating the strength of the nonlinearity. Taking a geometric view upon this equation, we consider the structure of the corresponding solution manifold $\{(u, f, \lambda) \,|\, L_\lambda[u] = f\}$. First looking at its symmetries, we note that it is invariant under

- translating x, i. e. under $z \mapsto (u(z+\cdot), f(z+\cdot), \lambda)$ for $z \in \mathbb{R}^d$,
- rescaling x, i. e. under $r \mapsto (u(r\cdot), r^2 f(r\cdot), r^2\lambda)$ for $r > 0$,
- rescaling u, i. e. under $\mu \mapsto (\mu u, \mu f, \mu^{1-m}\lambda)$ for $\mu > 0$.

We suppose that we are given a space $\mathcal{F} \ni f$ of forcings which itself is invariant under translating x and under the rescaling $r^{2-\alpha} f(r\cdot)$ for $r > 0$ and some fixed $\alpha \in \mathbb{R}$. From the above symmetries we then infer that the solution manifold is invariant under $r \mapsto (r^{-\alpha}u(r\cdot), r^{2-\alpha}f(r\cdot), r^{2-(1-m)\alpha}\lambda)$ for $r > 0$. This crucial scaling symmetry can be reformulated in terms of ambient Hölder spaces as $(u, f) \in \dot{C}^{\alpha} \times \dot{C}^{\alpha-2}$.

A 'thin' space $\mathcal{F}$ of right-hand sides f. Concerning our assumptions on the forcing, we consider the singular ($\alpha < 0$) but sub-critical ($(m-1)\alpha+2 > 0$) regime, where the former condition means that the solution u is genuinely distributional (and hence the nonlinearity is classically ill-defined), while the latter condition indicates that the effective strength of the nonlinearity however vanishes at small scales i. e. under blow up. Furthermore, we are interested in the case when 'the tangent space is less rough than the ambient space', i. e. is contained in a Sobolev space of higher regularity:

$$\delta f \in \mathsf{T}_f\mathcal{F} \subseteq \dot{H}^s \text{ while } f \in \mathcal{F} \subseteq \dot{C}^{\alpha-2} \text{ for some } s > \alpha - 2;$$

hence the tangent space has more derivatives (however measured in L^2 instead of the C^0 scale). This property is characteristic of stationary Gaussian ensembles with support contained in $\dot{C}^{\alpha-2}$, where the Gaussian measure is preserved under the shift $f \mapsto f + \delta f$ iff $\delta f \in \dot{H}^s$ with $s = \alpha - 2 + d/2$. Note the identical scaling of $\dot{H}^s$ and $\dot{C}^{\alpha-2}$, and that white noise corresponds to $s = 0$.

Linearizing the equation. This suggests, as is standard in regularity theory, to differentiate the equation (1), here w.r.t f:

$$(\mathsf{T}_u L_\lambda)\delta u = (-\Delta + \lambda m u^{m-1})\delta u = \delta f.$$

At this linearized level, we note that under the self-consistent assumptions $u^{m-1} \in \dot{C}^{(m-1)\alpha}$ and $\delta u \in \dot{H}^{2+s}$, the product $u^{m-1}\delta u$ is controllable iff $(m-1)\alpha+2+s > 0$. We may now describe at a high level the main result of [2].

Theorem ([2, Theorem 1] paraphrased). *Suppose* $(m-1)\alpha + 2 + \min\{s, 0\} > 0$. *Then one can construct a (renormalized) solution manifold.*

Charts U and transition maps P for the solution manifold. To make the statement more precise, we continue to describe the geometry of the solution manifold. In the linear case where $\lambda = 0$ we observe that $\{u \,|\, L_0[u] = f\}$ is affine over $\{p \,|\, L_0[p] = 0\} \subseteq \mathcal{P} := \{\text{analytic functions}\}$, and hence $\{u \,|\, L_0[u] \in f + \mathcal{P}\}$ is affine over the entire $\mathcal{P}$. In the general case where $\lambda \in \mathbb{R}$, we aim at constructing an (inverse) chart $u = U_\lambda[f, p]$ with domain $\mathcal{F} \times \mathcal{P} \ni (f, p)$ such that

$$L_\lambda[U_\lambda[f, p]] = f + L_\lambda[p], \tag{2}$$

and which is affine for $\lambda = 0$: $U_0[f, p] = U_0[f, 0] + p$. As it turns out, such a chart is unique (in a term-by-term sense, see below), upon imposing compatibility with our scaling invariances of the form

$$U_{\mu^{1-m}\lambda}[\mu f, \mu p] = \mu U_\lambda[f, p] \quad \text{and} \quad U_{r^2\lambda}[r^2 f(r\cdot), p(r\cdot)] = U_\lambda[p, f](r\cdot). \tag{3}$$

However it is incompatible with translation invariance (unless $\lambda = 0$), i. e.

$$U_\lambda[f(z + \cdot), p(z + \cdot)] \neq U_\lambda[f, p](z + \cdot).$$

In fact, the second item in (3) implicitly relies on a non-canonical choice of 'origin' in d-dimensional space. Hence for every point x in space we obtain a chart $U_{x\lambda}$. We therefore aim as well at constructing transition maps $P_{xy\lambda}[f,\cdot]$ on $\mathcal{P}$ characterized by

$$U_{x\lambda}[f,p] = U_{y\lambda}[f, P_{xy\lambda}[f,p]], \tag{4}$$

which incidentally allows to recover compatibility with translation invariance in the form of

$$P_{xy\lambda}[f(z+\cdot), p(z+\cdot)] = P_{z+x\,z+y\,\lambda}[f,p].$$

A robust characterization of the charts U. The key observation is a characterization of the charts U which is robust and hence eliminates the non-robust operator L. Indeed, taking derivatives of (2) w.r.t. $f \in \mathcal{F}$ and $p \in \mathcal{P}$, respectively, yields

$$(\mathsf{T}_{U_\lambda} L_\lambda)(\mathsf{T}_f U_\lambda)\delta f = \delta f,$$
$$(\mathsf{T}_{U_\lambda} L_\lambda)(\mathsf{T}_p U_\lambda)\delta p = (\mathsf{T}_p L_\lambda)\delta p.$$

If $\mathsf{T}_f\mathcal{F}$ were contained in $\mathcal{P}$, we could hope to equate the right-hand sides and hence construct a linear map $A_\lambda[p]$ from $\mathsf{T}_f\mathcal{F}$ to $\mathcal{P}$ such that

$$\mathsf{T}_f U_\lambda = (\mathsf{T}_p U_\lambda) A_\lambda \quad \text{and} \quad \mathrm{id} = (\mathsf{T}_p L_\lambda) A_\lambda.$$

However, this construction cannot be applied here since we only have $\mathsf{T}_f\mathcal{F} \subseteq \dot{H}^s \not\subseteq \mathcal{P}$. The transition maps are essential to tackle this issue: taking also a derivative w.r.t. $p \in \mathcal{P}$ of (4) yields

$$\mathsf{T}_p U_{x\lambda} = (\mathsf{T}_{P_{xy\lambda}} U_{y\lambda}) \mathsf{T}_p P_{xy\lambda}.$$

This motivates to make the ansatz that for some linear map $A_{xy\lambda}[f,p]$ from $\mathsf{T}_f\mathcal{F}$ to $\mathcal{P}$,

$$(\mathsf{T}_f U_{x\lambda})\delta f = (\mathsf{T}_{P_{xy\lambda}} U_{y\lambda}) A_{xy\lambda}\delta f + \mathcal{O}(|\cdot - y|^{2+s}) \quad \text{and}$$
$$\mathrm{im} A_{xy\lambda} \subseteq \{q \in \mathcal{P} \,|\, \text{degree of } q < 2+s\}.$$

A 'term-by-term' ansatz. The construction of U, P, and A is performed 'term-by-term' and algebrized as follows. Introducing on $\mathcal{P}$ the coordinates $\mathsf{z}_{\mathbf{n}}[p] := \frac{1}{\mathbf{n}!}\frac{d^{\mathbf{n}}p}{dx^{\mathbf{n}}}(0)$ for $\mathbf{n} \in \mathbb{N}_0^d$ allows to expand functions on $\mathbb{R} \times \mathcal{P} \ni (\lambda, p)$ in terms of the monomials

$$\mathsf{z}^\beta[\lambda,p] := \lambda^{\beta(m)} \prod_{\mathbf{n}\in\mathbb{N}_0^d} \mathsf{z}_{\mathbf{n}}^{\beta(\mathbf{n})}[p]$$

for multi-indices β on $\{m\} \cup \mathbb{N}_0^d$. One can then interpret P_{xy}, A_{xy}, and U_x as formal power series in λ and $\mathsf{z}_{\mathbf{n}}$:

$$\mathsf{z}^\gamma[\lambda, P_{xy\lambda}[p]] = \sum_\beta (P_{xy})_\beta^\gamma \, \mathsf{z}^\beta[\lambda,p] \quad \text{where } (P_{xy})_\beta^\gamma \text{ is a function on } \mathcal{F},$$

$$\mathsf{z}_{\mathbf{n}}[A_{xy\lambda}[p]] = \sum_\beta (A_{xy})_\beta^{(\mathbf{n})} \, \mathsf{z}^\beta[\lambda,p] \quad \text{where } (A_x)_\beta^{(\mathbf{n})} \text{ is a section of } \mathsf{T}^*\mathcal{F},$$

$$U_{x\lambda}[p] = \sum_{\beta} U_{x\beta}\, \mathsf{z}^{\beta}[\lambda, p] \quad \text{where } U_\beta \text{ is a section of the fiber bundle over } \mathcal{F} \text{ with base space Schwartz distributions.}$$

The main advantage of this term-by-term formulation is that we gain a triangular structure which only involves inverting the linear operator L_0.

Theorem ([2, Theorem 1] less paraphrased)**.** *One can inductively construct $U_{x\beta}$, $(P_{xy})_\beta^\gamma$, and $(A_{xy})_\beta^{(\mathbf{n})}$.*

Similar ideas can be applied to other equations, including parabolic and quasi-linear variants of (1), see [3, 4]. Although these formal power series are not expected to converge, such a term-by-term ansatz is used in [1] to develop a well-posedness theory for (1) based on a continuity method.

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Regularity vs. singularity for the special Lagrangian equation

Connor Mooney

(joint work with Ovidiu Savin and Ravi Shankar)

The special Lagrangian equation is

$$F(D^2u) := \sum_{i=1}^{n} \tan^{-1}(\lambda_i) = \Theta. \tag{1}$$

Here u is a function on a domain in $\mathbb{R}^n$, λ_i are the eigenvalues of D^2u, and $\Theta \in (-n\pi/2,\, n\pi/2)$ is a constant.

The geometric significance of (1) is that the graph

$$\Sigma := \{(x,\, \nabla u(x))\} \subset \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{C}^n$$

is calibrated (and therefore area-minimizing), at least when u is smooth [3]. More precisely, Σ is calibrated by the n-form

$$\mathrm{Re}(e^{-i\Theta} dz_1 \wedge \ldots \wedge dz_n).$$

Submanifolds that are calibrated by such a form are called special Lagrangians. Therefore, one strategy to produce special Lagrangians is to find smooth solutions to (1). However, the regularity theory for (1) is difficult, since the equation is both degenerate elliptic and, in general, neither convex nor concave.

It is well-known that there is a unique viscosity solution $u \in C\left(\overline{B_1}\right)$ to the Dirichlet problem for (1) on B_1 with boundary data $g \in C(\partial B_1)$. Wang and Yuan proved in [8] that u is smooth (and therefore has minimal gradient graph) if

$$|\Theta| \geq (n-2)\pi/2. \tag{2}$$

If in addition $g \in C^\infty$, then u is smooth up to the boundary [1]. These results are related to the fact that $\{F = \Theta\}$ is convex if and only if (2) holds [10].

In contrast, for any $n \geq 3$, $|\Theta| < (n-2)\pi/2$, and $\alpha > 0$, there exist viscosity solutions to (1) that are C^1 but not $C^{1,\,\alpha}$ ([7], [9]). However, these examples have smooth and minimal gradient graph. This left open the questions whether u is necessarily C^1, and has minimal gradient graph. The first result we presented, which is joint work with O. Savin [4], answers both questions in the negative:

Theorem 1. *There exist Lipschitz but non-C^1 viscosity solutions to (1) in dimension $n \geq 3$ that have non-minimal and non-smooth gradient graph.*

Another condition known to guarantee the regularity of a viscosity solution to (1) is that u is convex [2]. The second result we presented, which is joint with R. Shankar [5], is an extension of this result to semi-convex solutions, with optimal assumptions and conclusions:

Theorem 2. *Assume that $\Theta \in (-(n-2)\pi/2,\, \pi/2)$, and let $\theta := (\pi/2-\Theta)/(n-1) \in (0,\, \pi/2)$. If u is a viscosity solution to (1) such that $u + \tan(\theta)|x|^2/2$ is convex, then u is smooth, and moreover*

$$|D^2u(0)| \leq e^{C(n,\,\Theta)(1+\|\nabla u\|_{L^\infty(B_1)})}. \tag{3}$$

The assumptions are optimal in the following sense: if $\Theta \in [\pi/2,\, (n-2)\pi/2)$, then for $\epsilon > 0$ arbitrarily small, there exist singular solutions u such that $u + \epsilon|x|^2/2$ is convex. Likewise, if $\Theta \in (-(n-2)\pi/2,\, \pi/2)$, then for $\epsilon > 0$ arbitrarily small, there exist singular solutions u such that $u + (\tan(\theta) + \epsilon)|x|^2/2$ is convex. Moreover, in both cases, the examples can be taken to be Lipschitz but not C^1 and have non-minimal gradient graph. The conclusion is optimal in the sense that the exponential dependence on $\|\nabla u\|_{L^\infty(B_1)}$ cannot be improved.

The example from Theorem 1 is obtained by solving the degenerate Bellman equation

$$\max\{\det D^2w,\ F(D^2w) - c\} = 0 \tag{4}$$

with compact free boundary, and then taking the Legendre transform u of w. We interpret the gradient graph of u as a rotation of the gradient graph of w. The graph of ∇w is $C^{1,\,1}$ but not C^2, and is non-minimal on a compact set (where $\det D^2w = 0$). The solution u is in fact singular on part of a smooth hypersurface, and is also semi-convex.

The proof of Theorem 2 is based on the observation that, under our assumptions, certain $U(n)$ rotations of the gradient graph of u (interpreted in a multi-valued sense) have a $C^{1,\,1}$ potential that is also a viscosity solution of (1). Such rotations can be done for the gradient graphs of arbitrary semi-convex solutions, but the example from Theorem 1 illustrates that one can lose the viscosity solution property when doing so.

To conclude we note that all examples mentioned above have unbounded Hessian. It is natural to ask whether $C^{1,1}$ viscosity solutions to (1) (which have calibrated and therefore area-minimizing gradient graph, [3]) are smooth. The monotonicity formula for minimal surfaces reduces this question to the following

Problem 1. *Do there exist non-flat graphical special Lagrangian cones?*

A general result about fully nonlinear elliptic PDEs gives a negative answer in dimensions $n \leq 4$ [6]. Problem 1 remains open in dimensions $n \geq 5$.

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A simpler characterization of *GBD*

Antonin Chambolle
(joint work with Vito Crismale)

The space "*GBD*" was introduced by G. Dal Maso in [6] in order to define a proper energy space for tackling variational problems in the theory of crack growth in linearized elasticity [7]. A typical such problem is the minimization of the Griffith energy:

$$\min_{u,K} \mathcal{E}(u,K) := \int_\Omega |e(u)|^2\,dx + \mathcal{H}^{d-1}(K) \qquad (Gr)$$

where here, $\Omega \subset \mathbb{R}^d$ ($d \geq 2$, in applications $d \in \{2,3\}$) is the reference configuration of an elastic material, $u : \Omega \to \mathbb{R}^d$ the displacement of the object, smooth in $\Omega \setminus K$, K a possible $(d-1)$-dimensional crack set, of $(d-1)$-dimensional Hausdorff measure $\mathcal{H}^{d-1}(K)$, and $e(u) = (Du + Du^T)/2$ the symmetric part of the gradient of u (away from the set K, where u is allowed to jump). The displacement u may be subject to constraints (in general, a Dirichlet condition on some part of $\partial\Omega \setminus K$, corresponding to a *load* of the material) or further penalization.

One difficulty in tackling such problems is that the energy does not control the amplitude of the jump of u accross the set K, and thus is not coercive in $BD(\Omega) := \{u \in L^1(\Omega;\mathbb{R}^d) : Du + Du^T \text{ is a bounded Radon measure}\}$, which is the natural variant of BV in this context [8, 1]. In that space, one can define properly $e(u)$ as the absolutely continuous part of the measure $Eu := (Du + Du^T)/2$, the jump set J_u (as the set of points where blowups limits of u take two different values), and a Cantor part Cu. ($e(u)dx$ is the d-dimensional part of the measure Eu, the $(d-1)$-dimensional part is carried on J_u, and Cu is "in between", supported by a Lebesgue-negligible set and vanishing on (σ-finite) $(d-1)$-dimensional sets.) Then, a weak energy is naturally defined by replacing K with J_u in (Gr), and is shown to be lower semi-continuous in $SBD(\Omega) := \{u \in BD(\Omega) : Cu = 0\}$, [1, 2]. Yet, as said, one lacks compactness to show existence of minimizers of the weak energy.

To resolve these issues, in [6], the space GBD was defined as the space of vector measurable functions $u : \Omega \to \mathbb{R}^d$ such that for any direction $\xi \in \mathbb{S}^{d-1}$ and any base point $z \in \xi^\perp$, their *one-dimensional slicings* $u_\xi^z : s \mapsto \xi \cdot u(z + s\xi)$, defined in $\Omega_\xi^z := \{s \in \mathbb{R} : z + s\xi \in \Omega\}$, are in $BV(\Omega_\xi^z)$ and such that there exists a bounded positive measure $\lambda \in \mathcal{M}(\Omega)$ which satisfies:

$$(1)\quad \int_{\xi^\perp} \left(|Du_\xi^z|(A_\xi^z \setminus J_{u_\xi^z}) + \sum_{s \in J_{u_\xi^z} \cap A_\xi^z} |u_\xi^z(s+0) - u_\xi^z(s-0)| \wedge 1 \right) d\mathcal{H}^{d-1}(z) \le \lambda(A)$$

for any open set $A \subset \Omega$, where $u_\xi^z(s \pm 0)$ are the left and right limits of u_ξ^z at the jump (discontinuity) point $s \in J_{u_\xi^z}$. In the definition, the "min" $(|[u_\xi^z(s)]| \wedge 1)$ accounts for the fact that jumps of large amplitude are not penalized by the energy, and therefore these amplitudes should not be controlled in the definition of the space.

The talk reviewed rapidly the base techniques introduced to show that with this relatively weak definition, one can obtain stronger properties, compactness for the weak energy, existence of minimizers, and eventually that these weak minimizers are strong, see the short review [4] and the references therein.

It then focussed on a small technical problem addressed in [5]. For BD functions, it is easy to check that controlling the BV norms of the u_ξ^z for $d(d+1)/2$ directions is enough to ensure that u is BD: more precisely, given $(e_i)_{i=1}^d$ a basis, one has

$$\int_{\xi^\perp} |Du_\xi^z|(\Omega_\xi^z) d\mathcal{H}^{d-1}(z) = \int_\Omega |\xi \cdot (Eu \cdot \xi)|$$

and if this is finite for $\xi \in W := \{e_i : i = 1, \dots, d\} \cup \{e_i + e_j : 1 \le i < j \le d\}$ then one controls all coefficients of the (symmetric) matrix-valued measure Eu. Yet the definition of GBD in [6] requires that (1) holds for all (or, in fact, a dense subset of) $\xi \in \mathbb{S}^{d-1}$. The main theorem in [5] is as follows:

Theorem ([5, Corollary 1]). *Let $u : \Omega \to \mathbb{R}^d$ measurable such that for all $\xi \in W$,*

$$\int_{\xi^\perp} \left(|Du_\xi^z|(\Omega_\xi^z \setminus J_{u_\xi^z}) + \sum_{s \in J_{u_\xi^z}} |u_\xi^z(s+0) - u_\xi^z(s-0)| \wedge 1 \right) d\mathcal{H}^{d-1}(z) < +\infty.$$

Then $u \in GBD(\Omega)$.

In the talk, a sketch of proof was given: it relies on a very elementary argument (already used in a more basic form in [3]) which consists in finding for $\varepsilon > 0$ small appropriate discretizations $(u(\varepsilon y_\varepsilon + i))_{i \in \varepsilon\mathbb{Z}^d}$ of u, for suitable choices of $y_\varepsilon \in [0,1)^d$, which, after reinterpolation, (i) converge to u as $\varepsilon \to 0$ and (ii) satisfy a global "bound" in GBD, which ensures by lower semi-continuity that also the limit u is in GBD.

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The Harnack inequality without convexity for curve shortening flow

Peter Topping
(joint work with Arjun Sobnack)

In previous work [3] the authors have proposed, and partially realised, a principle of delayed parabolic regularity for curve shortening flow. This asserts that given a regular curve shortening flow such as a static (proper) straight line in the plane, and a disjoint proper curve shortening flow γ that is known only to enclose an area A in between it and the straight line, the quantified regularity of γ is completely uncontrolled until time $\frac{A}{\pi}$, at which precise time parabolic regularity is switched on and the flow and all its derivatives are controlled in terms only of A and $t > \frac{A}{\pi}$.

One of the key methods in that work was the introduction of new Harnack inequalities that were inspired by ideas in Ricci flow and Kähler Ricci flow [4, 2]. In this talk, these ideas were developed further, making close contact with earlier work of Neves [1] in Lagrangian mean curvature flow. In particular, a

Harnack inequality was given for curve shortening flow that does not require the convexity assumption of Hamilton's famous Harnack estimate, but which can be used to similar effect (giving the same pointwise curvature bound) when convexity is additionally assumed.

One of the applications of the general Harnack inequality is to quantifiably control a curve shortening flow in terms of very weak information about the initial curve (analogous to the enclosed area A discussed above). In particular, after a specific threshold time it can be argued that the flow must become graphical in several precise senses, making contact with our earlier delayed parabolic regularity theory.

Acknowledgements: Thanks to Felix Schulze for important insights into connections with the existing literature. The author was partly supported by EPSRC grant EP/T019824/1.

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Classification of global solutions to the one phase free boundary problem asymptotic to singular minimizing cones

Zihui Zhao

(joint work with Max Engelstein, Daniel Restrepo)

Let Ω_0 be a fixed domain in $\mathbb{R}^d$, and φ be a non-negative function on $\partial\Omega_0$. We consider the following *one phase free boundary problem*

$$\min\left\{\int_{\Omega_0} |\nabla u|^2 + \chi_{\{u>0\}}\,dx \ \mid u=\varphi \text{ on } \partial\Omega_0\right\},$$

first studied by Alt and Caffarelli in [1]. A function u is a critical point to this functional, if it satisfies the following Euler-Lagrange equation

$$\text{(EL)} \qquad \begin{cases} -\Delta u = 0, & \text{in } \Omega := \{u>0\}, \\ |\nabla u| = 1, & \text{on } \partial\Omega \cap \Omega_0. \end{cases}$$

Clearly, the regularity of the solution u is closely linked to the regularity of its free boundary $\Gamma := \partial\{u>0\}$. More precisely, by a blow-up analysis, the regularity of Γ is determined by 1-homogeneous solutions to (EL) in the entire space $\mathbb{R}^d$, namely

$$(1)\qquad \begin{cases} -\Delta b = 0, & \text{in } \Omega := \{b > 0\}, \\ |\nabla b| = 1, & \text{on } \partial\Omega, \\ b(r\omega) = rb(\omega), & \text{where } r > 0, \omega \in \mathbb{S}^{d-1}. \end{cases}$$

Examples of 1-homogeneous solutions. A trivial solution to (1) is $b_f(x) = \max\{x_1, 0\}$, whose free boundary is a flat hyperplane. So far, the only other homogeneous solution to (1) proven to be *energy-minimizing* is an axially symmetric solution b_{as} in $\mathbb{R}^d$ for dimensions $d \geq 7$, see [2].

The topic of this talk concerns solutions to one phase free boundary problem that are close to this singular minimizing solution b_{as}. This will have implications about ways to perturb away such singularities, and our work also lays down the foundation for studying the behavior near other singular minimizing solutions, if they exist. In particular, we are able to fully classify global solutions that are asymptotic to b_{as} at infinity.

Theorem 1. *Let b_{as} be the axially symmetric minimizing solution in $\mathbb{R}^d$, $d \geq 7$. Suppose that u is a global solution to* (EL) *which is asymptotic to b_{as} at infinity, namely*

$$\text{there exists a sequence } R_k \to +\infty \text{ s.t. } \frac{u(R_k x)}{R_k} \to b_{as}(x) \text{ locally uniformly.}$$

Then modulo translations in $\mathbb{R}^d$, u either agrees with b_{as}, or it is a homothetic rescaling of the functions $u_\pm$ constructed in [3, 4].

The functions $u_\pm$ enjoy several nice properties. But for the sake of the presentation, we only highlight that they are smooth global solutions to (EL), they are ordered as $u_- \leq b \leq u_+$, and the graphs of their rescalings $u_\lambda^\pm(x) := \frac{u_\pm(\lambda x)}{\lambda}$ form a foliation of the space $\mathbb{R}^d \times [0, +\infty)$.

Our key observation is a linear analysis based at the singular solution b_{as}. For any 1-homogeneous solution b, the linearized operator based at b is

$$(\text{L})\qquad \begin{cases} -\Delta v = 0, & \text{in } \Omega = \{b > 0\}, \\ \partial_\nu v + Hv = 0, & \text{on } \partial\Omega. \end{cases}$$

where ν denotes the unit normal vector on the free boundary Γ pointing towards Ω, and $H > 0$ denotes the mean curvature of Γ oriented towards the complement of Ω. By the homogeneity of b, the domain $\Omega = \{b > 0\}$ is a cone. Hence solutions to (L) have the separation of variables and are of the form $|x|^\gamma v(\frac{x}{|x|})$, where $v|_{\mathbb{S}^{d-1}}$ is an eigenfunction on the sphere

$$\begin{cases} -\Delta_{\mathbb{S}^{d-1}} v = \lambda v, & \text{in } \Sigma := \Omega \cap \mathbb{S}^{d-1}, \\ \partial_\nu v + Hv = 0, & \text{on } \partial\Sigma, \end{cases}$$

and the radial exponent γ is related to the eigenvalue by

$$\gamma^2 + (d-2)\gamma - \lambda = 0.$$

By PDE arguments, we conclude that the kernel elements of the linearized operator which result in solutions asymptotic to b are the ones with radial exponent $\gamma < 1$; moreover, we can fully classify that those kernel elements correspond to the foliation and translations. This confirms an analogy between the singular minimizing solution b_{as} and quadratic cones in the setting of area-minimizing hypersurfaces, which eventually allows us to prove Theorem 1 analogously as Simon and Solomon for complete minimal hypersurfaces asymptotic to quadratic cones in [5].

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Index and intersections of min-max geodesics

Douglas Stryker

(joint work with Jared Marx-Kuo and Lorenzo Sarnataro)

The length spectrum of a Riemannian surface is a sequence of geometric invariants called p-widths. The p-widths are special critical values for the length functional on the space of closed curves, analogous to the Dirichlet energy min-max characterization of the eigenvalues of the Laplacian.

By the recent breakthrough work of Chodosh and Mantoulidis [2], it is known that each p-width equals the length of a union (not necessarily disjoint) of closed immersed geodesics (with multiplicity). This result is a nontrivial analogue of the min-max regularity theory due to Almgren and Pitts [5], which guarantees in higher dimensions that each p-width equals the area of a disjoint union of closed embedded minimal hypersurfaces (with multiplicity).

In this talk, I will discuss joint work with Jared Marx-Kuo and Lorenzo Sarnataro [4] that investigates the geometric properties of these associated geodesics. It is expected that the collection of geodesics $\{\gamma_i\}_{i=1}^{N_p}$ achieving the p-width can be chosen to satisfy the index bound

$$\sum_{i=1}^{N_p} \mathrm{ind}(\gamma_i) + \sum_{v \in \mathrm{Vert}(\{\gamma_i\})} \binom{\mathrm{ord}(v)}{2} \leq p,$$

where $\mathrm{ind}(\gamma_i)$ is the Morse index of the closed geodesic γ_i for the length functional, $\mathrm{Vert}(\{\gamma_i\})$ is the set of vertices of the graph given by the union of the curves γ_i, and $\mathrm{ord}(v)$ is the number of transverse curves in the graph intersecting at the point v. By proving a generic structure theorem for the vertices of a union of closed geodesics, we can show the weaker result

$$\sum_{i=1}^{N_p} \mathrm{ind}(\gamma_i) \leq p \qquad \text{and} \qquad \sum_{v \in \mathrm{Vert}(\{\gamma_i\})} \binom{\mathrm{ord}(v)}{2} \leq p.$$

This result is the first progress towards the conjectured index bound beyond $p = 1$ (see [1, 3]).

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Non-uniqueness of mean curvature flow

Tang-Kai Lee

(joint work with Alec Payne)

Mean curvature flow (MCF) provides a way to deform submanifolds in a canonical way, and as a result, it has many applications in geometry and topology. The most important property of codimension one MCF is that it satisfies the avoidance principle, which says that two smooth mean curvature flows of hypersurfaces remain disjoint if they are disjoint initially. The avoidance principle is ubiquitously used throughout the MCF literature, especially for controlling the location of an MCF by comparison with a well-chosen disjoint flow. However, very little is known about how to compare two MCFs if they are not disjoint.

In the joint work [11] with Payne, we describe the general behavior of the Hausdorff dimension and measure of the intersection of MCFs, both before and after the first singular time. In the smooth case, we first prove that the dimension of the intersection of two properly embedded smooth MCFs is non-increasing in time if one of the flows is compact. This is based on a recent work by Huang–Jiang [5]. Next, we extend this monotonicity result to level set flow, a weak formulation for mean curvature flow [2, 3], when certain localizability condition holds. As a consequence, we characterize the non-fattening and the non-discrepancy of level set flow by an "intersection principle." We note that both fattening and discrepancy reflect non-uniqueness of the MCF evolution passing singular times. Examples of fattening flows starting from smooth, closed, embedded initial data have been constructed [8, 9, 10].

Theorem 1. *Let M_t be a compact level set flow starting from a smooth, closed, embedded hypersurface in $\mathbb{R}^{n+1}$. Suppose M_t has finitely many singularities. Then, the following are equivalent:*

(1) M_t is non-fattening.
(2) M_t is non-discrepant. That is, the inner and outer flows coincide with M_t.
(3) M_t satisfies the "intersection principle" with respect to smooth MCF. Specifically, if N_t is a smooth closed MCF such that $N_t \not\subseteq M_t$ for each $t \in [0, \infty)$, then $t \mapsto \dim(M_t \cap N_t)$ is non-increasing. Moreover, if $\dim(M_s \cap N_s) < n - 1$ for some s, then $M_t \cap N_t = \emptyset$ for all $t > s$.

If a level set flow M_t, starting from a smooth initial condition, satisfies property (3) from Theorem 1, then M_t is non-fattening without any other additional assumptions. In other words, a sufficient condition for a level set flow to be non-fattening is that it satisfies the intersection principle with respect to smooth flows. It is a well-known problem to characterize when a level set flow is non-fattening (see Ilmanen's Conjecture G in [7, Appendix]). Theorem 1 answers this conjecture when the flow has finitely many singularities. Just as Ilmanen used the avoidance principle to characterize level set flow [6, 7], Theorem 1 suggests that the intersection principle could be used to characterize the fattening of level set flows in general. It is an open question to what extent non-fattening of level set flows is equivalent to the intersection principle.

There are two main issues with proving Theorem 1. The first is that when two level set flows intersect each other on a set of codimension greater than two, it is not necessarily true that one of them lies on one side of the other, in contrast with the case when both flows are smooth. A typical example is a round shrinking sphere intersecting a shrinking dumbbell at its neck singularity.

One-sidedness of flows with small intersection dimension is crucial for proving monotonicity of the dimension over time. To deal with the lack of one-sidedness, we prove a localization result for level set flows with finitely many singularities. This means that we find a natural way to decompose the flow into subsets, such that the union of the level set flows of the subsets is the whole flow. This is a very special property, since level set flow is fundamentally non-local. For example, if M is a smooth, connected, and closed hypersurface and M_1 and M_2 are two smooth hypersurfaces with nonempty boundary such that $M = M_1 \cup M_2$, then the level set flow M_t will be a smooth MCF yet $(M_1)_t$ and $(M_2)_t$ instantaneously vanish under LSF [3, Theorem 8.1]. Only in special cases does the union of level set flows of subsets give the level set flow of the whole set.

The second issue with proving Theorem 1 is the conjecture that non-fattening level set flows must coincide with both their inner and outer flows (see [4, Conjecture 2.4]). This is an important conjecture which, if true, would confirm that nonfattening level set flows are unique in a strong sense. As part of Theorem 1, we prove this conjecture in the case of finitely many singularities. Hershkovits–White showed that this conjecture is true for flows with mean convex neighborhoods of singularities [4], and Bamler–Kleiner proved it in general for MCFs in $\mathbb{R}^3$ [1, Theorem 1.8]. Agreement between the inner and outer flows is used in our proof of Theorem 1 in order to understand the singularities of the level set flow via an associated Brakke flow.

There are more general conditions than what is stated in Theorem 1 which guarantee that a level set flow satisfies the intersection principle. We define a general class of "localizable" level set flows, which loosely means that the flow has no singularities which are locally disconnected at a singular time yet which subsequently flow to become locally connected. For example, a one-sheeted flow desingularizing a two-sheeted cone would not be localizable. Our main result, given in [11, Theorem 4.22], is roughly stated as follows:

> *A non-fattening, localizable level set flow with no higher multiplicity planar tangent flows satisfies the intersection principle.*

Non-localizable flows, such as one-sheeted flows desingularizing a two-sheeted cone, are a primary source of examples for fattening level set flows. Thus, localizability is closed related to fattening, and hence the intersection principle. Although localizability does not necessarily imply non-fattening, under reasonable assumptions, a level set flow is localizable if and only if both the inner and outer flows satisfy the intersection principle with respect to smooth closed MCFs. It is an interesting open problem what restrictions on singularities ensure that a level set flow is localizable. It is plausible that flows with only cylindrical or spherical singularities would have this property, and hence we expect that generic MCF would satisfy the intersection principle, i.e. item (3) from Theorem 1.

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The size of the singular set of stationary integral varifolds obeying ϵ-regularity

Paul Minter
(joint work with Brian Krummel, Neshan Wickramasekera)

One of the basic problems in geometric measure theory is controlling the size of the singular set of weak, measure-theoretic, solutions to geometric variational problems. A standard example is a critical point of the area functional, known as a stationary integral varifold. For these, we do not yet even know if the singular set can have positive measure. For absolute minimisers of area, the infamous work of Almgren shows that the singular set has codimension at least 2 [2].

The main difficulty in both situations is the analysis of so-called *branch points*, which are (non-immersed) singular points where one tangent cone is a plane with (integer) multiplicity $Q \in \{2, 3, \dots\}$. The Hausdorff dimension of other singular points can be handled by soft stratification arguments by looking at tangent cones. Whilst it is still unknown whether the structure about a typical branch point is topologically simple, the prototypical case to consider is when the local structure of the stationary integral varifold is given by the graph of a multi-valued function. Examples to bear in mind include complex algebraic varieties, such as

$$\{(z,w) \in \mathbb{C}^2 : z^2 = w^3\} \qquad \text{and} \qquad \{(z,w,u) \in \mathbb{C}^3 : z^2 = w^3 u\}.$$

In both these examples, the origin is a branch point with $\{z = 0\}$ being the unique tangent cone there (occurring with multiplicity 2). The surfaces are then determined by the graphs of the 2-valued functions $w \mapsto \pm w^{3/2}$ and $(w,u) \mapsto \pm w^{3/2}u^{1/2}$, respectively. An important distinction between these two examples is that in the latter case the regularity of the graphing function is only $C^{0,1/2}$, whilst in the former it is $C^{1,1/2}$.

In [1], we investigate the size of the singular set of stationary integral n-varifolds in $B_2^{n+k}(0) \subset \mathbb{R}^{n+k}$ which satisfy an ϵ-regularity property close to planes with multiplicity Q (more precisely, $\leq Q$). Loosely speaking, the ϵ-regularity property is the assumption that, if the varifold V is close in a cylinder to a plane with multiplicity Q in the varifold topology, then:

- There is a Lipschitz multi-valued function u whose graph coincides with V in the half-cylinder;
- The function u furthermore satisfies estimates of a $C^{1,\alpha}$-type nature (in fact, it is *generalised-*$C^{1,\alpha}$);
- If one rescales u by the L^2 height excess of V to the plane then the resulting multi-valued function is close *strongly* in $W^{1,2}_{\text{loc}}$ to another multi-valued function (known as a *coarse blow-up*).

In particular, for such V the local structure about branch points is determined by graphs of *Lipschitz* multi-valued functions. Note that for the complex variety examples above, this regularity is true in the first example but not the second, so certainly this assumption does not hold unconditionally.

Nonetheless, by now there are several classes of stationary integral varifold which are known to satisfy such an ϵ-regularity property, including:

(1) Stationary integral n-varifolds in $B_2^{n+1}(0) \subset \mathbb{R}^{n+1}$ which have stable regular part and no classical singularities of density $< Q$ (this includes as a special case area minimising hypersurfaces mod p) [3];
(2) Lipschitz 2-valued functions whose graphs are stationary [4].

In fact, by the results in [4] the local structure about a density 2 branch point X in an *arbitrary* stationary integral varifold V is given by the graph of a Lipschitz 2-valued function *provided* X is not a limit point of triple junction singularities or multiplicity one regular points where the tangent plane is 'vertical' relative to the (unique) tangent plane to V at X. Note this is exactly what happens in the second complex variety example above.

The main result of [1] can then be summarised as follows:

Theorem. The density $\leq Q$ branch set of a stationary integral varifold as above has Hausdorff dimension $\leq n-2$.

Consequently the full singular set in the region $\{\Theta_V < Q+1\}$ must have Hausdorff dimension $\leq n-1$. In the special case of area minimising hypersurfaces mod p this recovers the result in [5], and for Lipschitz 2-valued functions whose graphs are stationary this improves on the result in [6].

A key tool in our analysis is the *planar frequency function* introduced by Krummel–Wickramasekera [7]. We are able to show that it is approximately monotone at *every* branch point x in question, and its limiting frequency value $\mathcal{N}(x)$ at radius 0 is valued in $[1+\alpha, \infty)$ for some $\alpha > 0$. Consequently, we can use soft stratification arguments based on looking at suitable tangent maps (i.e. linearising with respect to the *tangent plane* only) to show that the set

$$\{x : \mathcal{N}(x) \neq 2\}$$

has Hausdorff dimension at most $n-2$. For the remaining part of the branch set, namely $\{x : \mathcal{N}(x) = 2\}$, we are able to build a *single* center manifold $\mathcal{M}$ which touches V at all such points. Thus, to control this remaining part of the branch set it suffices to control the touching set of V and $\mathcal{M}$, reducing the problem to a question in unique continuation. By then proving approximate monotonicity of a suitable frequency function of V relative to this (single) center manifold, we are then able to stratify the set $\{x : \mathcal{N}(x) = 2\}$ in an analogous fashion, except now by looking at tangent maps of V relative to the center manifold. As in [6], to circumvent issues regarding strong convergence in $W^{1,2}_{\text{loc}}$ when taking tangent maps relative to the center manifold, we may view the tangent maps as a type of multi-valued Young measure.

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Instability of peanut solution in the mean curvature flow

Natasa Sesum
(joint work with Sigurd Angenent, Panagiota Daskalopoulos)

We consider families of compact hypersurfaces $\hat{M}_\theta(t) \subset \mathbb{R}^{n+1}$ that evolve by Mean Curvature Flow, and which depend continuously on the parameter $\theta \in \Theta$; the parameter belongs to some topological space Θ, which in our examples will always be an open subset of $\mathbb{R}^m$ for some $m \geq 1$. These solutions become singular at a finite time $T(\theta)$ which may vary with the parameter $\theta \in \Theta$. Such solutions have a parametrization $(p,t,\theta) \in \mathcal{M}^n \times [0,\infty) \times \Theta \mapsto \hat{F}(p,t,\theta) \in \mathbb{R}^{n+1}$ whose domain is an open subset of $\mathcal{M}^n \times [0,\infty) \times \Theta$ given by

$$\mathcal{O} = \big\{(p,t,\theta) \in \mathcal{M}^n \times [0,\infty) \times \Theta \mid 0 \leq t < T(\theta)\big\}.$$

For each $\theta \in \Theta$ the immersion $p \mapsto \hat{F}(p,t,\theta)$ satisfies the Mean Curvature Flow equation

$$\text{(MCF)} \qquad \big(\partial_t \hat{F}\big)^\perp = \Delta_{\hat{F}}(\hat{F}),$$

in which $(\partial_t \hat{F})^\perp$ is the component perpendicular to $T_{\hat{F}(p,t,\theta)}\hat{M}_\theta(t)$ of $\partial_t \hat{F}(p,t,\theta) \in T_{\hat{F}(p,t,\theta)}\mathbb{R}^{n+1}$, and $\Delta_{\hat{F}}$ is the Laplacian of the pullback of the Euclidean metric under the immersion $p \mapsto \hat{F}(p,t,\theta)$.

There have been many works towards understanding the formation of singularities in the mean curvature flow, that is classifying all possible singularity models. It is a very hard, if not even impossible question to answer in its full generality. To understand the singularities, which inevitably happen for closed mean curvature flows, one parabolically dilates around the singularity in space and time. Huisken's monotonicity formula guarantees that a subsequential limit of such dilations will weakly limit to a tangent flow which will be a weak solution to (MCF), evolving only by homothety. These solutions are called self-shrinking solutions. We call singularities modeled on generalized cylinders $\Sigma^k := \mathbb{R}S^{n-k} \times \mathbb{R}R^k$ are called in short *neckpinch* singularities. Even among neckpinch singularities, there are different types of neckpinch singularities, i.e. the nondegenerate and degenerate neckpinches. We expect the former one to be generic, that is, if a mean curvature flow starting with some initial hypersurface M_0 develops a degenerate neckpinch singularity, we expect to find a sequence of perturbations converging to M_0, whose mean curvature flows all develop nondegenerate neckpinch singularities. A neckpinch singularity is called **nondegenerate** if every pointed singularity model, that is, a smooth limit of any sequence of blow ups around $(x_i, t_i) \to (0, 0)$, is a round cylinder Σ^1, and is called **degenerate** if there is at least one blowup sequence around some $(x_i, t_i) \to (0, 0)$ with a pointed limit that is not Σ^1.

In this talk we focus on so-called *peanut solutions* whose existence was first suggested by Richard Hamilton, and then established in [1, 2]. In [2] the asymptotics of these solutions have been also established. These are examples of closed mean curvature flow solutions that contract to a point at the singular time, without ever becoming convex prior to that. At the same time these are examples of degenerate neckpinches.

If Θ is a two dimensional set of parameters, we will consider a two parameter family of solutions $\{\hat{M}_\theta(t) \mid \theta \in \Theta\}$ so that each $M_\theta(t)$ is a smooth MCF solution for $t \in [0, T(\theta))$, and so that for $\theta = \mathbf{0} := (0, 0)$ we have that $\hat{M}_{\mathbf{0}}(t)$ is one of the peanut solutions.

We show that degenerate neckpinch type behavior exhibited by any of peanut solutions in consideration is highly *unstable*, in the sense that there exist θ' arbitrarily close to $\mathbf{0}$ for which $\hat{M}_{\theta'}(t)$ forms a qualitatively different kind of singularity than $\hat{M}_\theta(t)$. More precisely, our goal in the talk is to prove the following result.

Theorem. Let $\hat{M}_{\mathbf{0}}(t)$ be the peanut solution as discussed above, and let $\bar{T}$ be its first singular time. There exists a t_0 close to $\bar{T}$, so that in every sufficiently small neighborhood of $\hat{M}_{\mathbf{0}}(t_0)$, there exist perturbations $\hat{M}_{\theta_s}(t_0)$ and $\hat{M}_{\theta_c}(t_0)$ with the following property. The MCF starting at $\hat{M}_{\theta_s}(t_0)$ as its initial data develops a spherical singularity, while at the same time the MCF starting at $\hat{M}_{\theta_c}(t_0)$ as its initial data develops a nondegenerate neckpinch singularity. Here θ_s and θ_c can be chosen arbitrarily small.

In [3] the authors showed that the ancient ovals occur as a limit flow of a closed MCF $\{M_t\}$ if and only if there is a sequence of spherical singularities converging to a cylindrical singularity. As a corollary of Theorem we show an analogous result for a blow up limit of our families of MCF solutions that can be seen as perturbations of peanut solution. More precisely, we have the following result.

Theorem. Appropriately rescaled subsequence of any sequence of solutions which belong to one of our families of solutions, whose initial data converge to the peanut solution, and all of which develop spherical singularities, converges to the Ancient oval solution.

We choose a parameter $K_0 > 0$ and let $\{\bar{M}_t : 0 < t < T\}$ denote a corresponding peanut solution. The profile function $\bar{U}(x, t)$ of this solution becomes *singular* at time T. The parabolic Type I rescaling $\bar{u}(y,\tau)$ is a solution of

$$u_\tau = \frac{u_{yy}}{1+u_y^2} - \frac{y}{2}u_y - \frac{n-1}{u} + \frac{u}{2}. \tag{1}$$

Its deviation from the cylinder is given by

$$\bar{u}(y,\tau) = \sqrt{2(n-1)} - K_0 Hm_4(y)e^{-\tau} + o(e^{-\tau}) \qquad (|y| \le 2\rho e^{\gamma\tau}, \tau \ge \tau_0). \tag{2}$$

The cylinder soliton corresponds to the constant solution $u = \sqrt{2(n-1)}$. The peanut solutions from [2] are perturbations of the cylinder given by $u(y,\tau) = \sqrt{2(n-1)} - v(y,\tau)$, where $v(y,\tau)$ satisfies

$$v_\tau = \mathcal{L}v - \frac{v_y^2 v_{yy}}{1+u_y^2} - \frac{1}{2}\frac{v^2}{\sqrt{2(n-1)}-v} = \mathcal{L}v + \mathcal{O}(v^2, v_y^2, v_{yy}^2), \tag{3}$$

where $\mathcal{L}$ is the *drift Laplacian*

$$\mathcal{L}v := v_{yy} - \frac{y}{2}\,v_y + v. \tag{4}$$

This operator is self adjoint in the Hilbert space

$$\mathcal{H} = \big\{ f \in L^2(\mathbb{R}; e^{-y^2/4}\,dy) \mid \forall y : f(-y) = f(y) \big\}.$$

We only have to consider even functions because of reflection symmetry of peanut solutions. Its spectrum is given by the sequence of simple eigenvalues

$$\lambda_k = 1 - \frac{k}{2}, \qquad k = 0, 2, 4, 6, \ldots$$

and the corresponding eigenfunctions are Hermite polynomials Hm_k.

To introduce a family of perturbations of $\bar{u}(y,\tau_0)$ in the direction of the lower eigenfunctions Hm_0, Hm_2 we let $\eta : \mathbb{R} \to \mathbb{R}$ be a smooth even cutoff function satisfying

$$\eta(y) = \begin{cases} 0 & \text{for } |y| \ge 2 \\ 1 & \text{for } |y| \le 1, \end{cases}$$

For any given $\epsilon > 0$, $\ell_0 > 0$, and $\Omega = (\Omega_0, \Omega_2) \in \mathbb{S}^1$ we then define our perturbations as

$$u_{\epsilon,\Omega}(y,\tau_0) = \bar{u}(y,\tau_0) + \epsilon\,\eta\big(\frac{y}{\ell_0}\big)\big\{\Omega_0 Hm_0(y) + \Omega_2 Hm_2(y)\big\}. \tag{5}$$

Let $u_{\epsilon,\Omega}(y,\tau)$ be the rescaled mean curvature flow solution starting at $u_{\epsilon,\Omega}(y,\tau_0)$. Using barriers, L^2 theory type of arguments for the system of ODEs of projections of difference of our solution from the round cylinder, and shooting type of arguments which are based on degree theory, we show that if we run the mean curvature flow sufficiently long, the $Hm_2(y)$ mode will start dominating. We show that depending on a sign of the coefficient in front of the $Hm_2(y)$ mode at much later time we have either the spherical or the nondegenerate neckpinch singularity as claimed.

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Tracing the contour of a cylindrical mean curvature flow with eigenfunctions

Kyeongsu Choi

The mean curvature flow (MCF) develops generically only spherical or cylindrical singularities in $\mathbb{R}^3$, and it is conjectured to be true in higher dimensions as well. Indeed, as stable singularities, the cylindrical singularities naturally appears in many examples, including the marriage ring, peanut solutions, dumbbell solutions, and degenerate necks. Of course, after Type I blow-up, one can find the same cylinder $\Sigma = \mathbb{R}^1 \times S^1(\sqrt{2}\,)$ at those singularities. However, their detailed shapes are all different. Indeed, they are locally the graph of a function u over the cylinder Σ, and u is close to an eigenfunction of the Ornstein-Uhlenbeck operator

$$L = \frac{\partial^2}{\partial y^2} + \frac{1}{2}\frac{\partial^2}{\partial \theta^2} - \frac{y}{2}\frac{\partial}{\partial y} + 1.$$

This is because the MCF can be written as $u_\tau = Lu + E(u)$ with an quadratic error $|E(u)| \leq C\|u\|^2_{C^2}$. Hence, as u approaches to zero, it behaves as like a solution to the linear equation $u_\tau = Lu$. Thus, it is asymptotic to $a(\tau)\phi(y,\theta)$ for some eigenfunction ϕ. On the other hand, it gives a local shape of the flow over the cylinder around the center of rescaling. Therefore, we can not find the global shape. However, we can choose another center of rescaling over the cylinder, we can discover local shapes around random center points. To be specific, any two center of rescaling will converges to each other as time to goes back after rescaling. Hence, we can observe that their local shape in the past time was similar. Then, considering their radius of the asymptotic cylinder in the future as their scales, we can guess their local shape. Then, we can get the global shape of the flow.

Non-uniqueness of locally minimizing clusters via singular cones

Anna Skorobogatova

(joint work with Lia Bronsard, Robin Neumayer, Michael Novack)

The classical multiple bubble problem in $\mathbb{R}^n$ concerns the existence and structure of configurations of N chambers, i.e. sets of locally finite perimeter, of prescribed finite volumes, together with an exterior chamber of infinite volume, that minimize interfacial area. This problem has received much attention in recent decades, with an abundance of work concerning the existence [4], structure and regularity [21, 7], and characterization [8, 9, 10, 18, 17, 22, 16, 13, 12] of minimizers and critical points. This includes recent breakthrough work in which Milman & Neeman [11] gave a complete classification of minimizers in the case $N \leq \min\{4, n\}$, resolving a long-standing conjecture of Sullivan [20].

Such questions have recently been extended to configurations with multiple infinite chambers, following the introduction of the (N, M)-cluster problem, by Alama, Bronsard & Vriend in [3], involving partitions of $\mathbb{R}^n$ into M chambers of finite volume and N chambers of infinite volume. Our focus is on the simplest case of $(1,2)$-*clusters* $\mathcal{X} = (\mathcal{X}(1), \mathcal{X}(2), \mathcal{X}(3))$, where $\mathcal{X}(1)$ denotes the finite-volume chamber. Such a cluster $\mathcal{X}$ is said to be locally minimizing if, for every $\rho > 0$,

$$\mathcal{P}(\mathcal{X}; B_\rho(0)) \leq \mathcal{P}(\mathcal{X}'; B_\rho(0))$$

whenever $\mathcal{X}(i)\Delta\mathcal{X}'(i) \Subset B_\rho(0)$ and $|\mathcal{X}(i)| = |\mathcal{X}'(i)|$ for $i = 1, 2, 3$, where

$$\mathcal{P}(\mathcal{X}; B_\rho(x)) := \frac{1}{2}\sum_{i=1}^{3} P(\mathcal{X}(i); B_\rho(x)),$$

and $P(\mathcal{X}(i); B_\rho(x))$ denotes the relative perimeter of $\mathcal{X}(i)$ in $B_\rho(x)$. Clusters with more than one infinite chamber arise in the models of tri-block copolymers [1, 2] in the small-volume "droplet" regime for multiple phases. Recent progress has been made toward existence, classification, and stability for $(1,2)$-clusters (and (M,N)-clusters more generally) in [3, 14, 6, 5, 13].

In [6], Bronsard & Novack demonstrate that when $n \leq 7$ (see also [3] for the case $n = 2$), the *standard lens cluster* $\mathcal{X}_{\text{lens}}$ is the unique locally minimizing $(1,2)$-cluster in $\mathbb{R}^n$ with $|\mathcal{X}(1)| = 1$ modulo rigid motions. When $n \geq 8$, they obtain such uniqueness only among clusters with planar growth at infinity. The standard lens cluster, which is locally minimizing in every dimension, is characterized by the properties that $|\mathcal{X}_{\text{lens}}(1)| = 1$, $\partial\mathcal{X}_{\text{lens}}(2) \cap \partial\mathcal{X}_{\text{lens}}(3) \subset \{x_n = 0\}$ and $\partial\mathcal{X}_{\text{lens}}(1)$ is the union of pair of equal-radii spherical caps meeting on $\{x_n = 0\}$ with equal angles of $\frac{2\pi}{3}$ between the three interfaces.

The aforementioned classification in $n \leq 7$ relies on the fact that planes are the only area-minimizing hypercones in these dimensions [19]. In view of the existence of non-planar area-minimizing hypercones when $n \geq 8$, it is natural to ask about the existence of locally minimizing $(1,2)$-clusters besides the standard lens when $n \geq 8$. We provide an affirmative answer this open question in a large number of dimensions starting from 8.

Theorem 1 (Bronsard-Neumayer-Novack-S.). *Let* $n \in \{8, \ldots, 2700\}$. *There exists a locally minimizing* $(1,2)$-*cluster* $\mathcal{X}$ *that is not a standard lens.*

Remark 1. *Novaga, Paolini & Tortorelli* [15] *have independently proven non-uniqueness of the standard lens cluster as a locally minimizing* $(1,2)$-*cluster in the case* $n = 8$.

The basic scheme to construct the local minimizing $(1,2)$-clusters of Theorem 1 goes as follows. Fix $n \geq 8$ and let K be a perimeter minimizing cone in $\mathbb{R}^n$ whose boundary is not a plane, e.g., the region $K = \{x_1^2 + \cdots + x_4^2 < x_5^2 + \cdots + x_8^2\}$ bounded by the Simons cone in $\mathbb{R}^8$.

For $R > 0$ large, we set up an energy minimization problem in the class of $(1, 2)$-clusters $\mathcal{X}$ with $\mathcal{X}(1) \subset B_{3R}$ of volume 1 for which, outside of B_{3R}, the chambers $\mathcal{X}(2)$ and $\mathcal{X}(3)$ coincide with K and K^c respectively. Morally one wants to minimize the cluster perimeter $\mathcal{P}(\mathcal{X}; B_{4R})$, though by instead minimizing the energy $\mathcal{P}(\mathcal{X}; B_{4R}) + \mathcal{G}_R(\mathcal{X}(1))$ for a carefully constructed penalization potential $\mathcal{G}_R$, we circumvent technical challenges that would arise from the possibility of $\mathcal{X}(1)$ saturating the constraint $\mathcal{X}(1) \subset B_{3R}$.

For a sequence $R_k \to \infty$, take a sequence of minimizers $\mathcal{X}_k$ to this minimization problem with $R = R_k$. We wish to obtain a local minimizer of $\mathcal{P}$ as a limit of the $\mathcal{X}_k$, and to this end we use a "partial concentration compactness approach", namely:

Step 1: Characterize precisely how a sequence can lose compactness and the cost in energy to exhibit this behavior.

Step 2: Establish compactness for a (minimizing) sequence by showing its energy lies below this loss-of-compactness threshold.

In the present setting, Step 1 as stated seems out of reach, as there are in principle many possible asymptotic behaviors and energy costs of $\{\mathcal{X}_k\}$ if the sequence loses compactness due to volume loss at infinity of $\mathcal{X}_k(1)$. The key issue is an asymptotically infinite contribution to the energy coming from the minimal surface $\partial\mathcal{X}_k(2) \cap \partial\mathcal{X}_k(3)$, whose behavior at large scales that are still infinitesimal relative to R_k is highly difficult to characterize in general. This is the main difficulty in the problem, and the obstruction to characterizing the minimizers obtained in Theorem 1.

One possible way for a piece of $\mathcal{X}_k(1)$ to escape to infinity is along approximately planar portions of the minimal surface $\partial\mathcal{X}_k(2) \cap \partial\mathcal{X}_k(3)$. Using the rigidity result in [6], we deduce that in this case, $\mathcal{X}_k$ locally looks like a rescaling of the standard lens cluster. In particular, its renormalized local energy contribution in a large ball is approximately equal to sum of the area of an equatorial disk in this ball and the *renormalized lens energy*

$$\Lambda_{\mathrm{plane}}(n) := P(\mathcal{X}_{\mathrm{lens}}(1)) - \omega_{n-1}\rho_n^{n-1}\,,$$

where ρ_n denotes the radius of the disc $\mathcal{X}_{\mathrm{lens}}(1) \cap \{x_n = 0\}$.

As described above, this is not the only way that $\mathcal{X}_k(1)$ can lose mass at infinity. Indeed, a piece of $\mathcal{X}_k(1)$ may drift off to infinity along a non-planar portion of the minimal surface $\partial\mathcal{X}_k(2) \cap \partial\mathcal{X}_k(3)$, but the key observation is that this will still produce a local minimizer as in Theorem 1 with a singular blowdown cone. Thus we only need to rule out the possibility that all of the mass of $\mathcal{X}_k(1)$ is lost via asymptotic lens behavior, which is the one asymptotic behavior whose limiting energy we can characterize. Given an area-minimizing hypercone C, consider the class

$$\mathcal{F}(C) := \{\text{clusters } \mathcal{X} \text{ with } |\mathcal{X}(1)| = 1,\ \partial\mathcal{X}(2) \cap \partial\mathcal{X}(3) \subset C\}\,,$$

and the associated energy

$$\Lambda(C) := \inf\{P(\mathcal{X}(1)) - \mathcal{H}^{n-1}(C \cap \mathcal{X}(1)^{(1)}) :\ \mathcal{X} \in \mathcal{F}(C)\}\,. \tag{1}$$

In this notation, $\Lambda_{\rm plane}(n) = \Lambda(\mathbb{R}^{n-1} \times \{0\})$ and we use the notation $\Lambda_{\rm plane}(n)$ because it is more concise. We then reduce the validity of Theorem 1 to the validity of the strict inequality

$$\Lambda(C) < \Lambda_{\rm plane}(n) \tag{2}$$

for some area-minimizing cone C, which precisely rules out the possibility of all mass escaping in the form of a lens and thus proving that a piece of the limit of the $\mathcal{X}_k$ yields a locally minimizing $(1,2)$-cluster for which all possible blowdowns have interfaces that are area-minimizing cones. Such a cluster is therefore not the standard lens cluster, since it does not have planar growth.

We then proceed to verify that the strict inequality (2) holds for certain choices of quadratic (Lawson) cones

$$C_{k,l} := \left\{ (x,y) \in \mathbb{R}^{k+1} \times \mathbb{R}^{l+1} : |x|^2 = \frac{k}{l}|y|^2 \right\} .$$

for $k,l \in \mathbf{N}$, which are area-minimizing cones with an isolated singularity in $\mathbb{R}^{k+l+2}$ when $k+l > 6$, while the cones $C_{3,3}$ (Simons' cone) and $C_{2,4}$ are the only two area-minimizing quadratic cones in $\mathbb{R}^8$. For $n \in \{8,\dots,2700\}$, we verify that $\Lambda(C_{n/2-1,n/2-1}) < \Lambda_{\rm plane}(n)$ if n is even, and $\Lambda(C_{(n-1)/2-1,(n-1)/2}) < \Lambda_{\rm plane}(n)$ if n is odd, via an explicit construction of competitor $(1,2)$-clusters modeled on these cones, therefore allowing us to conclude.

These computations may be reduced to one-dimensional integrals which are not easy to estimate by hand to a desirable level of precision in all dimensions. We therefore use the FLINT C library to provide us with a suitably precise calculation of these values in general, making this aspect of our proof computer-assisted.

In even dimensions $n \in 2\mathbb{N}$ we are able to obtain a reasonably clean closed formula for these integrals, thus allowing one in principle to compute both $P(\mathcal{X}(1)) - \mathcal{H}^{n-1}(C \cap \mathcal{X}(1))$ for our choice of competitor and $\Lambda_{\rm plane}(n)$ by hand and avoid the need for computer assistance, and indeed we perform the by-hand computations for the case $n = 8$.

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Some uniqueness related to Ricci flow smoothing

Man-Chun Lee

Let $C(X)$ be a metric cone with link X which is possibly non-smooth. These arise naturally as tangent cones of Gromov-Hausdorff limits of sequences of non-collapsing smooth manifolds with uniform Ricci curvature lower bounds due to the celebrated work of Cheeger-Colding [5]. Particularly, any blow-down limit a complete non-compact manifold (M, g) with $\mathrm{Ric}(g) \geq 0$ and Euclidean volume growth, in the pointed Gromov-Hausdorff sense is a metric cone. We are preliminarily interested in studying metric cones by Ricci flow. This is a one parameter family of metrics $g(t)$ satisfying $\partial_t g(t) = -2\mathrm{Ric}(g(t))$ for $t > 0$ and the initial condition is understood in the sense of Gromov-Hausdorff topology:

Definition 1. Given a smooth manifold M, $x_0 \in M$ and a Ricci flow $g(t)$ on $M \times (0,T]$, we say that $g(t)$ is coming out of a metric cone $C(X)$ if

$$(M, d_{g(t)}, x_0) \to (C(X), d_c, o_{tips})$$

in the pointed Gromov-Hausdorff sense as $t \to 0$, where d_c denotes the conical distance metric.

We are interested in the existence, uniqueness and stability of Ricci flow in the sense of Definition 1, which was found to be powerful in studying the rigidity of geometric structure. For instances when $n = 3$, a simple consequence of Simon-Topping [19] (building on the work of Hochard [10]) shows that for any three-manifolds with $\mathrm{Ric} \geq 0$ and Euclidean volume growth, any metric cone $C(X)$ is bi-Hölder to a smooth manifold. This is based on constructing quantitative Ricci flow along blow-down procedure. Along the way, the estimates of the Ricci flow enables us to show that M^3 is diffeomorphic to $\mathbb{R}^3$, giving an alternative proof of a result by Zhu [20]. Indeed, this circle of idea can be generalized to manifolds with non-negative 1-isotropic curvature, i.e. $\mathrm{Rm}^{\mathbb{C}}(\omega, \bar{\omega}) \geq 0$ for all simple and isotropic $\omega \in \Lambda^2(\mathbb{C}^n)$. This can be viewed as a generalization of $\mathrm{Ric} \geq 0$ in higher dimension in the sense that it coincides for $n = 3$. In this case, the quantitative existence of Ricci flow was established by Lai [12], see also [9] for the application. We also refer interested readers to [2, 18] for an overview on related topics and its important role in differentiable sphere Theorem.

Given the existence result established in many situations, it is asked by Schulze that how rigid the Ricci flow is, if it is coming out of a cone:

Conjecture 1 (Schulze)**.** *The Ricci flow coming out of a cone $C(X)$ is unique within the class of solutions $g(t)$ with $0 \leq \mathrm{Rm}(g(t)) \leq \alpha t^{-1}$ for some $\alpha > 0$.*

This is partially motivated by a result of Deruelle-Schulze-Simon [6] who shows that if the initial distance metric space is Reifenberg and locally bi-Lipschitz to Euclidean space, then two solutions to the Ricci flow whose Ricci curvature is uniformly bounded from below and whose curvature is bounded by αt^{-1} converge to one another at an exponential rate once they have been appropriately gauged. This plays an important role in the resolution of pinching conjecture by Hamilton-Lott, see also [17, 15]. Indeed since a metric cone is self-similar under dilation, it follows heuristically by uniqueness that a Ricci flow coming out of a metric cone should itself be self-similar, in which the geometry is more rigid.

Motivated by this, in a joint work with Chan and Peachey [4], we show that those Ricci flows must be self-similar.

Theorem 2 (Theorem 1.3 in [4])**.** *If $g(t)$ is a Ricci flow smoothing coming out of cone with non-negative 1-isotropic curvature, then it must be a expanding gradient Ricci soliton. That is $2t \cdot \mathrm{Ric}(g(t)) + g(t) = \nabla^2 u(t)$ for some smooth function u in space-time.*

This is built upon the work [7] of Deruelle-Schulze-Simon who shows in case the initial distance metric is Reifenberg, then the solution behaves asymptotically like a gradient expander. Using this, they are able to extend partially the Hamilton-Lott's pinching conjecture to higher dimension in the sense of 1-isotropic curvature. Together with the works by Lott [17] the author and Topping [15, 16], the following is well-understood as the applications of existence, uniqueness and stability of Ricci flow smoothing:

Theorem 3. *Suppose* (M^n, g_0) *is a complete non-compact such that*

$$\mathrm{Rm}(g_0) - \varepsilon \cdot \mathrm{scal} \cdot g_0 \in \mathrm{C}_{WPIC1}$$

for some $0 < \varepsilon << 1$*, then* (M, g_0) *is flat if one of the following hold:*

(i) $n = 3$*;*
(ii) $\mathrm{Rm}(g_0) \in \mathrm{C}_{WPIC2}$*;*
(iii) (M, g_0) *has Euclidean volume growth.*

Most of the result doesn't rely on the strong curvature condition: $\mathrm{Rm} \geq 0$. This raise the question of whether the conjecture of Schulze is optimally stated or not. Unfortunately without any curvature lower bound, a general uniqueness was shown to be impossible by the work of Angenent-Knopf [1] who constructed explicit examples of metric cones in dimensions five and higher which can be smoothed out by an arbitrary finite number of geometrically distinct expanding gradient Ricci solitons. It was also shown by the author and Topping [14] that even if we restrict $n = 2$, fixed topology and with scaling invariant control, uniqueness is impossible if the initial data of the Ricci flow is attained weakly.

In contrast with the development of mean curvature flow, the subtlety lies on the gauge fixing which is relatively complicated in Ricci flow. Indeed given two Ricci flow solutions $g(t)$ and $\tilde{g}(t)$, it is more natural to gauge between them by solving the harmonic map heat flow

$$\partial_t F = \Delta_{g(t),\tilde{g}(t)} F \tag{1}$$

using the connection induced by $g(t)$ and $\tilde{g}(t)$. In case $\tilde{g}(t)$ is static, the flow tends to be the harmonic map between $g(t)$ and the reference geometry $\tilde{g}$. It was first studied by Eells-Sampson [8]. In Ricci flow content, it transforms one Ricci flow $g(t)$ into a strictly parabolic flow, called Ricci-DeTurck flow, as long as F remains diffeomorphism. The Ricci-DeTurck flow can be regarded as a gauged fixed version of Ricci flow, and the stability in bounded geometry case has been studied by Koch-Lamm [11], Burkhardt-Guim [3] and many others.

By utilizing the ideas, in [13] we construct local solution to (1) starting from identity when initial data is attained smoothly but without quantitative geometric control. Using this, we discuss how gauge can be fixed under scaling invariant estimate and thus prove uniqueness in that case:

Theorem 4 (Theorem 1.1 in [13])**.** *Ricci flow is unique within the class of solutions with* $|\mathrm{Rm}(g(t))| \leq \alpha t^{-1}$ *and* $\lim_{t\to 0} g(t) = g_0$ *in* C^∞_{loc}*.*

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Evanescence of minimal submanifolds with isolated singularities

Alessandro Carlotto
(joint work with Yangyang Li, Zhihan Wang)

In this lecture I have given a (fairly non–technical) account of the main results in [1], ensuring that minimal submanifolds with only strongly isolated singularities are non–generic, except for certain borderline cases which on the one hand can be geometrically characterized and, on the other hand, cannot possibly occur in the codimension one case. Our study stemmed from a question posed by André Neves about the landscape of minimal hyperspheres in nearly-round metrics, in relation to Hsiang's resolution, in the negative, to Chern's spherical Bernstein conjecture.

1. Context and setup

Let (M^N, g) be a compact (smooth) Riemannian manifold, which we shall typically assume to have empty boundary, and let V denote a (boundaryless) n-dimensional stationary integral varifold thereof, with $2 \leq n < N$. We shall say that a singular point of V, $p \in \mathrm{spt}(V) \setminus \mathrm{reg}(V)$, is strongly isolated if some tangent cone to V at p is regular (namely: has smooth link) and has multiplicity one.

If that is the case, we know, by pioneering work of Leon Simon [7], that (i) any such p is topologically isolated, (ii) the aforementioned tangent cone is actually unique, and (iii) in a neighborhood of the point in question the support of V can be written as the image through the exponential map of a normal graph (a section of the normal bundle), with a defining section subject to suitable decay estimates on approach to the singularity. In what follows, we will employ the acronym **MSI** to denote any such stationary varifold having only strongly isolated singularities. For the sake of notational convenience, we can just identify any such varifold with its regular part, that is an open smooth minimal submanifold.

While the monotonicity formula allows one to rule out the presence of **MSI** in Euclidean spaces, it is possible to obtain plenty of examples in round spheres by means of the following construction: for any closed, smooth minimal hypersurface Σ_0 in $\mathbb{S}^{N-1}$ one can consider the spherical suspension of Σ_0 in $\mathbb{S}^N$, which shall indeed exhibit two strongly isolated singularities at the north and south poles unless Σ_0 is totally geodesic. A particular case that warrants special attention is the one corresponding to the spherical suspension of the standard Clifford torus $\mathbb{S}^1(1/\sqrt{2})\times\mathbb{S}^1(1/\sqrt{2})$ in $\mathbb{S}^3 \subset \mathbb{R}^4$: the corresponding **MSI** in $\mathbb{S}^4$, henceforth referred to as Clifford football, turns out to be the (multiplicity one) varifold limit of the sequence of closed minimal hyperpsheres constructed by Hsiang in round $\mathbb{S}^4$ to disprove Chern's spherical Bernstein conjecture [5].

It is natural to wonder about the possibility of constructing **MSI** in ambient manifolds more general than space forms. As a concrete instance, one wonders whether this is feasible by means of perturbative methods, in analogy with what is routinely done for closed smooth minimal submanifolds (cf. [8]) and – perhaps more closely – for singular special Lagrangian conifolds, see e. g. [3].

2. The evanescence result and related comments

The main result in [1] gives a strong negative answer to the previous question:

Main Theorem. Given a closed manifold M of dimension $N \geq 3$, there exists a generic subset $\mathscr{G}_0$ of the space of smooth metrics on M with the following property: for every $g \in \mathscr{G}_0$, any g-stationary integral n-varifold in (M, g), $2 \leq n < N$, will:

(i) either be entirely smooth, or
(ii) have at least one singular point that is not strongly isolated, or
(iii) have only strongly isolated singular points all having links with Morse index equal to N.

As a very important special case, we note that in the codimension one scenario (namely for $N = n + 1$) the third alternative cannot possibly occur, and so we derive the following remarkable conclusion:

" for any $n \geq 2$ non-smooth MSI are only observable at non-generic metrics."

In particular, this suffices to rule out the potential persistence of the Clifford football mentioned at the end of the previous section.

Remark. The third alternative (i. e. links with Morse index is equal to N) may occur:

- in the case when the link is disconnected:

  if and only if $2n = N$ and the link is the disjoint union of exactly two equatorial $\mathbb{S}^{n-1} \subset \mathbb{S}^{2n}$;

- in the case when the link is connected:

  certainly for the Veronese embedding of $\mathbb{RP}^2 \subset \mathbb{S}^4$, and conjecturally for the Veronese-type embeddings of $\mathbb{CP}^2 \subset \mathbb{S}^7$ and $\mathbb{HP}^2 \subset \mathbb{S}^{13}$, although a full classification is still open.

We expect all such singularities (listed above) to be persistent: for instance considering their respective spherical suspensions one should see them for a full neighborhood of the round metric.

The previous claim can be easily justified in the first case, namely when the singularity is a "cross". Indeed, for $d \leq N/2-1$ let Σ_0 be the image of an equatorial (totally geodesic) immersion

$$\varphi = \varphi_1 \sqcup \varphi_2 : \ \mathbb{S}^d_1 \sqcup \mathbb{S}^d_2 \ \longrightarrow \ \mathbb{S}^{N-1}$$

and let $\Sigma \subset \mathbb{S}^N$ be the spherical suspension of Σ_0. If $d+1 = N/2$ then by a classical estimate in [6] Σ is a **MSI** with two singularities each having Morse index equal to N (equivalently: effective index 0). By the implicit function theorem (applied to each minimal sphere of Σ separately) – in fact possibly by a Lyapunov-Schmidt reduction in presence of Jacobi fields (e. g. at the round metric) – one proves persistence of the picture in an open set of metrics, thus sharpness of our generic regularity result. The analysis of the borderline case when the link is connected is a lot more delicate, and will be the object of forthcoming work by the same authors.

3. A geometric application: spherical Bernstein conjecture and nearly–round metrics

Starting from the preceding statement, with some extra work one can actually show the following result:

Corollary. For every $\varepsilon > 0$, there exists a neighborhood $\mathscr{N}(\varepsilon)$ of the round metric on S^4 such that for every $g \in \mathscr{N}(\varepsilon)$, every mod 2 cyclic g-stationary integral 3-varifold V with total mass $\leq 4\pi^2 - \varepsilon$ has only strongly isolated singularities. Hence, generically, any such varifold is entirely smooth.

In turn, this easily implies (arguing by contradiction, and ultimately relying again also on the bumpy metric theorem in [8]) the following generic finiteness assertion:

Corollary. Given any $\varepsilon > 0$ there exists a neighborhood $\mathscr{N}(\varepsilon)$ of the round metric on S^4 such that for a generic choice of $g \in \mathscr{N}(\varepsilon)$ the Riemannian manifold (S^4, g) shall contain only finitely many closed, embedded minimal hypersurfaces of area less that $4\pi^2 - \varepsilon$.

Hence, since the Clifford football has three-dimensional Hausdorff measure equal to π^3, we reach the geometric conclusion that Hsiang's sequence of closed minimal hyperspheres cannot be possibly persist for all nearly-round metrics as an infinite family.

Our analysis should be compared to the striking advances that have recently been obtained in the study of generic properties both for minimal surfaces (partly including also the non-trivial case of geodesic nets) and for the mean curvature flow. Although we add a structural assumption on the nature of singularities we deal with, there are two peculiar aspects in our results:

- we do not restrict to area-minimizing (or min-max) submanifolds, thus the limit cones at the singularities do not need to be stable;
- we do not place any restrictions to the dimension and codimension of the submanifolds in question.

While works like [2], which follows the previous breakthroughs by Chodosh, Mantoulidis and Schulze about (unconditional) generic regularity up to ambient dimension 10, ultimately connect to the pioneering work by Hardt and Simon [4], and thus build upon the design of minimal (in fact: minimizing) foliations, our main result is rather achieved as a consequence of the fine analysis of the Fredholm index of the Jacobi operator of an **MSI**. More specifically, we prove on the one hand an exact formula relating that number to the Morse indices of the conical links at the singular points, while on the other hand we show that the same number is non-negative for all such varifolds if the ambient metric is generic.

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On the boundary branching set of the one-phase problem

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(joint work with L. Ferreri and B. Velichkov)

Let $\mathcal{A}$ be a bounded open set in $\mathbb{R}^d$, $d \geq 2$, and $u_0 : \partial\mathcal{A} \to \mathbb{R}$ be a given nonnegative function. The one-phase Bernoulli problem consists in minimizing the functional

$$J_1(u, \mathcal{A}) := \int_{\mathcal{A}} |\nabla u|^2 \, dx + |\Omega_u \cap \mathcal{A}| \tag{1}$$

among all nonnegative functions $u : \overline{\mathcal{A}} \to \mathbb{R}$ that agree with u_0 at $\partial\mathcal{A}$, where for any function $u \in H^1(\mathcal{A})$ we denote by Ω_u its positivity set

$$\Omega_u := \{u > 0\} \cap \mathcal{A}.$$

It is well-known, due to the works of Alt, Caffarelli, De Silva, Jerison, Kenig, Savin and Weiss among others, that the optimal regularity for minimizers u of J is Lipschitz and that the free boundary $\partial\Omega_u \cap \mathcal{A}$ inside $\mathcal{A}$ is analytic up to a closed singular set of Hausdorff dimension at most $d - 5$. The behavior of the free boundary $\partial\Omega_u \cap \mathcal{A}$ and the solution u up to the boundary of $\partial\mathcal{A}$ was first studied in [1] around points $z_0 \in \partial A$ at which the boundary datum u_0 vanishes identically:

$$u_0 \equiv 0 \quad \text{in} \quad B_r(z_0) \cap \mathcal{A},$$

for some $r > 0$. Without loss of generality, we can take $B_r(z_0) = B_1$. It is immediate to check that if $u_0 \equiv 0$ in $\partial\mathcal{A} \cap B_1$, then the minimizer u (trivially extended in $B_1 \setminus \mathcal{A}$) is a solution to the variational problem

$$\min\Big\{ J_1(v, B_1) \ : \ v \geq 0,\ v = u \text{ on } \partial B_1,\ \{v > 0\} \subset \mathcal{A} \Big\}. \tag{2}$$

In [1] it was shown that if u is a solution to the above variational problem and $\partial\mathcal{A}$ is $C^{1,\frac{1}{2}}$ smooth in a neighborhood of z_0, then the free boundary $\partial\Omega_u \cap \mathcal{A}$ is $C^{1,\frac{1}{2}}$ regular in a neighborhood of any point on $\partial\Omega_u \cap \partial\mathcal{A}$; this regularity is also optimal in the sense that even for analytic $\partial\mathcal{A}$, there are solutions u whose boundary $\partial\Omega_u$ is no more than $C^{1,\frac{1}{2}}$. Moreover, any minimizer u is a (classical) solution to the problem

$$\begin{cases} \Delta u = 0 & \text{in} & \Omega_u \cap \mathcal{A} \\ u = 0 & \text{on} & \partial\Omega_u \\ |\nabla u| = 1 & \text{on} & \partial\Omega_u \cap \mathcal{A} \\ |\nabla u| \geq 1 & \text{on} & \partial\Omega_u \cap \partial\mathcal{A}, \end{cases} \tag{3}$$

in a neighborhood of the contact set $B_1 \cap \partial\Omega_u \cap \partial\mathcal{A}$.

The points where the two boundaries $\partial\Omega_u$ meet $\partial\mathcal{A}$ are the so called *points of branching*:

$$\mathcal{B}_1(u) := \Big\{x \in B_1 \cap \partial\Omega_u \cap \partial\mathcal{A} \ : \ |B_r(x) \cap (\mathcal{A} \setminus \Omega_u)| \neq 0 \ \text{ for every } \ r > 0\Big\}. \tag{4}$$

The $C^{1,\frac{1}{2}}$ regularity of $\partial\mathcal{A}$, $\partial\Omega_u$ and u provides that at the points of branching the two boundaries $\partial\Omega_u$ and $\partial\mathcal{A}$ are tangent and the minimizer u satisfies

$$u = 0 \quad \text{and} \quad |\nabla u| = 1 \quad \text{at} \quad \mathcal{B}_1(u),$$

so at first order these points are indistinguishable from the interior points $\partial\Omega_u \cap \mathcal{A}$. Thus, the ($C^{1,\frac{1}{2}}$-)regularity of $\partial\Omega_u$ and $\partial\mathcal{A}$ by itself does not provide any a priori information on the contact set $\partial\Omega_u \cap \partial\mathcal{A}$ and the way $\partial\Omega_u$ is approaching $\partial\mathcal{A}$.

This branching behavior is a common issue in the free boundary regularity theory and in geometric analysis; it appears naturally at points at which the first order blow-up analysis of the solutions does not provide any geometric information about the set of points itself.

Determining the fine structure of these branching points is not an easy task even in dimension $d = 2$ and only few results are available in this direction. Concerning the Bernoulli problem, De Philippis together with the second and third named authors recently proved that, in dimension $d = 2$, the set of branching points is locally finite through a quasi-conformal map argument (later generalized by the authors of this paper to analytic boundaries).

The analysis of the branching sets in higher dimension is strongly related to a unique continuation type problem, and therefore to Almgren's frequency function. In the framework of area-minimizing currents, the upper bound on the dimension of the branching set in the interior is known thanks to the monumental work of Almgren and the subsequent works of De Lellis and Spadaro, while the branching behavior at the boundary, for analytic boundaries, is currently an open problem. For the linear thin-obstacle problem, where the Almgren's frequency function is well-known to hold , a $(d-2)$-rectifiability result was recently obtained by Focardi and Spadaro. For the obstacle problem in dimension $d > 2$ it is known, thanks to the works of Caffarelli and Monneau, that the set of singular points (which exhibits the same type of branching behavior) is contained in a C^1 manifold of dimension $(d-1)$. While for generic free boundaries finer results were obtained by Figalli, Ros-Oton and Serra, the optimal dimension of the branching set is still open.

Our result gives a more precise description of the set of branching points (4) in any dimension $d \geq 2$ for the one-phase Bernoulli problem at the boundary: in any dimension, when $\partial\mathcal{A}$ is a $(d-1)$-dimensional analytic manifold, the set of branching points $\mathcal{B}_1(u)$ has Hausdorff dimension at most $(d-2)$. We notice that the analyticity of $\partial\mathcal{A}$ is fundamental for this result. Indeed, one can easily produce examples of wildly behaving contact sets by taking a half-plane solution $u = (x_d)^+$ and then constructing a set $\mathcal{A}$ with C^∞ boundary $\partial\mathcal{A}$ touching $\partial\Omega_u = \{x_d = 0\}$ on an arbitrary closed set.

In what follows, we assume that the origin is a branching point, $0 \in \mathcal{B}_1(u)$. We will denote by x' the points in $\mathbb{R}^{d-1}$, so that $B_1' := B_1 \cap (\mathbb{R}^{d-1} \times \{0\})$, and we assume that the boundary of $\mathcal{A}$ is the graph of an analytic function $\phi : B_1' \to \mathbb{R}$, precisely:

$$\mathcal{A} :=\{(x', x_d) \in B_1 \ : \ x_d > \phi(x')\}.$$

Thanks to the $C^{1,\frac{1}{2}}$ regularity of $\partial\Omega_u$ we may assume that there is a $C^{1,\frac{1}{2}}$ function

$$f : B_1' \to \mathbb{R}, \quad f \geq \phi \ \text{ on } \ B_1',$$

such that, up to a rotation and translation of the coordinate system, we have

$$\begin{cases} u(x) > 0 & \text{for} \quad x \in (x', x_d) \in B_1 \quad \text{such that} \quad x_d > f(x'); \\ u(x) = 0 & \text{for} \quad x \in (x', x_d) \in B_1 \quad \text{such that} \quad x_d \leq f(x'). \end{cases} \tag{5}$$

In terms of the functions f and ϕ the contact set of the two boundaries $\partial\Omega_u$ and $\partial\mathcal{A}$ reads as

$$\mathcal{C}_1(u) := B_1 \cap \partial\Omega_u \cap \partial\mathcal{A} = \{(x', x_d) \in B_1 \ : \ x_d = \phi(x') = f(x')\}. \tag{6}$$

We also introduce the set of points

$$\mathcal{S}_1(u) := \{x \in \mathcal{C}_1(u) \ : \ |\nabla u|(x) = 1\}, \tag{7}$$

which is a closed subset of $\{x_d = \phi(x')\}$ and contains all points of branching, that is:

$$\mathcal{B}_1(u) \subset \mathcal{S}_1(u).$$

The following is the main result of the paper.

Theorem 1 (Dimension of the boundary branching set)**.** *Let $B_1 \subset \mathbb{R}^d$ and let $u \in H^1(B_1)$ be a solution of* (2)*. Suppose, moreover, that the function ϕ describing $\partial\mathcal{A}$ is analytic. Then, either*

$$\partial\Omega_u \cap B_1 \equiv \mathrm{graph}(\phi) \cap B_1$$

or

$$\dim_{\mathcal{H}}(\mathcal{S}_1(u)) \leq d - 2.$$

Moreover, in the second case, if $d = 2$ then $\mathcal{S}_1(u)$ is locally finite.

Remark 1. *We notice that the estimate of Theorem 1 is optimal in all dimensions. Indeed, in dimension two, examples of solutions with isolated branching points were constructed by De Philippis and the second and third named authors. These 2D solutions can be then used to build sharp examples in any dimension d, by extending them to functions invariant with respect to the remaining $d-2$ variables. The extensions obtained this way are still solutions to* (3) *and minimizers to* (2).

As a consequence of our analysis, we also obtain three results of independent interest:

(1) a boundary unique continuation result for quasilinear elliptic operators;
(2) an estimate on the dimension of the free boundary in quasilinear thin-obstacle problems;

(3) an estimate on the dimension of the set of branching points in the two-phase problem, under the assumption that there exists an analytic manifold lying between the two free boundaries.

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Passing through nondegenerate singularities of mean curvature flow

Zhihan Wang

(joint work with Ao Sun, Jinxin Xue)

A 1-parameter family of smooth hypersurfaces $t \mapsto \mathbf{M}(t) \subset \mathbb{R}^{n+1}$ is called a *mean curvature flow (MCF)* if it solves

$$(\partial_t X)^\perp = \vec{H}_{\mathbf{M}(t)} \,,$$

where $\perp$ is the projection onto normal bundle of $\mathbf{M}(t)$, and $\vec{H}_{\mathbf{M}(t)}$ denotes the mean curvature of $\mathbf{M}(t)$. It is well-known that any closed smooth hypersurface $M_0 \subset \mathbb{R}^{n+1}$ starts a MCF for short time, and it always develop singularities in finite time. Ilmanen [10] constructed a distributional measure theoretic subsolution $t \mapsto \mathbf{M}(t)$, known as a *Brakke flow*, to the MCF equation, which agrees with the smooth solution before first singular time. A central question is then to understand the geometric and topological behavior of $\mathbf{M}$ near each singularity.

A common way to study singularities is via a blow-up process: given a singularity $p_\circ = (X_\circ, t_\circ) \in \mathbb{R}^{n+1} \times \mathbb{R}$ of $\mathbf{M}$ in spacetime, Huisken [8] introduced a new family of hypersurfaces called the *rescaled mean curvature flow (RMCF)* of $\mathbf{M}$ based at $p_\circ$:

$$\mathcal{M}^{p_\circ}(\tau) = e^{\tau/2}(\mathbf{M}(t_\circ - e^{-\tau}) - X_\circ) \,,$$

and shown that it's a negative gradient flow of the Gaussian area functional

$$\mathcal{F}[\Sigma] := \int_\Sigma (4\pi)^{-n/2} e^{-|X|^2/4} \, dX \,,$$

and hence has long time subsequential (weak) limits S solving the *self-shrinker equation* $\vec{H}_S + X^\perp/2 = 0$ (where $X^\perp$ denotes the projection of position vector onto the normal direction of S). Any such limit S models the infinitesimal behavior of the flow $\mathbf{M}$ near $p_\circ$ and hence is called a *tangent flow* of $\mathbf{M}$ at $p_\circ$.

The most commonly appearing family of self-shrinkers are the *round cylinders*: $\mathcal{C}_{n,k} := S^{n-k}(\sqrt{2(n-k)}) \times \mathbb{R}^k$, defined for $1 \le k \le n-1$. Besides the hyperplane and round sphere $S^n(\sqrt{2n})$, they are known to be the only mean convex self-shrinkers [9], "linearly stable" self-shrinkers [3] and genus 0 self-shrinkers when $n = 2$ [2]. Suppose MCF $\mathbf{M}$ has a rotation of $\mathcal{C}_{n,k}$ to be a tangent flow at $p_\circ$ with multiplicity 1 (call such $p_\circ$ a *k-cylindrical singularity*), then by [18, 4], after a rotation, its RMCF $\mathcal{M}^{p_\circ}(\tau)$ based at $p_\circ$ converges locally smoothly to $\mathcal{C}_{n,k}$. Sun-Xue [12] obtained a refined convergence: when $\tau \gg 1$, in the ball of radius $\sim \sqrt{\tau}$,

$\mathcal{M}^{p_\circ}(\tau)$ is a C^2 graph of some function $u(\cdot,\tau)$ over $\mathcal{C}_{n,k}$, and $\exists\,\mathcal{I}\subset\{1,2,\dots,k\}$ (possibly empty) such that after a rotation in $\mathbb{R}^k$ direction,

$$u(\theta,y;\tau)=\frac{\sqrt{2(n-k)}}{4\tau}\sum_{i\in\mathcal{I}}(y_i^2-2)+o(\tau^{-1})\,.$$

We call $p_\circ$ a *nondegenerate singularity* if $\mathcal{I}=\{1,\dots,k\}$ in the asymptotic above. Call it *(fully) degenerate* if $\mathcal{I}=\emptyset$. Nondegenerate singularities were proved to be locally generic and stable under initial data perturbation by Sun-Xue [12].

Our main theorem in [13] describes the geometry and topology of MCF $\mathbf{M}$ near a nondegenerate cylindrical singularity, which implies the following.

Theorem 1. *Let $1\le k\le n-1$, $t\mapsto\mathbf{M}(t)$ be a time-translation of MCF in $\mathbb{R}^{n+1}$ constructed by Ilmanen above; suppose $(0,0)$ is nondegenerate singularity of $\mathbf{M}$ with tangent flow $\mathcal{C}_{n,k}$. Let $Q_r:=B_r^{n-k+1}\times B_r^k$. Then $\exists\,r_\circ,t_\circ\in(0,1)$ such that*

(i) *$(0,0)$ is the only singularity of $\mathbf{M}$ in $Q_{r_\circ}\times[-t_\circ,t_\circ]$;*
(ii) *$\forall\,t\in(0,t_\circ]$, within $Q_{r_\circ}$, topologically $\mathbf{M}(t)$ is obtained by an $(n-k)$-surgery on $\mathbf{M}(-t)\cap Q_{r_\circ}$.*

Note that those properties in the theorem may fail without assuming "nondegeneracy". In fact, the famous "marriage ring" example is a MCF in $\mathbb{R}^3$ with a curve of cylindrical singularities (violating (i)), and there are degenerate neck pinch examples [1] of MCF with an isolated cylindrical singularity but no topology change after singular time (violating (ii)).

In general, we let $\mathcal{S}^k(\mathbf{M})$ be the set of k-cylindrical singularities of $\mathbf{M}$. White [17] estimated the parabolic Hausdorff dimension $\dim_{\mathcal{P}}\mathcal{S}^k(\mathbf{M})\le k$; Colding-Minicozzi [5] shown that $\mathcal{S}^k(\mathbf{M})$ is locally contained in a k-dimensional C^1 submanifold in $\mathbb{R}^{n+1}\times\mathbb{R}$. Our main theorem in [14] gives a improved partial regularity:

Theorem 2. *Let $\mathbf{M}$ be a MCF constructed by Ilmanen above; $\mathcal{S}^k_+(\mathbf{M})\subset\mathcal{S}^k(\mathbf{M})$ be the set of (fully) degenerate k-cylindrical singularities of $\mathbf{M}$. Then,*

(i) *$\dim_{\mathcal{P}}(\mathcal{S}^k(\mathbf{M})\setminus\mathcal{S}^k_+(\mathbf{M}))\le k-1$;*
(ii) *$\mathcal{S}^k_+(\mathbf{M})$ is relatively closed in $\mathcal{S}^k(\mathbf{M})$ and is locally contained in a $C^{2,\alpha}$ k-dimensional submanifold in $\mathbb{R}^{n+1}\times\mathbb{R}$, $\forall\,\alpha\in(0,\min\{1,\frac{2}{n-k}\})$.*

In particular, the theorem implies that a C^1 but not $C^{2,\alpha}$ curve can't be realized as the singular set of a MCF starting from a mean convex surface in $\mathbb{R}^3$.

One key ingredient in the proof of both theorems is the following novel L^2 non-concentration lemma, which allows us to reduce the study of RMCF near $\mathcal{C}_{n,k}$ to the study of functions solving certain parabolic equations on $\mathcal{C}_{n,k}\times\mathbb{R}$ without worrying about infinity:

Lemma 1 ([13, Corollary 3.3])**.** *Let $\tau\mapsto\mathcal{M}(\tau)$ be a RMCF with finite entropy. Then there exist dimensional constants $K_n,C_n>1$ such that $\forall\,\tau>0$,*

$$\int_{\mathcal{M}(\tau)}\overline{\mathrm{dist}}_{n,k}(X)^2(1+\tau|X|^2)\;d\mu\le C_ne^{K_n\tau}\int_{\mathcal{M}(0)}\overline{\mathrm{dist}}_{n,k}(X)^2\;d\mu$$

where $\overline{\mathrm{dist}}_{n,k}(X) := \min\{\mathrm{dist}(X, \mathcal{C}_{n,k}), 1\}$ *is the truncated distance function, and* $d\mu := (4\pi)^{-n/2} e^{-|X|^2/4}\, dX$ *is the Gaussian area form.*

Non-concentration results of similar flavor have been proved in the minimal surface setting by Simon [11] and Szekelyhidi [15] and used to extract refined information near cylindrical singularity models by [11, 15, 16, 7, 6].

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A min-max variational approach to the existence of gravity water waves

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(joint work with Dennis Kriventsov)

We establish the existence of gravity water waves by applying a mountain pass theorem to a singular perturbation of the Alt-Caffarelli functional associated with the two-dimensional water wave equations. Our approach is formulated entirely in physical coordinates and does not require the air phase to be connected. Nor does it rely on symmetry in the x or monotonicity in the x- or y-direction.

In this result, we focus on the existence of traveling waves which satisfy

$$\begin{cases} \Delta\psi = 0 & \text{in } D, \\ \psi = 0 & \text{on } \partial_a D, \\ |\nabla\psi(x,y)|^2 = A - By & \text{on } \partial_a D; \end{cases} \tag{1}$$

here ψ is the stream function of the velocity vector field.

One approach to finding non-trivial solutions of the fluid equilibrium problems with lateral inflow and outflow in a bounded domain is to minimize the Alt-Caffarelli energy with a gravity term

$$E[\psi] = \int_{\mathbb{T}\times[0,\infty)} \left(|\nabla\psi|^2 + \chi_{\{\psi>0\}}(A - By)_+\right). \tag{2}$$

The Euler-Lagrange equation for this functional is precisely (1). However, naive minimization of this energy with boundary condition $\psi(x,0) = c > 0$ will only lead to the trivial flat wave. This was observed and studied in [5], where the authors then also study minimizers with nonconstant boundary conditions and other configurations. In [14], a different approach is taken to get non-flat solutions: roughly speaking, ψ is constrained to be 0 along a line segment $\{1/2\} \times [l,\infty)$ in a way which precludes the flat wave from being a solution. The authors then study the behavior of the minimizers, including near the point $(1/2,l)$. It is not, however, clear that for some parameter l the resulting constrained minimizer is truly a solution of (1) at the point $(1/2,l)$.

On the other hand, existence results for large-amplitude smooth waves have been obtained by completely different methods by Krasovskii [17], and by Keady and Norbury [16]. The existence of large-amplitude smooth solitary waves and of extreme solitary waves has been shown by Amick and Toland [4]. All of these existence results use an equivalent formulation of the problem as a non-linear singular integral equation due to Nekrasov (derived via conformal mapping). John Toland posed the following question to the second author:

Question 1. *Can one obtain any (even small amplitude) existence results for* (1) *by variational methods in the original variables?*

Our main results are:

Theorem 2. *Assume that* $B < 2(\frac{A}{3})^{3/2}$ *and also*

$$2\frac{A}{B}2\pi\left(1-\frac{1}{3}\left(1+2\cos\left(\frac{1}{3}\arccos\left(1-\frac{27B^2}{2A^3}\right)\right)\right)\right)\times \\ \coth\left(2\pi\frac{1}{3}\frac{A}{B}\left(1+2\cos\left(\frac{1}{3}\arccos\left(1-\frac{27B^2}{2A^3}\right)\right)\right)\right)<1.$$

Then there exists a domain variation critical point of E *(defined on* $\mathbb{T}\times[0,\infty)$ *and* 1*-periodic in* x*) that is not independent of* x*.*

The conditions on A, B describe precisely the set of parameters when there are two distinct flat waves, with one locally minimal while the other sufficiently unstable.

Let us emphasize that for our existence approach neither symmetry nor monotonicity in the x- or y-direction are necessary. This may be of interest, as numerical results indicate the existence of non-symmetric waves ([10], [21], [24]) as well as water waves non-monotone in the y-direction ([11], [23]). We can, however, also produce waves with symmetries, which have enhanced regularity properties.

Theorem 3. *Assume that* $B < 2(\frac{A}{3})^{3/2}$ *and also*

$$2\frac{A}{B}2\pi\left(1-\frac{1}{3}\left(1+2\cos\left(\frac{1}{3}\arccos\left(1-\frac{27B^2}{2A^3}\right)\right)\right)\right)\times \\ \coth\left(2\pi\frac{1}{3}\frac{A}{B}\left(1+2\cos\left(\frac{1}{3}\arccos\left(1-\frac{27B^2}{2A^3}\right)\right)\right)\right)<1.$$

Then there exists a domain variation critical point u *of* E *(defined on* $\mathbb{T}\times[0,\infty)$ *and* 1*-periodic in* x*) that is not independent of* x*,* $u(x,y) = u(-x,y)$*, and* u *is symmetrically decreasing, that is* $u_x(x,y) \le 0$ *for* $x \in (0,1/2)$*. The free boundary* $\partial\{u>0\}$ *is the graph of a function of* y*, that is,* $\partial\{u>0\} = \{(f(y),y) : y \in S\}$*, where* S *is a closed subset of* $[0, A/B]$*. The water surface* $\mathcal{S} := \{(f(y),y) : y \in I\}$*, where* I *is the first/leftmost connected component of* S *is regular in the sense that* $\mathcal{S}\setminus((0,A/B)\cup\{|x|=1/2\})$ *is locally the graph of an analytic function. Moreover, either* $\mathcal{S}\setminus(0,A/B)$ *is locally the graph of an analytic function, or there is a downward-pointing cusp of* $\mathcal{S}$ *at* $|x| = 1/2$ *at which non-*$\mathcal{S}$ *free boundary points must exist that converge to the cusp point.*

The basic idea of the proof is, in some sense, straightforward, but presents challenges in the execution. We begin by studying the energy structure of the Alt-Caffarelli functional (2) (as was, in fact, already done in [5]): for the values of A, B under consideration, there are only three one-dimensional critical points, with two of them local minimizers and one being unstable. The key further observation we make is that, again for the parameters as above, the unstable solution has Morse index at least 2. Formally, then, one should be able to apply a mountain pass theorem to curves connecting the two local minimizers to obtain a critical point of Morse index at most 1, which is then not any of these three flat solutions.

The main issue with making this rigorous is that there is no mountain pass theorem available in the literature for functionals like (2), which are not differentiable.

If one attempts to use classic versions like [3], it will be impossible to verify the Palais-Smale condition. An analogy can be made with the minimal surface functional, where an extensive min-max theory has been developed (and is an area of active study), but is extremely non-trivial and requires somewhat different ideas from the traditional semilinear context. Bernoulli-type free boundary problems like (1) often exhibit similar difficulties to minimal surfaces.

In this paper, we present an elementary approach to min-max arguments for Bernoulli problems. First, we regularize (2) to E_ε by smoothing out $\chi_{\{\psi>0\}}$ to a mollified $\mathcal{B}_\varepsilon(\psi)$. This is a classic strategy in free boundaries, and it is easy to see that e.g. E_ε γ-converges to E. In particular, the energy landscape of E_ε is similar to that of E. Unlike E, E_ε is smooth, satisfies the assumptions of standard mountain pass theorems like [3], and we successfully find the critical points we wanted. Then we "simply" take a limit of these critical points, to get a critical point of E itself. This strategy is reminiscent of the Allen-Cahn approach to min-max for minimal surfaces, albeit with a different semilinear approximation.

The main problem with this strategy would be that it is not at all clear that a limit of critical points to E_ε is actually a critical point to E. This was an open question in the literature for a long time, but in a recent work [18], we have been able to prove exactly such a compactness result. Moreover, in the Bernoulli context it is not difficult to pass second (inner) variation to the limit as well, and so the limiting critical point has Morse index at most one (this is different from the situation with minimal surfaces). We would like to emphasize that up to this point, the method is extremely general and requires minimal a priori knowledge of qualitative structure or regularity.

To prove Theorem 3, we first produce symmetric and monotone min-max solutions by performing a Steiner symmetrization to the min-max setup. Then we use free boundary arguments to obtain the regularity stated. As our goal here is to present this overall strategy and its application to the water waves problem (1), we do not attempt to obtain the strongest possible regularity results here. We intend to explore that point in future work.

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Parabolic equations with rough initial data

Tobias Lamm
(joint work with Herbert Koch)

In this talk I presented several results on the existence of weak solutions of parabolic equations with rough initial data. This is an ongoing joint project with Herbert Koch.

I mostly spoke about the following situation: Assume that the coefficients $a^{ij} : [0,1] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are elliptic in the sense that

$$\inf_{0 \le t \le 1, x \in \mathbb{R}^n, u} \sum_{i,j=1}^{n} a^{ij}(t,x,u)\xi_i \xi_j \ge \gamma |\xi|^2,$$

for some $\gamma > 0$. Moreover, let

$$c_{\sup a} := \|a^{ij}\|_{L^\infty([0,1]\times\mathbb{R}^n\times\mathbb{R})} < \infty$$

and assume that

$$\sup_{t \le 1, x} |a^{ij}(t,x,u) - a^{ij}(t,x,v)| \le c_L |u - v|$$

for some $0 \le c_L < \infty$. Then the following is the main Theorem.

Theorem 1. *There exists $\varepsilon > 0$, depending only on $c_{\sup a}$, γ, n and c_L, such that the following is true: If the coefficients $a^{ij} : [0,1] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ satisfy the above assumptions and the condition*

$$\sup_{0 \le s,t \le 1,\, |x-y| \le 1,\, u \in \mathbb{R}} |a^{ij}(t,x,u) - a^{ij}(s,y,u)| \le \varepsilon$$

and if $f : [0,1] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies the bound

$$\sup_{0 \le t \le 1, x} |f(t,x,u,p) - f(t,x,v,q)| \le c_f(|u-v|(\varepsilon + |p|^2 + |q|^2) + (|p| + |q|)|p - q|)$$

for some $c_f < \infty$. Then for every $u_0 : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\sup_{|x-y| \le 1} |u_0(x) - u_0(y)| + \sup_{t \le 1, x, u} |f(t,x,u,0)| =: d < \varepsilon,$$

there exists a unique weak solution $u : (0,1) \times \mathbb{R}^n \to \mathbb{R}$ of

$$u_t - \sum_{i,j=1}^{n} \partial_i (a^{ij}(t,x,u)\partial_j u) = f(t,x,u,Du)$$

which satisfies

$$\begin{aligned}
&\sup_{|x-y| \le 1, s,t \le 1} |u(t,x) - u(s,y)| + \sup_{x, r \le 1} \left(r^{-n} \int_0^{r^2} \int_{B_r(x)} |Du|^2 dydt \right)^{\frac{1}{2}} \\
&\quad + \sup_{x, r \le 1} \left(r^2 \int_{\frac{r^2}{2}}^{r^2} \int_{B_r(x)} |Du|^{n+4} dydt \right)^{\frac{1}{n+4}} \\
&\le C \left(\sup_{|x-y| \le 1} |u_0(x) - u_0(y)| + \sup_{t \le 1, x, u} |f(t,x,u,0)| \right).
\end{aligned}$$

for some constant $C < \infty$.

This result can be generalized in several directions:

(1) We can consider general manifolds with a metric structure with uniform bounds.

(2) We can consider fairly general parabolic systems involving higher order derivatives.
(3) We can relax the conditions on the initial data, allowing small perturbations in L^∞, $C^{0,1}$ resp. $C^{1,1}$.
(4) For a large class of systems we are able consider small BMO_{loc} perturbations of continuous functions in the setting of the above Theorem.

On the moduli spaces of mean convex spheres and tori in three-manifolds

Reto Buzano
(joint work with Sylvain Maillot)

Given a closed manifold M^n, a classical problem in Riemannian geometry is to investigate the homotopy type of the moduli space of positive scalar curvature metrics on M. This problem has its origins in an over 100 years old result of Weyl [8] who showed that for $n = 2$ the moduli space is path-connected (if non-empty). Rosenberg-Stolz [7] proved that it is in fact contractible. When $n = 3$, it is only much more recently that Marques [6] proved path-connectedness and Bamler-Kleiner [1] proved contractibility. These results are in sharp contrast to the situation in higher dimensions where the moduli spaces can have very complicated topology. An illustrative example is given by spheres $\mathbb{S}^{4k+3}$, $k \geq 1$, for which the moduli spaces are known to have infinitely many path components [5].

Here, we are concerned with the following extrinsic version of this problem: given a Riemannian three-manifold (M^3, g) and a closed surface Σ^2, investigate the homotopy type of the moduli space of mean convex two-sided embeddings

$$\mathcal{M}_{H>0}(\Sigma, M) := \{\Sigma \hookrightarrow M \text{ smooth two-sided embedding with } H > 0\}/\operatorname{Diff}(\Sigma).$$

This problem was first studied in joint work with Haslhofer and Hershkovits in the case where $(M^3, g) = (\mathbb{R}^3, g_{Eucl.})$ is Euclidean space. In [2, 3], we show that for $\Sigma = \mathbb{S}^2$ the moduli space is path-connected while for $\Sigma = T^2 = \mathbb{S}^1 \times \mathbb{S}^1$ the connected components of the moduli space are in bijective correspondence with the knot classes of closed embedded curves.[a]

It is natural to ask how these results extend to more general ambient three-manifolds (M^3, g). In this context we say that a smoothly embedded, two-sided surface $\Sigma \subset M$ has *positive mean curvature* (or is *mean convex*) if the mean curvature vector always points to the same side of Σ. Clearly this definition only makes sense if Σ is two-sided. For $\Sigma = \mathbb{S}^2$, we obtain the following.

Theorem 1 (Theorem 1.1 of [4]). *Let (M, g) be a complete, orientable Riemannian three-manifold with nonnegative Ricci curvature. Then the moduli space $\mathcal{M}_{H>0}(\mathbb{S}^2, M)$ of mean convex embedded two-spheres in (M, g) is path-connected.*

[a] In fact, [2, 3] deal with moduli spaces of *two-convex* embeddings of hypersurfaces Σ^{n-1} into $(\mathbb{R}^n, g_{Eucl.})$, but we only focus on dimension $n = 3$ here.

Theorem 1 is sharp in the sense that it does not hold if the condition $H > 0$ is weakened to $H \geq 0$, or if $\mathrm{Ric}_g \geq 0$ is weakened to $\mathrm{Ric}_g \geq -C$ for some $C > 0$ (or replaced by $1 \geq \mathrm{Ric}_g \geq -\varepsilon$ for some $\varepsilon > 0$), or if the completeness condition on M is dropped.

For $\Sigma = T^2$, we restrict the problem to *Heegaard surfaces* in order to guarantee that Σ bounds a solid (unknotted) torus. It is well known that the moduli space of *all* Heegaard tori is path-connected, but surprisingly, in contrast to the Euclidean case, the moduli space of *mean convex* Heegaard tori can have more than one path component. Before stating the results, we give some definitions and fix notation.

If M is a closed, orientable three-manifold, and Σ a closed, orientable surface smoothly embedded in M, we say that Σ is a *Heegaard surface* if the closure of each component of $M \setminus \Sigma$ is a handlebody. Three-manifolds admitting a Heegaard surface of genus one are called *lens spaces.* They can be constructed as follows. Let p, q be coprime integers with $p \geq 0$. The lens space $L(p, q)$ is obtained by gluing together two solid tori V_1 and V_2 along their boundaries in such a way that the meridian of the first torus goes to a curve wrapping around the longitude p times and around the meridian q times on the second torus. Special cases of lens spaces include $\mathbb{S}^3 = L(1, 0)$, $\mathbb{RP}^3 = L(2, 1)$, and $\mathbb{S}^1 \times \mathbb{S}^2 = L(0, 1)$. All lens spaces carry a Riemannian metric with nonnegative Ricci curvature (in fact, if $p \neq 0$, then $L(p, q)$ is a quotient of $\mathbb{S}^3$ and thus has a positive constant curvature metric.)

We now denote by

$$\mathcal{M}_{H>0}(p, q) = \mathcal{M}_{H>0}(T^2, L(p, q))$$

the *moduli space of Heegaard tori* in $L(p, q)$ that have positive mean curvature with respect to a chosen background metric g on $L(p, q)$.

Theorem 2 (Theorems 1.4 and 1.5 of [4])**.** *Let p, q be coprime integers with $p \geq 0$. Fix a background metric g on the lens space $L(p, q)$ with nonnegative Ricci curvature. If $q \cong \pm 1 \mod p$, then $\mathcal{M}_{H>0}(p, q)$ is path-connected, otherwise it has exactly two path-components. Moreover, if $H > 0$ is weakened to $H \geq 0$, then the corresponding moduli space is always path-connected.*

An interesting aspect of Theorem 2 is that it is not true that the homotopy type of the moduli spaces $\mathcal{M}_{H>0}(p, q)$ depends only on the homotopy type of the ambient space $L(p, q)$. An illustrative example is given by $L(7, 1)$ and $L(7, 2)$ which have the same homotopy type (and therefore also isomorphic fundamental groups and the same homology). Nevertheless, by Theorem 2, $\mathcal{M}_{H>0}(7, 1)$ is path-connected while $\mathcal{M}_{H>0}(7, 2)$ is not.

Towards the proofs: We first show that every mean convex 2-sphere as in Theorem 1 bounds a mean convex, compact domain. Already this step fails if any of the assumptions of the theorem are weakened or dropped as described after the theorem. For both Theorem 1 and Theorem 2, we then evolve the surfaces (respectively the mean convex domains bounded by them) by *mean curvature flow with surgery.* If the ambient manifold has nonnegative Ricci curvature, then mean curvature flow with surgery will always become extinct in finite time. We then extend the *gluing result* from [2] from Euclidean space to the ambient manifold setting, allowing us to connect mean convex domains along thin "strings" in a way that preserves mean convexity. This is used to show that any element of $\mathcal{M}_{H>0}(\mathbb{S}^2, M)$ can be deformed to a *marble tree* which then itself can be deformed to a small geodesic sphere. Similarly, any element of $\mathcal{M}_{H>0}(p, q)$ can be deformed to a *marble circuit* which then itself can be deformed further to the boundary of the ε-neighbourhood of a simple closed curve $T_\varepsilon = \partial N_\varepsilon(\gamma)$. Finally, the proof of Theorem 2 is completed by additional topological arguments, showing first that $N_\varepsilon(\gamma)$ must be mean-convex ambient isotopic to the tubular neighbourhood of at least one of the two cores of the solid tori used to construct the lens space. The conclusion then follows by showing that for $q \cong \pm 1 \mod p$ these tubular neighbourhoods are themselves mean-convex ambient isotopic, while otherwise they are not.

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A Nash-Kuiper theorem for isometric immersions beyond Borisov's exponent

Jonas Hirsch

(joint work with Wentao Cao, Dominik Inauen)

Abstract. Given any short immersion from an n-dimensional bounded and simply connected domain into $\mathbb{R}^{n+1}$ and any Hölder exponent $\alpha < (1 + n^2 - n)^{-1}$, we construct a $C^{1,\alpha}$ isometric immersion arbitrarily close in the C^0 topology. This extends the classical Nash–Kuiper theorem and shows the flexibility of $C^{1,\alpha}$ isometric immersions beyond Borisov's exponent. In particular, for $n = 2$, the regularity threshold aligns with the Onsager exponent 1/3 for the incompressible Euler equations. Our proof relies on three novelties that allow for the cancellation of leading-order error terms in the convex integration scheme: a new corrugation ansatz, an integration by parts procedure, and an adapted algebraic decomposition of these errors.

1. Introduction

The isometric immersion problem is a fundamental problem in differential geometry. It seeks an immersion $u : (\mathcal{M}, g) \to \mathbb{R}^m$ from an n-dimensional Riemannian manifold $(\mathcal{M}, g)$ into m-dimensional Euclidean space that preserves the length of any C^1 curve. This condition is equivalent to the equality of the induced metric $u^\sharp e$ and the intrinsic metric g. In local coordinates, this translates to the system of $n_* = n(n+1)/2$ nonlinear partial differential equations:

$$g_{ij} = \partial_i u \cdot \partial_j u, \quad 1 \le i, j \le n, \tag{1}$$

in m unknowns.

The classical Nash–Kuiper theorem [47, 43] establishes that for $m \ge n+1$, any short immersion (or embedding) $u : \mathcal{M} \to \mathbb{R}^m$ can be uniformly approximated by C^1 isometric immersions (resp. embeddings). Here, an immersion u is called *short* if $\partial_i u \cdot \partial_j u \le g_{ij}$ as quadratic forms. In particular, if the manifold is compact, any immersion can be made short by a homothety. The Nash–Kuiper theorem therefore demonstrates that when there are no topological obstructions to immersing the manifold into $\mathbb{R}^m$ (a condition satisfied for $m \ge 2n-1$), there exists an abundance of C^1 solutions to (1). This abundance, often termed the *flexibility* of isometric immersions, is especially striking given the overdetermined nature of (1) for $m \ge n+1$ and large n.

In contrast, classical *rigidity* results show that *smooth* isometric embeddings into Euclidean spaces of such low codimension are unique under appropriate geometric conditions. A prominent example is the rigidity theorem for the Weyl problem due to Cohn-Vossen [12] and Herglotz [31], which states that a C^2 isometric embedding $u : (\mathbb{S}^2, g) \to \mathbb{R}^3$, where g has positive Gaussian curvature, is unique up to rigid motions.

This strong contrast between flexibility in C^1 and rigidity in C^2 raises the natural question: is there a critical Hölder regularity threshold α_0 that separates flexibility from rigidity? That is, does there exist a threshold α_0 such that

- if $\alpha > \alpha_0$, isometric immersions $u \in C^{1,\alpha}$ exhibit (some form of) rigidity;
- if $\alpha < \alpha_0$, the Nash–Kuiper theorem extends to $C^{1,\alpha}$?

The precise value of α_0 remains unresolved. Similarities between the iteration processes used to construct both isometric immersions or embeddings of regularity $C^{1,\alpha}$ and Hölder continuous weak solutions to the incompressible Euler equations (see e.g. [34]) suggest the Onsager exponent $\alpha_0 = 1/3$ as a potential threshold. Indeed, the Onsager conjecture, which describes a similar phenomenon, states that for C^α weak solutions to the incompressible Euler equations

(1) if $\alpha > 1/3$, it preserves the energy;
(2) if $\alpha < 1/3$, the energy identity might be violated.

The rigidity result (1) was established in [19], while the flexibility result (2) was ultimately resolved in [33] following a sequence of works [22, 23, 4, 6] that built on the groundbreaking approach of [24], where the authors introduced a Nash-type iteration scheme to construct continuous weak solutions violating the energy identity.

On the other hand, [21, 13] demonstrate that for isometric embeddings, $C^{1,1/2}$ is a critical space *in a suitable sense*, suggesting $\alpha_0 = 1/2$ (see also [28, Question 36], [34, Section 10]).

The presented result focuses on the flexibility side of this dichotomy for general dimension n and codimension one.

1.1. **Flexibility of $C^{1,\alpha}$ isometries: known results.** The study of $C^{1,\alpha}$ isometric immersions dates back to the pioneering work of Yu. F. Borisov in the 1950s. Building on results of Pogorelov, Borisov proved in [8] that the Cohn–Vossen–Herglotz rigidity theorem extends to immersions of class $C^{1,\alpha}$ when $\alpha > 2/3$ (see [18] for an alternative proof). On flexibility, Borisov announced in [10] that the Nash–Kuiper theorem extends to $C^{1,\alpha}$ for

$$\alpha < \frac{1}{1+n^2+n}\,, \tag{2}$$

the *Borisov exponent*, when $\mathcal{M}$ is an n-dimensional ball, with a potential improvement to $\alpha < 1/5$ for $n = 2$. He provided a proof for $n = 2$ and $\alpha < 1/7$ with an analytic metric in [11]. More recently, [18] confirmed Borisov's claims, and in [20] it was shown that for $n = 2$ one can indeed achieve the exponent $\alpha < 1/5$. These results were extended to general compact manifolds in [17].

Remark 1 (High codimension)**.** *If the codimension is large, more regular isometric immersions can be constructed. The breakthrough result in this direction is also due to Nash* [38]*, who proved that any (M, g) with $g \in C^k$, $k \geq 3$, admits a C^k-regular isometric immersion into $\mathbb{R}^m$ for sufficiently large m. Gromov* [37] *and Günther* [35] *improved the codimension bounds, the latter simplifying Nash's intricate iteration (now known as the* hard implicit function theorem*). For less regular metrics $g \in C^{l,\beta}$ with $0 < l + \beta < 2$, Källén* [42] *demonstrated that if m is large enough (one can show that $m \geq 2n_* + n$ suffices), there exists a $C^{1,\alpha}$ immersion for any $\alpha < (l + \beta)/2$. See also* [21, 13, 14] *for further results in high codimension.*

1.2. **Statement of the result.** Our result improves the achievable Hölder exponent in the codimension one setting. Our main result is:

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be any smooth bounded and simply connected domain and $g \in C^2(\overline{\Omega}, \mathrm{Sym}_n^+)$ a Riemannian metric. For any short immersion $\underline{u} \in C^1(\overline{\Omega}, \mathbb{R}^{n+1})$, any $\epsilon > 0$, and any*

$$\alpha < \frac{1}{1+n^2-n}, \tag{3}$$

there exists an immersion $u \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^{n+1})$ such that

$$Du^t Du = g \quad \text{and} \quad \|u - \underline{u}\|_0 < \epsilon.$$

Remark 2. *For $n = 2$, Theorem 1 yields flexibility of $C^{1,1/3-}$ isometric immersions from surfaces to $\mathbb{R}^3$, aligning with the Hölder exponent of Onsager's conjecture. Given the parallels between convex integration solutions for Onsager's conjecture and isometric immersions (see e.g.* [1, 34]*) and the heuristic discussion* *about the Hölder exponent* $\alpha < (1+2N)^{-1}(N \geq 1)$ *in Section 1.3.2,* $1/3$ *may be optimal for codimension-one flexibility.*

Remark 3. *Following classical arguments (see e.g.* [20, Section 8]*), one can show that if the short map* $\underline{u}$ *in Theorem 1 is an embedding, then* u *can be chosen to be an embedding as well.*

Remark 4. *With additional effort, the key iteration can be adapted to the framework of* [17]*, leading to a global version of Theorem 1. We prioritize the local version here for clarity. Moreover, our approach is easily adaptable to prove analogous results for very weak solutions to the Monge–Ampère equation and system (see* [46, 15, 40, 44, 36, 2, 3]*).*

1.3. **Main ideas.** In this subsection, we briefly recall the classical construction procedure–Nash's iteration for isometric immersions–and provide a heuristic explanation of how Borisov's exponent (2) is obtained in [18]. We then outline the main strategy for proving Theorem 1 and highlight how our approach differs from previous methods. To focus on the core ideas, we adopt the local setting of Theorem 1: the manifold $\mathcal{M}$ is described by a single coordinate chart Ω, the Riemannian metric g is a matrix-valued function, and the induced metric $u^\sharp e$ is given by the matrix field Du^tDu.

1.3.1. *Nash's iteration.* Following Nash [47], the isometric immersion is obtained as the limit of an iteratively constructed sequence $\{u_q\}$ of strictly short immersions, whose induced metric converges to the intrinsic metric g while $\{u_q\}$ remains Cauchy in some $C^{1,\alpha}$ space.

The construction of u_{q+1} from u_q, referred to as a *stage* (same terminology as in [47] and subsequent works), consists of a finite number of *steps* designed to correct the metric deficit $g - Du_q^tDu_q$. This deficit is first decomposed into a finite sum of *primitive metrics*, i.e., rank-one tensors with positive coefficients:

$$g - Du_q^tDu_q = \sum_{i=1}^{N} a_i^2 \nu_i \otimes \nu_i, \tag{4}$$

where the directions $\nu_i \in \mathbb{S}^{n-1}$ are constant and the coefficients a_i are smooth functions. After this decomposition, the short immersion u_q is perturbed by N *steps* as follows:

Setting $u_{q,0} = u_q$, one defines iteratively

$$u_{q,i} = u_{q,i-1} + W_{q+1,i}, \text{ for } i = 1, \dots, N, \quad u_{q+1} = u_{q,N}\,.$$

Each function $W_{q+1,i}$ is a highly oscillatory perturbation, and is chosen so that the corresponding update increases the induced metric approximately by $a_i^2\nu_i \otimes \nu_i$, yielding

$$Du_{q,i}^tDu_{q,i} = Du_{q,i-1}^tDu_{q,i-1} + a_i^2\nu_i \otimes \nu_i + E_i,$$

where the error term E_i can be made arbitrarily small by selecting a sufficiently high oscillation frequency λ_i of $W_{q+1,i}$. Heuristically, the ansatz for $W_{q+1,i}$ has the form

$$W_{q+1,i} = \frac{a_i}{\lambda_i}\left(\gamma_1(\lambda_i x \cdot \nu_i)t_i + \gamma_2(\lambda_i x \cdot \nu_i)\zeta_i\right) \tag{5}$$

where the frequency $\lambda_i \gg 1$, t_i is a suitable *tangent vector* (or normal to $u_{q,i-1}$ in the case $m \geq n+2$ as in [47]), ζ_i is a unit normal vector to $u_{q,i-1}$ and γ_1, γ_2 are suitable periodic functions.[a]

By choosing the oscillation frequency large enough, one can ensure a geometric decay of the metric deficit:

$$\|g - Du_{q+1}^t Du_{q+1}\|_0 \leq \frac{1}{K}\|g - Du_q^t Du_q\|_0.$$

Meanwhile, the step-wise corrections yield a bound on the C^1-norm growth:

$$\|u_{q+1} - u_q\|_1 \leq C\sum_{i=1}^{N}\|a_i\|_0 \leq C\|g - Du_q^t Du_q\|_0^{1/2} \tag{6}$$

where the last estimate follows from (4). Given the geometric decay in $\|g - Du_q^t Du_q\|_0$, this ensures that the sequence u_q remains Cauchy in C^1, converging to a limiting function that is an isometric immersion.

1.3.2. *Borisov's exponent and the result of* [18]. To achieve convergence in $C^{1,\alpha}$, a refined choice of frequencies is required to control the blow-up of the sequence $\{\|u_q\|_2\}$. The error term E_i in the i-th step can be shown to satisfy

$$\|E_i\|_0 \leq C\|g - Du_q^t Du_q\|_0 \frac{\lambda_{i-1}}{\lambda_i}.$$

Setting $\lambda_i = K\lambda_{i-1}$, we achieve the geometric decay

$$\|g - Du_{q+1}^t Du_{q+1}\|_0 \leq CK^{-1}\|g - Du_q^t Du_q\|_0,$$

but at the cost of a C^2-norm growth of order K^N, where N is the number of primitive metrics required in the decomposition (4). This leads to

$$\|u_{q+1} - u_q\|_2 \leq C(K^N)^q\|u_1 - u_0\|_2.$$

Using (6) and interpolation estimates, the sequence converges in $C^{1,\alpha}$ for any

$$\alpha < \frac{1}{1+2N}.$$

If the metric deficit $g - Du_q^t Du_q$ is close to a constant positive definite matrix (which can be achieved by a rescaling), it can be decomposed in exactly $N = n(n+1)/2$ primitive metrics. This provides a heuristic explanation of Borisov's exponent (2) and captures the core idea of the proof in [18]. A technical challenge in turning this idea into a rigorous proof lies in the loss of derivatives appearing in

[a]In fact, the corrugations used in [43] and [18] have the more complicated expression

$$W_{q+1,i} = \frac{1}{\lambda_i}\left(\Gamma_1(a_i, \lambda_i x \cdot \nu_i)t_i + \Gamma_2(a_i, \lambda_i x \cdot \nu_i)\zeta_i\right)$$

for some suitable functions Γ_1, Γ_2 periodic in the second component, see also [39].

the above estimates. In [18] and subsequent works, this loss is managed through an additional mollification step at the beginning of each stage.

1.3.3. *Prior improvements.* Borisov's exponent (2) has been improved previously via the following three approaches. We will discuss and show below the obstacles applying them to attain Theorem 1.

(1) *Reducing the number N in the decomposition* (4). In [20], the authors studied the case $n = 2$, where isothermal coordinates can be used to diagonalize the metric deficit $g - Du_q^t Du_q$. This reduces N from 3 to 2, leading to the improved Hölder exponent $\alpha < 1/5$. However, for $n \geq 3$, no analogous coordinate transformation is known.

(2) *Increasing codimension.* When the codimension $m-n$ is large, multiple perturbations can be applied simultaneously at the same frequency, thereby reducing the growth of the C^2 norm of $\{u_q\}$. A procedure analogous to that used in [44] in the context of the Monge-Ampère system would yield the improved exponent

$$\alpha < \frac{1}{1 + 2\frac{n_*}{m-n}}.$$

But our setting is codimension one.

(3) *Absorbing the error terms E_i directly into the decomposition* (4). For large codimension $m-n \geq 2n_*$, from the second approach, it seems that at most $\alpha < 1/2$ can be obtained. However, Källén [42] introduced a new method that incorporates error terms into the decomposition:

$$g - Du_q^t Du_q = \sum_{i=1}^{N} a_i^2 \nu_i \otimes \nu_i + \sum_{i=1}^{N} E_i.$$

Since the error terms E_i depend on a_i and its derivatives, this absorption is nontrivial. However, by applying a suitable Picard iteration to this equation, the decomposition can be achieved up to an arbitrarily small error

$$\|E\|_0 \leq C\|g - Du_q^t Du_q\|_0 \left(\frac{\lambda_0}{\lambda_1}\right)^J.$$

Consequently, by choosing a very slow increase in frequency, $\lambda_1 = K^{1/J}\lambda_0$, the exponent improves to

$$\alpha < \frac{1}{1 + \frac{2}{J}}.$$

By taking J arbitrarily large, one can achieve any $\alpha < 1$.

When codimension $m - n \geq 1$, absorbing error terms into the decomposition does not work, since the perturbations take the form of corrugations rather than Nash's spirals as in [42], and the structure of the error terms prevents the Picard iteration procedure from working. Nevertheless, in the context of the 2-dimensional Monge-Ampère equation (corresponding to $n = 2, m = 3$), [40] observed that the Picard scheme still works if applied only to E_1 with its 22 component removed. This modification allows for the trade of a fast derivative for a slow one, making the scheme work. The missing 22 component is then cancelled exactly by the second perturbation. To apply the analogous strategy to codimension one isometric immersions is significantly more challenging due to the nonlinear nature of the required equation. Furthermore, although the approach seems to generalize to higher dimensions, the resulting improvement on the Hölder exponent is minor.

1.3.4. *Our approach–iterative integration by parts.* In this paper, we improve the Hölder exponent in the codimension one setting *without* reducing the number of primitive metrics in the decomposition of the metric deficit and (almost) without absorbing error terms into the decomposition. Instead, we modify the ansatz (5) for the corrugation in the first n perturbations, achieving an error with the very small bound:

$$\|E_i\|_0 \leq C\|g - Du_q^t Du_q\|_0 \left(\frac{\lambda_{i-1}}{\lambda_i}\right)^J$$

up to a component that belongs to an $(n_* - n)$-dimensional subspace of the space of symmetric matrices. This remaining part is then cancelled exactly in the last $n_* - n$ perturbations. By choosing J arbitrarily large, the contribution of the first n perturbations to the growth of the C^2 norm becomes negligible, effectively reducing the number of contributing perturbations to $n_* - n$. This explains the Hölder threshold claimed in Theorem 1.

The key observation behind modifying the ansatz (5) is that, at each step i, all but one of the leading-order error terms in E_i take the form

$$\gamma(\lambda_i x \cdot \nu_i)M,$$

where M is a symmetric matrix oscillating at the lower frequency λ_{i-1}, and γ is a periodic function with zero mean. Another observation is that for a fixed $\nu \in \mathbb{S}^{n-1}$, any symmetric matrix can be decomposed as

$$M = \mathrm{sym}(\alpha(M) \otimes \nu) + F,$$

where the "remainder" F belongs to an $(n_* - n)$-dimensional subspace of the space of symmetric matrices. Using this decomposition, we can rewrite

$$\gamma(\lambda_i x \cdot \nu_i)M = 2\mathrm{sym}\left(D\left(\frac{\gamma^{(1)}(\lambda_i x \cdot \nu_i)}{\lambda_i}\alpha(M)\right)\right) - 2\frac{\gamma^{(1)}}{\lambda_i}\mathrm{sym}(D\alpha(M)) + \gamma F.$$

Here, $\gamma^{(1)}$ is an antiderivative of γ with zero mean. We refer to this process as *integration by parts.* If M and γ are sufficiently smooth, we can iterate this process J times, and construct fields w^J, E^J, F^J such that

$$\gamma M = 2\mathrm{sym}(Dw^J) + \left(\frac{\lambda_{i-1}}{\lambda_i}\right)^J E^J + F^J,$$

where the remainder F^J remains in a lower-dimensional subspace. Consequently, introducing vector field w^J in the step ansatz so that $2\,\mathrm{sym}(Dw^J)$ appears in the induced metric allows us to cancel γM up to a very small error $\left(\frac{\lambda_{i-1}}{\lambda_i}\right)^J E^J$ and the large but lower-dimensional remainder F^J. Furthermore, by selecting a suitable basis $\{\nu_i \otimes \nu_i\}$ for the decomposition (4) we ensure that, when applied to the first n perturbations (i.e., for $\nu = \nu_i$, $i = 1, \ldots, n$), the integration by parts procedure produces errors F^J that all remain within the same subspace $\mathcal{V} = \text{span}\{\nu_j \otimes \nu_j : j = n+1, \ldots, n_*\}$. These errors in $\mathcal{V}$ can then be canceled exactly by appropriately adjusting the effective amplitude a_j of the perturbation $W_{q,j}$ for $j \geq n+1$.

A complication arises due to a specific leading-order error term in E_i of the form

$$\gamma^2 M,$$

which prevents a direct application of integration by parts technique since γ^2 does not have zero mean. However, we can decompose

$$\gamma^2 M = (\gamma^2 - \fint \gamma^2) M + \fint \gamma^2 M.$$

The prefactor of the first term has zero mean, allowing the first term to be handled using the integration by parts process described above. On the other hand, the remaining term $\fint \gamma^2 M$ oscillates slowly, so that can be treated by absorbing it into the decomposition via a simple Picard iteration, similar to Källén's approach.

The details can be found in [41].

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