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Mini-Workshop: Renormalisation and Randomness

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Abstract. In recent decades, renormalisation, which has long been a workman's tool for theoretical physicists, has also become an essential mathematical tool that appears in many guises. Within mathematics, renormalisation bridges across topics such as combinatorics and stochastic analysis. Yet, in part due to a lack of a common language, the advances on the mathematical side do not seem to fully reach out to the theoretical physicist. Conversely, the mathematician rarely benefits from the physicist's expertise in renormalisation. The goal of this workshop is to bridge this gap and provide a platform for communication and exchange of ideas, a first step in the direction of increased interaction and cross fertilisation between the two communities.

Mathematics Subject Classification (2020): 60L20, 60L30, 81T15, 81T16, 81T17, 60L30, 40G10, 18G85.

Introduction by the Organizers

There is a plethora of field theories – including marginal ones, which seem to be the Nemesis of many a mathematical approach to renormalisation – for which the physicist and the mathematician have similar questions: make sense of the path integral measure, compute correlation functions, identify fixed points, establish conformal invariance, and so on. What can the physicist and the mathematician learn from each other in such cases?

As reflected by the abstracts, the workshop brought together a group of experts, both mathematicians and physicists. The workshop made space for ample discussions between the participants so as to foster interactions between the communities, Ultimately one would wish for a full control of QFT, encompassing both existence proofs and constructions as well as effective numerical techniques allowing one to compute physically relevant quantities. This workshop addressed the former, leaving the latter question for future work.

Scientific interactions during the workshop were geared around several questions including:

- *Rigorous mathematical results of relevance for physically interesting theories.*
- *Upgrading a perturbative result to a non-perturbative approach.*
- *Mathematical structures underlying field theory and renormalisation, including geometric and algebraic structures.*

The first day was organised around 5 introductory talks to various topics of interest for the workshop. The rest of the week was organised around more specialised 50 minutes talks and shorter 15 minutes talks by junior researchers. The speakers who had been asked to be as pedagogical as possible, gave remarkable talks which were understandable to non experts.

This joint effort to share various approaches and tools, whether mathematical or physical, enabled very fruitful discussions, some of which took place within the workshop schedule, others more informally.

Acknowledgement: The organisers are very thankful to the MFO for hosting this interdisciplinary workshop.

Mini-Workshop: Renormalisation and Randomness

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Abstracts

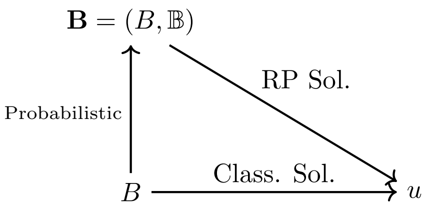
Rough analysis and renormalisation

Ilya Chevyrev

Rough analysis is a collection of methods that allows one to solve singular stochastic differential equations in an essentially pathwise manner. It was initiated by Lyons' theory of rough paths [1] which gives a solution theory for rough differential equations of the form

$$\mathrm{d}u = f(u)\,\mathrm{d}B\ ,$$

for a function $u\colon [0,T] \to \mathbb{R}^m$, an input $B\colon [0,T] \to \mathbb{R}^n$, and driving function $f\colon \mathbb{R}^m \to L(\mathbb{R}^n, \mathbb{R}^m)$. In its applications to stochastic differential equations, where B is, e.g. a (fractional) Brownian motion, the key idea of the theory is to factor the solution map $B \mapsto u$ into two steps. The first step is probabilistic, in which one enhances the path B to a richer object $\mathbf{B} = (B, \mathbb{B})$, where concretely $\mathbb{B} = \int B \otimes \mathrm{d}B$ is the iterated integral of B; this step is typically discontinuous but has the advantage that it is explicit and simple. In the second step , which is analytic and entirely deterministic, one applies a *continuous* 'Rough Path' solution map $\mathbf{B} \to u$. The result is depicted in a commuting diagram in the figure below, in which the enhancement $B \mapsto \mathbf{B}$ and continuous RP solution map commute with the Classical Solution Map whenever B is smooth.



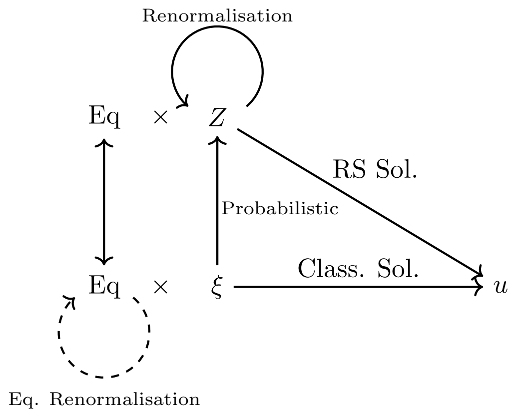
In a major breakthrough, the philosophy of rough paths was extended to singular stochastic *partial* differential equation (SPDEs) by Hairer's theory of regularity structures [2]. The prototypical example of equation handled by this theory is

$$(\partial_t - \Delta)u = f(u, \nabla u, \xi)$$

where $u \in \mathcal{D}'([0,T] \times \mathbb{R}^d)$ and $\xi \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^d)$ are, in general, distributions. The steps in the theory are similar to rough paths, but one has the major complication that, when constructing the aforementioned enhancements, which are generalisations of iterated integrals, one requires a renormalisation procedure. To wit, for $f = -u^3 + \xi$ and ξ a white noise (the dynamical Φ^4 model), one of the objects that needs to be constructed is $[(\partial_t - \Delta)^{-1}\xi]^3$ (the cube of the stochastic heat equation). If $\Psi_\varepsilon = (\partial_t - \Delta)^{-1}\xi_\varepsilon$ is a smooth approximation, then the cube Ψ_ε^3 for dimensions $d \geq 2$ is known to diverge. But for $d < 4$, one can obtain a meaningful limit as $\varepsilon \downarrow 0$ by considering the Wick cube

$$\Psi_\varepsilon^{:3:} = \Psi_\varepsilon^3 - 3C_\varepsilon \Psi_\varepsilon\ ,$$

where $C_\varepsilon = \mathbb{E}\Psi_\varepsilon^2(0)$ is the so-called Wick constant that diverges as $\varepsilon \downarrow 0$. The shift $\Psi_\varepsilon^3 \to \Psi_\varepsilon^{:3:}$ is an example of *renormalisation*, and it is a crucial step in the regularity structures solution theory. One can view this step as the action of a renormalisation group $\mathfrak{R}$ on the space of enhanced inputs Z (called 'models' in regularity structures). The algebraic and analytic tools necessary to enahce and renormalise a wide class of random distributions ξ is are given in [3, 4]. We depict this step by 'Renormalisation' in the figure below.



The effect of shifting polynomial functions of ξ has a corresponding effect on the equation, e.g. returning to the Φ^4 model, it is solutions to the *renormalised* PDE

$$(\partial_t - \Delta)u_\varepsilon = -u_\varepsilon^3 + 3C_\varepsilon u_\varepsilon + \xi_\varepsilon$$

that converge as $\varepsilon \downarrow 0$. The first proof that one can always renormalise an equation with suitable counterterms to obtain a limit as $\varepsilon \downarrow 0$ was shown [5]. This step is labelled 'Eq. Renormalisation' in above diagram. The commuting diagram depicts that, enhancing $\xi \mapsto Z$, applying a renormalisation group element $M \in \mathfrak{R}$, $M\colon Z \mapsto \hat{Z}$, followed by the *continuous* Regularity Structure Solution Map, is the same as applying an adjoint action of the renormalisation group element $M^*\colon \mathrm{Eq} \mapsto \widehat{\mathrm{Eq}}$ to the equation (the effect of which is to add counterterms), and then applying the Classical Solution Map $\xi \mapsto u$ for the *renormalised equation.*

This initial programme was completed in [2, 3, 4, 5] and is able to treat a wide class of subcritical SPDEs (corresponding to super-renormalisable regime in QFT). Different approaches to regularity structures, and singular SPDEs in general, have since appeared, e.g. paracontrolled calculus [6], multi-index regularity structures [7], and Wilsonian renormalisation group [8, 9].

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Wilsonian renormalization group

Manfred Salmhofer

Introduction. Wilsonian renormalization group (RG), as invented by Kadanoff [1], Wilson[1] [2, 3], and Wegner [4], has become ubiquitous in physics (for reviews, see [5, 6, 7, 8, 9]). Since the early 1980s it has also been developed and used as a mathematical tool in mathematically rigorous, non-perturbative constructions of models of quantum field theory, such as [10, 11, 12, 13, 14, 15, 16]. Since the 1990s, it has also been applied to quantum many-body theory and quantum statistical mechanics, yielding many rigorous results, see, e.g. [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. Here I give a brief overview of some basic concepts, their implementation via flow equations for effective actions, and the solution of these flow equations by formal and by convergent expansions, a few applications and open questions.

Given a quantum field theory in its representation as an integral over space-time-dependent fields (sketched below), the idea of the Kadanoff-Wilson-Wegner approach is to implement a combination of local averaging and scale transformations in real space on the algebra of fields, and thereby define a flow of effective interactions. Since local averaging smoothes out details, it is not injective, and therefore the Wilsonian renormalization "group" is really a semigroup. (In a setting of formal perturbation theory, and restricted to the so-called relevant parameters (the unstable manifold), it can be inverted, so that it is a group.) Using a regularized, hence mathematically well-defined version of the functional integral of quantum field theory, this semigroup can be set up as a heat flow on a space of action functionals. Its solution is the Wilsonian effective action. Polchinski [36] used this setup to give a simplified proof of perturbative renormalizability. His method was further simplified, in some respects clarified, and extended to general renormalization conditions, in [37]. Based on this, the method was developed

[1]Kenneth Wilson received the 1981 Nobel prize for his contributions to this theory and its applications

further, and it has by now yielded the strongest results in perturbative renormalization, see [38, 26, 39, 40, 41] for a selection of results. A detailed exposition of the method is given in [42], a short version is in [43].

Already a few years after Polchinski, Brydges and Kennedy found an explicit solution to Polchinski's equation and used it to prove convergence of Mayer expansions [44]. Their tree formula can also be used to reorganize quantum field theoretical perturbation theory as a tree expansion. A reorganization via (possibly augmented) tree expansions can be done also without flow equations, and it has been used to prove convergence of perturbation expansions in fermionic quantum field theories [45, 46, 48]. Because the Brydges-Kennedy formula is based on the same PDE as the perturbative proofs, it connects seemingly different approaches in a transparent way. Below, I describe it in a bit more detail and highlight some basic structural results.

The context. A natural setting for Wilsonian RG is that of functional integrals. Wilson's effective action is defined by the equation $e^{-W(C,V)} = \mu_C * e^{-V}$, that is

$$e^{-W(C,V)(\phi)} = \int d\mu_C(\varphi)\, e^{-V(\varphi+\phi)} \tag{1}$$

where μ_C is a normalized, centered Gaussian measure with covariance C and V is bounded below. A prototypical model is scalar field theory ϕ^4_d, where $C = (-\Delta+m^2)^{-1}$ (with the d-dimensional Laplacian Δ) and $V(\phi) = \int \phi(x)^4\, d^dx$. More generally, using the Feynman-Kac formula or coherent-state functional integrals, any quantum field theory can be cast into this form where the integration variables are real, complex, or Grassmann-valued, fields. The effective action contains all information about the theory.

As it stands, (1) is ill-defined because the covariance $C(x,y)$ is singular at coinciding points and V is local in ϕ. There are many ways how to give a well-defined version of such integrals, e.g. by replacing continuous space by a discrete lattice, or by smoothing out the singularity, considering e.g. $C_{\Lambda_0} = C\ e^{\Delta/\Lambda_0^2}$. In the latter case, renormalization can then be described as finding a suitable modification V_{Λ_0} of V such that the limit $\Lambda_0 \to \infty$ of $W(C_{\Lambda_0}, V_{\Lambda_0})$ exists and has properties that imply, e.g., the Osterwalder-Schrader axioms [49, 50]. In theories that have more structure than the scalar field theory, such as gauge theories, there are additional requirements, namely gauge covariance, which pose further constraints and significantly complicate the analysis.

Polchinski's equation. In the following, we assume that some regularization has been chosen, so that the integral is well-defined. For $s \geq 0$ define $\Lambda_s = \Lambda_0\ e^{-s}$, and $C(s) = C_{\Lambda_s,\Lambda_0} = C(e^{\Delta/\Lambda_0^2} - e^{\Delta/\Lambda_s^2})$. Then $W(s,\phi) = W(C_{\Lambda_s,\Lambda_0}, V_{\Lambda_0})$ solves the initial-value problem

$$\dot W = \frac{1}{2}\Delta W - \frac{1}{2}(\nabla W,\ \dot C\ \nabla W)\ , \qquad W(0) = V_{\Lambda_0}\ . \tag{2}$$

In quantum field theory, this equation is known as Polchinski's equation. In this equation, the dot denotes the s-derivative, and gradient and Laplacian are now operators in field space., i.e. $\nabla = \frac{\delta}{\delta\phi}$ and $\Delta = (\nabla, C\nabla) = \int_{x,y} C(x,y) \frac{\delta}{\delta\phi(x)} \frac{\delta}{\delta\phi(y)}$. For a mathematically careful setup and details, see, e.g. [42]. The idea is now to study the quantum field theory by showing existence and uniqueness of the solution to (2) and studying its properties in the limit $\Lambda_0 \to \infty$.

Perturbative renormalization. Polchinski did this in his paper [36] in the framework of a formal perturbative expansion in the renormalized coupling constant. In the sense of formal power series, his method provides a rigorous, yet simple, inductive scheme for proving finiteness of W in the limit $\Lambda_0 \to \infty$. Specifically, when replacing V by λV with a formal parameter λ, the effective action becomes a formal power series in λ,

$$W(C_{\Lambda,\Lambda_0}, \lambda V_{\Lambda_0})(\phi) = \sum_{r\geq 1} \lambda^r \sum_{m=0}^{2r+2} W_{m,r}^{\Lambda,\Lambda_0}(\phi) \tag{3}$$

where $W_{m,r}$ is homogeneous of degree m in the fields ϕ, and (2) then implies

$$\dot{W}_{m,r}^{\Lambda,\Lambda_0} = \frac{1}{2}\Delta W_{m+2,r}^{\Lambda,\Lambda_0} - \frac{1}{2}\sum_{\mu=1}^{m+1}\sum_{\rho=1}^{r-1}\left(\nabla W_{\mu,\rho}^{\Lambda,\Lambda_0}\ ,\ \dot{C}_{\Lambda,\Lambda_0}\ \nabla W_{m+2-\mu,r-\rho}^{\Lambda,\Lambda_0}\right) \tag{4}$$

This equation has a recursive structure: in the second term, ρ and $r-\rho$ are both strictly smaller than r, and in the first, $W_{m+2,r}$ appears in the equation for $\dot{W}_{m_r}$. Thus, the $W_{m,r}$ can be obtained recursively from this equation, proceeding upwards in r and at fixed r downwards in m (starting at $m = 2r+2$). Similarly, one can derive differential inequalities for the coefficient functions of $W_{m,r}^{\Lambda,\Lambda_0}(\phi)$, and prove Λ-dependent bounds for suitable norms of these functions in an inductive scheme tailored to the above recursive structure. An ansatz with canonical scaling then works, provided that the two- and four-point terms are renormalized by an appropriate final condition on W. In four dimensions, this gives bounds of the form $\Lambda^{4-m}P_{m,r}(\log\Lambda)$, where $P_{m,r}$ is a polynomial. Details for ϕ^4 theories are given, e.g., in [37, 42]; as mentioned the method applies in great generality.

Tree formula. The Brydges-Kennedy formula regroups the expansion in V into an expansion over trees. Here we present it only for the simplest case when the s-dependence is $C(s) = sC$, $s \in [0,1]$. The $O(V^p)$ term in an expansion of W in $W_0 = V$ is $W^{(1)}(s,\phi) = (\mu_C * V)(\phi)$ for $p = 1$, and for $p \geq 2$

$$W^{(p)}(s,\phi) = \frac{(-1)^{p-1}}{p!} \sum_{T\in\mathcal{T}_p} \int_{[0,s)^{p-1}} d^T\sigma \left[\!\left[e^{\frac{1}{2}\Delta_C[M_T(s,\sigma)]} \prod_{t\in T} \tfrac{1}{2}\Delta_C^{(t)} \prod_{q=1}^{p} V(\phi^{(q)}) \right]\!\right] \tag{5}$$

where $\mathcal{T}_p$ is the set of trees on p vertices,

$$\begin{aligned} \Delta_C[M] &= \sum_{q,q'=1}^{p} M_{q,q'} \Delta_C^{(q,q')} \qquad \Delta_C^{(q,q')} = (\nabla^{(q)}, C\,\nabla^{(q')}) \\ (M_T)_{q,q'}(s,\sigma) &= \int_0^s d\rho\, 1\,(\sigma_l < \rho\ \forall l \in P_{q,q'} \subset T) \end{aligned} \tag{6}$$

$[\![F(\phi^{(1)},\ldots,\phi^{(p)})]\!] = F(\phi,\ldots,\phi)$, and $d^T\sigma = \prod_{t\in T} d\sigma_t$. The matrix $(M_T)(s,\sigma)$ is nonnegative (i.e. all its eigenvalues are nonnegative).

Formula (5) can be proven simply by inserting it into the differential equation (2). In general, it holds as a formal expansion in V, as before. It can be used to generate the Feynman graph expansion in two steps: The p vertices of the graph correspond to the factor $\prod_q V(\phi^{(q)})$. The operator $\prod_{t\in T} \frac{1}{2}\Delta_C^{(t)}$ connects these vertices by a tree. When fullly expanded, the remaining operator $e^{\frac{1}{2}\Delta_C[M_T(s,\sigma)]}$ produces additional loop lines and hence a summation over the corresponding Feynman graphs. The matrix $M_T(s,\sigma)$ implies a weighting of loop lines of the Feynman graphs by combinations of the parameters $(\sigma_t)_{t\in T}$, so that the integration over the σ_t's prevents an overcounting of graphs (this is nontrivial because the spanning tree of a connected graph G is not unique if G is not itself a tree, hence an unlabelled Feynman graph will appear multiple times in the above expansion).

In the case of fermionic field theories, where the fields are Grassmann variables and the integrals are Berezin integrals, this formula is key to convergent bounds. In this case, one can express the effect of $e^{\frac{1}{2}\Delta_C[M_T(s,\sigma)]}$ in terms of determinants (or Pfaffians) associated to the tree, hence avoiding a factorial growth that prevents convergence. Using Cayley's theorems for tree counting, a convergence proof can be given, provided that the covariance has both a finite determinant constant δ_C and a finite decay constant α_C. They are defined as $\alpha_C = |C|_{1,\infty} = \sup_x \int |C(x,y|\ dy$, and (in the translation-invariant case) $\delta_C^2 \le \|\hat{C}\|_1$ where the hat denotes the Fourier transform. This is explained in more detail, and precise convergence theorems are stated, in [45, 46, 47, 48].

The idea how to get from these bounds to a multiscale construction of the model is now as follows. The continuous flow $s \to C(s)$ is integrated over intervals to give a discrete iteration: in $C = \int_0^\infty \dot{C}(s)ds$ with $\Lambda_s = \Lambda_0\ e^{-s}$, decompose

$$[0,\infty) = [0,s_1) \cup [s_1,s_2) \cup \ldots, \cup [s_j,s_{j+1}) \cup \ldots \tag{7}$$

with some increasing sequence $0 = s_1 < s_2 < s_3 < \ldots < s_j \ldots$. Then

$$C = \sum_{j=0}^{\infty} C_j\ , \qquad C_j = \int_{s_j}^{s_{s+1}} \dot{C}(s)\ ds \tag{8}$$

The discrete RG flow is then obtained as an iterated convolution (take $s_j = \sigma j$, $\sigma > 0$ fixed)

$$\mu_C * e^{-V} = \lim_{J\to\infty} \mu_{C_J} * \left(\mu_{C_{J-1}} * \ldots * \left(\mu_{C_1} * (\mu_{C_0} * e^{-V})\right) \ldots\right) \tag{9}$$

This is a recursion $W_0 = V$ and

$$e^{-W_j} = \mu_{C_j} * e^{-W_{j-1}} \tag{10}$$

With the above technique, every W_j can then be constructed by convergent expansions from W_{j-1}. An iteration, with appropriate modifications (extraction of 'relevant' terms, i.e. ones that grow in the iteration), then yields convergence of the effective action. An example where this is used are Luttinger fermions in four dimensions with a local four-fermion interaction [51]. Major open problems are about Fermi-liquid behaviour in the three-dimensional many-fermion model, and constructions of models at real time.

In the bosonic case, i.e. when the fields are real or complex variables, a direct application of the tree formula does not yield convergence because the determinants (or Pfaffians) of the fermionic case are replaced by permanents (or Hafnians), which exhibit a factorial growth. A powerful, but technically more involved, procedure is to split field space into a small-field and a large-field region. This is the method employed in many of the classics of constructive quantum field theory, such as [11, 12, 13, 17] and [52]. In some cases, such as quantum many-boson systems at negative chemical potential, one can change variables so that this factorial growth does not occur and a convergent expansion results [53, 54].

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Introduction to perturbative algebraic quantum field theory (pAQFT)

Kasia Rejzner

In this introductory talk, I presented the basic principles underlying the pAQFT framework, which can be used to construct models that satisfy *Isotony*, *Causality* and the *Time-slice axiom* of the Haag-Kastler framework [5], but the local algebras are *topological *-algebras* rather than C^* or von Neumann algebras. For details, see for example [3, 2, 7].

Classical Field Theory in the pAQFT Framework. In the pAQFT approach, the starting point is the *classical theory*, formulated geometrically on an appropriate infinite-dimensional space of field configurations $\mathcal{E}$, typically chosen as a space of smooth sections $\mathcal{E} = \Gamma(E \to M)$ for some vector bundle over M. Observables are represented as functionals on $\mathcal{E}$. To ensure sufficient regularity, we work with smooth functionals $F : \mathcal{E} \to \mathbb{C}$ whose derivatives are compactly supported. Important classes of functionlals are: *local functionals* $\mathcal{F}_{\text{loc}}$ (can be expressed as integrals of smooth functions on the jet bundle), *multilocal functionals* $\mathcal{F}$ (sums of products of local ones), *regular polynomial functionals* $\mathcal{F}_{\text{reg}}$ (derivatives above a fixed order N vanish and all the derivatives are smooth sections) and *equicausal functionals* $\mathcal{F}_{\text{ec}}$ (defined via wavefront set conditions on their derivatives, with an additional equicontinuity requirement [6]).

The dynamics of the classical theory are described through a *1-form* dL on the configuration space (Euler-Lagrange derivative of a local Lagrangian L). The equation $dL = 0$ determines the classical solutions. The space of solutions can also be described homologically using the *Koszul complex*, which captures the algebraic structure of functionals modulo the equations of motion.

Quantisation via Deformation Quantisation. Quantisation in pAQFT proceeds in two steps: one first quantises the *free theory* (given by dL_0 which is linear in the fields, so can be written in terms of a differential operator P as $dL_0(\varphi) = P\varphi$) and then incorporates the *interaction* perturbatively. We assume P to be Green hyperbolic [1], so there exist unique retarded and advanced Green functions $\Delta^{R/A}$.

The quantisation of the free theory is performed using the method of *formal deformation quantisation.* The idea is to deform the commutative pointwise product of functionals into a non-commutative *star product*, defined as

$$F \star G = m \circ e^{\hbar\langle W, \frac{\delta^2}{\delta\varphi\otimes\delta\varphi}\rangle}(F \otimes G),$$

where W is a Hadamard function of the free theory, and m denotes the pointwise multiplication map. We have that $W = \frac{i}{2}\Delta + H$, with the antisymmetric part $\Delta := \Delta^R - \Delta^A$, and with symmetric, non-unique part H, chosen such that W is of positive type and it satisfies a wavefront set condition motivated by the wafefront set of the Wightman 2-point function on Minkowski spacetime in the vacuum state. This star product encodes the canonical commutation relations in the algebraic structure of functionals and defines the *quantum algebra of observables.*

Time-Ordered Products and the Path Integral Heuristics. Once the free theory is quantised, interactions are introduced using perturbation theory. A central structure in this process is the *time-ordered product*, denoted $\cdot_{\mathcal{T}}$ and constructed using the *Feynman propagator* $\Delta^F := \frac{i}{2}(\Delta^R + \Delta^A) + H$ by

$$F \cdot_{\mathcal{T}} G := m \circ e^{\hbar\langle \Delta^F, \frac{\delta^2}{\delta\varphi\otimes\delta\varphi}\rangle}(F \otimes G)\,.$$

The core challenge is that such products become ill-defined for $F, G \in \mathcal{F}_{\text{loc}}$, due to singularities of the Feynman propagator Δ^F. Epstein–Glaser renormalisation [4] provides a way to define these products consistently by extending distributions that are initially defined only away from the diagonal (i.e., for non-coincident spacetime points). As written, $\cdot_{\mathcal{T}}$ is well-defined for $F, G \in \mathcal{F}_{\text{reg}}$ and can also be expressed as $F \cdot_{\mathcal{T}} G = \mathcal{T}(\mathcal{T}^{-1}F \cdot \mathcal{T}^{-1}G)$, with the *time-ordering operator*

$$\mathcal{T} := e^{\hbar\langle \Delta^F, \frac{\delta^2}{\delta\varphi^2}\rangle}\,,$$

which can be interpreted as the algebraic counterpart of the path integral, i.e.

$$(\mathcal{T}F)(0) \overset{\text{formal}}{=} \int F(\varphi)\, e^{\frac{i}{\hbar}\langle\varphi, P\varphi\rangle}\, d\varphi = \int F(\varphi)\, e^{\frac{i}{\hbar}L_0}\, d\varphi\,.$$

Interacting observables. Interactions are introduced perturbatively via the *Bogoliubov formula*, using the *retarded Møller operator* R_V associated with the interaction functional V. The interacting fields are expressed as

$$R_V(F) = (\mathcal{T}e^{\frac{i}{\hbar}V})^{-1} \star (\mathcal{T}e^{\frac{i}{\hbar}V}F),$$

which provides a mathematically precise version of the Gell–Mann–Low formula from physics. Heuristically, we have

$$R_V(F)(0) \overset{\text{formal}}{=} \frac{\int F(\varphi)\, e^{\frac{i}{\hbar}(L_0+V)}\, d\varphi}{\int e^{\frac{i}{\hbar}(L_0+V)}\, d\varphi}\,.$$

For a relatively compact region $\mathcal{O}$, its interacting (with interaction V) local algebra $\mathfrak{A}_V(\mathcal{O})$ is generated with respect to the free star product $\star$ by all $R_V(F)$ such that the support of F is contained in $\mathcal{O}$. The resulting net satisfies *Isotony*, *Causality* and *the Time-slice axiom*, as required.

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From Faá di Bruno to Connes–Kreimer — BPHZ renormalisation scheme

Joscha Diehl, Kurusch Ebrahimi-Fard

In this talk, we discuss Delamotte's formulation of renormalisation in perturbative quantum field theory (QFT) as a reparametrisation procedure [9] in terms of the Faà di Bruno Hopf algebra [13], and trace its relation to the BPHZ renormalisation scheme expressed through the Connes–Kreimer Hopf algebra of rooted trees [4]. The bridge between the two perspectives is provided by a specific Hopf algebra isomorphism, established in Foissy's work [14], which realizes the Faà di Bruno Hopf algebra as a Hopf subalgebra in the Hopf algebra of rooted trees, showing that the two viewpoints are two facets of the same algebraic process.

Renormalisation [3] is a systematic procedure developed to remove (ultraviolet) divergences that appear in perturbative calculations in QFT. These infinities arise because the original perturbative expansions are expressed in terms of the wrong (unrenormalised or "bare") parameters. The central result of renormalisation theory asserts that, order by order in perturbation theory, the divergences of a QFT can be eliminated by consistently absorbing them into redefinitions of its parameters in terms of physical couplings, all while preserving locality, unitarity, and Lorentz invariance.

At its core, renormalisation can be viewed as a reparametrisation procedure that expresses perturbative expansions in terms of measurable physical quantities instead of unphysical "bare" parameters. This serves as the starting point for Delamotte [9], who emphasises this simple characterisation of renormalisation by distinguishing the aspects specific to models in QFT from those intrinsic to the renormalisation process itself. Indeed, he emphasizes that formulating renormalisation as a reparametrisation procedure is independent of the Feynman graph expansions tied to a specific QFT model, thereby sidestepping the associated graph-theoretic combinatorics. The underlying structure of the inductive renormalisation procedure is instead that of the Faà di Bruno formula which governs the combinatorics of nested derivatives and function compositions via partitions of non-negative integers.

It might therefore be instructive to compare Delamotte's reparametrization perspective with the Bogoliubov–Parasiuk–Hepp–Zimmermann (BPHZ) renormalisation procedure which provides a formulation of renormalisation in perturbative QFT based on replacing ill-defined integrals by finite, well-defined expressions [2]. Its combinatorial backbone is the structure of Feynman graphs. The central idea is to systematically subtract ultraviolet divergences from Feynman amplitudes using a recursive procedure based on the forest formula. Given a Feynman graph, one identifies all divergent subgraphs, including overlapping ones, and assigns countert-erms that remove their divergences. Zimmermann's forest formula then organises these subtractions in a combinatorial manner, ensuring locality and consistency.

Connes and Kreimer reformulated the BPHZ renormalisation procedure in terms of a combinatorial Hopf algebra defined over (polynomials of 1PI) Feynman graphs [7]. Its coproduct encodes the purely combinatorial operation of extracting (1PI) divergent subgraphs and contracting them inside larger graphs. The recursive subtraction of divergences is an avatar of the antipode of this Hopf algebra "twisted" by a subtraction map. The latter is required to be a Rota–Baxter operator [11].

Thus, what once looked like an intricate analysis-driven combinatorial procedure emerges as an instance of the general theory of combinatorial Hopf algebras. One of the key insights of Connes and Kreimer arising from this approach is that BPHZ renormalisation can be naturally formulated as a factorisation problem in the group of characters over their combinatorial Hopf algebra of Feynman graphs, that is, in the group of unital linear multiplicative functions: the Feynman rules define a particular character on the Hopf algebra, and an algebraic Birkhoff decomposition of this character yields two Hopf algebra characters, one encoding counterterms and the other giving renormalised amplitudes [7]. A closer look reveals that this decomposition is a manifestation of a universal combinatorial structure governed by the facorisation properties of Rota–Baxter algebras [10].

Before considering Feynman graphs, Connes and Kreimer showed that the combinatorics of perturbative renormalisation can be understood in terms of non-planar rooted trees [4]. The latter provide a universal combinatorial model for nested and overlapping subdivergencies: each node in a tree represents an insertion of a divergent subgraph, and the branching encodes how divergences in a Feynman graph are nested inside one another. The coproduct on trees precisely mirrors the coproduct on Feynman graphs, and one may say that renormalisation appears as an instance of a general phenomenon: the Hopf algebra of rooted trees captures the essential combinatorics, while Feynman graphs are one concrete representation.

Connes and Kreimer [6] highlighted the connection between their work and that of Butcher in numerical analysis. Brouder further developed profound links with the combinatorics underlying numerical analysis [1].

The Hopf algebras of rooted trees and of Feynman graphs are deeply connected to the Faà di Bruno Hopf algebra [5, 8]. The latter encodes the algebra of formal diffeomorphisms under composition, and its coproduct mirrors the Faà di Bruno formula [12, 13]. Connes and Kreimer observed a common combinatorial principle: the Faà di Bruno coproduct acts by partitioning positive integers, whereas the coproduct in renormalisation acts by partitioning combinatorial objects such as trees or graphs. In this sense, the Faà di Bruno Hopf algebra provides the archetypal template for renormalisation Hopf algebras.

Building on the insight of Connes and Kreimer, several authors clarified in detail the precise relationship between the Faà di Bruno Hopf algebra and the Hopf algebra of rooted trees. We mention in particular Manchon and Frabetti [15]. In [14], Foissy further developed and deepened this connection by giving a systematic account of how Faà di Bruno Hopf subalgebras arise inside the Hopf algebra of non-planar rooted trees.

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The extension to higher codimension of local field theories

Francesco Bonechi

The Batalin-Vilkovisky (BV) method was introduced to deal with theories with general gauge invariance, evolving the BRST method used for theories as Yang-Mills or Chern-Simons. It is properly formulated in the language of graded symplectic geometry [4]. Given the gauge invariant classical action, the construction of the solution of the Classical Master Equation (CME) is understood as an homological resolution of the space of solutions of the equations of motions modulo gauge transformations; it involves tools of homological algebra [5]. The hamiltonian counterpart, called Batalin-Fradkin-Vilkovisky (BFV), is an homological resolution of the coisotropic reduction defined by the constraints. The AKSZ construction (see [1],[6]) defines the solution of the CME in terms of very simple geometrical data so avoiding the rather cumbersome construction provided by homological perturbation theory. It is a wide class of interesting topological field theories, including Poisson Sigma Model (PSM) in $2d$, Chern Simons in $3d$ and BF theories in all dimensions.

This setting provides a concrete framework for extending a local field theory to nonzero codimensions. In the Cattaneo-Mnev-Reshetikhin (CMR) approach (see [3]), the theory extends to codimension k if one can associate to each codimension k stratum of the spacetime a compatible BF^kV theory. For example for $k = 0$ (the bulk) one has a Batalin Vilkovisky (BV) theory and for k=1 (the boundary) a Batalin-Fradkin-Vilkovisky (BFV) theory describing the hamiltonian theory. The AKSZ solutions can be canonically extended to the points. In this talk I will discuss PSM. In general there are obstructions to this extension whose nature should be investigated. An important example to be understood is General Relativity (GR) and its supersymmetric versions. So far it has been proved that GR extends to codimension one [2].

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Stochastic Partial Differential Equations and Renormalization à la Epstein-Glaser

Claudio Dappiaggi
(joint work with A. Bonicelli, B. Costeri, N. Drago, P. Rinaldi & L. Zambotti)

We present a novel framework for the study of a large class of nonlinear stochastic partial differential equations (SPDEs), which is inspired by the algebraic approach to quantum field theory [4]. The main merit is that, by realising random fields within a suitable algebra of functional-valued distributions, we are able to use specific techniques proper of microlocal analysis. These allow us to deal with renormalization using an Epstein–Glaser perspective, hence without resorting to any specific regularisation scheme. In addition, we are able to devise an algorithmic and diagrammatic procedure to derive, at a perturbative level, the expectation value and the correlation functions of nonlinear SPDEs, establishing case by case whether the underlying equation lies in the subcritical regime. While, in this talk, to discuss a concrete example, we focus on the stochastic Φ^3_d model, this procedure has been applied to several other models such as the stochastic nonlinear Schrödinger equation [2] and the stochastic Thirring model [1].

It is worth stressing that our construction, being inspired by algebraic quantum field theory, is inherently perturbative. Although this might appear as a potential limitation, we have shown in [3] that convergence can indeed be achieved in specific instances, most notably for the hyperbolic stochastic sine-Gordon model. This result provides further evidence that the perturbative algebraic framework can, in suitable settings, capture genuine nonperturbative features. Nevertheless, further work is required to connect this approach with other modern frameworks for SPDEs, such as regularity structures, paracontrolled calculus, and flow equations, in order to fully elucidate its scope and potential.

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A strong–weak duality for the 1d long-range Ising model

Dario Benedetti

(joint work with Edoardo Lauria, Dalimil Mazac, Philine van Vliet)

The 1d Ising model with long-range interactions is described by the classical Hamiltonian

$$H_{\text{LRI}} = \frac{J}{2} \sum_{i \neq j} \frac{(\sigma_i - \sigma_j)^2}{|i-j|^{1+s}}, \quad J > 0,$$

where $\sigma_i = \pm 1$ is the Ising variables at lattice sites i. It is known since a long time that this model admits a phase transition for $0 \leq s \leq 1$, described by mean-field theory (MFT) for $s \leq 1/2$, and by a nontrivial family of 1d conformal field theories (CFTs) for $1/2 \leq s \leq 1$. Above $s = 1$, there is no phase transition at finite temperature, as in the short-range 1d Ising model. Therefore, the point $s = 1$ corresponds to a crossover between long-range and short-range behavior. The critical regime can be studied perturbatively near the MFT end at $s = 1/2$, as a generalized free scalar field perturbed by a quartic interaction, but such description becomes strongly coupled near the short-range crossover at $s = 1$.

We propose a dual weakly-coupled description near $s = 1$. The partition function Z of our model is constructed from a generalized free field ϕ of negative mass-dimension $(s-1)/2$ and compact target space of radius $1/b$, perturbed by an impurity-type interaction,

$$Z = \left\langle \text{trPexp}\left\{ \int \left[g\left(\hat{\sigma}_+ e^{ib\phi} + \hat{\sigma}_- e^{-ib\phi}\right) + h\, \frac{i\hat{\sigma}_3}{\sqrt{2}} \partial_x \phi \right] dx \right\} \right\rangle_{GFF_\phi},$$

where $\{\hat{\sigma}_\pm, \hat{\sigma}_3\}$ is the $\mathfrak{sl}_2$-triplet and trPexp stands for the trace of the path-ordered exponential. The interaction term proportional to the coupling g is interpreted as generating kink and antikink configurations in the original Ising spins, while h shifts the compactification radius and is needed for renormalization.

We have performed a number of consistency checks of our proposal, in particular calculating the perturbative CFT data around $s = 1$ analytically using both our proposed field theory and the analytic conformal bootstrap. Our results show complete agreement between the two methods.

It would also be interesting to study the CFT data in the full range $1/2 < s < 1$, interpolating between the near-MFT regime and our dual one. It would also be interesting to study similar duality problems for other 1d long-range models, such as multicritical models, Potts, percolation, or models with long-range disorder.

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Discreteness and Randomness: Quantum Field Theory on Causal Sets

Sumati Surya
(joint work with Emma Albertini and Kasia Rejzner)

The fundamental field in General Relativity is a Lorentzian geometry g with signature $(-,+,+\ldots+)$ defined over a manifold M. It is a unique feature of this signature alone that the sets of future and past directed lightcones in the tangent space are topologically disjoint. If one considers a causal spacetime (M,g) then this lightcone structure implies that there is a globally determined partially ordered set $(M,\prec)$ underlying (M,g), where $\prec$ satisfies (i) acyclicity: $x \prec y \Rightarrow y \not\prec x$ and (ii) transitivity: $x \prec y, y \prec z \Rightarrow x \prec z$. It is of interest to assess how much of (M,g) is captured by $(M,\prec)$. Assuming very weak global causality conditions, a set of theorems due to Hawking, King, McCarthy [1] and Malament [2], supplemented by results due to Kronheimer and Penrose [3] shows that a causal bijection between two spacetimes exists iff they are conformally isometric. Thus, $(M,\prec)$ encodes the entire conformal geometry of the spacetime. The only missing ingredient is the conformal factor, or equivalently, the local volume element.

This motivates the causal set approach to quantising gravity where, instead of promoting the spacetime geometry g directly to a quantum field, one begins by discretising $(M,\prec)$ [4, 5, 6, 7]. This introduces the third condition: (iii) local finiteness: $\forall (e,e') = \{e''|e \preceq e" \preceq e'\}$, $|(e,e')| < \infty$, and the resulting locally finite partially ordered set is called a causal set. While there is a loss of continuum information, in an approximate sense the counting measure given by $|(e,e')|$ provides the missing local volume)element. Causal set theory thus views the continuum spacetime (M,g) as an approximation of an underlying random causal set $\mathbf{C}$. The elements of $\mathbf{C}$ can be generated from a Poisson point process (PPP) into (M,g), where the probability $P_n(V)$ of finding n elements in a spacetime region of volume V is

$$P_n(V) = \frac{(\rho V)^n}{n!} \exp(-\rho V), \tag{1}$$

where ρ^{-1} is the discreteness scale. The order relations in $\mathbf{C}$ are then induced by the causal ordering of events in (M,g). It can be shown that $\mathbf{C}$ does not violate Lorentz invariance in Minkowski spacetime, but is non-local in the sense that it is not a finite or fixed valency graph.

While the main goal of causal set theory is quantum gravity, it can also be used as a covariant regularisation of non-gravitational quantum field theory (QFT). Conversely, causal set modifications to QFT can give us signatures of new UV physics. In order to construct QFT on causal sets directly, one needs a fundamentally spacetime and causal formulation, which is provided by the AQFT framework [8, 9, 10, 11, 12, 13]. Consider the massive Klein Gordon scalar field satisfying $\widehat{P}\phi(x) = 0$ on (M,g) where $\widehat{P} = \Box + m^2 + \xi R$ is a hyperbolic operator and $G^{R,A}(x,x')$ the associated retarded and advanced Green's functions respectively. The Peierls bracket quantisation condition is

$$[\Phi(x), \Phi(x')] = i\Delta(x,x'), \tag{2}$$

where the Pauli Jordan function $i\Delta(x,x') \equiv i(G^R(x,x') - G^A(x,x'))$. For a function $f \in C_0^k$ (compactly supported functions with $k > 2$), the integral operator $i\widehat{\Delta} \circ f(x) \equiv i\int dV \Delta(x,x') f(x')$, is self adjoint. Since $Ker(\widehat{P}) = \mathcal{I}m(\widehat{\Delta})$, the eigenmodes of $i\widehat{\Delta}$ (the Sorkin-Johnston or SJ modes) can be used to define the SJ vacuum state or Wightmann operator $\widehat{W} = \mathrm{Pos}(i\Delta)$ [11, 12, 13, 14].

This formalism can be be directly implemented on the random causal set $\mathbf{C}$, starting from the discrete versions $K^{R,A}(e,e')$ of the scalar Green's functions $G^{R,A}(x,x')$. The $K^{R,A}(e,e')$ have been constructed in random causal sets approximated by various $d = 2,4$ spacetime regions [10, 15]. The causal set SJ vacuum $\mathcal{W}_{SJ}(e,e')$ can thus be constructed and matches the continuum version for large spacetime separations [11, 12, 16]. However there can be significant features in $\mathcal{W}_{SJ}(e,e')$ at small separations, even those that are significantly larger than the discreteness scale. Such UV modifications can change the behaviour of renormalised quantities. For example, in $\lambda\phi^4$ theory in Minkowski spacetime, the renormalised mass is

$$m_r^2 = m^2 + \frac{1}{2} i\lambda\Delta_F(0), \tag{3}$$

where $\Delta_F(x - x')$ is the Feynman propagator. In dimension $d = 2$,

$$\Delta_F(0) = \frac{1}{4\pi} \ln\left(\frac{\Lambda_c^2 + m^2}{m^2}\right), \tag{4}$$

where Λ_c is the UV cut-off. In the causal set, $\Delta_F(0)(= \mathcal{W}_{SJ}(0))$ is built out of the SJ modes. The effect of discreteness is that the SJ eigenspectrum has a characteristic "knee": at IR scales, i.e., before the knee the spectrum mimics that of the continuum, but in the UV, beyond the knee, it is distinctively non-continuumlike. Using a simple substitution of the causal set $\mathcal{W}_{SJ}(0)$ into the renormalised mass formula Eqn. (3) we find a significant modification from the continuum behavior Eqn. (4). Continuumlike results are recovered when a truncation is performed at the knee. Our preliminary results therefore suggest that causal set discreteness modifies the broad features of renormalisation and renormalisation flows in the deep UV.

Our results are suggestive and need further study. The pAQFT approach [17, 18] offers insights into renormalisation from a Lorentzian and causal perspective and has recently been adapted to causal sets [19, 20]. Obtaining the renormalised parameters and flows on causal sets using these techniques is the subject of ongoing research.

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A multiplicative surface signature through its Magnus expansion

Joscha Diehl

(joint work with Ilya Chevyrev, Kurusch Ebrahimi-Fard, and Nikolas Tapia)

The *path signature* - introduced by K.-T. Chen - associates to a curve $X : [0,1] \to \mathbb{R}^n$ a sequence of iterated integrals forming a group-like element in the tensor algebra $T((\mathbb{R}^n))$. This algebraic structure, together with Chen's identity (also known as the cocycle property)

$$S(X)_{s,t}\, S(X)_{t,u} = S(X)_{s,u},$$

is key to both its mathematical universality and its efficient (linear-time) computation. Its logarithm, the *Magnus expansion*, lives in the free Lie algebra, provides a redundancy-free description of paths (up to tree-like equivalence and reparameterization) and is characterized by a nonlinear differential equation.

We present a generalization of the path signature to surfaces $X : [0,1]^2 \to \mathbb{R}^n$ which shares many of the key properties of the path signature, in particular a 2-dimensional analog of Chen's identity, [2]. Algebraically, it builds on the *free crossed module of Lie algebras* introduced by A. Kapranov [4] as a 2-dimensional analog of the free Lie algebra. Analytically, we apply a novel 2-dimensional sewing lemma, which is of independent interest and allows to deal with non-smooth surfaces. Moreover, we "stay in the Lie algebra" by constructing a *Magnus expansion for surfaces*. (We note that an alternative approach working in the enveloping algebra is developed in [3].)

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BV pushforward and equivariant Yang Mills theory

Francesco Bonechi

The BV pushforward of solutions of the Quantum Master Equation is a very simple construction in the Batalin Vilkovisky (BV) formalism: given a symplectic splitting of the BV space of fields, the fibrewise integration along one factor defines a new solution of the CME on the other factor. At tree level, it corresponds to the homotopy transfer of the ℓ_∞-algebras encoded in the solution of the CME. In this form, it first appeared in [5] as a way to construct simplicial BF theory, see also [3] for Chern-Simons.

This construction can be extended to solutions of the equivariant extension of the CME. In [2] the BV formalism has been extended to work equivariantly with respect to a Lie algebra action on the spacetime; in particular an equivariant extension of CME has been introduced. In general there is no canonical equivariant extension of a given solution of CME, but, as usual, AKSZ theories behave very nicely and display such extension: in this case the full Cartan calculus can be defined on the space of fields so that the equivariant extension is defined in the Cartan model for equivariant cohomology.

In this talk we discuss the case of topological Yang Mills (TYM_4) in four dimensions. There exists a splitting of the space of BV fields such that the restriction to one symplectic submanifold coincides with the BV formulation of physical Yang-Mills (YM_4) introduced in [4]. The BV pushforward then will give a solution of the CME that deforms YM theory. In this talk we discuss the equivariant extension. Since TYM_4 is AKSZ, its equivariant extension is straightforward. Physical YM_4 is not anymore AKSZ, in particular the space of fields does not have the full Cartan calculus, the contraction operator missing. The extension can be obtained by means of the BV pushforward from TYM_4. We computed so far the abelian case: the result is an equivariant extension of the CME that involves higher order contraction operators. The non abelian case is under study.

Finally, there are several other examples of BV pushforward of a topological theory that deforms a physical (non topological) one. For instance the Topological Yang Mills theory in $2d$ TYM_2 gives physical YM_2 decorated with zero modes; the Poisson Sigma Model with symplectic target gives the complex scalar. These examples are easier to compute giving us the possibility to clarify the nature of this relation. This is joint work with A.Cattaneo and M. Zabzine.

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Observables and gauge covariant renormalisation of stochastic 3D Yang–Mills

Ilya Chevyrev

(joint work with Hao Shen)

One of the goals of Euclidean quantum Yang–Mills theory is to give meaning and study the *Yang–Mills probability measure*. For a compact Lie group $G \subset U(N)$ with Lie algebra $\mathfrak{g}$, this measure is heuristically given by

$$\mu(\mathrm{d}A) \propto e^{-S(A)}\,\mathrm{d}A$$

where A is a $\mathfrak{g}$-valued 1-form on $\mathbb{R}^d$ written in coordinates as $A = (A_1, \ldots, A_d)\colon \mathbb{R}^d \to \mathfrak{g}^d$ and S is the Yang–Mills action

$$S(A) = \int_{\mathbb{R}^d} \|F\|^2 = -\sum_{i<j} \int \mathrm{Tr}\, F_{ij}^2$$

where $F = F(A)$ is the curvature 2-form

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]\ .$$

An approach to quantum Yang–Mills theory was initiated in [2, 1] that relies on the Yang–Mills Langevin dynamic, proposed by Parisi–Wu, in the DeTurck gauge,

$$\text{(SYM)} \qquad \partial_t A = -\,\mathrm{d}_A F(A) - \mathrm{d}_A\, \mathrm{d}^* A + \xi\ .$$

For dimensions $d = 2, 3$, this is a singular subcritical stochastic PDE, and the main results of [2, 1] is that, if we mollify the noise $\xi \rightsquigarrow \xi^\varepsilon$ and work on the torus $\mathbb{T}^d$, then there exist linear operators $C^\varepsilon \in L(\mathfrak{g}, \mathfrak{g})$ such that the renormalised equation

$$\text{(RSYM)} \qquad \partial_t A^\varepsilon = -\,\mathrm{d}_{A^\varepsilon} F_{A^\varepsilon} - \mathrm{d}_{A^\varepsilon}\, \mathrm{d}^* A^\varepsilon + C^\varepsilon A^\varepsilon + \xi^\varepsilon$$

admits a unique limiting solution $A\colon [0,T]\times\mathbb{T}^d\to\mathfrak{g}^d$ (which may blow up in finite time). This solution is independent of the mollifier and possesses a gauge covariance property: if we let A^a, A^b denote the solutions started from two gauge equivalent configurations $a\sim b$, then A^a_t and A^b_t remain gauge equivalent in *distribution* for all $t\geq 0$. Here, gauge equivalence $a\sim b$ means that there exists a 'gauge transformation' $g\colon \mathbb{T}^d\to G$ such that

$$g\cdot a := gag^{-1} - \mathrm{d}gg^{-1} = b\;.$$

In dimension $d=2$, by studying the equation (SYM) on the lattice, it was moreover shown in [3] that the limiting dynamic A has a unique invariant measure on the space of gauge orbits, and this measure coincides with the 2D Yang–Mills measure for the trivial principle bundle [4].

It turns out that, for $d=2$, the renormalisation constant $C=C^\varepsilon$ can be taken independent of ε and only depending on the shape of the mollifier. (For $d=3$, the constant C^ε in general diverges as $\varepsilon\downarrow 0$.) A crucial step in the proof in [3] is to show that C is the *unique* constant that leads a gauge covariant limiting dynamic – roughly speaking, this allows one to promote a compactness result to a convergence result.

In a recent work [5], we showed that the same uniqueness result for C^ε in $d=3$. More precisely, consider another renormalisation constant $\bar C^\varepsilon\in L(\mathfrak{g},\mathfrak{g})$ such that

$$\lim_{\varepsilon\downarrow 0}\{\bar C^\varepsilon - C^\varepsilon\}\neq 0\;,$$

and let A^a denote the limiting dynamic to (RSYM) with C^ε replaced by $\bar C^\varepsilon$ and with initial condition a. Consider the smooth (non-contractible) loop $\ell\colon S^1\to\mathbb{T}^3$, $\ell_t=(t,0,0)$, where we identify S^1 and $\mathbb{T}^3$ with $[0,1)$ and $[0,1)^3$ respectively as sets. Then there exists $g\in\mathcal{C}^\infty(\mathbb{T}^3,G)$ such that for all $t\ll 1$, there exists an initial condition $a\in\mathcal{C}^\infty(\mathbb{T}^3,\mathfrak{g}^3)$ and $s=t^\beta$ for some $\beta>0$ for which

$$|\mathbb{E}W_\ell\{\mathcal{F}_s[A^a_t]\} - \mathbb{E}W_\ell\{\mathcal{F}_s[A^{g\cdot a}_t]\}|\gtrsim t^{10/9}\;. \tag{1}$$

The exponent 10/9 is an arbitrary number larger than 1. One should compare this to the case $\bar C = C$ for which one has an *upper bound* for the left-hand side of (1) of order t^M for any $M>0$.

Above, W_ℓ is the well-known Wilson loop observable of ℓ that maps sufficiently regular (say smooth) 1-forms a to $W_\ell(a)=\mathrm{Tr}\,y_1\in\mathbb{C}$, where $y\colon[0,1]\to G$ solves the linear ODE

$$\mathrm{d}y_t = y_t\langle a(\ell_t),\dot\ell_t\rangle\,\mathrm{d}t\;,\qquad y_0=1\in G\;.$$

The map $\mathcal{F}_s(A)$ is a form of gauge covariant regularisation of A. It is given by the time-s solution of the Yang–Mills heat flow $\partial_s B = -\,\mathrm{d}_B^* F_B - \mathrm{d}_B\,\mathrm{d}^* B$ with initial condition $B(0)=A$. An important feature of $\mathcal{F}_s$ is that, for smooth a,b, $\mathcal{F}_s(a)\sim\mathcal{F}_s(b)$ if and only if $a\sim b$. The advantage of $\mathcal{F}_s$ is that $\mathcal{F}_s(a)$ makes sense for a class of distributions a which are too rough for $W_\ell(a)$ to be well-defined – this is the case for the Gaussian free field in dimension $d\in[3,4)$ [6]. One can then use $\mathcal{F}_s(a)\sim\mathcal{F}_s(b)$ as a *definition* of gauge equivalence for such rough a,b.

The composed observables $A\mapsto W_\ell\{\mathcal{F}_s(A)\}$ were proposed in [7, 1] as a way to define a state space for 3D Yang–Mills measure. The main result of [5] gives an application of these observables to the 3D Yang–Mills Langevin dynamic. The key steps in the proof are short time expansions of the SPDE (RSYM), of the Yang–Mills heat flow $\mathcal{F}_s$, and of Wilson loops. The choice of the initial condition a is based on the Chow–Rashevskii theorem from sub-Riemannian geometry.

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Covariant transport and Poincaré symmetry with groupoids and direct connections

Alessandra Frabetti

(joint work with S. Azzali, Y. Boutaib, S. Paycha and S. Amiel, M. D'Agostino, A. Miti)

Proper sources as in the example above

Transport maps, in Regularity Structures, relate generalized Taylor expansion of distributions centered at different points. Poincaré symmetry, in Special Relativity, ensures the invariance of the laws of relativistic physics for inertial frames, including those related by translations. Because of "translations", these two topics typically require to work on a flat base manifold: the Euclidian space on one side and the Minkowski space-time on the other. A covariant presentation of transport maps on general Riemannian manifolds was given in [A. Dahlqvist, J. Diehl, B. Driver, Probability Theory and Related Fields, 1-2 (2018)] with the help of a transport precision. Similarly, a covariant presentation of Poincaré symmetry on pseudo-Riemannian manifolds was given in [B.T. Costa, M. Forger, L.H. Pegas, J. Geom. Phys. 131 (2018)] in terms of a jet interpretation of the orthogonal frame groupoid of the manifold. In this talk we explain how groupoids relate these results, and how transport maps lead to a generalization of gauge fields as direct connections on jet groupoids.

More precisely, after a brief introduction to the two motivating topics, we present the leading idea of the talk, namely the extension of some key ingredients of a field theory with coupled gauge theory:

- Given (matter) fields $\psi : M \to E$ as sections of a vector bundle, the structure Lie group G ruling the transition functions of E and the gauge transformations is extended to a *Lie groupoid* $\mathcal{G}$ acting on E (that is, a subgroupoid of the *frame groupoid* of E), together with its *group of bisections* acting on the fields.
- The Lie algebra $\mathfrak{g}$ of G, which encodes the infinitesimal gauge transformations, is then replaced by the *Lie algebroid* of the Lie groupoid $\mathcal{G}$.
- The gauge fields A typically coupeld to ψ, which are 1-form connections with values in $\mathfrak{g}$, were already generalized to *Lie algebroid connections* by several authors. We further promote them to *direct connections* on the groupoid $\mathcal{G}$. Lie groupoids endowed with a direct connection are *gauge groupoids* $\mathcal{G}(P)$ canonically associated to a principal bundle P. Compared to the usual connection 1-forms on P, there are many more direct connections on $\mathcal{G}(P)$, carrying interesting higher terms which get lost in their infinitesimal versions.
- Finally, the need of jet bundles to describe the Lagrangian of ψ and classical observables leads to consider *jet groupoids* and to study the *jet prolongation of direct connections.*

All the new ingredients are sketched in the talk, together with some relevant examples. We report on specific results we obtained on the jet prolongation of direct connections. We conclude with an explicit *geometric polynomial model* of Regularity Structures on a manifold M, very similar to that defined by Dahlqvist, Diehl and Driver but with transport maps given by direct connections on jet groupoids.

The talk is based on works in progress with S. Azzali, Y. Boutaib, S. Paycha and S. Amiel, M. D'Agostino, A. Miti, and on the global understanding of covariant field theory developed with O. Kravchenko and L. Ryvkin in [arxiv:2407.15287].

An Introduction to Higher Differential Geometry

Konrad Waldorf

We have reviewed the passage from Lie groups to Lie 2-groups, which forms the foundation of higher differential geometry. Lie groups appear as symmetry groups–of space and time, of internal state spaces, or combinations thereof–while Lie 2-groups describe second-order symmetries, or symmetries of symmetries. Typical examples arise in the study of B-fields in string theory [13], T-duality [9, 2], and categorified representation theory [1, 5].

Extending Lie groups to Lie 2-groups entails corresponding generalizations of differential-geometric structures such as principal bundles, connections, and representations. These generalizations require an abstract, conceptual understanding from first principles. For instance, principal G-bundles with connections can be reconstructed from their local model–connection 1-forms and gauge transformations–via sheafification; this local model itself follows from the requirement that parallel transport be well-defined [10]. Analogously, local models for principal 2-bundles arise from a notion of surface parallel transport [11, 12], and a concise, conceptually grounded definition of higher principal bundles follows by sheafification [7, 8].

As an example of a categorified representation, we discussed the stringor representation of the string 2-group on a 2-vector space given by the hyperfinite type III_1 von Neumann algebra [4]. It plays the role of the spinor representation of the spin group, but for strings rather than point particles. Recently, the stringor bundle–a structure long anticipated by Stolz and Teichner–has been constructed rigorously in this framework [3]. This was achieved through analytical work relating Connes fusion of Fock spaces to the fusion of loops in a manifold [6].

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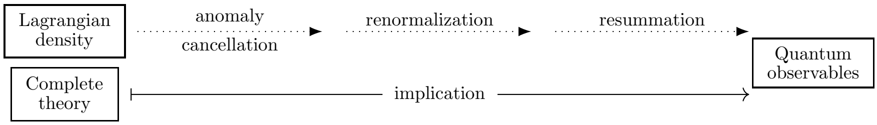
Renormalization and Complete QFTs

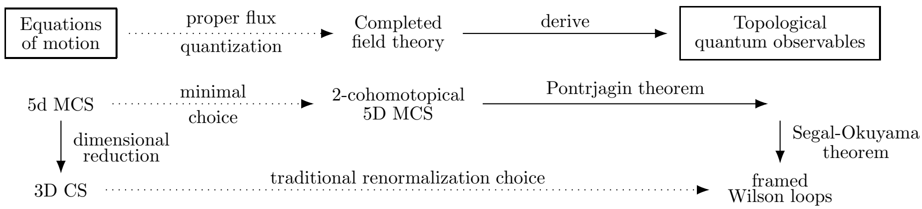
Urs Schreiber

(joint work with Hisham Sati)

In the practice of physics model building, the process of *renormalization*, *resummation*, and *anomaly cancellation* is to incrementally repair initially ill-defined *Lagrangian* quantum field theories by a successive choice of partial fixes. Impressive as this is, one would rather have concisely defined complete theories to begin with, and understand these choices as emergent from fundamental principles:



As an instructive example, we recall renormalization choices for Wilson loop observables in abelian Chern-Simons theory. Then we show that these emerge in a novel non-Lagrangian topological completion of 5D Maxwell-Chern-Simons QFT, by means of proper flux quantization in 2-Cohomotopy [1, 2, 3]:



Here it is the classical *Pontrjagin theorem* (refined by Segal '73 and, underappreciatedly, by Okuyama '05) — identifying n-Cohomotopy cocycles of a manifold with *normally framed* codim$=n$ submanifolds — which makes emerge [4] the *writhe* of *framed Wilson loops* that is traditionally their *ad hoc* renormalization choice:

$$\left\{\begin{matrix}\text{Quantum observables}\\ \text{on 2-cohomotopical}\\ \text{solitonic flux}\end{matrix}\right\} \xrightarrow[\text{[1, 2, 3]}]{\sim} \mathrm{Map}^*\big(\mathbb{R}^3_{\cup\{\infty\}}, S^2\big) \xrightarrow[\text{[4, §2]}]{\sim} \left\{\begin{matrix}\text{Framed}\\ \text{oriented}\\ \text{links}\end{matrix}\right\} \xrightarrow[\text{[4, §3]}]{e^{\frac{\pi i}{K}\mathrm{wrth}(-)}} \mathbb{C}\,.$$

This result is a modest cousin, with applications to topological quantum materials, of a completion of 11D supergravity by proper flux-quantization in 4-Cohomotopy ("Hypothesis H"). Details are at: `ncatlab.org/schreiber/show/MFO2539b`.

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Composite observables and nonperturbative operator product expansion via stochastic analysis

Peter Paulovics
(joint work with M. Gubinelli)

Recent activity in the rigorous study of Euclidean QFTs via stochastic analysis techniques has been successful in the construction and description of many aspects of basic fields [1]. However, nontrivial composite observables in this framework are still poorly understood, e.g. it has been shown [2] that even the Wick square of the free field cannot exist as an $\mathcal{S}'$-valued random variable in $d = 4$. We propose an approach (see also [3]), that circumvents these issues and also allows for a detailed study of nonperturbative properties of observables of an EQFT.

Let $\varphi = (\varphi_t)_{t\in\mathbb{R}_+}$ be a scale-by-scale coupling between the free field and the target EQFT φ_∞; the measures $\mathrm{Law}(\varphi_t)$ satisfy the Wilsonian exact RG equation. We think of φ as an $\mathcal{H}$-valued Markov diffusion in its canonical filtration $\mathcal{F}$ with parabolic generator $\mathcal{L}^V$, where $\mathcal{H}$ is a function space of smooth field configurations. Observables, identified with their effective description, are defined to be $\mathcal{F}$-martingales. For a systematic construction and good control of an observable $\mathfrak{Q}$, one looks for so-called germs $q : \mathbb{R}_+ \times \mathcal{H} \to \mathbb{R}$, i.e. approximate solutions of the PDE $\mathcal{L}^V q = 0$ for which $t \mapsto \mathbb{E}[|q_t(\varphi_t)|] \in L^1(\mathbb{R}_+)$. Then by Ito formula q uniquely determines a remainder process r^q such that $\mathfrak{Q}_t = q_t(\varphi_t) + r^q_t$. Germs and bounds on them are obtained via a Polchinski-like solution theory of $\mathcal{L}^V q = 0$.

For the massive sine–Gordon model on the plane for $\beta^2 < 6\pi$, the coupling φ has been constructed in [4]. Using this, the above procedure yields the construction of various observables. Moreover, by finding an exact relation between the germs and controlling the remainders, we can prove the scale-by-scale OPE

$$\mathfrak{Q}^{(a)}(x+y)\mathfrak{Q}^{(b)}(x-y) = \sum_k \mathcal{C}^{(k)}(y)\mathfrak{Q}^{(k)}(x) + \mathfrak{R}$$

where $\mathfrak{Q}^{(a)}, \mathfrak{Q}^{(b)}, \mathfrak{Q}^{(k)}$ are certain local observables (e.g. their germs are polynomials of point evaluations of the field φ), $\mathcal{C}^{(k)}$ are deterministic and scale-independent distributions, and $\mathfrak{R}$ is martingale vanishing as $y \to 0$ in a “spectator” topology. Open questions include extending perturbative OPE results of Hollands et al, and the study of Ward identities and Coleman correspondence in our setting.

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Introduction to q-Mezdons

Martin Peev

q-Gaussian processes, first introduced by Bożejko and Speicher, form an interesting family of noncommutative algebras interpolating between Fermions and Bosons, including Free Probability. In this talk, we will give a brief overview of their definition and their ultracontractivity property. Using this, we will show how to equip the q-Gaussian processes with the novel q-Mezdonic topology and how to derive operator insertion estimates for multilinear functionals of q-Gaussian noises. These are crucial in solving Mezdonic singular SPDEs.

Geometric and analytic properties of the renormalisation group in perturbative Algebraic Quantum Field Theory

Fabrizio Zanello

The most rigorous approach to perturbative quantum field theory is represented by perturbative Algebraic Quantum Field Theory (pAQFT). Beside being fully equivalent to more traditional approaches when considering Minkowski spacetime, pAQFT is naturally well-suited for a formulation also on globally hyperbolic spacetimes and Riemannian manifolds. The renormalisation picture is directly inspired by the Epstein-Glaser renormalisation scheme [7], later generalised to globally hyperbolic spacetimes [1, 8, 9] and Riemannian manifolds [10, 5].

Epstein and Glaser's original breakthrough consisted in the identification of two crucial aspects. The first one is a *factorisation property* which allows to reduce the construction of the scattering matrix to an inductive procedure over the order n of the time-ordered product T_n. Exploiting the fact that the time-ordered product can be directly defined for generic observables when their supports are (causally) disjoint and assuming that the time-ordered products T_k, $\forall k < n$, have already been constructed, it is possible to show that the time-ordered product of order n is then uniquely determined as a distribution defined on $M^{\times n} \setminus \Delta_n$, where M denotes spacetime and $\Delta_n = \{(x, \ldots, x) \,|\, x \in M\}$ is the thin diagonal. The second crucial aspect is the reformulation of the renormalisation problem as the problem of finding *extensions of distributions*, which can then be solved employing microlocal analysis techniques. In fact, to conclude the inductive step at order n, it is sufficient to find an adequate extension to the whole space $M^{\times n}$ of the time-ordered product T_n. The fundamental criterion to select adequate extensions is given by the notion of scaling degree [2, 6]. In particular, adequate extensions are required to preserve the scaling degree of the time-ordered product.

The extension process is in general ambiguous and the freedom in the choice of an extension is reflected exactly in the freedom in the choice of local counter terms. All the possible renormalisation choices are related to each other by the Stückelberg-Petermann renormalisation group, denoted by $\mathcal{G}_{SP}$.

Some relevant aspects of the theory still remain to be clarified. This is the case of the characterisation of the geometric and analytic properties of the Stückelberg-Petermann renormalisation group. It is well-known that $\mathcal{G}_{SP}$ is a subgroup of the group of analytic maps on the space of local observables. As a consequence, the topology of the space of local observables [3] naturally induces a topology on the space of analytic maps on it. The main difficulty is in understanding the role of the various renormalisation conditions and of the additional constraints imposed by the presence of symmetries in the determination of the topological (or, possibly, smooth) structure of $\mathcal{G}_{SP}$.

I plan to resolve this conundrum by combining the most refined results on the geometric and topological properties of the algebras of observables of pAQFT [3] with the most recent and detailed accounts of the Epstein-Glaser renormalisation framework, both in the Euclidean [10, 5] and the Lorentzian [4] settings.

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Higher gauge theory

Branislav Jurčo

(joint work with Leron Borsten, Hyungrok Kim, Mehran Jalali Farahani, Jiří Nárožný, Christian Saemann, Martin Wolf)

This talk was about an ongoing collaboration [1] which aims, from the first principles, to exhaustively describe higher principal bundles over higher spaces whose gauge symmetries are encoded in higher Lie groupoids, including their differential geometry. The most general accessible geometric model for both higher spaces and higher groupoids seems to be that of Kan simplicial manifolds. This framework is very general, and it contains, for example, the smooth 2-groups employed in the string 2-group. Moreover, it leads for a straightforward approach to the computation of all the kinematical ingredients to corresponding higher gauge theories given by higher Deligne cocycles and coboundaries. In particular, it allows to describe higher gauge potentials, their higher curvatures, finite gauge transformations, and the globalisation of this data by gluing. This work is a continuation and a significant extension of [2].

The approach is based on the non-abelian version of Dold-Kan correspondence between simplicial groups and strict higher groups (hypercrossed complexes) [3]. This leads to a correspondence between simplicial principal bundles (twisted Cartesian products) and principal bundles with higher structure groups, the former one being very-well understood, see e.g., [4].

Starting from the simplicial description, one can naturally develop the theory of corresponding Atiyah groupoids and their bisection groups (inner automorphisms) of twisted Cartesian products. In the smooth case, this approach leads to a description of the corresponding simplicial Atiyah algebroids in therms of simplicial Lie-Rinehart pairs. Finally, using the Quillen's correspondence [5] between simplicial Lie algebras and differential graded Lie algebras it gives a natural way to define higher connections as splitting of the corresponding Atiyah sequences.

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Towards a Noncommutative Marcinkiewicz Theorem

Matteo Ravot Licheri

(joint work with I. Chevyrev, P. K. Friz, F. Medwed, S. Paycha)

The moments of a random variable X on a finite-dimensional vector space V (if they exist) are encoded by $\mathbb{E}[\exp_\otimes(X)]$ in the tensor series algebra $T((V)) = \varprojlim T^{(n)}V = \varprojlim \bigoplus_{k=0}^{n} V^{\otimes k}$. $T((V))$ is naturally a complete Hopf algebra, hence possesses a well-defined logarithm function, which makes sense of the notion of cumulants as $\log_\otimes \mathbb{E}[\exp_\otimes(X)]$. As the grouplike elements of the symmetric tensor series algebra $S((V))$ are all of the form $\exp_\otimes(v)$ for some $v \in V$, Marcinkiewicz's theorem characterizes all possible cumulants of measures on the grouplike elements $\mathcal{G}(S((V)))$ as being (possibly degenerate) positive-semidefinite polynomials of degree 2.

Our goal is to characterise elements of the form $\log_\otimes \mathbb{E}[X]$, where X is a random variable on $\mathcal{G}(T((V)))$ (the grouplike elements of $T((V))$). If $\log \mathbb{E}[X] \in T(V)$, the cumulants of grouplike-valued random variables are identified with differential operators on the grouplike elements. It is well-known that, in the case when they correspond to positive-semidefinite semielliptic operators of degree ≤ 2, they are identified with signature cumulants of Brownian paths with drift. Connections to semigroup-theory and PDE are explored to present current work in progress to prove that those positive-semidefinite semielliptic operators of degree ≤ 2 are the only differential operators on $\mathcal{G}(T((V)))$ which can arise as cumulants of $\mathcal{G}(T((V)))$-valued random variables.

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A geometric perspective on the signature group

Felix Medwed

(joint work with P. K. Friz, S. Paycha, M. Ravot Licheri, A. Schmeding)

The signature S of a weakly geometric p-rough path [4] over $\mathbb{R}^n$, and throughout assumed to be continuous, is a group-like element in the extended tensor algebra and determines the path up to tree-like equivalence $\sim_\tau$, reparametrisation and fixing a starting point. First shown for $p = 1$ (bounded variation) in [3] and later generalised to the $p > 1$ case in [5], the image under the signature of equivalence classes $[X]_\tau$ (tree reduced paths), under start point fixing at the identity, form

a proper subgroup G^p of the group-like elements, the *p-signature group*. Chen's relation can then be understood as the group homomorphism property of S. Using Lyons' lift $\mathcal{L}$ [2, 6] of weakly geometric rough paths, the signature can be written as $S = \mathrm{ev}_1 |_{\mathrm{imag}_{\mathcal{L}}} \circ \mathcal{L}$, where $\mathcal{L}([X]_\tau)$ corresponds to a unique injective path, where the set of all such injective paths with starting point at the identity shall be $\mathcal{I}$. For convenience, we suppose that ev_1 is restricted to the image of $\mathcal{L}$, ref. [5].
In the $p = 1$ case, it is known [4], that the truncations up to level k of the image of the signature give the full free nilpotent Lie group $G_k(\mathbb{R}^n)$ of step k and rank n, which carries the natural notion of a Carnot group [10] under the Carnot-Carathéodory (CC) metric. While the lift of a bounded variation path to any $G_k(\mathbb{R}^n)$ is independent of the choice of (homogeneous) metric [4], this property breaks on G^1 and it was shown in [7], that under the CC-metric the signature can be written as a projective limit in the category of pointed metric groups with point-preserving submetries $\mathsf{MetGrp}_{\mathsf{Submet},\star}$, where the limiting topology on G^1 is that of an $\mathbb{R}$-tree, which fails to make G^1 give rise to a topological group. In [8] the question is raised if one can save this construction and find a desireable smooth structure making G^1 into a Lie group. Another question is the generalisation of the result shown in [7] to the p-variation case for $p > 1$ building a bridge to the results in [5]. As a first step towards the answers, we show that the connection between [3] and [7] is captured by the following diagram

$$\begin{array}{ccc}
(WG\Omega_p/\sim_\tau, \delta_p, o) & \xrightarrow{\mathcal{L},\sim} & (\mathcal{I}, d_p, O) \\
\Big\downarrow{\scriptstyle \sim} & \overset{S,\sim}{\searrow} & \Big\downarrow{\scriptstyle \mathrm{ev}_1,\sim} \\
(G^p_\infty, d_{p,\infty}, 1_\infty) & \xrightarrow{\Pi_p,\sim} & (G^p, D_p, 1)
\end{array}$$

in the $p = 1$ case and choosing the CC-metric, which is part of the categories groupoid. This gives a straight-up generalisation of the results of [7] for $p > 1$ measuring p-variation via the metric

$$\rho_k(g, h) = \max_{1 \le j \le k} ((k/p)! || \mathrm{proj}_j(g^{-1}h)||)^{p/k}$$

leading not only to corresponding metrics on the space of tree reduced paths, but – to the best of the authors knowledge – to an open classification problem of the metrics as well as the question if all levelwise metric constructions will ultimately end up in an $\mathbb{R}$-tree topology, fully characterised in [1].

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Relativistic Luttinger Fermions as models for rigorous QFT in four dimensions

Hannes Keppler

(joint work with Razvan Gurau, Louis Jussios, Manfred Salmhofer)

One way to construct well-defined interacting Euclidean quantum field theories (QFT) goes through the Wilson–Polchinski renormalization group. Within this framework, a crucial step is the removal of the ultraviolet (UV) cutoff. This requires a theory to be well-defined up to arbitrary high energy scales/small length scales. This can be achieved if the theory becomes effectively non-interacting at high energy scales. This is called asymptotic freedom in the UV. A prime example for this behavior is Yang–Mills theory or quantum chromodynamics, which nevertheless has not yet been rigorously constructed in four dimensions. In contrast, in simple ϕ_4^4 scalar field theory the coupling grows towards higher energies and the UV cutoff can not be removed without rendering the theory trivial (non-interacting).

It is therefore of great interest to find other examples of four-dimensional asymptotically free QFTs. In this talk, I presented such an example: theories of relativistic Luttinger Fermions; and advertised them as a playground for other approaches to rigorous QFT. These Fermions were first introduced by J. M. Luttinger [7] as non-relativistic effective degrees of freedom in solid-state physics. Very recently, the authors of [4] introduced Luttinger Fermions into high-energy physics by constructing a four dimensional relativistic action and demonstrating asymptotic freedom by calculating the β functions up to second order. In a mean field approximation, [5] showed that these systems can have a symmetry-broken phase at low energies and non-perturbatively generate a mass—this makes these theories remarkably similar to the two-dimensional Gross–Neveu model that has been rigorously studied in, e.g., [3, 2, 6]. Our work aims to rigorously establish renormalizability and asymptotic freedom in these models using an integrated version of Polchinski's flow equation, as presented in the talk by M. Salmhofer. Being a Fermionic field theory, perturbation theory converges if UV and infrared cutoffs are in place [8].

Fields and action. Let $\bar{\psi}(x)$, $\psi(x)$ be anticommuting Grassmann fields and $x \in \mathbb{R}^4$. Each $\psi(x)$ is a vector (spinor) with 32 components in $d = 4$ space-time dimensions. Let $\mu, \nu \in \{0,1,2,3\}$, the kinetic part of the action of relativistic Luttinger Fermions reads

$$S(\bar{\psi}, \psi) = -\int \bar{\psi}(x) G_{\mu\nu} \partial_\mu \partial_\nu \psi(x)\ d^4x\ ,$$

where an implicit sum over μ, ν is always understood, and, $\forall \mu, \nu$, the $G_{\mu\nu}$ are hermitian 32×32 matrices, called Luttinger matrices. Crucially, the kinetic term is of second order in derivatives of the fields. Therefore, the canonical scaling dimension of ψ is one and agrees with that of a four dimensional scalar field. This means that a quartic interaction term, such as

$$\int \lambda \left(\bar{\psi}\psi\right)^2(x)\ d^4x\ ,$$

is marginal and the theory would be just-renormalizable (critical in the stochastic quantization language).

Luttinger matrices. The $G_{\mu\nu}$ matrices are hermitian, traceless, with

$$G_{\mu\nu} = G_{\nu\mu}\ , \qquad \sum_{\mu=0}^{3} G_{\mu\mu} = 0\ ,$$

and they fulfill the Abrikosov algebra [1]:

$$\{G_{\mu\nu}, G_{\rho\sigma}\} = -\frac{2}{d-1}\delta_{\mu\nu}\delta_{\rho\sigma} + \frac{d}{d-1}(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho})\ .$$

Due to this anti-commutation relation, the second order differential operator $(G_{\mu\nu}\partial_\mu\partial_\nu)^2 = \Delta^2$ squares to the square of the Laplace operator. The $G_{\mu\nu}$ can be built from linear combinations of nine gamma matrices $\gamma_1, \ldots, \gamma_9$, taken from a ten-dimensional Clifford algebra $\{\gamma_a \mid a, b = 1, \ldots, 10,\ \{\gamma_a, \gamma_b\} = \delta_{ab}\}$. The remaining matrices γ_{10}, $\gamma_{11} := i\gamma_1 \cdots \gamma_{10}$, and $\gamma := i\gamma_{10}\gamma_{11}$ anticommute or commute with all $G_{\mu\nu}$ and can be used to construct various symmetry transformations and different quadratic (mass) terms in the action, such as

$$\int im^2 \left(\bar{\psi}\gamma\psi\right)(x)\ d^4x\ .$$

Using this mass term, the propagator of the model reads in Fourier space

$$\hat{C}(p) = \frac{G_{\mu\nu}p_\mu p_\nu - i\gamma m^2}{p^4 + m^4}\ ,$$

or for $m = 0$ in position space

$$C(x) = -\frac{1}{(2\pi)^2}\frac{G_{\mu\nu}x_\mu x_\nu}{x^4}\ .$$

Renormalization and asymptotic freedom. We introduce a sequence of scales $\Lambda_J < \Lambda_{J-1} < \cdots < \Lambda_1 < \Lambda_0$, and for $j \in \{1, 2, \ldots, J\}$ the regularized propagator

$$\hat{C}_j(p) = \hat{C}(p) \left(e^{-\frac{p^2+m^2}{\Lambda_j^2}} - e^{-\frac{p^2+m^2}{\Lambda_{j-1}^2}} \right) .$$

The effective interactions $W_j(\bar{\psi}, \psi)$ are computed by iterated Gaussian convolution:

$$e^{-W_{j+1}(\bar{\psi},\psi)} = \int d\mu_{C_j}(\bar{\Phi}, \Phi)\, e^{-W_{j+1}(\bar{\psi}+\bar{\Phi},\psi+\Phi)} ,$$

where $d\mu_{C_j}$ denotes the normalized Gaussian measure with covariance C_j. The renormalization procedure amounts to adjusting $W_0(\bar{\psi}, \psi)$, such that the UV limit $\Lambda_0, J \to \infty$ exists.

To study the flow of the four-point vertex, we set

$$W_j(\bar{\psi}, \psi) = \int \left[\lambda_{0,j} \left(\bar{\psi} \gamma \psi \right)^2 (x) + \lambda_{t,j} \left(\bar{\psi} G_{\mu\nu} \psi \right)^2 (x) \right] d^4x + R_j(\bar{\psi}, \psi, \nabla\bar{\psi}, \nabla\psi) ,$$

with two scale dependent coupling constants $\lambda_{0,j}$ and $\lambda_{t,j}$, and a rest term R_j, that vanishes at the initial conditions $j = 0$. Controlling the rest term, one can then show that the resulting discrete dynamical system has non-trivial solutions with $|\lambda_{0,0}|, |\lambda_{t,0}| \to 0$, as $\Lambda_0, J \to \infty$. This means that the system is indeed asymptotically free in the UV and the theory is a natural candidate for an interacting four-dimensional QFT.

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Renormalisation group flow approach to singular stochastic PDEs

Leonard Ferdinand
(joint work with Ajay Chandra)

The flow approach to singular parabolic stochastic PDEs is a strategy to develop a solution theory for a certain class of equations that are classically ill-posed due to the roughness of their driving term. It was first introduced by Kupiainen [7], in the same years in which regularity structures [6] and paracontrolled calculus [5] were developed. Common to these three approaches is the idea that the rough driver alone is not sufficient to describe the solution and that, in order to construct the solution, it is necessary to enrich the driver by a finite collection of its polynomials. In the flow approach, these polynomials are built up using some ideas coming from the renormalisation group.

Singular parabolic equations regularised at a scale ε can be recast in mild formulation under the form

$$u_\varepsilon = GF_\varepsilon[u_\varepsilon]\,, \tag{1}$$

where G is the solution operator to the linear term, a parabolic operator, and F denotes the non-linearity, including the regularised rough driver, alongside all the terms coming from boundary conditions.

The aim of the flow approach is to show that while the RHS of (1) diverges as $\varepsilon \downarrow 0$, the RHS $F_{\varepsilon,\mu}$ of the equation solved by the regularised version of the approximate solution $u_{\varepsilon,\mu} := \rho_\mu u_\varepsilon$ (here ρ_μ is a smooth approximation of a Dirac distribution at scale $\mu > \varepsilon$) should converge as $\varepsilon \downarrow 0$ for any $\mu > 0$.

In [3, 4], Duch constructed $F_{\varepsilon,\mu}$ inductively, using a hierarchy of differential equations in which the flow parameter μ is seen as a continuous parameter. In these works, he deals with equations with polynomial nonlinearities. With Ajay Chandra [1, 2], we generalised this approach to equations with non-polynomial non-linearities, such as the generalised KPZ equation.

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Renormalising Feynman diagrams with multi-indices

Yvain Bruned
(joint work with Yingtong Hou)

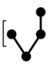
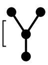
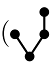
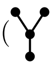
We start by considering the scalar-valued ordinary differential equation

$$y'(t) = f(y), \quad y(0) = y_0 \in \mathbb{R}.$$

One way to describe numerical schemes for this equation is to use B-series coming from [10]

$$B(a, h, f, y_0) = a(\emptyset)y_0 + \sum_{\tau \in \mathbf{T}} \frac{h^{|\tau|}a(\tau)}{S(\tau)} F_f[\tau](y_0)$$

where $\mathbf{T}$ are rooted trees, $\emptyset$ is the empty tree, $S(\cdot)$ is the symmetry factor, $|\tau|$ is the number of nodes in τ, h is the step-size of the numerical method, $a(\cdot)$ are its coefficients and $F_f[\cdot]$ are the so-called elementary differentials. One observes that certain trees have the same elementary differentials. As an example, one has

$F_f[$  $] = F_f[$  $] = f^2 f^{(1)} f^{(2)}, \quad \pi($  $) = \pi($  $) = z_0^2 z_1 z_2.$

Each node corresponds to a f and the number of derivatives is the number of entering edges on that node in the rooted tree. One needs only to have the arity of each node for computing the elementary differentials. One can define another combinatorial set called multi-indices which are monomials of the form $\prod_{k\in\mathbb{N}} z_k^{\beta(k)}$ such that z_k is a node of arity k. One has a surjection π from the trees to a certain class of multi-indices. Multi-indices first appeared for quasi-linear equations in singular SPDEs [17] with the idea of giving a more appropriate combinatorial description of the local expansion of the solution, replacing decorated trees [12, 6, 2] arising from Regularity Structures. Multi-indices are also very natural in numerical analysis [16, 3]. Recently, it has been shown in [8] that one cannot expect via a naive approach to get an intermediate combinatorial set between rooted trees and multi-indices. We want to see the equivalent of multi-indices in the context of renormalisation for Quantum Field Theory. Let Λ be the d-dimensional torus, we consider

$$\mu_0 = \frac{1}{Z(0)} e^{-\int_\Lambda \frac{1}{2}\|\nabla\phi(x)\|^2 + \frac{1}{2}:\phi(x)^2:dx} d\phi, \quad Z(\alpha) = Z(0)\mathbb{E}\left[e^{-\int_\Lambda \sum_{k\in\mathbb{N}} \alpha_k :\phi(x)^k:dx}\right].$$

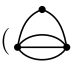
where $:\phi(x)^k:$ is the Wick product, $Z(\alpha)$ is the partition function and the expectation $\mathbb{E}$ is taken with respect to the measure μ_0. One seeks values of γ_k such that the following diagram commutes

$$\begin{array}{ccccc} \mathbb{E}\left[e^{-\int_\Lambda \sum_{k\in\mathbb{N}} \alpha_k :\phi(x)^k:dx}\right] & \longrightarrow & \sum_{z^\beta\in\mathbf{M}} \frac{\Upsilon^\alpha_{\mathrm{M}}(z^\beta)}{\hat{S}_{\mathrm{M}}(z^\beta)} \Pi_{\mathrm{M}}(z^\beta) & \longrightarrow & \sum_{z^\beta\in\mathbf{M}} \frac{\Upsilon^\alpha_{\mathrm{M}}(z^\beta)}{\hat{S}_{\mathrm{M}}(z^\beta)} \Pi_{\mathrm{M}}(\hat{M}_{\mathrm{M}} z^\beta) \\ \downarrow & & & \nearrow & \\ \mathbb{E}\left[e^{-\int_\Lambda \sum_{k\in\mathbb{N}} (\alpha_k+\gamma_k) :\phi(x)^k:dx}\right] & & & & \end{array}$$

Here, this diagram is written with multi-indices $\mathbf{M}$ that we will introduce later. When one moves from the expectation to a series on multi-indices $\mathbf{M}$, one performs a cumulant expansion. The map $\hat{M}_{\text{M}}$ implements the BPHZ renormalisation [9, 14, 19] on the multi-indices. We suppose given some abstract variables $(z_k)_{k\in\mathbb{N}}$, one sets

$$\Phi(\Gamma) := \prod_{v\in\mathcal{V}(\Gamma)} z_{k(v)}$$

where $k(v)$ is the number of edges attached to the vertex v and $\mathcal{V}(\Gamma)$ is the vertex set of the Feynman diagram Γ. One has

$\Phi($  $) = z_2 z_4^2.$

We define a multi-indice $z^\beta \in \mathbf{M}$ with $z^\beta \in \text{Im}(\Phi)$ and its symmetry factor as

$$z^\beta := \prod_{k\in\mathbb{N}} z_k^{\beta(k)}, \quad S_{\text{M}}(z^\beta) := \prod_{k\in\mathbb{N}} \beta(k)!(k!)^{\beta(k)}.$$

These multi-indices appear in the literature under the name of pre-Feynman diagrams [1] and it is a generalisation of the Hopf algebraic approach coming from [7]. We denote by $\mathbf{F}$ the set of Feynman diagrams.

Proposition 1 ([5]). *For a Feynman diagram $\Gamma \in \mathbf{F}$, one has*

$$S_{\text{M}}(\Phi(\Gamma)) = N(\Gamma) S_{\text{F}}(\Gamma),$$

where $N(\Gamma)$ is the number of distinct pairings of half-edges in $\Phi(\Gamma)$ that can form Feynman diagrams isomorphic to Γ and $S_{\text{F}}(\Gamma)$ is the symmetry factor of Γ.

Then, one defines the injective map $\mathcal{P} : \mathbf{M} \to \langle \mathbf{F} \rangle$

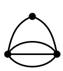
$$\mathcal{P}(z^\beta) := \sum_{\Gamma:\Phi(\Gamma)=z^\beta} \frac{S_{\text{M}}(z^\beta)}{S_{\text{F}}(\Gamma)} \Gamma.$$

In the next theorem, one connects the reduced extraction-contraction coproduct Δ_{F} on Feynman diagrams introduced in [11] with the reduced extraction-contraction coproduct Δ_{M} on multi-indices.

Theorem 1 ([5]). *For any multi-indice $z^\beta \in \mathbf{M}$*

$$(\mathcal{P} \otimes \mathcal{P}) \circ \Delta_{\text{M}}(z^\beta) = \Delta_{\text{F}} \circ \mathcal{P}(z^\beta).$$

The valuation of a Feynman diagram is given as

$\Pi_{\text{F}}($  $) = \int_{\Lambda^3} K(x_1 - x_2) K(x_2 - x_3)^3 K(x_3 - x_1) dx_1 dx_2 dx_3$

where $K(x-y) = \mathbb{E}(\Phi(x)\Phi(y))$. For multi-indices, one does not have an explicit formula but has to proceed recursively

$$\begin{aligned}\Pi_{\text{M}}(\prod_{k\in\mathbb{N}} z_k^{\beta(k)}) :=& \mathbb{E}\left[\prod_{k\in\mathbb{N}}\left(\int_\Lambda :\Phi(x)^k : dx\right)^{\beta(k)}\right] \\ &- \sum_{n\geq 2}\sum_{\sum_{i=1}^n \beta_i=\beta}\prod_{k\in\mathbb{N}}\binom{\beta(k)}{\beta_1(k),\ldots,\beta_n(k)}\prod_{i=1}^n \Pi_{\text{M}}(z^{\beta_i}).\end{aligned}$$

With the previous definitions, one can define the BPHZ renormalisation map on both combinatorial sets:

$$\hat{M}_{\text{F}} := (\Pi_{\text{F}}(\mathcal{A}_{\text{F}}(\cdot))\otimes \mathrm{id})\,\Delta^-_{\text{F}}, \quad \hat{M}_{\text{M}} := (\Pi_{\text{M}}(\mathcal{A}_{\text{M}}(\cdot))\otimes \mathrm{id})\,\Delta^-_{\text{M}}$$

where the $\Delta^-_{\text{F}}, \Delta^-_{\text{M}}$ are the extraction-contraction coproducts and $\mathcal{A}_{\text{F}}, \mathcal{A}_{\text{M}}$ are twisted antipodes. This Hopf algebraic approach with a twisted antipode is inspired from [11, 13]. The two renormalisations are connected in the next theorem

Theorem 2 ([5]). *The two renormalisations agree*

$$\Pi_{\text{M}} \circ \hat{M}_{\text{M}} = \Pi_{\text{F}} \circ \hat{M}_{\text{F}} \circ \mathcal{P}.$$

The cumulant expansion of $Z(\alpha)/Z(0)$ admits the following Feynman diagram representation

$$\log \mathbb{E}\left[e^{-\int_\Lambda \sum_{k\in\mathbb{N}}\alpha_k :\Phi(x)^k: dx}\right] = \sum_{\Gamma\in\mathbf{F}} \frac{\Upsilon^\alpha_{\text{F}}(\Gamma)N(\Gamma)}{\hat{S}_{\text{F}}(\Gamma)}\Pi_{\text{F}}(\Gamma)$$

where one has explicit expressions for the coefficients $\Upsilon^\alpha_{\text{F}}(\Gamma)$ and $\hat{S}_{\text{F}}(\Gamma)$. Then, one can rewrite this series using multi-indices and gets

Theorem 3 ([5]). *The diagram described above commutes and one has explicit formulae for the γ_k.*

We finish with some perspectives:

- One can define a general extraction-contraction coproduct on multi-indices suitable for encoding higher order renormalisations. It has been performed for SPDEs in [4].
- One may want to understand symmetries with this formalism like Ward identities in [18].
- One can hope to get new ideas for the convergence of the renormalisation like what happened in [15] for singular SPDEs.

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Random Renormalisation of Singular SPDEs with Correlated Coefficients

Harprit Singh
(joint work with L. Broux, N. Clozeau, R. Steele)

The area of stochastic partial differential equations has seen rapid progress over the past decade, spurred by the introduction of the theory of regularity structures [Hai14] and of para-controlled calculus [GIP15]. Despite the close connections of singular SPDEs to physical phenomena, the theory of singular SPDEs has to date been developed primarily in homogeneous settings, involving constant-coefficient operators.

First I shall motivate the study of SPDEs in heterogeneous environments using the examples of Φ^4 as a model of ferromagnetism and the parabolic Anderson model as it relates to branching processes. Then I shall discuss the renormalisation of the g-PAM equation on $\mathbb{T}^2 \times \mathbb{R}$, formally given by

$$\partial_t u - \nabla \cdot a \nabla u = \sum_{i,j=1}^{2} f_{i,j}(u) \partial_i u \partial_j u + \sigma(u)\xi \tag{1}$$

where ξ is a spatial white noise as well as the Φ^4_d equation on $\mathbb{T}^d \times \mathbb{R}$ for $d = 2, 3$ formally given by

$$\partial_t u - \nabla \cdot a \nabla u = -u^3 + \xi \ , \tag{2}$$

where ξ is a space-time white noise. The remainder of the talk will be divided into the following parts.

(1) I shall recall the renormalisation of these equations in the by now classical setting of [Hai14] where the matrix a is constant and positive definite.
(2) I shall explain the necessary changes needed to go to sufficiently regular, positive definite coefficient fields $a = A(x, t)$ in (1)&(2) following [Sin25]. Here I shall explain a particularly natural choice of renormalisation functions, which is local and, for sufficiently covariant regularisations of the noise, very explicit. I shall also mention the more recently established general convergence results of [BSS25] which are applicable to a very general class of subcritical variable coefficient singular SPDEs.
(3) Finally, I shall discuss results of [CS25] which apply to the case when the coefficient field is itself random and correlated to the driving noise, i.e. $a(x, t) = A(g(x, t))$ for $g = \sigma * \xi + \mu$ for $\sigma, \mu \in C^2_c(\mathbb{T}^d \times \mathbb{R})$ and $A : \mathbb{R} \to \mathbb{R}^{d\times d}$ bounded, positive definite and smooth.

In the third setting the renormalisation counterterms have themselves to be chosen random – hence the title of the talk.

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What Fractional Brownian Motion Can Teach Us About Renormalization

Peter K. Friz

Fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ is a centered Gaussian process with stationary increments and covariance

$$\mathbb{E}[B_t B_s] = \tfrac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big).$$

It is almost surely α–Hölder continuous for all $\alpha < H$ and can be realized as $K \star \xi$, Volterra integral against Gaussian white noise ξ. One can give (Itô integral) meaning to $f(B)\xi$, while Wong-Zakai type approximations $f(B^\varepsilon)\xi^\varepsilon$ in general requires Wick-type renormalization, effectively treated in (and a model case example for) Hairer's regularity structures, discussed in detailed in [8]. Ramifications of this example are connected to a major development in quantitative finance, dubbed rough volatility, e.g. [3, 2, 1] and references therein.

In a classical multivariate setting,with IID fBm components, for $H > \frac{1}{4}$, results of Coutin–Qian [6] and subsequent refinements show that smooth mollifications B^ε admit a canonical rough path lift which converges, yielding a fractional Brownian rough path. This allows one to define stochastic integration and differential equations driven by fBm pathwise, via the rough paths framework.

For $H > \frac{1}{3}$, level–2 rough paths suffice and one obtains well–posed rough differential equations

$$dx_t = \sum_{i=1}^{d} V_i(x_t) \circ dB^i_t,$$

together with a rich stochastic analysis, including Hörmander–type density results, support theorems, and large deviation and Laplace principles, much of which can be found in [7].

The rough path situation changes drastically as $H \downarrow \frac{1}{4}$. While level–1 objects remain well defined, higher iterated integrals develop diverging variances. After earlier works, including Nualart-Tindel [10], and in particular a series of papers by Unterberger (see [11] and references therein), Hairer [9] shows that after introducing suitable ε–dependent rescalings of the driving signal, solutions of ODEs driven by mollified fBm converge in law to a genuinely new limiting object. In the critical and subcritical regimes $H \leq \frac{1}{4}$, the limit is no longer driven by fBm itself but by a diffusion generated by Lie brackets of the vector fields.

The proof strategy relies on the weak convergence of high–order (up to level–4) rough path lifts and delicate moment estimates for iterated integrals of fBm. An intriguing open perspective is to reinterpret these results in terms of expected signatures, which by [5] characterize laws of rough paths.

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