Mathematisches Forschungsinstitut Oberwolfach

Report No. 30/2025

DOI: 10.4171/OWR/2025/30

Differentialgeometrie im Großen

Organized by
Richard Bamler, Berkeley
Otis Chodosh, Stanford
Urs Lang, Zürich
Lu Wang, New Haven

29 June – 4 July 2025

Abstract. This workshop brought together researchers working in various areas of differential geometry, with an emphasis on problems that connect local geometric analysis with global structure and topology. Talks covered a broad range of topics, including geometric flows, scalar curvature, minimal surfaces, singular spaces, and large-scale geometric phenomena. While the methods and settings varied widely, a recurring theme was the use of analytic tools to understand spaces with curvature or topological constraints, especially in the presence of singularities. The format encouraged informal discussion and exchange of ideas across different subfields.

Mathematics Subject Classification (2020): 53-XX.

Introduction by the Organizers

The workshop "Differentialgeometrie im Großen" brought together a broad group of mathematicians working in various areas of differential geometry. The focus was on geometric questions that link local and global behavior, with topics ranging from geometric flows and scalar curvature to variational problems, singular spaces, and large-scale geometry.

A significant number of talks were devoted to geometric flows, particularly mean curvature flow and related evolution equations. Several presentations addressed ancient solutions and classification of singularities, including work on flows in dimension four and connections. The resolution of the Multiplicity One Conjecture and its consequences for understanding the structure of singularities in mean curvature flow also featured in one of the talks.

Scalar curvature appeared in many forms throughout the week. Talks addressed the existence and structure of 3- and 4-manifolds with positive scalar curvature, rigidity phenomena, and smoothing of singular metrics. Inverse mean curvature flow was used in one talk as a tool to study scalar curvature and geometric inequalities on 3-manifolds.

Minimal surfaces and variational methods formed another core area of the workshop. Topics included the construction of minimal surfaces of prescribed genus in 3-manifolds with positive Ricci curvature, regularity of capillary hypersurfaces, and generic regularity results for minimizing hypersurfaces in dimension 11. Another talk discussed the fine structure of two-dimensional area-minimizing currents near branch points. Related work explored energy-minimizing harmonic spheres in singular metric spaces.

One talk focused on the large-scale geometry of complete manifolds with nonnegative Ricci curvature and Euclidean volume growth, combining tools from analysis and metric geometry. Other contributions dealt with width-type invariants and p-sweepouts in the sense of Gromov, Einstein manifolds, isoperimetric gaps in nonpositive curvature, and aspects of special holonomy, including constructions of G_2 and Calabi–Yau monopoles and the use of geometric flows in G_2-geometry.

The program included 21 talks in total. Most were standard research talks of around 50 minutes, and three shorter talks of about 30 minutes were given by junior participants. The schedule left room for informal discussions and collaboration throughout the week. As usual, the Wednesday afternoon hike was planned, but due to an ongoing heat wave, only a small and hardy group of participants took part.

In summary, the workshop covered a broad and lively range of topics. While the techniques and settings varied widely, many talks reflected a shared interest in how analytic and geometric tools can be used to study spaces with rich structure, especially in connection with singularities, curvature, and topology. We hope the discussions and connections from this week will continue to develop into future work.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-2230648, “US Junior Oberwolfach Fellows”.

Workshop: Differentialgeometrie im Großen

Table of Contents

Abstracts

Smoothing L^∞ Riemannian metrics with nonnegative scalar curvature outside of a singular set

Paula Burkhardt-Guim

We show that any L^∞ Riemannian metric g on $\mathbb{R}^n$ that is smooth with nonnegative scalar curvature away from a singular set of finite $(n-\alpha)$-dimensional Minkowski content, for some $\alpha > 2$, admits an approximation by smooth Riemannian metrics with nonnegative scalar curvature, provided that g is sufficiently close in L^∞ to the Euclidean metric. The approximation is given by time slices of the Ricci-DeTurck flow, which converge locally in C^∞ to g away from the singular set. We also identify conditions under which a smooth Ricci-DeTurck flow starting from a L^∞ metric that is uniformly bilipschitz to Euclidean space and smooth with nonnegative scalar curvature away from a finite set of points must have nonnegative scalar curvature for positive times. The work described here is carried out in [2].

For a Riemannian metric g, let $R(g)$ denote the scalar curvature of g. Let δ denote the Euclidean metric on $\mathbb{R}^n$. We show the following:

Theorem 1. *For all $\alpha > 2$ and $n \geq 3$ there exists $\bar{\varepsilon}(\alpha, n)$ such that the following is true:*

Suppose that g is a measurable metric on $\mathbb{R}^n$ such that $||g-\delta||_{L^\infty(\mathbb{R}^n)} < \bar{\varepsilon}$, where $||\cdot||_{L^\infty(\mathbb{R}^n)}$ is measured with respect to δ. Suppose that g is smooth on $\mathbb{R}^n \setminus \mathcal{S}$, where $\mathcal{S} \subset \mathbb{R}^n$ is a set of finite $(n-\alpha)$-dimensional Minkowski content, and that $R(g) \geq 0$ on $\mathbb{R}^n \setminus \mathcal{S}$. Then there exists a smooth Ricci-DeTurck flow $(g_t)_{t\in(0,\infty)}$ with respect to the background metric δ such that

$$R(g_t) \geq 0 \text{ for all } t > 0$$

and

$$g_t \xrightarrow[t\searrow 0]{C^\infty_{loc}(\mathbb{R}^n\setminus\mathcal{S})} g.$$

In particular, g admits an approximation in $C^\infty_{loc}(\mathbb{R}^n \setminus \mathcal{S})$ by smooth metrics with nonnegative scalar curvature.

Note that in Theorem 1, g need not be continuous, and that, aside from the Minkowski content requirement, we do not impose geometric conditions on $\mathcal{S}$.

Question 1. *For $n \geq 3$, does there exist a L^∞ metric on $\mathbb{R}^n$ that is uniformly bilipschitz to the Euclidean metric smooth outside of a singular set of finite $(n-2)$-dimensional Minkowski content, for which the conclusion of Theorem 1 fails? Does there exist such a metric for which the $(n-2)$-dimensional Minkowski content of the singular set is 0?*

Question 2. *Suppose that in the setting of Theorem 1, $\mathcal{S}$ has Hausdorff dimension equal to $n-\alpha$ for some $\alpha > 2$, rather than finite $(n-\alpha)$-dimensional Minkowski content. Does the conclusion of Theorem 1 still hold?*

Given that the scalar curvature under Ricci flow is a supersolution to the heat equation, Theorem 1 may seem surprising by analogy: consider a negative Gaussian evolving by the heat equation on $\mathbb{R}^n \times (0, \infty)$, which tends to 0 everywhere except at the origin as $t \searrow 0$. This example demonstrates that the conclusion of Theorem 1 is false when $R(g_t)$ is replaced with a solution to the heat equation with respect to δ. The key differences between this example and the statement of Theorem 1 are that the evolution of the volume form under Ricci flow is also influenced by the scalar curvature and also that there is a positive source term in the evolution equation for the scalar curvature under Ricci flow.

We expect results analogous to Theorem 1 to hold on manifolds for ε perturbations of complete smooth metrics with bounded curvature, in view of [6] and [3]. The condition that the metric g be $(1+\varepsilon)$-bilipschitz to a fixed complete smooth background metric of bounded curvature is used in two ways:

(1) To guarantee the existence of a Ricci-DeTurck flow starting from g, as in the work of [5] or [6], and
(2) To ensure an a priori bound of the form $R(g_t) \geq -c\varepsilon/t$ for some $c > 0$, for all $t > 0$, which in turn is used to derive an upper bound for a backwards heat kernel, as in [1, Theorem 2.3].

Interestingly, the second use seems to be somewhat inessential in the case that the singular set consists of finitely many points:

Theorem 2. *Suppose that $\mathcal{S} \subset \mathbb{R}^n$ has finite 0-dimensional Minkowski content. Suppose that g is a measurable Riemannian metric on $\mathbb{R}^n$ that is smooth on $\mathbb{R}^n \setminus \mathcal{S}$ and satisfies $R(g) \geq 0$ on this region. Suppose that there exists a smooth Ricci-DeTurck flow $(g_t)_{t\in(0,T)}$, defined for some $T > 0$, on $\mathbb{R}^n$ with respect to the background metric δ, satisfying:x*

(1) $g_t \xrightarrow[t\searrow 0]{C^2_{loc}(\mathbb{R}^n\setminus\mathcal{S})} g$,
(2) there exists some $c > 0$ such that for $k = 1, 2$, $|\nabla^k(g_t)|_\delta \leq c/t^{k/2}$, where ∇ is taken with respect to δ,
(3) there exists some $b > 0$ such that g_t is $(1 + b)$-bilipschitz to δ for all $t \in (0, T)$, and
(4) there exists some $0 < c_0 < n/2$ such that for all $t \in (0, T)$, $R(g_t) \geq -c_0/t$.

Then $R(g_t) \geq 0$ for all $t \in (0, T)$.

We note that any Ricci-DeTurck flow $(g_t)_{t\in(0,T)}$ satisfies a universal lower scalar curvature bound of the form given by item (4) with $c_0 = n/2$. Theorem 2 does not address the edge case $c_0 = n/2$.

In a previous draft of this paper that was posted on the arXiv, we posed the following question concerning the sharpness of the $(1 + \varepsilon)$-bilipschitz condition:

Question 3. *Is the $(1+\varepsilon)$-bilipschitz condition necessary? That is, are is Theorem 1 also true for metrics that are merely uniformly bilipschitz to some fixed complete smooth background metric of bounded curvature?*

Question 3 has since been answered by Cecchini – Frenck – Zeidler [4, Theorem B]. They show that the $(1+\varepsilon)$-bilipschitz condition is indeed necessary: for certain

$n \geq 8$ there exists a metric g on $\mathbb{R}^n$ that is uniformly bilipschitz to the Euclidean metric and smooth with positive scalar curvature on $\mathbb{R}^n \setminus \{0\}$, but for which there exists no smooth family of Riemannian metrics $(g_t)_{t\in(0,T)}$ satisfying both

$$R(g_t) \geq 0 \text{ for all } t \in (0,T)$$

and

$$g_t \xrightarrow[t\searrow 0]{C^2_{loc}(\mathbb{R}^n\setminus\{0\})} g.$$

In particular, Theorem 2 places restrictions on possible Ricci-DeTurck flows, with the background metric δ, starting from these metrics. We note that the optimal constant ε needed in Theorem 1 is not known.

References

[1] R. Bamler and E. Chen, *Degree theory for 4-dimensional asymptotically conical gradient expanding solitons*, arXiv:2305.03154 [math.DG] (2023).

[2] P. Burkhardt-Guim, *Smoothing L^∞ Riemannian metrics with nonnegative scalar curvature outside of a singular set*, arXiv:2406.04564 [math.DG] (2024).

[3] P. Burkhardt-Guim, *Pointwise lower scalar curvature bounds for C^0 metrics via regularizing Ricci flow*, Geometric and Functional Analysis **29** (2019), 1703–1772.

[4] S. Cecchini and G. Frenck and R. Zeidler, *Positive scalar curvature with point singularities*, arXiv:2407.20163 [math.DG] (2024).

[5] H. Koch and T. Lamm, *Geometric flows with rough initial data*, Asian Journal of Mathematics **16** (2012), 209 – 236.

[6] M. Simon, *Deformation of C^0 Riemannian metrics in the direction of their Ricci curvature*, Communications in Analysis and Geometry **10** (202), 1033–1074.

Thom's gradient conjecture for geometric evolution equations

Beomjun Choi

(joint work with Pei-Ken Hung)

This work is motivated by a conjecture of R. Thom [7], proved affirmatively by Kurdyka–Mostowski–Parusinski [4]:

Theorem 1 (Thom's gradient conjecture)**.** *Let $x(t)$ be a convergent gradient flow of an analytic potential $f : \mathbb{R}^n \to \mathbb{R}$, i.e. $\dot{x} = -\nabla f(x)$ with $\lim_{t\to\infty} x(t) = x_\infty$. Then the secant direction*

$$\theta(t) = \frac{x(t) - x_\infty}{|x(t) - x_\infty|}$$

converges to a limit $\theta_\infty \in \mathbb{S}^{n-1}$ as $t \to \infty$.

Prior to this, Łojasiewicz [5] had addressed convergence of gradient flows for analytic potentials. Thom's conjecture concerns the finer asymptotics of the *direction* of convergence.

Many geometric evolution equations, for example the mean curvature flow, Yamabe flow, and Yang-Mills flow, can be viewed (at least formally) as gradient flows on infinite-dimensional spaces of surfaces, metrics, or connections. Simon's pioneering work [6] adopted the Łojasiewicz inequality into this setting and triggered a long line of research on the convergence of solution. Recent progress in geometric flows, including the classification of ancient solutions, the theory of generic perturbations, and flows through singularities, hinges crucially on understanding finer asymptotics of solutions, going beyond mere convergence results.

Our goal is to characterize the possible *rates* and *directions* of convergence for a class of geometric evolution equations by exploiting their analytic gradient flow structure. Working on a closed manifold Σ as in Simon's framework, let a solution $u(t,\cdot)$ satisfies either

$$\dot{u} + \nabla\mathcal{F}(u) = \mathcal{N}(u), \tag{1}$$

or

$$\ddot{u} + \dot{u} + \nabla\mathcal{F}(u) = \mathcal{N}(u). \tag{2}$$

Here $\mathcal{F}$ is an analytic functional on sections of a vector bundle over Σ such that $\nabla\mathcal{F}(u)$ is the Euler-Lagrange operator whose minus is an elliptic operator, and $\mathcal{N}(u)$ collects lower order nonlinear terms (could be thought as zero for a sake of simplicity). For the precise assumptions, we refer the reader to our preprint [3]. For convergent solution, after recentering, we may assume that $u(t) \to 0$ and 0 is a stationary solution to the equation. We state the main theorem by dividing it into two cases.

Theorem 2 (fast convergence, C.-Hung [3])**.** *If a solution to* (1) *satisfies* $\|u(t)\| = O(e^{-\delta t})$ *for some* $\delta > 0$*, then either* $u \equiv 0$ *or there exist* $\lambda > 0$*,* $C \neq 0$*, and a smooth eigensection* φ *with* $\nabla^2\mathcal{F}(0)\varphi = \lambda\varphi$ *such that*

$$u(t) = C\, e^{-\lambda t}\varphi + o\big(e^{-\lambda t}\big).$$

Thus exponential decay occurs only along a stable eigendirection of $\nabla^2\mathcal{F}(0)$, with rate determined by its eigenvalue. Note there holds a corresponding theorem for solutions to (2).

Next, when exponential decay fails, dynamics in the kernel of $\nabla^2\mathcal{F}(0)$ and analyticity of $\mathcal{F}$ become decisive.

Theorem 3 (slow convergence, C.-Hung [3])**.** *There exists a finite set* $\mathcal{Z}(\mathcal{F}) \subset \mathbb{Q}_{\geq 3} \times (0,\infty)$ *such that if* $\|u(t)\|e^{\delta t} \to \infty$ *for every* $\delta > 0$*, then for some* $(\ell,\alpha) \in \mathcal{Z}$ *and* $\varphi \in \ker \nabla^2\mathcal{F}(0)$ *the solution* $u(t)$ *satisfies*

$$u(t) = \big[\alpha\ell(\ell-2)t\big]^{-\frac{1}{\ell-2}}\varphi + o\big(t^{-\frac{1}{\ell-2}}\big).$$

A guiding example for above slow convergence is the 1-D flow $\dot{x} = -(\alpha x^{\ell})'$, whose solutions obey $x(t) \sim \operatorname{sign}(\alpha)[\alpha\ell(\ell-2)t]^{-1/(\ell-2)}$. Via subtle reduction method and a refinement of [4], we show any solution eventually falls into finitely many such model cases.

Combining two theorems, we settle Thom's gradient conjecture

Corollary 1. *For Simon's class of equations* [6]*, Thom's gradient conjecture holds:* $u(t)/\|u(t)\| \to \varphi$ *smoothly for some* φ*.*

Remarks. (i) Our convergence rate classification is new even for finite dimensional gradient flows. (ii) Previous constructions of slowly converging solutions in [1][2] show the existence of solutions which decay at rate $t^{-\frac{1}{p-2}}$ for some integer $p \geq 3$. In terms of our result, this p indeed corresponds to the smallest possible ℓ.

Our results naturally suggest extensions to more general settings, such as non-compact manifolds or singular spaces. Intriguing future directions include classifying ancient solutions when the kernel of second variation operator is non-integrable, and exploring V. Arnold's conjecture concerning convergence of the tangent direction $\dot{u}/|\dot{u}|$, which is stronger than Thom's conjecture and, in the case of mean curvature flow, implies convergence results for the mean curvature vector profile.

References

[1] D. Adams, and L. Simon, *Rates of asymptotic convergence near isolated singularities of geometric extrema*, Indiana University mathematics journal **37(2)** (1988), 225–254.
[2] A. Carlotto, O. Chodosh, Y. Rubinstein, *Slowly converging Yamabe flows*, Geometry & Topology **19(3)** (2015), 1523–1568.
[3] B. Choi, and P.-K. Hung, *Thom's gradient conjecture for nonlinear evolution equations*, arXiv preprint arXiv:2405.17510.
[4] K. Kurdyka, T. Mostowski, and A. Parusinski, *Proof of the gradient conjecture of R. Thom*, Ann. Math. (2000), 763–792.
[5] S. Łojasiewicz, *Une propriété topologique des sous-ensembles analytiques réels*, Les équations aux dérivées partielles **117** (1963).
[6] L. Simon, *Asymptotics for a class of non-linear evolution equations, with applications to geometric problems*, Ann. Math. **118(3)** (1983), 525–571.
[7] R. Thom, *Problemes rencontrés dans mon parcours mathématique: un bilan*, Publ. Math. IHÉS **70** (1989), 199–214.

Improved regularity of capillary minimizing hypersurfaces

Nick Edelen

(joint work with Otis Chodosh, Chao Li)

If you observe liquid in a container you'll notice the surface will meet the container at a particular angle. This is called a capillary angle condition, and is determined by the cohesive/adhesive forces in/between the liquid and container. The total energy of the liquid is captured by the Gauss free energy: if Ω^{n+1} is a smooth $(n+1)$-manifold representing the container, and $E \subset \Omega$ a subset representing the liquid, then the Gauss free energy of E is

$$\mathcal{G}(E) = \mathcal{H}^n(\partial E \cap \text{int}\Omega) + \int_{\partial E \cap \partial \Omega} \sigma d\mathcal{H}^n + \int_E g d\mathcal{H}^{n+1}.$$

Here the first term is the surface tension (with $\mathcal{H}^k$ being the k-dimensional Hausdorff measure), the second term is the wetting energy (with energy density $\sigma : \partial\Omega \to (-1, 1)$), and the third term is the gravitational energy.

The liquid configuration E will be a minimizer or critical point for $\mathcal{G}$ subject to a volume constraint. Geometrically, being critical for $\mathcal{G}$ is equivalent to ∂E having mean curvature $g + (const)$ inside Ω, and meeting $\partial\Omega$ with angle $\arccos(\sigma)$, both understood in a distributional sense. One can think of the interface where E meets $\partial\Omega$ as a kind of non-linear free-boundary problem.

Strictly inside the container, minimizers E of $\mathcal{G}$ behave like perimeter-minimizers, so by well-known theory $\partial E \cap \mathrm{int}\Omega$ is smooth away from a singular set of dimension $\leq n-7$, where we recall n is the surface dimension of ∂E. Regularity at the boundary is less well understood. Works of [1], [3] have shown the singular set of the boundary interface $\partial E \cap \partial\Omega$ is at most $(n-3)$-dimensional. In our paper [2] we improve on this estimate, and for certain ranges of angles (near 0°, 90°, 180°) we get even better bounds.

Theorem 1. *There is an $\epsilon(n)$ so that the following holds. Let $E \subset \Omega^{n+1}$ minimize $\mathcal{G}$, subject to a possible volume constraint. Then $M := \partial E \cap \mathrm{int}\Omega$ is a smooth hypsurface away from a singular set* $\mathrm{sing}\, M$ *satisfying* $\dim(\mathrm{sing}\, M \cap \mathrm{int}\Omega) \leq n-7$ *and:*

- $\dim(\mathrm{sing}\, M \cap \partial\Omega) \leq n-4$*;*
- $\dim(\mathrm{sing}\, M \cap \partial\Omega) \leq n-7$ *where* $|\sigma| < \epsilon$*;*
- $\dim(\mathrm{sing}\, M \cap \partial\Omega) \leq n-5$ *where* $|\sigma - 1| < \epsilon$ *or* $|\sigma + 1| < \epsilon$*.*

Like in the interior setting, there are good compactness, monotonicity, and ϵ-regularity theorems, which allow you to take tangent cones at the capillary boundary, and to apply the principle of dimension reduction. Therefore proving Theorem 1 boils down to classifying low-dimensional capillary minimizing cones (under possible angle restrictions) in a half-space as planar. The improved estimate $n-4$ comes from adapting an argument of Almgren to show that stable 3-dimensional capillary cones are planar. The improved estimate $n-7$ comes from pertubing the argument of Simons classifying stable 6-dimensional stable cones as planar.

Arguably the most interesting case of Theorem 1 is when the angle is close to 0° or 180°. We show that capillary minimizing surfaces in a half-space with very small angle can be rigorously approximated by minimizers $u : \mathbb{R}^n \to \mathbb{R}$ of the Alt-Caffarelli functional

$$J(u) = \int_{\{u>0\}} |Du|^2 + 1.$$

(Such u are often referred to as solutions of the one-phase Bernoulli problem.) The idea of J being the linearization of capillary has been well-known to experts, but we make this approximation precise, and moreover we prove a regularity theorem that says whenever the corresponding u is smooth, then the original capillary surface is smooth also. So regularity of the small-angle capillary problem is dictated by the regularity of the one-phase Bernoulli problem. The sharp regularity bound for one-phase Bernoulli is a hard open question, but is know to be $\in \{n-5, n-6, n-7\}$.

Almost certainly our bounds are not sharp, which leads to the obvious question: what is the optimal dimension bound for the singular set of capillary minimizers? Could this optimal dimension change with the angle? The hopeful guess is that the best dimension bound is $n-7$ for all angles, but unfortunately we would expect this problem to be at least as hard as the corresponding one-phase Bernoulli problem. We also remark also that while [3], [2] give various upper bounds on the singular dimension, currently there are no known lower bound for general angles, as we do not yet have any rigorous examples of singular minimizing capillary cones with contact angle $\neq 90°$.

References

[1] Guido De Philippis, Francesco Maggi, *Regularity of free boundaries in anisotropic capillarity prob- lems and the validity of Young's law*, Arch. Ration. Mech. Anal. 216 (2015), no. 2, 473–568.

[2] Otis Chodosh, Nick Edelen, Chao Li, *Improved regularity for minimizing capillary hypersurfaces*, Ars Inven. Anal. April (2025).

[3] Jean Taylor, *Boundary regularity for solutions to various capillarity and free boundary problems*, Comm. Partial Differential Equations 2 (1977), no. 4, 323–357.

Contractibility of spaces of positive scalar curvature metrics with symmetry

Bernhard Hanke

(joint work with Christian Bär)

Studying spaces of Riemannian metrics of positive scalar curvature on compact smooth manifolds has been a major research topic during the past decades. More specifically, one asks: When are these spaces nonempty? If so, what are their topological types? Both questions are interesting and nontrivial. We will focus on the second one here.

Suppose that M is an orientable compact connected smooth manifold without boundary. If $\dim M = 2$, then, by the uniformization theorem, the space of positive scalar curvature metrics on M is either empty or contractible. The same is true if $\dim M = 3$, as shown by Bamler and Kleiner [1], using the Ricci flow in families. If $\dim M = 4$, Ruberman [14] showed that this space can be disconnected, using a 1-parameter version of Seiberg-Witten theory.

Suppose that M is a compact smooth manifold of dimension at least 5, possibly with boundary. The space of Riemannian metrics with positive scalar curvature on M - with suitable boundary conditions if applicable - can have a very rich topology. This has been demonstrated by the work of Hitchin [11]; Gromov and Lawson [9]; Hanke, Schick, and Steimle [10]; Botvinnik, Ebert, and Randal–Williams [5]; Ebert and Randal–Williams [7]; and Bär and Hanke [3], using methods from differential and geometric topology. In fact, on a fixed manifold, this space may have nonzero homotopy groups in infinitely many degrees. Furthermore, these homotopy groups may not be finitely generated.

The picture changes significantly under symmetry assumptions. If Γ is a compact, possibly nonconnected, Lie group that acts smoothly and effectively on a compact connected smooth manifold M, then we denote by $\mathscr{R}^{\Gamma}_{>0}(M)$ the space of Γ-invariant Riemannian metrics of positive scalar curvature on M. This space is equipped with the C^∞-topology.

The following Theorems, 1 and 2, appear in our recent preprint [4].

Theorem 1. *Let Γ be a compact Lie group and M be a compact connected smooth Γ-manifold of dimension at least 2 with nonempty (not necessarily connected) boundary. Then $\mathscr{R}^{\Gamma}_{>0}(M)$ is contractible.*

For $\Gamma = \{1\}$, this follows from Gromov's h-principle [8]. An equivariant version of Gromov's h-principle by Bierstone [6] applies, if for every closed subgroup $H < \Gamma$, each connected component of the union of all Γ-orbits with isotropy group conjugate to H has a nonempty intersection with the boundary of M.

Our proof of Theorem 1, which is independent of h-principle techniques, is based on equivariant Morse theory and conformal deformations around unstable manifolds. In particular, it makes the construction of the relevant contracting homotopies quite explicit.

Now assume that M is a compact connected smooth Γ-manifold without boundary. Lawson and Yau [12] showed that if Γ is a compact connected non-abelian Lie group, then $\mathscr{R}^{\Gamma}_{>0}(M) \neq \emptyset$. Wiemeler [15] showed that if Γ is a compact Lie group containing a normal S^1-subgroup with fixed-point components of codimension 2 in M (and possibly fixed-point components of higher codimension), then $\mathscr{R}^{\Gamma}_{>0}(M) \neq \emptyset$. This is not true without the codimension-2 assumption. These results differ from existence results for positive scalar curvature metrics based on bordism-theoretic methods in that they are independent of spin and fundamental group assumptions.

Theorem 2. *Let Γ be a compact Lie group and let M be a compact connected smooth Γ-manifold without boundary. Suppose that Γ contains a normal S^1-subgroup with fixed-point components of codimension 2 in M. Then $\mathscr{R}^{\Gamma}_{>0}(M)$ is contractible.*

This provides the first complete description of homotopy types of spaces of positive scalar curvature metrics on closed manifolds in dimensions larger than 3. Note that Theorem 2 strengthens Wiemeler's existence result.

We give a rough outline of our argument for proving Theorem 2. According to a classical result by Palais [13], it is sufficient to show that $\mathscr{R}^{\Gamma}_{>0}(M)$ is weakly contractible, i.e., it is path-connected and has trivial homotopy groups in all degrees. Now, let $m \geq 0$, let D^m be the closed unit m-ball and let

$$g\colon \partial D^m \to \mathscr{R}^{\Gamma}_{>0}(M)$$

be a continuous map. We have to show that g extends to a continuous map

$$G\colon D^m \to \mathscr{R}^{\Gamma}_{>0}(M).$$

(For $m = 0$, this means that $\mathscr{R}^{\Gamma}_{>0}(M)$ is nonempty.)

Let S^1 be a normal subgroup in Γ such that M^{S^1} contains components of codimension 2 in M. Let S be the union of these components. This is a Γ-invariant, possibly disconnected submanifold of M.

Pulling back the metrics $g(\xi)$ along appropriate Γ-equivariant diffeomorphisms of M, one can assume that for sufficiently small $\rho > 0$, the closed tubular neighborhood $S \subset B_\rho(S) \subset M$ of radius ρ with respect to $g(\xi)$ is independent of $\xi \in \partial D^m$.

Let $\breve{M}$ be the complement of the interior of $B_\rho(S)$ in M. Both $B_\rho(S)$ and $\breve{M}$ are compact Γ-manifolds with nonempty boundaries and $\breve{M}$ is connected.

Let $\breve{g}\colon \partial D^m \to \mathscr{R}^\Gamma_{>0}(\breve{M})$ be induced by g. By Theorem 1, we can extend $\breve{g}$ to a continuous map $\breve{G}\colon D^m \to \mathscr{R}^\Gamma_{>0}(\breve{M})$. The challenge is extending the map $g_{B_\rho(S)}\colon \partial D^m \to \mathscr{R}^\Gamma_{>0}(B_\rho(S))$ induced by g to a continuous map $G_{B_\rho(S)}\colon D^m \to \mathscr{R}^\Gamma_{>0}(B_\rho(S))$ in such a way that the union of $\breve{G}(\xi)$ and $G_{B_\rho(S)}(\xi)$ defines a smooth metric on M for all $\xi \in D^m$.

To achieve this, using an equivariant version of the local flexibility lemma [2], we can assume that the metrics $g_{B_\rho(S)}(\xi)$ on $B_\rho(S)$ are Riemannian submersion metrics projecting onto S. From this, we can construct a continuous extension $G_{B_\rho(S)}\colon D^m \to \mathscr{R}^\Gamma_{>0}(B_\rho(S))$ consisting of Riemannian submersion metrics. We may need to pass to a smaller ρ in these steps.

To smoothly glue the metrics $\breve{G}(\xi)$ and $G_{B_\rho(S)}(\xi)$ along $\partial\breve{M} = \partial B_\rho(S)$, we shrink the S^1-orbits near $\partial\breve{M} \subset \breve{M}$ with respect to $\breve{G}(\xi)$ and the fibers of the Riemannian submersion metrics $G_{B_\rho(S)}(\xi)$. This ensures that the sum of the mean curvatures of $\partial\breve{M} \subset \breve{M}$ and of $\partial B_\rho(S) \subset B_\rho(S)$ with respect to these metrics, for each $\xi \in D^m$, is non-negative, while preserving the positivity of the scalar curvature. By a further deformation, one can ensure that, for each $\xi \in D^m$, the metrics $\breve{G}(\xi)$ and $G_{B_\rho(S)}(\xi)$ induce the same metric on $\partial\breve{M} = \partial B_\rho(S)$. In this situation, the smoothing of mean-convex singularities [3] can be applied to produce the required smooth Γ-invariant metrics on M. All of these constructions must be performed with continuous dependence on $\xi \in D^m$ and without altering the given metrics $g(\xi)$ for $\xi \in \partial D^m$.

We conclude with an application of Theorem 2. Let $T^n = (S^1)^n$ be the n-torus. Recall that a *torus manifold* is a closed connected smooth $2n$-dimensional effective T^n-manifold with nonempty fixed-point set. Smooth compact toric varieties are examples of torus manifolds.

Corollary 1. *Let M be a torus manifold of dimension $2n$ and let $H < T^n$ be a closed subgroup of dimension at least 1. Then $\mathscr{R}^H_{>0}(M)$ is contractible.*

References

[1] Richard Bamler, Bruce Kleiner, *Ricci flow and contracibility of spaces of metrics*, Preprint (2019) available at `https://arxiv.org/pdf/1909.08710`

[2] Christian Bär, Bernhard Hanke, *Local flexibility for open partial differential relations*, Commun. Pure Appl. Math. **75** (2022), 1377–1415

[3] Christian Bär, Bernhard Hanke, *Boundary conditions for scalar curvature*, Perspectives in scalar curvature. Vol. 2, 325–377, World Sci. Publ., Hackensack, NJ, 2023

[4] Christian Bär, Bernhard Hanke, *Contractibility of spaces of positive scalar curvature metrics with symmetry*, Preprint (2025) available at `https://arxiv.org/abs/2503.16232`

[5] Boris Botvinnik, *Infinite loop spaces and positive scalar curvature*, Invent. Math. **209** (2017), 749–835

[6] Edward Bierstone, *Equivariant Gromov theory*, Topology **13** (1974), 327–345

[7] Johannes Ebert, Oscar Randal-Williams, *The positive scalar curvature cobordism category*, Duke Math. J. **171** (2022), 2275–2406

[8] Mikhael Gromov, *Partial differential relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Vol. 9, Springer-Verlag. Berlin, 1986

[9] Mikhael Gromov, H. Blaine Lawson, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Inst. Hautes Études Sci. Publ. Math. **58**, 83–196
[10] Bernhard Hanke, Thomas Schick, Wolfgang Steimle, *The space of metrics of positive scalar curvature*, Publ. Math. Inst. Hautes Études Sci. **120** (2014), 335–367
[11] Nigel Hitchin, *Harmonic spinors*, Advances in Math. **14** (1974), 1–55
[12] H. Blaine Lawson, Shing-Tung Yau, *Scalar curvature, non-abelian group actions and the degree of symmetry of exotic spheres*, Comment. Math. Helv. **49** (1974), 232–244
[13] Richard Palais, *Homotopy theory of infinite dimensional manifolds*, Topology **5** (1966), 1–16
[14] Daniel Ruberman, *Positive scalar curvature, diffeomorphisms and the Seiberg-Witten invariants*, Geom. Topol. **5** (2001), 895–924
[15] Michael Wiemeler, *Circle actions and scalar curvature*, Trans. Amer. Math. Soc. **368** (2016), 2939–2966

Classification of ancient noncollapsed flows in $\mathbb{R}^4$

Robert Haslhofer
(joint work with Kyeongsu Choi)

In this note, we discuss our recent classification of all noncollapsed singularities for the mean curvature flow in $\mathbb{R}^4$. In stark contrast to $\mathbb{R}^3$, a classification in $\mathbb{R}^4$ until recently seemed out of reach. Fundamentally, this is because of the existence of examples with reduced symmetry. More precisely, Hoffman-Ilmanen-Martin-White constructed a 1-parameter family of 3d-translators that interpolate between the round 3d-bowl and $\mathbb{R}\times$2d-bowl, and are only $\mathbb{Z}_2 \times \mathrm{O}_2$-symmetric [13]. Similarly, in joint work with Du we constructed a 1-parameter family of 3d-ovals that interpolate between the $\mathrm{O}_2 \times \mathrm{O}_2$-symmetric 3d-oval and $\mathbb{R}\times$2d-oval, and are only $\mathbb{Z}_2^2 \times \mathrm{O}_2$-symmetric [10]. Recently, in joint work with K. Choi, building also on our earlier collaborations with B. Choi, Daskalopoulos, Du, Hershkovits, and Sesum, we obtained a complete classification of all noncollapsed singularities:

Theorem 1 (classification [6, 7, 9, 8, 4, 5])**.** *Any ancient noncollapsed flow in $\mathbb{R}^4$ is, up to scaling and rigid motion,*

- *either one of the standard shrinkers S^3, $\mathbb{R} \times S^2$, $\mathbb{R}^2 \times S^1$ or $\mathbb{R}^3$,*
- *or the 3d-bowl, or $\mathbb{R}\times$2d-bowl, or belongs to the 1-parameter family of $\mathbb{Z}_2 \times \mathrm{O}_2$-symmetric translators from* [13]*,*
- *or the $\mathbb{Z}_2 \times \mathrm{O}_3$-symmetric 3d-oval, or the $\mathrm{O}_2 \times \mathrm{O}_2$-symmetric 3d-oval, or $\mathbb{R}\times$2d-oval, or belongs to the 1-parameter family of $\mathbb{Z}_2^2 \times \mathrm{O}_2$-symmetric 3d-ovals from* [10]*.*

In addition to the 1-parameter families of translators and ovals discussed above, our list of course also contains all classical historical examples, in particular the two examples of cohomogeneity-one 3d-ovals from [14] and [12], respectively. As an immediate consequence we obtain a classification of all potential blowup limits (and thus a canonical neighborhood theorem) for mean-convex flows in $\mathbb{R}^4$:

Corollary 1 (canonical neighborhoods)**.** *For the mean curvature flow of mean-convex hypersurfaces in $\mathbb{R}^4$ (or in a 4-manifold) every blowup limit is given by one of the solutions from the above list. In particular, for every $\varepsilon > 0$ there is an*

$H_\varepsilon = H_\varepsilon(M_0) < \infty$, such that around any space-time point (p,t) with $H(p,t) \geq H_\varepsilon$ the flow is ε-close (after rescaling) to one of the solutions from the above list.

More generally, in light of Ilmanen's multiplicity-one and mean-convex neighborhood conjecture, the conclusion of the corollary is also expected to hold for blowup limits near any generic singularity.

To outline our approach, let $\mathcal{M} = \{M_t\}$ be an ancient noncollapsed mean curvature flow in $\mathbb{R}^4$ that is neither a static plane nor a round shrinking sphere. By general theory the tangent flow at $-\infty$ is either a neck or a bubble-sheet. Since the neck-case has already been dealt with in the fundamental work by Brendle-Choi [2, 3] and Angenent-Daskalopoulos-Sesum [1], we can from now assume that

$$\lim_{\lambda\to 0} \mathcal{D}_\lambda \mathcal{M} = \{\mathbb{R}^2 \times S^1(\sqrt{2|t|})\}_{t<0}\,. \tag{1}$$

The analysis of such ancient solutions starts by considering the bubble-sheet function $u = u(y,\vartheta,\tau)$, which captures the deviation of the renormalized flow $\bar{M}_\tau = e^{\tau/2} M_{-e^{-\tau}}$ from the round bubble-sheet $\mathbb{R}^2 \times S^1(\sqrt{2})$. The evolution of u is governed by the Ornstein-Uhlenbeck type operator

$$\mathcal{L} = \partial_{y_1}^2 + \partial_{y_2}^2 - \tfrac{y_1}{2}\partial_{y_1} - \tfrac{y_2}{2}\partial_{y_2} + \tfrac{1}{2}\partial_\vartheta^2 + 1, \tag{2}$$

which has the unstable eigenfunctions $1, y_1, y_2, \cos\vartheta, \sin\vartheta$, and the neutral eigenfunctions $y_1^2-2, y_2^2-2, y_1y_2, y_1\cos\vartheta, y_1\sin\vartheta, y_2\cos\vartheta, y_2\sin\vartheta$. Based on these spectral properties, and taking also into account that the ϑ-dependence is tiny thanks to Zhu's symmetry improvement result [15], in joint work with Du we proved:

Theorem 2 (normal form [8, 9]). *For $\tau \to -\infty$, in suitable coordinates, in Gaussian L^2-norm we have*

$$u = O(e^{\tau/2}) \quad \text{or} \quad u = \tfrac{4-y_1^2-y_2^2}{\sqrt{8}|\tau|} + o(\tfrac{1}{|\tau|}) \quad \text{or} \quad u = \tfrac{2-y_2^2}{\sqrt{8}|\tau|} + o(\tfrac{1}{|\tau|}). \tag{3}$$

Accordingly, the classification problem can be split up into 3 cases, which we call the case of fast convergence, slow convergence, and mixed convergence, respectively. In the case of fast convergence, which is easiest case, we have:

Theorem 3 (no wings [6]). *There are no wing-like ancient noncollapsed flows in $\mathbb{R}^4$. In particular, if the convergence is fast, then $\mathcal{M}$ is either a round shrinking $\mathbb{R}^2 \times S^1$ or a translating $\mathbb{R}\times$2d-bowl.*

To prove this, we showed that

$$u^X = (a_1 y_1 + a_2 y_2) e^{\tau/2} + o(e^{\tau/2}) \tag{4}$$

for all τ negative enough depending only on the bubble-sheet scale. Analyzing this expansion along potential different edges, we concluded that $\mathcal{M}$ in fact splits off a line (hence is not wing-like) and is selfsimilar. The case of slow convergence has been settled in joint work with B. Choi, Daskalopoulos, Du and Sesum:

Theorem 4 (bubble-sheet ovals [4]). *If the convergence is slow, then $\mathcal{M}$ is either the $\mathrm{O}_2\times\mathrm{O}_2$-symmetric 3d-oval, or belongs to the one-parameter family of $\mathbb{Z}_2^2\times\mathrm{O}_2$-symmetric 3d-ovals from* [10].

Regarding the proof, let us just mention that (3) in the case of slow convergence means inwards quadratic bending in all directions, which yields that M_t is compact with axes of length approximately $\sqrt{2|t|\log|t|}$. Hence, up to technical challenges, the problem turned out to be amenable to the techniques from [1]. Finally, in joint work with Choi, we settled the most difficult case of mixed convergence:

Theorem 5 (mixed convergence [5])**.** *If the convergence is mixed, then $\mathcal{M}$ is either $\mathbb{R}\times$2d-oval or is selfsimilarly translating (and hence by* [7] *is either $\mathbb{R}\times$2d-bowl, or belongs to the 1-parameter family of $\mathbb{Z}_2 \times \mathrm{O}_2$-symmetric translators from* [13]*).*

Loosely speaking, to capture the (dauntingly small) slope in y_1-direction, we consider the derivative $u_1^X = \partial u^X/\partial y_1$, which kills the leading order dependence on y_2, and prove that

$$u_1^X = ae^{\tau/2} + o(e^{\tau/2}). \tag{5}$$

This differential neck theorem, which goes vastly beyond (4), can then be used to conclude that $\mathcal{M}$ is noncompact (hence there are no exotic ovals) and either splits off a line or is selfsimilarly translating.

Finally, for the related problem for 4d Ricci flow we conjecture:

Conjecture 1 ([11])**.** *Any κ-solution in 4d Ricci flow is, up to scaling and finite quotients, given by one of the following solutions.*

- *shrinkers: S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, $\mathbb{R} \times S^3$ or $\mathbb{R}^2 \times S^2$.*
- *steadies: 4d Bryant soliton, the 3d Bryant soliton times a line, or belongs to the 1-parameter family of $\mathbb{Z}_2\times\mathrm{O}_3$-symmetric steady solitons constructed by Lai.*
- *ovals: the $\mathbb{Z}_2\times\mathrm{O}_4$-symmetric 4d ovals from Perelman, the 3d ovals times a line, the $\mathrm{O}_2\times\mathrm{O}_3$-symmetric 4d ovals constructed by Buttsworth, or belongs to the 1-parameter family of $\mathbb{Z}_2^2 \times \mathrm{O}_3$-symmetric ovals from* [11]*.*

References

[1] S. Angenent, P. Daskalopoulos, N. Sesum, *Uniqueness of two-convex closed ancient solutions to the mean curvature flow*, Ann. of Math. **192** (2020), 353–436.

[2] S. Brendle, K. Choi, *Uniqueness of convex ancient solutions to mean curvature flow in $\mathbb{R}^3$*, Invent. Math. **217** (2019), 35–76.

[3] S. Brendle, K. Choi, *Uniqueness of convex ancient solutions to mean curvature flow in higher dimensions*, Geom. Topol. **25** (2021), 2195–2234.

[4] B. Choi, P. Daskalopoulos, W. Du, R. Haslhofer, N. Sesum, *Classification of bubble-sheet ovals in $\mathbb{R}^4$*, Geom. Topol. **29** (2025), 931–1016.

[5] K. Choi, R. Haslhofer, *Classification of ancient noncollapsed flows in $\mathbb{R}^4$*, preprint.

[6] K. Choi, R. Haslhofer, O. Hershkovits, *A nonexistence result for wing-like mean curvature flows in $\mathbb{R}^4$*, Geom. Topol. **28** (2024), 3095–3134.

[7] K. Choi, R. Haslhofer, O. Hershkovits, *Classification of noncollapsed translators in $\mathbb{R}^4$*, Camb. J. Math. **11** (2023), 563–698.

[8] W. Du, R. Haslhofer, *A nonexistence result for rotating mean curvature flows in $\mathbb{R}^4$*, J. Reine Angew. Math. **802** (2023), 275–285.

[9] W. Du, R. Haslhofer, *Hearing the shape of ancient noncollapsed flows in $\mathbb{R}^4$*, Comm. Pure Appl. Math. **77** (2024), 543–582.

[10] W. Du, R. Haslhofer, *On uniqueness and nonuniqueness of ancient ovals*, Amer. J. Math. (to appear).
[11] R. Haslhofer, *On kappa-solutions and canonical neighborhoods in 4d Ricci flow*, J. Reine Angew. Math. **811** (2024), 257–265.
[12] R. Haslhofer, O. Hershkovits, *Ancient solutions of the mean curvature flow*, Comm. Anal. Geom. **24** (2016), 593–604.
[13] D. Hoffman, T. Ilmanen, F. Martin, B. White, *Graphical translators for mean curvature flow*, Calc. Var. Partial Differential Equations **58** (2019), Art. 117.
[14] B. White, *The nature of singularities in mean curvature flow of mean-convex sets*, J. Amer. Math. Soc. **16** (2003), 123–138.
[15] J. Zhu, SO(2)-*symmetry of translating solitons of the mean curvature flow in* $\mathbb{R}^4$, preprint.

Closed Einstein manifolds of negative curvature

Frieder Jäckel
(joint work with Ursula Hamenstaedt)

A Riemannian metric g on a smooth manifold M is called *Einstein* if for some $\lambda \in \mathbb{R}$, called the *Einstein constant*,

$$\mathrm{Ric}(g) = \lambda g,$$

that is, if (M, g) has constant Ricci curvature. The study of Einstein metrics has a long and rich history in Riemannian Geometry (see for example [Bes08]). However, it is extremely difficult to construct examples of Einstein metrics on closed manifolds.

On the other hand, the following philosophy has proven to be fruitful:

There is an abundance of closed manifolds with negative sectional *curvature.*

In fact, some experts even say that "most" closed manifolds are negatively curved (for example [So24]). In view of this philosophy, and the fact that Einstein metrics are objects of high interest in Riemannian Geometry, we believe that the following question is very natural.

Question 1. *Are there, in some sense, many closed manifolds admitting an Einstein metric with negative sectional curvature?*

There is a handful of examples of closed manifolds admitting Einstein metrics with negative Einstein constant $\lambda < 0$, including:

(1) locally symmetric spaces of non-compact type, e.g., hyperbolic or complex-hyperbolic manifolds;
(2) compact Kähler manifolds with $c_1 < 0$ admit a Kähler–Einstein metric with Einstein constant $\lambda < 0$ due to the work of Aubin [Aub78] and Yau [Yau78];
(3) manifolds obtained by generalized Dehn filling of hyperbolic cusps in dimensions $n \geq 4$, due to Anderson [And06] and Bamler [Bam12].

Out of these, only the examples in (1) are known to have negative sectional curvature, and these examples have been known for more than a century. In fact, until recently, locally symmetric spaces of non-compact type were the only known examples of Einstein metrics with negative sectional curvature on a closed manifold (in the non-compact case the existence of negatively curved non-symmetric Einstein metrics has been known for a long time - see for example [GL91]). This changed a few years ago with the following breakthrough result of Fine–Premoselli [FP20].

Theorem 1 (Fine–Premoselli)**.** *There are infinitely many closed* 4*-manifolds that admit an Einstein metric with negative sectional curvature, but that do* not *admit any locally symmetric metric (e.g., no hyperbolic or complex-hyperbolic metric).*

We extend the result of Fine–Premoselli to all dimensions $n \geq 4$, also greatly simplifying the proof.

Theorem 2 (Hamenstädt-J.)**.** *For all* $n \geq 4$ *there exist infinitely many closed* n*-manifolds admitting an Einstein metric with negative sectional curvature, but that do* not *admit any locally symmetric metric.*

In dimensions at least five, these are the first non-trivial examples of closed Einstein manifolds with negative sectional curvature. The proof builds on the original construction of Fine–Premoselli but exploits an algebraic property of arithmetic hyperbolic manifolds, called *subgroup separability*, in order to greatly simplify the involved analytic arguments, allowing for an extension to all dimensions. Very recently, this construction was also extended to the Kähler setting by Guenancia–Hamenstädt [GH25].

References

[And06] M. T. Anderson. Dehn Filling and Einstein Metrics in Higher Dimensions. *Journal of Differential Geometry*, 73(2):219–261, 2006.

[Aub78] T. Aubin. Équations du type Monge–Ampère sur les variétés kählériennes compactes. *Bulletin des Sciences Mathématiques*, 102:63–95, 1978.

[Bam12] R. H. Bamler. Construction of Einstein metrics by generalized Dehn filling. *Journal of the European Mathematical Society*, 14:887–909, 2012.

[Bes08] A. L. Besse. *Einstein Manifolds.* Classics in Mathematics. Springer, 2008. Reprint of the 1987 edition.

[FP20] J. Fine and B. Premoselli. Examples of compact Einstein four-manifolds with negative curvature. *Journal of the American Mathematical Society*, 33(4):991–1038, 2020.

[GL91] C. R. Graham and J. M. Lee. Einstein Metrics with Prescribed Conformal Infinity on the Ball. *Advances in Mathematics*, 87(2):186–225, 1991.

[GH25] H. Guenancia and U. Hamenstädt. *Kähler–Einstein metrics of negative curvature.* Preprint, arXiv:2503.02838, 2025.

[HJ24] U. Hamenstädt and F. Jäckel. *Negatively curved Einstein metrics on Gromov–Thurston manifolds.* Preprint, arXiv:2411.12956, 2024.

[So24] A. Song. Minimal surfaces in infinite dimensional spheres. Talk at the conference *Metric Geometry and Geometric Measure Theory* in Fribourg, 2024.

[Yau78] S.-T. Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I. *Communications on Pure and Applied Mathematics*, 31(3):339–411, 1978.

Mean curvature flow in R^3 and the Multiplicity One Conjecture

Bruce Kleiner
(joint work with Richard Bamler)

We introduce *almost regular mean curvature flow*, a new notion of singular mean curvature flow in $\mathbb{R}^3$. We use this as a framework for addressing several longstanding conjectures, building on a series of recent advances [Bre16, HW20, CHH22, CCS23].

Theorem 1 (Multiplicity One Conjecture). *If $\mathcal{M}$ is an almost regular mean curvature flow, then any tangent flow is a multiplicity one shrinking soliton. In fact, any blow-up sequence has a subsequential limit which is an almost regular flow, and has multiplicity one.*

Theorem 2 (Existence). *Any outermost or innermost flow in the sense of* [HW20] *– in particular any nonfattening level set flow – is an almost regular flow.*

Theorem 3 (Uniqueness iff nonfattening). *If $\mathcal{K}$ is a level set flow, then $\mathcal{K}$ is nonfattening iff there is a unique almost regular flow with initial condition $\mathcal{K}_0$.*

Theorem 4 (Partial regularity). *The spacetime singular set of an almost regular flow has spacetime dimension at most* 1*.*

Theorem 5 (Generic singularities). *Suppose $M \subset \mathbb{R}^3$ is a compact smooth surface. Then there is a sequence*

$$M^j \xrightarrow{C^\infty} M$$

such that if $\mathcal{K}^j$ is the level set flow starting from M^j, then:

- *$\mathcal{K}^j$ is nonfattening.*
- *All tangent flows of $\mathcal{K}^j$ are round spheres or cylinders.*

Theorem 6 (Uniqueness in the S^2 case). *If $\mathcal{K}$ is a level set flow with initial condition diffeomorphic to S^2, then:*

- *$\mathcal{K}$ is nonfattening.*
- *All tangent flows of $\mathcal{K}$ are round spheres or cylinders.*

References

[Bre16] Simon Brendle, *Embedded self-similar shrinkers of genus 0*, Ann. of Math. (2) **183** (2016), no. 2, 715–728. MR 3450486

[HW20] Or Hershkovits and Brian White, *Nonfattening of mean curvature flow at singularities of mean convex type*, Comm. Pure Appl. Math. **73** (2020), no. 3, 558–580. MR 4057901

[CHH22] Kyeongsu Choi, Robert Haslhofer, and Or Hershkovits, *Ancient low-entropy flows, mean-convex neighborhoods, and uniqueness*, Acta Math. **228** (2022), no. 2, 217–301. MR 4448681

[CCS23] Otis Chodosh, Kyeongsu Choi, and Felix Schulze, *Mean curvature flow with generic initial data II*, `https://arxiv.org/abs/2302.08409` (2023).

3-Manifolds with positive scalar curvature and bounded geometry

Yi Lai
(joint work with Otis Chodosh, Kai Xu)

Yau asked the question of classifying 3-manifolds admitting complete Riemannian metrics with positive scalar curvature; see Question 27 in [1]. A fundamental observation of Schoen–Yau relates scalar curvature to the stability of minimal surfaces [2], which ultimately leads to several topological obstructions to the existence of complete nonnegative scalar curvature metrics on a 3-manifold M. J. Wang [3, 4] proved that if M is contractible and admits an exhaustion by solid tori, then $M \cong \mathbb{R}^3$. In particular, the Whitehead manifold does not admit such a metric. The general classification is widely open. In particular, we note the following special cases:

- If M is a contractible 3-manifold and admits a complete metric of nonnegative scalar curvature, do we have $M \cong \mathbb{R}^3$? (Asked by J. Wang [3, 4], cf. [5].)
- If M_γ is an open handlebody of genus γ, and admits a complete metric of non-negative scalar curvature, do we have $\gamma \leq 1$? (Asked by Gromov [6, §3.10.2].)

In joint work with O. chodosh and K. Xu, we resolve these two questions under the *additional* assumption that the metric has bounded geometry:

$$\text{(BG)} \qquad |\mathrm{Rm}| \leq \Lambda, \qquad \mathrm{inj} \geq \Lambda^{-1},$$

We use R to denote the scalar curvature.

Theorem 1 ([7]). *Let (M, g) be a complete, connected, contractible Riemannian 3-manifold satisfying $R \geq 0$ and* (BG). *Then M is diffeomorphic to $\mathbb{R}^3$.*

Theorem 2 ([7]). *Let M_γ denote the interior of the handlebody of genus γ. If (M_γ, g) is a complete Riemannian 3-manifold satisfying $R \geq 0$ and* (BG), *then $\gamma \leq 1$.*

Note that $\mathbb{R}^3$ and $\mathbb{R}^2 \times \mathbb{S}^1$ (corresponding to $\gamma = 0, 1$ in Theorem 2) both admit complete metrics with $R \geq 0$ and bounded geometry. Concrete examples are a capped-off half-cylinder (which actually has $R \geq 1$) and the product of Cigar soliton and $\mathbb{S}^1$, respectively.

For the stronger uniformly positive scalar curvature condition $R \geq 1$, J. Wang has obtained a complete classification [8]: these 3-manifolds are infinite connect sums of spherical space forms and $S^2 \times S^1$. In particular, the only contractible manifold or handlebody admitting such a metric is $\mathbb{R}^3$. We note that earlier work of Bessières–Besson–Maillot [9] used Ricci flow to prove such a classification with an additional bounded geometry assumption.

The key novelty introduced in this work is the use of inverse mean curvature flow as a replacement for μ-bubbles in topological applications. A family of hypersurfaces is a smooth inverse mean curvature flow (IMCF) if it evolves in the outwards pointing direction with speed $\frac{1}{H}$, where H denotes the mean curvature.

The relevance of IMCF to scalar curvature is the following: if (M, g) is a Riemannian 3-manifold with nonnegative scalar curvature, and $\Sigma_t \subset M$ is a compact family evolving by the smooth IMCF, then $|\Sigma_t| = e^t|\Sigma_0|$ and

$$\frac{d}{dt}\int_{\Sigma_t} H^2 \leq -\frac{1}{2}\int_{\Sigma_t} H^2 + 4\pi\chi(\Sigma_t). \tag{1}$$

This is known as the Geroch monotonicity formula [10]. In particular, if the flow exists for all time $t \in [0, \infty)$ then Σ_t cannot have genus ≥ 2 for all large t, since otherwise (1) would force $\int_{\Sigma_t} H^2$ to be negative for $t \gg 1$, which is impossible. In particular, this implies that M admits an exhaustion by regions with sphere or torus boundaries, strongly constraining its topology.

However, there are major issues with the assumption of long-time existence in practice. First, singularities are likely to develop along the flow. Secondly, it's possible that the flow rushes to infinity in finite time if the infinity is not "large" enough.

To allow for singularities, we can use the notion of weak IMCF introduced by Huisken–Ilmanen [11] en route to their proof of the Riemannian Penrose inequality. Intuitively, this solution can be described as running the smooth flow except at each time replacing Σ_t by its least area enclosure. As proven by Huisken–Ilmanen, the Geroch monotonicity (1) remains true for weak solutions as long as they exist.

A weak IMCF that does not rush to infinity in finite time is called proper. In our current setting, we inevitably encounter non-proper weak IMCFs (i.e. weak IMCFs that rush to infinity within finite time). We make essential use of K. Xu's recent work [12], which shows that (M, g) always admits a "maximal" (or "innermost", "slowest") weak IMCF. Assuming bounded geometry and one-endedness of M, we show that the maximal weak solution satisfies exactly one of the following three possibilities:

(i) Proper: The solution exists and remains bounded for all time.
(ii) Sweeping: The solution entirely moves to infinity at some time $T \in (0, \infty)$.
(iii) Escaping: The solution exists until a time $T \in (0, \infty)$, then "jumps" to infinity.

In the proper case (i), we can obtain a topological obstruction using the monotonicity formula (1) as above. Now we consider the remaining cases (ii) (iii); see Figure 1 for examples of each of these cases.

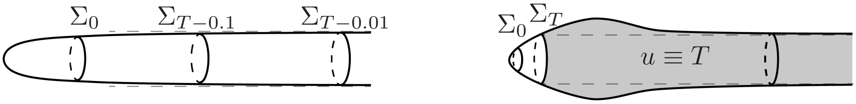

FIGURE 1. An IMCF sweeping out the manifold at $t = T$ (left) and one that escapes at $t = T$ (right).

First, we consider the case of sweeping flow. We can show that for a sequence of times $t_i \nearrow T$, the surfaces Σ_{t_i} have uniformly bounded diameters, are uniformly

$C^{1,\alpha}$-smooth, and are "almost" area-minimizing. Taking a subsequential limit, we obtain an area-minimizing hypersurface in some limit of M at infinity. By the scalar curvature lower bound, this limiting hypersurface must be $\mathbb{S}^2$ or $\mathbb{T}^2$, which in turn implies that all but finitely many Σ_{t_i} are $\mathbb{S}^2$ or $\mathbb{T}^2$. This again puts strong constraints on the topology of M.

Finally, we consider the case of escaping flow. In order to find a nice exhausting sequence and perform a limiting argument, we make a small perturbation of the metric so that it becomes "larger at infinity". This will delay the escape time of the maximal IMCF, thus some new level set will form in the edited region. Letting the edited region diverge to infinity and making the perturbation smaller and smaller, we obtain another diverging sequence of hypersurfaces which are "almost" area-minimizing as well. Then the limiting argument in case (ii) is employed to prove the main theorems.

References

[1] Yau, S. Problem section. *Seminar On Differential Geometry.* **102** pp. 669-706 (1982)
[2] Schoen, R. & Yau, S. Existence of Incompressible Minimal Surfaces and the Topology of Three Dimensional Manifolds with Non-Negative Scalar Curvature. *Annals Of Mathematics.* **110**, 127-142 (1979), http://www.jstor.org/stable/1971247
[3] Wang, J. Contractible 3-manifold and Positive scalar curvature (I). *J. Differential Geom..* **127**, 1267-1304 (2024), https://doi.org/10.4310/jdg/1721071503
[4] Wang, J. Contractible 3-manifolds and positive scalar curvature (II). *J. Eur. Math. Soc. (JEMS).* **26**, 537-572 (2024), https://doi.org/10.4171/jems/1366
[5] Chang, S., Weinberger, S. & Yu, G. Taming 3-manifolds using scalar curvature. *Geom. Dedicata.* **148** pp. 3-14 (2010), https://doi.org/10.1007/s10711-009-9402-1
[6] Gromov, M. Four lectures on scalar curvature. *Perspectives In Scalar Curvature. Vol. 1.* pp. 1-514
[7] Chodosh, O., Lai, Y. & Xu, K. 3-manifolds with positive scalar curvature and bounded geometry. *ArXiv Preprint ArXiv:2502.09727.* (2025)
[8] Wang, J. Topology of 3-manifolds with uniformly positive scalar curvature. (2023), https://arxiv.org/abs/2212.14383
[9] Bessières, L., Besson, G. & Maillot, S. Ricci flow on open 3-manifolds and positive scalar curvature. *Geom. Topol..* **15**, 927-975 (2011)
[10] Geroch, R. Energy Extraction. *Annals Of The New York Academy Of Sciences.* **224**, 108-117 (1973)
[11] Huisken, G. & Ilmanen, T. The inverse mean curvature flow and the Riemannian Penrose inequality. *J. Differential Geom..* **59**, 353-437 (2001), http://projecteuclid.org/euclid.jdg/1090349447
[12] Xu, K. Inverse mean curvature flow with outer obstacle. (2024), https://arxiv.org/abs/2405.15181

Einstein Constants and Differential Topology

Claude LeBrun

Recall that a Riemannian metric g is said to be *Einstein* if its Ricci curvature, considered as a function on the unit tangent bundle, is constant. This is of course equivalent to requiring that the Ricci tensor r of g satisfy

$$r = \lambda g \tag{1}$$

for a real number λ that is called the *Einstein constant* of g. In what follows, the term *Einstein manifold* will always mean a compact connected n-manifold M with empty boundary that is equipped with a Riemannian metric satisfying (1).

In dimension $n = 2$ or 3, an Einstein manifold necessarily has constant *sectional* curvature $\lambda/(n-1)$, and, so, in these low dimensions, the sign of λ is consequently predetermined by the size and structure of the fundamental group of M. Motivated in part by this observation, the introduction to the highly influential book *Einstein Manifolds* by the pseudonymous A.L. Besse concluded by asking [3, p. 19] whether the sign of λ is quite generally determined by the diffeomorphism type of M:

> "Can Einstein metrics with [Einstein] constants of opposite signs exist on the same manifold? If this is impossible, it would add weight to the remark... that positive and negative Einstein metrics belong to essentially distinct families."

Given the negative cast of the final sentence, this passage was often understood to be conjecturing that the answer to the question would turn out to be no. However, this expectation actually turned out to be incorrect. Indeed, roughly a decade after the appearance of Besse's book, a sequence of counter-examples was constructed by Fabrizio Catanese and the speaker [5]:

Theorem 1 (Catanese-LeBrun)**.** *For each $k \geq 2$, there is a closed simply-connected $4k$-manifold M that admits a pair of Einstein metrics with Einstein constants of opposite signs.*

A decade later, Rareş Răsdeaconu and Ioana Şuvaina then proved [14] a beautiful improvement of Theorem 1, by means of essentially the same strategy:

Theorem 2 (Răsdeaconu-Şuvaina)**.** *For every $k \geq 2$, there are at least $\binom{k+3}{3}$ distinct smooth closed simply-connected $4k$-manifolds that admit both $\lambda > 0$ and $\lambda < 0$ Einstein metrics.*

These results are proved by first constructing homotopy-equivalent pairs (X_ℓ, Y_ℓ) of compact complex surfaces, where $c_1(X_\ell) < 0$, but where $c_1(Y_\ell) > 0$. In fact, by refining a series of breakthrough results by various algebraic geometers [2, 10, 12, 13], the cited authors succeeded in constructing four such pairs (X_ℓ, Y_ℓ), where $c_1^2(X_\ell) = c_1^2(Y_\ell) = \ell$ for $\ell = 1, 2, 3, 4$. For each ℓ, the manifold Y_ℓ is diffeomorphic to the connected sum $\mathbb{CP}_2 \# (9-\ell)\overline{\mathbb{CP}}_2$, while X_ℓ is homeomorphic, but not diffeomorphic, to Y_ℓ; nonetheless, by a theorem of Wall [17], the 4-manifolds X_ℓ and Y_ℓ are still *h-cobordant* for every ℓ. The Cartesian products $X_{\ell_1} \times \cdots \times X_{\ell_k}$ and $Y_{\ell_1} \times \cdots \times Y_{\ell_k}$ are consequently also h-cobordant, so that, for any $k \geq 2$, Smale's h-cobordism theorem [15] implies that these products are actually diffeomorphic. However, the X_ℓ admit $\lambda = -1$ Einstein metrics (that are, incidentally, Kähler) by the Aubin-Yau theorem [1, 18], while the Y_ℓ all admit $\lambda = +1$ Einstein metrics (which are again, incidentally, Kähler) by the work of Tian-Yau [16]. Endowing $X_{\ell_1} \times \cdots \times X_{\ell_k}$ and $Y_{\ell_1} \times \cdots \times Y_{\ell_k}$ with the corresponding product Einstein metrics then yields the result, since these products have been shown to be diffeomorphic.

These examples certainly demonstrate that $\lambda > 0$ and $\lambda < 0$ Einstein metrics can coexist on specific smooth compact manifolds. But the list of manifolds where this phenomenon has actually been proved to occur remains surprisingly limited, and consists of spaces than share many rare and peculiar features. One of my key goals in giving this talk at Oberwolfach was therefore to challenge the community to try to find entirely new examples that would broaden and deepen our understanding of this phenomenon. For example, the examples of Theorems 1 and 2 only occur in even dimensions. Can one construct odd-dimensional examples of coexistence? Moreover, the constructed Einstein metrics occurring in Theorems 1 and 2 are never Ricci-flat. What can be proved, for example, about the coexistence of Einstein metrics with $\lambda = 0$ and $\lambda > 0$?

Of course, all the Einstein metrics used to prove Theorems 1 and 2 actually had special holonomy, and this naturally reflects the degree to which the majority of our most powerful methods for constructing Einstein metrics have essentially arisen in connection with special holonomy. Can any light be shed on the questions we have just raised by means of ideas related to special holonomy?

Well, any compact, odd-dimensional, non-locally-symmetric Riemannian manifold with irreducible special holonomy is [8] necessarily a 7-manifold of holonomy G_2, and every such manifold is automatically Ricci-flat. Fortunately, hundreds of thousands of diffeotypes of such closed simply-connected $\lambda = 0$ Einstein 7-manifolds are currently known [6]. On the other hand, there are also infinitely many diffeotypes of compact, simply-connected 7-manifolds that are now known to admit $\lambda > 0$ Einstein metrics [4], even among the examples that arise as *Sasaki-Einstein* manifolds. While the latter class of 7-manifolds *do not* have special holonomy, they are nonetheless characterized by the property that their metric cones have holonomy $\mathbf{SU}(4)$, and so are Calabi-Yau manifolds of real dimension eight.

All of this might seem to augur well, and one might therefore hope to find some compact simply-connected 7-manifolds that admitted special Einstein metrics of both these flavors. However, such hopes are, alas, misguided. These two types of Einstein metric can never coexist [9] on any smooth compact 7-manifold!

Theorem 3 (L '25)**.** *No smooth compact 7-manifold can admit both a Sasaki-Einstein metric g_1 and a metric g_2 of holonomy G_2.*

The key to proving this is the following:

Proposition 1. *If M is a smooth compact 7-manifold that carries a Sasaki-Einstein metric g_1, then the first Pontrjagin class $p_1(M) \in H^4(M,\mathbb{Z})$ is a torsion class.*

Indeed, any Sasaki-Einstein (M^7, g) carries a unit-length Killing field ξ, and the flow lines of ξ are then the leaves of the *Reeb foliation* $\mathfrak{F}$ of M. To prove the proposition, one first shows that the deRham version $p_1^{\mathbb{R}}(M)$ of the first Pontrjagin class $p_1(M)$ is represented by a closed 4-form that is *basic*, in the sense that its contraction with ξ is identically zero. This means that $p_1^{\mathbb{R}}(M)$ belongs to the image of the basic cohomology $H^4_B(M,\mathfrak{F})$ in the deRham cohomology $H^4_{dR}(M)$. However, because the local geometry of the leaf space $M/\mathfrak{F}$ is Kähler-Einstein, a transverse version [7] of the Hard Lefschetz Theorem holds, and this can then be used to show that the image of $H^4_B(M, \mathfrak{F}) \to H^4_{dR}(M)$ is actually zero. The proposition is therefore an immediate consequence.

On the other hand, if M^7 is a compact 7-manifold that admits a metric g_2 of holonomy G_2, and if φ is the fundamental closed 3-form corresponding to g_2, then one has the remarkable identity [8] that

$$\langle p_1^{\mathbb{R}}(M) \cup [\varphi], [M] \rangle = -\frac{1}{8\pi^2} \int_M |\mathcal{R}|^2 d\mu_{g_2} < 0 \tag{2}$$

where $\mathcal{R}$ denotes the Riemann curvature tensor of g_2. Thus, the existence of such a metric g_2 implies that $p_1^{\mathbb{R}}(M) \neq 0$, and Theorem 3 therefore becomes an immediate consequence of the Proposition.

For more details, along with various other related results, see [9].

References

[1] T. Aubin, *Équations du type Monge-Ampère sur les variétés kählleriennes compactes*, C. R. Acad. Sci. Paris Sér. A-B, 283 (1976), pp. Aiii, A119–A121.

[2] R. Barlow, *A simply connected surface of general type with $p_g = 0$*, Inv. Math., 79 (1985), pp. 293–301.

[3] A. L. Besse, *Einstein Manifolds*, vol. 10 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1987.

[4] C. P. Boyer and K. Galicki, *Sasakian Geometry*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2008.

[5] F. Catanese and C. LeBrun, *On the scalar curvature of Einstein manifolds*, Math. Res. Lett., 4 (1997), pp. 843–854.

[6] A. Corti, M. Haskins, J. Nordström, and T. Pacini, *G_2-manifolds and associative submanifolds via semi-Fano 3-folds*, Duke Math. J., 164 (2015), pp. 1971–2092.

[7] A. El Kacimi-Alaoui, *Opérateurs transversalement elliptiques sur un feuilletage riemannien et applications*, Compositio Math., 73 (1990), pp. 57–106.

[8] D. D. Joyce, *Compact Manifolds with Special Holonomy*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[9] C. LeBrun, *Einstein Constants and smooth topology, in* Real and Complex Geometry: in honour of Paul Gauduchon, edited by L. Ornea, Springer, 2025, pp. 219-235.

[10] Y. Lee and J. Park, *A simply connected surface of general type with $p_g = 0$ and $K^2 = 2$*, Invent. Math., 170 (2007), pp. 483–505.

[11] Y. Odaka, C. Spotti, and S. Sun, *Compact moduli spaces of del Pezzo surfaces and Kähler-Einstein metrics*, J. Differential Geom., 102 (2016), pp. 127–172.

[12] H. Park, J. Park, and D. Shin, *A simply connected surface of general type with $p_g = 0$ and $K^2 = 3$*, Geom. Topol., 13 (2009), pp. 743–767.

[13] ———, *A simply connected surface of general type with $p_g = 0$ and $K^2 = 4$*, Geom. Topol., 13 (2009), pp. 1483–1494.

[14] R. Răsdeaconu and I. Şuvaina, *Smooth structures and Einstein metrics on $\mathbb{CP}^2\#5, 6, 7\overline{\mathbb{CP}^2}$*, Math. Proc. Cambridge Philos. Soc., 147 (2009), pp. 409–417.

[15] S. Smale, *On the structure of manifolds*, Ann. Math., 87 (1964), pp. 387–399.

[16] G. Tian and S.-T. Yau, *Kähler-Einstein metrics on complex surfaces with $C_1 > 0$*, Comm. Math. Phys., 112 (1987), pp. 175–203.

[17] C. T. C. Wall, *On simply connected 4-manifolds*, J. Lond. Math. Soc., 39 (1964), pp. 141–149.

[18] S. T. Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. U.S.A., 74 (1977), pp. 1798–1799.

Large mass limit of G_2 and Calabi-Yau monopoles

Yang Li

Our general theme lies in the intersection between higher dimensional gauge theory and calibrated submanifolds. The gauge theory side may be viewed as the generalization of the classical theory of monopoles on $\mathbb{R}^3$. On a G_2 manifold (M, ϕ) with $\psi = *\phi$ (resp. Calabi-Yau 3-fold (M, ω, Ω)), let A be a connection on a principal $SU(2)$-bundle, and Φ be an adjoint valued section, called the Higgs field. We say (A, Φ) is a G_2-monopole (resp. Calabi-Yau monopole), if

$$F_A \wedge \psi = *\nabla\Phi,$$

resp.

$$F_A \wedge \mathrm{Re}\Omega = *\nabla\Phi, \quad F_A \wedge \omega^2 = 0.$$

An integration by parts argument shows that Φ is parallel on compact M. In the setting of asymptotically conical M, the previous work of Oliveira et al. [2] determined the asymptotic boundary condition under mild hypotheses: the mass

$$m := \lim_{x\to\infty} |\Phi|.$$

exists, the structure group reduces to $U(1)$ asymptotically, the connection converges to a pseudo-HYM connection on the link at infinity, and the Higgs field is asymptotically parallel. The upshot is that the asymptotic boundary condition is specified by the topology (which is fixed), except for the mass parameter $m > 0$, and the Donaldson-Segal programme [3] asks what happens in the limit $m \to +\infty$.

The answer can be summarized in the following slogans:

- In some L^1_{loc}-sense, the solutions converge to some $U(1)$ G_2 (resp. Calabi-Yau) monopole with Dirac singularity along a coassociative/special Lagrangian cycle Q;
- The part of curvature orthogonal to Φ is small in the L^1_{loc}-sense, and the part parallel to Φ dominates;
- The curvature density $\frac{1}{2\pi m}|F|^2 dvol$ concentrates on the support of Q, and almost all the energy can be accounted for by monopole bubbling transverse to Q.

The most striking consequence is that assuming the existence of the sequence (A, Φ), then it produces a coassociative (resp. special Lagrangian) cycle, within a prescribed homology class. This reduces the highly non-perturbative existence question for these calibrated cycles, to a question in gauge theory. This strategy is morally analogous to producing holomorphic curves by showing the non-triviality of some Seiberg-Witten invariant, using Taubes's famous work that $GR = SW$.

The big open question is whether we can define counting invariants for both the gauge theory and the calibrated submanifolds, and then prove some equality between the two invariants.

References

[1] Y. Li, *The large mass limit of G_2 and Calabi-Yau monopoles*, arXiv:2503.12075

[2] Fadel, Daniel; Nagy, Ákos; Oliveira, Goncalo. *The asymptotic geometry of* G_2*-monopoles.* Mem. Amer. Math. Soc. 303 (2024), no. 1521, v+85 pp. ISBN: 978-1-4704-7126-2; 978-1-4704-7990-9

[3] Donaldson, S. K.; Thomas, R. P. *Gauge theory in higher dimensions*. The geometric universe (Oxford, 1996), 31–47, Oxford Univ. Press, Oxford, 1998.

Constructing minimal surfaces of prescribed genus in closed Riemannian 3-spheres with positive Ricci curvature

Yangyang Li

(joint work with Adrian Chun-Pong Chu, Zhihan Wang)

In the round 3-sphere $\mathbb{S}^3$, there exists an $\mathbb{RP}^3$-family of minimal 2-spheres (the equatorial spheres) and an $\mathbb{RP}^2 \times \mathbb{RP}^2$-family of minimal tori (the Clifford tori). Almgren (1966)[1] and Calabi (1967)[2] later proved that the equatorial spheres are the only minimal 2-spheres, and Brendle (2013)[3] confirmed that the Clifford tori are the only minimal tori, thereby resolving the Lawson conjecture. The topology of these moduli spaces motivated Yau (1982)[4] and White (1989)[5] to conjecture the existence of at least four minimal spheres and five minimal tori, respectively, in any closed Riemannian 3-sphere. For other topological types, Lawson (1970)[6] constructed minimal surfaces of arbitrary genus in $\mathbb{S}^3$, now known as Lawson surfaces. Inspired by Yau's and White's conjectures, it is further expected that for any genus g, there exist multiple genus g minimal surfaces in any closed Riemannian 3-sphere, with the number related to the topology of the space of genus g Lawson surfaces.

In this talk, I present the resolution of Yau's conjecture by Wang-Zhou (2023)[7] and of White's conjecture by the work of Adrian and myself (2024)[8], in the setting of positive Ricci curvature. I then discuss how the techniques we developed can also be applied to construct minimal surfaces of higher genus, particularly genus 2. This is based on joint work with Adrian Chu and Zhihan Wang[9].

References

[1] F. J. Almgren Jr., Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem, Ann. of Math. (2) **84** (1966), 277–292; MR0200816

[2] E. Calabi, Minimal immersions of surfaces in Euclidean spheres, J. Differential Geometry **1** (1967), 111–125; MR0233294

[3] S. Brendle, Embedded minimal tori in S^3 and the Lawson conjecture, Acta Math. **211** (2013), no. 2, 177–190; MR3143888

[4] S.-T. Yau, Problem section, in *Seminar on Differential Geometry*, pp. 669–706, Ann. of Math. Stud., No. 102, Princeton Univ. Press, Princeton, NJ, ; MR0645762

[5] B. C. White, Every three-sphere of positive Ricci curvature contains a minimal embedded torus, Bull. Amer. Math. Soc. (N.S.) **21** (1989), no. 1, 71–75; MR0994891

[6] H. B. Lawson Jr., Complete minimal surfaces in S^3, Ann. of Math. (2) **92** (1970), 335–374; MR0270280

[7] Z. Wang, X. Zhou, Existence of four minimal spheres in S^3 with a bumpy metric, arXiv preprint arXiv:2305.08755 (2023)
[8] A. C.P. Chu, and Y. Li, Existence of 5 minimal tori in 3-spheres of positive Ricci curvature, arXiv preprint arXiv:2409.09315 (2024)
[9] A. C.P. Chu, Y. Li, and Z. Wang, Existence of genus 2 minimal surfaces in 3-spheres. I, in preparation

Einstein metrics, G_2 geometry and geometric flows

Jason D. Lotay
(joint work with Aaron Kennon, Jakob Stein)

Einstein metrics on compact Riemannian manifolds are of considerable interest but are currently poorly understood outside of Kähler geometry, particularly those which are Ricci-flat. As a case in point, the only non-trivial odd-dimensional compact Ricci-flat manifolds are G_2*-manifolds*: that is, 7-dimensional Riemannian manifolds with holonomy G_2. On the other hand, there are many infinitely many compact *nearly* G_2*-manifolds*, which have positive Einstein metrics.

Given the success of geometric flows in studying both Riemannian geometry and special structures, especially Kähler structures, it is natural to ask whether one can use such methods in G_2 geometry. This geometry on a 7-manifold is encoded by a G_2-structure 3-form φ, which defines a metric and orientation, thus a dual 4-form $\psi = *\varphi$. A geometric flow, called the G_2*-Laplacian coflow*, was proposed [3] which evolves *closed* 4-forms ψ by the Hodge Laplacian defined by φ:

$$\text{(LF)} \qquad \frac{\partial \psi}{\partial t} = \Delta_\varphi \psi \quad \text{and} \quad \mathrm{d}\psi = 0.$$

This is the gradient flow of the Hitchin volume functional on the cohomology class $[\psi_0]$ of the initial condition and has G_2-manifolds as its critical points, which are strict maxima (modulo diffeomorphisms). Moreover, nearly G_2-manifolds are solitons for (LF) or, equivalently, critical points for a suitably normalized (LF).

A rich source of G_2-structures comes from 3-Sasakian geometry (which includes the round 7-sphere $\mathcal{S}^7$): 3-Sasakian 7-manifolds always admit two natural continuous 3-parameter families $\Psi^\pm$ of closed 4-forms dual to G_2-structure 3-forms, and each family contains a 1-parameter family, depending on $\kappa > 0$, of nearly G_2-manifolds defined by 4-forms $\psi^\pm_\kappa$. (In particular, one sees that all 3-Sasakian 7-manifolds admit two canonical positive Einstein metrics.) The first result shows that the $\psi^\pm_\kappa$ are global attractors for (LF) in $\Psi^\pm$ after rescaling.

Theorem 1 (Kennon–L. [4])**.** *Let $\kappa > 0$ and let $\psi^\pm \in \Psi^\pm$ on a 3-Sasakian 7-manifold. Then, after suitable normalization, the* G_2*-Laplacian coflow* (LF) *exists in $\Psi^\pm$ for all time and converges to the nearly* G_2 $\psi^\pm_\kappa$*.*

This theorem is proved using methods inspired by dynamical systems, as the problem reduces to a nonlinear ODE system, with some care required as the system degenerates near the boundary of the parameter space defining $\Psi^\pm$. Theorem 1 in particular shows that the nearly G_2 $\psi^\pm_\kappa$ are stable for normalized (LF), at least in $\Psi^{\pm}$; work in [5] indicates that potentially all nearly G_2 manifolds are stable as solitons for (LF), not just those in $\Psi^{\pm}$ on 3-Sasakian 7-manifolds.

Despite the good geometric properties of (LF) and the convergence result in Theorem 1, unfortunately (LF) is not known to even have short-time existence. As a consequence a *modified* G_2-*Laplacian coflow* was introduced [2] which now has short-time existence, enjoys Shi-type estimates and can be defined for any $\kappa \geq 0$:

$$\text{(MF)} \qquad \frac{\partial \psi}{\partial t} = \Delta_{\varphi}\psi + \tfrac{1}{2}\mathrm{d}\big((5\kappa - 7\tau_0)\varphi\big) \quad \text{and} \quad \mathrm{d}\psi = 0,$$

where $7\tau_0 = *(\varphi \wedge \mathrm{d}\varphi)$. We have that G_2-manifolds are still critical points of (MF) and nearly G_2-manifolds are solitons (or critical points for a normalized flow). It therefore seems important to understand what behaviour nearly G_2-manifolds have for the modified G_2-Laplacian coflow (MF).

Theorem 2 (L.–Stein [5]). *Let $\kappa > 0$ and recall the nearly G_2 $\psi_\kappa^{\pm} \in \Psi^{\pm}$ on a 3-Sasakian 7-manifold. Then $\psi_\kappa^{\pm}$ is unstable as a critical point of normalized modified G_2-Laplacian coflow* (MF)*, with index 1 in $\Psi^{\pm}$.*

Theorem 2 is proved again using ODE systems techniques yet it already contrasts with Theorem 1. One is naturally motivated to ask further whether Theorem 2 provides all of the unstable directions for the nearly G_2-manifolds given by $\psi_\kappa^{\pm}$. This turns out to very much not to be the case, as the following result shows for the round 7-sphere (which, we recall, is a particular case of $\psi_\kappa^{\pm}$ on a 3-Sasakian 7-manifold).

Theorem 3 (L.–Stein [5]). *Consider the round 7-sphere $\mathcal{S}^7$ with its canonical nearly G_2-structure with dual 4-form ψ. Then ψ is unstable as a critical point for normalized modified G_2-Laplacian coflow* (MF) *with index at least* 7047.

The key to proving Theorem 3 is to identify the unstable directions with eigenforms for $\mathrm{d}*$ acting on certain exact 4-forms with particular eigenvalues. Then the result follows from representation theory, building on [1], since $\mathrm{d}*$ and the space on which it is acting are invariant under the isometry group of $\mathcal{S}^7$.

The results show that (MF) has both positive and negative features in comparison to (LF). One can interpret the instability results Theorem 2–3 for nearly G_2-manifolds positively as saying that one should be able to perturb initial conditions so that these should not appear along the flow (MF), which might make (MF) useful as a means to find G_2-manifolds. However, the flow (LF) is more natural and has very good geometric features, so it seems worthwhile to try to obtain the analytic results one needs to make (LF) viable as a means to study Einstein metrics and G_2 geometry.

References

[1] C. Bär, *The curl operator on odd-dimensional manifolds*, J. Math. Phys. **60** (2019), no. 3, 031501, 16.

[2] S. Grigorian, *Short-time behaviour of a modified Laplacian coflow of* G_2*-structures*, Adv. Math. **248** (2013), 378–415.

[3] S. Karigiannis, B. McKay, and M-P. Tsui, *Soliton solutions for the Laplacian co-flow of some* G_2 *structures with symmetry*, Differential Geom. Appl. **30** (2012), no. 4, 318–333.

[4] A. Kennon and J.D. Lotay, *Geometric flows of* G_2*-structures on* 3*-Sasakian* 7*-manifolds*, J. Geom. Phys. **187** (2023), 104793.

[5] J.D. Lotay and J. Stein, *Nearly* G_2*-manifolds and* G_2*-Laplacian co-flows*, arXiv:2505.14121.

A finiteness theorem for isoparametric foliations

Alexander Lytchak

(joint work with Manuel Krannich, Marco Radeschi)

Isoparametric submanifolds in space forms have been introduced by Levi-Civita and studied for almost a century from differential geometric, topological and algebraic perspectives by Segre, Cartan, Muenzner, Ferus, Karcher, Abresch, Palais, Terng, Thorbergsson, Stolz, Chi, Cecil and many others; we refer for an overview to [4]. The concept has been generalized to symmetric spaces and arbitrary Riemannian manifolds by Palais, Terng, Thorbergsson, Wang and Alexandrino, we refer to another survey [5] for the description and the history of this development. While in space forms and in symmetric spaces one might hope and sometimes achieve a classification, in more general situation only qualitative topological-geometric questions are meaningful.

The talk is devoted to the following finiteness theorem obtained jointly with Manuel Krannich and Marco Radeschi.

Theorem 1. *Given* n, v, κ, D *there exists at most finitely many isoparametric foliations up to homeomorphisms on compact simply connected Riemannian manifolds of dimension* n*, volume at least* v*, diameter at most* D *and sectional curvatures bounded in absolute values by* κ*.*

If $n \neq 5$*, the finiteness statement holds up to diffeomorphisms.*

The theorem is obtained by reformulating the result in terms of submetries onto intervals, using compactness of such objects and analyzing the limiting procedure in great details. While many parts of the proof are purely topological, the statement is not. Moreover, being foliated homeomorphic turns our to be a much finer invariant than having pairwise diffeomorphic fibers as the following result shows. It is based on results in geometric topology distinguishing between concordance and isotopies:

Theorem 2. *For* $n \geq 5$ *there exists infinitely many Riemannian metrics on the sphere* $\mathbb{S}^n$ *each equipped with an isoparametric foliation whose singular fibers are the canonical* $\mathbb{S}^{n-2}$ *and* $\mathbb{S}^1$ *and whose regular fibers are diffeomorphic to* $\mathbb{S}^{n-2} \times \mathbb{S}^1$*, such that the foliations are pairwise not foliated homeomorphic.*

We explain and verify in the talk why some assumption on the fundamental group is needed in the statement.

An important step in the proof of the main result is the observation that for a manifold submetry from a simply connected manifold M of bounded geometry onto an interval, the interval is uniformly non-collapsed. The second step, the actual stability argument, verifies the validity of the following conjecture in the case of one-dimensional base spaces. Other special cases of this theorem are provided by Perelman's stability theorem [2], the equivariant stability theorem of Harvey [1] and the case of Riemannian submersions verified by Tapp [3].

Conjecture 1. *Let $P : M_i \to Y_i$ be manifold submetries. Assume that M_i are compact, of uniformly bounded geometry and converge in the Gromov–Hausdroff topology to M. Assume that P_i converge to a submetry P. If the sequences M_i and Y_i do not collapse, then, for all i, j large enough, P_i and P_j are equivalent up to homeomorphisms between M_i and M_j and Y_i and Y_j, respectively.*

References

[1] J. Harvey, *Equivariant Alexandrov geometry and orbifold finiteness*, J. Geom. Anal. **26** (2016), no. 3, 1925–1945. MR 3511464

[2] V. Kapovitch, *Perelman's stability theorem*, Surveys in differential geometry. Vol. XI, (2007), pp. 103–136.

[3] K. Tapp, *Bounded Riemannian submersions*, Indiana Univ. Math. J. **49** (2000), no. 2, 637–654.

[4] G. Thorbegrsson, *A survey on isoparametric hypersurfaces and their generalizations*, Handbook of differential geometry, **1** (2000), 963–995. .

[5] G. Thorbegrsson, *From isoparametric submanifolds to polar foliations*, Sao Paolo J. Math. Sci. **16** (2020), 459–472.

An Overview of Gromov's p-widths and Their Applications

Jared Marx-Kuo

For (M^{n+1}, g) a Riemannian manifold, the "p-widths" (also known as the *volume spectrum*), $\{\omega_p\}$, are a sequence of invariants introduced by Gromov [8, 9]. Let $\mathcal{P}_p$ denote the set of p-sweepouts, $\Phi : X^p \to \mathcal{Z}_n(M, \mathbb{Z}/2\mathbb{Z})$ (see [6] for formal definitions) such that $\Phi^*(\lambda^p) \neq 0 \in H^p(X)$. Then we define

$$\omega_p = \inf_{\Phi \in \mathcal{P}_p} \sup_{x \in \mathrm{Dom}(\Phi)} \mathbf{M}(\Phi(x))$$

The reader may think of Φ as a map from some p-dimensional manifold, $X = \mathrm{Dom}(\Phi)$, to the space of hypersurfaces, replacing $\mathbf{M}(\Phi(x)) \to \mathrm{Area}(\Phi(x))$. In particular, we compare this with the following definition of λ_p, the pth eigenvalue of the laplacian on a closed manifold. Let $\tilde{P}_p$ denote the space of all at most p-dimensional subspaces of $H^1(M)$. Then

$$\lambda_p = \inf_{V \in \tilde{P}_p} \sup_{0 \neq f \in V} \frac{\int_M |\nabla f|^2}{\int_M f^2}$$

If $\Delta u_p = \lambda_p u_p$, then u_p is critical for the L^2 normalized dirichlet energy.

Originally posed as a non-linear spectrum of the volume functional, the p-widths have been essential in the study of the existence of minimal surfaces over the past decade. Given the same min-max formulation as the eigenvalues of the Laplacian, we expect ω_p to yield critical points of the area functional on hypersurfaces. Indeed, we have

Theorem 1 ([16]). *For $3 \leq n+1 \leq 7$ and each $p \in \mathbb{Z}^+$,*

$$\omega_p = \sum_{i=1}^{N_p} m_i^p A(\Sigma_i^p) \tag{1}$$

for some collection of $\{\Sigma_i^p\}$, smooth, embedded, disjoint minimal surfaces, and some multiplicities, $m_i^p \in \mathbb{Z}^+$.

The p-widths were an essential tool in the resolution of Yau's conjecture ([23, Problem Section]):

Theorem 2 ([3, 22, 17, 11, 24, 12]). *On any closed (M^{n+1}, g) with $n+1 \geq 3$, there exist infinitely many embedded minimal hypersurfaces.*

When $m_i^p = 1$ for all i, p, i.e. the so called "multiplicity one" setting, we can deduce that there are infinitely many minimal surfaces if $\lim_{p\to\infty} \omega_p = \infty$. Indeed, this holds due to the following Weyl law:

Theorem 3 ([9, 10, 14]). *For all $n+1 \geq 2$, there exists a constant $a(n+1)$ such that*

$$\lim_{p\to\infty} \omega_p p^{-1/(n+1)} = a(n+1) Vol(M)^{n/(n+1)}$$

The reader may compare this with an analogous Weyl law for the eigenvalues of the laplacian on a closed manifold. We note that even when there is multiplicity (i.e. $m_i^p > 1$ in equation (1)), the *sublinear* growth of $\{\omega_p\}$ was extremely useful in showing the existence of infinitely many minimal surfaces in [17, 22].

In addition to their existence, the minimal surfaces from equation (1) record the parameter p in their index, in the "multiplicity one" setting

Theorem 4 ([24]). *For g a generic metric on M^{n+1}, $3 \leq n+1 \leq 7$, $m_i^p = 1$ in equation*

This leads to sharp index estimates for Σ_i^p:

Theorem 5 ([16, 24]). *For g a generic metric as above*

$$p = \sum_{i=1}^{N_p} Ind(\Sigma_i^p)$$

We also note that the p-widths have been applied to construct constant mean curvature (CMC) hypersurfaces using mountainpass constructions [5], as well as CMC hypersurfaces which bound a set of half volume [21]. More recently, there have been applications to showing the existence of prescribed mean curvature (PMC) hypersurfaces for certain prescribing functions and ambient manifolds [7].

In the setting of surfaces, the p-widths behave differently.

Theorem 6 ([4]). *For (M^2, g) closed and each $p \in \mathbb{Z}^+$,*

$$\omega_p = \sum_{i=1}^{N_p} m_i^p \ell(\gamma_i^p)$$

for some collection of $\{\gamma_i^p\}$, smooth, immersed, (potentially non-disjoint) geodesics, and some multiplicities $m_i^p \in \mathbb{Z}^+$.

We emphasize the lack of disjointness and embeddedness. In this setting, it is natural ask

a) Do there exist surfaces for which ω_p are realized by non-embedded geodesics?
b) Do index bounds for the $\{\gamma_i^p\}$ hold?
c) Does multiplicity one hold generically?

In joint work with Mantoulidis [15], the author showed that the answer to a) is "no" by constructing a realization of Almgren's starfish such that $\omega_1 = \ell(\gamma)$, for γ a topological figure eight. See also related work by Lima [13] for an example on hyperbolic surfaces of genus at least 2. In joint with Sarnataro and Stryker [20], the author showed that the index of the $\{\gamma_i^p\}$ and the number of intersections (including self-intersections) are bounded above by p. In addition, it was shown in [20] that multiplicity one does not hold generically, building off of work of Chodosh–Mantoulidis [4] and Aiex [1].

Surfaces are an interesting setting for the p-widths in that they provide the simplest examples for which we can compute the p-widths for all p. In fact, the p-widths are *only known for all p* in the following three cases:

(1) (S^2, g_{round}) due to deep work of Chodosh–Mantoulidis [4]
(2) $(\mathbb{RP}^2, \overline{g}_{round})$ due to an adaption by the author [19]
(3) (S^2, g) where g is any Zoll metric connected to g_{round} [18] due to a short argument by the author.

The author's construction with Zoll metrics in [18] was the first step towards the *isospectral problem* for the p-widths: for which manifolds do the values of $\{\omega_p\}$ determine (M, g)? While the author showed that there is no such rigidity for the sphere, Ambrozio–Marques–Neves showed that $\mathbb{RP}^2$ is rigid:

Theorem 7 ([2]). *Given (M^{n+1}, g) closed, suppose that*

$$\omega_p = \omega_p(\mathbb{RP}^2, \overline{g}_{round})$$

for all p. Then $(M, g) \cong (\mathbb{RP}^2, \overline{g}_{round})$.

In general, it would be extremely interesting to compute $\{\omega_p\}$ for all values of p for the simplest manifolds possible. It would also be of great interest if other manifolds are p-width isospectrally rigid.

References

[1] Nicolau Sarquis Aiex. The width of ellipsoids. *arXiv preprint arXiv:1601.01032*, 2016.

[2] Lucas Ambrozio, Fernando C Marques, and André Neves. Rigidity theorems for the area widths of riemannian manifolds. *arXiv preprint arXiv:2408.14375*, 2024.

[3] Otis Chodosh and Christos Mantoulidis. Minimal surfaces and the allen–cahn equation on 3-manifolds: Index, multiplicity, and curvature estimates. *Annals of Mathematics*, 191(1):213–328, 2020.

[4] Otis Chodosh and Christos Mantoulidis. The p-widths of a surface. *Publications mathématiques de l'IHÉS*, 137(1):245–342, 2023.

[5] Akashdeep Dey. Existence of multiple closed cmc hypersurfaces with small mean curvature. *Journal of Differential Geometry*, 125(2):379–403, 2023.

[6] Ailana Fraser, André Neves, Peter M Topping, Paul C Yang, Fernando C Marques, and André Neves. Applications of min–max methods to geometry. *Geometric Analysis: Cetraro, Italy 2018*, pages 41–77, 2020.

[7] Pedro Gaspar and Jared Marx-Kuo. Infinitely many surfaces with prescribed mean curvature in the presence of a strictly stable minimal surface. *arXiv preprint arXiv:2502.07098*, 2025.

[8] Mikhail Gromov. Isoperimetry of waists and concentration of maps. Technical report, SIS-2003-246, 2002.

[9] Mikhail Gromov. Dimension, non-linear spectra and width. In *Geometric Aspects of Functional Analysis: Israel Seminar (GAFA) 1986–87*, pages 132–184. Springer, 2006.

[10] Larry Guth. Minimax problems related to cup powers and steenrod squares. *Geometric And Functional Analysis*, 18:1917–1987, 2009.

[11] Kei Irie, Fernando C Marques, and André Neves. Density of minimal hypersurfaces for generic metrics. *Annals of Mathematics*, 187(3):963–972, 2018.

[12] Yangyang Li. Existence of infinitely many minimal hypersurfaces in higher-dimensional closed manifolds with generic metrics. *Journal of Differential Geometry*, 124(2):381–395, 2023.

[13] Vanderson Lima. On the first width of hyperbolic surfaces: multiplicity and lower bounds. *arXiv preprint arXiv:2503.14846*, 2025.

[14] Yevgeny Liokumovich, Fernando C Marques, and André Neves. Weyl law for the volume spectrum. *Annals of mathematics*, 187(3):933–961, 2018.

[15] Christos Mantoulidis and Jared Marx-Kuo. Almgren's three-legged starfish. *arXiv preprint arXiv:2504.20895*, 2025.

[16] Fernando C Marques and André Neves. Morse index and multiplicity of min-max minimal hypersurfaces. *arXiv preprint arXiv:1512.06460*, 2015.

[17] Fernando C Marques and André Neves. Existence of infinitely many minimal hypersurfaces in positive ricci curvature. *Inventiones mathematicae*, 209(2):577–616, 2017.

[18] Jared Marx-Kuo. The isospectral problem for p-widths: an application of zoll metrics. *Comptes Rendus. Mathématique*, 363(G6):565–570, 2025.

[19] Jared Marx-Kuo. The p-widths of rp^2. *arXiv preprint arXiv:2501.13311*, 2025.

[20] Jared Marx-Kuo, Lorenzo Sarnataro, and Douglas Stryker. Index, intersections, and multiplicity of min-max geodesics. *arXiv preprint arXiv:2410.02580*, 2024.

[21] Liam Mazurowski and Xin Zhou. Infinitely many half-volume constant mean curvature hypersurfaces via min-max theory. *arXiv preprint arXiv:2405.00595*, 2024.

[22] Antoine Song. Existence of infinitely many minimal hypersurfaces in closed manifolds. *Annals of Mathematics*, 197(3):859–895, 2023.

[23] Shing-Tung Yau. *Seminar on differential geometry*. Number 102. Princeton University Press, 1982.

[24] Xin Zhou. On the multiplicity one conjecture in min-max theory. *Annals of Mathematics*, 192(3):767–820, 2020.

Energy minimizing harmonic 2-spheres in metric spaces

Damaris Meier

(joint work with Noa Vikman, Stefan Wenger)

The classical existence problem for harmonic maps between a closed surface M, equipped with a fixed Riemannian metric, and a compact Riemannian manifold N asks whether a given continuous map $\varphi\colon M \to N$ can be deformed, via homotopy, into a harmonic map $u\colon M \to N$. A result of Lemaire [1], Schoen-Yau [7] and Sacks-Uhlenbeck [6] shows that if the second homotopy group $\pi_2(N)$ is trivial, then every homotopy class of maps from M to N contains an energy minimizing harmonic map. However, when $\pi_2(N)$ is non-trivial, the situation is more involved, and a phenomenon known as "bubbling" may occur. The remarkable work of Sacks-Uhlenbeck [6] shows that if N is a compact Riemannian manifold with $\pi_2(N) \neq 0$, then there exists a non-contractible map $u\colon S^2 \to N$ minimizing energy and area within its homotopy class. Moreover, every such u is a conformal branched immersion.

In joint work with Vikman and Wenger [5], we explore a novel and conceptually simple approach to proving existence of energy minimizers in homotopy classes. Unlike the method used in [6], our approach does not rely on PDE results and is applicable to a wide range of metric space targets X. It is based on the existence and regularity of energy and area-minimizing Sobolev maps in proper metric spaces satisfying a local quadratic isoperimetric inequality proven in a series of works by Lytchak-Wenger, see e.g. [2, 3, 4]. Additionally, we assume that the target space X is compact and quasiconvex. The last requirement is that every continuous map from S^2 to X of sufficiently small diameter is null-homotopic. These conditions are satisfied by a broad class of spaces, including compact Riemannian manifolds, compact Finsler manifolds, more generally, compact Lipschitz manifolds, compact locally CAT(κ) spaces for $\kappa \in \mathbb{R}$, and many more.

Fix a suitable notion of energy and denote by $e(\varphi)$ the infimal energy over all Sobolev maps contained in the homotopy class of a given continuous map φ. The following main theorem of our work [5, Theorem 1.3] generalizes the above mentioned results from [1, 7, 6].

Theorem 1. *Let M and X be as above. Then every continuous map $\varphi\colon M \to X$ has an iterated decomposition $\varphi_0\colon M \to X$ and $\varphi_1, \dots, \varphi_k\colon S^2 \to X$ satisfying*

$$e(\varphi_0) + e(\varphi_1) + \cdots + e(\varphi_k) = e(\varphi)$$

and such that every φ_i contains an energy minimizer in its homotopy class.

The proof of Theorem 1 follows a direct variational approach and crucially depends on the following two key ingredients: A convergence result for minimizing sequences of uniformly distributed energy [5, Theorem 8.1] and a result [5, Propositions 9.1 and 9.2] showing that (up to possibly precomposing with a conformal diffeomorphism) every almost energy minimizer in the homotopy class of an ε-indecomposable map has small energy on balls of small radius. In particular, after suitably decomposing the initial map φ, the latter result implies the applicability of the convergence result.

In addition to the existence result from Theorem 1, we show the following regularity properties of homotopic energy minimizers [5, Theorem 6.3 and Corollary 3.3]: Every continuous Sobolev map $u\colon M \to X$ that minimizes energy in its homotopy class is harmonic (i.e. locally minimizes energy) and Hölder continuous. If $M = S^2$, then u is also infinitesimally quasiconformal. We do not know whether the same holds true in the more general context of harmonic spheres.

Question 1. *Let X be as above and let $u\colon S^2 \to X$ be a harmonic map. Is it true that u is infinitesimally quasiconformal?*

Using the techniques developed in [6], Sacks-Uhlenbeck moreover showed that every closed Riemannian manifold with non-trivial k-th homotopy group for some $k \geq 2$ contains a non-trivial harmonic 2-sphere, see [6, Theorem 5.7]. It would be interesting to know whether a similar result holds in a metric setting.

Question 2. *Let X be a compact metric space with non-trivial k-th homotopy group for some $k \geq 2$. Under what additional conditions does X admit a non-trivial harmonic 2-sphere?*

References

[1] L. Lemaire, *Applications harmoniques de surfaces riemanniennes*, J. Differential Geom., 13(1):51–78, 1978.

[2] A. Lytchak and S. Wenger, *Regularity of harmonic discs in spaces with quadratic isoperimetric inequality*, Calc. Var. Partial Differential Equations, 55(4):55:98, 2016.

[3] A. Lytchak and S. Wenger, *Area minimizing discs in metric spaces*, Arch. Ration. Mech. Anal., 223(3):1123–1182, 2017.

[4] A. Lytchak and S. Wenger, *Energy and area minimizers in metric spaces*, Adv. Calc. Var., 10(4):407–421, 2017.

[5] D. Meier, N. Vikman and S. Wenger, *Energy minimizing harmonic 2-spheres in metric spaces*, Preprint arXiv:2503.08553, 2025.

[6] J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of 2-spheres*, Ann. of Math. (2), 113(1):1–24, 1981.

[7] R. Schoen and Shing Tung Yau, *Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature*, Ann. of Math. (2), 110(1):127–142, 1979.

Complete 3-manifolds of positive scalar curvature with quadratic decay

Stéphane Sabourau

(joint work with Florent Balacheff, Teo Gil Moreno de Mora Sardà)

In his Problem Section [10], Yau asked for a classification of 3-manifolds that admit a Riemannian metric of positive scalar curvature. The closed case was addressed by Schoen–Yau [6, 7] using minimal surfaces and in parallel by Gromov–Lawson [3, 4, 5] using both minimal surfaces and the Dirac operator method, and finally concluded in the light of Perelman's work. They proved that a closed orientable 3-manifold which admits a Riemannian metric with positive scalar curvature decomposes as a connected sum of spherical manifolds and $\mathbb{S}^2 \times \mathbb{S}^1$ summands. A similar decomposition theorem has recently been proved for open manifolds admitting complete Riemannian metrics of uniformly positive scalar curvature independently by Gromov [2] and Wang [9], using μ-bubble theory.

Theorem 1 ([2, 9]). *Let M be a complete orientable Riemannian 3-manifold with uniformly positive scalar curvature. Then M decomposes as a possibly infinite connected sum of spherical manifolds and $\mathbb{S}^2 \times \mathbb{S}^1$.*

We will consider 3-manifolds admitting a complete Riemannian metric of positive scalar curvature with at most a quadratic decay at infinity.

Definition 1. *Let M be a complete Riemannian n-dimensional manifold. Fix a basepoint $x \in M$, and denote by $r_x(y) = d(x, y)$ the distance function to x. The scalar curvature of M has a* at most C-quadratic decay at infinity *with $C > 0$ if there exists a constant $R_0 > 0$ such that for every $y \in M$ with $r_x(y) \geq R_0$,*

$$\mathrm{scal}(y) > \frac{C}{r_x(y)^2}.$$

Our main theorem extends the topological decomposition of Theorem 1 to complete Riemannian 3-manifolds of positive scalar curvature with at most a quadratic decay at infinity for some constant $C > 64\pi^2$.

Theorem 2 ([1])**.** *Let M be a complete orientable Riemannian 3-manifold. Suppose that M has positive scalar curvature with at most C-quadratic decay at infinity for some $C > 64\pi^2$. Then M decomposes as a possibly infinite connected sum of spherical manifolds and $\mathbb{S}^2 \times \mathbb{S}^1$ summands.*

One may wonder whether the conclusion of Theorem 2 holds under a weaker decay rate. The example of the manifold $\mathbb{R}^2 \times \mathbb{S}^1$ shows this is impossible. Indeed, the manifold $\mathbb{R}^2 \times \mathbb{S}^1$ admits a complete metric of positive scalar curvature decaying $\frac{1}{2}$-quadratically at infinity, but it does not decompose as an infinite connected sum of spherical manifolds and $\mathbb{S}^2 \times \mathbb{S}^1$. Therefore, the decay rate in Theorem 2 is optimal. As for the optimal value of the decay constant C under which the conclusion of Theorem 2 holds, this last example shows that we cannot hope for more than $C > \frac{1}{2}$ (while our result holds for $C > 64\pi^2$).

More generally, Gromov conjectured the following [2, Section 3.6.1].

Conjecture 1 (Critical Rate of Decay Conjecture [2])**.** *There exists a dimensional constant $C_n > 0$ such that the following holds. Let M be an orientable n-manifold that admits a complete Riemannian metric of positive scalar curvature.*

(1) For every $C < C_n$, there exists a complete Riemannian metric on M of positive scalar curvature with at most C-quadratic decay at infinity.

(2) If M admits a complete Riemannian metric with positive scalar curvature with C-quadratic decay at infinity for $C > C_n$, then M admits a complete Riemannian metric with uniformly positive scalar curvature.

The following rigidity result, which addresses the case (2) of Conjecture 1, is a direct consequence of Theorem 2 and an adaptation of Gromov–Lawson's Surgery Theorem.

Corollary 1 ([1])**.** *Let M be an orientable 3-manifold. If M admits a complete Riemannian metric of positive scalar curvature with at most C-quadratic decay at infinity for some $C > 64\pi^2$, it also admits a complete Riemannian metric with uniformly positive scalar curvature.*

Actually, we will deduce Theorem 2 from a more general statement, which involves the following notion of fill radius.

Definition 2. *Let M be a Riemannian n-manifold with possibly nonempty boundary. The* fill radius *of a contractible closed curve γ in M is defined as*

$$\operatorname{fillrad}(\gamma) := \sup\{R \geq 0 \mid d(\gamma, \partial M) > R \text{ and } [\gamma] \neq 0 \in \pi_1(U(\gamma, R))\}$$

where $U(\gamma, R)$ denotes the closed R-neighborhood of γ in M. Define also

$$\operatorname{fillrad}(M) := \sup\{\operatorname{fillrad}(\gamma) \mid \gamma \text{ contractible closed curve of } M\}.$$

The following bound on the fill radius has been established in [5, 8]: If M is a complete orientable 3-manifold with bounded geometry and uniformly positive scalar curvature $\operatorname{scal} \geq s_0 > 0$, then the universal Riemannian cover $\tilde{M}$ of M satisfies $\operatorname{fillrad}(\tilde{M}) \leq 2\pi/\sqrt{s_0}$. Therefore, an upper bound on the fill radius of the universal cover provides a generalization of the notion of uniformly positive scalar curvature.

If a complete orientable 3-manifold M has positive scalar curvature decaying at infinity, then the fill radius is not necessarily bounded in general. Still, if the decay is not too pronounced, one can control the growth of the fill radius of the lifts to the universal cover of the closed curves contractible in M. This property will serve as a generalization of the notion of positive scalar curvature with at most C-quadratic decay at infinity.

Definition 3. *Let M be a complete Riemannian manifold, and denote by $\tilde{M}$ its universal Riemannian cover. Fix a basepoint $x \in M$. Denote by $B(x, R)$ the closed metric ball of radius R centered at x. The fill radius of $\tilde{M}$ has* at most c-linear growth at infinity *with $c > 0$ if there is a constant $R'_0 \geq 0$ such that if $R \geq R'_0$, then for every closed curve γ lying in $B(x, R)$ and contractible in M, any of its lifts $\tilde{\gamma}$ to $\tilde{M}$ satisfies*

$$\operatorname{fillrad}(\tilde{\gamma}) < cR.$$

We will prove the topological decomposition of Theorem 2 by replacing the scalar curvature assumption with a weaker condition about the filling disks of the lifts of contractible closed curves, namely that the fill radius of $\tilde{M}$ has at most c-linear growth at infinity with $c < \frac{1}{3}$.

Theorem 3 ([1]). *Let M be an orientable complete Riemannian 3-manifold, and denote by $\tilde{M}$ its universal Riemannian cover. Suppose that the fill radius of $\tilde{M}$ has at most c-linear growth at infinity for some $c < \frac{1}{3}$. Then M decomposes as a possibly infinite connected sum of spherical manifolds and $\mathbb{S}^2 \times \mathbb{S}^1$.*

This result yields the same topological decomposition as Gromov–Wang's theorem under a weaker, more robust, assumption. In particular, it applies to metrics of positive scalar curvature with at most C-quadratic decay at infinity for some $C > 64\pi^2$, and not just of uniformly positive scalar curvature. More generally, Theorem 3 does not require any curvature assumption and relies on $\mathcal{C}^0$ topological arguments, rather than on $\mathcal{C}^2$ analytical ones. In particular, the proof of Theorem 3 relies neither on the μ-bubble theory, nor on the minimal surface approach. Interestingly, and somewhat surprisingly, this general approach leads to an optimal statement in the decay rate at infinity despite the lack of analytical tools.

References

[1] F. Balacheff, T. Gil Moreno de Mora Sardà, S. Sabourau, *Complete 3-manifolds of positive scalar curvature with quadratic decay*, Mathematische Annalen (2025). https://doi.org/10.1007/s00208-025-03192-9

[2] M. Gromov, *Four lectures on scalar curvature*, In *Perspectives in scalar curvature. Vol. 1*, pages 1–514. World Sci. Publ., Hackensack, NJ, 2023.

[3] M. Gromov, B. Lawson, *Spin and scalar curvature in the presence of a fundamental group. I.* Annals of Mathematics (2) **111** (1980), 209–230.

[4] M. Gromov, B. Lawson, *The classification of simply connected manifolds of positive scalar curvature*, Annals of Mathematics (2) **111** (1980), 423–434.

[5] M. Gromov, B. Lawson, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Publications Mathématiques de l'IHÉS **58** (1983), 83–196.

[6] R. Schoen, S.-T. Yau, *Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature*, Annals of Mathematics (2) **110** (1979), 127–142.

[7] R. Schoen, S.-T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscripta Mathenatica **28** (1979) 159–183.

[8] R. Schoen, S.-T. Yau, *The existence of a black hole due to condensation of matter*, Communications in Mathematical Physics **90** (1983), 575–579.

[9] J. Wang, *Topology of 3-manifolds with uniformly positive scalar curvature*, Preprint arXiv:2212.14383v2, 2023.

[10] S.-T. Yau, Problem section. In *Seminar on Differential Geometry*, volume 102 of *Ann. of Math. Stud.*, pages 669–706. Princeton Univ. Press, Princeton, NJ, 1982.

Generic regularity for minimizing hypersurfaces in dimension 11

Felix Schulze

(joint work with Otis Chodosh, Christos Mantoulidis, Zhihan Wang)

Overview. Consider a smooth, closed, oriented $(n-1)$-dimensional submanifold $\Gamma \subset \mathbb{R}^{n+1}$. We are interested in Plateau's problem. Among all smooth, compact hypersurfaces $M \subset \mathbb{R}^{n+1}$ with $\partial M = \Gamma$, we want to find one of least area. It's now well-known that such M always exists for $n+1 \leq 7$, while when $n+1 \geq 8$ and for certain choices of Γ, no minimizer M can be found among smooth hypersurfaces. Using geometric measure theory, one can prove the existence of a minimizer among a wider class of objects which are smooth hypersurfaces except perhaps along an $(n-7)$-dimensional singular set. See [Fed69, MM84, Giu84, Mag12, Fle62, DG65, Alm66, Sim68, BDGG69, HS79].

In $\mathbb{R}^8$, the first dimension that singularities can appear, a fundamental result of Hardt–Simon [HS85] shows that for a generic choice of Plateau boundary Γ, there does exist a smooth M minimizing area. An analogous result in 8-dimensional manifolds was proven by Smale [Sma93]. These generic regularity results were recently extended to cover $\mathbb{R}^9$ and $\mathbb{R}^{10}$ in [CMS23a] using new ideas from the works of the first three authors with K. Choi on generic mean curvature flows, and specifically [CCMS24a, CCMS24b].

In this work we prove that solutions to Plateau's problem in $\mathbb{R}^{11}$ are generically smooth. We also prove that in any $\mathbb{R}^{n+1}$, an area-minimizing M will have a $\leq n - 10 - \epsilon_n$ dimensional singular set after perhaps a C^∞-perturbation of the

Plateau boundary. We had previously obtained the upper bound $\leq n-9-\epsilon'_n$ in [CMS24].

We also prove the analogous results in the context of area-minimization in integral homology classes of a closed oriented manifold (N^{n+1}, g). As is well-known, this extends Schoen–Yau's stable minimal hypersurface obstruction to positive scalar curvature up to dimension 11 and also implies the positive mass theorem in these dimensions after a well-known reduction of Lohkamp. See also the work of Schoen–Yau and Lohkamp [SY22, Loh23].

The results. We obtain the following generalizations of the main result of [CMS23a, CMS24]. All submanifolds are considered smoothly embedded, and if Σ is such, we denote $\operatorname{sing}\Sigma = \bar{\Sigma} \setminus \Sigma$. All singular set dimensions are Hausdorff dimensions.

For the Plateau problem in $\mathbb{R}^{n+1}$ we show:

Theorem 1. *Consider a smooth, closed, oriented, $(n-1)$-dimensional submanifold $\Gamma \subset \mathbb{R}^{n+1}$. There exist C^∞-small perturbations Γ' of Γ (in the space of C^∞ submanifolds) such that every minimizing integral n-current with boundary $[\![\Gamma']\!]$ is of the form $[\![\Sigma']\!]$ for a smooth, precompact, oriented hypersurface $\Sigma' \subset \mathbb{R}^{n+1}$ with $\partial\Sigma' = \Gamma'$, and*

$$\operatorname{sing}\Sigma' = \emptyset \textit{ if } n+1 \leq 11, \textit{ else } \dim \operatorname{sing}\Sigma' \leq n-10-\epsilon_n,$$

where $\varepsilon_n > 0$ is a dimensional constant.

For the homological Plateau problem in a manifold we have:

Theorem 2. *Consider a closed, oriented, $(n+1)$-dimensional Riemannian manifold (N, g). Let $[\alpha] \in H_n(N, \mathbb{Z}) \setminus \{[0]\}$. There exist C^∞-small perturbations g' of g such that every g'-minimizing integral n-current in $[\alpha]$ is of the form $\sum_{i=1}^{Q} k'_i [\![\Sigma'_i]\!]$ for disjoint, smooth, precompact, oriented hypersurfaces $\Sigma'_1, \ldots, \Sigma'_Q \subset N$ without boundary and*

$$\operatorname{sing}\Sigma'_i = \emptyset \textit{ if } n+1 \leq 11, \textit{ else } \dim \operatorname{sing}\Sigma'_i \leq n-10-\epsilon_n,$$

and multiplicities $k'_1, \ldots, k'_Q \in \mathbb{Z}$; again, $\varepsilon_n > 0$ is a dimensional constant.

Remark. It is a well-known consequence of Allard's interior regularity theorem [All72] and Hardt–Simon's boundary regularity theorem [HS79] that $\operatorname{sing}\Sigma' = \emptyset$ is an open condition in such a multiplicity-one setting. Therefore, when $n+1 \leq 11$, the set of Γ', g' for which the corresponding minimizers are smooth objects is simultaneously open (by this observation) and dense (by Theorems 1 and 2), and thus Baire generic.

References

[All72] William K. Allard. On the first variation of a varifold. *Ann. of Math. (2)*, 95:417–491, 1972.

[Alm66] F. J. Almgren, Jr. Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem. *Ann. of Math. (2)*, 84:277–292, 1966.

[BDGG69] E. Bombieri, E. De Giorgi, and E. Giusti. Minimal cones and the Bernstein problem. *Invent. Math.*, 7:243–268, 1969.

[CCMS24a] Otis Chodosh, Kyeongsu Choi, Christos Mantoulidis, and Felix Schulze. Mean curvature flow with generic initial data. *Invent. Math.*, 237(1):121–220, 2024.

[CCMS24b] Otis Chodosh, Kyeongsu Choi, Christos Mantoulidis, and Felix Schulze. Mean curvature flow with generic low-entropy initial data. *Duke Math. J.*, 173(7):1269–1290, 2024.

[CMS23a] Otis Chodosh, Christos Mantoulidis, and Felix Schulze. Generic regularity for minimizing hypersurfaces in dimensions 9 and 10. *Preprint*, arXiv:2302.02253, 2023.

[CMS24] Otis Chodosh, Christos Mantoulidis, and Felix Schulze. Improved generic regularity of codimension-1 minimizing integral currents. *Ars Inven. Anal.*, pages Paper No. 3, 16, 2024.

[DG65] Ennio De Giorgi. Una estensione del teorema di Bernstein. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 19:79–85, 1965.

[Fed69] Herbert Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York, Inc., New York, 1969.

[Fle62] Wendell H. Fleming. On the oriented Plateau problem. *Rend. Circ. Mat. Palermo (2)*, 11:69–90, 1962.

[Giu84] Enrico Giusti. *Minimal surfaces and functions of bounded variation*, volume 80 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1984.

[HS79] Robert Hardt and Leon Simon. Boundary regularity and embedded solutions for the oriented Plateau problem. *Ann. of Math. (2)*, 110(3):439–486, 1979.

[HS85] Robert Hardt and Leon Simon. Area minimizing hypersurfaces with isolated singularities. *J. Reine Angew. Math.*, 362:102–129, 1985.

[Loh23] Joachim Lohkamp. The secret hyperbolic life of positive scalar curvature. In *Perspectives in scalar curvature. Vol. 1*, pages 611–642. World Sci. Publ., Hackensack, NJ, [2023] ©2023.

[Mag12] Francesco Maggi. *Sets of finite perimeter and geometric variational problems*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.

[MM84] Umberto Massari and Mario Miranda. *Minimal surfaces of codimension one*, volume 91 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1984. Notas de Matemática [Mathematical Notes], 95.

[Sim68] James Simons. Minimal varieties in riemannian manifolds. *Ann. of Math. (2)*, 88:62–105, 1968.

[Sma93] Nathan Smale. Generic regularity of homologically area minimizing hypersurfaces in eight-dimensional manifolds. *Comm. Anal. Geom.*, 1(2):217–228, 1993.

[SY22] Richard Schoen and Shing-Tung Yau. Positive scalar curvature and minimal hypersurface singularities. In *Surveys in differential geometry 2019. Differential geometry, Calabi-Yau theory, and general relativity. Part 2*, volume 24 of *Surv. Differ. Geom.*, pages 441–480. Int. Press, Boston, MA, [2022] ©2022.

The large-scale structure of complete 4-manifolds with nonnegative Ricci curvature and Euclidean volume growth

Daniele Semola

(joint work with Elia Bruè, Alessandro Pigati)

A smooth complete 4-manifold (M^4, g) with $\mathrm{Ric} \geq 0$ is said to have Euclidean volume growth if there exists $c > 0$ such that for some $p \in M^4$ there holds

$$\frac{\mathrm{vol}(B_r(p))}{r^4} \geq c \quad \text{for all } r > 0\,. \tag{1}$$

In joint work with E. Bruè and A. Pigati we prove that for every such (M^4, g) there exists a spherical space form S^3/Γ such that every blow-down of (M^4, g) is a cone with cross-section homeomorphic to S^3/Γ. Our main theorem is inspired by the following result obtained by J. Cheeger and A. Naber earlier in [6]:

Theorem 1. *Let (M^4, g) be a Ricci-flat 4-manifold satisfying (1). There exists a finite group $\Gamma < \mathrm{O}(4)$ acting freely on S^3 such that $(M^4, r^{-2}g, p) \to (\mathbb{R}^4/\Gamma, g_{\mathrm{eucl}}, 0)$ as $r \to \infty$ in the pointed Gromov-Hausdorff sense and in C^∞_{loc} away from p and 0.*

The goal of the talk was to discuss which aspects of Theorem 1 continue to hold and which ones fail when the Ricci-flat assumption is weakened to $\mathrm{Ric} \geq 0$.

Let (M^4, g) have $\mathrm{Ric} \geq 0$ and satisfy (1). By Gromov's precompactness theorem, for any sequence $r_i \to \infty$, up to the extraction of a subsequence that we do not relabel, $(M^4, r_i^{-2}g, p) \to (Y, \mathrm{d}_Y, q)$ in the pointed Gromov-Hausdorff sense (from now on abbreviated as pGH), where (Y, d_Y, q) is a complete and pointed metric space. Any such metric space is called a blow-down of (M^4, g). Note that neither the dimension nor the Euclidean volume growth condition play any role for the moment. Without further assumptions, the metric structure of blow-downs is poorly understood. On the other hand, if the manifold has Euclidean volume growth, Cheeger and T.-H. Colding proved in [4, Theorem 7.6] that every blow-down is a metric cone. More precisely, there exists a compact metric space (Z, d_Z) (the cross-section of the cone) with diam(Z) $\leq \pi$ such that

$$Y = [0, +\infty) \times Z/_{\{0\}\times Z}\,, \tag{2}$$

and for every $(r_1, z_1), (r_2, z_2) \in Y$ there holds

$$\mathrm{d}_Y^2\left((r_1, z_1), (r_2, z_2)\right) = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\mathrm{d}_Z(z_1, z_2))\,. \tag{3}$$

In [9], G. Perelman constructed a manifold (M^4, g) with $\mathrm{Ric} \geq 0$ and Euclidean volume growth whose blow-down is not unique. Letting

$$\mathcal{C}_\infty := \{(Z, \mathrm{d}_Z) \,:\, (Z, \mathrm{d}_Z)\,\text{is the cross-section of a blow-down of}\ (M^4, g)\}$$

be the collection of cross-sections of blow-downs, $\mathcal{C}_\infty$ is compact and connected with respect to the Gromov-Hausdorff topology. Moreover, by volume convergence [5, Theorem 5.4], the 3-dimensional Hausdorff measure $\mathcal{H}^3$ is constant on $\mathcal{C}_\infty$.

The moral behind our main result can be easily illustrated under the additional assumption that all the elements of $\mathcal{C}_\infty$ are smooth Riemannian manifolds. Under such assumption, each blow-down $C(Z)$ of (M^4, g) has $\mathrm{Ric} \geq 0$ in the smooth part, i.e., in the complement of the vertex. An elementary computation shows this holds if and only if $\mathrm{Ric}_Z \geq 2$. Thanks to R. Hamilton's work, any cross-section must be diffeomorphic to a spherical space form. Moreover, by Cheeger and Colding's stability [5, Theorem A.1.3], the diffeomorphism type is constant on $\mathcal{C}_\infty$.

In general, no smoothness can be expected for the cross-sections $(Z, \mathrm{d}_Z) \in \mathcal{C}_\infty$. Nevertheless, a combination of the main results obtained in [3] allows us to make the previous formal argument fully rigorous also in the general case. More precisely, we prove:

Theorem 2. *Let (M^4, g) be smooth, complete, with* $\mathrm{Ric} \geq 0$ *and satisfying* (1). *There exists a finite group $\Gamma < \mathrm{O}(4)$ acting freely on S^3 such that for every cross-section of some blow-down $(Z, \mathrm{d}_Z) \in \mathcal{C}_\infty$, $(Z, \mathrm{d}_Z, \mathcal{H}^3)$ is an* $\mathrm{RCD}(2,3)$ *space with Z homeomorphic to S^3/Γ.*

There are three upshots for Theorem 2:

i) The cross-section of every blow-down is a topological manifold;
ii) The possible topologies of the cross-sections are restricted;
iii) For a fixed (M^4, g) the homeomorphism type is unique on $\mathcal{C}_\infty$.

Both i) and iii) might fail in higher dimensions. For i), this can be understood by considering the blow-down of the product metric $g_{\mathrm{EH}} + \mathrm{d}r^2$ on $T^*S^2 \times \mathbb{R}$, where g_{EH} denotes the Eguchi-Hanson metric on the cotangent bundle of S^2. Such metric is Ricci-flat with Euclidean volume growth. Its (unique) blow-down is isometric to $C\left(S^4/(\mathbb{Z}/2\mathbb{Z})\right)$, where $\mathbb{Z}/2\mathbb{Z}$ acts isometrically by involution with two fixed points on S^4. In particular, the cross-section is an orbifold which is not a topological manifold. For ii), the potential failure of the uniqueness of the topological type on $\mathcal{C}_\infty$ is illustrated by the examples constructed by Colding and Naber in [7]. In the recent [10], S. Zhou constructed examples of complete (M^4, g) with $\mathrm{Ric} \geq 0$ and Euclidean volume growth asymptotic to $C(S^3_\delta/\Gamma)$ for every finite $\Gamma < \mathrm{O}(4)$ acting freely on S^3. Here, $0 < \delta = \delta(\Gamma) \leq 1$ denotes the radius of S^3, which is endowed with a round metric. This result shows that every admissible topology for the cross-section of some blow-down according to Theorem 2 can arise. On the other hand, it is open whether every $\mathrm{RCD}(2,3)$ metric on some spherical space form can arise in this way.

The main ingredients for the proof of Theorem 2 are:

i) A statement ruling out the existence of noncollapsed Ricci limit spaces of the form $\mathbb{R} \times C(W^2)$ with W^2 homeomorphic to $\mathbb{RP}^2$, see [3, Thm. 1.6];
ii) A manifold recognition theorem for $\mathrm{RCD}(2,3)$ spaces, see [3, Thm. 1.8];
iii) A topological stability theorem for noncollapsing sequences of $\mathrm{RCD}(2,3)$ spaces, see [3, Thm. 1.11].

Theorem 2 has been recently used to obtain some restrictions on the topology of complete 4-manifolds (M^4, g) with $\mathrm{Ric} \geq 0$ and Euclidean volume growth:

i) C. Brena, Bruè and Pigati proved in [2] that M^4 must be orientable;
ii) H. Huang and X.-T. Huang proved in [8] that $\pi_1(M^4)$ is isomorphic to a quotient of the fundamental group of a spherical space form.

Cheeger and Naber's Theorem 1 can be used to show that a contractible, Ricci-flat (M^4, g) with Euclidean volume growth must be isometric to the Euclidean space with the flat metric, see [1, Lemma 6.3]. In the context of Theorem 2, we raise the following:

Conjecture 1. *Let (M^4, g) be a smooth, complete, contractible 4-manifold with* $\mathrm{Ric} \geq 0$ *and Euclidean volume growth. Then the following hold:*

i) M^4 is homeomorphic to $\mathbb{R}^4$;
ii) for every $(Z, \mathrm{d}_Z) \in \mathcal{C}_\infty$, Z is homeomorphic to S^3.

References

[1] M.-T. Anderson, *Ricci curvature bounds and Einstein metrics on compact manifolds.* J. Am. Math. Soc. **2** (1989), no. 3, 455–490.
[2] C. Brena, E. Bruè, A. Pigati, *Lower Ricci Curvature Bounds and the Orientability of Spaces.* Preprint arXiv:2412.19288v1 (2024).
[3] E. Bruè, A. Pigati, D. Semola, *Topological regularity and stability of noncollapsed spaces with Ricci curvature bounded below.* Preprint arXiv:2405.03839 (2024).
[4] J. Cheeger, T.-H. Colding, *Lower bounds on Ricci curvature and the almost rigidity of warped products.* Ann. of Math. (2) **144** (1996), no. 1, 189–237.
[5] J. Cheeger, T.-H. Colding, *On the structure of spaces with Ricci curvature bounded below. I.* J. Differential Geom. **46** (1997), no. 3, 406–480.
[6] J. Cheeger, A. Naber, *Regularity of Einstein manifolds and the codimension 4 conjecture.* Ann. of Math. (2) **182** (2015), no. 3, 1093–1165.
[7] T.-H. Colding, A. Naber, *Characterization of tangent cones of noncollapsed limits with lower Ricci bounds and applications,* Geom. Funct. Anal. **23** (2013), no. 1, 134–148.
[8] H. Huang, X.-T. Huang, *Nonnegative Ricci Curvature, Euclidean Volume Growth, and the Fundamental Groups of Open 4-Manifolds.* Preprint arXiv:2502.03259 (2025).
[9] G. Perelman, *A complete Riemannian manifold of positive Ricci curvature with Euclidean volume growth and nonunique asymptotic cone.* Grove, Karsten (ed.) et al., Comparison geometry. Math. Sci. Res. Inst. Publ. **30**, 165-166 (1997).
[10] S. Zhou, *A family of 4-manifolds with nonnegative Ricci curvature and prescribed asymptotic cone.* Preprint arXiv:2406.02279 (2024).

Fine structure of two-dimensional mod(q) area-minimizing hypersurfaces near branch points

Anna Skorobogatova

(joint work with Luca Spolaor, Salvatore Stuvard)

Let Σ^m be a minimal surface in a smooth Riemannian manifold (M^{m+n}, g). Aside from existence of such surfaces Σ, a major question concerning the behavior of such surfaces Σ is their regularity. In general, minimal surfaces can exhibit singularities, and in full generality, one does not even know if the dimension of the singular set is strictly smaller than m. In codimension one, under the additional assumption of stability and some other structural conditions, there has been some significant progress in recent years (see e.g. [13, 8]). However, particularly in higher codimension, very little is known. On the other hand, the regularity theory for *area-minimizing* surfaces of arbitrary dimension and codimension is much more approachable. Indeed, in the framework of integral currents, where surfaces can have integer multiplicies, Almgren's celebrated Big Regularity Theorem [1] yields a singular set of dimension $m-2$. However, this is *better* than the regularity that physical soap films exhibit. A natural and more physically reasonable framework for the study of the Plateau problem is via *mod*(q) *currents*, for a given positive integer q. Such currents typically have an $(m-1)$-dimensional singular set, and the regularity theory seems to agree with that expected for more general stable minimal surfaces, while still maintaining a minimization property. They are defined as follows.

For integer rectifiable currents T^m and S^m, recall the mod(q) flat distance

$$\mathcal{F}^q(T-S) = \inf\{\mathbf{M}(R) + \mathbf{M}(W) : T - S = R^m + \partial(W^{m+1}) + qZ^m\}.$$

Definition 1. *An m-dimensional* mod(q) area-minimizing current T *in* (M^{m+n}, g) *is a representative in its $\mathcal{F}^q$-equivalence class $[T]_q$ with*

$$\mathbf{M}(T) \leq \mathbf{M}(S),$$

for every $S \equiv (T + \partial R)$ mod(q) for some $(m+1)$-dimensional integer rectifiable current R in $\mathbb{R}^{m+n}$.

One of the first groundbreaking results in the structure of the singularities of codimension 1 area-minimizing mod(q) currents was the work of Taylor [9], which showed that a two-dimensional area-minimizing mod(3) surface in $\mathbb{R}^3$ is locally a $C^{1,\alpha}$-perturbation of a two-dimensional $\mathbf{Y}$-singularity near any singular point. In pariticular, its singular set is locally an embedded $C^{1,\alpha}$-submanifold. Another important early result for area-minimizing mod(q) hypersurfaces of any dimension was due to White [12], who demonstrated that any points of density strictly lower than $\frac{q}{2}$ are regular points, but this is very specific to the codimension being one, and no longer holds for surfaces of higher codimension, which may have *classical branch points* such as $\{w^2 = z^3\}$ in $\mathbb{C}^2 \cong \mathbb{R}^4$.

A combination of the recent works [2, 3, 4, 5, 6, 8, 10, 13] yields the following general structural theorem for area-minimizing mod(q) currents of arbitrary dimension and codimension.

Theorem 1. *Let $q \in \mathbb{N}_{\geq 2}$. If T^m is a mod(q) area-minimizing current in a smooth Riemannian manifold (M^{m+n}, g), then the interior singular set decomposes as*

$$\mathrm{Sing}(T) = \underbrace{\mathrm{Sing}^{branch}(T)}_{(m-2)\text{-rectifiable}} \sqcup \underbrace{\mathcal{S}^{m-1}(T) \setminus \mathcal{S}^{m-2}(T)}_{\substack{\text{loc. } (m-1)\text{-dim.} \\ C^{1,\alpha} \text{ mfds}}} \sqcup \underbrace{\mathcal{S}^{\mathrm{m}-2}(\mathrm{T})}_{(m-2)\text{-rectifiable}},$$

where $\mathrm{Sing}^{branch}(T)$ denotes the set of branch points, and $\mathcal{S}^k(T)$ denotes the k-th stratum T, characterizing the maximal number of translation-invariant directions that tangent cones may have.

If $n = 1$, then locally around every $x \in \mathcal{S}^{m-1}(T) \setminus \mathcal{S}^{m-2}(T)$, T is a $C^{1,\alpha}$-perturbation of an open book singularity model, supported by a finite collection of half-spaces meeting at a common $(m-1)$-dimensional interface.

This in particular recovers an analogous structure to that of Taylor for *codimension one* area-minimizing mod(q) currents near $(m-1)$-invariant cones.

A natural follow-up question concerns the structure of such currents near other singularity models. Particularly, one would like to understand the behavior near branch points, where the surface has at least one tangent cone supported in an m-dimensional plane with multiplicity. In joint work [11] with Luca Spolaor and Salvatore Stuvard, we answer this question in the case when it is two-dimensional and the codimension is one:

Theorem 2. *Let T be a 2-dimensional area-minimizing current mod(q) in a smooth Riemannian 3-manifold. Then, for any density $\frac{q}{2}$ flat singular point p of T, there exists $r_0(p) > 0$ and $\alpha > 0$ such that $T \llcorner \mathbf{B}_{r_0}(p)$ is a $C^{1,\alpha}$-perturbation of the multigraph of a special $\frac{q}{2}$-valued function (see [2]) arising from a superposition of homogeneous harmonic polynomials of the same degree.*

We additionally demonstrate that top density branch points (namely, the "genuine mod(q) ones") are isolated for two-dimensional surfaces of any codimension. The key idea is to demonstrate a power law decay for *Almgren's frequency function*, which in this context is used to measure the order of collapsing of such surfaces near branch points, relative to their (smoothed out) average known as the *center manifold*. Such a decay may be obtained by a suitable competitor argument for the (multi-valued) Dirichlet energy for a suitably strong graphical approximation to the surface near a given branch point, since the latter approximation is almost energy-minimizing in a quantitative sense.

The higher codimension case of such perturbative results for two dimensional surfaces both near branch points and near the $(m-1)$-symmetric "classical singularities" described above is ongoing work, and requires us to rule out the possibility of classical branch points accumulating to top density branch points.

References

[1] F. J. Almgren Jr., *Almgren's big regularity paper*, World Scientific Monograph Series in Mathematics, 1, World Sci. Publ., River Edge, NJ, 2000; MR1777737

[2] C. De Lellis, J. Hirsch, A. Marchese and S. Stuvard, Area-minimizing currents mod $2Q$: linear regularity theory, Comm. Pure Appl. Math. **75** (2022), no. 1, 83–127; MR4373168

[3] C. De Lellis, J. Hirsch, A. Marchese and S. Stuvard, Regularity of area minimizing currents mod p, Geom. Funct. Anal. **30** (2020), no. 5, 1224–1336; MR4181825

[4] C. De Lellis, J. Hirsch, A. Marchese, L. Spoalor and S. Stuvard, Area minimizing hypersurfaces modulo p: a geometric free-boundary problem, arXiv preprint (2021), available at `https://arxiv.org/abs/2105.08135`

[5] C. De Lellis, J. Hirsch, A. Marchese, L. Spoalor and S. Stuvard, Excess decay for minimizing hypercurrents mod $2Q$, Nonlinear Anal. **247** (2024), Paper No. 113606, 47 pp.; MR4771017

[6] C. De Lellis, J. Hirsch, A. Marchese, L. Spoalor and S. Stuvard, Fine structure of the singular set of area minimizing hypersurfaces modulo p, arXiv preprint (2022), available at `https://arxiv.org/abs/2201.10204`

[7] C. De Lellis, P. Minter and A. Skorobogatova, Fine Structure of Singularities in Area-Minimizing Currents Mod(q), arXiv preprint (2024), available at `https://arxiv.org/abs/2403.15889`

[8] P. Minter and N. Wickramasekera, A structure theory for stable codimension 1 integral varifolds with applications to area minimising hypersurfaces mod p, J. Amer. Math. Soc. **37** (2024), no. 3, 861–927; MR4736529

[9] J. E. Taylor, Regularity of the singular sets of two-dimensional area-minimizing flat chains modulo 3 in R^3, Invent. Math. **22** (1973), 119–159; MR0333903

[10] A. Skorobogatova, Rectifiability of flat singular points for area-minimizing mod$(2Q)$ hypercurrents, Int. Math. Res. Not. IMRN **2024**, no. 11, 9237–9255; MR4756111

[11] A. Skorobogatova, L. Spolaor and S. Stuvard, Structure of two-dimensional mod(q) area-minimizing currents near flat singularities: the codimension one case, arXiv preprint (2025), available at `https://arxiv.org/abs/2506.17813`

[12] B. C. White, A regularity theorem for minimizing hypersurfaces modulo p, in *Geometric measure theory and the calculus of variations (Arcata, Calif., 1984)*, 413–427, Proc. Sympos. Pure Math., 44, Amer. Math. Soc., Providence, RI, ; MR0840290
[13] N. Wickramasekera, A general regularity theory for stable codimension 1 integral varifolds, Ann. of Math. (2) **179** (2014), no. 3, 843–1007; MR3171756

Isoperimetric gaps in non-positive curvature

Stephan Stadler
(joint work with Cornelia Druţu, Urs Lang, Panos Papasoglu, David Urech)

Isoperimetric filling inequalities in dimension $n+1$ control the volume needed in order to fill an n-cycle S by an $(n+1)$-chain V. Their specific form is intimately related to the geometry of the underlying metric space. A central role is played by isoperimetric inequalites of the Euclidean type

$$\mathrm{vol}_{n+1}(V) \leq \mathrm{const} \cdot \mathrm{vol}_n(S)^{1+1/n}.$$

By a fundamental result of Wenger, which builds on earlier work of Gromov, such inequalities hold in particular in all Banach spaces and CAT(0) spaces. It is natural to expect that for CAT(0) spaces, Euclidean isoperimetric inequalities continue to hold when the classes of admissible cycles and fillings are restricted to Lipschitz spheres and balls, respectively. In the case $n = 1$ – when circles are filled by discs – we have a good understanding. It has recently been shown that non-positive curvature is equivalent to a Euclidean isoperimetric inequality with the sharp constant $1/(4\pi)$. Moreover, a length space that admits a quadratic isoperimetric inequality for curves with a constant strictly smaller than $1/(4\pi)$ is necessarily Gromov hyperbolic. These sharp results were predated by the general observation, originally due to Gromov, that length spaces with a subquadratic isoperimetric inequality must in fact satisfy a linear isoperimetric inequality.

In joint work with Cornelia Druţu, Urs Lang and Panos Papasoglu we established a sharp isoperimetric gap theorem for fillings of 2-spheres by 3-balls in CAT(0) spaces.

Theorem 1 ([1, Theorem A]). *For a proper CAT(0) space X, the following are equivalent:*

(1) There exists a constant $c < 1/(6\sqrt{\pi})$ such that every Lipschitz 2-sphere $\hat{S} \subset X$ of large area admits a filling by a Lipschitz 3-ball $\hat{B} \subset X$ with volume

$$\mathrm{vol}(\hat{B}) \leq c \cdot \mathrm{area}(\hat{S})^{3/2}.$$

(2) For every $\delta > 0$ there exists a constant $C = C(\delta)$ such that every Lipschitz 2-sphere $S \subset X$ extends to a Lipschitz 3-ball $B \subset X$ with volume

$$\mathrm{vol}(B) \leq C \cdot \mathrm{area}(S)^{1+\delta}.$$

(3) The asymptotic rank of X is at most 2.

The notion of *asymptotic rank* appearing in the last item is due to Gromov and plays an important role in large scale geometry. For proper CAT(0) spaces with cocompact isometry group, the asymptotic rank equals the maximal n such that X contains an *n-flat*, that is, an isometric copy of $\mathbb{R}^n$. In particular, the isoperimetric inequality in the second item of Theorem 1 holds for the universal cover X of any compact manifold of non-positive curvature provided that X contains no 3-flat. For a general CAT(0) space X, the asymptotic rank is at most n if and only if no asymptotic cone of X contains an $(n+1)$-flat, and it is at most 1 if and only if X is Gromov hyperbolic.

Gromov conjectured that proper cocompact CAT(0) spaces of asymptotic rank at most n admit linear isoperimetric inequalities

$$\mathrm{vol}_{n+1}(V) \leq \mathrm{const} \cdot \mathrm{vol}_n(S)$$

for fillings of n-cycles by $(n+1)$-chains. For general CAT(0) spaces of asymptotic rank at most n, where $n \geq 2$, the best known result in this direction is Wenger's sub-Euclidean inequality, stating that every n-cycle of mass s admits a filling with mass at most $o(s^{1+1/n})$ as $s \to \infty$.

For closed Lipschitz surfaces of higher genus we can prove the following.

Corollary 1 ([1, Corollary B])**.** *Let X be a proper CAT(0) space of asymptotic rank at most 2. For every $\delta > 0$ and every integer $g \geq 0$ there exists a constant $C_g = C_g(\delta)$ such that every closed Lipschitz surface $\Sigma \subset X$ of genus g extends to a Lipschitz handlebody $H \subset X$ with volume*

$$\mathrm{vol}(H) \leq C_g \cdot \mathrm{area}(\Sigma)^{1+\delta}.$$

In joint work with Urs Lang and David Urech we address the case of general k-cycles for $k \geq 2$ in a (not necessarily proper) CAT(0) space X of asymptotic rank 2. We consider the chain complex $\mathbf{I}_{*,\mathrm{c}}(X)$ of metric integral currents with compact support, which comprises all Lipschitz singular chains. The proof of Theorem 1 made use of the topology of surfaces. To cope with the missing topological control, we assume that X has finite *asymptotic Nagata dimension*, a variant of Gromov's asymptotic dimension.

Theorem 2 ([2, Theorem 1.1])**.** *Let X be a CAT(0) space of asymptotic rank and most 2 and of finite asymptotic Nagata dimension. Then for every cycle $T \in \mathbf{I}_{k,\mathrm{c}}(X)$ in X of dimension $k \geq 2$ and every $\delta > 0$ there exists a $V \in \mathbf{I}_{k+1,\mathrm{c}}(X)$ with boundary $\partial V = T$ and mass*

$$\mathbf{M}(V) \leq C \cdot \mathbf{M}(T)^{1+\delta}$$

for some constant C depending only on X, k, and δ.

References

[1] C. Druţu, U. Lang, P. Papasoglu, S. Stadler, *Minimal tetrahedra and an isoperimetric gap theorem in non-positive curvature*, arXiv:2502.03389 [math.MG].

[2] U. Lang, S. Stadler, D. Urech, *Isoperimetric inequalities in Hadamard spaces of asymptotic rank two*, arXiv:2506.04882 [math.MG].

Scalar Curvature Rigidity and the Higher Mapping Degree

Thomas Tony

A foundational result in scalar curvature comparison geometry is Llarull's rigidity theorem [2]. It states that every smooth area non-increasing map $f\colon M \to S^n$ from an n-dimensional closed connected Riemannian spin manifold onto the round sphere, $n \geq 3$, with non-zero degree and $\mathrm{scal}_M \geq n(n-1) = \mathrm{scal}_{S^n}$, is an isometry. Goette and Semmelmann [1] generalized this result to area non-increasing spin maps $f\colon M \to N$ of non-zero $\hat{A}$-degree onto a closed connected oriented Riemannian manifold of non-vanishing Euler characteristic and non-negative curvature operator.

In this report, we present a recent generalization of the extremality and rigidity statement of Goette and Semmelmann [1, Theorem 2.4] to spin maps between not necessarily orientable manifolds where the topological condition on the $\hat{A}$-degree is replaced by a less restrictive index-theoretical condition involving the so-called higher mapping degree (Theorem 1). The proof is based on the Dirac operator method. While a non-zero classical index always implies a non-trivial kernel of the corresponding Dirac operator, this is in general no longer true for a non-vanishing higher index. To overcome this difficulty, a new method is developed by the author in [5] that extracts from a non-vanishing higher index a geometrically useful family of almost harmonic sections (Lemma 1).

Theorem 1 ([5, Theorem A]). *Let $f\colon (M, g_M) \to (N, g_N)$ be an area non-increasing spin map between two closed connected Riemannian manifolds of dimension $n+k$ and n, respectively. Suppose that the curvature operator of N is non-negative and*

$$\chi(N) \cdot \deg_{\mathrm{hi}}(f) \neq 0 \in \mathrm{KO}_k(\mathrm{C}^*\pi_1(M)). \tag{1}$$

Then $\mathrm{scal}_M \geq \mathrm{scal}_N \circ f$ on M implies $\mathrm{scal}_M = \mathrm{scal}_N \circ f$. If, moreover, $\mathrm{scal}_N > 2\,\mathrm{Ric}_N > 0$ (or f is distance non-increasing and $\mathrm{Ric}_N > 0$), then $\mathrm{scal}_M \geq \mathrm{scal}_N \circ f$ implies that f is a Riemannian submersion.

Here, we call the map f *area non-increasing* if $g_M \geq f^* g_N$ holds on $\Lambda^2 TM$, and *spin* if $w_1(TM) = f^* w_1(TN)$ and $w_2(TM) = f^* w_2(TN)$. Moreover, $\chi(N)$ denotes the Euler characteristic of N, and the higher mapping degree is defined as follows.

Definition 1. *The higher degree of the map f is defined via*

$$\deg_{\mathrm{hi}}(f) := \mathrm{ind}(\mathcal{D}_{\mathcal{S}M_p \otimes \mathcal{L}(M)|_{M_p}}) \in \mathrm{KO}_k(\mathrm{C}^*\pi_1(M))$$

for a regular value p of the map f. Here $\mathcal{S}M_p$ denotes the Cl_k-linear spinor bundle of $M_p := f^{-1}(p)$, $\mathcal{L}(M)$ the Mishchenko bundle of M [3], $\mathrm{C}^*\pi_1(M)$ the maximal group C^*-algebra of the fundamental group of M, and $\mathrm{KO}_k(\mathrm{C}^*\pi_1(M))$ its k-th Real K-theory group.

Theorem 1 applies to the projections $\mathrm{pr}_1\colon S^{2n} \times T^k \to S^{2n}$ and $\mathrm{pr}_1\colon \mathbb{RP}^{2n} \times \Sigma^{8k+j} \to \mathbb{RP}^{2n}$ for $j \in \{1, 2\}$ and Σ^{8k+j} an exotic sphere with non-vanishing

Hitchin invariant. In both examples the $\hat{A}$-degree vanishes, hence Theorem 1 is a proper extension of the classical rigidity statement by Goette and Semmelmann [1, Theorem 2.4].

The generalization in Theorem 1 is motivated by the fact that the Rosenberg index [4] of a closed connected spin manifold M is the most general known index-theoretical obstruction to the existence of a positive scalar curvature metric on M. Since the Rosenberg index of the n-torus does not vanish, there exists no positive scalar curvature metric on the n-torus. This information cannot be read off the classical index of the spin Dirac operator. This is the same phenomenon as that the classical result by Goette and Semmelmann [1] does not apply to $\mathrm{pr}_1\colon S^{2n} \times T^n \to S^{2n}$ but the higher version in Theorem 1 does. The Rosenberg index is known to be non-zero for many closed connected spin manifolds, including (area-)enlargeable manifolds, those admitting metrics of non-positive sectional curvature, and aspherical manifolds whose fundamental groups satisfy the Novikov conjecture.

We now provide an outline of the proof of Theorem 1. Since the map f is spin, there exists an indefinite spin structure on the vector bundle $TM \oplus f^*TN$ equipped with the indefinite metric $g_M \oplus (-f^*g_N)$. We fix such an indefinite spin structure, and twist its induced $\mathbb{C}\mathrm{l}_{n+k,n}$-linear spinor bundle $\mathcal{S}$ by the Mishchenko bundle of M. The induced Dirac operator $\mathcal{D}_{\mathcal{L}}$ satisfies, as in the classical proof by Goette and Semmelmann [1], the Schrödinger-Lichnerowicz type formula

$$\mathcal{D}_{\mathcal{L}} \geq \nabla^*\nabla + \frac{1}{4}(\mathrm{scal}_M - \mathrm{scal}_N \circ f). \tag{2}$$

Moreover, the following index theorem holds.

Theorem 2 ([5, Theorem 5.4])**.** *The higher index of $\mathcal{D}_{\mathcal{L}}$ satisfies*

$$\mathrm{ind}(\mathcal{D}_{\mathcal{L}}) = \chi(N) \cdot \deg_{\mathrm{hi}}(f) \in \mathrm{KO}_k(\mathrm{C}^*\pi_1(M)).$$

We obtain by equation 1 and Theorem 2 that the higher index of $\mathcal{D}_{\mathcal{L}}$ does not vanish. A main difficulty in the proof of the extremality and rigidity statement of Theorem 1 is that a non-vanishing higher index does in general not give rise to a non-trivial kernel of the corresponding Dirac operator. The following lemma establishes a new method that extracts from a non-vanishing higher index a family of geometrically useful sections of the corresponding Dirac bundle.

Lemma 1 ([5, Lemma D])**.** *Let M be a closed Riemannian manifold, $\mathcal{A}$ a graded Real unital C^*-algebra, and $\$ \to M$ a graded Real $\mathcal{A}$-linear Dirac bundle with induced $\mathcal{A}$-linear Dirac operator $\not{D}$. If the higher index of $\not{D}$ does not vanish, the following holds.*

(1) There exists a family $\{u_\epsilon\}_{\epsilon>0}$ of almost $\not{D}$-harmonic sections of $\$$, i.e. $\|u_\epsilon\|_{L^2} = 1$ and $\|\not{D}^i u_\epsilon\|_{L^2} < \epsilon^i$ for all $i \geq 1$ and all $\epsilon > 0$.

(2) If, moreover, $\|\nabla u_\epsilon\|_{L^2} < \epsilon$ for all $\epsilon > 0$, then there exist positive constants C, r such that $\|\nabla u_\epsilon\|_\infty < C\epsilon^r$ for all $\epsilon \in (0, 1)$.

The first part of Lemma 1 is well-known and follows from a standard procedure using the functional calculus of the Dirac operator. Combined with the

Schrödinger-Lichnerowicz formula, it provides a proof of the fact that a non-vanishing Rosenberg index is an obstruction to a positive scalar curvature metric on closed connected spin manifolds. The proof of the second statement in Lemma 1 is mainly based on Moser iteration as well as the Sobolev embedding and the elliptic estimates for the Dirac operator. Lemma 1 part (2) is the key observation that makes it possible to use higher index theory in the context of scalar curvature rigidity. In addition to the application in Theorem 1, it also yields a spinorial proof of the rigidity statement that every closed connected Riemannian spin manifold of non-vanishing Rosenberg index and non-negative scalar curvature is already Ricci-flat.

We proceed with the proof of Theorem 1. By Theorem 2, Lemma 1 and the Schrödinger-Lichnerowicz type formula in Equation 2, there exists a family $\{u_\epsilon\}_{\epsilon>0}$ of sections of $\mathcal{S} \otimes \mathcal{L}(M)$ which is almost $\mathcal{D}_{\mathcal{L}}$-harmonic and almost constant. This means there exist positive constants C, r such that

$$\|\overline{u}_\epsilon - \langle u_\epsilon, u_\epsilon \rangle_p\| < C\epsilon^r, \quad \text{with} \quad \overline{u}_\epsilon \coloneqq \frac{1}{\mathrm{vol}(M)} \int_M \langle u_\epsilon, u_\epsilon \rangle_p dp,$$

for all $p \in M$ and all $\epsilon \in (0,1)$. Finally, we obtain

$$\big\| \mathrm{scal}_M - \mathrm{scal}_N \circ f \big\|_{L^1} \lesssim \big\| \big\langle (\mathrm{scal}_M - \mathrm{scal}_N \circ f) u_\epsilon, u_\epsilon \big\rangle_{L^2} \big\| \lesssim \epsilon$$

for all sufficiently small $\epsilon > 0$. Taking the limit $\epsilon \to 0$ yields $\mathrm{scal}_M = \mathrm{scal}_N \circ f$, and the extremality statement in Theorem 1 is proved. A similar consideration generalizes the classical proof of the rigidity statement in [1, Section 1.c] to its higher version in Theorem 1.

References

[1] S. Goette and U. Semmelmann, *Scalar curvature estimates for compact symmetric spaces*, In: Diff. Geom. Appl. 16.1 (2002).

[2] M. Llarull, *Sharp estimates and the Dirac operator*, In: Mathematische Annalen 310.1 (1998), pp. 55–71.

[3] A. S. Mishchenko and A. T. Fomenko, *The index of elliptic operators over C*-algebras*, In: Mathematics of the USSR. Izvestiya 15 (1980), pp. 87–112.

[4] J. Rosenberg, *C*-algebras, positive scalar curvature, and the Novikov conjecture*, In: Publications Mathématiques 58 (1983), pp. 409–424.

[5] T. Tony, *Scalar curvature rigidity and the higher mapping degree*, In: Jounal of Functional Analysis 288.3 (2025).

Reporter: Jared Marx-Kuo

Participants

Prof. Dr. Richard Bamler
Department of Mathematics
University of California at Berkeley
970 Evans Hall
Berkeley, CA 94720-3840
UNITED STATES

Prof. Dr. Jacob Bernstein
Department of Mathematics
Johns Hopkins University
3400 N. Charles Street
Baltimore, MD 21218
UNITED STATES

Prof. Dr. Gérard Besson
Département de Mathématiques
Université Grenoble Alpes
Institut Fourier
CS 40700
38058 Grenoble Cedex 09
FRANCE

Dr. Paula Burkhardt-Guim
Department of Mathematics
Stony Brook University
Stony Brook, NY 11794-3651
UNITED STATES

Shuli Chen
Department of Mathematics
University of Chicago
5734 S. University Avenue, Eckhart Hall
Chicago, IL 60637
UNITED STATES

Dr. Otis Chodosh
Department of Mathematics
Stanford University
Stanford CA 94305
UNITED STATES

Prof. Dr. Beomjun Choi
Department of Mathematics
POSTECH
77 Cheongam-Ro. Nam-Gu
Pohang, Gyungbuk
37673 Pohang
KOREA, REPUBLIC OF

Prof. Dr. Kyeongsu Choi
School of Mathematics
KIAS
Hoegiro 85
Seoul 02455
KOREA, REPUBLIC OF

Prof. Dr. Alix Deruelle
Institut de Mathématiques d'Orsay
Université Paris-Saclay
Batiment 307, rue Michel Magat
91405 Orsay
FRANCE

Dr. Nick Edelen
Department of Mathematics
University of Notre Dame
283 Hurley Building
Notre Dame, IN 46556-4618
UNITED STATES

Dr. Federico Franceschini
Institute for Advanced Study
177 WEYL LN
08540-4900 Princeton 08540-4900
UNITED STATES

Giada Franz
Laboratoire d'Analyse et de
Mathématiques Appliquées, UMR 8050
Université Gustave Eiffel
5, Boulevard Descartes,
Champs-sur-Marne
77454 Marne-la-Vallée Cedex 2
FRANCE

Dr. Max Hallgren
Department of Mathematics
Rutgers University
Busch Campus, Hill Center
New Brunswick, NJ 08854-8019
UNITED STATES

Prof. Dr. Bernhard Hanke
Institut für Mathematik
Universität Augsburg
Universitätsstraße 14
86159 Augsburg
GERMANY

Prof. Dr. Robert Haslhofer
Department of Mathematics
University of Toronto
Room BA 6208
40 St George Street
Toronto ONT M5S 2E4
CANADA

Prof. Dr. Or Hershkovits
Einstein Institute of Mathematics
Edmond J. Safra Campus
The Hebrew University of Jerusalem
Givat Ram
9190401 Jerusalem
ISRAEL

Prof. Dr. Tom Ilmanen
Departement Mathematik
ETH Zürich
Rämistrasse 101
8092 Zürich
SWITZERLAND

Frieder Jäckel
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Daniel Ketover
Department of Mathematics
Rutgers University
Busch Campus, Hill Center
New Brunswick, NJ 08854-8019
UNITED STATES

Bruce Kleiner
Courant Institute of Mathematical
Sciences
New York University
251, Mercer Street
New York NY 10012-1110
UNITED STATES

Prof. Dr. Yi Lai
Department of Mathematics
UC Irvine
Irvine, CA 94305-2125
UNITED STATES

Prof. Dr. Urs Lang
Departement Mathematik
ETH Zürich
Rämistrasse 101
8092 Zürich
SWITZERLAND

Prof. Dr. Claude LeBrun
Department of Mathematics
Stony Brook University
Z-3651 Stony Brook NY 11794-3651
UNITED STATES

Yang Li
Department of Mathematics, Cambridge
University
CB21DP Cambridge
UNITED KINGDOM

Dr. Yangyang Li
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago IL 60637-1514
UNITED STATES

Prof. Dr. Jason D. Lotay
Mathematical Institute
University of Oxford
Andrew Wiles Building
Radcliffe Observatory Quarter
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM

Prof. Dr. John Lott
Department of Mathematics
University of California, Berkeley
970 Evans Hall
Berkeley, CA 94720-3840
UNITED STATES

Prof. Dr. Alexander Lytchak
Karlsruher Institut für Technologie
Fakultät für Mathematik
Institut für Algebra und Geometrie
Karslruhe Institute of Technology
Englerstr. 2
76131 Köln
GERMANY

Prof. Dr. Elena Mäder-Baumdicker
Fachbereich Mathematik
Technische Universität Darmstadt
Schloßgartenstrasse 7
64289 Darmstadt
GERMANY

Jared Marx-Kuo
Department of Mathematics
Rice University
6100 Main St
77005 Houston, TX 77005-1892
UNITED STATES

Damaris Meier
Département de Mathématiques
Université de Fribourg
Perolles
Chemin du Musee 23
1700 Fribourg
SWITZERLAND

Dr. Paul Minter
Department of Pure Mathematics and
Mathematical Statistics
University of Cambridge
Wilberforce Road
Cambridge CB3 0WA
UNITED KINGDOM

Dr. Jiewon Park
Department of Mathematical Sciences
Korea Advanced Institute of Science and
Technology (KAIST)
291 Daehak-ro, Yuseong-gu
34141 Daejeon
KOREA, REPUBLIC OF

Dr. Stephane Sabourau
LAMA
Université Paris-Est Créteil
61 Avenue du Général de Gaulle
94010 Créteil Cedex
FRANCE

Prof. Mariel Saez Trumper
Facultad de Matematicas
Universidad Catolica de Chile
Vicuña Mackenna 4860
7820436 Santiago
CHILE

Prof. Dr. Felix Schulze
Mathematics Institute
University of Warwick
Gibbet Hill Road
Coventry CV4 7AL
UNITED KINGDOM

Dr. Daniele Semola
Fakultät für Mathematik
Universität Wien
Oskar-Morgenstern-Platz 1
1090 Wien
AUSTRIA

Prof. Dr. Miles Simon
Institut für Analysis und Numerik
Otto-von-Guericke-Universität
Magdeburg
Universitätsplatz 2
39106 Magdeburg
GERMANY

Dr. Anna Skorobogatova
Department of Mathematics,
Institute for Theoretical Sciences,
ETH Zurich
Rämistrasse 101
8092 Zürich
SWITZERLAND

Dr. Stephan Stadler
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Dr. Douglas Stryker
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Thomas Tony
Institut für Mathematik
Universität Potsdam
Karl-Liebknecht-Str.24-25
14476 Potsdam
GERMANY

Dr. Lu Wang
Department of Mathematics
Yale University
P.O. Box 208283
New Haven CT 06520-8283
UNITED STATES

Dr. Zhichao Wang
Shanghai Center for Mathematical
Sciences
Fudan University
Shanghai Shi 200438
CHINA

Prof. Dr. Stefan Wenger
Département de Mathématiques
Université de Fribourg
Perolles
Chemin du Musee 23
1700 Fribourg
SWITZERLAND

Prof. Dr. Brian White
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
UNITED STATES

Prof. Dr. Burkhard Wilking
Mathematisches Institut
Universität Münster
Einsteinstraße 62
48149 Münster
GERMANY

Dr. Yujie Wu
Department of Mathematics
Stanford University
Stanford
UNITED STATES

Dr. Kai Xu
Department of Mathematics
Duke University
P.O. Box 90320
Durham, NC 27708-0320
UNITED STATES

Prof. Dr. Rudolf Zeidler
Institut für Mathematik
Universität Potsdam
Karl-Liebknecht-Str. 24-25
14476 Potsdam
GERMANY

Prof. Dr. Xin Zhou
Department of Mathematics
Cornell University
531 Malott Hall
Ithaca, NY 14853-4201
UNITED STATES