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C^*-Algebras

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Abstract. The subject of operator algebras is a very active area of mathematics which, since its inception in the 1940s, has always been driven by interactions with other fields of mathematics and physics. The scope of these interactions is very wide, ranging over dynamical systems, (non-commutative) geometry, functional analysis, (geometric) group theory, topology, random matrices, harmonic analysis and quantum information theory.

The goals of this workshop were to stimulate new collaborations across these fields of mathematics, to disseminate recent progress by giving participants a global view on the subject and to specially focus on several important developments, including progress on Connes' rigidity conjecture for property (T) groups, a deeper understanding of the analogs of group boundaries in both C^* and von Neumann algebra theory, C^*-simplicity and selflessness of groups, and progress in (equivariant) classification of C^*-algebras.

Mathematics Subject Classification (2020): 46Lxx.

Introduction by the Organizers

The Oberwolfach C^*-algebra workshops hold a key role in the operator algebra research landscape worldwide. In contrast to the numerous specialized conferences, it is one of the rare meetings where participants learn about all the major developments in C^*-algebras *and* von Neumann algebras, *and* their interactions with other fields of mathematics. The workshop also has an important community building role. We therefore pay particular attention to a systematic renewal of the team of organizers, to inviting promising junior researchers and to gender diversity.

The 2025 workshop gave a broad overview over recent exciting developments in the field, with particular emphasis on recent results related to strict comparison of group C^*-algebras, on equivariant classification and new insights into the structure and classification of Cartan subalgebras of nuclear C^*-algebras, and on results around W^*-superrigidity.

In terms of talks an effort was made to follow the Oberwolfach tradition to balance a substantial program with generous space for informal interaction. There were 27 regular talks, plus one informal evening talk (by Marius Junge) and one (also informal) open problem session (moderated by Stuart White).

As particular highlights illustrating the stimulating atmosphere at Oberwolfach workshops, we would like to mention two recent papers which to a large part originated at this workshop and which were written directly afterwards: In arXiv:2508.07938, Narutaka Ozawa proves that the reduced group C^*-algebras of infinite countable discrete groups having topologically-free extreme boundaries are selfless in the sense of Robert. This generalizes previous (and also quite recent) results of Amrutam, Gao, Kunnawalkam Elayavalli, and Patchell, and of Vigdorovich; it was at least in part inspired by Patchell's talk on selflessness and strict comparison.

In arXiv:2508.05834, David Jekel proved that the unitary groups of SOT-separable II_1 factors are SOT-contractible. The proof was conceived during the workshop and the paper written directly afterwards; it was accepted by Mathematische Annalen only a month or so after the workshop.

In other news, we are happy to report that Ilijas Farah has agreed to write an Oberwolfach Snapshot entitled "Games in the Matrix".

Workshop: C*-Algebras

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Abstracts

W*-superrigidity for group and quantum group von Neumann algebras

Stefaan Vaes
(joint work with Milan Donvil)

A discrete group G is said to be *W*-superrigid* if the group von Neumann algebra $L(G)$, generated by the regular representation of G, fully remembers the group G. The first instances of W*-superrigid groups G were found in [6]. They are given by a generalized wreath product construction, i.e. of the form $G = (\mathbb{Z}/2\mathbb{Z})^{(I)} \rtimes \Gamma$, given specific actions of a countable group Γ on a countable set I.

If for such a group G, the group von Neumann algebra $M = L(G)$ has another group von Neumann algebra decomposition $M = L(\Lambda)$, generated by unitary elements $(v_s)_{s\in\Lambda}$, one finds the *comultiplication embedding* $\Delta : M \to M \overline{\otimes} M : \Delta(v_s) = v_s \otimes v_s$. In specific situations, Popa's deformation/rigidity theory allows to describe all such embeddings Δ, which ultimately leads to proving that the unitaries $(v_s)_{s\in\Lambda}$ can be unitarily conjugated to (multiples of) the unitaries $(u_g)_{g\in G}$ that canonically generate $L(G)$.

Over the years, the methods to classify all comultiplication type embeddings of classes of II_1 factors M into their tensor powers have been refined. In particular in [5], this led to the first instances of W*-superrigid groups with Kazhdan's property (T). A wide open conjecture in this direction is due to Connes and predicts that all lattices in higher rank simple Lie groups are W*-superrigid.

In [1], a classification theorem of comultiplication embeddings led to the following new types of W*-superrigidity theorems for groups G of the form $G = (\mathbb{Z}/2\mathbb{Z})^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$ given by a group Γ that belongs to the so-called class $\mathcal{C}$. These are groups that are nonamenable, weakly amenable, biexact and have the property that the centralizer of any nontrivial element is amenable. This class includes all torsion free hyperbolic groups, all free groups, and all free products of infinite amenable groups.

In [1], we proved on the one hand that these groups G remain W*-superrigid when their group von Neumann algebras are twisted by a scalar 2-cocycle: if $\mu \in H^2(G, \mathbb{T})$ is any scalar 2-cocycle, and (Λ, ω) is any other pair of a discrete group Λ with scalar 2-cocycle $\omega \in H^2(\Lambda, \mathbb{T})$, and if the twisted group von Neumann algebras $L_\mu(G)$ and $L_\omega(\Lambda)$ are isomorphic, then the pairs (G, μ) and (Λ, ω) must be isomorphic. This means that there must exist a group isomorphism $\delta : G \to \Lambda$ such that the 2-cocycles $\omega \circ \delta$ and μ are cohomologous.

Moreover in [1], we prove the same result up to *virtual isomorphisms:* if $L_\mu(G)$ is virtually isomorphic to $L_\omega(\Lambda)$, which can be expressed by the existence of a finite index Hilbert bimodule, or equivalently by the existence of a finite index embedding of one into an amplification of the other, then the pairs (G, μ) and (Λ, ω) are virtually isomorphic. The latter means that the groups are isomorphic up to passage to finite index subgroups and quotients by finite normal subgroups, and that such a virtual group isomorphism preserves the 2-cocycles, up to 2-cocycles given by finite-dimensional projective representations.

In the talk, we discussed two other applications of [1]: the construction from [2] of groups $\widetilde{G}$ that are W*-superrigid and have *infinite center*, and the new notion from [3] of W*-superrigidity in the larger category of *discrete quantum groups*.

Assume that $e \to C \to \widetilde{G} \to G \to e$ is a central extension of G by the abelian group C. Denote by $\Omega \in H^2(G, C)$ the associated 2-cocycle. By definition, the group von Neumann algebra $L(C)$ is contained in the center of $L(\widetilde{G})$. Identifying $L(C) \cong L^\infty(\widehat{C})$ by Pontryagin duality, the corresponding direct integral decomposition is given by

$$L(\widetilde{G}) = \int_{\widehat{C}}^{\oplus} L_{\mu\circ\Omega}(G)\, d\mu\ .$$

This immediately leads to two obstructions to W*-superrigidity of $\widetilde{G}$. First, if $\widetilde{G}'$ is another central extension of G by C such that the associated 2-cocycle $\Omega' \in H^2(G, C)$ has the property that $\mu \circ \Omega = \mu \circ \Omega'$ in $H^2(G, \mathbb{T})$ for all $\mu \in \widehat{C}$, then $L(\widetilde{G}) \cong L(\widetilde{G}')$ and the group $\widetilde{G}$ is not W*-superrigid. In this case, $\Omega_0 = \Omega - \Omega'$ is a 2-cocycle with the property that $\mu \circ \Omega_0 = 1$ for all $\mu \in \widehat{C}$. By the universal coefficient theorem, such 2-cocycles Ω_0 come from an abelian extension of the abelianization G_{ab} by C. So to avoid the first obstruction, one has to assume that $\mathrm{Ext}^1(G_{\mathrm{ab}}, C)$ is trivial.

A second obstruction to W*-superrigidity of $\widetilde{G}$ arises as follows: if the continuous group homomorphism $\widehat{C} \to H^2(G, \mathbb{T}) : \mu \mapsto \mu \circ \Omega$ has a nontrivial kernel, we find a proper subgroup $C_0 < C$ such that this kernel is given by $\widehat{C/C_0}$. Then,

$$L(\widetilde{G}) \cong L(C/C_0)\, \overline{\otimes} \int_{\widehat{C_0}}^{\oplus} L_{\mu\circ\Omega}(G)\, d\mu\ .$$

Since the abelian group C/C_0 is typically not W*-superrigid, this prevents $\widetilde{G}$ from being W*-superrigid.

The main result of [2] says that for groups $G = (\mathbb{Z}/2\mathbb{Z})^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$, where Γ belongs to class $\mathcal{C}$ and $\mathrm{Aut}\,\Gamma$ is countable, and arbitrary central extensions $e \to C \to \widetilde{G} \to G \to e$ with C torsion free, the two obvious obstructions to W*-superrigidity for $\widetilde{G}$ are the only obstructions. This then leads to numerous W*-superrigid groups with infinite center. Note here that in [4], W*-superrigid groups with infinite center and property (T) were obtained.

Every discrete group G is also a discrete quantum group, and every discrete quantum group generates a von Neumann algebra using the regular representation. It is thus quite natural to wonder if some of the W*-superrigid groups G remain superrigid in this broader category of quantum groups. In [3], we coined this notion as *quantum W*-superrigidity.* By definition, this notion is stronger than W*-superrigidity. Much to our surprise, we proved in [3] that it is often strictly stronger: none of generalized wreath product groups $G = (\mathbb{Z}/2\mathbb{Z})^{(I)} \rtimes \Gamma$ are quantum W*-superrigid! The reason is that the dual compact quantum group $(L(G), \Delta)$ always admits nontrivial (dual) 2-cocycles. These are unitaries $\Omega \in L(G)\, \overline{\otimes}\, L(G)$

satisfying

$$(\Omega \otimes 1)(\Delta \otimes \mathrm{id})(\Omega) = (1 \otimes \Omega)(\mathrm{id} \otimes \Delta)(\Omega) \ .$$

It follows that the twisted comultiplication $a \mapsto \Omega\Delta(a)\Omega^*$ defines a new quantum group structure on the same von Neumann algebra $L(G)$.

To avoid this obstruction to quantum W*-superrigidity, we thus introduce in [3] a broader class of wreath-like product (quantum) groups. Given a Kac type compact quantum group (A_0, Δ_0) (in particular, given a group von Neumann algebra $(L(H_0), \Delta_0)$), and given an action $\Gamma \curvearrowright^{\beta} (A_0, \Delta_0)$ by quantum group automorphisms, we consider the tensor product $(A, \Delta) = (A_0, \Delta_0)^{\overline{\otimes}\Gamma}$ with embeddings $\pi_k : A_0 \to A$ for all $k \in \Gamma$, and define the action $\Gamma \times \Gamma \curvearrowright^{\alpha} A$ given by

$$\alpha_{(g,h)}\pi_k(a) = \pi_{gkh^{-1}}(\beta_g(a)) \quad \text{for all } g, h, k \in \Gamma,\ a \in A_0.$$

Then the von Neumann algebra $M = A \rtimes_\alpha (\Gamma \times \Gamma)$ carries a natural quantum group structure $\Delta : M \to M \mathbin{\overline{\otimes}} M$. Note that when $(A_0, \Delta_0) = (L(H_0), \Delta_0)$ and β is the trivial action, then $M = L(G)$ where $G = H_0^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$.

The main result of [3] now says that under the appropriate nontriviality and rigidity assumption on $\Gamma \curvearrowright^{\beta} (A_0, \Delta_0)$ and under the appropriate conditions ensuring that (M, Δ) has no nontrivial dual 2-cocycles, the quantum group algebra (M, Δ) is indeed quantum W*-superrigid. This then leads to numerous examples of discrete groups G that are quantum W*-superrigid, and also to numerous instances of quantum W*-superrigidity for quantum groups that are neither commutative, nor cocommutative.

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Tensorial primeness of ultraproducts of tracial von Neumann algebras

Ilijas Farah

(joint work with Andrea Vaccaro)

A C*-algebra is called (tensorially) prime if it is not a tensor product of two infinite-dimensional C*-algebras. Ghasemi proved in [13] that prime C*-algebras include all ultraproducts, asymptotic sequence algebras, coronas of σ-unital C*-algebras, and relative commutants of separable C*-subalgebras of such algebras.

More precisely, all countably degree-1 saturated C*-algebras (in particular, all SAW*-algebras) are prime; see [6, §15.3]. This was motivated by a question of Simon Wasserman, who asked whether the Calkin algebra $\mathcal{Q}(H)$ (defined as the quotient $\mathcal{B}(H)/\mathcal{K}(H)$ for a separable, infinite-dimensional, Hilbert space H) is isomorphic to $\mathcal{Q}(H) \otimes \mathcal{Q}(H)$ (here and below $\otimes$ is the minimal tensor product). The following is still open.

Question 1. Is $\mathcal{Q}(H) \otimes \mathcal{Q}(H) \cong \mathcal{Q}(H) \otimes \mathcal{Q}(H) \otimes \mathcal{Q}(H)$?

A considerably more ambitious question is the following.

Question 2. Suppose that m and n are in $\mathbb{N}$ and that A_i, for $i < m$, B_j, for $j < n$, are SAW*-algebras such that $\bigotimes_{i<m} A_i \cong \bigotimes_{j<n} B_j$.

Can we conclude that $m = n$?

If so, is there a permutation π of m such that A_i and $B_{\pi(i)}$ are stably isomorphic?

If all of the A_i and B_j, are abelian, then the answer to each of the parts of Question 2 is 'yes'. This is a consequence of [5, Theorem 3] (see [6, Notes for Chapter 15], also [3] for the origins of this study).

From the primality of norm ultrapowers of C*-algebras we move on to the question of primality of (tracial) ultraproducts of tracial von Neumann algebras. In [16], Popa proved a remarkable lemma about orthogonality in tracial von Neumann algebras. Soon after it was noticed independently by Popa and others that this lemma easily implies that the ultrapower of any II_1 factor is not a tensor product of two II_1 factors ([17], [4], [14], [10]). It is however unclear what the correct definition of 'prime' is for tracial von Neumann algebras.

Example. Let M be a type II_1 tracial von Neumann algebra with diffuse center. Then M absorbs $\ell_\infty(\mathbb{N})$ tensorially. This is because for every $\varepsilon > 0$ there is a central projection p in M such that $pM \cong M$.[1] The proof relies on [7, Corollary 5.3], showing that the first-order theory of a direct integral can be computed from the distribution of the theories of the fibers (see also [11]).

This example motivates the following.

Definition 1. A tracial von Neumann algebra is *prime* if it is not isomorphic to a (von Neumann) tensor product of two von Neumann algebras whose unit balls are not $\|\cdot\|_2$-compact. Equivalently, it is not a tensor product of two diffuse von Neumann algebras.

Even with this definition, no ultrapower of a diffuse type I tracial von Neumann algebra is tensorially prime. This is a consequence of Maharam's theorem from measure theory (see e.g., [9, §331]), which implies that every diffuse type I von Neumann algebra absorbs $L_\infty[0,1]$ tensorially; see the discussion in [8, §2]. The following question was motivated by our (unsuccessful) attempts to solve Ozawa's notorious "exercise" on von Neumann algebras ([15]), and it in turn motivated the study presented here.

[1] Our proof of this fact uses the Continuum Hypothesis.

Question 3. Take an ultrapower of $L_\infty[0,1]\bar{\otimes}M$ for a II_1 factor M. Is it prime?

By [1, §29] such ultrapower is a direct integral of a measurable field of II_1 factors, but it is not clear whether this field can be taken to be constant. The following are [8, Theorem 4 and Corollary 4].

Theorem 1. *An ultraproduct $\prod^{\mathcal{U}} M_n$ of diffuse tracial von Neumann algebras is prime if and only if it is not of type I. If N has separable predual and $\prod^{\mathcal{U}} M_n \cap N'$ is diffuse, then it is tensorially prime if and only if it is diffuse.*

Corollary 1. *Suppose that M is a II_1 factor and $\mathcal{U}$ is a nonprincipal ultrafilter over $\mathbb{N}$. Then $(L^\infty[0,1]\bar{\otimes}M)^{\mathcal{U}}$ is not isomorphic to $L^\infty(\Omega,\mu)\bar{\otimes}N$ for any type II_1 tracial von Neumann algebra N and probability measure space (Ω,μ).*

Our original proof of Theorem 1 used Cohen's method of forcing introduced to prove the independence of the Continuum Hypothesis from ZFC ([2]). Stefaan Vaes suggested an improvement (Definition 2 below), and Adrian Ioana found a proof that uses only ideas from the standard deformation rigidity theory in addition to a standard consequence of countable quanifier-free saturation. In the following $E_{Z(M)}$ is the conditional expectation of M to its center $Z(M)$.

Definition 2. A unitary w in a tracial von Neumann algebra is *uniformly Haar* if $E_{Z(M)}(w^k)=0$ for all $k\geq 1$.

If M is a factor, then every Haar unitary is uniformly Haar. If M is type II_1 then it has a unital copy of the hyperfinite II_1 factor R and a Haar unitary in R is uniformly Haar in M.

We can now sketch a proof of the nontrivial implication in Theorem 1. Suppose that M an ultraproduct or a relative commutant of a separable subalgebra of an ultraproduct, diffuse, and not of type I. By passing to a corner, we may assume that M is of type II_1. Suppose $M=P\bar{\otimes}Q$, both P and Q are diffuse, and P not of type I.

Let w_0, w_1 be uniformly Haar unitaries in P and Q. Then $w_0\otimes w_1$ and $w_0\otimes 1_Q$ are uniformly Haar in M. It is straightforward to see that every two uniformly Haar unitaries are approximately unitarily equivalent. Therefore the system of equations (type) $\|x(w_0\otimes w_1)x^*-w_0\otimes 1_Q\|_2=0$, $\|xx^*-1\|_2=0$ has approximate solutions. By [6, Corollary 16.4.7], M is countably quantifier-free saturated (*cqfs*), meaning that for any system of countably many equations of the form $\|P_n(x)\|_2=r_n$ its finite subsystems have approximate solutions if and only if the system has an exact solution ([6, Definition 15.2.1]). Hence this system has an exact solution u. Let u be such that $u(w_0\otimes w_1)u^*=w_0\otimes 1_Q$. Note that $E_{P\otimes 1_Q}(w_0^k\otimes 1_Q)=w_0\otimes 1_Q$, a unitary, while $\lim_{k\to\infty}\|E_{P\otimes 1_Q}(a(w_0^k\otimes w_1^k)b)\|_2=0$ for all a,b; contradiction.

The following permanence property is straightforwaard (see Proposition 16.5.6 in the second edition of [6]).

Lemma 1. *A tracial von Neumann algebra M is cqfs if and only if M^t is cqfs for some (all) $t>0$. If M is cqfs and N is a von Neumann subalgebra of M with separable predual, then $M\cap N'$ is cqfs.*

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Almost almost periodic type III_1 factors and their 3-cohomological obstructions

Amine Marrakchi

Let M be a factor with separable predual. The Tomita-Takesaki theory associates to every faithful normal state φ on M a continuous one-parameter automorphism group $\sigma^\varphi : \mathbb{R} \to \mathrm{Aut}(M)$ called the *modular flow* of φ. We say that φ is *almost periodic* [Co74] if the closure of $\sigma^\varphi(\mathbb{R})$ is a compact subgroup of $\mathrm{Aut}(M)$. Almost periodic states play a fundamental role in the classification of type III factors because their modular theory is much simpler to handle.

Since the celebrated work of Connes [Co72], it is known that the modular flow σ^φ does not depend on the choice of the state φ up to inner automorphisms. In other words, if we let $\mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Inn}(M)$ be the quotient of $\mathrm{Aut}(M)$ by the subgroup of inner automorphisms, then the modular flow σ^φ descends to one-parameter group $\sigma^M : \mathbb{R} \to \mathrm{Out}(M)$, called the *outer modular flow* of M, that does not depend on the choice of φ. If we moreover assume that M is *full*, that is if $\mathrm{Inn}(M)$ is closed in $\mathrm{Aut}(M)$, then $\mathrm{Out}(M)$ inherits a well-behaved Hausdorff quotient topology for which σ^M is continuous.

Connes observed that if the full factor M admits some almost periodic state, then σ^M must also be almost periodic. What about the converse? If σ^M is almost periodic does it mean that M admits an almost periodic state?

Surprinsingly, we show that the answer to this question is *negative* in general and that the counter-examples are quite subtle. Indeed, we prove that σ^M is almost periodic if and only if $M \overline{\otimes} R$ admits an almost periodic state, where R is the hyperfinite II_1 factor. Moreover, we give a complete and precise description of the 3-cohomological obstruction to the existence of an almost periodic state on M itself. We interpret this obstruction in terms of an *integral quadratic form* on the Lie algebra of the compactification $K = \overline{\sigma^M(\mathbb{R})}$, we show that the infinitesimal generator of the outer modular flow $\sigma^M : \mathbb{R} \to K$ must be an *isotropy* vector for this integral quadratic form and that this is the only restriction.

This talk is based on [Ma25].

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Flows on stably projectionless C*-algebras: Exploring uncharted territory

Gábor Szabó

(joint work with Johannes Christensen & Robert Neagu)

In this talk I motivated ongoing work in progress with my coauthors from the point of view of the broader theme of trying to understand flows on C*-algebras, i.e., continuous actions of $\mathbb{R}$. Roughly speaking, the two conceptual main problems about flows one can work on are:

(1) Understand rigidity or classification phenomena for well-behaved flows on well-understood C*-algebras.
(2) Understand how rich the class of flows really is on well-understood C*-algebras.

These two goals are of course interrelated and progress on either can lead to progress on both. The term "well-understood C*-algebra" can in practice be taken to mean that it falls within the scope of Elliott's classification program for simple amenable C*-algebras. The term "well-behaved flow" may have an ambiguous meaning (and may continue to be ambiguous as the state-of-the-art evolves), but with the current methodology, this typically refers to flows satisfying Kishimoto's Rokhlin property [5].

Indeed, Kishimoto had conjectured early on that these flows ought to be accessible for classification and he obtained various partial results that served as convincing evidence for his suspicion. In my prior work [8], I demonstrated that this is indeed the case. Firstly, I showed that Rokhlin flows on Kirchberg algebras are unique up to cocycle conjugacy (confirming a conjecture of Kishimoto), where the existence of such flows on all Kirchberg algebras was shown in [1]. Secondly, I showed in the same article that when A is a stable classifiable KK-contractible C*-algebra, then any Rokhlin flow $\alpha : \mathbb{R} \curvearrowright A$ is determined by the induced trace action $T(\alpha) : \mathbb{R} \curvearrowright T^+(A)$ up to cocycle conjugacy. As of writing this report, this remains the only classification theorem covering Rokhlin flows on finite C*-algebras, although one may expect further progress in the future.

If one considers the Elliott invariant consisting of K-theory and traces, then the only very obvious classification invariant for flows would be the aforementioned induced (continuous affine) action of $\mathbb{R}$ on the traces. In the unital case, the tracial invariant would consist of the Choquet simplex of tracial states, but for the purpose of this talk we also want to utilize the tracial cone of a (possibly nonunital) C*-algebra, i.e., the set of densely defined lower semicontinuous tracial weights. A less obvious but important cocycle conjugacy invariant for flows $\alpha : \mathbb{R} \curvearrowright A$ is Connes' rotation map defined in [2]. This provides a dynamically induced interaction between K-theory and traces, and can be viewed as a group homomorphism $r^\alpha : K_1(A) \to \mathrm{Aff}(T^+(A)^\alpha, \mathbb{R})$.

Along the lines of the second goal above, there are mainly two prior results to overview concerning these two dynamical invariants. In general, it is not well understood what actions of $\mathbb{R}$ on $T^+(A)$ are induced from a flow, but Kishimoto–Kumjian have shown in [7] that certain stably projectionless C*-algebras admit so called trace-scaling flows (with a fixed scaling constant). Concerning the rotation map, Kishimoto observed for any Rokhlin flow $\alpha : \mathbb{R} \curvearrowright A$ on a unital C*-algebra that evaluating on any given trace $\tau \in T(A)^\alpha$ yields a homomorphism $K_1(A) \to \mathbb{R}$ with dense range. This provides a concise conceptual justification why Rokhlin flows cannot exist on unital AF algebras. In the opposite direction, he proved in [6] that if A is unital simple A$\mathbb{T}$, has real rank zero and is monotracial, then any homomorphism $K_1(A) \to \mathbb{R}$ with dense range is the rotation map of some Rokhlin flow on A. Although the class of such C*-algebras is not vast from today's point of view, it covers numerous examples such as noncommutative tori. To my knowledge, there has been little progress since these results in the 1990s on the question of realizing more abstract invariants as (Rokhlin) flows.

Before we state the main result, let us introduce some ad-hoc terminology:

(a) A lattice cone C is the positive part $C = E_+$ in a locally compact real vector lattice E.
(b) We call a continuous affine action $\sigma : \mathbb{R} \curvearrowright C$ a scaling action if σ preserves all the extremal rays in C.

We note that a scaling action σ has the property that for any $\tau \in C$ belonging to an extremal ray, there exists some $b > 0$ such that $\sigma_t(\tau) = b^t \cdot \tau$ for all $t \in \mathbb{R}$. However, the scaling constant b is allowed to depend on τ. A special case where it is easy to describe all the scaling actions is when the lattice cone C is of "Bauer type", i.e., C is affinely homeomorphic to $\mathfrak{M}_f(X)$, the set of finite Borel measures on a compact metrizable space X. In that case, it is easy to see that scaling actions are in a canonical one-to-one correspondence with continuous maps $X \to \mathbb{R}^{>0}$.

Our main result (in progress) then goes as follows. Note that the class of C*-algebras determined by the assumptions stated below was classified in [4].

Theorem 1. *Let A be a separable simple amenable C*-algebra that is projectionless, stable, satisfies the UCT, and $A \cong A \otimes \mathcal{Z}_0$. Let $\sigma : \mathbb{R} \curvearrowright T^+(A)$ be a scaling action and let $r : K_1(A) \to \mathrm{Aff}(T(A)^\sigma, \mathbb{R})$ be any homomorphism. Then there exists a flow $\alpha : \mathbb{R} \curvearrowright A$ such that*

- *$T(\alpha) = \sigma$ as actions of $\mathbb{R}$ on $T^+(A)$.*
- *$r^\alpha = r$.*
- *α has the rational Rokhlin property, i.e., $\alpha \otimes \mathrm{id}_{\mathcal{U}}$ has the Rokhlin property for any infinite-dimensional UHF algebra $\mathcal{U}$.*

As a not so hard corollary of this theorem (using also [3]), one gets that every stable classifiable C*-algebra B has some crossed product decomposition $B \cong A \rtimes \mathbb{R}$ for A belonging to the above class.

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On Hausdorff covers for non-Hausdorff groupoids

Xin Li

(joint work with Kevin Aguyar Brix, Julian Gonzales, Jeremy B. Hume)

In my talk, I reported on the paper [2]. The main objects of study are topological groupoids, which arise naturally in a variety of areas such as dynamics, topology, geometry, group theory and C^*-algebras, building bridges between all these areas of mathematics. The link between topological groupoids and C^*-algebras is established by the construction of convolution algebras [19, 5, 6]. Many important classes of C^*-algebras arise from groupoids (for example AF-algebras, Cuntz-Krieger algebras, graph algebras etc., see also [16]), and groupoid models are very helpful for studying structural properties or interesting invariants of the corresponding C^*-algebras. For topological groupoids which are Hausdorff (as well as locally compact and étale), there are satisfactory results characterising fundamental properties of groupoid C^*-algebras such as simplicity, the ideal intersection property or nuclearity in terms of the underlying groupoids (see [7, 3, 12, 1]), and the notion of Cartan subalgebras allows to reconstruct groupoids from their C^*-algebras [14, 21]. However, several of these results, such as characterisations of simplicity, the ideal intersection property, or reconstruction results using Cartan subalgebras, do not carry over to non-Hausdorff groupoids (see [7] and also a related example due to Skandalis in [20]), and new challenges arise in the non-Hausdorff setting because we have to work with compact sets which are not closed and functions which are not continuous. At the same time, important examples of groupoids arising as groupoid of germs, from foliations or self-similar groups typically are non-Hausdorff (see [8, 17, 18] for some examples), which provides a strong motivation to systematically develop a better understanding of non-Hausdorff groupoids and their C^*-algebras.

In this context, a new construction called essential groupoid C^*-algebras has been introduced for non-Hausdorff groupoids [9, 15], and simplicity as well as the ideal intersection property have been characterised for this new construction in terms of the underlying groupoids [4, 12]. While these recent developments and their algebraic analogues in the setting of Steinberg algebras have led to progress in our understanding of non-Hausdorff groupoids, it remains an open question when essential groupoid C^*-algebras agree with the more classical reduced groupoid C^*-algebras, and there is an analogous open question for Steinberg algebras (see [23, 24, 27, 11] for progress in the case of groupoids arising from self-similar groups).

The goal of [2] is to develop a new approach to non-Hausdorff groupoids and their algebras, based on the construction of Hausdorff covers as in [25] (similar constructions appeared in [13, 26]). Given a non-Hausdorff étale groupoid G with Hausdorff unit space, its Hausdorff cover $\tilde{G}$ is a Hausdorff étale groupoid given by the closure of G in the space of non-empty closed subsets of G with respect to the Fell topology [10]. Alternatively, we can think of $\tilde{G}$ as the Gelfand spectrum of the smallest commutative C^*-algebra of bounded, complex-valued functions on G containing $\mathscr{C}_c(G) \coloneqq \mathrm{span}(\{C_c(U)\colon U \subseteq G \text{ open, Hausdorff}\})$. Here $C_c(U)$ is the algebra of continuous functions $U \to \mathbb{C}$ with compact support contained in U, and we extend a function in $C_c(U)$ by zero to view it as a function on G. Using the Hausdorff cover, we develop new tools in the non-Hausdorff setting and reduce questions about non-Hausdorff groupoids to the Hausdorff case. Here is a summary of our main achievements.

- We establish a complete characterisation when singular ideals vanish in Steinberg algebras over arbitrary rings in terms of a groupoid property. In combination with [23, Theorem A], this completely characterises simplicity of Steinberg algebras over fields, which answers [4, Question 1].

- We also completely characterise when the C^*-algebraic singular ideal of a non-Hausdorff étale groupoid G has trivial intersection with $\mathscr{C}_c(G)$. This leads to a characterisation when the C^*-algebraic singular ideal vanishes, under the assumption that the closure of the unit space $G^{(0)}$ of G has finite fibres over $G^{(0)}$. In combination with existing simplicity characterisations for essential groupoid C^*-algebras in [12], this yields a characterisation in terms of a groupoid property when the reduced groupoid C^*-algebra of G is simple, which provides a partial answer to [4, Question 2]. Another consequence is that, for a non-Hausdorff ample groupoid G satisfying our finiteness condition, simplicity of the complex Steinberg algebra of G implies simplicity of the reduced groupoid C^*-algebra of G, which provides a partial answer to [4, Question 3]. We also obtain a conceptual explanation for the results on groupoids of contracting self-similar groups in [11].
- We show, for a non-Hausdorff étale groupoid G which can be covered by countably many open bisections, that the ideal intersection property for the essential groupoid C^*-algebra $C^*_{\mathrm{ess}}(G)$ is equivalent to the ideal intersection property of the reduced C^*-algebra $C^*_r(\tilde{G}_{\mathrm{ess}})$, where $\tilde{G}_{\mathrm{ess}}$ is the restriction of $\tilde{G}$ to a closed invariant subset of the unit space of $\tilde{G}$. Since $\tilde{G}_{\mathrm{ess}}$ is Hausdorff, we obtain a conceptual explanation for results on simplicity or the ideal intersection property for essential C^*-algebras of non-Hausdorff étale groupoids in [4, 9, 15, 12].
- We show that nuclearity of the reduced C^*-algebra of a non-Hausdorff étale groupoid G is equivalent to topological amenability of its Hausdorff cover $\tilde{G}$. If G is σ-compact, results in [22] imply that this is equivalent to amenability of G itself (in the sense of [22, Definition 2.7]). This reduces questions about nuclearity and amenability to the Hausdorff case.

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Uniqueness theorems for maps into II_1-factors and ultraproducts of matrices

Shanshan Hua
(joint work with Stuart White)

Starting with Elliott's classification of approximately finite dimensional C^*-algebras and over the past few decades, the classification results for C^*-algebras have been built on those of morphisms. Recently, to recapture the unital classification theorem obtained in 2015, Carrión, Gabe, Schafhauser, Tikuisis and White classified, via an abstract approach, unital and full *-homomorphisms $A \to B_\omega$ up to unitary equivalence in [1], by enriched K-theoretical and tracial data. Their underlying assumptions are: A is nuclear and satisfies the universal coefficient theorem (UCT), and B is simple and satisfies a regularity condition called *$\mathcal{Z}$-stability.* This regularity property resembles the McDuff property for II_1-factors and means that $A \cong A \otimes \mathcal{Z}$, where $\mathcal{Z}$ is the Jiang-Su algebra. See [6] for a survey of this abstract classification approach.

The only place where the full force of $\mathcal{Z}$-stability is needed in [1] instead of its consequences, such as strict comparison and uniform property Γ, is in the proof of the uniqueness statement. A proper "$\mathcal{Z}$-stability" notion, called separable $\mathcal{Z}$-stability, for B_ω is involved in the proof. If B is $\mathcal{Z}$-stable, then B_ω is usually not $\mathcal{Z}$-stable, but are *separably $\mathcal{Z}$-stable*, which means that for any separable C^*-subalgebra, there exists a separable $\mathcal{Z}$-stable C^*-subalgebra containing it.

In collaboration with Stuart White, we target to prove uniqueness statements for more general maps lacking such $\mathcal{Z}$-stability assumption. One important example is the class of unital embeddings from a separable and nuclear C^*-algebras into a II_1-factor, which is the starting point of the modern abstract classification approach pioneered by Schafhauser in [5] and further developed in [1]. The uniqueness of such maps in the $\|\cdot\|_2$-norm topology induced by the unique trace of the II_1-factor is well-known as a consequence of Connes' characterizations for hyperfiniteness for von Neumann algebras.

Theorem 1 (Connes; c.f. [5, Proposition 1.1])**.** *Let A be a separable and nuclear C^*-algebra and $\mathcal{M}$ is a II_1-factor. Let $\phi, \psi : A \to \mathcal{M}$ be unital $*$-homomorphisms such that $\tau_{\mathcal{M}} \circ \phi = \tau_{\mathcal{M}} \circ \psi$, then there exists a sequence of unitaries $(u_n)_n$ in $\mathcal{M}$ such that*

$$\|u_n\phi(a)u_n^* - \psi(a)\|_{2,\tau_{\mathcal{M}}} \to 0, \quad a \in A. \tag{1}$$

We manage to prove a similar uniqueness theorem in the norm topology, with the additional assumption that the domain C^*-algebra satisfies the UCT.

Theorem 2 (H., White)**.** *Let A be a separable, unital and nuclear C^*-algebra satisfying the UCT and $\mathcal{M}$ be a II_1-factor. Let $\phi, \psi : A \to \mathcal{M}$ be unital and faithful $*$-homomorphisms such that $\tau_{\mathcal{M}} \circ \phi = \tau_{\mathcal{M}} \circ \psi$, then there exists a sequence of unitaries $(u_n)_n$ in $\mathcal{M}$ such that*

$$\|u_n\phi(a)u_n^* - \psi(a)\| \to 0, \quad a \in A. \tag{2}$$

Theorem 2 is not covered by techniques in [1], since for instance it is open whether $\mathcal{R}$ is separably $\mathcal{Z}$-stable. This result is new and interesting even when the domain is commutative, in which case the UCT is automatic and the faithfulness of maps is not needed. The UCT assumption is needed in general to access the KK-machinery. It is conjectured that the UCT holds for every separable and nuclear C^*-algebra, and all such concrete examples that people have checked indeed have the UCT. Thus, Theorem 2 is widely applicable in practice.

Another class of maps of interest can be considered as the uniqueness counterpart of quasidiagonality for nuclear C^*-algebras. A separable, unital and nuclear C^*-algebra A is *quasidiagonal* if there exists a unital map from A into $\prod_\omega M_{k_n}$

for some sequence of natural numbers $(k_n)_n$, or equivalently into $\mathcal{Q}_\omega$, where $\mathcal{Q}$ is the universal UHF-algebra. Maps into $\mathcal{Q}_\omega$ are classified by Schafhauser in [5]. Comparing to $\mathcal{Q}_\omega$, which is separably $\mathcal{Z}$-stable since $\mathcal{Q}$ is $\mathcal{Z}$-stable, the algebra $\prod_\omega M_{k_n}$ is not separably $\mathcal{Z}$-stable, since the corresponding tracial ultraproduct does not have the property Γ. The following theorem generalizes Lin's result in [4] for commutative A and for certain AH-algebras.

Theorem 3 (H., White)**.** *Let A be a separable, unital and nuclear C^*-algebra satisfying the UCT. Let $\phi, \psi : A \to \prod_\omega M_{k_n}$ be full and unital $*$-homomorphisms such that $\underline{K}(\phi) = \underline{K}(\psi)$ and $\tau_\omega \circ \phi = \tau_\omega \circ \psi$, where τ_ω is the unique limit trace on $\prod_\omega M_{k_n}$. Then there exists a unitary $u \in \prod_\omega M_{k_n}$ such that $\phi = Ad(u) \circ \psi$.*

To prove Theorem 2 and Theorem 3, appropriate KK-uniqueness theorems for maps are needed, the proof of which follows the same strategy as in the work [2] of Dadarlat and Eilers. The Paschke duality gives an isomorphism between KK-group and K_1-group of the so-called *Paschke dual algebra*. To obtain the KK-uniqueness theorem, it suffices to show that the Paschke dual algebra is K_1-*injective*, meaning that every unitary with the trivial K_1-class is path connected in the unitary group to the identity. When codomain C^*-algebras are $\mathcal{Z}$-stable, the proof of the so-called $\mathcal{Z}$-stable KK-uniqueness theorems in [1] instead works with the $\mathcal{Z}$-stabilization of the Paschke dual algebra, which is automatically K_1-injective by Jiang's result ([3]). For cases that we are interested in, due to the absence of $\mathcal{Z}$-stability, we show K_1-injectivity results directly.

Combining such KK-uniqueness theorems with the abstract approach in [5] and [1], we prove the main uniqueness theorem, which covers Theorem 2 and Theorem 3 as spacial cases and goes beyond the previous scope of classification.

Theorem 4 (H., White)**.** *Let A be a separable, unital and nuclear C^*-algebra satisfying the UCT. Let $(B_n)_{n=1}^\infty$ be a sequence of simple and unital C^*-algebras which have real rank zero, stable rank one, tracial states τ_n, totally ordered Murray–von Neumann semigroups $V(B_n)$ and $K_1(B_n) = 0$. We write B_ω for $\prod_\omega B_n$.*

Given full and unital *-homomorphisms $\phi, \psi \colon A \to B_\omega$ with $\tau_{B_\omega} \circ \phi = \tau_{B_\omega} \circ \psi$ and $\underline{K}(\phi) = \underline{K}(\psi)$, there exists a unitary $u \in B_\omega$ with $\psi = (\mathrm{Ad}\, u) \circ \phi$.

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Selflessness of reduced free product C^*-algebras

Ben Hayes
(joint work with Srivatsav Kunnawalkam Elayavalli, and Leonel Robert)

Let A be a C^*-algebra equipped with a state ρ. Robert in [12] defined (A,ρ) is selflessness if there exists a C^*-algebra C with a state κ so that $(A,\rho)*(C,\kappa)$ has a state-preserving embedding into $(A^{\omega,\|\cdot\|},\rho^{\omega})$ for some cofinal ultrafilter ω on a directed set I. Robert (see [12, Theorem 2.6]) showed that if (A,ρ) is selfless, then whenever B is a C^*-algebra with a state ψ such that (B,ψ) has a state-preserving embedding into $(A^{\omega,\|\cdot\|},\rho^{\omega})$, it necessarily follows that $(A,\rho)*(B,\psi)$ has a state-preserving embedding into $(A^{\omega,\|\cdot\|},\rho^{\omega})$. Note that for any such (B,ψ) selflessness is equivalent to saying that $A\hookrightarrow A*B$ is a sequentially split inclusion in the sense of [2]. This definition can be taken as an analogue of a theorem of Popa in [11], showing that the ultrapower of any II_1-factor contains a Haar unitary which is freely independent of the diagonal embedding.

The crucial significance of this is that by [2, Theorem 2.9],[12, Theorem 3.1] it follows that selfless C^*-algebras automatically enjoy several desirable properties, e.g. they are all simple. Additionally, if ρ is not a trace, they are automatically purely infinite and have real rank zero. When ρ is a trace, they have stable rank one and have strict comparison. Strict comparison is a desirable property recently for the classification theory of C^*-alegbras (see e.g [3, 4, 5, 6, 13, 14, 15]), and has been open for a while for many non-nuclear C^*-algebras, such as the reduced free group C^*-algebra, and the C^*-algebra generated by r free semicirculars for $r\geq 2$ (this last question being a problem of Gabe-Phillips).

In the breakthrough work [1], the authors show selflessness for a broad class of reduced group C^*-algebras, including free groups and more generally finitely generated acylindrically hyperbolic groups with trivial finite radical (see also [8] for rapidly changing and exciting developments). It required as a crucial input the rapid decay property for these groups. Despite this work being of fundamental importance, it nevertheless left open the question of selflessness for C^*-algebras of more general free products.

In order to address this question, we first develop a general notion of rapid decay property, which encompasses the group setting and in the commutative case reduces to the classical theory of orthogonal polynomials.

Definition 1. Let A be a unital C^*-algebra equipped. A *filtration* of A is a sequence $(V_n)_{n=0}^{\infty}$ of subspaces of A so that:

- $V_0=\mathbb{C}1$,
- $V_n\subseteq V_{n+1}$ for every natural number n,
- V_n is closed under $*$,
- $\bigcup_n V_n$ is norm dense in A.

For example, if G is a finitely generated group and S is a finite generating set, the for an integer $n\geq 0$ we use

$$B(e,n)=\{s_1\cdots s_r : s_i\in S\cup S^{-1}\cup\{e\}\}.$$

For $A = C^*_\lambda(G)$, we then have a filtration given by $V_n = \mathrm{Span}(\{\lambda_g : g \in B(e,n)\})$.

For $A = C([a,b])$ with $a < b$ real numbers, we may consider $V_n = \mathrm{Span}(\{x^k : 0 \leq k \leq n\})$.

Definition 2. Let A be a C^*-algebra and ρ a state on A. Given a filtration $(V_n)_{n=0}^\infty$ of A, we say that $(V_n)_{n=0}^\infty$ has the *rapid decay property* if there constants $C, \alpha \geq 0$ so that

$$\|x\| \leq C(n+1)^\alpha \|x\|_2, \quad \text{for all } x \in V_n.$$

The reader can check that reduces to the definition for groups given in [9], if one uses the filtration coming from balls in a Cayley graph as above.

Using fundamental results of Ricard-Xu [10], we show the following.

Theorem 1. *Let $(A_j, \rho_j), j = 1, \cdots, m$ be unital C^*-algebras equipped with states. For all $j = 1, \cdots, m$ let $(V_{n,j})_{n=0}^\infty$ be filtrations of A_j. For integers $0 \leq t \leq n$, let $W_{t,n}$ be the span of all words of the form $x_1 \cdots x_t$ where:*

- $x_i \in V_{n,j(i)} \ominus \mathbb{C}1$,
- $j \colon \{1, \cdots, t\} \to \{1, \cdots, m\}$ *has* $j(i) \neq j(i+1)$ *for all* $1 \leq i \leq t-1$.

Set $V_n = \sum_{t=0}^n W_{t,n}$. Then:

(1) $(V_n)_{n=0}^\infty$ is a filtration of $A_1 * \cdots * A_m$,
(2) if each $(V_{n,j})_{n=0}^\infty$ has the rapid decay property with respect to ρ_j, then $(V_n)_{n=0}^\infty$ has the rapid decay property with respect to $\rho_1 * \rho_2 * \cdots * \rho_m$.

Using this, we prove the following theorem, settling the question of Gabe-Phillips, and also establishing selfness for a large family of reduced free products.

Theorem 2. *Let $(A_j, \rho_j), j = 1, 2$ be unital C^*-algebras equipped with states $\rho_j, j = 1, 2$. Suppose each (A_j, ρ_j) has a filtration with the rapid decay property, and that ρ_1 is a trace. Let M be the GNS completion of A_1 with respect to ρ_1. Assume that one of the following conditions hold:*

(1) M is a II_1-factor, or
(2) $M^{\omega, \|\cdot\|_2} \cap M'$ is diffuse (e.g. if M has diffuse center).

Then $(A_1, \rho_1) * (A_2, \rho_2)$ is selfless.

We remark here that, while we take the strategy of [1] as a roadmap, there are crucial differences. Namely, in [1] the authors conjugate part of the free product $A_1 * A_2 * A_1$ by unitaries that come from the filtration of A_1 to define maps which prove selflessness. Note that for the filtration of $(C([-2,2]), \int \cdot \frac{dx}{2\pi\sqrt{4-x^2}})$ given by polynomials does not have any nonconstant untiaries. So we cannot follow this strategy and are led to conjugate by unitaries outside of the filtration. In order to preserve the appropriate rapid decay estimates, we need to better analyze the combinatorial structure of this map and the theorem of Ricard-Xu. In the case that M is a II_1-factor, we use Popa's theorem [11] to find a sequence of unitaries in A_1 asymptotically free from M. In the case that $M^{\omega, \|\cdot\|_2} \cap M'$ is diffuse, we use a nontrivial theorem of Kirchberg-Rørdam to find unitaries in A which asymptotically commute *in norm* with every element of A and are asymptotically orthogonal to all of A. With these crucial ingredients, we can then modify the strategy of [1] to prove the above theorem.

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New Examples of Strict Comparison in C^*-Algebras

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(joint work with Tattwamasi Amrutam, David Gao, and Srivatsav Kunnawalkam Elayavalli)

One of the most fundamental ways to compare matrices is via their rank. For two matrices X and Y, rank(X) is less than or equal to rank(Y) if and only if there are matrices S and T such that X = SYT. The rank can be generalized to C^*-algebras using dimension functions and the latter algebraic condition can be generalized to a condition known as Cuntz subequivalence. C^*-algebras for which the dimension functions recover Cuntz subequivalence are said to have strict comparison, see [5, 36, 14, 2]. Strict comparison is known to have applications to classification of *-homomorphisms of C^*-algebras, including existence and uniqueness of embeddings of the Jiang–Su algebra [34, 19, 15, 32]; to the calculation of the Cuntz semigroup [8, 19, 7]; and to a breakthrough in the C^*-algebraic analogue of Tarski's problem determining non-elementary equivalence of the reduced group C^*-algebras of free groups, strikingly contrasting Sela's results to elementary equivalence of free groups [38, 39]. In the nuclear setting, strict comparison is equivalent to tensorial absorption of the Jiang–Su algebra [27]. However, previously there was a severe lack of non-nuclear examples of strict comparison in the setting of reduced group C*-algebras. In 1991, Anderson–Blackadar–Haagerup showed strict comparison for free products of finite-dimensional abelian algebras [1]; in 1998 Dykema–Rørdam showed that infinite reduced free products have strict comparison [13]; but, even for the free group on two generators, strict comparison of the reduced group C^*-algebra was a long-standing open problem. In our work we show that the reduced group C^*-algebra of the free group on two generators has strict comparison. Our methods are very general and lead to proving strict comparison (and the stronger property of selflessness, due to Robert [35]) for all acylindrically hyperbolic groups with the rapid decay property.

Our methods combine two properties for groups, the rapid decay property and a quantitative version of the mixed-identity free property. A group G is said to have *rapid decay* if there is some polynomial P such that for all $\varphi \in \mathbb{C}[G]$ supported on the radius N ball, $\|\varphi\| \leq P(N)\|\varphi\|_2$. The rapid decay property (see the survey [37]) was first proved by Haagerup for the free groups [18]. It was then generalized to Gromov hyperbolic groups [23, 20], relatively hyperbolic groups with peripheral subgroups having rapid decay [37], cocompact lattices in $SL_3(\mathbb{R})$ or $SL_3(\mathbb{C})$ [25, 9], mapping class groups of surfaces [4], large type Artin groups [10], hierarchically hyperbolic groups [3], and more. Rapid decay is extremely influential and has been used to prove several important problems, including the Novikov conjecture [11], the Baum-Connes conjecture [26], and others including [17]. A group G is said to be *mixed-identity free* if for all nontrivial words $w(x) \in G * \langle x \rangle$ there is $g \in G$ such that $w(g) \neq e$. This is equivalent to G embedding existentially in $G * \mathbb{Z}$, see Section 5 in [22]. For our purposes, we require something stronger, namely that there is a polynomial Q such that every word $w(x)$ as before of length N can be violated by a group element g of length at most $Q(N)$. We are able to show this strengthened version of the mixed-identity free property for all free groups as well as the much broader class of acylindrically hyperbolic groups with trivial finite radical [29]. A group is said to be *acylindrically hyperbolic* if it admits a non-elementary acylindrical action on a Gromov hyperbolic metric space. See [28] for context, examples, and references. Recall also the finite radical for a group is the largest finite normal subgroup. Within the class of acylindrically hyperbolic groups G, it turns out that the finite radical being trivial is equivalent to $C_r^*(G)$ having unique trace, which is in turn equivalent to $C_r^*(G)$ being simple (see Theorem 1.4 of [7] and Theorem 2.35 of [12]).

Our methods show that in fact $C_r^*(G)$ is selfless for G having rapid decay and being acylindrically hyperbolic with trivial finite radical. A C^*-algebra (A, ϕ) is *selfless* if (A, ϕ) embeds existentially in $(A, \phi) *_r (C(\mathbb{T}), \lambda)$. This can be seen as a C^*-algebraic analogue of Popa's theorem that all II_1 factors are freely complemented in their tracial ultrapowers [31]. As seen in [35], all tracial selfless C^*-algebras are simple, have stable rank 1, and have strict comparison.

Since the time of our work, there have been several developments in selfless C^*-algebras. Vigdorovich showed that cocompact lattices in $PSL_d(\mathbb{K})$ have linear size violators of mixed identities for any local field $\mathbb{K}$ and any $d \geq 2$ [40]. In particular, in combination with Lafforgue's result that cocompact lattices in $SL_3(\mathbb{R})$ and $SL_3(\mathbb{C})$ have rapid decay [25], this shows that the reduced group C^*-algebras of such groups are selfless. Bradford–Sisto were able to greatly improve upon our estimates of mixed-identity violation in acylindrically hyperbolic groups with trivial finite radical and show that, in fact, the violators can be chosen with linear growth [6]. Hayes–Kunnawalkam Elayavalli–Robert showed that free products of C^*-algebras with filtrations admitting rapid decay are selfless under some mild assumptions [21]. This allowed them to prove selflessness for C^*-algebras not arising from groups, such as the reduced free products of continuous functions on the interval. Raum–Thiel–Vilalta show that twisted reduced group C^*-algebras arising from the same groups as we considered (quantitative mixed-identity free and rapid decay) are selfless. They also note that selflessness implies pureness and use this to prove strict comparison for the reduced group C^*-algebras of acylindrically hyperbolic groups with non-trivial finite radical and rapid decay [33]. Most recently, Ozawa found both a dynamical criterion (existence of a topologically free extremely proximal minimal action) which implies selflessness and showed stability of selflessness under minimal tensor products in the presence of exactness [30]. In particular, this implies that non mixed-identity free groups can have selfless reduced group C^*-algebras, such as $\mathbb{F}_2 \times \mathbb{F}_2$.

Several questions remain regarding strict comparison and selflessness. For instance, Robert asks whether every C^*-simple group has selfless reduced group C^*-algebra [35]. Whether strict comparison for $C^*(G)$ is equivalent to C^*-simplicity or whether selflessness is equivalent to strict comparison plus C^*-simplicity are also open problems. Whether the torsion Tarski monsters are selfless is also open; notably, these have not just a mixed identity but an identity and are C^*-simple. A similar question can be asked about free Burnside groups. In light of Ozawa's result on extremely proximal actions [30], is there a dynamical condition characterizing selflessness? There are also many questions regarding crossed products. Giol–Kerr constructed a crossed product of $\mathbb{Z}$ on $C(X)$ which is simple but fails to have comparison [16]. Of course, $C^*_r(\mathbb{Z})$ is not simple, much less selfless. But are there minimal actions of C^*-selfless groups giving rise to non-selfless reduced crossed products? To the author's knowledge, this is open even for minimal actions of $\mathbb{F}_n$ at the moment. Note that all such crossed products are simple by Kawabe [24]. More generally, can one characterize selflessness of reduced crossed products either through a dynamical condition or a condition on the stabilizer subgroups?

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Selfless higher rank lattices

Itamar Vigdorovich

Given a unital C^*-algebra A, several structural properties play a central role as regularity inputs for classification. Very roughly:

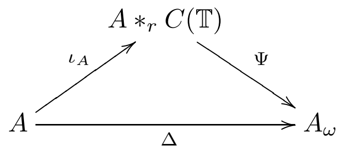
(1) *Cuntz subequivalence:* a relation on positive elements $a, b \in A_+$ asserting that a can be approximately cut out of b by inner conjugation; this gives rise to the Cuntz semigroup $\mathrm{Cu}(A)$.
(2) *Strict comparison:* traces detect Cuntz subequivalence for positive elements (if $d_\tau(a) < d_\tau(b)$ for all traces τ, then $a \precsim b$).
(3) *Stable rank* 1*:* invertible elements are dense. This is a noncommutative low-dimensionality condition that drastically simplifies the K-theory.
(4) *The Jiang–Su algebra* $\mathcal{Z}$*:* embeddings of $\mathcal{Z}$ into A and their uniqueness up to approximate unitary equivalence are key regularity features.
(5) *Selflessness:* existence in A of a sequence of Haar unitaries that is asymptotically freely independent from A.

The last property, selflessness, was introduced by L. Robert and, in the settings relevant here, it implies the other regularity properties above, so we focus on it.

Definition 1 (Selfless C^*-algebra [5])**.** Let (A, φ) be a unital C^*-algebra with a faithful state φ, and fix a free ultrafilter ω on $\mathbb{N}$. Write $\Delta\colon A \to A_\omega$ for the diagonal embedding into the C^*-ultrapower A_ω. Let m be Haar measure on $\mathbb{T}$, and denote by $(A,\varphi) *_r \big(C(\mathbb{T}), m\big)$ the reduced free product. We say that A is *selfless* if there exists a unital $*$-monomorphism

$$\Psi\colon A *_r C(\mathbb{T}) \longrightarrow A_\omega$$

such that the diagram



commutes; i.e. $\Psi \circ \iota_A = \Delta$.

A typical example of a selfless C^*-algebra is an *infinite* free product, e.g. $C^*_r(\mathbb{F}_\infty)$ (see [5]). Beyond this, natural examples are harder to come by. For instance, until recently it was open whether $C^*_r(\mathbb{F}_2)$ has strict comparison. A recent breakthrough by Amrutam–Gao–Kunnawalkam Elayavalli–Patchell settled this positively by proving selflessness (and hence strict comparison) for $C^*_r(G)$ for large classes of groups, including all hyperbolic groups and many acylindrically hyperbolic groups [1]. However, their argument crucially exploits hyperbolic geometry, and they asked for examples of a different nature, specifically for higher rank lattices.

Theorem 1. *If Γ is a cocompact lattice in $\mathrm{SL}_3(\mathbb{R})$, then $C^*_r(\Gamma)$ is selfless.*

Idea of the proof (following the general AGKEP strategy).

(1) *Effective selflessness at the group level.* Produce a sequence $\gamma_n \in \Gamma$ that is, in a quantitative sense, free from the radius-n ball $B_\Gamma(n)$, with word-length $|\gamma_n|$ growing at most linearly (or, at worst, subexponentially).
(2) *Upgrade via (RD).* Use the effectiveness above together with the rapid decay property to pass from the group level to $C^*_r(\Gamma)$.

Luckily, the groups in Theorem 1 have property (RD): for $\mathrm{SL}_3(\mathbb{R})$ and $\mathrm{SL}_3(\mathbb{C})$ by V. Lafforgue [2], and for groups acting on $\widetilde{A}_2$-buildings by Ramagge–Robertson–Steger [4]. In contrast, (RD) remains open for many nonuniform lattices (e.g. $\mathrm{SL}_3(\mathbb{Z})$), and it is widely conjectured to hold for further cocompact lattices in simple Lie groups but remains notoriously hard.

The theorem above follows from the following group-theoretic statement.

Theorem 2 (Group-level effective freeness; see [6]). *Let Γ be a cocompact lattice in $\mathrm{PSL}_n(\mathbb{R})$ and fix a finite generating set S. There exists $C > 0$ and a sequence $\gamma_n \in \Gamma$ with $|\gamma_n|_S \leq Cn$ such that γ_n is "freely independent from the ball $B_\Gamma(n)$" in the sense of effective mixed-identity-freeness defined in* [6].

Open question. Is $C^*_r(\mathrm{SL}_3(\mathbb{Z}))$ selfless? In the absence of (RD), new ideas seem necessary. In this direction, see the recent work of Ozawa on extremely proximal boundaries and selflessness [3], which arose from this very Oberwolfach meeting.

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W*-superrigidity for property (T) groups with infinite center

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(joint work with Ionuţ Chifan, Denis Osin and Hui Tan)

To any countable group G, one can associate its von Neumann algebra $\mathcal{L}(G)$ [10], defined as the weak operator closure of the complex group algebra $\mathbb{C}[G]$, acting by left convolution on the Hilbert space ℓ^2G of square-summable functions on G. A major research theme in the field of von Neumann algebras, which has garnered considerable attention over the years, is understanding the extent to which $\mathcal{L}(G)$ retains algebraic information about the underlying group G.

Recall that a group G is said to have the *ICC property* if the conjugacy class of every non-trivial element of G is infinite. In [4] A. Connes discovered that II_1 factors associated with ICC Kazhdan property (T) groups exhibit strong rigidity under small perturbations; in particular, he showed that both the fundamental group and the outer automorphism group of every such II_1-factor is countable. These result and their fairly conceptual proofs motivated Connes to conjecture that *if G,H are ICC property (T) groups with $\mathcal{L}(G) \cong \mathcal{L}(H)$, then $G \cong H$*, [5].

The first W*-rigidity results for von Neumann algebras $\mathcal{L}(G)$ groups G with infinite center were obtained in [2], focusing on direct products $G = A \times K$, where A is an infinite abelian group and K is an ICC property (T) wreath-like product group as in [3]. Such product groups cannot be W*-superrigid as their center cannot be reconstructed from their von Neumann algebra. However, in [2] it was shown that this is the only obstruction to their W*-superrigidity: *any group H with $\mathcal{L}(G) \cong \mathcal{L}(H)$ must be a direct product of the form $H \cong B \times K$, where B is an infinite abelian group.*

The main goal of this paper is to construct examples of property (T) groups with infinite center that are W*-superrigid. Our approach builds on methods from [2] and uses a generalization of the notion of wreath-like products of groups introduced in [3]. These, along with the non-property (T) examples involving left-right wreath product groups obtained by Donvil and Vaes in parallel, independent work [8], constitute the first known W*-superrigid groups with an infinite center.

For given groups A and B and an action $B \curvearrowright I$ on a set I, we denote by $\mathcal{WR}(A, B \curvearrowright I)$ the class of *wreath-like products of A and B corresponding to the action $B \curvearrowright I$* introduced in [3]. Recall $G \in \mathcal{WR}(A, B \curvearrowright I)$ if there exists a short exact sequence

$$1 \to \oplus_{i\in I} A_i \hookrightarrow G \overset{\varepsilon}{\twoheadrightarrow} B \to 1$$

such that $A_i \cong A$ for all $i \in I$ and $gA_ig^{-1} = A_{\varepsilon(g)\cdot i}$ for every $g \in G$, where A_i is the i-th copy of A in $\oplus_{i\in I} A_i$. When $I = B$, this class is denoted simply by $\mathcal{WR}(A, B)$ and its elements are called regular wreath-like products of A and B. Our first result is the following.

Theorem 1. Let A be a nontrivial free abelian group, B a nontrivial ICC subgroup of a hyperbolic group, $B \curvearrowright I$ an action on a countable set I with amenable stabilizers. Every property (T) group G with infinite center such that $G/Z(G) \in \mathcal{WR}(A, B \curvearrowright I)$ is virtually W*-superrigid. That is, if H is an arbitrary countable group for which $\mathcal{L}(H) \cong \mathcal{L}(G)$, then the groups G and H are virtually isomorphic.

We note that the virtual W*-superrigidity in the conclusion of Theorem A cannot be promoted to the genuine W*-superrigidity. Indeed, let Q be any infinite property (T) group and let A and B be any nonisomorphic finite abelian groups of the same order (e.g., we can take $A = \mathbb{Z}_{nm}$ and $B = \mathbb{Z}_n \times \mathbb{Z}_m$ for some positive integers $n, m \geq 2$ which are not co-prime). Consider $G = A \times Q$ and $H = B \times Q$. Since G and H are finite index extensions of Q and the latter has property (T), it follows that G and H have property (T). Clearly, we have $G \ncong H$. However, $\mathcal{L}(G) \cong \mathbb{C}^{|B|} \overline{\otimes} \mathcal{L}(Q) \cong \mathcal{L}(H)$.

As with the majority of previous rigidity results for group factors [9, 1, 3, 7], it is desirable to provide a complete description of the $*$-isomorphism between $\mathcal{L}(G) \cong \mathcal{L}(H)$ in terms of the virtual isomorphism between the underlying groups, $G \cong_v H$, along with other relevant data about these groups, such as their multiplicative characters, or more generally, their finite-dimensional representations, inductions from their finite-index subgroups, etc.

Theorem 2. Let $W \in \mathcal{WR}(A, B \curvearrowright I)$ and $1 \to Z(G) \hookrightarrow G \to W \twoheadrightarrow 1$ be groups as in the statement of Theorem 1. Additionally, assume that G has trivial abelianization and $\mathrm{Out}(G) = \{1\}$. If H is an arbitrary countable group for which $\Theta : \mathcal{L}(G) \to \mathcal{L}(H)$ is a von Neumann algebra isomorphism, then there exists a group isomorphism $\delta : G \to H$ and a unitary $w \in \mathcal{L}(H)$ for which

$$\Theta(u_g) = wv_{\delta(g)}w^*, \text{ for all } g \in G.$$

Using an approach that combines our construction of central extensions with the control of outer automorphisms of Dehn fillings from [6], we are able to obtain examples of groups satisfying the assumptions of Theorem 2.

Theorem 3. For each $n \in \mathbb{N}$, there is a central extension $1 \to Z(Q) \hookrightarrow Q \to W \twoheadrightarrow 1$ satisfying

(a) $Z(Q) \cong \mathbb{Z}^n$;

(b) $W \in \mathcal{WR}(\mathbb{Z}, B \curvearrowright I)$ is an ICC group where B is ICC hyperbolic, $B \curvearrowright I$ is transitive with finite stabilizers and $\mathrm{Out}(B) = \{1\}$; and
(c) Q has property (T), trivial abelianization and $\mathrm{Out}(Q) = \{1\}$.

We also mention in passing that the previous two results yield, in particular, property (T) groups G with infinite centers extensions for which $\mathrm{Out}(\mathcal{L}(G)) = 1$. To the best of our knowledge, this is the first concrete computation of outer automorphisms of property (T) von Neumann algebras with diffuse center.

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Strong convergence of unitary representations

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(joint work with Michael Magee)

Let Γ be a finitely generated group. Every unitary representation of Γ gives rise to a semi-norm on $C^*(\Gamma)$. A sequence of unitary representations π_n is said to converge strongly to π_∞ if the corresponding semi-norms converge pointwise:

$$\forall a \in C^*(\Gamma),\ \lim_n \|\pi_n(a)\| = \|\pi_\infty(a)\|.$$

In this talk, we were mainly interested in the case when:

- $\pi_\infty = \lambda$ is the left-regular representation of Γ on $\ell_2(\Gamma)$: $\lambda(\gamma)f = f(\gamma^{-1}\cdot)$,
- π_n is a finite-dimensional unitary representation.

When such a sequence exists, we say [14] that Γ is PMF (Purely Matricial Field). The terminology comes from the fact that, in that case, we obtain an embedding of $C^*_\lambda(\Gamma)$ into a C^*-algebraic ultraproduct of matrix algebras (sending

$\lambda(\gamma)$ to $(\pi_n(\gamma))_n$): $C^*_\lambda(\Gamma)$ is MF (Marticial Field) in the vocabulary of Blackadar and Kirchberg [1].

PMF groups are necessary residually finite by Mal'cev's theorem, and every amenable residually finite group is PMF. Finding examples and non-examples apart from these almost obvious cases turned out to be a challenge. The first examples were free groups, as proven by Haagerup and Thorbjørnsen [9] by means of random matrices. Since then, many other examples of random matrix models have been shown to strongly converge to the left-regular representation of the free group (for example [8, 13, 2, 5, 3, 4, 7, 6, 15], the list is far from complete). These results have important applications [9, 12, 10].

Michael Magee, with Lars Louder and Joe Thomas, has initiated the project of finding other examples of PMF groups, and succeeded [11, 16].

In my talk, I gave a survey of the known results and provided a detailed proof of the following result, that gives the first examples of residually finite groups that are not PMF.

Theorem 1. [14] *Every nonzero finite-dimensional unitary representation of* $\mathrm{SL}_4(\mathbf{Z})$ *has a nonzero vector that is fixed by the subgroup* $\begin{pmatrix}\mathrm{SL}_2(\mathbf{Z}) & 0\\ 0 & \mathrm{id}_2\end{pmatrix}$.

The reason why this theorem implies that $\Gamma = \mathrm{SL}_4(\mathbf{Z})$ is not PMF is because the subgroup $\Lambda = \begin{pmatrix}\mathrm{SL}_2(\mathbf{Z}) & 0\\ 0 & \mathrm{id}_2\end{pmatrix}$ is not amenable: if S is a finite symmetric generating set of Λ and $a = \sum_{s\in S} s$, then we know by Kesten's theorem that $\|\lambda(a)\| < 1$, whereas the theorem implies that $\|\pi(a)\| = 1$ for every finite-dimensional unitary representation of Γ.

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Operator System Limits of C*-Algebras

Kristin Courtney
(joint work with Niklas Galke, Lauritz van Luijk, Alexander Stottmeister, Wilhelm Winter)

We consider inductive sequences of C*-algebras in the operator system category, meaning the usual *-homomorphic connecting maps are replaced with completely positive contractive (c.p.c.) maps (so-called *c.p.c. systems.* These sequences arise naturally in a number of settings. For example, the dual statement (over $\mathbb{C}$) of Lazar and Lindenstrauss' theorem [4] that any metrizable Choquet simplex is the projective limit of finite-dimensional simplices, says that the affine function space on a metrizable simplex is the limit of a c.p.c. system $\mathbb{C} \to \mathbb{C}^2 \to \mathbb{C}^3 \to \dots$.

Given a finite-dimensional c.p.c. system, one can ask what properties of the sequences, such as nuclearity, are transferred to the limit.

Theorem 1 (C.–Galke–van Luijk–Stottmeister, Ding–Peterson)**.** *The limit of a finite-dimensional c.p.c. system is nuclear.*

Another question is whether or not it is (completely order isomorphic to) a C*-algebra. For Choquet simplices, this is asking whether the limit is Bauer.

Theorem 2 (C.,Blackadar–Kirchberg, C.–Winter)**.** *For a finite-dimensional c.p.c. system, consider the following.*

(1) The limit of the system is a C**-algebra.*
(2) The system is NF (in the sense of [1]*).**
(3) The system is CPC* *(in the sense of* [3]*).**
(4) The system is C**-encoding (in the sense of* [2]*).* *

Then we have $(2) \implies (3) \implies (4) \iff (1)$*. (Where* $(*)$ *indicates the claim holds up to passing to cofinal subsystems.)*

Condition (4) characterizes when a nuclear operator system is a C*-algebra (and can also tell whether a simplex is Bauer). With this, we give new examples of operator systems that are not C*-algebras, which are sparse in the literature. On the other hand, to capture structural information of the limit, e.g., its K-theoretic or tracial information, one needs the structure inherent in (2) and (3).

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Relative solidity results and their applications to computations of some II_1-factor invariants

Ionuţ Chifan

(joint work with J.F. Ariza Mejia, N. Amaraweera Kahitotage, J. Lim, K. Khan)

For this talk I will show that whenever G is a group that is hyperbolic relative to a family of exact residually finite subgroups $\{H_1, H_2, \ldots, H_n\}$, the corresponding von Neumann algebra $L(G)$ is solid relative to the family of subalgebras $\{L(H_1), L(H_2), \ldots, L(H_n)\}$. Building on this result and combining it with findings from geometric group theory, I will construct a continuum of icc property (T) relative hyperbolic groups that give rise to pairwise non-isomorphic factors, each of which has trivial one-sided fundamental group. In addition, I will explain the construction of the first property (T) groups G such that the Jones index set of $L(G)$ consists of all positive integers.

Biexact von Neumann algebras

Jesse Peterson

(joint work with Changying Ding)

We introduce the notion of biexactness for von Neumann algebras. This generalizes the notion for groups as introduced by Ozawa. I.e., a discrete group Γ is biexact if and only if $L\Gamma$ is biexact. As a consequence it follows that if G and H are semisimple real Lie groups with G having real rank 1 and H having real rank greater than 1, then for any lattices $\Gamma \subset G$ and $\Lambda \subset H$ we have $L\Gamma \not\hookrightarrow L\Lambda$. Other biexactness results will also be presented.

Contractibility of the automorphism group of a von Neumann algebra

Narutaka Ozawa

I first talked about the contractibility problem of the unitary group $\mathcal{U}(M)$ of a von Neumann algebra M (with separable predual) equipped with the ultraweak topology, particular emphasis on type II_1 factors, the last case still unsolved. (After the talk, during the workshop, David Jekel ([1]) found a proof that they are all contractible.) I then moved to the contractibility problem of the automorphism group $\mathrm{Aut}(M)$, equipped with the u-topology, which makes $\mathrm{Aut}(M)$ a Polish group. Thanks to the progress in the classification theory of type II_1 factors, we know a large variety of groups that appear as $\mathrm{Out}(M)$ (e.g., all countable discrete groups or more generally locally compact second countable groups were realized by Popa and Vaes). The study of the homotopy type of the automorphism group of a von Neumann algebra was initiated by Popa and Takesaki (1993), where the homotopy types of the hyperfinite factor (with separable predual) of type I, II_1, II_∞, and III_λ $(0 < \lambda < 1)$ were determined, leaving the cases of type III_0 and III_1 unsettled. I outlined my proof of contractibility of the approximate inner automorphism group $\overline{\mathrm{Inn}}(M)$ of a strongly stable von Neumann algebra M. Since the hyperfinite factors are strongly stable and $\mathrm{Aut}(R_{\mathrm{III}_1}) = \overline{\mathrm{Inn}}(R_{\mathrm{III}_1})$ for the hyperfinite factor R_{III_1} of type III_1, this settles the contractibility problem for $\mathrm{Aut}(R_{\mathrm{III}_1})$ and reduces the problem for the remaining type III_0 case to a problem on ergodic flows (which looks very complicated). The talk was based on [2].

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On a question of Blecher and Quantum Games

Marius Junge

In this joint work with Roy Araiza and Carlos Palazuelos we solve a problem raised by David Blecher [Ble92] approximately 30 years ago and re-emphasised by seminal work by Pisier and Shlyakhtenko [PS02] on the operator space version of Grothendieck's inequality. The tool for this analysis is the tensor theory of quantum games and an example of a so-called non-signaling strategy for a game introduced by Bavarian and Shor [BS14].

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Basic homotopy lemmas via abstract classification

Aaron Tikuisis
(joint work with José Carrión, Jamie Gabe, Chris Schafhauser, Stuart White)

Basic homotopy lemmas, as a first approximation, are results saying that an approximately central unitary in a C*-algebra can be connected to the unit by a continuous path of approximately central unitaries, provided K-theoretic obstructions vanish; they go back to work of Bratteli, Elliott, Evans, and Kishimoto from the '90s. Such lemmas are often very technical in order to deal with the approximation and spell out the obstruction; for simplicity, I presented variations which are more conceptual (and shorter), and also I mostly stuck to the real rank zero case where additional obstructions do not arise. With this in mind, here is a recasting of one of the original results of Bratteli, Elliott, Evans, and Kishimoto.

Theorem 1 (Bratteli, Elliott, Evans, Kishimoto [2, Theorem 8.1] and Kishimoto [5, Theorem 4.4])**.** *Let B be a real rank zero simple unital $A\mathbb{T}$ algebra and let u be a unitary in $C([0,1],B)_{\mathcal{U}} \cap B'$. Then $u(0)$ is homotopic to $u(1)$ in $B_{\mathcal{U}} \cap B'$.*

This result was generalized significantly by Matui [7, Lemma 3.9] and Lin [6, Theorem 8.1]. Our further generalization is as follows.

Theorem 2 (Carrión, Gabe, Schafhouser, Tikuisis, White [1])**.** *Let A be a separable unital exact C*-algebra satisfying the UCT, let B be a unital $\mathcal{Z}$-stable C*-algebra with strict comparison with respect to $T(B)$ and with real rank zero. Let $\phi\colon A \to B_{\mathcal{U}}$ be full, unital, and exact. If $u \in C([0,1],B)_{\mathcal{U}} \cap \phi(A)'$ and $u(0) = 1_{B_{\mathcal{U}}}$ then $u(1) = e^{ih}e^{ik}$ for self-adjoint elements $h,k \in B_{\mathcal{U}} \cap \phi(A)'$ of norm $\leq \pi$.*

There have been a number of applications of basic homotopy lemmas in the past. It allows a vanishing cohomology result (by a technique due to Kishimoto) which feeds into the Evans–Kishimoto intertwining technique [3]. It has also been used to classify certain homomorphisms up to asymptotic unitary equivalence (for example in [4]), as demanded to use Winter's classification-by-localization technique from [8].

A further application of the basic homotopy lemma is a K-theoretic computation for relative commutants as follows

Theorem 3 (Carrión, Gabe, Schafhouser, Tikuisis, White [1])**.** *Let A be a separable exact C*-algebra satisfying the UCT, let B be a unital $\mathcal{Z}$-stable C*-algebra with strict comparison with respect to $T(B)$ and with real rank zero. Let $\phi\colon A \to B_{\mathcal{U}}$ be full, unital, and exact. Then*

$$K_1(B_{\mathcal{U}} \cap \phi(A)') \cong \mathrm{Hom}_{\Lambda}(\underline{K}(SA), \underline{K}(B_{\mathcal{U}})).$$

This application illustrates the K-theoretic obstruction (which appears as an approximate obstruction in the actual basic homotopy lemma). Our proof relies on a classification of (full nuclear) *-homomorphisms, where we take the domain to be $C(\mathbb{T},A)$ (the universal C*-algebra generated by a copy of A and a commuting unitary), and the codomain to be $B_{\mathcal{U}}$. It is crucial that our classification of *-homomorphisms allows non-simple domains.

Beyond the real rank zero case, the path component of 1 can have infinite path-diameter, making a result as strong as Theorem 2 unrealistic. This can be rectified by including Hausdorffized unitary algebraic K_1-data in the invariant, and yielding a computation of the Hausdorffized unitary algebraic K_1 of the relative commutant, as a generalization of Theorem 3.

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Paper-folding models for the CAR algebra

Grigoris Kopsacheilis

(joint work with Wilhelm Winter)

The Connes–Feldman–Weiss theorem [4] asserts that any two Cartan subalgebras (in the sense of Vershik and Feldman–Moore [5]) in the hyperfinite II_1 factor $\mathcal{R}$ are *conjugate*, i.e. there is an automorphism carrying the one onto the other. It is natural to ask what is the analogous picture on the side of C*-algebras, i.e. what is the structure of Cartan subalgebras and C*-diagonals (this time in the sense of Renault and Kumjian [13, 7]) for uniformly hyperfinite (UHF) C*-algebras, and more generally approximately finite (AF) C*-algebras. Any AF algebra

$$A = \varinjlim (F_1 \xrightarrow{\varphi_1} F_2 \xrightarrow{\varphi_2} \dots)$$

(where the F_i are finite-dimensional C*-algebras) admits a natural C*-diagonal with totally disconnected spectrum, namely $D_A := \varinjlim D_{F_n}$, where $D_{F_n} \subset F_n$ are maximal abelian *-subalgebras (masas) for all $n \geq 1$ such that the connecting maps preserve normalizers (recursively choosing D_{F_n} to obtain such a sequence of masas is always possible; cf. [15]). It is well-known that any two Cartan subalgebras in A arising in this fashion are conjugate by an (approximately inner) automorphism [12, Theorem 5.7] – these are called *AF diagonals* or *standard diagonals*. The motivating question at this point becomes if all Cartan subalgebras of A are necessarily AF diagonals.

Obviously, for a Cartan subalgebra to be an AF diagonal, it is necessary that it has the same spectrum as that of the AF diagonals, and to be a diagonal (as opposed to being only Cartan). It is not at all obvious that there do exist Cartan subalgebras in AF algebras that fail each of these necessary conditions, but that is indeed the case:

- In [2], Blackadar showed that the CAR algebra (the UHF algebra of type 2^∞) admits a C*-diagonal with spectrum $S^1 \times \Omega$, where Ω is the Cantor space.
- In [10], Mitscher and Spielberg showed that the continued fraction AF algebras of Effros and Shen admit non-diagonal Cartan subalgebras with Cantor spectrum.

The question thus becomes: in an AF algebra, are all Cantor spectrum diagonals AF diagonals? We show that this is not the case, and thus the topological picture is more complicated than the measure-theoretic one.

Theorem 1. [6, Theorems A & B]. *The CAR algebra admits a Cantor spectrum* C^**-diagonal that is not (conjugate to) an AF diagonal. In fact, there are (at least) countably many pairwise non-conjugate Cantor spectrum* C^**-diagonals with Cantor spectrum in the CAR algebra.*

As in [2, 10], the main idea in our approach is to identify a dynamical object with the desired features, for which we can show that the associated C*-algebra lies in the domain of a classification theorem and has computable invariant, with the chance of being the same as that of the C*-algebra for which we wish to give a dynamical presentation (in our case the CAR algebra); see the discussion around Problem XLVII in [14], that asks for a criterion that determines when a Cartan subalgebra of an AF algebra is an AF-diagonal.

The dynamical system that is key in our construction involves the well-known *paper-folding sequence* described as follows: take a strip of paper and fold it infinitely many times, every time by making a fold to the right. Unfold it, and observe the resulting crest: whenever a right turn appears, mark 1, and whenever a left turn appears, mark 0. The resulting sequence is

$$110110011100100111011000110010\ldots$$

The acting group in the free minimal action that we construct is the product of the infinite dihedral group $\mathbb{Z} \rtimes \mathbb{Z}_2$ together with the locally finite group $\bigoplus_{\mathbb{N}} \mathbb{Z}$. After K-theoretic calculations (using Thomsen's work in [16] that builds on [3] and is using Natsume's exact sequence [11]), it follows by classification theory that the crossed product C*-algebra of our Cantor system $(\mathbb{Z} \rtimes \mathbb{Z}_2) \times \bigoplus_{\mathbb{N}} \mathbb{Z}_2 \curvearrowright X$ is the CAR algebra. To conclude that the Cantor spectrum diagonal $C(X)$ is not an AF diagonal, we employ a result of Archbold and Kumjian [1], that entails that any C*-algebra sitting between an AF diagonal and the ambient AF algebra must be

AF. This condition is not satisfied in our construction: there is a C*-algebra with non-trivial K_1-group between the constructed diagonal and the CAR algebra.

The countably many pairwise non-conjugate Cantor spectrum diagonals arise as tensor powers of the diagonal described above (using that the CAR algebra is strongly self-absorbing), and we distinguish these by the different values of their *diagonal dimension*, as defined by Li, Liao and Winter in [9].

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A Logical Classification Theorem for C*-Algebras

Jennifer Pi

(joint work with Michał Szachniewicz, Mira Tartarotti)

In the paper "Games on AF-Algebras", the authors demonstrate a logical analogue of Elliott's initial classification result for AF-algebras (see [2, Theorem A]). They additionally ask if a similar result holds for unital Kirchberg algebras satisfying the UCT. Our joint ongoing work answers not only this question, but also uses tools from descriptive set theory to obtain an analogous theorem for all classifiable C*-algebras.

We first establish a general notion of Ehrenfeucht-Fraïssé games, following the presentation of [1, § 3] and the partial isomorphism games of [2]. As a vague notion, the moves of the game between two structures A and B involve trying to extend partial isomorphisms between substructures of A and B one step at a time. Under this framework, we say that two structures A, B are equivalent up to rank α, denoted $A \equiv_\alpha B$, if any game with starting clock α has a winning strategy under which the plays of the game generates an isomorphism. (Details can be found in [2, Definition 2.1 and 2.4]).

Our main result is that there is a function $\theta : \omega_1 \to \omega_1$ such that, for any classifiable C*-algebras A and B, we have

$$\mathrm{KT}_u(A) \equiv_{\theta(\alpha)} \mathrm{KT}_u(B) \implies A \equiv_\alpha B,$$

where by "classifiable" we mean a unital, simple, separable, nuclear, $\mathcal{Z}$-stable C*-algebra satisfying the UCT, and $\mathrm{KT}_u(\cdot)$ denotes the invariant given in [3, Definition 2.3]. The crux of the argument, after some reductions using standard machinery from descriptive set theory, is to show that the class of separable C*-algebras satisfying the UCT is an analytic set. We do this by developing a general first-order model theory for functors on Borel categories (i.e. categories equipped with a standard Borel space structure on the objects and morphisms, and Borel composition map on the morphisms). We then leverage the well-known facts that a C*-algebra satisfies the UCT if and only if it is KK-equivalent to a commutative C*-algebra, along with the characterization of KK as a universal functor satisfying certain properties [6]. After establishing a way of interpreting the category of separable C*-algebras as a Borel category, following the presentation of [5], this allows us to say that there is a set of proofs (in our model theory for functors over the Borel category of C*-algebras) characterizing those algebras which have the UCT. By combining with the fact that the map taking a separable C*-algebra to its invariant is a Borel map (see [4, Theorem 3.3]), we are able to conclude our main result.

One notable thing about our approach is that we do not implement a "transfer of strategies" proof using standard intertwining arguments, as is done in [2]. While this gives a novel approach to the problem, it also means we are unable to control or compute what form the function θ takes. Additionally, our proof that the set of separable C*-algebras satisfying the UCT is analytic begs the question: is it Borel, or strictly analytic? Finally, our work develops an interesting general model-theoretic approach for functors over suitably nice categories of objects, and a natural direction of further study is other applications of this framework.

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On Chern classes of almost representations

Marius Dadarlat
(joint work with Forrest Glebe)

Let Γ be a discrete group and let (A_n) be a sequence of unital Banach algebras. A sequence of unital maps $\{\rho_n : \Gamma \to GL\,(A_n)\}_n$ is called an asymptotic homomorphism if

$$\lim_{n\to\infty} \|\rho_n(a)\rho_n(b) - \rho_n(ab)\| = 0 \text{ and } \sup_n ||\rho_n(a)|| < \infty, \quad \text{for all } \; a, b \in \Gamma.$$

We abelianize the kernel of the extension associated with an asymptotic homomorphism to tracial Banach algebras (A_n, τ_n) using the de la Harpe-Skandalis pre-determinant. Using the tracial property, we obtain in fact a central extension leading to a 2-cohomology class

$$[(\rho_n)] \in H^2(\Gamma, Q(\mathbb{C})) \cong \mathrm{Hom}(H_2(\Gamma, \mathbb{Z}), Q(\mathbb{C})),$$

where $Q(\mathbb{C}) = c_0(\mathbb{N}, \mathbb{C})/c_{00}(\mathbb{N}, \mathbb{C})$. If A_n are tracial C^*-algebras and $\rho_n : \Gamma \to U\,(A_n)$, then $[(\rho_n)] \in H^2(\Gamma, Q(\mathbb{R}))$. If $\|\rho_n(a) - \rho'_n(a)\| \to 0$ for all $a \in \Gamma$, then $[(\rho_n)] = [(\rho'_n)]$ and if all ρ_n are homomorphisms, then $[(\rho_n)] = 0$. Consequently, we can use nonvanishing of 2-cohomology to show non-stability by this method. For matricial algebras $A_n = M_{k_n}(\mathbb{C})$, the pairing between $[(\rho_n)]$ and each homology class $c \in H_2(\Gamma, \mathbb{Z})$ is an element $\langle \rho, c\rangle$ of $Q(\mathbb{C})$ whose components are eventually equal to winding numbers of Kazhdan and Exel-Loring type, divided by k_n. For general tracial C^*-algebras, these components belong to $(\tau_n)_*(K_0(A_n))$.

To illustrate our method, let $\sigma \in Z^2(\Gamma, \mathbb{R})$ be a 2-cocycle, set $\omega_n = e^{2\pi i\sigma/n} \in Z^2(\Gamma, \mathbb{T})$, and consider the canonical sequence of unital maps $\rho_n : \Gamma \to U(L(\Gamma, \omega_n))$ to the unitary groups of twisted group von Neumann algebras $L(\Gamma, \omega_n)$.

These maps factor through both the full and the reduced twisted C^*-algebras $C^*(\Gamma, \omega_n)$ and $C^*_r(\Gamma, \omega_n)$ and they constitute an asymptotic homomorphism. Indeed, since $\rho_n(s)\rho_n(t)\rho_n(st)^{-1} = e^{2\pi i\sigma(s,t)/n}1_n$ we have that

$$\lim_{n\to\infty} \|\rho_n(s)\rho_n(t) - \rho_n(st)\| = 0, \quad \forall\, s, t \in \Gamma.$$

Moreover, if the 2-cocycle $\sigma : \Gamma \times \Gamma \to \mathbb{R}$ is a bounded function, then

$$\lim_{n\to\infty} \sup_{s,t\in\Gamma} \|\rho_n(s)\rho_n(t) - \rho_n(st)\| = 0.$$

Theorem 1 ([1]). *Let Γ be a discrete countable group. Let σ be a normalized 2-cocycle with $[\sigma] \in H^2(\Gamma, \mathbb{R}) \setminus \{0\}$. For the canonical sequence of maps $\rho_n : \Gamma \to U(L(\Gamma, e^{2\pi i\sigma/n}))$, there exists no sequence of group homomorphisms $\pi_n : \Gamma \to GL(L(\Gamma, e^{2\pi i\sigma/n}))$ such that $\lim_{n\to\infty} \|\rho_n(s) - \pi_n(s)\| = 0$, for all $s \in \Gamma$.*

In particular, this shows that the full group C^*-algebra $C^*(\Gamma)$ of a discrete group Γ is not C^*-stable if $H^2(\Gamma, \mathbb{R}) \neq 0$ and in fact, Γ is not stable in operator norm with respect to tracial von Neumann algebras. The comparison map $H_b^2(\Gamma, \mathbb{R}) \to H^2(\Gamma, \mathbb{R})$ is known to be surjective for all hyperbolic groups, by work of Mineyev. Therefore, if Γ is a hyperbolic group with $H^2(\Gamma, \mathbb{R}) \neq 0$, then Γ is not uniform-to-local stable with respect to the class of unital tracial C^*-algebras and with respect to the class of separable von Neumann algebras endowed with a faithful trace.

Suppose now that we consider asymptotic homomorphisms $\{\rho_n : \Gamma \to U(k_n)\}_n$. One can obtained higher dimensional obstructions to group stability with respect to the operator norm by using vector bundles on compact subspaces of the classifying space $B\Gamma$. For simplicity, we will work with individual unital maps $\rho : \Gamma \to U(n)$ such that $\|\rho(st) - \rho(s)\rho(t)\| < \varepsilon$ for s, t in a "large" finite subset F of Γ for a "very small" ε. We say that ρ is an (F, ε)-almost representation.

Atiyah-Segal's map which associates to a finite dimensional representation of a discrete group Γ, a flat vector bundle on $B\Gamma$, generalizes to almost representations. For any compact subspace Y of $B\Gamma$, one can associate a rank n vector bundle $E\rho$ on Y, to each map $\rho : \Gamma \to U(n)$ which is sufficiently approximately multiplicative. We compute the first Chern class of E_ρ in terms of ρ.

Theorem 2 ([2]). *Let Γ be a discrete countable group. For every compact subspace of $B\Gamma$ there are F and ε such that for any (F, ε)-almost representation $\rho : \Gamma \to U(n)$. the formula $\omega(a, b) := \frac{1}{2\pi i}\mathrm{Tr}(\log(\rho(a)\rho(b)\rho(ab)^{-1}))$, defines a local 2-cocycle on Γ and*

$$c_1(E_\rho) = [\omega] \in H^2(Y, \mathbb{R}).$$

If moreover, ρ is both a projective representation and an (F, ε)-almost representation, then the Chern character of E_ρ is given by

$$\mathrm{ch}(E_\rho) = ne^{\frac{1}{n}c_1(E_\rho)} = n\, e^{\frac{1}{n}[\omega]}.$$

As an application, we find invariants that obstruct perturbation of almost representations to almost representations constructed algebraically from projective representations. These invariants are rationally complete for residually finite amenable groups, in a stable sense.

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On the (Local) Lifting Property

Tatiana Shulman
(joint work with Dominic Enders)

The celebrated Choi-Effros Theorem [6] states that if A is a nuclear C*-algebra, then any contractive completely positive map from A to a quotient C*-algebra admits a contractive completely positive lifting:

$$\begin{array}{ccc} & & B \\ & \nearrow & \downarrow \\ A & \longrightarrow & B/I \end{array}$$

In [10] Kirchberg introduced the following properties.

Definition. A C*-algebra A has the (Local) Lifting Property ((L)LP) if any c.c.p. map from A to a quotient C*-algebra lifts (locally) to a c.c.p. map.

In general ccp maps lift more often than $*$-homomorphisms. A C*-algebra A such that any $*$-homomorphism from A to a quotient C*-algebra lifts to a $*$-homomorphism is called *projective*, and projectivity is a very strong and therefore rare property. That for liftability of ccp maps it is sufficient to lift $*$-homomorphisms was observed already in the proof that $C^*(F_n)$ has the LP ([5, Th. 13.1.3]). This observation was stated explicitly in [8, Prop. 6.6] and [7, Cor. 2.9].

In [9] we prove that to be able to lift ccp maps, a certain lifting condition for $*$-homomorphisms is not only sufficient but also necessary.

Theorem ([9]). *Let A be separable. The following are equivalent:*

(i) A has the LP;

(ii) for any σ-unital C-algebra B, its ideal I and a $*$-homomorphism $f : A \to B/I$ there is a $*$-homomorphism $g : A \to M(B/I \otimes K)$ such that $f \oplus g$ lifts to a $*$-homomorphism $A \to M(B \otimes K)$;*

(iii) for any σ-untal C-algebra B, its ideal I and a $*$-homomorphism $f : A \to B/I$ there is a $*$-homomorphism $g : A \to M(B/I \otimes K)$ such that $f \oplus g$ lifts to a ccp map $A \to M(B \otimes K)$.*

The LLP also can be reformulated in terms of lifting $*$-homomorphisms ([9, Cor. 3.3]).

Despite of the fact that the properties LP and LLP are of central importance, there are not many examples of C*-algebras with the (L)LP outside the class of nuclear C*-algebras. In [10] Kirchberg proved that the (L)LP is preserved under certain operations, e.g. both LP and LLP are closed under tensoring with nuclear C*-algebras, and the LLP is closed under extensions. Boca proved that the LP is closed under free products [3] and Pisier proved the same for the LLP [14]. The latter result was generalized by Ozawa who showed that the LLP is closed under an amalgamated free product over a finite dimensional C*-subalgebra [12]. Ozawa states that it is not known whether the LP is preserved under an amalgamated free product over a finite dimensional C*-subalgebra ([12, p. 15]).

Our characterization of the LP above is a tool that allows to settle the aforementioned question about free products amalgamted over a finite-dimensional C*-subalgebra.

Theorem ([9]). *Let F be a finite-dimensional C*-subalgebra of A and B. If A and B have the LP, then $A *_F B$ has the LP.*

As a consequence we obtain that finite tree products with finite edge groups and in particular finitely generated virtually free groups have full group C*-algebras with the LP. Our technique for amalgamated free products applies also to some other lifting properties. We give a new proof of Blackadar's result stating that semiprojectivity passes to free products amalgamated over a finite-dimensional subalgebra. We give a new proof of Li and Shen's characterization of when unital free products amalgamated over a finite-dimensional subalgebra are RFD and obtain a characterization in the non-unital case.

Currently it is not many examples of C*-algebras with the LP besides nuclear C*-algebras and C*-algebras obtained from them by the constructions mentioned above. The following theorem provides examples of different nature. Recall that the soft torus $C(\mathbb{T}^2)_\epsilon$ is the universal C*-algebra generated by two unitaries commmuting up to ϵ:

$$C(\mathbb{T}^2)_\epsilon = C^* \langle u, v \mid u \text{ and } v \text{ are unitaries and } \|[u, v]\| \leq \epsilon \rangle.$$

Theorem ([9]). Soft tori have the LP.

The full group C*-algebra $C^*(F_2 \times F_2)$ plays an important role in the C*-algebra theory. The Connes Embeding Problem is equivalent to the question of whether or not $C^*(F_2 \times F_2)$ is residually finite-dimensional (RFD). The following is a long-standing open question.

Question. Does $C^*(F_2 \times F_2)$ have the (L)LP?

Using techniques developed in the proof of previous theorem we obtain the following result.

Theorem ([9]). $C^*(F_2 \times F_2)$ is inductive limit of RFD C*-algebras with the LP.

We further consider the semigroup *Ext* of extensions by compact operators. The semigroup *Ext* was introduced in the celebrated work of Brown-Douglas-Fillmore [4]. The first example of a C*-algebra A such that $Ext(A)$ is not a group was constructed by J. Anderson [1]. Since then more examples were found but still there is no clear understanding of when *Ext* is a group. It is known that the LLP implies that *Ext* is a group and it is not known whether these properties are equivalent.

Question. Does A have the LLP if and only if $Ext(A)$ is a group?

Here we prove that for an interesting class of C*-algebras, including $C^*(F_n \times F_n)$ and all contractible C*-algebras, they are equivalent.

Theorem. *Suppose A is an inductive limit, with surjective connecting maps, of separable C*-algebras that are RFD and have the LP (e.g. $A = C^*(F_n \times F_n)$ or A is any contractible C*-algebra). Then A has the LLP if and only if $Ext(A)$ is a group.*

Kirchberg proved that a separable C*-algebra A has the LLP if and only if $Ext(cone(A))$ is a group [10]. Since $cone(A)$ has the LLP if and only if A has, and cones are contractible, the theorem above generalizes Kirchberg's theorem.

As mentioned above it is not known whether $C^*(F_2 \times F_2)$ has the LLP, but now we can say that the problem is equivalent to the question of whether $Ext(C^*(F_2 \times F_2))$ is a group.

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Representation stability and classification of ∗-homomorphisms

Rufus Willett

For a discrete group Γ, an approximate representation of Γ is a map from Γ to a finite-dimensional linear group (or some other 'well-understood' group like a symmetric group) that approximately satisfies the relations needed to be a representation. The motivating question for this talk (and a well-studied problem in general) is whether a given approximate representation can be perturbed to an actual representation.

There are several different settings where this has been studied: in particular, one has to specify what sort of (approximate) representations one is interested in, and what metric to put on the target group to measure how good an approximation one has. For us, we will be interested in unitary representations, and the norm used will be the operator norm. We make everything precise as follows.

Definition 1. Let $\Gamma = \langle S \mid R \rangle$ be a finitely presented group. For a tuple $(u_s)_{s\in S}$ in $GL(n,\mathbb{C})$ indexed by the generating set, and for $r = s_1^{\epsilon_1} \cdots s_m^{\epsilon_m} \in R$ (with $s_i \in S$ and $\epsilon_i \in \{\pm 1\}$), write $u_r := u_{s_1}^{\epsilon_1} \cdots u_{s_m}^{\epsilon_m} \in GL(n,\mathbb{C})$.

For $\delta \geq 0$, a *δ-representation* of Γ is a tuple $\pi = (u_s)_{s\in S}$ in some $GL(n,\mathbb{C})$ such that $\|1 - u_r\| \leq \delta$ for all $r \in R$ and $\|u_s^* u_s - 1\| \leq \delta$ for all $s \in S$.

Note that for $\delta = 0$, giving a δ-representation is the same as giving an actual representation, i.e. a homomorphism $\Gamma \to U(n,\mathbb{C})$. We will just say 'representation' for a tuple $(u_s)_{s\in S}$ as above satisfying $u_r = 1$ for all $r \in R$ and $u_s^* u_s = 1$ for all $s \in S$.

Definition 2. Let $\Gamma = \langle S \mid R \rangle$ be a finitely presented group, and let $\delta, \epsilon \geq 0$. A δ-representation $(u_s)_{s\in S}$ with values in $GL(n,\mathbb{C})$ *can be ϵ-perturbed to a representation* if there is a representation $(w_s)_{s\in S}$ with values in $GL(n,\mathbb{C})$ such that $\|u_s - w_s\| \leq \epsilon$ for all $s \in S$.

More generally, $(u_s)_{s\in S}$ can be *stably ϵ-perturbed to a representation* if there are representations (v_s) with values in $GL(m,\mathbb{C})$ for some $m \geq 0$ and (w_s) with values in $GL(n+m,\mathbb{C})$ such that $\|(u_s \oplus v_s) - w_s\| < \epsilon$ for all $s \in S$.

Our question is then whether for any ϵ, there is a δ such that any δ-representation is (stably) ϵ-close to an actual representation. The answer is no in general, due to the existence of K-theoretic obstructions. This is conceptually easiest to explain when Γ satisfies the following assumptions, which will be in force for the rest of this note:

(1) Γ admits a finite CW complex model for its classifying space $B\Gamma$;
(2) if $C^*\Gamma$ is the maximal group C^*-algebra, then the Baum-Connes-Kasparov assembly map $\mu : K_*(B\Gamma) \to K_*(C^*\Gamma)$ is an isomorphism, and $C^*(\Gamma)$ satisfies the universal coefficient theorem.

The second condition above is satisfied for all a-T-menable groups, and in particular all amenable, groups due to work of Higson-Kasparov [10].

Now, under these assumptions, for any suitably small δ, a δ representation π induces a map $\pi_* : K_0(C^*\Gamma) \to \mathbb{Z}$. We define also

$$\widetilde{K}_0(C^*\Gamma) := \mathrm{Kernel}(t_* : K_0(C^*\Gamma) \to \mathbb{Z}).$$

where $t : C^*\Gamma \to \mathbb{C}$ is the trivial representation.

The following theorem, which combines work of many authors stretching from 1982 to 1999, is the jumping-off point for our work.

Theorem 1 (Kazhdan [11], Voiculescu [14], Loring [13], Gong-Lin [8], Eilers-Loring Pedersen [6, 7])**.** *Let Γ be either $\mathbb{Z}\oplus\mathbb{Z}$, or $\mathbb{Z}\rtimes\mathbb{Z}$ (the semidirect product for the* *unique non-trivial action of $\mathbb{Z}$ on itself), i.e. the fundamental groups of the torus and Klein bottle respectively. Then for any $\epsilon > 0$, there exists $\delta > 0$ such that for a δ-representation π, the following are equivalent:*

(1) π can be ϵ-perturbed to a representation;
(2) π can be stably ϵ-perturbed to a representation;
(3) $\pi_*(\widetilde{K}_0(C^*\Gamma)) = 0$.

Note that for $\Gamma = \mathbb{Z} \rtimes \mathbb{Z}$, $\widetilde{K}_0(C^*\Gamma) = 0$, so the third condition above is vacuous for this group. On the other hand, for $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$, $\widetilde{K}_0(C^*\Gamma) \cong \mathbb{Z}$, and combining the work of Kazhdan, Voiculescu, and Loring cited above shows that for any $\delta > 0$, there is a δ-representation π such that $\pi_*(\widetilde{K}_0(C^*\Gamma)) \neq 0$, so the third condition is not vacuous in this case.

Our main motivation was to generalize the above result to the fundamental group of any closed surface: one would hope for precise analogues of the results for $\mathbb{Z} \oplus \mathbb{Z}$ (in the orientable case, when $\widetilde{K}_0(C^*\Gamma) \cong \mathbb{Z}$) and for $\mathbb{Z} \rtimes \mathbb{Z}$ (in the non-orientable case, when $\widetilde{K}_0(C^*\Gamma) = 0$). However, there are serious technical difficulties arising from non-amenability of such groups. Our main theorem is a generalization of the equivalence of the second and third points above to a much wider class of groups.

Theorem 2. *In addition to our standing assumptions on Γ, assume the following:*
(a) $C^*\Gamma$ satisfies Kirchberg's LLP;
(b) $C^*\Gamma$ is RFD;
(c) the torsion subgroup of the K-homology group $K^0(C^*\Gamma)$ is generated by formal differences $\sigma_* - \rho_*$ of the classes associated to finite-dimensional representations.

Then for any $\epsilon > 0$, there exists $\delta > 0$ such that for a δ-representation π, the following are equivalent:

(1) π can be stably ϵ-perturbed to a representation;
(2) $\pi_*(\widetilde{K}_0(C^*\Gamma)) = 0$.

Let us comment briefly on these assumptions. The LLP assumption is a consequence of amenability by the Choi-Effros lifting theorem, and is known to hold more generally: for example, it holds for free groups and free-by-cyclic groups. The RFD assumption follows if the group is amenable and residually finite, and again is known somewhat more generally including for free-by-cyclic groups and surface groups. The assumption on torsion classes is the most mysterious, but is at least satisfied if $K^0(C^*\Gamma)$ is torsion free (or equivalently, if $K_1(C^*\Gamma)$ is torsion free), which is often the case; one can also (we say more about this below) establish it if $B\Gamma$ has dimension at most three, which covers the case of surface groups. Unfortunately, the theorem does not cover the case of general surface groups, as the LLP is not yet known for such groups (the consensus of experts seems to be that it is likely to hold, however).

Let us say with a few words on the strategy of the proof of the main theorem. The key steps are as follows.

(1) Use the LLP assumption to show that a δ-representation π can be approximated by the restriction of an almost multiplicative ucp map $C^*\Gamma \to M_n(\mathbb{C})$. It thus suffices to consider such maps.
(2) Following Lin [12] and Dadarlat-Eilers [4], establish a 'stable uniqueness' theorem for almost multiplicative ucp maps $C^*\Gamma \to M_n(\mathbb{C})$, which says roughly that two such maps are approximately unitally equivalent (up to adding an auxiliary almost-multipliciative ucp map) if and only if they define the same class in K-homology. They main difficulty here is that $C^*\Gamma$ is not exact, so one cannot use the same arguments as Lin and Dadarlat-Eilers. Instead, we base our arguments on the controlled K-homology groups developed by Yu and the speaker [15].
(3) Use the RFD assumption to show that the auxiliary almost multiplicative ucp map one has to add in the previous stage can be taken to be a representation.
(4) Use the UMCT of Dadarlat-Loring [5] to show there is a short exact sequence

$$0 \to \mathrm{Hom}(\mathrm{Tor}(K_1(C^*\Gamma)), \mathbb{Q}/\mathbb{Z}) \to K^0(C^*\Gamma) \to \mathrm{Hom}(K_0(C^*\Gamma), \mathbb{Z}) \to 0.$$

The assumption that $\pi_*(\widetilde{K}_0(C^*\Gamma)) = 0$ for an approximate representation π implies that the image of π_* in $\mathrm{Hom}(K_0(C^*\Gamma), \mathbb{Z})$ agrees with the class ρ_* of an actual representation. Hence the difference $\pi_* - \rho_*$ lies in the subgroup $\mathrm{Hom}(\mathrm{Tor}(K_1(C^*\Gamma)), \mathbb{Q}/\mathbb{Z})$ of $K^0(C^*\Gamma)$. This subgroup is generated by formal differences of representations by the assumption (c) on the torsion subgroup of $K^0(C^*\Gamma)$, which completes the proof.

Finally, let us conclude with a few words about establishing the validity of the relatively mysterious condition (c) above. Conceivably, this could hold for all groups satisfying the other assumptions, but this seems quite far out of reach. We know three different methods to show that it holds for groups with $B\Gamma$ of dimension at most three (and somewhat more generally):

(1) using the ζ map of Gong-Lin-Niu [9] and Carrión-Gabe-Schafhauser-Tikuisis-White [3] and Hausdorffized unitary algebraic K_1;
(2) using the relative eta invariants and index theory for flat bundles of Atiyah-Patodi-Singer [1];
(3) using the theory of flat bundles and the Atiyah-Segal completion theorem [2].

The computations needed suggest new relations between the ingredients in (1), (2), and (3) above, which seem interesting in of themselves: there seems likely to be much more to say here.

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Rigidity for graph product von Neumann algebras

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(joint work with Adrian Ioana)

Over the past two decades, Popa's deformation/rigidity theory has led to major advances in the classification of von Neumann algebras, including striking rigidity results, often accompanied by the computations of the symmetries of the von Neumann algebras considered. The goal of the talk was to extend the scope of these rigidity phenomena to the context of graph products of tracial von Neumann algebras.

Graph products were originally defined by Green [Gr90] in the context of groups, and later extended to von Neumann algebras by Caspers–Fima [CF17].

Let Γ be a finite simple graph, i.e. Γ has no loop-edge and no multiple edges between vertices, and let $(G_v)_{v\in V\Gamma}$ be a family of groups, one per vertex of Γ. The *graph product* $*_{v,\Gamma}G_v$ is the group obtained from the free product of the groups G_v by adding as only additional relations that G_v and G_w commute whenever v, w are adjacent. Graph products of groups encompass direct products (when Γ is a complete graph), free products (when Γ is edgeless) and right-angled Artin groups (when all vertex groups are isomorphic to $\mathbb{Z}$). A lot of work has revolved around the computation of the automorphism group of a graph product, e.g. [GM19, Ge24]. In particular, if A_Γ is a right-angled Artin group associated to a finite simple graph Γ, a theorem of Laurence [La95] (confirming a conjecture of Servatius [Se89]) asserts that $\mathrm{Out}(A_\Gamma)$ is finite if and only if

(1) Γ is *transvection-free*, i.e. there are no distinct vertices $v, w \in \Gamma$ such that $B(v,1) \setminus \{v\} \subseteq B(w,1)$ (otherwise $v \mapsto vw$ gives an infinite-order outer automorphism);
(2) Γ has *no partial conjugation*, i.e. for every vertex $v \in \Gamma$, the ball $B(v,1)$ does not disconnect Γ (otherwise, conjugating one complementary component by v gives an infinite-order outer automorphism).

Likewise, given a family of tracial von Neumann algebras (M_v, τ_v), their graph product $M_\Gamma := *_{v,\Gamma}(M_v, \tau_v)$ is a tracial von Neumann algebra that contains all M_v as subalgebras, in which M_v, M_w generate their tensor product if v, w are adjacent, and their free product otherwise.

Our main theorem is as follows.

Theorem 1 (Horbez–Ioana [HI25]). *Let Γ, Λ be two finite simple graphs which are transvection-free, do not contain a square, and are not reduced to a vertex. Let $(M_v, \tau_v)_{v\in\Gamma}$ and $(N_w, \tau_w)_{w\in\Lambda}$ be families of diffuse tracial von Neumann algebras.*

If $\theta : M_\Gamma \to N_\Lambda$ is any $*$-isomorphism, then the graphs Γ and Λ are isomorphic and there exists a graph isomorphism $\alpha : \Gamma \to \Lambda$ such that $\theta(M_v) \prec^s_{N_\Lambda} N_{\alpha(v)}$ and $N_{\alpha(v)} \prec^s_{N_\Lambda} \theta(M_v)$, for every $v \in \Gamma$.

Here $\prec^s$ denotes strong intertwining in the sense of Popa. In particular, a consequence of the conclusion is that $\theta(M_v)$ has a corner that embeds as a finite-index subalgebra in a corner of $N_{\alpha(v)}$, and vice versa.

Previous works giving rigidity theorems for graph product von Neumann algebras include [CDD25, BCC24, DV25], but with stronger assumptions on the vertex algebras. In novel fashion, our main theorem covers non-factorial vertex algebras.

In different regimes for the vertex algebras, we obtain variations over our main theorem: the same conclusion holds if

(1) Γ, Λ are only assumed to be transvection-free (and not reduced to one vertex), but the algebras M_v, N_w are assumed to be diffuse and amenable;
(2) Γ, Λ are only assumed to be transvection-free and strongly reduced (and not reduced to one vertex), but the algebras M_v, N_w are assumed to be II_1 factors – *strongly reduced* means that no proper subgraph of Γ, Λ can be collapsed to a vertex to give a new graph product structure for M_Γ, N_Λ.

As a consequence of the first point, we get the following W^*-classification theorem for right-angled Artin groups.

Corollary 1 (Horbez–Ioana [HI25]). *Let Γ, Λ be transvection-free finite simple graphs. Then $L(A_\Gamma) \simeq L(A_\Lambda)$ if and only if $A_\Gamma \simeq A_\Lambda$ (if and only if $\Gamma \simeq \Lambda$).*

When the vertex algebras are II_1 factors, with more assumptions on the graphs, we reach a stronger conclusion and manage to compute the fundamental group and outer automorphism group in some cases.

Theorem 2 (Horbez–Ioana [HI25]). *Let Γ, Λ be finite simple graphs of girth at least 5, with no vertex of valence 0 or 1. Let $(M_v, \tau_v)_{v\in\Gamma}$ and $(N_w, \tau_w)_{w\in\Gamma}$ be families of II$_1$ factors. Let $M_\Gamma = *_{v,\Gamma}(M_v, \tau_v)$ and $N_\Lambda = *_{w,\Lambda}(N_w, \tau_w)$. Then*

(1) If $\theta : M_\Gamma \to N_\Lambda$ is an isomorphism, then there exist a graph isomorphism $\alpha : \Gamma \to \Lambda$ and unitaries $u_v \in N_\Lambda$, such that $\theta(M_v) = u_v N_{\alpha(v)} u_v^*$ for every $v \in \Gamma$.

(2) The fundamental group of M_Γ is trivial.

(3) If in addition Γ has no separating star, then

$$\mathrm{Out}(M_\Gamma) \simeq (\oplus_{v\in\Gamma}\mathrm{Aut}(M_v)) \rtimes \mathrm{Aut}(\Gamma; M_\Gamma),$$

where $\mathrm{Aut}(\Gamma; M_\Gamma) \subseteq \mathrm{Aut}(\Gamma)$ is the subgroup preserving the isomorphism types of the vertex algebras.

The first class of II$_1$ factors with trivial fundamental group was given by Popa in his breakthrough work [Po06a]. We obtain new examples coming from graph products. In the special case where the vertex algebras are group von Neumann algebras, we reach the following corollary.

Corollary 2 (Horbez–Ioana [HI25]). *Let Γ, Λ be two finite simple graphs of girth at least 5, which contain no vertices of valence 0 or 1. Let $(G_v)_{v\in\Gamma}$ and $(H_w)_{w\in\Lambda}$ be families of ICC groups. Let $G_\Gamma = *_{v,\Gamma}G_v$ and $H_\Lambda = *_{w,\Lambda}H_w$ be the associated graph product groups. Then the following conditions are equivalent:*

(1) $L(G_\Gamma) \simeq L(H_\Lambda)$.

(2) $L(G_\Gamma)$ and $L(H_\Lambda)$ are stably isomorphic.

(3) There exists a graph isomorphism $\alpha : \Gamma \to \Lambda$ such that G_v is W-equivalent to $H_{\alpha(v)}$, for every $v \in \Gamma$.*

This yields a von Neumann algebraic analogue to a similar classification theorem in measured group theory, obtained with Huang for right-angled Artin groups [HH22], and with Escalier for general graph products [EH24]. Recall that two countable groups are *orbit equivalent* if they have essentially free, measure-preserving actions on a standard probability space with (essentially) the same orbits.

Theorem 3 (Escalier–Horbez [EH24]). *Let Γ, Λ be two finite simple graphs, not reduced to one vertex, which are transvection-free and have no partial conjugations. Let $(G_v)_{v\in\Gamma}$ and $(H_w)_{w\in\Lambda}$ be families of infinite groups. Let $G_\Gamma = *_{v,\Gamma}G_v$ and $H_\Lambda = *_{w,\Lambda}H_w$ be the associated graph product groups.*

Then G_Γ and H_Λ are orbit equivalent if and only if there exists a graph isomorphism $\alpha : \Gamma \to \Lambda$ such that G_v is orbit equivalent to $H_{\alpha(v)}$, for every $v \in \Gamma$.

This is also equivalent to G_Γ and H_Λ being measure equivalent. In fact this theorem, when combined with uniqueness of Cartan subalgebras obtained in the realm of deformation/rigidity theory (see specifically [CKE24] for graph products) also yields structural information about cross-product von Neumann algebras associated to free, ergodic, probability measure-preserving actions of graph products of groups as in the statement, like the triviality of their fundamental group.

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Strong convergence to operator-valued semicirculars

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(joint work with Yoonkyeong Lee, Brent Nelson, Jennifer Pi)

In [3], we establish a framework for weak and strong convergence of Gaussian matrix models to the operator-valued semicircular systems introduced by Speicher [5] and Shlyakhtenko [7]. Strong convergence (or non-commutative laws) refers to convergence of the operator norms of polynomials in random matrices (or other operators) to the norms of the corresponding polynomial in the limiting operator tuple X. While the weak convergence of non-commutative laws produces an embedding of $\mathrm{W}^*(X)$ into the W^*-ultraproduct of matrices, strong convergence enables an embedding into the C^*-ultraproduct. Our results in particular give examples of natural generators for interpolated free group factors that admit strongly convergence random matrix models, and hence generate an MF C^*-algebra.

Strong convergence was first studied by the seminal work of Haagerup and Thorbjørnsen in 2005, and it has been a topic of much recent research. Notably, Hayes showed that strong convergence of certain tensor product matrix models could be used to solve the Peterson–Thom conjecture on the free group von Neumann algebras; the tensor strong convergence conjecture was subsequently addressed by Belinschi and Capitaine, Bordenave and Collins, Magee and de la Salle, Parraud, and Chen–Garza Vargas–van Handel. The depth of this result is illustrated by the diversity of different techniques used in the proof, which offer different kinds of robustness. For instance, the "polynomial method" of Chen, Garza Vargas, and van Handel allows the largest size of coefficient matrices to be used with the GUE matrices, but heavily exploits that the expected trace of a polynomial is an analytic function of $1/n$. Meanwhile, Parraud's approach can be extended to the setting of free Gibbs laws with sufficiently smooth potentials, while Magee and de la Salle investigate strongly convergent approximations of larger classes of groups. The idea of strong convergence has also motivated some of the recent work on selfless C*-algebras used to obtain strict comparison in other results discussed in this conference. We use results and techniques of Bandeira, Boedihardjo, and van Handel which gives sharp estimates comparing the spectrum of a general Gaussian matrix with the spectrum of a corresponding semicircular; although this work did not prove the Peterson–Thom conjecture, it is well-suited to the study of general Gaussian matrices.

Our work focuses on extending strong convergence to general Gaussian random matrix models (the entrywise covariances can be essentially arbitrary) where the limiting models are given by operator-valued semicirculars. These are an extension of free semicircular families that serve as central limit distributions for free independence with amalgamation over a subalgebra B. B-valued semicircular families $(X_i)_{i\in I}$ (with mean zero) are specified in terms of the maps $\eta_{i,j}(b) = E_B[X_i b X_j]$ for $b \in B$, which together form an *operator-valued covariance matrix* $(\eta_{i,j})_{i,j\in I}$. In a parallel way, a mean-zero jointly Gaussian family of $n \times n$ self-adjoint random matrices $(X_i^{(n)})_{i\in I}$ can be specified in terms of an $\mathbb{M}_n$-valued covariance matrix $(\eta_{i,j}^{(n)})_{i,j\in I}$ where $\eta_{i,j}^{(n)}(b) = \mathbb{E}[X_i^{(n)} b X_j^{(n)}]$ where $\mathbb{E}$ is the classical expectation applied to a matrix-valued random variable. The matrix giving the covariances of the individual entries $(X_i^{(n)})_{i\in F}$ for a finite index subset F, turns out to be nothing but the Choi matrix of the completely positive map $(\eta_{i,j}^{(n)})_{i,j\in F}$. To adapt the notions of weak and strong convergence to the B-valued setting, we replace the ordinary non-commutative polynomials with "covariance polynomials," which are expressions depending on operators $(b_\omega)_{\omega\in\Omega}$ as well as a covariance matrix $\eta = (\eta_{i,j})$, which involve algebraic operations on the variables together with applications of the maps $\eta_{i,j}$. An example would be

$$f(\eta, b) = \eta_{1,2}[b_1 \eta_{3,1}(b_2) b_3] b_2 + b_3 \eta_{2,3}(b_2).$$

Our result in [3] thus considers the following setup:

- A tracial von Neumann algebra B and generators $(b_\omega)_{\omega\in\Omega}$.
- A B-valued covariance $\eta = (\eta_{i,j})_{i,j\in I}$.
- A (B,η)-semicircular family $(X_i)_{i\in I}$.
- Matrix approximations $(b_\omega^{(n)})_{\omega\in\Omega}$.
- An $\mathbb{M}_n$-valued covariance $\eta^{(n)} = (\eta_{i,j}^{(n)})$.
- An $\eta^{(n)}$-Gaussian matrix ensemble $X^{(n)} = (X_i^{(n)})$.

Assuming that the operator norm of the Choi matrix for $\eta^{(n)}$ (associated to each finite subset of the indices) vanishes as $n \to \infty$, and that that $(\eta^{(n)}, b^{(n)})$ converge weakly to (η, b), we obtain weak convergence of $(\eta^{(n)}, b^{(n)}, X^{(n)})$ to (η, b, X) using ideas from [1].

For strong convergence, we assume that the norm of the Choi matrix vanishes *faster* than $1/(\log n)^3$ to apply further results from [1], which then allows us to compare the norms of ordinary polynomials in the Gaussians to corresponding operator-valued semicirculars–specifically the $(\mathbb{M}_n, \eta^{(n)})$ semicircular families that depend on n. We next need to reduce the study of the operator norm for covariance polynomials to the operator norm of ordinary polynomials. To accomplish this, we use an averaging trick based on the free law of large numbers where we simulate the application of $\eta_{i,j}(b)$ by an average of $X_{i,k} b X_{j,k}$ where $(X_{i,k})$ are free copies of (X_i) with amalgamation over B. We thus need to assume in our hypotheses that infinitely many free copies of the $(\mathbb{M}_n, \eta^{(n)})$-semicircular families strongly converge to the $(B, \eta^{(n)})$-semicircular family. We do not know convenient sufficient conditions for this in general, but we remark that if the base algebra is fixed, then the free exactness theorem of Skoufranis and Pisier can be applied.

Finally, we apply these results to the case of Gaussian matrices with independent entries, but with variances weighed using a discretization of some continuous function. This in particular includes smoothed out versions of Gaussian block matrices [2] and Gaussian band matrices [6]. More specifically, it includes matrix approximations for Rădulescu's model of interpolated free group factors [4].

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Actions of free groups and stable rank one

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(joint work with Jamie Bell, Shirly Geffen)

The notion of stable rank was introduced by Rieffel in the 1980s in the context of nonstable K-theory [11]. It has also appeared more recently in connection with phenomena around classification theory such as rank realization, $\mathcal{Z}$-stability, strict comparison, and divisibility [12, 13, 8]. For simple separable unital finite C*-algebras, the property of stable rank one (which boils down in the unital case to the density of invertible elements) is strictly weaker than $\mathcal{Z}$-stability (the operative regularity hypothesis in classification theory), even in the nuclear setting, although it can fail there as well [12, 14, 1, 7]. Stable rank one is thus not in complete alignment with the regularity properties associated to classification and can hold under more general structural conditions, and for this reason it can serve as a touchstone for understanding and distinguishing various general classes of C*-algebras. Most interesting in this regard are those C*-algebras that arise as crossed products of topologically free minimal actions of countably infinite groups G on compact metrizable spaces X for which the space $M_G(X)$ of invariant Borel probability measures is nonempty. It is not known whether stable rank one ever fails in this case, and it always holds when $G = \mathbb{Z}^d$ [7], despite the fact that $\mathcal{Z}$-stability sometimes fails for such actions [5]. Beyond $\mathbb{Z}^d$, however, we do not know much about when the properties of stable rank one and $\mathcal{Z}$-stability diverge, although $\mathcal{Z}$-stability occurs quite frequently for amenable G, for example when G is elementary amenable and X is finite-dimensional [6, 9].

In the present work we venture into the world of nonnuclear tracial reduced crossed products and show that stable rank one is generic within two natural spaces of minimal actions of the free group F_d on the Cantor set X with $M_{F_d}(X) \neq \emptyset$. Our approach is inspired by Li and Niu's stable rank one theorem in the amenable setting (which assumes the uniform Rokhlin property and a weak form of dynamical comparison) [7] and is driven by the discovery, for a generic action, that at some scale the effects of amenability of one generator of F_d (specifically, dynamical tilings with approximate invariance properties) overpower the relation of freeness with the other generators. The two spaces in question, each equipped with the natural Polish topology, are (i) the set of all weakly mixing topologically free minimal actions $F_d \curvearrowright X$ with $M_{F_d}(X) \neq \emptyset$, and (ii) the set of all topologically free actions $F_d \curvearrowright X$ with $M_{F_d}(X) \neq \emptyset$ that are spectrally aperiodic and minimal on each standard generator. If we consider all topologically free minimal actions $F_d \curvearrowright X$ with $M_{F_d}(X) \neq \emptyset$ then there is in fact a generic action, the universal odometer [4], and so we have imposed conditions that preclude periodic behaviour. Nevertheless, in case (i) we have not been able to rule out the existence of a generic action. One the other hand, using a theorem of Ormes on ergodic realization up to orbit equivalence in minimal Cantor systems [10] along with Blanchard's local entropy technique for establishing disjointness [3], we have been able to verify that conjugacy classes are meagre in the second space.

Our methods also show that if $G \curvearrowright X$ and $H \curvearrowright Y$ are topologically free minimal actions of countable groups on compact metrizable spaces with G infinite, and the first action has the uniform Rokhlin property and dynamical comparison, then the reduced crossed product of the product action has stable rank one. In fact the genericity results in the previous paragraph hinge in a similar kind of product construction, with the key difference that this must be performed internally as a diagonal product using a common acting group, which leads to a lengthy analysis involving tools from ergodic theory like measure disjointness and the Jewett–Krieger theorem. Taking a diagonal product acts like a type of stabilization that allows one to avoid having to use dynamical comparison, which we do not know to hold for any of the actions in question. It is in fact still an unresolved problem whether all free minimal actions (even those on the Cantor set) have dynamical comparison. We also do know whether any of the crossed products at play have strict comparison, although for the reduced C*-algebra of F_d itself this was shown recently in [2] using very different techniques, notably the rapid decay property.

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Ultrapowers of reduced free group C*-algebras

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(joint work with Srivatsav Kunnawalkam Elayavalli)

Murray and von Neumann's free group factor remains problem one of the oldest open questions in operator algebras: if F_n denotes the free group on $n \geq 2$ generators, does the free group factor $L(F_n)$ depend on n? A natural variation of this question, which appears to be just as difficult, asks which free group factors are elementarily equivalent, or equivalently, have isomorphic (tracial) ultrapowers. Modifying these questions in another way, one can ask about the isomorphism and elementary equivalence classes of the reduced free group C*-algebras $C^*_r(F_n)$.

A fundamental result of Pimsner and Voiculescu ([8]) provides a solution to the isomorphism problem: the reduced free group C*-algebras $C^*_r(F_n)$ are mutually non-isomorphic. In fact, as a by-product of their solution to one of Kadison's Baton Rouge problems, showing $C^*_r(F_n)$ has no non-trivial projections, they compute $K_1(C^*_r(F_n)) \cong \mathbb{Z}^n$. This talk focused on the elementary equivalence problem for these C*-algebras, recently solved in [4].

It is easy to see that free abelian groups of different (countable) rank are not elementarily equivalent. For example, using ω to denote a free ultrafilter on $\mathbb{N}$ and $\mathbb{Z}^n_\omega$ to denote the discrete group ultrapower of $\mathbb{Z}^n$, $[\mathbb{Z}^n_\omega : 2\mathbb{Z}^n_\omega] = [\mathbb{Z}^n : 2\mathbb{Z}^n] = 2^n$. So the naive approach to the elementary equivalence problem for reduced free group C*-algebras is to show that for a C*-algebra A, the elementary equivalence class of $K_1(A)$ (as a discrete group) depends only on the elementary equivalence class of A (as a C*-algebra). This, however, is false, with the first counterexamples due to Phillips in [7]. The main technical result in [4] is that the statement is true if one restricts to unital simple C*-algebras with unique trace and strict comparison, including $C^*_r(F_n)$, where strict comparison was recently shown in [1] (see also [6, 10]).

In more detail, for each C*-algebra A, using A_ω to denote the operator norm ultrapower of A, there is a natural group homomorphism

$$\eta_A \colon K_1(A_\omega) \to K_1(A)_\omega, \tag{1}$$

given on representatives by sending a unitary $u \in M_\infty(A_\omega)$ to a representing sequence of unitaries $(u_n)_{n=1}^\infty \subseteq M_\infty(A)$. The goal becomes to show η_A is an isomorphism when $A = C^*_r(F_n)$, $n \geq 2$, or more generally, η_A is an isomorphism whenever A is a unital simple C*-algebra with unique trace and strict comparison ([4, Corollary 4.2]). This readily implies the C*-algebras $C^*_r(F_n)$ have non-isomorphic ultrapowers, as outlined in the previous paragraph.

In general, the map η_A in (1) is neither injective nor surjective. The issue with surjectivity is easier to see. Elements in $K_1(A)$ are represented by unitaries in matrix algebras $M_d(A)$, and in general, one needs arbitrarily large integers d to realize all K_1 elements. When A has stable rank one, Rieffel ([9]) has shown one can in fact realize all K_1-elements with $d = 1$, making η_A surjective. This covers the case of reduced free group C*-algebras by a result of Dykema, Haagerup, and Rørdam ([3]), and the more general class of A discussed above have stable rank one by a recent result of Lin ([5]). For injectivity, there is a related problem. Two unitaries $u, v \in A$, agree in $K_1(A)$ if and only if $u \oplus 1_A^{\oplus(d-1)}$ and $v \oplus 1_A^{\oplus(d-1)}$ are in the same path component of the unitary group of $M_d(A)$, but again, there is generally no upper bound on d. Rieffel also solved this problem in the stable rank one case showing one can take $d = 1$. This shows that when A has stable rank one, $K_1(A)$ is precisely the group of path components of the unitary group of A That is, $K_1(A) = U(A)/U^0(A)$, where $U(A)$ is the unitary group of A and $U^0(A)$ is the path component of $U(A)$ containing 1_A.

There is another more subtle issue regarding the injectivity of η_A. In fact, even in the stable rank one case, the map η_A need not be surjective. It is an elementary fact that if $u \in U_0(A)$, there are self-adjoint $h_1, \ldots, h_n \in A$ with $\|h_i\| \leq \pi$ such that

$$u = e^{ih_1} \cdots e^{ih_n}. \tag{2}$$

Further, up to increasing n, one may assume $\|h_i\| \leq \pi$ for all $i = 1, \ldots, n$ (replace e^{ih} with $e^{ih/2}e^{ih/2}$ as needed). However, in general, n cannot be chosen uniformly over $u \in U^0(A)$, which prevents η_A from being surjective. In fact, the de la Harpe–Skandalis determinant provides lower bounds on n for a given $u \in U^0(A)$: if $u = e^{ik_1} \cdots e^{ik_m}$ for self-adjoint $k_i \in A$ (possibly or large norm), then for any other such decomposition $u = e^{ih_1} \cdots e^{ih_n}$,

$$\sum_{i=1}^{n} \|h_i\| \geq \inf_{x \in K_0(A)} \sup_{\tau \in T(A)} \Big| \sum_{i=1}^{m} \tau(k_i) - 2\pi\tau(x) \Big|, \tag{3}$$

where $T(A)$ denotes the set of tracial states on A.

When A is a unital C*-algebra with unique trace, the right hand side of (3) is at most π, so there are no determinant obstructions. Combining the vanishing of the determinant obstruction with trace-kernel techniques (see [2], for example) produces the following result. Roughly, the strict comparison hypothesis gives suitable regularity results on the trace-kernel ideal $J_A \trianglelefteq A_\omega$, and the determinant obstructions appear in the group $K_1(J_A)$. The theorem does not quite bound n in (2), but this slightly weaker result is still enough for K-theoretic computations.

Theorem 1 ([4, Theorem 1.2]). *If A is a unital simple C^*-algebra with unique trace and strict comparison (e.g. $A = \mathrm{C}^*_r(F_n)$, $n \geq 2$), then for $u \in U^0(A)$, there are self-adjoint $h, k, l \in M_2(A)$ of norm at most π such that $u \oplus 1_A = e^{ih}e^{ik}e^{il}$. Further, one can take l to have arbitrarily small norm.*

In particular, this implies the map η_A from (1) is an isomorphism. Then restricting to the case when A is a reduced free group C*-algebra shows that $\mathrm{C}^*_r(F_m)$ and $\mathrm{C}^*_r(F_n)$ have isomorphic ultrapowers if and only if $m = n$.

The statement in Theorem 1 is presumably not optimal. It is not clear to me what the optimal bound on the number of exponentials should be, but I strongly expect that the 2×2 matrix amplification is not needed. As far as hypotheses on the algebra, it is reasonable to suspect that the conclusion of Theorem 1 may hold for all unital simple C*-algebras with strict comparison for which projections separate (quasi)traces, although the techniques in [4] do use the unique trace assumption in a crucial way.

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Open Problem Session

Collected by Stuart White
(joint work with all and sundry)

This informal session was devoted to collect and present a number of open problems in the area. The problems were suggested by the participants, collected by Stuart White, and briefly discussed during the session.

Problem 1. Let $\mathbb{Z} \curvearrowright A$ be a continuous action on a separable C*-algebra A. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Does the continuous part of the induced action $\mathbb{Z} \curvearrowright A^{\mathcal{U}}$ lift to $\ell^\infty(A)$?

Problem 2. Let $p \in \mathcal{R}$ be a projection with irrational trace in the hyperfinite II_1 factor $\mathcal{R}$. Is $p \otimes 1_{\mathcal{R}} \sim_{\mathrm{MvN}} 1_{\mathcal{R}} \otimes p$ in $\mathcal{R} \otimes \mathcal{R}$?

Problem 3. Is there an autormorphism α on $\mathcal{O}_\infty \otimes \mathcal{K}$ which fixes the diagonal and satisfies $K_0(\alpha) = -id$?

Problem 4. Does there exist a relatively free group G such that $C^*_r(G)$ is selfless?

Problem 5. Let $\alpha : \mathbb{R} \curvearrowright \mathcal{W} \otimes \mathcal{K}$ satisfy $\tau \circ \alpha_t = e^t \tau$, where $\mathcal{W} \otimes \mathcal{K}$ denotes the stabilised Razak-Jacelon algebra. Does α have the Rokhlin property?

Problem 6. Consider the embeddings $\iota_1, \iota_2 : \mathcal{O}_\infty \to \mathcal{O}_\infty * \mathcal{O}_\infty$ into the unital full free product $\mathcal{O}_\infty * \mathcal{O}_\infty$. Do there exist non-trivial projections $p, q \in \mathcal{O}_\infty$ such that $\iota_1(p) \sim_h \iota_2(q)$?

Problem 7. Does there exist a simple C*-algebra which is not K_1-injective?

Problem 8. Let G be a locally compact group. When is $L(G)$ almost almost unimodular?

Problem 9. Is every ε-C*-algebra close to a C*-algebra?

Problem 10. Let $\varphi : A \to A$ be a completely positive map such that $\|\varphi - \varphi^2\|_{cb}$ is small. Is φ $\|.\|_{cb}$-close to a conditional expectation?

Problem 11. Consider the construction of the CAR algebra M_{2^∞} via the canonical anticommutation relations. Then, for some separable Hilbert space $\mathcal{H}$ where the relations are satisfied, one has $A : \mathcal{H} \to B(\mathcal{H})$ and $C^*(A(\mathcal{H})) \cong M_{2^\infty}$.

Consider $\{F_{ijk} \mid i,j = 1,\dots,n, k = 1,\dots,d\}$, where F_{ijk} are pairwise orthogonal. Define $X_k = (\frac{1}{\sqrt{n}}A(F_{ijk}))_{i,j=1}^n \in M_n(M_{2^\infty})$. Does X_k converge strongly to a free circular distribution?

Problem 12. Are q-Gaussian C*-algebras selfless for $-1 < q < 1$, $q \neq 0$?

Problem 13. Given a factor M, find a localization for $M \cong M\bar{\otimes}P$ where P is a hyperfinite III$_0$ factor.

Problem 14. Consider $L(SL_2(\mathbb{Z}))^{\mathcal{U}}, \prod_{\mathcal{U}} M_n$ and $L(\mathbb{F}_2)^{\mathcal{U}}$ for a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Are any two of these algebras isomorphic?

Problem 15. Does a group G exist which is not mixed identity-free and such that $C_r^*(G)$ is selfless?

Problem 16. Does $C^*(\mathbb{F}_2 \times \mathbb{F}_2)$ have the (local) lifting property?

Problem 17. Does there exist a non-amenable group G without free subgroups such that $C_r^*(G)$ has the rapid decay property?

Problem 18. Let K be compact and convex. Does $O_{min} - \mathrm{Aff}(K) = O_{max} - \mathrm{Aff}(K)$ imply that K is a Choquet simplex?

Problem 19. Is $L(\mathbb{F}_2) \ncong L(\mathbb{F}_3)$?

Problem 20. Is $\mathcal{Z}^{*n}_{(r,\tau)} \cong \mathcal{Z}^{*m}_{(r,\tau)}$? Is there a dichotomy?

Problem 21. Is every unital simple separable C*-algebra singly generated?

Problem 22. Do classifiable C*-algebras admit filtrations such that the rapid decay property is satisfied?

Problem 23. Is $\mathbb{F}_2$ C_r^*-(super)rigid (within hyperbolic groups)?

Problem 24. Let M be a von Neumann algebra with a faithful state φ. When is (M,φ) selfless as a von Neumann algebra?

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