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# Cohomology Theories for Automorphic Forms and Enumerative Algebra

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Abstract. Cohomology theories have long proven to be powerful, unifying tools in numerous areas of mathematics, specifically in algebra, geometry, and number theory. Historically, the invention of the "right" cohomology theory often proved to be the key to uniform, conceptual proofs of conjectures and explanations of heuristic phenomena, bringing together seemingly unrelated mathematical areas. This workshop concentrated on the areas of automorphic forms and enumerative algebra, two fields in which cohomology theories already had a particular impact. It brought together mathematicians from various subdirections of these areas, both junior and senior.

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## Introduction by the Organizers

The workshop *Cohomology theories for automorphic forms and enumerative algebra*, organized by Claudia Alfes (U Bielefeld), YoungJu Choie (POSTECH, Pohang), Anke Pohl (U Bremen), and Christopher Voll (U Bielefeld), was well attended with 25 participants (2 online) providing a gender-balanced blend of senior and junior researchers of various research directions within the overarching theme of cohomology theories for automorphic forms and enumerative algebra. It featured 20 presentations as well as ample and highly appreciated opportunities for individual scientific collaborations and discussions.

A recurrent overall theme in cohomology theory is to forge connections between different objects via suitable cohomology theories, hereby creating novel research tools, re-interpretating existing results, and creating vantage points for further research. Within the rich realm of mathematical cohomology theories, the workshop focused on those in the research areas of automorphic forms and enumerative algebra.

Roelof Bruggeman explained the interpretation of Maass cusp forms in parabolic cohomology that was developed by him, Lewis and Zagier, and reported on ongoing developments. Complementary to his talk, Anke Pohl surveyed the relation of parabolic cohomology with the dynamics of hyperbolic orbisurfaces and the Selberg zeta function. Roberto Miatello discussed Poincaré series for semisimple Lie groups of real rank one and showed the state of art in regard to the question of which automorphic forms on these Lie groups can be obtained from these series. Jens Funke focused in his survey talk on harmonic weak Maass forms and showed that the cohomological periods of two differential operators applied to the forms coincide. Martin Raum presented the theory of vector-valued modular forms and introduced the notion of modular forms of virtually-arithmetic type which he developed in work with Michael Mertens and Tobias Magnusson.

Nikolaos Diamantis considered iterated integrals, a generalization of Manin's modular symbols. He discussed an extension of standard cohomology reflecting structures emerging in the study of modular symbols and false theta functions, of which a characterization of classes of iterated integrals is an application. Also in the spirit of understanding iterated integrals as generalizations of modular symbols and related to multiple zeta values, Morten Risager discussed the limiting distributions of iterated integrals and showed computer-generated figures that indicated the difficulties for higher length situations.

For certain Laurent polynomials, the Mahler measure is related to special values of $L$-functions. An important proof is by Deninger's cohomological method. Jungwon Lee described generalizations of this method obtained by her and Wei He. Dohyeong Kim analyzed the behavior of Dedekind zeta functions over $\mathbb{Z}_p$-extension with a focus on the Euler–Kronecker constants.

In his survey talk on zeta functions in enumerative algebra, Joshua Maglione pointed to a number of "cohomological shadows" in this area, suggesting a cohomological explanation for non-negativity and self-reciprocity phenomena seen in various local zeta functions associated with groups, rings, and modules. In a similar vain, Tobias Rossmann surveyed phenomena observed in and conjectured about reduced and topological versions of such zeta functions, pointing towards connections with Hilbert series of Cohen–Macaulay graded rings and modules. Bianca Marchionna talked about her recent work on a conjecture of Rossmann's on residues of local zeta functions associated with pattern algebras, using multivariate $p$-adic integrals.

Mima Stanojkovski discussed joint work with Oihana Garaialde on isomorphism classes of extensions of finite $p$-groups. Paul Kiefer reported on ongoing joint work with Lennart Gehrmann in which they construct $\Lambda$-adic families of Funke–Millson cycles. Min Lee presented joint work with Jonathan Bober, Andrew R. Booker, Claire Burrin, Vivian Kuperberg, David Lowry-Duda, Catinca Mujdei, and Hsin-Yi Yang in which they prove the existence of murmurations for elliptic curves which was observed by AI techniques.

Bo-Hae Im presented the proof of the Zagier–Hoffman conjectures for $N$th multiple zeta values in positive characteristic which she obtained in joint work with Hojin Kim, Khac Nhuan Le, Tuan Ngo Dac, and Lan Huong Pham. In her online talk Winnie Li reported on her results with Jerome Hoffmann, Ling Long, and Fang-Ting Tu on the computation of traces of Hecke operators via hypergeometric character sums. Gabriele Bogo discussed deformations of modular forms and extensions of symmetric tensor representations motivated by the uniformization theorem for Riemann surfaces. Lakshmi Ramesh closed the workshop with a presentation of her joint work with Janko Böhm and Santosh Gnawali on the computation of cohomology of coherent sheaves. Integral to this work is an algorithm, implemented in the computer algebra system SINGULAR.

Wednesday afternoon featured a lively problem session. Participants confirmed the organizers' impression of an enjoyable and highly productive workshop.

## Workshop: Cohomology Theories for Automorphic Forms and Enumerative Algebra

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# Abstracts

## Cohomological shadows from two zeta functions from algebra

Joshua Maglione

### 1. Subalgebra zeta functions

Let $A$ be a finitely generated $\mathbb{Z}$-algebra. For us, this means that $(A, +)$ is an abelian group, and its product is $\mathbb{Z}$-bilinear. So algebras need not have a multiplicative unit nor be associative. The *subalgebra zeta function* of $A$ is

$$\zeta_A(s) = \sum_{B \leqslant A} |A : B|^{-s},$$

where the sum runs over all finite index subalgebras $B$ in $A$. By the structure theorem of abelian groups, the finite quotient of $\mathbb{Z}$-modules $A/B$ decomposes into cyclic groups. This leads to an Euler product decomposition of the subalgebra zeta function counting subalgebras of $p$-power index:

$$\zeta_A(s) = \prod_{p \text{ prime}} \zeta_{A,p}(s).$$

Let us look at a few examples of the subalgebra zeta function. First, let $A = \mathbb{Z}$; then there is a unique subalgebra of index $n$ for each $n \in \mathbb{N}$. Hence, $\zeta_{\mathbb{Z}}(s) = \zeta(s)$ is the Riemann zeta function. Now consider $A = \mathbb{Z}^2$ with component-wise multiplication. By Datskovsky and Wright [1],

$$\zeta_{\mathbb{Z}^2}(s) = \zeta(s)^3\zeta(2s)^{-2}\zeta(3s-1).$$

Lastly, we consider $A = \mathbb{Z}^3$ with component-wise multiplication. Nakagawa [6] and Liu [3] independently showed

$$\zeta_{\mathbb{Z}^3}(s) = \prod_{p \text{ prime}} \frac{1 + 4p^{-s} + 2p^{-2s} + \cdots - 2p^{2-7s} - 4p^{2-8s} - p^{2-9s}}{(1-p^{-s})^2(1-p^{2-4s})(1-p^{3-6s})}. \tag{1}$$

A remarkable feature of the local zeta functions in (1) is that the coefficients of the numerators are palindromic, up to multiplying by $-1$. The reason for this is that these local subalgebra zeta functions satisfy a functional equation. The following deep theorem tells us that this always happens.

**Theorem 1** (Voll [8]). *Assume $A$ is torsion free and rank $n$. For almost all primes $p$,*

$$\zeta_{A,p}(s)\big|_{p\to p^{-1}} = (-1)^n p^{\binom{n}{2}-ns}\zeta_{A,p}(s).$$

We consider a slight variant of the subalgebra zeta function that will exemplify the two cohomology theories in Theorems 1. There is a way to encode the elliptic curve $E = y^2 + x^3 - x$ into a Lie algebra $L_E$, see for example [4]. The ideal zeta function of $L_E$ is

$$\zeta^{\triangleleft}_{L_E}(s) = W_1(p, p^{-s}) + \#E(\mathbb{F}_p) \cdot W_2(p, p^{-s}),$$

such that $W_1(X^{-1}, Y^{-1}) = -X^{36}Y^{15}$ and $W_2(X^{-1}, Y^{-1}) = -X^{37}Y^{15}$.

We have two cohomology theories in the proof of Theorem 1: the $\ell$-adic cohomology from the Weil conjectures and simplicial cohomology from counting integral points on polyhedral cones.

**Question 1.** *Is there a bespoke cohomology theory for subalgebra zeta functions that unifies these two theories?*

If there is such a cohomology theory that answers Question 1, it might also provide a way understand the poles arising in these zeta functions. Currently, we have no way of getting even a finite set of candidate poles.

## 2. Flag Hilbert–Poincaré series

Now we explore a more combinatorial zeta function within enumerative algebra. Let $\mathcal{A}$ be a finite set of hyperplanes in $\mathbb{A}^n_K$ with $K$ a number field. Write $\mathcal{L}(\mathcal{A})$ for the set of all possible intersections of the hyperplanes of $\mathcal{A}$, excluding $\mathbb{A}^n_K$. The *flag Hilbert–Poincaré series* of $\mathcal{A}$ is

$$\mathrm{fHP}_{\mathcal{A}}\left(Y, (T_X)_{X\in\mathcal{L}(\mathcal{A})}\right) = \sum_{F} \pi_F(Y) \prod_{X\in F} \frac{T_X}{1-T_X},$$

where the sum runs over all flags $F$ of subspaces in $\mathcal{L}(\mathcal{A})$ and $\pi_F(Y)$ is a product of Poincaré polynomials like those for the complex manifold $\mathbb{C}^n \setminus \bigcup_{H\in\mathcal{A}} H$.

**Theorem 2** (M.–Voll [5])**.** *There exist substitutions of* $\mathrm{fHP}_{\mathcal{A}}$ *yielding*

- *the Igusa local zeta function associated with $\mathcal{A}$,*
- *the motivic zeta function associated with $\mathcal{A}$,*
- *the ask zeta function of co-graphical group schemes.*

*If* $\hat{1} := \bigcap_{H\in\mathcal{A}} H \neq \varnothing$, *then*

$$\mathrm{fHP}_{\mathcal{A}}(Y^{-1}, (T_X^{-1})_X) = (-Y)^{-\mathrm{rk}(\mathcal{A})} T_{\hat{1}} \cdot \mathrm{fHP}_{\mathcal{A}}(Y, (T_X)_X).$$

Similar to Theorem 1, two cohomology theories come together in Theorem 2. We can observe more shadows by simplifying $\mathrm{fHP}_{\mathcal{A}}$; define the bivariate coarsening

$$\mathrm{cfHP}_{\mathcal{A}}(Y, T) = \mathrm{fHP}_{\mathcal{A}}(Y, (T)_X) = \frac{\mathcal{N}_{\mathcal{A}}(Y, T)}{(1-T)^{\mathrm{rk}(\mathcal{A})}}.$$

It is not difficult to show that $\mathcal{N}_{\mathcal{A}}(Y, 0) = \pi_{\mathcal{A}}(Y)$, the Poincaré polynomial of the complex manifold $\mathbb{C}^n \setminus \bigcup_{H\in\mathcal{A}} H$. Additionally,

$$\mathcal{N}_{\mathcal{A}}(0, T) = \mathrm{Hilb}(\mathrm{SR}_{\mathcal{A}}, T),$$

where $\mathrm{SR}_{\mathcal{A}}$ is a Stanley–Reisner ring associated with $\mathcal{A}$. The polynomial $\mathcal{N}_{\mathcal{A}}(Y,T)$ seems to record more cohomological information.

**Theorem 3** (Dorpalen-Barry–M.–Stump [2]). *The coefficients of $\mathcal{N}_{\mathcal{A}}(Y,T)$ are non-negative.*

**Theorem 4** (Stump [7]). *Letting* $\mathrm{CH}(\mathcal{A})$ *be the Chow ring associated with* $\mathcal{A}$*,*

$$\mathrm{Hilb}(\mathrm{CH}(\mathcal{A}),t) = \mathcal{N}_{\mathcal{A}}(t,-t).$$

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# (Strong) isomorphism of $p$-groups and orbit counting


Mima Stanojkovski

(joint work with Oihana Garaialde)


Let $G$ and $N$ be finite groups. A group extension

$$1 \longrightarrow N \xrightarrow{\iota} E \xrightarrow{\pi} G \longrightarrow 1 \tag{1}$$

of $G$ by $N$ is called *central* if $\iota(N)$ is a central subgroup of $E$: in this case, the action of $G$ on $N$ that is induced by $\iota$ and $\pi$ is trivial, i.e. $N$ is a trivial $\mathbb{Z}G$-module.

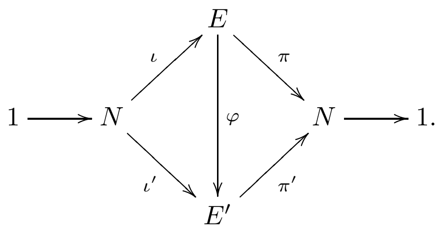
Since non-trivial groups of prime power order have non-trivial center, every such group can be realized as a central extension as in (1) where $N$ is a trivial $\mathbb{F}_pG$-module and $p$ is the unique prime number dividing the order of $G$.

Let now $p$ be a prime number and recall that a finite group is called a $p$-group if its order equals a power of $p$. Assume, moreover, that $G$ is an abelian $p$-group and that $N$ is cyclic of order $p$. Then an extension of $G$ by $N$ is either abelian or with commutator subgroup of order $p$. Thanks to the classification of finite abelian groups, the first family is easily described up to isomorphism. Instead, the latter has been classified in [2] with respect to the group order and relies on the classification of bilinear forms. In [3], we study these extensions and employ cohomological tools to classify them as we now explain.

Classically, the second cohomology group $\mathrm{H}^2(G;N)$ parametrizes the extensions of $G$ by $N$ up to *equivalence*: If

$$1 \longrightarrow N \xrightarrow{\iota'} E' \xrightarrow{\pi'} G \longrightarrow 1$$

is another extension like (1), then their equivalence translates to the existence of an isomorphism $\varphi : E \to E'$ such that the following diagram commutes:




Relaxing the requirement on $\varphi$ to simply induce, by restriction, an isomorphism $\iota(N) \to \iota'(N)$, one speaks of *strong isomorphism classes* of extensions of $G$ by $N$. The following is a weaker version of Theorem 4.7[1] from [1]:

**Proposition 1.** *Let $p$ be a prime, $G$ a finite group, and $N$ a trivial $\mathbb{F}_pG$-module. Then the set of strong isomorphism classes of extensions of $G$ by $N$ is in bijection with the orbits of the natural action of* $\mathrm{Aut}(G) \times \mathrm{Aut}(N)$ *on* $\mathrm{H}^2(G;N)$.

Assume now the following: $p$ is a prime number and $G$ is a finite abelian $p$-group with no summands of order 2. Let, moreover, $A = \mathrm{Aut}(G) \times \mathbb{Z}_p^*$ and consider the natural action of $A$ on $\mathrm{H}^2(G;\mathbb{F}_p)$. Under these assumptions, $\mathrm{H}^2(G;\mathbb{F}_p)$ is an $\mathbb{F}_p$-vector space with a canonical split short exact sequence of $A$-modules:

$$0 \longrightarrow \mathrm{Ext}^1(G;\mathbb{F}_p) \longrightarrow \mathrm{H}^2(G;\mathbb{F}_p) \longrightarrow \mathrm{Hom}(\Lambda^2(G/pG),\mathbb{F}_p) \longrightarrow 0.$$

Since $\mathrm{Hom}(\Lambda^2(G/pG),\mathbb{F}_p)$ equals the span of the cup product

$$\begin{aligned}\cup : \mathrm{Hom}(G,\mathbb{F}_p) \times \mathrm{Hom}(G;\mathbb{F}_p) &\longrightarrow \mathrm{H}^2(G;\mathbb{F}_p),\\ (f,g) &\longmapsto f \cup g \;=\; ((x,y) \mapsto f(x)g(y)),\end{aligned}$$

the following equality holds: $\mathrm{H}^2(G;\mathbb{F}_p) = \mathrm{Ext}^1(G;\mathbb{F}_p) \oplus \langle \mathrm{im}\cup \rangle$. Moreover, $\mathrm{im}\cup$ being $A$-stable, the following is sound:

**Proposition 2.** ([3, Cor. 3.11]) *For $i = 1,2$ define $\mathcal{S}^i(G) = \{M \leq G : G/M \cong \mathbb{F}_p^i\}$. Then there is an isomorphism of $A$-sets*

$$\psi : \mathbb{P}\mathrm{Ext}^1(G;\mathbb{F}_p) \times \mathbb{P}\mathrm{im}\cup \longrightarrow \mathcal{S}^1(G) \times \mathcal{S}^2(G).$$

In the next definition, for a subgroup $T$ of $G$ and a positive integer $s$, denote by $T[s]$ the $s$-torsion subgroup of $T$.

[1] The most general version of this theorem allows for $N$ to be any $\mathbb{F}_pG$-module and parametrizes strong isomorphism classes in terms of an action of the *compatible pairs* of $A = \mathrm{Aut}(G) \times \mathrm{Aut}(N)$ (in our case, all elements of $A$).

**Definition 3.** *Let $M$ and $T$ be subgroups of $G$.*

*(1) The* $T$-levels *of $M$ are given by the pair* $\ell\mathrm{L}_T(M) = (\ell_T(M), \mathrm{L}_T(M))$ *where*
- $\ell_T(M) = 1 + \max\{0 \leq i \leq \log_p \exp(T) : T[p^i] \subseteq M \cap T\}$,
- $\mathrm{L}_T(M) = \min\{j \in \mathbb{Z}_{\geq 0} : T[p^j] + (M \cap T) = T\}$.

*(2) The* index *of $M$ in $T$ is*

$$i(T : M) = \begin{cases} 0 & \text{if } M \subseteq T, \\ 1 & \text{otherwise.} \end{cases}$$

If $T = G$, simply write $\ell\mathrm{L}(M)$ for $\ell\mathrm{L}_G(M)$.

**Theorem 1.** ([3, Thm. 6.1]) *Let $c, d \in \mathrm{Ext}^1(G; \mathbb{F}_p)$ and let $\omega, \vartheta \in \mathrm{im}\cup$. Denote $(T, M) = \psi(c, \omega)$ and $(S, N) = \psi(d, \vartheta)$. Then the following are equivalent:*

*(1) $c + \omega$ and $d + \vartheta$ are in the same $A$-orbit,*
*(2) $(\ell\mathrm{L}(M)), \ell\mathrm{L}(T), \ell\mathrm{L}_T(M), i(T : M)) = (\ell\mathrm{L}(N)), \ell\mathrm{L}(S), \ell\mathrm{L}_S(N), i(S : N))$.*

As applications of the last theorem we deduce the numbers and sizes of all orbits of $A$ acting on $\mathrm{H}^2(G; \mathbb{F}_p)$ when $G$ is generated by at most 3 elements. We hope to come back to the study of higher rank tensors, and thus to a full classification of orbits independently on the number of generators, in the future.

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# Order complexes, nested permutohedra, and ask zeta functions


Alec Schmutz


## 1. Combinatorial Denef formulae

For a finite poset $P$, a *combinatorial Denef formula* is loosely defined to be a multivariate rational function expressible in the following form:

$$\sum_{F \in \Delta(P)} \Phi_F(X) \prod_{x \in F} \frac{T_x}{1 - T_x},$$

where $\Delta(P)$ denotes the order complex of $P$, that is the abstract simplicial complex whose faces consist of chains of $P$, and where $\Phi_F(X) \in \mathbb{Z}[X]$ is a polynomial encoding combinatorial information about the summand. Such finitary sums arise in numerous enumeration problems, where upon suitable substitutions of the variables, one recovers a local factor for some zeta function. A prototypical example is given by the $n$-th Igusa zeta function

$$I_n(X,(T_i)_{i\in[n]}) = \sum_{I\subset[n]} \binom{n}{I}_X \prod_{i\in I} \frac{T_i}{1-T_i},$$

where $\binom{n}{I}_X$ denotes the Gaussian multinomial coefficient. Upon suitable substitutions of the variables, namely by evaluating $I_n(p^{-1},(p^{(n-i-s)i})_{i=0}^{n})$ for a prime $p$, one recovers the subgroup zeta function $\zeta^{\leq}_{\mathbb{Z}_p^n}(s)$ (cf. [6, Example 2.20]).

In some cases, such combinatorial Denef formulae are malleable to cohomological methods, which in turn clarify the behaviours of the coefficients appearing in the numerator of these rational functions. For instance in [3], Cohen-Macaulayness of the Stanley-Reisner ring $\mathbb{F}[\Delta(\mathcal{L}(\mathcal{A}))]$, where $\mathcal{L}(\mathcal{A})$ denotes the intersection lattice of some hyperplane arrangement $\mathcal{A}$, is leveraged in order to deduce non-negativity of the coefficients of the corresponding Hilbert series.

## 2. Ask zeta functions

Introduced in [4], ask zeta functions are rational generating functions which enumerate average sizes of kernels of matrices of linear forms defined over finite rings. More precisely, fixing a ring $R$ and an $R$-algebra $S$, we let $A(\mathbf{X}) \in M_{d\times e}(R[X_1,\dots, X_m])$ be a matrix whose entries consist of *linear forms*. Whenever $S$ is finite, the **a**verage **s**ize **k**ernel of $A(\mathbf{X})$ is

$$\mathrm{ask}_S(A(\mathbf{X})) = \frac{1}{|S|^m} \sum_{x\in S^m} |\ker(A(x))|.$$

Setting $R = \mathfrak{O}$ to be a compact DVR, the (analytic) ask zeta function of $A(\mathbf{X})$ is then given by the following Dirichlet series:

$$\zeta^{\mathrm{ask}}_{A/\mathfrak{O}}(s) = \sum_{k\geq 0} \mathrm{ask}_{\mathfrak{O}/\mathfrak{P}^k}(A(\mathbf{X}))\ q^{-ks}.$$

Such zeta functions are of group-theoretic interest, as they relate to the (conjugacy) class-counting zeta functions of unipotent groups associated with graphs, namely graphical group schemes; cf. [5, Propositon 1.1]. This enables one to tackle problems reminiscent of G. Higman's conjecture [2], postulating that the number of conjugacy classes (class number) of the full upper-unitriangular matrix group $U_n(\mathbb{F}_q)$ is given by a polynomial in $q$.

For a graph $\Gamma = (\{v_1,\dots,v_n\},E)$, we build the (antisymmetric) matrix of linear forms $A^-_\Gamma \in M_n(\mathfrak{O}[X_E])$ by "linearising" the adjacency matrix of $\Gamma$, viz.

$$(A^-_\Gamma)_{ij} = \begin{cases} X_e & \text{if } v_i \sim_e v_j \text{ and } i\leq j,\\ -X_e & \text{if } v_i \sim_e v_j \text{ and } i > j,\\ 0 & \text{otherwise.}\end{cases}$$

It turns out that whenever $\Gamma$ is a cograph, i.e., a $P_4$-free graph, its associated ask zeta function can be recovered by a combinatorial formula of Denef type:

**Theorem 1** (Rossmann-Voll [5, Theorems C & D]). *For any cograph $\Gamma = (V, E)$, there exists a modelling hypergraph $H = (V, (\mu_I)_{I \subset V})$ such that for any compact DVR $\mathfrak{O}$ with residue field size $q$, one has that $\zeta^{\mathrm{ask}}_{A^-_\Gamma/\mathfrak{O}}(s) = W_{H_\Gamma}(q, q^{-s})$, where*

$$(1) \qquad W_{H_\Gamma}(X, T) = \sum_{F \in \Delta(2^{[n]})} (1 - X^{-1})^{|\sup(F)|} \prod_{J \in F} \frac{X^{|J| - \sum_{I \cap J \neq \varnothing} \mu_I} T}{1 - X^{|J| - \sum_{I \cap J \neq \varnothing} \mu_I} T}.$$

By virtue of [5, Proposition 1.1], the previous theorem yields polynomial expressions in $q$ for the class numbers of the cographical groups $\mathbf{G}_\Gamma(\mathfrak{O}/\mathfrak{P}^k)$. In the spirit of previously discussed non-negativity results, the combinatorial nature of Theorem 1 moreover imposes the non-negativity of the coefficients of these polynomials, when expressed in $q - 1$; cf. [5, Theorem E]. It is of natural interest to determine whether Theorem 1 can be extended to a larger family of graphs:

**Question 1** (Rossmann-Voll [5, Question 1.8 (iii)]). *Is there a meaningful combinatorial formula (in the spirit of Theorem 1) for the functions $W^-_\Gamma(X, T)$ which is valid for all graphs on a given vertex set?*

One attempt at answering the previous question is to relate the order complex $\Delta(2^{[n]})$ appearing in (1) to the face lattice of the permutohedron, which we denote by $\mathcal{L}(\mathcal{P}_n)$. Identifying the permutohedron with its dual fan, the cones of the Braid fan $\mathrm{Br}_n$ determine regions where one has a total ordering on the $n$ vertices of the given graph $\Gamma$. An obstruction to obtaining a formula akin to (1) for arbitrary graphs is, in essence, due to "sandwiches": for a given cone of the Braid fan where

$$v_{\pi^{-1}(1)} \leq v_{\pi^{-1}(2)} \leq \cdots \leq v_{\pi^{-1}(n)},$$

one often has to compare "two-sums" of the form $v_{\pi^{-1}(i)} + v_{\pi^{-1}(l)}$ and $v_{\pi^{-1}(j)} + v_{\pi^{-1}(k)}$, where $1 \leq i < j < k < l \leq n$. Such a comparison is unnecessary when dealing with cographs, but are prevalent amongst non-cographs.

Introduced in [1], the nested Braid fan $\mathrm{Br}^2_n$ (dual to the nested permutohedron) resolves in some situations the aforementioned pesky sandwiches. Moreover, the nested Braid fan admits a combinatorial description in terms of the ordered (set) partition poset; cf. [1, Proposition 4.10]. These observations prompt the following question:

**Question 2.** *For which family of graphs do the corresponding ask zeta functions admit a combinatorial Denef formula in terms of the nested braid fan $\mathrm{Br}^2_n$?*

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## Maass cusp forms and cohomology

Roelof Bruggeman

I gave an introduction to the relation between Maass cusp forms and cohomology groups, as developed in [1]. I kept mainly to the context of the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$.

For holomorphic cusp forms the relation with cohomology goes back to Eichler's paper [2]. Changing the polynomial kernel function $(\tau - X)^{k-2}$ to a more complicated kernel, built with the Poisson kernel $R(t,z)^s$, we arrive at cocycles

$$\gamma \mapsto \int_{z=\gamma^{-1}\tau_0}^{\tau_0} \big[u(z), R(t,z)^s\big]$$

with values in the space $\mathcal{V}_s^\omega$ of analytic vectors in the principal series representation with a spectral parameter $s$ such that $s(1-s)$ is the eigenvalue of the hyperbolic Laplace operator on the Maass form $u$. Changing the base point $z_0$ in the upper half-plane does not influence the cohomology class. For a cusp form $u$ the exponential decay at cusps allows us to use a cusp $\xi \in \mathbb{R} \cup \{\infty\}$ as the base point. Then the cocycle is only a $C^\infty$-function at the points $\xi$ and $\gamma^{-1}\xi$. In this way we arrive at a linear map from the space of Maass cusp forms with eigenvalue $s(1-s)$ to the cohomology group $H^1_{\mathrm{pb}}(\Gamma; \mathcal{V}_\nu^{\omega^0,\infty})$. By $\omega^0$ we indicate that the principal series vectors are analytic outside a finite number of cusps, and by $\infty$ that the vectors are smooth at those cusps.

This map is not surjective. We have to impose an additional property at the cusps where the principal series vectors are not analytic. With the smaller $\Gamma$-module $\mathcal{V}_s^{\omega^0,\infty,\mathrm{exc}}$ we arrive at a linear bijection between the space of Maass cusp forms with spectral parameter $s$ satisfying $0 < \mathrm{Re}\, s < 1$ and the parabolic cohomology group $H^1_{\mathrm{pb}}\big(\Gamma; \mathcal{V}_s^{\omega^0,\infty,\mathrm{exc}}\big)$. Theorem B in [1] gives this isomorphism, and some more cohomology groups isomorphic to the same space of Maass cusp forms.

A parabolic cocycle $\psi$ representing a class in $H^1_{\mathrm{pb}}\big(\Gamma; \mathcal{V}_s^{\omega^0,\infty,\mathrm{exc}}\big)$ is determined by its values on the generators $S = \pm\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$ and $T = \pm\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}$ of $\Gamma$. There is a unique choice such that $\psi_T = 0$. Then the cohomology class is determined by the function $\psi_S$, which satisfies

$$\psi_S(-1/t) = -|t|^{-2s}\psi_S(t) \qquad \text{for } t \in \mathbb{R} \setminus \{0\}\,,$$
$$\psi_S(t) = \psi_S(t+1) + (t+1)^{-2s}\, \psi_S\big(t/(t+1)\big) \qquad \text{for } t \in (0,\infty)\,,$$
$$\psi_S \in C^\infty(\mathbb{R}) \text{ and } \psi \text{ is real-analytic on } \mathbb{R} \setminus \{0\}\,.$$

John Lewis [3] found functions satisfying equivalent relations associated to even Maass cusp forms. He and Don Zagier gave a thorough treatment, including the relation to cohomology, in [4]. This is extended to all cofinite discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$ in [1].

It turned out that functions with similar properties had arisen in the study of the Gauss map by Mayer, [5]. In this way we may consider parabolic cohomology groups like $H^1_{\mathrm{pb}}\bigl(\Gamma; \mathcal{V}_s^{\omega^0,\infty,\mathrm{exc}}\bigr)$ as intermediate between Maass cusp forms and eigenspaces of a transfer operator arising in the study of a dynamical system.

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## From geodesics to period functions, parabolic cohomology and the Selberg zeta function: a survey

Anke Pohl

The development of cohomological interpretations of automorphic functions and forms was initiated with the highly influential work by Eichler [5] and Shimura [13], and it continues till today. A few recent examples are [2, 6, 1, 3, 11], which were partly discussed in the talks by R. Bruggeman, N. Diamantis and myself. In addition, J. Funke presented a (yet unpublished) cohomological interpretation of weakly holomorphic modular forms.

For this report we focus on (classical) Maass cusp forms for Fuchsian groups. For these functions, Bruggeman, Lewis and Zagier [2] provide an interpretation in parabolic 1-cohomology. Using tools of parabolic cohomology only, for a few special Fuchsian groups, such as $\mathrm{PSL}_2(\mathbb{Z})$, the parabolic cocycle classes were seen to be in linear isomorphism with sufficiently regular solutions of certain explicit finite-term functional equations. These solutions are indeed period functions, the functional equation arises from a certain change of path of integration, and the period functions serve in a very precise way as building blocks for parabolic cocycle classes. (The history of these discoveries is highly involved and included important work by D. Mayer, Chang and Mayer [8, 9, 4] as well as by Lewis and Zagier [7].) One of the difficult steps in establishing such an isomorphism is to find the "correct" functional equation. Solving this problem for generic Fuchsian groups seemed (and still seems) to be out of reach without further insights.

However, the intimate relation between geometric and dynamical entities of hyperbolic orbisurfaces on the one side and spectral entities on the other side allowed us to develop such an insight. The Selberg zeta function is one, arguably one of the most important, objects that realize such a relation by connecting the geodesic length spectrum to the Laplace spectrum and, in particular, to the spectral parameters of Maass cusp forms. However, this zeta function is based on the static geometry only.

Taking advantage of the dynamics of the geodesics as well, a careful construction of symbolic dynamics for the geodesic flow in combination with transfer operator techniques provide an algorithm to find the requested functional equations. Here, the determining equations of eigenfunctions with eigenvalue 1 of the arising transfer operators are precisely those functional equations. This construction further helps to detect the necessary regularity and growth properties of period functions, and it provides a geometric approach to parabolic cohomology that allowed us to extend investigations from cofinite Fuchsian groups to non-cofinite ones. In addition, a parabolic induction (cuspidal acceleration) algorithm yields closely related "companion" transfer operator families whose Fredholm determinant equals the Selberg zeta function. More details can be found in, e.g., [10, 3], and informal explanations are provided, e.g., by the survey article [12] and [3, Chapter 8].

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### Cohomolgical aspects of weakly holomorphic modular forms

Jens Funke

Let $\Gamma$ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. For $k \geq 0$ a non-negative integer, we let $M^!_{2k+2}(\Gamma)$ be the space of weakly holomorphic modular forms, and let $S^!_{2k+2}(\Gamma)$ be the subspace of "weak cusp forms", consisting of those forms whose constant term vanishes at all cusps. We also let $S^{!,\perp}_{2k}(\Gamma) \subset S^!_{2k+2}(\Gamma)$ be the orthogonal complement of the space of cusp forms $S_{2k+2}(\Gamma)$ under the (suitably) regularized Petersson scalar product. We have

$$M^!_{2k+2}(\Gamma) = S_{2k+2}(\Gamma) \oplus \mathrm{Eis}_{2k+2}(\Gamma) \oplus S^{!,\perp}_{2k+2}(\Gamma).$$

Here $\mathrm{Eis}_{2k+2}(\Gamma)$ denotes the space of Eisenstein series of weight $2k+2$ for $\Gamma$.

We let $H^!_{-2k}(\Gamma)$ be the space of harmonic weak Maass forms of weight $-2k$ for $\Gamma$. So in particular,

$$\xi_{-2k}(f) \in M^!_{2k+2}(\Gamma).$$

Here (with $z = x+iy$) $\xi_{-2k}f(z) = y^{2k-2}\overline{L_{-2k}f(z)}$ with $L_r = -2iy^2\frac{\partial}{\partial \bar{z}}$, the Maass lowering operator. We let $H_{-2k}(\Gamma)$ be the subspace of those forms such that $\xi_{-2k}(f) \in S_{2k+2}(\Gamma)$. By [4], section 3, we have

$$\xi_{-2k} : H_{-2k}(\Gamma)/M^!_{-2k}(\Gamma) \simeq S_{2k+2}(\Gamma).$$

Recall that we also have the operator $D = \frac{1}{2\pi i}\frac{\partial}{\partial z}$. By Bol's Lemma we have

$$D^{2k+1}(f) \in M^!_{2k+2}(\Gamma)$$

for $f \in H^!_{-2k}(\Gamma)$. By [5], section 4, the image of $D^{2k+1}$ is equal to $S^{!,\perp}_{2k}(\Gamma)$. In fact, using Proposition 3.19 in [4], we have the isomorphism

$$D^{2k+1} : H_{-2k}(\Gamma) \simeq S^{!,\perp}_{2k+2}(\Gamma). \tag{1}$$

We therefore see that the images of $H_{-2k}(\Gamma)$ under $\xi_{-2k}$ and $D^{2k+1}$ both lie in $S^!_{2k+2}(\Gamma)$ but are disjoint and perpendicular.

We let $W = W_1$ be the standard (complex) representation of $\Gamma$ with standard basis $\mathfrak{e}_1$ and $\mathfrak{e}_2$. We let $W_m = \mathrm{Sym}^m W$ be the irreducible self-dual representation of dimension $m+1$ of highest weight $m$.

Let $X = \Gamma\backslash\mathbb{H}$ be the modular curve associated to $\Gamma$, and let $\mathcal{W}_m$ be the local system on $X$ associated to $W_m$. We then can consider $H^\bullet(X, \mathcal{W}_m)$, the (de Rham) cohomology of $X$ with coefficients in $\mathcal{W}_m$, which is isomorphic to the group cohomology $H^\bullet(\Gamma, W_m)$. The cohomology groups are in a natural fashion Hecke-modules. Note we also have homology groups $H_\bullet(X, \mathcal{W}_m)$.

Let $f(z) \in M^!_{2k+2}(\Gamma)$. Then

$$\eta_f := f(z)dz \otimes (z\mathfrak{e}_1 + \mathfrak{e}_2)^{2k} \tag{2}$$

defines a holomorphic 1-form on $\mathbb{H}$ which descends to a holomorphic 1-form on $X$ with values in the local system $\mathcal{W}_{2k}$. Thus the assignment $f \mapsto [\eta_f]$ gives a map

$$[\eta] : M^!_{2k+2}(\Gamma) \longrightarrow H^1(X, \mathcal{W}_{2k}). \tag{3}$$

It is well-known that $\eta$ when restricted to regular holomorphic forms induces the Eichler-Shimura isomorphisms

$$\mathrm{Eis}_{2k+2}(\Gamma) \oplus S_{2k+2}(\Gamma) \oplus \overline{S_{2k+2}(\Gamma)} \simeq H^1(X, \mathcal{W}_{2k}), \tag{4}$$

$$S_{2k+2}(\Gamma) \oplus \overline{S_{2k+2}(\Gamma)} \simeq H^1_!(X, \mathcal{W}_{2k}). \tag{5}$$

Here $H^1_!(X, \mathcal{W}_{2k+2})$ is the inner cohomology of X, that is, the image of the compactly supported cohomology $H^1_c(X, \mathcal{W}_{2k})$ in the absolute cohomology $H^1(X, \mathcal{W}_{2k})$. Further, the restriction of $\eta$ to weak cusp forms yields a map

$$[\eta] : S^!_{2k+2}(\Gamma) \longrightarrow H^1_!(X, \mathcal{W}_{2k}). \tag{6}$$

The isomorphisms (4)and (5) are Hecke-equivariant under the map (3), that is,

$$[\eta_{T_m f}] = T_m[\eta_f].$$

for any Hecke operator $T_m$ and any modular form $f \in M_{2k+2}(\Gamma)$.

The main result presented in the lecture is

**Theorem 1.** *Let $f \in H^!_{-2k}(\Gamma)$ be a weak Maass form. Then there is an explicit non-zero constant $c_k$ such that*

$$[\eta_{\xi_{-2k}(f)}] = c_k[\overline{\eta_{D^{2k+1}(f)}}]$$

*as classes in $H^1(X, \mathcal{W}_{2k})$. In particular, if $f \in M^!_{-2k}(\Gamma)$ is weakly holomorphic,*

$$[\eta_{D^{2k+1}(f)}] = 0.$$

The theorem is proved by constructing an explicit coboundary relating the two classes. We note that Brown [3] obtained this result as well in a more general context.

Guerzhoy [7] observed that the subspace $D^{2k+1}(M^!_{-2k}(\Gamma))$ of $M^!_{2k+2}(\Gamma)$ is preserved by the action of the Hecke algebra, and hence considered the quotient $M^!_{2k+2}(\Gamma)/D^{2k+1}(M^!_{-2k}(\Gamma))$ as a Hecke module. In particular, he calls $f \in M^!_{2k+2}(\Gamma)$ an eigenform under $T_m$ with eigenvalue $\lambda_m$ if there exists a $g \in M^!_{-2k}(\Gamma)$

$$T_m f = \lambda_m f + D^{2k+1}(g).$$

Theorem 1 gives a cohomological interpretation for Guerzhoy's definition, since for $f$ as above we have

$$T_m[\eta_f] = [\eta_{T_m f}] = \lambda_m[\eta_f].$$

Theorem 1 together with (1) also immediately implies

**Theorem 2.** *The assignment $f \mapsto [\eta_f]$ induces the following isomorphisms of Hecke modules:*

*(i)* $M^!_{2k+2}(\Gamma)/D^{2k+1}(M^!_{-2k}(\Gamma)) \simeq H^1(X, \mathcal{W}_{2k})$,

*(ii)* $S^!_{2k+2}(\Gamma)/D^{2k+1}(M^!_{-2k}(\Gamma)) \simeq H^1_!(X, \mathcal{W}_{2k}) \simeq S_{2k+2}(\Gamma) \oplus \overline{S_{2k+2}(\Gamma)}$,

*(iii)* $S^{!,\perp}_{2k+2}(\Gamma)/D^{2k+1}(M^!_{-2k}(\Gamma)) \simeq H^{0,1}_!(X, \mathcal{W}_{2k}) \simeq \overline{S_{2k+2}(\Gamma)}$.

This recovers and extends Theorem 1.5 in [1] to arbitrary congruence subgroups while also giving a cohomological interpretation for their "multiplicity 2 statement" for Hecke eigenvalues (in the sense of Guerzhoy) in the space $S^!_{2k+2}(\Gamma)$.

**Remark 3.** *Theorem 2 (iii) shows that $H^{0,1}_!(X, \mathcal{W}_{2k})$ can be realized by meromorphic differential 1-forms with poles at the cusps. Using the Riemann-Roch theorem this is a very natural result in Riemann surface theory, and we can view this aspect of Theorem 1 as an explicit version of this statement.*

Theorem 1 also implies that (co)homological periods of $\xi_{-2k}(f)$ and $c_k D^{2k+1}(f)$ coincide, that is, pairings of $[\eta_{\xi_{-2k}(f)}]$ and $c_k[\overline{\eta_{D^{2k+1}(f)}}]$ with homology classes in $H_1(X, \mathcal{W}_{2k})$ coincide.

Given $\mathbf{x} = [a, b, c] \in \mathbb{Z}^3$ with discriminant $D = b^2 - 4ac > 0$, we consider the corresponding geodesic $D_\mathbf{x}$ in the upper half plane by

$$D_\mathbf{x} = \{z \in \mathbb{H};\ a|z|^2 + bRe(z) + c = 0\}.$$

Then $D_\mathbf{x}$ defines a closed geodesic $C_\mathbf{x}$ in $X$ if $D$ is not an integral square. One can then equip these special cycles with coefficients to define classes $[C_{\mathbf{x},[k]}]$ in $H_1(X, \mathcal{W}_{2k})$, see eg [6]. We conclude

**Theorem 4.** *Let $f \in H^!_{-2k}(\Gamma)$. Then*

$$\langle [\eta_{\xi_{-2k} f}], [C_{\mathbf{x},[k]}] \rangle = c_k \overline{\langle [\eta_{D^{2k+1} f}], [C_{\mathbf{x},[k]}\rangle}.$$

*Explicitly,*

$$\int_{C_\mathbf{x}} (\xi_{-2k} f)(z)(az^2 + bz + c)^k dz = c_k \overline{\int_{C_\mathbf{x}} (D^{2k+1} f)(z)(az^2 + bz + c)^k dz}.$$

For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, this result was obtained independently by Bringmann, Guerzhoy, and Kent [2] using a completely different approach.

Furthermore, the approach outlined here can be used to interpret special values of an appropriately defined L-function for weakly holomorphic forms.

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## Iterated integrals and cohomology

Nikolaos Diamantis
(joint work with Kathrin Bringmann)

The presented work is motivated by the different behaviour certain iterated integrals exhibit when considered over different domains, in [2] and in [3] respectively.

Specifically, if $f_1, f_2$ are cusp forms for $\Gamma_0(N)$ of half-integral weights $k_1, k_2$ and multiplier systems $\chi_1, \chi_2$ respectively, we consider the iterated integral:

$$I_{f_1,f_2}(\tau) := \int_{\bar{\tau}}^{i\infty} f_1(w_1)(w_1-\tau)^{k_1-2} \int_{w_1}^{i\infty} f_2(w_2)(w_2-\tau)^{k_2-2} dw_2 dw_1.$$

In [2] $I_{f_1,f_2}$ is viewed as a function on the *lower* half-plane $\bar{\mathbb{H}}$ and the $\Gamma_0(N)$-action on it is studied. By contrast, in [3], a 1-dimensional analogue of $I_{f_1,f_2}$ is studied on the *upper* half-plane $\mathbb{H}$. In [3], the effect of the action was more complicated than in [2] due to the branch cut of a square root appearing in the integrand.

In our work [1] we reconcile the above differences in behaviour employing an approach used in [4] to associate a cohomology to arbitrary real weight modular forms. We first define the module $D^{\omega,\infty,\text{exc}}$: For $\mathfrak{a} \in \mathbb{Q} \cup \{i\infty\}$, $a_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} \in \mathbb{R}_+$ set

$$V_{\mathfrak{a}}(a_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}}) : \{M\tau \in \mathbb{H}; |\Re(\tau)| \leq a_{\mathfrak{a}}, \Im(\tau) > \varepsilon_{\mathfrak{a}}\},$$

where $M \in \mathrm{SL}_2(\mathbb{Z})$ is such that $M(i\infty) = \mathfrak{a}$. Then, if $E = \{\mathfrak{a}_1, \dots \mathfrak{a}_n\} \subset \mathbb{Q} \cup \{i\infty\}$, we call $\Omega \subset \mathbb{P}^1_{\mathbb{C}}$ a *$E$-excised neighbourhood of* $\bar{\mathbb{H}} \cup \mathbb{P}^1_{\mathbb{R}}$ if there is a neighbourhood $U$ of $\bar{\mathbb{H}} \cup \mathbb{P}^1_{\mathbb{R}}$ such that $U \setminus \cup_{i=1}^n V_{\mathfrak{a}_i}(a_{\mathfrak{a}_i}, \varepsilon_{\mathfrak{a}_i}) \subset \Omega$. We then set

$$D^{\omega,\text{exc}}[\mathfrak{a}_1, \dots \mathfrak{a}_n] = \varinjlim \mathcal{O}(\Omega)$$

where $\Omega$ ranges over all $E$-excised neighbourhoods of $\bar{\mathbb{H}} \cup \mathbb{P}^1_{\mathbb{R}}$ and $\mathcal{O}(\Omega)$ is the space of holomorphic functions on $\Omega$. With this notation, we define the space

$$D^{\omega,\infty,\text{exc}} := \varinjlim D^{\omega,\infty,\text{exc}}[\mathfrak{a}_1, \dots \mathfrak{a}_n] \cap C^\infty(\bar{\mathbb{H}} \cup \mathbb{P}^1_{\mathbb{R}}),$$

where $[\mathfrak{a}_1, \dots \mathfrak{a}_n]$ ranges over all $n$-tuples of elements of $\mathbb{Q} \cup \{i\infty\}$. Crucially, if $\chi$ is a weight $r$ multiplier system $D^{\omega,\infty,\text{exc}}$ is closed under the action $|_{r,\chi}$ given by

$$(g|_{r,\chi}\gamma)(\tau) := \overline{\chi(\gamma)}(c_\gamma z + d_\gamma)^{-r} g(\gamma\tau).$$

We can now interpret our objects. For $\gamma \in \Gamma_0(N)$, we define the period functions

$$r_{f_1}(\gamma)(\tau) := \int_{\gamma^{-1}i\infty}^{i\infty} f_1(w_1)(w_1-\tau)^{k_1-2} dw_1 \quad \text{and}$$

$$r_{f_1,f_2}(\gamma)(\tau) := \int_{\gamma^{-1}i\infty}^{i\infty} f_1(w_1)(w_1-\tau)^{k_1-2} \int_{w_1}^{i\infty} f_2(w_2)(w_2-\tau)^{k_2-2} dw_2 dw_1.$$

We then have (all propositions that follow are stated and proved in [1])

**Proposition 1.** *For all $\gamma_1, \gamma_2 \in \Gamma_0(N)$, we have*

$$I_{f_1,f_2}|_{4-k_1-k_2,\chi_1\chi_2}(\gamma - 1) = -r_{f_1,f_2}(\gamma) + r_{f_1}(\gamma) r_{f_2}(\gamma) - r_{f_2}(\gamma) I_{f_1},$$

*on* $\bar{\mathbb{H}}$*, where* $I_{f_1}$ *is the usual (1-dimensional) Eichler integral attached to* $f_1$ *and*

$$r_{f_1,f_2}(\gamma_2\gamma_1) - r_{f_1,f_2}(\gamma_2)|_{4-k_1-k_2,\chi_1\chi_2}\gamma_1 - r_{f_1,f_2}(\gamma_1) = r_{f_1}(\gamma_2)|_{2-k_1,\chi_1} \cdot r_{f_2}(\gamma_1).$$

This and the following propositions and definitions hold for the generalisations of $r_{f_1,f_2}$ associated to arbitrarily many cusp forms $f_i$.

**Proposition 2.** *For each* $\gamma \in \Gamma_0(N)$*, we have*

$$r_{f_1}(\gamma) \in D^{\omega,\infty,exc} \quad \text{and } r_{f_1,f_2}(\gamma) \in D^{\omega,\infty,exc} \otimes D^{\omega,\infty,exc}$$

Proposition 2 gives the coefficient module for the desired cohomological framework. To express the cocycle-like relation of Prop. 1 cohomologically, we have the following inductive definition. We present it here for only up to "length 2".

Let $G$ be a group. First, if $M, N$ are $G$-modules and $C^n(G,N)$ the group of $m$-cochains, we introduce the map $\mu : C^n(G,M) \otimes C^n(G,N) \to C^{n+m}(G, M \otimes N)$ induced by the assignment $\mu(\sigma_1 \otimes \sigma_2) = \sigma_1 \cup \sigma_2$ (the cup product of $\sigma_1$, $\sigma_2$.)

**Definition.** *Let* $M_1, M_2$ *be* $G$*-modules. For each* $m \in \mathbb{N}_0$ *set:*
*i.* $L^m_{(1)} = 0$, $\pi_{(1)}$ *=identity map on* $C^1(G,M_1)$ *and* $d^m_{(1)} = d^m$ *= standard differential on* $C^m(G,M_1)$*. We then set* $Z^m_{(1)}(G,M_1) = ker(d^m_{(1)})$, $B^m_{(1)}(G,M_1) = ker(d^{m-1}_{(1)})$ *and* $H^m_{(1)}(G,M_1) = Z^m_{(1)}(G,M_1)/B^m(1)(G,M_1)$.
*ii.* $L^m_{(2)} = Z^1_{(1)}(G,M_1) \otimes C^{m-1}(G,M_2)$, $\pi_{(2)}$ *=identity map on* $C^1(G,M_1 \otimes M_2)$ *and* $d^m_{(2)} : C^m(G,M_1 \otimes M_2)/\mu(L^m_{(2)}) \to C^{m+1}(G,M_1 \otimes M_2)/\mu(L^{m+1}_{(2)})$ *the map induced by* $d^m$*. We then set* $Z^m_{(2)}(G,M_1 \otimes M_2) = ker(d^m_{(2)})$, $B^m_{(2)}(G,M_1 \otimes M_2) = ker(d^{m-1}_{(2)})$ *and* $H^m_{(2)}(G,M_1 \otimes M_2) = Z^m_{(2)}(G,M_1 \otimes M_2)/B^m_{(2)}(G,M_1 \otimes M_2)$.

Since $H^m_{(1)}(G,M_1) = H^m(G,M_1)$, this extends the usual cohomology. We have:

**Theorem 1.** *Suppose that* $\Gamma_0(N)$ *acts diagonally on the tensor product in terms of the actions* $|_{2-k_1}$, $|_{2-k_2}$*. Then, the map* $r_{f_1,f_2}$ *induces an element* $[r_{f_1,f_2}]$ *of the group* $Z^1_{(2)}(\Gamma_0(N), D^{\omega,\infty,exc} \otimes D^{\omega,\infty,exc})$.

In the positive even weight case, Theorem 1 holds with the coefficient module replaced by the smaller space $\mathbb{C}_{k-2}[\tau]$ of polynomials of degree $\leq k-2$ over $\mathbb{C}$

**Corollary 1.** *Let* $k_1, k_2 \in 2\mathbb{N}$*. The map* $r_{f_1,f_2}$ *induces an element* $[r_{f_1,f_2}]$ *of* $Z^1_{(2)}(\Gamma_0(N), \mathbb{C}_{k_1-2}[\tau] \otimes \mathbb{C}_{k_2-2}[\tau])$.

## Applications to mulitple $L$-series

If $k_1, k_2 \in 2\mathbb{N}$ and $\frac{a}{b} \in \mathbb{Q}$, we define the *additive twist of the double* $L$*-series* by

$$L_{f_1,f_2}(a/b; s_1, s_2) = \sum_{n_1,n_2} \frac{c_1(n_1)c_2(n_2)e^{2\pi i(n_1+n_2)a/b}}{(n_1+n_2)^{s_1} n_2^{s_2}}$$

for all $\Re(s_1), \Re(s_2) \gg 1$, where $c_i(n)$ is the $n$-th Fourier coefficient of $f_i(\tau)$. Values of this function at $s_i \in \mathbb{N}$ in the special case of $a = 0$ have been studied by Manin, Choie, Provost and others. Because of the analogy of $L_{f_1,f_2}(0; s_1, s_2)$ with the double zeta function $\zeta(s_1,s_2)$, it is natural to ask if $L_{f_1,f_2}(a/b;s_1,s_2)$ satisfies relations analogous to those satisfied by $\zeta(s_1,s_2)$, e.g., for $2 \le j \le k/2$,

$$\mathrm{Z}(j) = \zeta(j)\zeta(k-j), \quad \text{where } \mathrm{Z}(j) := \sum_{\ell=2}^{k-1} \left( \binom{\ell-1}{j-1} + \binom{\ell-1}{k-j-1} \right) \zeta(k-\ell,\ell).$$

The following proposition allows us to deduce similar relations for $L_{f_1,f_2}$ from the cocycle relation of $r_{f_1,f_2}$:

**Proposition 3.** *i. For $\Re(s_1),\Re(s_2) \gg 1$, we have*

$$L_{f_1,f_2}(a/b;s_1,s_2) = \frac{(-2\pi i)^{s_1+s_2}}{\Gamma(s_1)\Gamma(s_2)} \Lambda_{f_1,f_2}(a/b;s_1,s_2) \quad \textit{where}$$

$$\Lambda_{f_1,f_2}\left(\frac{a}{b};s_1,s_2\right) = \int_{\frac{a}{b}}^{i\infty} f_1(w_1)\left(w_1 - \frac{a}{b}\right)^{s_1-1} \int_{w_1}^{i\infty} f_2(w_2)(w_2-w_1)^{s_2-1} dw_2 dw_1.$$

*ii. For each $\gamma \in \Gamma_0(N)$, we have*

$$r_{f_1,f_2}(\gamma)(\tau) = \sum_{n=0}^{k_1+k_2-4} \mathcal{L}_{f_1,f_2}(\gamma^{-1}i\infty, n)(\gamma^{-1}i\infty - \tau)^n$$

*where $\mathcal{L}_{f_1,f_2}\left(\frac{a}{b},n\right)$ is given by the sum*

$$\sum_{n_2=0}^{k_1+k_2-n-2} \binom{k_2-2}{n_2}\binom{k_1+k_2-n_2-4}{n} \Lambda_{f_1,f_2}\left(\frac{a}{b};k_1+k_2-n_2-n-3, n_2+1\right)$$

Combining with Corollary 1, we obtain

**Theorem 2.** *For each pair of $\gamma_1,\gamma_2 \in \Gamma_0(N)$, there are $k_1+k_2-3$ $\mathbb{Q}$-linear combinations of $\Lambda_{f_1,f_2}\left((\gamma_1\gamma_2)^{-1}i\infty;n,m\right)$, $\Lambda_{f_1,f_2}\left(\gamma_1^{-1}i\infty;n,m\right)$, $\Lambda_{f_1,f_2}\left(\gamma_2^{-1}i\infty;n,m\right)$ ($n \in [1,k_1-1], m \in [1,k_1+k_2-n-3]$), each of which equals a $\mathbb{Q}$-linear combination of products $\Lambda_{f_1}(\gamma_1^{-1}i\infty,n)\Lambda_{f_2}(\gamma_2^{-1}i\infty,m)$ ($1 \le n \le k_1-1$, $1 \le m \le k_2-1$).*

**Corollary 2.** *For each $k \in \mathbb{N}$, we have*

$$\mathrm{L}_{f_1,f_2}(k) = \left(\Lambda_{f_1}\left(\frac{-1}{N(k+1)},1\right) - \Lambda_{f_1}\left(\frac{-1}{Nk},1\right)\right)\Lambda_{f_2}\left(\frac{-1}{Nk},1\right), \quad \textit{where}$$

$$\mathrm{L}_{f_1,f_2}(k) := \mathcal{L}_{f_1,f_2}\left(\frac{-1}{N(k+1)};k_1+k_2-4\right)$$
$$-\sum_{n=0}^{k_1+k_2-4}(-k-1)^n(kN)^{k_1+k_2-4-n}\mathcal{L}_{f_1,f_2}\left(\frac{-1}{N};n\right) - \mathcal{L}_{f_1,f_2}\left(\frac{-1}{Nk};k_1+k_2-4\right).$$

*If $f_1,f_2$ are normalised Hecke eigenforms, then, all $\mathrm{L}_{f_1,f_2}(k)$ belong to a 4-dimensional vector space over the field generated by the Fourier coefficients of $f_1,f_2$.*

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## A Λ-adic Family of Funke–Millson Cycles and a Λ-adic Funke–Millson Lift


Paul Kiefer

(joint work with Lennart Gehrmann)


In my talk I reported on work in progress with Lennart Gehrmann about Λ-adic families of Funke–Millson cycles generalizing previous work by Glenn Stevens [6] in the case of the Shintani lift. Therefore, let $L$ be an even lattice of signature $(2, n)$ with quadratic form $q$ and

$$\mathcal{D} = \{z \subseteq L \otimes \mathbb{R} \mid \dim(z) = n, q|_z < 0\}$$

the Grassmannian of negative definite planes. It can be endowed with a complex structure and has complex dimension $n$. For a vector $v \in L$ with $q(v) > 0$ the real analytic submanifold

$$\mathcal{D}_v = \{z \in \mathcal{D} \mid z \perp v\}$$

has real dimension (and codimension) $n$. For a neat arithmetic subgroup $\Gamma \subseteq \mathrm{SO}(L)$, the quotient $\mathcal{C}_{v,0} = \Gamma_v \backslash \mathcal{D}_v$ can be embedded into $\Gamma\backslash\mathcal{D}$, so that we obtain a submanifold, which can be endowed with a natural orientation. Given a Schwartz function $\varphi \in \mathcal{S}(L \otimes \mathbb{A}_f)$ that is fixed by $\Gamma$, we define the weighted cycle

$$\mathcal{C}_{m,0} = \sum_{\substack{v \in \Gamma\backslash L\otimes\mathbb{Q} \\ q(v)=m}} \varphi(v)\mathcal{C}_{v,0},$$

called special cycle or Kudla–Millson cycle. The chosen Schwartz function $\varphi$ and the arithmetic subgroup $\Gamma$ will always be implicit in the notation. Integrating a compactly supported cohomology class $\eta \in H^n_c(\Gamma\backslash\mathcal{D})$ over this cycle yields, by Poincaré duality a cohomology class that we will also denote $\mathcal{C}_{m,0} \in H^n(\Gamma\backslash\mathcal{D})$ and we denote the corresponding pairing by $(\eta, \mathcal{C}_{m,0})$.

In [1], Funke and Millson promoted these special cycles for all $k \in \mathbb{N}$ to Funke–Millson cycles $\mathcal{C}_{m,k} \in H^n(\Gamma\backslash\mathcal{D}, V_k)$ with coefficients in the local system associated to a $\Gamma$-module $V_k$ and proved the following generalization of the results of Kudla–Millson [3].

**Theorem 1** ([1]). *The geometric theta function*

$$\Theta_k(\tau) = \sum_{m\in\mathbb{Q}_{>0}} \mathcal{C}_{m,k} e(m\tau), \qquad e(x) = e^{2\pi i x}, \tau \in \mathbb{H}$$

*is a cusp form for some congruence subgroup of* $\mathrm{SL}_2(\mathbb{Z})$ *with values in* $H^n(\Gamma\backslash\mathcal{D}, V_k)$*, i.e. for all* $\eta \in H^n_c(\Gamma\backslash\mathcal{D}, V_k^*)$ *the function*

$$(\eta, \Theta_k(\tau)) = \sum_{m\in\mathbb{Q}_{>0}} (\eta, \mathcal{C}_{m,k}) e(m\tau)$$

*is a modular form and we call the map* $\eta \mapsto (\eta, \Theta_k(\tau))$ *the Funke–Millson lift.*

This is a cohomological version of the result by Oda [4], who generalized work of Shintani [5] from signature $(2,1)$ to general signature $(2,n)$, see [2] for the signature $(2,1)$ case. The aim of the talk was to explain a $\Lambda$-adic version of this theorem.

Therefore, let $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]] = \varprojlim_n \mathbb{Z}_p[\mathbb{Z}_p^\times/(1+p^n\mathbb{Z}_p)]$. It has the universal property

$$\mathrm{Hom}_{\mathrm{cont.grp.}}(\mathbb{Z}_p^\times, \overline{\mathbb{Q}}_p^\times) = \mathrm{Hom}_{\mathrm{cont.alg.}}(\Lambda, \overline{\mathbb{Q}}_p).$$

In particular, the continuous group homomorphism $t \mapsto t^k$ induces a continuous algebra homomorphism $\lambda_k : \Lambda \to \overline{\mathbb{Q}}_p$ for all $k \in \mathbb{N}$. Moreover, we will fix an embedding $\overline{\mathbb{Q}}_p \subseteq \mathbb{C}$.

**Theorem 2.** *There is a* $\Lambda$*-module* $\mathbb{V}$ *with specialization maps*

$$H^n(\Gamma\backslash D, \mathbb{V}) \to H^n(\Gamma\backslash D, V_k)$$

*such that for a certain choice of Schwartz function* $\varphi \in \mathcal{S}(L\otimes\mathbb{A}_f)$ *and arithmetic subgroup* $\Gamma \subseteq \mathrm{SO}(L)$ *there is a cohomology class* $\mathcal{C}_m \in H^n(\Gamma\backslash D, \mathbb{V})$ *which is mapped to* $\mathcal{C}_{m,k}$ *under the specialization map for all* $k \in \mathbb{N}$*.*

We call the cohomology class $\mathcal{C}_m$ of the theorem a $\Lambda$-adic family of Funke–Millson cycles.

Further, there is another $\Lambda$-module $\mathbb{D}$ given by $\mathbb{Z}_p$-valued measures and specialization maps

$$H^n_c(\Gamma\backslash D, \mathbb{D}) \to H^n_c(\Gamma\backslash D, V_k^*), \qquad \eta \mapsto \eta_k$$

for all $k \in \mathbb{N}$. The following theorem yields a $\Lambda$-adic Funke–Millson lift.

**Theorem 3.** *There is a pairing*

$$H^n_c(\Gamma\backslash D, \mathbb{D}) \times H^n(\Gamma\backslash D, \mathbb{V}) \to \Lambda$$

*which satisfies*

$$\lambda_k((\eta, \mathcal{C}_m)) = (\eta_k, \mathcal{C}_{m,k}) \in \overline{\mathbb{Q}}_p \subseteq \mathbb{C}.$$

*In particular, the formal power series*

$$\Theta(\tau) = \sum_{m\in\mathbb{Q}_{>0}} \mathcal{C}_m e(m\tau) \in \Lambda[[e(\tau)]]$$

*is a* $\Lambda$*-adic family of modular forms with values in* $H^n(\Gamma\backslash\mathcal{D}, \mathbb{V})$ *in the sense that for all* $k \in \mathbb{N}$ *and all* $\eta \in H^n_c(\Gamma\backslash D, \mathbb{D})$ *the function*

$$\lambda_k((\eta, \Theta(\tau))) = \sum_{m\in\mathbb{Q}_{>0}} \lambda_k((\eta, \mathcal{C}_m)) e(m\tau) = \sum_{m\in\mathbb{Q}_{>0}} (\eta_k, \mathcal{C}_{m,k}) e(m\tau) = (\eta_k, \Theta_k(\tau))$$

*is a modular form.*

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## Murmurations


Min Lee

(joint work with Jonathan Bober, Andrew R. Booker, Claire Burrin, Vivian Kuperberg, David Lowry-Duda, Catinca Mujdei, Hsin-Yi Yang)


The notion of "Murmurations," originally describing the cloud-like movements of flocks of birds, is now used for phenomena related to elliptic curves and, more generally, families of $L$-functions. He, Lee, Oliver and Pozdnyakov considered the following average associated with elliptic curves and discovered that it shows cloud-like oscillating patterns [2]:

$$\frac{1}{\#\mathcal{E}_r[N_1, N_2]} \sum_{E \in \mathcal{E}_r[N_1, N_2]} a_p(E),$$

where $\mathcal{E}_r[N_1, N_2]$ is the set of elliptic curves (ordered by the size of the conductor) of rank $r$, conductor $N \in [N_1, N_2]$, and $a_p(E) = p + 1 - \#E(\mathbb{F}_p)$ for primes $p$ not dividing $N$. They named the pattern "murmurations" and sought mathematical explanations. Further numerical experiments followed [4], and the following were observed.

(1) There is a correlation of $a_p(E)$ with the root number of $E$ (not just rank).
(2) It is scale-invariant – the prime $p$ should be scaled relative to the conductor $N$.
(3) Murmuration is a more general phenomenon – it occurs in families of $L$-functions.

In his letter to Zubrilina and Sutherland [3], Sarnak explained the murmuration phenomena – a phase transition in the 1-level density of the low-lying zeros of these families.

For this talk, we focused on the murmurations of modular forms in different aspects.

For an even positive integer $k$ and a positive integer $N$, let $H_k(N)$ be an orthogonal basis of holomorphic cuspidal newforms of weight $k$ and level $N$, consisting of Hecke eigenfunctions. For primes $p$ not dividing $N$, let $\lambda_f(p)$ be the $p$th Hecke eigenvalue of $f \in H_k(N)$. By Weil, we know that the Ramanujan–Petersson conjecture is true in this case, so $|\lambda_f(p)| = O_f(1)$. Let $\epsilon_f \in \{-1, 1\}$ be the root number of $f$. The $L$-function of $f$ has conductor of size $N\big(\frac{k-1}{4\pi}\big)^2$.

In 2023 [5], Zubrilina achieved a breakthrough on the murmuration problem, proving the murmurations of modular forms in the level aspect. Fix $k \in 2\mathbb{Z}{>}\,0$. Let $X$, $Y$ and $P$ be parameters $\to \infty$ with $P$ prime. Assume that $Y = (1 + o(1))X^{1-\delta_2}$ and $P \ll X^{1+\delta_1}$ for $0 < 2\delta_1 < \delta_2 < 1$ and let $y = \frac{P}{X}$. Then

$$\frac{\sum_{\substack{N\in[X,X+Y]\\ \text{square-free}}} \sum_{f\in H_k(N)} \epsilon_f \lambda_f(P)\sqrt{P}}{\sum_{\substack{N\in[X,X+Y]\\ \text{square-free}}} \sum_{f\in H_k(N)} 1} = M_k(y) + O_\varepsilon\Big(X^{-\min\{\frac{\delta_2}{2}, \frac{1+\delta_2}{2}\}+\delta_1+\varepsilon} + P^{-1}\Big),$$

where

$$M_k(y) = \frac{12}{(k-1)\prod_p(1-(p^2+p)^{-1})}\bigg\{\prod_p\left(1+\frac{p}{(p+1)^2(p-1)}\right)\sqrt{y}$$
$$+(-1)^{\frac{k}{2}-1}\prod_p\left(1-\frac{p}{(p^2-1)^2}\right)\sum_{1\le r\le 2\sqrt{y}} c(r)\sqrt{4y-r^2}\,U_{k-2}\left(\frac{r}{2\sqrt{y}}\right) - \delta_{k=2}\pi y\bigg\}.$$

Here $U_{k-2}(\cos\theta) = \frac{\sin((k-1)\theta)}{\sin\theta}$ is the Chebyshev polynomial and

$$c(r) = \prod_{p|r}\left(1+\frac{p^2}{(p^2-1)^2-p}\right).$$

On the other hand, in 2023, Bober, Booker, Lowry-Duda and myself proved the murmurations of modular forms in the level aspect. Our motivation, suggested by Sarnak in a workshop at ICERM (July 2023), was to explore murmurations of non-arithmetic families of $L$-functions. As Sarnak observed in his letter [3], since the size of the family is small compared to the conductor in this case (in the level aspect $\sum_{N\le X} \#H_k(N) = O(X^2)$ and in the weight aspect $\sum_{k^2\le X} \#H_k(1) = O(X)$), we needed to take an extra average over primes $n$.

We proved the following result. Assume Generalised Riemann Hypothesis (GRH) for $L$-functions of Dirichlet characters and modular forms. Fix $\varepsilon \in (0, \frac{1}{12})$, $\delta \in \{0, 2\}$ (index for the root numbers, i.e., $\epsilon_f = (-1)^\delta$ for $f \in H_k(1)$ with $k \equiv 2\delta \bmod 4$) and compact interval $E \subset \mathbb{R}_{>0}$, $|E| > 0$. Let $K$, $H \in \mathbb{R}_{>0}$ with $K^{\frac{5}{6}+\varepsilon} < H < K^{1-\varepsilon}$ and set $N = \big(\frac{K-1}{4\pi}\big)^2$. As $K \to \infty$, we have

$$\frac{\sum_{\substack{n \text{ prime}\\ \frac{n}{N}\in E}} \log n \sum_{\substack{k\equiv 2\delta \bmod 4\\ |k-K|\le H}} \sum_{f\in H_k(1)} \lambda_f(p)}{\sum_{\substack{n \text{ prime}\\ \frac{n}{N}\in E}} \log n \sum_{\substack{k\equiv 2\delta \bmod 4\\ |k-K|\le H}} \sum_{f\in H_k(1)} 1} = \frac{(-1)^\delta}{\sqrt{N}}\left(\frac{\nu(E)}{|E|} + o_{E,\varepsilon}(1)\right),$$

where

$$\nu(E) = \frac{1}{\zeta(2)} \sideset{}{^*}\sum_{\substack{a,q\in\mathbb{Z},(a,q)=1,\\ \frac{q^2}{a^2}\in E}} \frac{\mu(q)^2}{\varphi(q)^2\sigma(q)}\frac{q^3}{a^3}$$
$$= \frac{1}{2}\sum_{t=-\infty}^{\infty}\prod_{p\mid t}\frac{p^2-p-1}{p^2-p}\int_E \cos\left(\frac{2\pi t}{\sqrt{y}}\right)\, dy.$$

Here the $*$ means terms are occurring at the end points of $E$ are halved.

Finally, and most recently, at the workshop WINE 5 (Women In Numbers in Europe 5) this August, Burrin, Kuperberg, Mujdei, Yang and myself proved the murmurations of modular forms in the "depth aspect," suggested by Booker in his talk at RIMS in 2023.

We proved the following result. Let $\ell > 2$ be a prime. Assume GRH for $L$-functions of Dirichlet characters. Fix a compact interval $E \subset \mathbb{R}_{>0}$ with $|E| > 0$. As $e \to \infty$, we have

$$\frac{\sum_{\substack{n \text{ prime}\\ \frac{n}{\ell^{2e+1}}\in E}} \log n \sum_{f\in H_k(\ell^{2e+1})} \epsilon_f \lambda_f(n)\sqrt{n}}{\sum_{\substack{n \text{ prime}\\ \frac{n}{\ell^{2e+1}}\in E}} \log n \sum_{f\in H_k(\ell^{2e+1})} 1} = \frac{1}{|E|}\frac{24(-1)^{\frac{k}{2}+1}}{\pi(k-1)}\prod_p \frac{p(p^2-p-1)}{(p^2-1)(p-1)}$$
$$\times \int_E \sum_{\substack{t\in\mathbb{Z}\\ |t|<2\ell x}} \frac{c_\ell(t)}{\ell-1}\prod_{p\mid \ell t}\left(\frac{p^2-p}{p^2-p-1}\right)\sqrt{4\ell^2 x - t^2}\,U_{k-2}\left(\frac{t}{2\sqrt{\ell x}}\right) dx$$
$$+ O_{\varepsilon,k,\ell,E}(\ell^{-\frac{e}{5}+\varepsilon}).$$

The methods are all based on the Eichler–Selberg trace formula.

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**Mahler measure and special values of $L$-function: Deninger's cohomological method**

Jungwon Lee
(joint work with Wei He)

The Mahler measure is defined, for a Laurent polynomial $P \in \mathbb{C}[x_1^{\pm}, \cdots, x_n^{\pm}]$ with complex coefficients, by

$$m(P) = \frac{1}{(2\pi i)^n} \int_{T^n} \log |P(x_1, \cdots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} \tag{1}$$

where $T^n$ denotes the real $n$-torus.

The quantity $m(P)$ naturally arises from the diverse contexts in number theory or dynamical system, for instance as a height function on polynomials or a topological entropy. Here, we focus on its mysterious connection to special values of $L$-functions.

One of the first relations between Mahler measure and special $L$-values goes back to Smyth and Boyd, who discovered

$$m(1 + x + y) = L'(-1, \chi_3),$$
$$m(1 + x + 1/x + y + 1/y) = r_E \cdot L'(0, E),$$

where $\chi_3 : (\mathbb{Z}/3\mathbb{Z})^\times \to \mathbb{C}^\times$ is the unique non-trivial Dirichlet character modulo 3, $E$ is an elliptic curve of conductor 15 defined by the given equation and $r_E \in \mathbb{Q}^\times$. There have been numerous such instances, which precisely take the form:

$$\frac{L'(0)}{m(P)} \in \mathbb{Q}^\times \tag{2}$$

where $P \in \mathbb{Q}[x_1^{\pm}, x_2^{\pm}]$, and either $L(s-1, \chi)$ or $L(s) = L(s, E)$. See [5, 2].

Naturally this lead to several questions, e.g. given a Laurent polynomial $P$, can we determine which $L$-function is related to $m(P)$? Conversely, given a Dirichlet character or an elliptic curve, is the special $L$-value always related to the Mahler measure of one or more polynomials? In general, these questions are out of reach.

In this note, we discuss the breakthrough work of Deninger, which partially and theoretically confirms an answer in the case of elliptic curves. We then describe our result that is a generalisation of Deninger's cohomological method for Dirichlet characters.

## 1. Deninger's method

We briefly outline the work of Deninger, later further refined by the work of Besser–Deninger.

In [3], Deninger showed that if $P$ does not vanish on $T^n$, then the Mahler measure of $P$ can be viewed as a Deligne period of the motive $Z_P$ associated to the variety defined by $P = 0$. This implied that the value is related to the Beilinson regulator map, hence to $L$-values.

Let $P \in \mathbb{C}[x_1^{\pm}, \cdots, x_n^{\pm}]$ be a non-zero Laurent polynomial and set $P^*$ to be the leading coefficient of $P$ as a polynomial in $x_n$. Let $Z_P \subset \mathbb{G}_{m,\mathbb{C}}^n$ be the zero locus of $P$ and $A_P := \{|z_1| = \cdots = |z_{n-1}| = 1, |z_n| \leq 1\}$ be the subspace.

- Suppose $P^* \neq 0$ on $T^{n-1}$, applying Jensen's formula, we have
$$m(P) = m(P^*) - \frac{1}{(2\pi i)^{n-1}} \int_{A_P} \log|x_n| \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}.$$
Denote by $\eta_P$ the smooth differential form in the integration.
- Reinterpret the integration as a period pairing of $\eta_P$ and $A_P$. To do so, one has to consider $Z_P^{\text{reg}} \subset Z_P$ on which $\eta_P$ is a closed form and $A_P$ is compactly embedded. Then the integral becomes
$$\langle r_D(\{P, x_1, \cdots, x_n\}), [A_P] \otimes (2\pi i)^{1-n} \rangle$$
where $r_D$ denotes the Beilinson regulator applied to a motivic cohomology class induced by $P$ and $\log|x_i|$. Abusing the notation, denote by $r_D$ this pairing in the below.

This suggests that given $P$ satisfying certain non-vanishing assumptions, the Mahler measure is given by the regulator map, accordingly related to certain $L$-function associated to the motive defined by $P$. Conversely, given an arithmetic object, one can try to find a corresponding polynomial in this cohomological context. Indeed, in [1], they proved that there exists $P_E \in \mathbb{Q}[x_1, x_2]$ that corresponds $E_{/\mathbb{Q}}$ with CM by imaginary quadratic fields of class number 1 and with technical assumptions such that

$$m(P_E) - m(P_E^*) = r_E \cdot L'(0, E), \quad r_E \in \mathbb{Q}^\times. \tag{3}$$

## 2. Result

We now state our main result, which is a $\mathrm{GL}_1$-analogue of (3) for Dirichlet $L$-values. The proof rests on Deninger's foundational work described in Section 1 with an appropriate choice of the polynomial and motive attached.

Let $H_M^*$ and $H_D^*$ denote the Motivic and Deligne cohomology group respectively.

**Theorem 1** (He.-L [4], 2025). *There exists $\Psi := \Psi_N \in \mathbb{Z}[x_1, x_2]$ associated to $N$-th cyclotomic polynomials such that the regulator map*
$$r_D : H_M^2(Z_\Psi, \partial A_\Psi, \mathbb{Q}(2)) \to H_D^2(Z_\Psi, \partial A_\Psi, \mathbb{R}(2)) \xrightarrow{\frac{1}{2\pi i}\int_{A_P}} \mathbb{C}$$
*is given by*
$$r_D(\{\Psi, x_1, x_2\}) = m(\Psi) - m(\Psi^*) = \sum_{\chi \in \widehat{G}} r_\chi \cdot L'(-\varepsilon, \chi)$$
*where $G = \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$, $\varepsilon = 0$ when $\chi$ is even and $\varepsilon = 1$ when $\chi$ is odd.*

Then we have a natural short exact sequence

$$0 \to H_M^1(X_\Psi, \mathbb{Q}(2)) \to H_M^2(Z_\Psi, \partial A_\Psi, \mathbb{Q}(2))^{i=-1} \to H_M^1(Y_\Psi, \mathbb{Q}(1)) \to 0 \tag{4}$$

induced from the long exact sequence of relative motivic cohomologies for some suitable integral models $X_\Psi$ and $Y_\Psi$ of $\mathrm{Spec}(\mathbb{Q}(\mu_N))$, where $i$ is induced by the involution on $(Z_\Psi, \partial A_P)$ from the interchange of two variables.

**Theorem 2** (He.-L [4], 2025)**.** *Under certain $\mathbb{Q}$-linear independence assumption on partial $L$-values at $0$ and $-1$, we have a canonical splitting*

$$H^2_M(Z_\Psi, \partial A_\Psi, \mathbb{Q}(2))^{i=-1} \cong H^1_M(X_\Psi, \mathbb{Q}(2)) \oplus H^1_M(Y_\Psi, \mathbb{Q}(1))$$

*of the exact sequence* (4) *that is compatible with the regulator maps.*

*Thus we have an induced $G$-module structure on $H^2_M$ and for each $\chi \in \widehat{G}$, we have $r_D(\{\Psi, x_1, x_2\}^\chi) = r_\chi \cdot L'(-\varepsilon, \chi)$.*

*Concluding remarks.*

In Theorem 1, we remark that it is given by a linear combination of $L$-values, contrast to the Besser–Deninger formula. This appearance is essentially due to the Galois conjugate structure in our $\mathrm{GL}_1$ setting.

In Theorem 2, we show that one can extract the single $L$-values by splitting the exact sequence under extra $\mathbb{Q}$-linear independent assumption. Hence it also strongly suggests an answer to the converse question; Given a Dirichlet character $\chi$ modulo $N$, it is likely to have some $\Psi_N^\chi \in L[x_1, x_2]$ such that $m(\Psi^\chi)$ is related to $L'(-1, \chi)$, where $L$ is a number field containing the Hecke field $\mathbb{Q}(\chi)$.

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## Analytic properties of the Dedekind zeta function over $\mathbb{Z}_p$-extensions

Dohyeong Kim

(joint work with Harin Jung)

Let $p$ be a prime and $L/K$ a $\mathbb{Z}_p$-extension of number fields. For each $n \geq 0$, the extension $L/K$ has a unique subextension of degree $p^n$ which we denote by $L^{(n)}$. The $p$-primary subgroup of the ideal class group of $L^{(n)}$ shall be denoted by $A_n$. Iwasawa's celebrated theorem asserts that there is a strong regularity among $A_n$'s. Precisely, he showed that there are three integers $\mu, \lambda$ and $\nu$, with $\mu, \nu \geq 0$, such that the size of $A_n$ is $p^{e_n}$ with $e_n = \mu p^n + \lambda n + \nu$, for all sufficiently large $n$.

Are there other regularities in $\mathbb{Z}_p$-extensions? Our aim is to exhibit a regularity among the Dedekind zeta functions of $L^{(n)}$'s by studying the asymptotic behavior of the Euler–Kronecker constants along $\mathbb{Z}_p$-extensions. The aforementioned constant generalizes the better-known constant $\gamma$ named after Euler and Mascheroni, which admits a concise definition:-

$$\gamma := \lim_{n\to\infty}\left(\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)-\log n\right).$$

Approximately, one has $\gamma \approx 0.577$. Ihara [1] introduced an invariant $\gamma_F \in \mathbb{R}$ for a number field $F$ in a way that implies $\gamma_{\mathbb{Q}} = \gamma$. The definition of $\gamma_F$ is as follows. Consider the Dedekind zeta function $\zeta_F(s)$, which is given by the Dirichlet series – convergent when the real part of $s$ is greater than 1 –

$$\zeta_F(s) = \sum_{\mathfrak{a}} \mathrm{N}(\mathfrak{a})^{-s}$$

where $\mathfrak{a}$ runs over the set of all nonzero integral ideals of $F$. Here, $\mathrm{N}(\mathfrak{a})$ denotes the norm of $\mathfrak{a}$. It is well-known that $\zeta_F(s)$ has a meromorphic continuation to $s \in \mathbb{C}$ and has a unique pole at $s = 1$ which is simple. Therefore, the Laurent series expansion at $s = 1$ takes the form

$$\zeta_F(s) = \frac{c_{-1}}{s-1} + c_0 + \cdots .$$

From the Laurent series expansion, one simply defines $\gamma_F := c_0/c_{-1}$, which we call the Euler–Kronecker constant of $F$.

What kind of arithmetic information does $\gamma_F$ encode? Ihara, based on his explicit formula for $\gamma_F$, could justify the phenomenon that $\gamma_F$ tends to be located near $-\infty$ when $F$ has many primes of small norms. At this point it is instructive to consider $\mathbb{Q}(\zeta_m)$, where $m \geq 1$ is an integer and $\zeta_m$ a primitive $m$-th root of unity. It seemed "fairly likely" that $\gamma_{\mathbb{Q}(\zeta_m)} > 0$ always holds [1]. However, it was discovered later [2] that $\gamma_{\mathbb{Q}(\zeta_m)} \approx -0.1823$ when $m = 964477901$. Moreover, the Hardy–Littlewood conjecture predicts that such negative values occur infinitely often [2]. Nevertheless, the same conjecture predicts that the positivity holds with density one. See [2] for a further discussion.

We would like to explore the connection between the sign of $\gamma_F$ and the presence of many primes of small norms from a different perspective. Instead of asking whether $\gamma_F > 0$ holds, one may consider two number fields $F_1$ and $F_2$ for which one asks whether the difference $\gamma_{F_1} - \gamma_{F_2}$ is positive. Here one may impose the condition that $F_1$ and $F_2$ have the same degree and their discriminants are comparable. Then one may put forward the working hypothesis: $\gamma_{F_1} - \gamma_{F_2}$ is positive if $F_2$ has more primes of small norm than $F_1$ has.

Our main result supports the above working hypothesis. To state it, we need to recall some standard notions about $\mathbb{Z}_p$-extensions. First, we recall that $\mathbb{Q}$ has a unique $\mathbb{Z}_p$-extension which we denote by $\mathbb{Q}_{\mathrm{cyc}}$ and call the cyclotomic $\mathbb{Z}_p$-extension. Its $n$-th layer $\mathbb{Q}_{\mathrm{cyc}}^{(n)}$ is the unique subextension of degree $p^n$ in $\mathbb{Q}(\zeta_{p^{n+1}})$. For any number field $K$, define $K_{\mathrm{cyc}}$ to be the compositum of $K$ and $\mathbb{Q}_{\mathrm{cyc}}$, which is a $\mathbb{Z}_p$-extension of $K$. We call it the cyclotomic $\mathbb{Z}_p$-extension of $K$. On the other hand, if $L/K$ is a $\mathbb{Z}_p$-extension and there is a subfield $K' \subset K$ of degree two such that $L/K'$ is Galois with a non-abelian Galois group, then we call $L/K$ a $\mathbb{Z}_p$-extension of anti-cyclotomic type. Such extensions exist, for example, if $K$ contains a totally imaginary extension of a totally real field, also known as a CM field. If $K$ is totally real, the Leopoldt conjecture for $K$ is equivalent to the assertion that there is only one $\mathbb{Z}_p$-extension which must be the cyclotomic $\mathbb{Z}_p$-extension. The Leopoldt conjecture is known if $K$ is abelian over $\mathbb{Q}$ but not known in general. Having recalled the necessary notions about $\mathbb{Z}_p$-extensions, we proceed to point out a key difference between the cyclotomic and anticyclotomic ones. In the cyclotomic $\mathbb{Z}_p$-extension, no primes split completely. In contrast, in any $\mathbb{Z}_p$-extension of anti-cyclotomic type half of the primes in $K$ split completely. Combining it with the working hypothesis, one obtains a concrete statement, which we were able to prove albeit conditionally.

**Theorem.** Assume the generalized Riemann hypothesis. If $K$ is a number field and $L/K$ is a $\mathbb{Z}_p$-extension of anti-cyclotomic type, then

$$\gamma_{K_{\mathrm{cyc}}^{(n)}} > \gamma_{L^{(n)}} \tag{1}$$

for all sufficiently large $n$.

Our proof uses the explicit formula of Ihara and, therefore, relies on the generalized Riemann hypothesis.

To conclude, we would like to remind the reader of the Iwasawa's formula for size of $p$-primary class groups along a $\mathbb{Z}_p$-extension. Our result shows that such regularities is not contrained to the $p$-primary class groups and can be found among the Euler–Kronecker constants. The author speculates that the observed regularity among the Euler–Kronecker constants is an evidence for other patterns to be found among the Dedekind zeta functions of layers of a $\mathbb{Z}_p$-extension.

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## Distribution of Manin's iterated integrals

Morten S. Risager
(joint work with Y. Petridis and Nils Matthes)

Consider $f_1, \dots, f_l$ holomorphic cusp forms of weight 2 for the congruence group $\Gamma_0(q)$. Manin [5] considered iterated integrals

$$I_a^b(f_1, \dots, f_l) = \int_a^b f_1(z_1) \int_a^{z_1} f_2(z_2) \int_a^{z_2} f_3(z_3) \cdots \int_a^{z_{l-1}} f_l(z_l) dz_l dz_{l-1} \cdots dz_1$$

These are fascinating generalizations of the classical modular symbols, and they have many properties analogous to multiple zeta values; for instance they satisfy the following shuffle product relation

$$I_a^b(f_1,\dots,f_l)\cdot I_a^b(f_{l+1},\dots,f_{l+m})=\sum_{\sigma\in\Sigma_{l,m}} I_a^b(f_{\sigma(1)},\dots,f_{\sigma(l+m)}) \tag{1}$$

where $\Sigma_{l,m}$ is the set of permutations $\sigma$ on $l+m$ elements satisfying

$$\sigma^{-1}(1)<\dots<\sigma^{-1}(l) \text{ and } \sigma^{-1}(1+l)<\dots<\sigma^{-1}(m+l).$$

One also shows that if $a=i\infty$ and $b=\gamma(i\infty)$ where $\gamma\in\Gamma_0(q)$ they are equal to the central value of a certain multiple $L$-series twisted by an additive character. Let

$$T(M)=\{\tfrac{a}{c}\in\mathbb{Q}\cap[0,1): c\le M, q|c, (a,c)=1\}.$$

We study the asymptotic behavior as $M$ goes to infinity of the random variable

$$Z_M(A)=\frac{\{\frac{a}{c}\in T(M):\widetilde{I_{i\infty}^{\frac{a}{c}}}(f_1,\dots,f_l)\in A\}}{\#T(M)} \tag{2}$$

where

$$\widetilde{I_{i\infty}^{\frac{a}{c}}}(f_1,\dots,f_l)=\left(\frac{\mathrm{vol}(\Gamma_0(q)\backslash\mathbb{H})}{8\log c}\right)^{l/2} I_{i\infty}^{\frac{a}{c}}(f_1,\dots,f_l)$$

is a normalization of the iterated integral, and $A$ is a Borel set.

If one specializes to $l=1$ which is the case of the classical modular symbols, then we prove the following results

**Theorem 1.** *Let $l=1$ and assume that the Petersson norm of $f$ equals $\|f\|=1$. Then $Z_M$ converges in distribution to the complex Gaussian, i.e. the random variable $Z$ with density $\frac{1}{\pi}e^{-|z|^2}$.*

This theorem was conjectured by Mazur and Rubin [7], and proved by Petridis and Risager [9]. Different proofs were given by Constantinescu [3], Sun and Lee [10], Bettin and Drappeau [1], Nordentoft [8], and Matthes and Risager [6]. It was furthermore extended to general weight by Nordentoft [8] and by Bettin and Drappeau [1], to Bianchi groups by Constantinescu [3], and to Maass forms by Drappeau and Nordentoft [4].

For $l=2$ the situation is not as precisely understood.

**Theorem 2** (Matthes-Risager [6])**.** *Let $l=2$. The random variable $Z_M$ converges in distribution to a radially symmetric distribution $Z_{f_1,f_2}$ which depends only on the Gram matrix $\{\langle f_i,f_j\rangle\}_{i,j=1,2}$ of the Petersson inner products of $f_1$, $f_2$.*

We know the precise form of $Z_{f_1,f_2}$ only in two extreme cases:

(1) If $f_1=f_2$ with $\|f_i\|=1$ then $Z_{f_1,f_2}$ has the Kotz-type distribution with density $\frac{1}{\pi|z|}e^{-2|z|}$. This follows from the shuffle relation which gives that in this case $I_a^b(f_1,f_1)=I_a^b(f_1)^2/2$, and the Kotz-type distribution is the square of the complex Gaussian.

(2) If $f_1$,$f_2$ forms an orthonormal set then $Z_{f_1,f_2}$ has distribution function

$$\frac{1}{4}\int_0^1 \frac{1}{y(1-y)} \frac{\sinh\left(\frac{\pi|z|}{2\sqrt{y(1-y)}}\right)}{\cosh^2\left(\frac{\pi|z|}{2\sqrt{y(1-y)}}\right)} dy.$$

Proving this involves proving a non-trivial identity relating shuffling coefficients to Euler numbers.

For $l = 3$ we understand even less

**Theorem 3** (Matthes-Risager [6])**.** *Let $l = 3$. All asymptotic moments of $Z_M$ exist and are finite, and there exists at least one (but possibly infinitely many) radially symmetric distributions with these moments.*

Unfortunately we do not know convergence in distribution to any of the possible limit distributions indicated in Theorem 3, except in the case of $f_1 = f_2 = f_3$ with $\|f_i\| = 1$ when $Z_M$ converges in distribution to the Kotz-type distribution with distribution function

$$\frac{12}{\pi} \frac{e^{-|6z|^{2/3}}}{|6z|^{4/3}}.$$

This distribution function is known to be indeterminate, i.e. there are infinitely many distribution functions with the same moments as this distribution.

To prove the above theorems we compute all asymptotic moments of $Z_M$, but for the method of moments to apply these asymptotic moments should determine a unique distribution; as the $f_1 = f_2 = f_3$ case shows this is not the case in general. It would be interesting to see if Stein's method would allow us to determine the distribution.

To compute the moments we investigate a twisted Eisenstein series defined by Chinta, Horozov, and O'Sullivan [2]. By analyzing the analytic properties and the Fourier coefficients of these twisted Eisenstein series we obtain asymptotic formulas for the moments of $Z_M$.

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## On completeness of Poincaré series for $SU(2,1)$

Roberto J. Miatello

In the paper [2] we define and study generalized Poincaré series on the Lie group $SU(2,1)$ and prove completeness results in spaces of cusp forms.

Classically, the idea of a Poincaré series is to take as a germ a simple function $h$ on a space $X$ and to form the sum of transformed functions $x \mapsto \sum_{\gamma\in\Gamma} h(\gamma x)$. If this sum converges absolutely one has a $\Gamma$-invariant function on $X$. Poincaré (1882) used this idea to construct what he called *fonctions thêtafuchsiennes*.

For the well-known cuspidal Poincaré series in the theory of holomorphic modular forms, the germ $h(z) = e^{2\pi i n z}$, $n \in Z_{\geq 1}$ on the upper half-plane is invariant under the transformation $z \mapsto z+1$. With a suitable automorphy factor, it leads to series that converge absolutely, where the sum is over $\Gamma_\infty \backslash SL_2(Z)$, and $\Gamma_\infty$ is generated by $\pm\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$. Petersson (1932) showed that the non-zero series of this type span the spaces of holomorphic cusp forms.

Now, the function

$$h_s(x+iy) = e^{2\pi i n x}\, y^{1/2}\, I_{s-1/2}(2\pi|n|y)$$

on the complex upper half-plane, with $I_{s-1/2}$ an exponentially increasing modified Bessel function, is an eigenfunction of the hyperbolic Laplace operator on the upper half-plane $H$ and, for $Re\, s > 1$, the sum $\sum_{\gamma\in\Gamma_\infty\backslash SL_2(Z)} h_s(\gamma z)$ converges absolutely and defines a real-analytic Maass form with exponential growth. The resulting family of functions, called real-analytic Poincaré series, have a meromorphic continuation to the complex plane and the spaces of Maass cusp forms are spanned by residues of these families. (See Neunhöffer [5], Niebur [6].)

In [8], by extending the work of Neunhöffer ([5]) and Niebur ([6]) for $G = SL(2,R)$ to semisimple Lie groups of real rank one, Miatello and Wallach defined families of Poincaré series attached to a unitary character $\chi$ of $\Gamma\cap N\setminus\Gamma$. For such a group $G = NAK$ and $\chi$ a unitary character of $\Gamma\cap N\setminus N$, set

$$\mathbb{M}^\chi(\xi,\nu,g,\phi) = \sum_{\gamma\in\Gamma\cap N\setminus\Gamma} M^\chi(\xi,\nu,\gamma g,\phi) \tag{1}$$

for $M^\chi(\xi,\nu,g,\phi)$ an eigenfunction of the Casimir operator on $G$, $\xi \in \hat{M}$ and $Re(\nu) > \rho$. It was proved that such a series is absolutely convergent for $Re(\nu) > \rho$ and extends meromorphically to $\mathbb{C}$, with possible simple poles at spectral values of $\nu$. By taking the residues at these poles and certain special values, one obtains square integrable automorphic forms. By computing the inner product of such a residue or value with a square integrable automorphic form $f$ one obtains a non-zero multiple of $c_\chi(f)$, the $\chi$-Fourier coefficient of $f$. As a consequence, by collecting the residues and values of the family $\mathbb{M}^\chi(\xi,\nu,g,\phi)$ one can detect all automorphic forms $f$ such that $c_\chi(f) \neq 0$ for the given $\chi$.

This method yields most square integrable automorphic forms for the group $G = SO(n,1)$, since the unipotent subgroup $N$ is abelian. However, for other real rank one groups such as $SU(n,1)$ and $Sp(n,1)$, there are many non-zero automorphic forms for which all Fourier coefficients $c_\chi(f)$ are equal to zero, and hence cannot be detected by using a series as above. This happens, for instance, for automorphic forms $f$ in the holomorphic and antiholomorphic discrete series for the group $\mathrm{SU}(n,1)$, for $n \geq 2$ (see Gelbart-Piatetski-Shapiro [4]).

In recent years, in collaboration with Roelof Bruggeman we have constructed, in the particular case of the group $G = \mathrm{SU}(2,1)$, a complete set of automorphic forms by means of generalized Poincaré series. For this purpose, we defined Poincaré series attached to each irreducible, unitary, infinite dimensional representation of the unipotent subgroup $N$ (not just unitary characters) with the goal of detecting all square integrable $\Gamma$-automorphic forms on $G$.

Then, for each $\mathcal{N}$ a realization of an irreducible unitary representation of $N$ in $L^2(\Gamma \cap N \setminus N)$, in [1] we constructed a family $M^{\mathcal{N}}(\xi,\nu,g,\phi)$ of eigenfunctions of the Casimir operator of $G$, and defined a Poincaré series of the form

$$\mathbb{M}^{\mathcal{N}}(\xi,\nu,g,\phi) = \sum_{\gamma\in\Gamma\cap N\setminus\Gamma} M^{\mathcal{N}}(\xi,\nu,\gamma g,\phi).$$

Furthermore, we give the meromorphic continuation of the family and we prove that the poles are simple (except possibly for $\nu = 0$, that is a double pole) and occur at spectral parameters of square integrable representations. These Poincaré series have in general exponential growth.

In [2] we use results on abelian and non-abelian Fourier term modules obtained in [1] to compute the inner product of truncations of these series with square integrable automorphic forms, in connection with their Fourier expansions. As a consequence, we obtain general completeness results for $\mathrm{SU}(2,1)$ that, in particular, generalize those valid for the classical holomorphic (and antiholomorphic) modular forms.

We do expect that an extension of this method applied to a similar family, should hold for $SU(n,1)$ for all values of $n$ and, possibly, also for all real rank one groups.

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## On bases of alternating and cyclotomic multiple zeta values in positive characteristic

Bo-Hae Im
(joint work with Bo-Hae Im, Hojin Kim, Khac Nhuan Le, Tuan Ngo Dac, Lan Huong Pham)

In this talk, we completely establish, for all positive integer N, Zagier-Hoffman's conjectures for $N$th cyclotomic multiple zeta values in positive characteristic. By working with the tower of all cyclotomic extensions, we present a proof that is uniform on $N$ and give an effective algorithm to express any cyclotomic multiple zeta value in the chosen basis. This generalizes all previous work on these conjectures for MZV's and alternating MZV's in positive characteristic [2].

Let $q$ be a power of a prime number $p$, and let $\mathbb{F}_q$ denote the finite field with $q$ elements. Carlitz introduced analogues of the classical Riemann zeta function in this setting, leading to the notion of MZVs in positive characteristic. These values have been extensively studied due to their connections to arithmetic geometry and transcendence theory.

Let $A = \mathbb{F}_q[\theta]$ be the polynomial ring in the variable $\theta$ over a finite field $k := \mathbb{F}_q$ of $q$ elements and characteristic $p > 0$. We denote by $A_+$ the set of monic polynomials in $A$. Let $K = \mathbb{F}_q(\theta)$ be the fraction field of $A$ equipped with the rational point $\infty$. Let $K_\infty = k((1/\theta))$ be the completion of $K$ at $\infty$, and let $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure $\bar{K}$ of $K$ at $\infty$.

We fix $N \in \mathbb{N}$ and denote by $k_N \subset \overline{k}$ the cyclotomic field over $k$ generated by a primitive $N$th root of unity $\zeta_N$. The group of $N$th roots of unity is denoted by $\Gamma_N$, whose cardinality is denoted by $\gamma_N$. We put

$$A_N = k_N[\theta], \quad K_N = k_N(\theta), \quad \text{and} \quad K_{N,\infty} = k_N((1/\theta)).$$

Cyclotomic MZVs extend classical MZVs by incorporating $N$th roots of unity. For $\epsilon_1, \dots, \epsilon_r \in \Gamma_N$, the group of $N$th roots of unity, and positive integers $n_1, \dots, n_r$ with $(n_r, \epsilon_r) \neq (1, 1)$, one defines

$$\zeta\begin{pmatrix}\epsilon_1 & \dots & \epsilon_r\\ n_1 & \dots & n_r\end{pmatrix} = \sum_{0<k_1<\dots<k_r} \frac{\epsilon_1^{k_1}\dots\epsilon_r^{k_r}}{k_1^{n_1}\dots k_r^{n_r}},$$

where $k_i$ are integers.

The main focus of this work is to provide a complete description of the vector spaces spanned by these cyclotomic MZVs, confirming the positive characteristic analogues of the Zagier-Hoffman conjectures.

**Hoffman-like Basis** ([3, Theorem A])**.** For each positive integer $N$ and weight $w$, let $CS_{N,w}$ denote the vector space over $\mathbb{K}_N$ spanned by all $N$th cyclotomic MZVs of weight $w$ and let $CS_{N,w}$ be the subset consisting of cyclotomic Carlitz multiple polylogarithms $\mathrm{Li}\begin{pmatrix}\epsilon_1 & \dots & \epsilon_r\\ n_1 & \dots & n_r\end{pmatrix}$ such that $q \nmid n_i$ for all $i$. Then, $CS_{N,w}$ forms a basis of $CN_{N,w}$.

**Dimension Formula - Zagier-type** ([3, Theorem B])**.** Let $d_N(w)$ denote the dimension of $CS_{N,w}$. Then we find an explicit recurrence formula for $d_N(w)$ as follows:

$$d_N(w) = \begin{cases} 1 & \text{if } w = 0,\\ \gamma_N(\gamma_N+1)^{w-1} & \text{if } 1 \le w < q,\\ \gamma_N((\gamma_N+1)^{w-1}-1) & \text{if } w = q,\\ \gamma_N\left(\sum_{i=1}^{q-1} d_N(w-i)\right) + d_N(w-q) & \text{if } w > q.\end{cases}$$

The main ingredients of the proofs include the following:

- **Anderson $t$-motives and Carlitz modules:** These structures provide a natural framework to encode the arithmetic of cyclotomic MZVs.
- **ABP criterion (Anderson–Brownawell–Papanikolas):** A central tool for establishing linear independence of MZVs. It can be stated as follows:

  **Theorem 1** (ABP Criterion [1, Theorem 3.1.1])**.** *Let $\Phi \in \mathrm{Mat}_\ell(\bar{K}[t])$ be a matrix such that*

  $$\det\Phi = c(t-\theta)^s$$

  *for some $c \in K$ and $s \in \mathbb{Z}_{\ge 0}$. Let $\psi \in \mathrm{Mat}_{\ell\times 1}(\mathcal{E})$ (see* [3, §1.1] *for $\mathcal{E}$) be a vector satisfying*

  $$\psi^{(-1)} = \Phi\psi$$

  *and let $\rho \in \mathrm{Mat}_{1\times\ell}(\bar{K})$ be such that*

  $$\rho\,\psi(\theta) = 0.$$

  *Then there exists a vector $P \in \mathrm{Mat}_{1\times\ell}(\bar{K}K[t])$ such that*

  $$P\psi = 0 \quad \textit{and} \quad P(\theta) = \rho.$$

Treating all cyclotomic extensions together allows a uniform approach to arbitrary $N$ and yields explicit formulas expressing any cyclotomic MZV in terms of the basis.

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## On local submodule zeta functions of nilpotent incidence algebras

Bianca Marchionna

**(Local) submodule zeta functions.** The submodule zeta functions of non-unital matrix algebras $\mathcal{E} \leq \mathrm{Mat}_n(\mathbb{Z})$ encode, within a Dirichlet series, how the number of $\mathcal{E}$-invariant $\mathbb{Z}$-submodules of $\mathbb{Z}^n$ grows by their index. In detail, for every $m \geq 1$, let $a_m(\mathcal{E} \curvearrowright \mathbb{Z}^n)$ be the number of $\mathbb{Z}$-submodules $L \leq \mathbb{Z}^n$ which are $\mathcal{E}$-invariant, i.e., $L \cdot \xi \subseteq L$ for every $\xi \in \mathcal{E}$, and have index $m$ in $\mathbb{Z}^n$. The *submodule zeta function of* $\mathcal{E} \curvearrowright \mathbb{Z}^n$ is the following Dirichlet series in the complex variable $s$:

$$\zeta_{\mathcal{E}\curvearrowright\mathbb{Z}^n}(s) := \sum_{m=1}^{\infty} a_m(\mathcal{E} \curvearrowright \mathbb{Z}^n) m^{-s}.$$

The polynomial growth of $\big(a_m(\mathcal{E} \curvearrowright \mathbb{Z}^n)\big)_{m\geq 1}$ in $m$ guarantees the absolute convergence of the series above in some non-empty half-plane $\{s \in \mathbb{C} \mid \Re(s) \geq \alpha\}$.

The previous notions can be extended verbatim by replacing the coefficient ring $\mathbb{Z}$ with any unital commutative ring $R$ for which $\big(a_m(\mathcal{E} \curvearrowright R^n)\big)_{m\geq 1}$ grows polynomially in $m$. A remarkable case in which this holds is when $R$ is the ring $\mathbb{Z}_p$ of $p$-adic integers. One refers to *local submodule zeta functions* whenever one deals with $\zeta_{\mathcal{E}\curvearrowright\mathbb{Z}_p^n}(s)$ for some $\mathcal{E} \leq \mathrm{Mat}_n(\mathbb{Z}_p)$ and some prime $p$. The Euler product decomposition

$$\zeta_{\mathcal{E}\curvearrowright\mathbb{Z}^n}(s) = \prod_{p\,\mathrm{prime}} \zeta_{\mathcal{E}\otimes\mathbb{Z}_p\curvearrowright\mathbb{Z}_p^n}(s), \tag{1}$$

links the relevant submodule zeta functions over $\mathbb{Z}$ and over every localisation of $\mathbb{Z}$ at a prime ideal, i.e., over $\mathbb{Z}_p$ for every prime $p$.

The study of local submodule zeta functions is generally more favourable than the corresponding "integral counterpart". A motivation is their expression via $p$-adic integrals. Following [3], for every $\mathbb{Z}_p$-algebra $\mathcal{E} \leq \mathrm{Mat}_n(\mathbb{Z}_p)$ one has

$$\zeta_{\mathcal{E}\curvearrowright\mathbb{Z}_p^n}(s) = \frac{1}{(1-p^{-1})^n} \int_{\mathcal{V}_\mathcal{E}} \prod_{i=1}^{n} |x_{ii}|^{s-i} \mathrm{d}\mu(x), \qquad \Re(s) \geq n \tag{2}$$

where $\mathcal{V}_\mathcal{E}$ is the set of all upper triangular matrices $x \in \mathrm{Tr}_n(\mathbb{Z}_p)$ such that $\mathbb{Z}_p^n x$ is $\mathcal{E}$-invariant in $\mathbb{Z}_p^n$, and $\mu$ is the probabilistic Haar measure on $\mathrm{Tr}_n(\mathbb{Z}_p)$. The fact before is heavily based on the compactness of $\mathbb{Z}_p$ and the presence of a Hermite normal form for matrices over $\mathbb{Z}_p$. The expression in (2) may be helpful for performing some explicit computations for small values of $n$. Du Sautoy and Grunewald [2] observed that $p$-adic cone integrals (as the one in (2)) determine rational functions in $p^{-s}$ with integral coefficients. In our case, this implies that $(a_m(\mathcal{E} \curvearrowright \mathbb{Z}_p^n))_{m\geq 1}$ satisfies a linear recurrence and provides a meromorphic continuation of $\zeta_{\mathcal{E}\curvearrowright\mathbb{Z}_p^n}(s)$. However, deriving an explicit formula for this rational function necessitates a resolution of singularities, which makes the problem consistently challenging in general. Alongside this approach, Rossmann [5, 6] and Voll [8] introduced other effective methods based on the theories of generating functions of rational cones and Bruhat–Tits buildings, respectively. These facilitate the production of explicit formulae when $\mathcal{E}$ varies within specific classes of (usually nilpotent) algebras (cf. [1, 5, 6, 7, 9]).

The difficulty of finding general formulae remains one of the primary obstacles in the study of submodule zeta functions. However, some general patterns have begun to surface. For instance, Rossmann [5, Conj. IV] conjectured that, whenever $\mathcal{E}$ is nilpotent, the value at $s = 0$ of $\zeta_{\mathcal{E}\curvearrowright\mathbb{Z}_p^n}(s)/\zeta_{\{0\}\curvearrowright\mathbb{Z}_p^n}(s)$ does not depend on $\mathcal{E}$. Below, we outline a proof of a weaker version of this conjecture for nilpotent incidence algebras, using an approach that avoids direct computations.

**The case of nilpotent incidence algebras.** Nilpotent incidence matrix algebras are defined by means of finite natural posets. A finite poset $\mathbb{P} = (P, \leq_P)$ with $P \subseteq \{1, \dots, n\}$ (possibly empty) and $n \in \mathbb{Z}_{\geq 1}$, is *natural* if $i \leq_P j$ implies $i \leq j$. Given $\mathbb{P}$ as before and a ring $R$, the *nilpotent incidence $R$-algebra $\mathcal{E}_{\mathbb{P},n}(R)$ associated to $\mathbb{P}$ and $n$* is the following algebra of strictly upper triangular matrices:

$$\mathcal{E}_{\mathbb{P},n}(R) := \{x \in \mathrm{Mat}_n(R) \mid x_{ij} \neq 0 \Rightarrow i \lneq_P j\}.$$

Inspired by (2), we introduce a multivariate version of $\zeta_{\mathcal{E}\curvearrowright\mathbb{Z}_p^n}(s)$ for any $\mathcal{E} \leq \mathrm{Mat}_n(\mathbb{Z}_p)$ via the $p$-adic cone integral

$$\zeta_{\mathcal{E}\curvearrowright\mathbb{Z}_p^n}(s_1, \dots, s_n) := \frac{1}{(1-p^{-1})^n} \int_{\mathcal{V}_\mathcal{E}} \prod_{i=1}^{n} |x_{ii}|^{s_i - i} \mathrm{d}\mu(x), \tag{3}$$

and obtain the following.

**Theorem 1** ([4])**.** *For every finite natural poset $\mathbb{P} = (P, \leq_P)$ with $P \subseteq \{1, \dots, n\}$ for some $n \geq 1$, and for every prime $p$, one has*

$$\lim_{s_n\to 0} \left( \cdots \left( \lim_{s_2\to 0} \left( \lim_{s_1\to 0} (1-p^{-s_1}) \zeta_{\mathcal{E}_{\mathbb{P},n}(\mathbb{Z}_p)\curvearrowright\mathbb{Z}_p^n}(s_1, \dots, s_n) \right) \right) \cdots \right) = \prod_{i=1}^{n-1} \frac{1}{1-p^i}. \tag{4}$$

To prove (4), we first recursively describe the set $\mathcal{V}_{\mathcal{E}_{\mathbb{P},n}(\mathbb{Z}_p)}$. Based on this, we introduce a sequence of functions $\big(f_i(s_i, \dots, s_n)\big)_{1\leq i\leq n}$, defined as appropriate $p$-adic cone integrals, such that $f_1(s_1, \dots, s_n)$ represents the general term of the limit in (4) and, for every $1 \leq i \leq n-1$,

$$f_i(s_i, \dots, s_n) = (1 - p^{i-s_i})^{-1} f_{i+1}(s_{i+1}, \dots, s_n).$$

A straightforward computation allows us to express the function $f_n(s_n)$ explicitly, leading to the conclusion. Finally, if the following limit exists

$$\lim_{(s_1,\ldots,s_n)\to(0,\ldots,0)}(1-p^{-s_1})\zeta_{\mathcal{E}_{\mathbb{P},n}(\mathbb{Z}_p)\curvearrowright\mathbb{Z}_p^n}(s_1,\ldots,s_n),$$

– an assumption supported by all the examples we can compute – then Theorem 1 implies that Rossmann's conjecture [5, Conj. IV] holds for $\mathcal{E}_{\mathbb{P},n}(\mathbb{Z}_p)$.

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## Reduced and topological zeta functions in enumerative algebra

Tobias Rossmann

**Setup.** Let $(a_n)_{n\geq 1}$ be a sequence of numbers attached to an instance of an algebraic counting problem. Classical examples are obtained by letting $a_n$ denote the number of subgroups or subalgebras of index $n$ in a given group $G$ or algebra $A$, respectively, or by taking $a_n$ to be the number of irreducible representations $G \to \mathrm{GL}_n(\mathbf{C})$, counted up to suitable equivalence. A common theme in enumerative algebra is to study such sequences by means of the associated *global zeta function* $\mathsf{Z}(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. Under suitable assumptions, this series will admit an Euler product decomposition $\mathsf{Z}(s) = \prod_p \mathsf{Z}_p(s)$, where $p$ ranges over primes and the *local zeta function* at $p$ is $\mathsf{Z}_p(s) = \sum_{k=0}^{\infty} a_{p^k} p^{-ks}$. Subject to further assumptions on the shapes of the $\mathsf{Z}_p(s)$, the reduced and topological zeta functions $\mathsf{Z}_{\mathrm{red}}(T)$ and $\mathsf{Z}_{\mathrm{top}}(s)$ are two related but subtly different rational functions obtained by taking limits of $\mathsf{Z}_p(s)$ as "$p \to 1$".

**Topological zeta functions.** Informally, the topological zeta function $\mathsf{Z}_{\mathrm{top}}(s) \in \mathbf{Q}(s)$ is the constant term of $(1-p^{-1})^e \mathsf{Z}_p(s)$ as a power series in $p-1$. Here, the exponent $e \in \mathbf{Z}$ depends both on the counting problem and on the particular instance. In a surprising number of cases of interest, the local zeta functions $\mathsf{Z}_p(s)$ are *(almost) uniform* in the sense that there exists a single rational function $W(X,T) \in \mathbf{Q}(X,T)$ such that $\mathsf{Z}_p(s) = W(p, p^{-s})$ for (almost) all primes $p$. In such cases, we formally expand $W(p,p^{-s})$ using the binomial series $p^{-s} = (1+(p-1))^{-s} = \sum_{m=0}^{\infty} \binom{-s}{m}(p-1)^m$ and we obtain $\mathsf{Z}_{\mathrm{top}}(s)$ as indicated above.

The key difficulty in rigorously defining topological zeta functions is to overcome the restriction to (almost) uniform cases. The dependence of $\mathsf{Z}_p(s)$ on $p$ is often governed by a formula of a type that first appeared in work of Denef [1, §3]. In these cases, there are schemes $V_1, \dots, V_r$ (over $\mathbf{Z}$) and rational functions $W_1(X,T), \dots, W_r(X,T)$ such that for almost all primes $p$, we have $\mathsf{Z}_p(s) = \sum_{i=1}^{r} \#V_i(\mathbf{F}_p) W_i(p, p^{-s})$. Using such a formula, Denef and Loeser [2] gave a rigorous definition of the topological zeta functions associated with a polynomial. The functions $W_i(p,p^{-s})$ are again expanded using the binomial series. Using $\ell$-adic interpolation arguments based on Grothendieck's trace formula, the limit of $\#V_i(\mathbf{F}_p)$ as "$p \to 1$" is $\chi(V_i(\mathbf{C}))$, the topological Euler characteristic of the complex analytic space attached to $V_i$. By [10, §3], in almost uniform cases, our informal approach agrees with the rigorous one.

In [3], Denef and Loeser gave another, independent description of the topological zeta function associated with a polynomial by means of a suitable specialisation of the associated *motivic zeta function*. In [5, §8], du Sautoy and Loeser defined topological subalgebra zeta functions by specialising motivic ones, having introduced the latter in the same paper. An $\ell$-adic approach to topological subobject zeta functions based on [2] was developed in [8, §5].

**Global and local subalgebra zeta functions.** Subalgebra zeta functions were introduced by Grunewald, Segal, and Smith [7, §3]. For the remainder of this abstract, let $A$ be a (not necessarily associative) $\mathbf{Z}$-algebra whose underlying $\mathbf{Z}$-module is free of finite rank $d$. Let $a_n(A)$ denote the number of subalgebras $B$ of $A$ of additive index $|A:B| = n$. Let $\mathsf{Z}^A(s) = \sum_{n=1}^{\infty} a_n(A) n^{-s}$ be the associated *(global) subalgebra zeta function*. By the Chinese remainder theorem, we obtain an Euler product $\mathsf{Z}^A(s) = \prod_p \mathsf{Z}_p^A(s)$ as above. The *local subalgebra zeta function* $\mathsf{Z}_p^A(s)$ enumerates subalgebras of the $\mathbf{Z}_p$-algebra $A \otimes \mathbf{Z}_p$. In [4], du Sautoy and Grunewald established Denef-style formulae for local subalgebra zeta functions associated with a fixed algebra $A$. Voll [12, Thm A] established a local functional equation of subalgebra zeta functions under "inversion of $p$" of the form

$$(\star) \qquad \mathsf{Z}_p^A(s)\Big|_{p \leftarrow p^{-1}} = (-1)^d p^{\binom{d}{2} - ds} \cdot \mathsf{Z}_p^A(s).$$

Voll's proof relied on a delicate interplay of (a) the functional equations satisfied by Hasse-Weil zeta functions of smooth projective varieties and (b) a self-reciprocity property of generating functions enumerating lattice points within certain cones. Part (a) is explained by Poincaré duality in $\ell$-adic cohomology while Stanley [11, Ch. I] elegantly explained (b) in terms of local cohomology.

**Reduced and topological subalgebra zeta functions.** Using our informal approach, the *topological subalgebra zeta function* $\mathsf{Z}^A_{\mathrm{top}}(s) \in \mathbf{Q}(s)$ is the constant term of $(1-p^{-1})^d \mathsf{Z}^A_p(s)$ as a series in $p-1$. Introduced by Evseev [6], the *reduced subalgebra zeta function* $\mathsf{Z}^A_{\mathrm{red}}(T) \in \mathbf{Q}[\![T]\!] \cap \mathbf{Q}(T)$ is obtained by viewing $\mathsf{Z}^A_p(s)$ as a series in $T = p^{-s}$ and applying a limit "$p \to 1$" to its coefficients. In (almost) uniform cases in which $\mathsf{Z}^A_p(s) = W(p, p^{-s})$ for (almost) all $p$, the reduced subalgebra zeta function is given by $\mathsf{Z}^A_{\mathrm{red}}(T) = W(1, T)$.

Upon taking the limit "$p \to 1$", Voll's local functional equation ($\star$) implies that $\mathsf{Z}^A_{\mathrm{red}}(T)$ satisfies the self-reciprocity identity $\mathsf{Z}^A_{\mathrm{red}}(T^{-1}) = (-1)^d T^d \cdot \mathsf{Z}^A_{\mathrm{red}}(T)$, a property reminiscent of Hilbert series of graded Gorenstein algebras.

**Conjectures.** Reduced and topological zeta functions are seemingly quite different invariants. Both constructions are, however, conjecturally related by the "coincidence conjecture" below. We first recall the following conjecture which predicts the vanishing order of topological subalgebra zeta functions at infinity.

**Degree conjecture** ([8, Conj. I])**.** $\deg(\mathsf{Z}^A_{\mathrm{top}}(s)) = -d$.

Evseev [6, Prop. 4.1] showed that if $A$ admits a particular type of basis, then there exists a $d$-dimensional cone $\mathcal{C} \subset \mathbf{R}^d_{\geq 0}$ such that the reduced zeta function $\mathsf{Z}^A_{\mathrm{red}}(T)$ is the (coarse) Hilbert series of the affine monoid algebra $\mathbf{Q}[\mathcal{C} \cap \mathbf{Z}^d]$. Using the explicit description of $\mathcal{C}$ by Evseev, results from combinatorial commutative algebra (see [11, Ch. 1]) show that $\mathbf{Q}[\mathcal{C} \cap \mathbf{Z}^d]$ is Gorenstein.

**Hilbert series conjecture** (Voll)**.** *There exists a (natural, meaningful)* $\mathbf{N}_0$*-graded Gorenstein algebra of dimension d whose Hilbert series is* $\mathsf{Z}^A_{\mathrm{red}}(T)$.

While Voll's Hilbert series conjecture has been informally shared with researchers for quite some time, it took more cautious forms in the published literature. Inspired by the preceding two conjectures, define

$$\mathrm{m}_{\mathrm{top}}(A) := s^{-d}\mathsf{Z}^A_{\mathrm{top}}(s^{-1})\Big|_{s=0}, \quad \mathrm{m}_{\mathrm{red}}(A) := (1-T)^d \mathsf{Z}^A_{\mathrm{red}}(T)\Big|_{T=1}.$$

The topological degree conjecture is equivalent to $\mathrm{m}_{\mathrm{top}}(A)$ being nonzero and finite. Similarly, if the Hilbert series conjecture is true, then $\mathsf{Z}^A_{\mathrm{red}}(T)$ has a pole of order $d$ at $T = 1$ whence $\mathrm{m}_{\mathrm{red}}(A)$ is nonzero and finite.

**Coincidence conjecture.** $\mathrm{m}_{\mathrm{red}}(A) = \mathrm{m}_{\mathrm{top}}(A)$ *and the common value is a positive rational number.*

While the preceding conjecture has been informally communicated for at least a decade, to the best of the author's knowledge, it too has not, so far, been formally stated as such in a published document. This notwithstanding, apart from the numerical evidence provided by computer calculations [8, 9], the coincidence conjecture has been verified for some families of algebras; see e.g. [13, §3.4].

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## Traces of Hecke operators via hypergeometric character sums

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(joint work with Jerome Hoffman, Ling Long, Fang-Ting Tu)

The eigenvalues of Hecke operators $T_p$ convey important arithmetic information about modular forms. We are interested in explicit formulae for the traces of $T_p$ on the space $S_{k+2}(\Gamma)$ of cusp forms of weight $k+2 \geq 3$ for a congruence subgroup $\Gamma$ of $\mathrm{SL}_2(\mathbb{R})$, assuming the Shimura canonical model for the modular curve $X_\Gamma$ is defined over $\mathbb{Q}$. This includes cusp forms for elliptic modular groups $\Gamma$ commensurable with $\mathrm{SL}_2(\mathbb{Z})$ and modular forms for cocompact groups $\Gamma$ commensurable with the norm 1 subgroup $\mathcal{O}_B^1$ of a maximal order $\mathcal{O}_B$ of an indefinite nonsplit quaternion algebra $B$ defined over $\mathbb{Q}$.

This was previously done for $\Gamma_0(2)$, $\Gamma_0(4)$, $\Gamma_0(8)$, $\mathrm{SL}_2(\mathbb{Z})$, $\Gamma_0(3)$, and $\Gamma_0(9)$ using the Selberg trace formula. We obtain explicit Hecke trace formulae in terms of hypergeometric character sums for certain arithmetic triangle groups $\Gamma$ of type a) or b) explained below. Our geometric approach gives a unified treatment for $\Gamma$ elliptic modular, including aforementioned groups, and $\Gamma$ arising from the indefinite quaternion algebra $B_6 = \left(\frac{-1,3}{\mathbb{Q}}\right)$ over $\mathbb{Q}$ with discriminant 6. The same method can also be applied to obtain eigenvalues of the Hecke operators. As an application, Hecke traces for congruence subgroups $\Gamma'$ not of type a) or b) can also be obtained as long as the modular curve $X_{\Gamma'}$ is defined over $\mathbb{Q}$ and there is an explicit $\mathbb{Q}$-rational covering map from $X_{\Gamma'}$ to $X_\Gamma$ for some $\Gamma$ of type a) or b).

**Geometric interpretation of traces of Hecke operators and our approach.** Using the moduli interpretation of $X_\Gamma$, Deligne [2] for $\Gamma$ non-cocompact and Ohta [6] for $\Gamma$ cocompact constructed, for each prime $\ell$, an automorphic $\ell$-adic sheaf $V^k(\Gamma)_\ell$ on $X_\Gamma \otimes \bar{\mathbb{Q}}$ which provided the following geometric interpretation of the Hecke traces. It was due to Deligne [2] for $\Gamma$ elliptic modular and Ohta [7] for $\Gamma$ quaternionic.

**Theorem 1.** *Given a prime $\ell$, for all primes $p \neq \ell$ where $X_\Gamma$ has good reduction, we have*

$$Tr(T_p \mid S_{k+2}(\Gamma)) = Tr(Frob_p \mid H^1_{\acute{e}t}(X_\Gamma \otimes \overline{\mathbb{Q}}, V^k(\Gamma)_\ell)).$$

Combined with the Grothendieck-Lefschetz fixed point formula, we obtain a geometric interpretation of the Hecke trace in terms of the sum of Frobenius traces:

$$-\mathrm{T}r(T_p \mid S_{k+2}(\Gamma)) = \sum_{\lambda \in X_\Gamma(\mathbb{F}_p)} \mathrm{T}r(Frob_\lambda \mid (V^k(\Gamma)_\ell)_{\bar{\lambda}}). \tag{1}$$

Here

$$\mathrm{T}r(Frob_\lambda \mid (V^k(\Gamma)_\ell)_{\bar{\lambda}}) = \mathrm{T}r(Frob_p \mid (V^k(\Gamma)_\ell)_{\bar{\lambda}'})$$

for any algebraic point $\lambda'$ on $X_\Gamma$ which reduces to $\lambda$ modulo a degree-1 prime $\mathfrak{P}$ above $p$.

For contributions in (1) from generic points $\lambda \in X_\Gamma(\mathbb{F}_p)$, the computation is further reduced to that on $V^1(\Gamma)_\ell$ for $\Gamma$ elliptic modular not containing -Id, and $V^2(\Gamma)_\ell$ for $\Gamma$ containing -Id.

Our strategy is to find suitable groups $\Gamma$ such that either $V^1(\Gamma)_\ell$ or $V^2(\Gamma)_\ell$ is isomorphic, up to a rank-1 twist, to the hypergeometric sheaf $\mathcal{H}(\mathrm{HD}(\Gamma))_\ell$ attach to a hypergeometric datum $\mathrm{HD}(\Gamma)$ introduced by Katz in [4, 5] and further extended by Beukers, Cohen and Mellit in [1] for which the Galois action on a stalk has Frobenius traces explicitly expressed by hypergeometric character sums.

**The groups we consider.** Let $e_1, e_2, e_3$ be elements in $\mathbb{Z}_{>0} \cup \{\infty\}$. Defined in terms of generators and relations, an arithmetic triangle group

$$(e_1, e_2, e_3) := \langle g_1, g_2, g_3 \mid g_1^{e_1} = g_2^{e_2} = g_3^{e_3} = g_1 g_2 g_3 = id \rangle$$

can be realized as a discrete subgroup $\Gamma$ of $\mathrm{PSL}_2(\mathbb{R})$ acting on $\mathfrak{H}$. The modular curve $X_\Gamma$ is a hyperbolic triangle with three vertices $v_i$ fixed by elements of order $e_i$. If $e_i = \infty$, then $v_i$ is a cusp of $\Gamma$, otherwise $v_i$ is an elliptic point of order $e_i$.

The concerns on the required properties of the hypergeometric sheaves over $\mathbb{Q}$ on $X_\Gamma$ led us to the seven $\Gamma = (e_1, e_2, e_3)$ listed below; those of type a) are isomorphic to a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ not containing -Id, while those of type b) are projective groups, which, for the sake of modular forms, may be regarded as subgroups of $\mathrm{SL}_2(\mathbb{R})$ containing - Id:

$$\begin{array}{ll} \text{Type } a) & (3,\infty,\infty)\simeq\Gamma_1(3), (\infty,\infty,\infty)\simeq\Gamma_1(4);\\ \text{Type } b) & (2,\infty,\infty)\simeq\Gamma_0(2)/\{\pm\mathrm{Id}\}, (2,3,\infty)\simeq\mathrm{PSL}_2(\mathbb{Z}),\\ & (2,4,\infty)\simeq\langle\Gamma_0(2),w_2\rangle/\{\pm\mathrm{Id}\}=\Gamma_0(2)^+/\{\pm\mathrm{Id}\},\\ & (2,6,\infty)\simeq\langle\Gamma_0(3),w_3\rangle/\{\pm\mathrm{Id}\}=\Gamma_0(3)^+/\{\pm\mathrm{Id}\},\\ & (2,4,6)=\langle\mathcal{O}^1_{B_6},w_2,w_3,w_6\rangle/\{\pm\mathrm{Id}\}. \end{array}$$

Here $w_2, w_3, w_6$ are Atkin-Lehner involutions.

**Statements of main results.** To each of the above seven groups $\Gamma$, we associate a primitive hypergeometric datum $\mathrm{HD}(\Gamma)=\{\alpha(\Gamma),\beta(\Gamma)\}$ as follows.

| $\Gamma$ | $(3,\infty,\infty)$ | $(\infty,\infty,\infty)$ | $(2,\infty,\infty)$ | $(2,3,\infty)$ | $(2,4,\infty)$ | $(2,6,\infty)$ | $(2,4,6)$ |
|---|---|---|---|---|---|---|---|
| $\alpha(\Gamma)$ | $\{\frac{1}{3},\frac{2}{3}\}$ | $\{\frac{1}{2},\frac{1}{2}\}$ | $\{\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$ | $\{\frac{1}{2},\frac{1}{6},\frac{5}{6}\}$ | $\{\frac{1}{2},\frac{1}{4},\frac{3}{4}\}$ | $\{\frac{1}{2},\frac{1}{3},\frac{2}{3}\}$ | $\{\frac{1}{2},\frac{1}{4},\frac{3}{4}\}$ |
| $\beta(\Gamma)$ | $\{1,1\}$ | $\{1,1\}$ | $\{1,1,1\}$ | $\{1,1,1\}$ | $\{1,1,1\}$ | $\{1,1,1\}$ | $\{1,\frac{5}{6},\frac{7}{6}\}$ |

Since each datum $\mathrm{HD}(\Gamma)$ is defined over $\mathbb{Q}$, for each prime $p$ and $\lambda\in\mathbb{F}_p^\times$, Beukers, Cohen and Mellit defined in [1] the hypergeometric character sum $H_p(\mathrm{HD}(\Gamma),\lambda)$ using Gauss sums. We shall express $\mathrm{Tr}(T_p,S_{k+2}(\Gamma))$ in terms of these hypergeometric character sums according to the type of $\Gamma$.

**Theorem 2.** [3] *Let $\Gamma\in\{(3,\infty,\infty),(\infty,\infty,\infty)\}$ be a group of type a). Then for all integers $k\geq 1$ and a prime $p$ where $X_\Gamma$ has good reduction, we have*

$$-\mathrm{Tr}(T_p,S_{k+2}(\Gamma))=\sum_{\lambda\in\mathbb{F}_p^\times,\lambda\neq 1}\sum_{j=0}^{\lfloor k/2\rfloor}(-1)^j\binom{k-j}{j}p^j\cdot H_p\left(HD(\Gamma);1/\lambda\right)^{k-2j}+\mathcal{E}_\Gamma(p,k),$$

*where*

$$\mathcal{E}_{(\infty,\infty,\infty)}(p,k)=1+\left(\frac{-1}{p}\right)^k+\frac{1+(-1)^k}{2},$$

*and*

$$\mathcal{E}_{(3,\infty,\infty)}(p,k)=1+\left(\frac{-3}{p}\right)^k+$$

$$+\begin{cases}0 & \text{if } p\equiv -1 \mod 3,\ k \text{ odd},\\ (-p)^{k/2} & \text{if } p\equiv -1 \mod 3,\ k \text{ even},\\ (-1)^k\displaystyle\sum_{\substack{0\leq i\leq k\\ k\equiv 2i \mod 3}}p^i\cdot J_\omega\left(\frac{1}{3},\frac{1}{3}\right)^{k-2i} & \text{if } p\equiv 1 \mod 3,\end{cases}$$

*in which $\widehat{\mathbb{F}_p^\times}=\langle\omega\rangle$ and $J_\omega(a,b)=\sum_{x\in\mathbb{F}_p}\omega^{(p-1)a}(x)\omega^{(p-1)b}(1-x)$ is a Jacobi sum.*

To state our results for groups of type b), for integers $m \geq 1$, let $F_m(S,T)$ be the degree$-m$ polynomial in $S$ and $T$ defined by the recursive relation

$$F_{m+1}(S,T) = (S-T)F_m(S,T) - T^2 F_{m-1}(S,T), \quad F_0(S,T) = 1, \quad F_1(S,T) = S.$$

**Theorem 3.** [3] *Let $\Gamma \in \{(2,\infty,\infty),(2,3,\infty),(2,4,\infty),(2,6,\infty),(2,4,6)\}$ be a group of type b). Then each elliptic point $z$ of $X_\Gamma$ is a CM point by an imaginary quadratic field $K_z = \mathbb{Q}(\sqrt{d_z})$ as follows:*

| $\Gamma$ | $(2,\infty,\infty)$ | $(2,3,\infty)$ | $(2,4,\infty)$ | $(2,6,\infty)$ | $(2,4,6)$ |
|---|---|---|---|---|---|
| $\sqrt{d_z}$ | $\sqrt{-4}$, -, - | $\sqrt{-4}, \sqrt{-3}$, - | $\sqrt{-8}, \sqrt{-4}$, - | $\sqrt{-3}, \sqrt{-3}$, - | $\sqrt{-24}, \sqrt{-4}, \sqrt{-3}$ |

*Denote by $N(z)$ its order. Let $p$ be a prime where $X_\Gamma$ has good reduction. Then for each even integer $k \geq 2$ we have*

$$-\mathrm{Tr}(T_p, S_{k+2}(\Gamma)) = \sum_{\lambda \in \mathbb{F}_p^\times, \lambda \neq 1} F_{k/2}(a_\Gamma(\lambda,p),p) + \sum_{z \in X_\Gamma(\mathbb{F}_p) \text{ cusp}} 1 + \sum_{z \in X_\Gamma(\mathbb{F}_p) \text{ elliptic}} \mathcal{E}_\Gamma(z,p,k),$$

*where*

$$a_\Gamma(\lambda,p) = \begin{cases} \left(\frac{1-1/\lambda}{p}\right) H_p(HD(\Gamma), 1/\lambda) & \text{if } \Gamma \neq (2,4,6), \\ \left(\frac{-3(1-1/\lambda)}{p}\right) pH_p(HD(\Gamma), 1/\lambda) & \text{if } \Gamma = (2,4,6), \end{cases}$$

*and*

$$\mathcal{E}_\Gamma(z,p,k) = \begin{cases} (-p)^{k/2} & \text{if } p \text{ is inert in } K_z, \\ \sum_{-\frac{k}{2N(z)} \leq i \leq \frac{k}{2N(z)}} p^{k/2}(\alpha_{N(z),p}^2/p)^{iN(z)} & \text{if } p \text{ splits in } K_z. \end{cases}$$

*In the latter case, upon picking any prime ideal $\mathfrak{p}$ of the ring of integers of $K_z$ above $p$, $\alpha_{N(z),p}$ can be chosen as any generator of the principal ideal $\mathfrak{p}$ when $N(z) > 2$. When $N(z) = 2$, $\alpha_{N(z),p}^2$ can be taken as any root of $T^2 - \left(\frac{-3}{p}\right)^u p^u H_p(HD(\Gamma);1)T + p^2 = 0$, where $u = 1$ if $\Gamma = (2,4,6)$, and $u = 0$ else.*

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**On deformations of modular forms and extensions of symmetric tensor representations**

Gabriele Bogo

In the 1880s, Poincaré envisioned the possibility of describing algebraic curves using curvilinear polygons in the complex plane–an intuition that would later (in 1907) lead to the uniformization theorem for Riemann surfaces. As a consequence of the uniformization theorem, every Riemann surface $X$ (except for $\mathbb{P}^1(\mathbb{C})$, $\mathbb{C}$, and $\mathbb{C}\setminus\{0\}$) has the upper half-plane $\mathbb{H}$ as its universal covering, and the curvilinear polygon associated with $X$ is a fundamental domain for the action of the (Fuchsian) deck group $\Gamma$ of the covering $\mathbb{H}\to X$. Poincaré's early attempts toward the uniformization theorem focused on the explicit construction of a (multivalued) inverse $\eta\colon X\to\mathbb{H}$ to the covering map $\mathbb{H}\to X$, based on the known relationship, explored by Schwarz, Klein, and others, between conformal maps and differential equations (see [5] for the history of the uniformization theorem)

Let us restrict to the case of genus zero Riemann surfaces, i.e., punctured spheres. Let

$$(1)\qquad X=\mathbb{P}^1(\mathbb{C})\setminus\{\infty,1,0,a_1,\dots,a_n\}\,,\quad a_i\in\mathbb{C}\setminus\{0,1\}\,,\quad a_i\neq a_j \text{ if } i\neq j\,,$$

The (multivalued) map $\eta\colon X\to\mathbb{H}$ must satisfy two assumptions: each single-valued determination of $\eta$ is a local biholomorphism; different branches of $\eta$ are interchanged by Möbius transformations. Such map arise as the ratio of linearly independent solutions of the linear differential equation

$$(2)\quad P(t)\frac{d^2y(t)}{dt^2}+P'(t)\frac{dy(t)}{dt}+\sum_{j=0}^{n}\lambda_j\cdot y(t)=0\,,\quad P(t)=t(t-1)\prod_{i=1}^{n}(t-a_i)\,,$$

where $\lambda_0,\dots,\lambda_n$ are complex parameters called *accessory parameters*. The *accessory parameters problem* asks to find the unique set $\lambda_F=(\lambda_{F_0},\dots,\lambda_{F_n})$ of accessory parameters such that the ratio of independent solutions of (2), seen as a single-valued function on the universal covering $\widetilde{X}$ of $X$, gives an identification $\widetilde{X}\simeq\mathbb{H}$. Up to now, nobody has been able to determine the correct choice of parameters in general, and the accessory parameters problem remains wide open.

Which are the consequences of the right choice of accessory parameters? First, the image of the monodromy representation $\pi_1(X,x_0)\to\mathrm{GL}_2(\mathbb{C})$ is a Fuchsian group $\Gamma\subset\mathrm{SL}_2(\mathbb{R})$ of the first kind (choices of accessory parameters not "too far" from the correct values still give an uniformization of $X$, but in terms of *quasi-Fuchsian groups* [1]). Second, the lift of a holomorphic solution of (2), seen as a function $f$ on $\mathbb{H}$, is a modular form of weight 1 (possibly with non-trivial multiplier system) on the group $\Gamma$. The $q$-expansion of $f$ at a cusp can be computed from the holomorphic solution of (2) via manipulation of power series. Let $y_1(\lambda,t)$ and $y_2(\lambda,t)=\log(t)y_1(\lambda,t)+\widetilde{y_2}(\lambda,t)$ be the local solutions in $t=0$, where we consider $\lambda=(\lambda_0,\dots,\lambda_n)$ as parameters. Define

$$Q(\lambda,t):=\exp(y_2(t)/y_1(t))=t+\cdots\,,\quad t(\lambda,Q)=Q+\cdots\,,$$

where $t(\lambda, Q)$ has been obtained by formally inverting the series $Q(\lambda, t)$ with respect to the variable $t$. Substitute finally $t(\lambda, Q)$ into $y_1(\lambda, Q)$ to get

$$f(\lambda, Q) = y_1(\lambda, t(\lambda, Q)) = \sum_{m=0}^{\infty} f_m(\lambda) Q^m \,, \tag{3}$$

where $f_m(\lambda)$ is a polynomial in $\lambda_0, \ldots, \lambda_n$ for every $m \geq 0$. By specializing to the correct values $\lambda_F$ of the accessory parameters in the $Q$-expansion of $f(\lambda, Q)$ we get the $q$-expansion of a modular form. For instance, if $X = \mathbb{P}^1(\mathbb{C}) \setminus \{\infty, 1, 0, 1/9\}$, then $n = 1$ and $\lambda_0 = 1$ and $\lambda_1 = 1/3$, and $Q = q = e^{2\pi i \tau}$ for $\tau \in \mathbb{H}$, and

$$f((1, 1/3), Q) = 1 + 3q + 3q^2 + 3q^3 + \cdots \in M_1(\Gamma_1(5)) \,. \tag{4}$$

In [2] a numerical method to compute the accessory parameter in the case of sphere with 4 punctures (i.e., the case $n = 1$) is described, based on the modularity of $f(\lambda, Q)$ for the correct determination of $\lambda$.

In my talk, I discussed certain deformation operators on the space of modular forms, defined in terms of accessory parameters. The deformation operators can be described in terms of deformation of complex structures and Teichmüller theory; the interested reader can find this description in Section 2.3 of [3].

Let $X$ be a punctured sphere as above, let $f(\lambda, Q)$ be as in (3) and let $f(\tau) = f(\lambda_F, Q) \in M_1(\Gamma)$. For $j \in \{1, \ldots, n\}$ (notice that the case $j = 0$ is not included) define

$$\partial_j f(\tau) := \frac{\partial f(\lambda, Q)}{\partial \lambda_j}\Big|_{\lambda = \lambda_F} \,. \tag{5}$$

The deformations can be extended to the space of modular forms $M_*(\Gamma)$. We first look at how the deformation $\partial = \partial_1$ acts on the example (4) related to $\Gamma_1(5)$. We find, for $f = f((1, 1/3), q)$ in (4), that

$$\partial f = 9q + \frac{153}{2} q^2 + 105 q^3 + \frac{543}{4} q^4 + \frac{36057}{200} q^5 + \cdots \,. \tag{6}$$

An immediately noticeable difference between (4) and (6) is the appearance of denominators in the coefficients of the latter. Where do they come from? From the Eichler integral of a cusp form $h \in S_4(\Gamma)$.

**Theorem 1** (Theorem 1, [3]). *Let $X = \mathbb{P}^1(\mathbb{C}) \setminus \{\infty, 1, 0, a_1, \ldots, a_n\}$ be as in* (1), *and let $\Gamma$ be its uniformizing Fuchsian group. For $j = 1, \ldots, n$ let $\partial_j$ denote the deformation operator* (5)*. There exist cusp forms $h_1, \ldots, h_n \in S_4(\Gamma)$ such that, for every $f \in M_k(\Gamma)$ it holds*

$$\partial_j f = k f \widetilde{h}_j' + 2 f' \widetilde{h} = [f, \widetilde{h}_j]_1 \,, \quad ' = q \frac{d}{dq}$$

*where $\widetilde{h}_j$ is the Eichler integral of the cusp form $h_j$, i.e., if $h_j = \sum_{m \geq 1} h_{j,m} q^m$, then $\widetilde{h}_j = \sum_{m \geq 1} \frac{h_{j,m}}{m^3} q^m$. Here $[\,,]_1$ denotes the first Rankin-Cohen bracket.*

Cusp forms of weight four should be interpreted as quadratic differentials on the Riemann surface $X$. The space $S_4(\Gamma)$ of quadratic differentials is the cotangent space to the Teichmüller space of $(n+3)$-punctured spheres at the point $X$. This space governs the deformation of complex structure of $X$, and this explains the appearance of weight four cusp forms in the action of the deformation operators.

Another aspect I discussed in my talk is the modularity of $\partial_j f$. The proof of Theorem 1 shows that deformations are related to differential operators of higher order obtained by composition of second-order linear differential operators. Their monodromy is an extension of standard representations of $\Gamma$. It is therefore reasonable to expect $\partial_j f$ to have some modularity property as a *vector-valued modular form* with respect to extensions of standard representations of $\Gamma$ (or their symmetric powers $\mathrm{Sym}^n(\mathbb{C}^2)$). This is precisely what happens.

Let $p_j(\gamma;\tau) = r_{j,2}(\gamma)\tau^2 + r_{j,1}(\gamma)\tau + r_{j,0}(\gamma)$ be the period polynomial of $\widetilde{h}_j$. The polynomial $p_j(\gamma,\tau)$ measures the failure of the modularity of $h_j$: for every $\gamma \in \Gamma$

$$\widetilde{h}_j(\gamma\tau)\cdot(c\tau+d)^{-2} - \widetilde{h}_j(\tau) = p_j(\gamma,\tau)\,, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Consider the class $[v^j_{0,2}] \in \mathrm{Ext}^1_\Gamma(\mathrm{Sym}^0(\mathbb{C}^2), \mathrm{Sym}^2(\mathbb{C}^2))$ represented by

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto v^j_{0,2}(\gamma) = \left(\begin{array}{c|ccc} 1 & r_{j,2}(\gamma) & r_{j,1}(\gamma) & r_{j,0}(\gamma) \\ \hline 0 & a^2 & 2ab & b^2 \\ 0 & ac & ad+bc & bd \\ 0 & c^2 & 2cd & d^2 \end{array}\right). \tag{7}$$

**Theorem 2** (Proposition 4 in [3])**.** *The vector* $\vec{\partial_j} f$

$$\vec{\partial_j} f \;=\; \begin{pmatrix} \partial_j f \\ \tau^2 f' + 2\tau f \\ \tau f' + f \\ f' \end{pmatrix},$$

*defines a vector-valued modular form for the representation* $v^j_{0,2}$.

*Final remarks.* The two theorems show that infinitesimal deformations of the underlying complex structure send modular forms to vector-valued modular forms attached to extensions defined by periods of cusp forms of weight four.

In a recent work [6], A. Keilthy and M. Raum defined and studied the deformations $\partial_j$ from a cohomological perspective, and relate them to motivic periods.

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**Vector-valued modular forms and two connections to cohomology**

Martin Raum
(joint work with Michael Mertens, Tobias Magnusson)

The transformation behavior of modular forms can be described by a representation. In the most classical setting, a modular form is fixed by the slash action of $\mathrm{SL}_2(\mathbb{Z})$ on functions $f : \mathbb{H} \to \mathbb{C}$:

$$f|_k\,\gamma = f.$$

In an alternative point of view, if $f \neq 0$ the one-dimensional space $\mathbb{C}\,f$ is a right-representation for $\mathrm{SL}_2(\mathbb{Z})$ that is trivial. Slightly more generally, if $f$ is a modular form for, say, a Dirichlet character $\chi$ modulo $N$, then

$$f|_k\,\gamma = \chi(\gamma)\,f, \quad \chi\big(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\big) = \chi(d).$$

In other words, if $f \neq 0$ then $\mathbb{C}\,f$ is isomorphic to the one-dimensional representation

$$\Gamma_0(N) \longrightarrow \mathrm{U}_1(\mathbb{R}) \subset \mathrm{GL}_2(\mathbb{C}), \quad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \longmapsto \chi(d).$$

Vector-valued modular forms $f : \mathbb{H} \to V(\rho)$ for a representation $\rho$ of $\mathrm{SL}_2(\mathbb{Z})$ with representation space $V(\rho)$ are associated with the modular covariance condition

$$f|_k\,\gamma = \rho(\gamma) \circ f.$$

The case of one-dimensional $\rho$ corresponds to classical, scalar-valued modular forms. Higher-dimensional representations that are commonly encountered include the Weyl representations associated with integral quadratic forms of even rank and inductions of characters.

The space of components of $f$

$$\big\{\lambda \circ f : \lambda \in V(\rho)^\vee\big\}$$

is isomorphic to a quotient of the dual $\rho^\vee$ of $\rho$. If $\rho$ has finite index kernel, it consists of modular forms for $\ker(\rho)$. That is, vector-valued modular forms in these cases help to organize relations among modular forms, but do not give rise to genuinely new kinds of modular forms.

This remains mostly true for the symmetric powers $\mathrm{sym}^{\mathsf{d}}$ of the standard representation of $\mathrm{SL}_2(\mathbb{R})$. We realize this representation on complex polynomials of degree at most $\mathsf{d}$ in a formal variable $X$ endowed with the slash action of weight $-\mathsf{d}$. In this realization, we find modular forms

$$(X - \tau)^{\mathsf{d}},$$

which are modular covariant of weight $-\mathsf{d}$ under the whole Lie group $\mathrm{SL}_2(\mathbb{R})$. This yields an injection

$$\mathrm{M}_{k+\mathsf{d}} \longrightarrow \mathrm{M}_k\big(\mathrm{sym}^{\mathsf{d}}\big), \quad f \longmapsto (X - \tau)^{\mathsf{d}} \cdot f.$$

We introduce a vector-valued raising operator that in combination with this embedding for $k > \mathsf{d}$ yields a decomposition

$$\mathrm{M}_k\big(\mathrm{sym}^{\mathsf{d}}\big) \cong \bigoplus_{\substack{j=-\mathsf{d} \\ j\equiv\mathsf{d} \pmod 2}}^{\mathsf{d}} \mathrm{M}_{k+j}.$$

The exceptional cases $k \le \mathsf{d}$ can be accommodated by the modular forms

$$\big((X-\tau)E_2 + \tfrac{6i}{\pi}\big)^{\mathsf{d}} \in \mathrm{M}_{\mathsf{d}}\big(\mathrm{sym}^{\mathsf{d}}\big),$$

where $E_2$ is the quasi-modular, holomorphic Eisenstein series of weight 2.

Building upon symmetric power representations, we investigate modular forms for their extension classes: virtually real-arithmetic types. Their simplest variant fits into a short exact sequence

$$0 \longrightarrow \mathrm{sym}^{\mathsf{d}} \longrightarrow \rho \longrightarrow \mathrm{sym}^{\mathsf{d}'} \longrightarrow 0.$$

We demonstrate that, in contrast to the other examples given above, they accommodate variations of modular forms that are usually not considered as modular forms. In particular, mock modular forms and higher order modular forms are components of modular forms of virtually real-arithmetic type. Further, all truncations of Brown's universal iterated integral are modular forms of virtually real-arithmetic type.

To illustrate the use of the presented formulation, we highlight a relation between multiple L-values, which appear as components of modular forms for two different virtually real-arithmetic types. We link them to each other via a suitable homomorphism of representations and deduce a relation between them from a vanishing statement for modular forms.

## Computing cohomologies of coherent sheaves

Lakshmi Ramesh
(joint work with Janko Böhm, Santosh Gnawali)

We aim to compute objects that are isomorphic to each graded part of the cohomology modules, $H^i(\mathbb{P}^n, \mathcal{F}(d))$, the $d$th graded part of the $i$th cohomology module. We are interested in obtaining the cohomology numbers

$$h^i(\mathbb{P}^n, \mathcal{F}(d)), \tag{1}$$

or the dimensions of each graded part.

In this talk, we discuss a correspondence, shown by Eisenbud, Fløystadt and Schreyer in [1], between the Tate resolution of a module $M$ over the graded polynomial ring and the cohomology of the corresponding coherent sheaf $\widetilde{M}$ over projective space. Thus, one can compute the cohomology numbers from 1 by computing the Tate resolution of a representing module.

The Tate resolution is a doubly infinite exact sequence of modules over the exterior algebra. The only non-trivial aspect of computing it is in the computation of the minimal free resolution of a module over the exterior algebra. I present a sketch of the algorithm presented by Janko Böhm and myself in [2], which uses a modification of the Schreyer theorem to a non-commutative case, and the theory of relative Gröbner bases. This algorithm has been implemented in Singular [3], and can be found in the library `sresext.lib`.

We adapted the refined Schreyer algorithm to compute free resolutions, presented by Erocal, Motsak and Schreyer in [4], to the case of non-commutative modules. Thus, the algorithm in [2] is in fact parallelisable. In ongoing work with Santosh Gnawali and Janko Böhm, we use the massively-parallel framework of GPI-Space and its interface with Singular [5] to implement an algorithm for parallel computations. This work is based on the algorithm for massively parallel computations of free resolutions of modules over a polynomial ring by Gnawali in [6].

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## Problem session

Chaired by Jens Funke. Notes by individual presenters.

### 1. Martin Raum: *Iterated integrals and polyharmonic Maass forms*

Mock modular forms yield important generating series of combinatorial origin. Many of their properties, such as asymptotic expansions or exact formulas, are derived via their modular completions. For instance, for a given mock modular form $f^+$, there exists a companion function $f^-$ (subject to natural analytic conditions) such that their sum $f^+ + f^-$ is modular invariant. These sums are known as harmonic weak Maass forms.

While the existence of modular completions is the standard way to define mock modular forms, a more intrinsic characterization exists: it is through the representations of the modular group that they generate under the slash action. For example, in integral weights these representations have finite length, and their composition factors extend to the real Lie group. This perspective naturally places mock modular forms within the context of iterated integrals, which conjecturally in even integral weights encompass all functions with this property. These can be viewed as mock modular forms of higher depth.

Harmonic weak Maass forms also admit a representation-theoretic interpretation via $(\mathfrak{g}, \mathrm{K})$-modules. Similar to mock modular forms, the resulting representations have finite length, and their composition factors are either finite-dimensional or (limits of) discrete series.

Moving to generalizations, in integral weight iterated integrals connect to motivic geometry. This connection can be harnessed to extend the concept of modular completions. From this viewpoint, mock modular forms and their generalizations are multi-valued periods on a suitable moduli space of elliptic curves, and their modular completions correspond to the images under the single-valued projection. This formalism has been explicitly developed in the physics literature. After a minor adjustment of weights using Maass operators, modular completions are found to be linear combinations of *products* of essentially holomorphic and antiholomorphic functions. Within the framework of $(\mathfrak{g}, \mathrm{K})$-modules, this shows that modular completions relate to tensor products of Harish-Chandra modules, which are notoriously difficult to study.

There is also a natural generalization for harmonic weak Maass forms: polyharmonic weak Maass forms. Even in the case of Eisenstein series, these forms reveal special values of derivatives of Dirichlet L-series, making them inherently interesting. They still offer a well-behaved interpretation in terms of $(\mathfrak{g}, \mathrm{K})$-modules, but the representations generated by the modular group are as mysterious as the tensor products of Harish-Chandra modules encountered with iterated integrals. At the level of Fourier coefficients, this is reflected by the limited existing results on the motivic nature of derivatives of L-series.

While at depth 1 we observe a remarkable parallelism between the representation theory of the modular group and that of the real Lie group associated with mock modular forms and harmonic weak Maass forms, this picture diverges for higher-depth examples.

**Problem:** *Connect higher-depth mixed mock modular forms and iterated integrals to polyharmonic weak Maass forms. Through this connection, explain the shift from a well-behaved representation theory for the modular group to a well-behaved representation theory for the real Lie group.*

**Remark:** As a starting point, iterated Eisenstein integrals (from the motivic perspective) and polyharmonic Eisenstein series (from the analytic perspective) present the fewest complications. However, it is currently unclear whether they are directly related, or whether cusp forms and weak forms interfere.

**Remark:** Generalizations of mock modular forms to higher depth exist in both integral and half-integral weights. One of the first examples in half-integral weights has been linked to class numbers for real-quadratic fields. It is therefore natural to extend the stated problem to the case of half-integral weights. However, even at depth one, the representation theory of the modular group is more complex as it involves infinite-dimensional composition factors.

2. Edgar Assing: *How many coefficients of cusp forms are needed?*

Let $S_{k+\frac{1}{2}}(4N)$ denote the space of holomorphic cusp forms of half-integral weight $k+\frac{1}{2}$ for the group $\Gamma_0(4N)$. Each $f \in S_{k+\frac{1}{2}}(4N)$ has a Fourier expansion of the form

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z}.$$

It is now natural to ask: *How many of the coefficients $a_f(n)$ are necessary to determine the form $f$ uniquely?* This question is rather vague and can be made precise in several ways. We have the following version in mind.

We fix a set $\mathcal{X} \subset \mathbb{N}$. Given two forms $f_1, f_2 \in S_{k+\frac{1}{2}}(4N)$ we write $f_1 \sim_{\mathcal{X}} f_2$ if $a_{f_1}(n) = a_{f_2}(n)$ for all $n \in \mathcal{X}$. We can restate the question as: *For which sets $\mathcal{X} \subseteq \mathbb{N}$ does $f_1 \sim_{\mathcal{X}} f_2$ imply $f_1 = f_2$?*

A case of historical importance arises when taking $\mathcal{X}$ to be the set

$$\mathcal{X}_{\mathrm{fd}} = \{|D| \colon D \text{ fundmental discriminant}, (-1)^k D > 0\}.$$

In this setting the question above has been raised by W. Kohnen (for Hecke eigenforms) in [2]. This case has been solved in subsequent works, most notably [1] and [3].

A very interesting question, which is still open, is if we can take $\mathcal{X}$ to be the set of primes

$$\mathcal{X}_{\mathrm{pr}} = \{p \colon \text{prime}\}.$$

In other words, we are asking wheter a form $f \in S_{k+\frac{1}{2}}(4N)$ is determined by its Fourier coefficients $a_f(p)$ at primes $p \in \mathcal{X}_{\mathrm{pr}}$. Solving this appears to be a hard problem. While the methods used in the case of $\mathcal{X}_{\mathrm{fd}}$ can be extended to handle almost primes, the prime case seems to require new ideas.

3. Lakshmi Ramesh: *Calculating cohomology*

Many participants of this workshop used group cohomology theories to understand automorphic forms and related geometric objects. For a finitely presented group $G$ and a $G-$vector space $M$, the software GAP can compute $H^1(G; M)$. However, for a general group $G$, checking if $H^k(G; M) = 0$ for $k \geq 2$ is shown to be undecidable. There are however special classes of groups for which combinatorial properties may be used to determine the cohomology numbers $h^k(G; M)$. The problem posed is to identify these groups and these counting problems, and to construct (efficient) algorithms to compute cohomology numbers.

4. Martin Raum: *Siegel modular generating series*

The theory of modular forms, particularly its analytic aspects, has been significantly impacted by generating series of combinatorial origin. For instance, asymptotic and exact formulas for Fourier coefficients of (weakly holomorphic) modular forms, as well as the study of congruences for Fourier coefficients, have been driven by connections to integer partitions. Most such generating series yield modular forms for subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

In the case of theta series (a specific type of generating series that tracks vector lengths in Euclidean lattices) generalization to sublattices yields Siegel modular forms associated with symplectic groups. The connection between Euclidean lattices and Siegel modular forms has significantly shaped the field, where the asymptotic behavior of Fourier coefficients has attracted considerable interest. In return, these results provide excellent bounds for the number of sublattices with prescribed geometry.

However, there are significantly fewer examples of Siegel modular generating series compared to classical modular generating series. Beyond theta series, examples include Borcherds–Kac–Weyl denominator formulas, Kudla generating series, generating series of Gromov–Witten invariants, and BPS counting functions from string theory.

**Problem:** *Find natural counting problems in enumerative algebra that yield Siegel modular forms, and formulate clear questions for the Siegel modular forms community regarding which properties should be proved about the coefficients.*

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