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The Principle of Limiting Amplitude for Wave Equations on a Star Graph

by

Kiyoshi Mochizuki, Hideo Nakazawa and Igor Trooshin

Abstract

We consider the dissipative wave propagation problem on an infinite star graph. High and low-frequency estimates and local Hölder conditions of the resolvent are studied for the reduced wave operator. Based on these results the principle of limiting amplitude is proved to hold for our wave propagation problem with oscillating external force.

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§0. Introduction

Let $\Gamma = \gamma_1 \times \cdots \times \gamma_n$ be a non-compact graph which consists of n semi-infinite rays $\gamma_j = \mathbf{R}_+ = \{x_j \in (0, \infty)\}$, with the origin of each ray identified with the single vertex of the graph (Figure 1). Each function $u(x)$ on $x \in \Gamma$ is identified as a vector $u(x) = (u_j(x_j))_{j=1}^n$. Let L be the Schrödinger operator

$$(0.1) \qquad Lu = -\frac{d^2u}{dx^2} + q(x)u = \Big(-\frac{d^2u_j}{dx_j^2} + q_j(x_j)u_j\Big)_{j=1}^n, \quad x_j \in \gamma_j,$$

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K. Mochizuki: Department of Mathematics, Chuo University, Kasuga, Bunkyo, Tokyo 112-8551, Japan (Emeritus Tokyo Metropolitan University);
e-mail: mochizuk@math.chuo-u.ac.jp
H. Nakazawa: Department of Mathematics, Nippon Medical School, Kyonan-chou, Musashino, Tokyo 180-0023, Japan;
e-mail: nakazawa-hideo@nms.ac.jp
I. Trooshin: Department of Mathematics, Shinshu University, Asahi, Matsumoto, Nagano 390-8621, Japan;
e-mail: trushin@shinshu-u.ac.jp

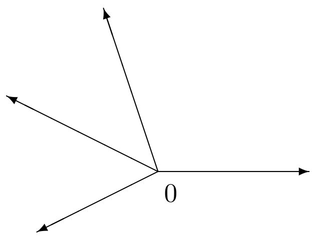

Figure 1.

on Γ defined for functions u satisfying the natural Kirchhoff boundary conditions on the vertex:

$$u_1(0) = u_2(0) = \cdots = u_n(0), \tag{0.2}$$

$$u_1'(0) + u_2'(0) + \cdots + u_n'(0) = 0, \tag{0.3}$$

where $u_j' = du_j/dx_j$.

Each potential $q_j(x)$ $(j = 1, \ldots, n)$ is assumed to be real on γ_j (hereafter we simply write x for each x_j if there is no possibility of confusion) and satisfies the condition

$$\int_{\gamma_j} (1+x)^\beta |q_j(x)|\, dx < \infty \quad \text{with } \beta \geq 1 \ (j = 1, \ldots, n). \tag{0.4}$$

Let $\mathcal{H} = L^2(\Gamma) = \prod_{j=1}^n L^2(\gamma_j)$ be the Hilbert space with norm

$$\|f\|_\Gamma = \left(\sum_{j=1}^n \|f_j\|_{\gamma_j}^2 \right)^{1/2}, \quad \|f_j\|_{\gamma_j}^2 = \int_{\gamma_j} |f_j(x)|^2\, dx.$$

Under the above conditions on $q_j(x)$ the operator L restricted to the domain

$$\begin{aligned} \mathcal{D}(L) = \big\{ u, \tfrac{du}{dx} \in \mathcal{H};\ \tfrac{du}{dx} \text{ being absolutely continuous,} \\ \text{satisfying (0.2), (0.3) and } (-\tfrac{d^2u}{dx^2} + q_j(x)u)_{j=1}^n \in \mathcal{H} \big\} \end{aligned} \tag{0.5}$$

forms a lower semi-bounded self-adjoint operator in $\mathcal{H}$, and the essential spectrum of L fills the non-negative half-line $[0, \infty)$. The discrete eigenvalues are in general finite on $(-\infty, 0)$.

In this paper, we are concerned with the asymptotic behavior as $t \to \infty$ of the solution $w(x,t) = (w_j(x,t))_{j=1}^n$ of the wave equation

$$\begin{aligned} \partial_t^2 w(x,t) + Lw(x,t) + B\partial_t w(x,t) = g(x)e^{-i\sigma_0 t}, \quad (x,t) \in \Gamma \times (0,\infty), \\ w(x,0) = \partial_t w(x,0) = 0, \quad x \in \Gamma, \end{aligned} \tag{0.6}$$

requiring that L is non-negative in $\mathcal{H}$, B is a non-negative multiplication operator with $Bf(x) = (b_j(x)f_j(x))_{j=1}^n$, $g(x) = (g_j(x))_{j=1}^n$ belongs to some weighted L^2-space on Γ, and σ_0 is a positive constant.

Problem (0.6) is a simplified mathematical model when one considers the propagation of waves of different natures in thin, tube-like domains. The simplest non-compact graphs are the positive half-line $\mathbf{R}_+ = (0, \infty)$ or the whole real line $\mathbf{R} = (-\infty, \infty)$. In these simple graphs, the steady state problem associated with (0.6) has been studied, when $B = 0$, by Marchenko [8] and Faddeev [2, 3] (see also Yafaev [17, Chapters 4 and 5]), and we can find there complete descriptions of the scattering theory. Their results are partly extended for example by Gerasimenko–Pavlov [5], Freiling–Ignatiev [4] and Mochizuki–Trooshin [12] (see also Mochizuki [10, Chapter 8]) to problems on general non-compact graphs. The behavior of the resolvent $(L - \kappa^2)^{-1}$, $\kappa \in \mathbf{C}_+ = \{\kappa \in \mathbf{C}; \operatorname{Im} \kappa > 0\}$ presented there is, with some addenda, applied in Sections 2 and 3 to examine necessary local regularity and low-frequency estimates of the resolvent.

We shall prove that under certain assumptions, the solution $w = (w_j)_{j=1}^n = (w_j(x,t))_{j=1}^n$ has the property

$$w_j(x,t) = v_j(x,\sigma_0)e^{-i\sigma_0 t} + \frac{1}{\sigma_0}[Pg]_j(x) + o(1) \quad (j = 1, 2, \ldots, n),\ t \to \infty$$

in an appropriate topology, where $v = (v_j)_{j=1}^n$ satisfies

$$-\frac{d^2 v_j}{dx^2} + q_j(x)v_j - i\sigma_0 b_j(x)v_j - \sigma_0^2 v_j = g_j \quad \text{in } \gamma_j,$$

with the Kirchhoff condition. The operator $P = (P_{kj})_{k,j=1}^n$ is determined in considering the steady state problem near the resonance 0 of L. This is the statement of the principle of limiting amplitude. The principle has been studied by many authors for waves in various infinite domains of $\mathbf{R}^n$, $n \geq 2$; see, e.g., [1, 6, 7, 9, 11, 13, 14, 15, 16] and references there. However, few results are available for waves on non-compact graphs. The complexity may be in the existence of the 0-resonance. This causes the appearance of the singularity when the resolvent of L approaches the resonance point. The existence of the friction term $-i\sigma_0 Bv$ may bring another complexity. For this problem, we have the proofs of [9, 11], which are also applicable to the present equation.

This paper will be organized as follows: In the next Section 1 we consider an abstract wave propagation problem with dissipation, and summarize conditions under which the validity of the principle of limiting amplitude is guaranteed. The results are applied in Section 2 to the Dirichlet problem of the concrete wave equation on the positive half-line $\mathbf{R}_+$. Finally, in Section 3 we extend the results of Section 2 to the problem on the non-compact star graph.

§1. An abstract setting of the principle of limiting amplitude

Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let L be a non-negative self-adjoint operator in $\mathcal{H}$ with dense domain $\mathcal{D}(L)$. Let B be a non-negative bounded operator in $\mathcal{H}$.

We consider the following problem:

$$\begin{aligned}&\partial_t^2 w(t) + Lw(t) + B\partial_t w(t) = ge^{-i\sigma_0 t}, \quad t>0,\\ &\text{with } w(0)=\partial_t w(0)=0,\end{aligned} \tag{1.1}$$

where σ_0 is a non-zero real number. In the following we put $\sigma_0>0$. The same treatment is possible when $\sigma_0<0$. For $\kappa\in\mathbf{C}_+$ let $R(\kappa^2)=(L-\kappa^2)^{-1}$ be the resolvent of the operator L.

There exists a Banach space X such that

$$X^* \hookrightarrow \mathcal{H} \hookrightarrow X,$$

where X^* is the dual space of X. In the following we write $B=A^*A$, where $A^*=A=\sqrt{B}$ in $\mathcal{H}$ and A is extended to an operator in $\mathcal{B}(X,\mathcal{H})$. Then A^* forms the operator in $\mathcal{B}(\mathcal{H},X^*)$.

Assumption 1 (Principle of limiting absorption). For $\kappa=\sigma+i\tau$ with $\sigma\in\mathbf{R}_\pm$ and $\tau>0$, the limit

$$R(\sigma^2\pm i0)=\lim_{\tau\downarrow 0} R(\{\sigma+i\tau\}^2)$$

exists in $\mathcal{B}(X^*,X)$. Thus, as a bounded operator from X^* to X, the resolvent $R(\kappa^2)$ is extended continuously in $\kappa\in\overline{\mathbf{C}_+}\setminus\{0\}$.

Assumption 2 (High-frequency estimates). There exists $C_0>0$ such that for all $\kappa\in\overline{\mathbf{C}_+}\setminus\{0\}$ we have

$$\|R(\kappa^2)\|_{\mathcal{B}(X^*,X)}\le C_0|\kappa|^{-1}.$$

Assumption 3 (Local Hölder continuity). Let $I=[a,b]\subset\mathbf{R}_\pm$. We set $\Lambda_\pm=\{\kappa=\sigma+i\tau;\ \sigma\in I,\ 0\le\tau<\nu_0\}$ for a small $\nu_0>0$. Then there exists $C_1=C_1(\Lambda_\pm)>0$ such that for any κ, $\kappa'\in\Lambda_\pm$ with $|\kappa-\kappa'|<1$ we have

$$\|R(\kappa^2)-R(\kappa'^2)\|_{\mathcal{B}(X^*,X)}\le C_1|\kappa-\kappa'|^{\delta_1} \quad \text{for some } 0<\delta_1\le 1.$$

Assumption 4 (Low-frequency estimates). There exist $P_0\in\mathcal{B}(X^*,X)$ and $C_2>0$ such that we have in $\Lambda_0=\{\kappa\in\overline{\mathbf{C}_+};\ 0<|\kappa|<1\}$,

$$\|\kappa R(\kappa^2)-P_0\|_{\mathcal{B}(X^*,X)}\le C_2|\kappa|^{\delta_2} \quad \text{for some } 0<\delta_2<1.$$

Assumption 5 (Compactness of $AR(\kappa^2)A^*$). For each $\kappa \in \overline{\mathbf{C}_+}\setminus\{0\}$, $AR(\kappa^2)A^*$ forms a compact operator in $\mathcal{H}$.

Under these assumptions, it is well known that for each $g \in \mathcal{H}$, solutions $w(t)$ of problem (1.1) exist and are unique in the class of differentiable functions of $t > 0$ with values in $\mathcal{H}$.

For this solution we shall prove the following theorem which establishes the principle of limiting amplitude.

Theorem 1. *Under Assumptions 1–5 consider the initial value problem (1.1) with $g \in X^*$. Then as $t \to \infty$ the solution $w(t)$ satisfies*

$$w(t) = v(\sigma_0)e^{-i\sigma_0 t} + \frac{1}{\sigma_0}Pg + o(1) \quad \textit{strongly in } X,$$

where $v(\sigma_0) = v(\sigma_0 + i0) \in X$ is the unique solution of the problem

$$Lv - i\kappa Bv - \kappa^2 v = g, \tag{1.2}$$

with $\kappa = \sigma_0 + i0$ and

$$P = P_0 + iP_0A^*\{1 - iAP_0A^*\}^{-1}AP_0.$$

Let

$$\widetilde{w} = \widetilde{w}(\kappa) = \int_0^\infty w(t)e^{i\kappa t}\,dt,$$

where $\kappa \in \overline{\mathbf{C}}_+$. Then $\widetilde{w}$ satisfies the reduced equation

$$L\widetilde{w} - i\kappa B\widetilde{w} - \kappa^2\widetilde{w} = \frac{-g}{i(\kappa - \sigma_0)}.$$

So, if v solves problem (1.2), then $v(\kappa) = -i(\kappa - \sigma_0)\widetilde{w}(\kappa)$ and the solution of (1.1) is given by

$$w(t) = \frac{1}{2\pi i}\lim_{\rho\to\infty}\int_{\rho+i\tau_0}^{-\rho+i\tau_0} \frac{v(\kappa)}{\kappa - \sigma_0}e^{-i\kappa t}\,d\kappa,$$

where τ_0 is a large positive number.

Let $\phi = Av$. Then (1.2) is reduced to the following equation for ϕ:

$$\phi - i\kappa AR(\kappa^2)A^*\phi = AR(\kappa^2)g. \tag{1.3}$$

Conversely, let $\phi \in \mathcal{H}$ satisfy this equation. Then the solution of (1.2) is given by

$$v = i\kappa R(\kappa^2)A^*\phi + R(\kappa^2)g. \tag{1.4}$$

In fact, if we denote the right by h, then we have $(L-\kappa^2)h = i\kappa A^*\phi + g$. By means of (1.3),

$$\begin{aligned} i\kappa A^*\phi &= i\kappa A^*\big\{i\kappa AR(\kappa^2)A^*\phi + AR(\kappa^2)g\big\} \\ &= i\kappa B\big\{i\kappa R(\kappa^2)A^*\phi + R(\kappa^2)g\big\} = i\kappa Bh. \end{aligned}$$

Thus, h satisfies the equation $(L-\kappa^2)h = i\kappa Bh + g$ showing $h = v$.

Lemma 1.1. *The operator $-i\kappa AR(\kappa^2)A^*$ is dissipative in $\mathcal{H}$: i.e., for each $\kappa = -\sigma + i\tau$ $(\tau \geq 0)$ and $f \in \mathcal{H}$ the following inequality holds:*

$$\mathrm{Re}[-i\kappa(AR(\kappa^2)A^*f, f)] \geq 0.$$

Proof. If $\sigma > 0$, we have

$$\begin{aligned} &\mathrm{Re}[-i\kappa(R(\kappa^2)A^*f, A^*f)] \\ &\quad = \int_0^\infty \frac{(\lambda+\sigma^2+\tau^2)\tau}{(\lambda-\sigma^2+\tau^2)^2+(2\tau\sigma)^2}\frac{d}{d\lambda}(E(\lambda)A^*f, Af)\,d\lambda \geq 0, \end{aligned}$$

where $E(\lambda)$ is the spectral family of the self-adjoint operator L. Further, by the limit procedure we have

$$\begin{aligned} &\lim_{\tau\downarrow 0}\mathrm{Re}[-i\kappa(R(\kappa^2)A^*f, A^*f)] \\ &\quad = -\frac{|\sigma|}{2i}(\{R(\sigma^2-i0) - R(\sigma^2+i0)\}A^*f, A^*f) \\ &\quad = \pi|\sigma|\frac{d}{d\lambda}(E(\lambda)A^*f, A^*f)|_{\lambda=\sigma^2} \geq 0. \end{aligned}$$

□

This lemma shows the uniqueness and the existence of solutions $\phi = \phi(x,\kappa)$ of (1.3) in $\mathcal{H}$.

Lemma 1.2. *For each $\kappa \in \overline{\mathbf{C}_+}\backslash\{0\}$ we have*

$$\|\phi(\kappa)\| \leq C_3|\kappa|^{-1}\|g\|_{X^*}, \tag{1.5}$$

where $C_3 = C_0\|A\|_{\mathcal{B}(X,\mathcal{H})}$. Moreover, for $\kappa, \kappa' \in \Lambda_\pm$,

$$\|\phi(\kappa) - \phi(\kappa')\| \leq C_4\|A^*\|_{\mathcal{B}(X,\mathcal{H})}|\kappa-\kappa'|^{\delta_1}\|g\|_{X^*}, \tag{1.6}$$

where $C_4 = \max_{\Lambda_\pm}\{|\kappa|^{-2}\}C_3^2 + C_1C_3\|A^*\|_{\mathcal{B}(\mathcal{H},X^*)} + C_1$.

Proof. By use of Lemma 1.1 we have from (1.3),

$$\|\phi(\kappa)\| \leq \|AR(\kappa^2)g\| \leq \|A\|_{\mathcal{B}(X,\mathcal{H})}\|R(\kappa^2)g\|_X.$$

Then (1.5) is direct from Assumption 2. Next note that

$$\begin{aligned}&\phi(\kappa)-\phi(\kappa')-i\kappa AR(\kappa^2)A^*\{\phi(\kappa)-\phi(\kappa')\}\\&\qquad=i(\kappa-\kappa')AR(\kappa^2)A^*\phi(\kappa')+i\kappa' A\{R(\kappa^2)-R(\kappa'^2)\}A^*\phi(\kappa')\\&\qquad\quad+A\{R(\kappa^2)-R(\kappa'^2)\}g.\end{aligned}$$

Then we can use Lemma 1.1 once more to obtain

$$\begin{aligned}\|\phi(\kappa)-\phi(\kappa')\|\le\|A\|_{\mathcal{B}(X,\mathcal{H})}\big\{&|\kappa-\kappa'|\,\|R(\kappa^2)A^*\phi(\kappa')\|_X\\&+|\kappa'|\,\|\{R(\kappa^2)-R(\kappa'^2)\}A^*\phi(\kappa')\|_X\\&+\|\{R(\kappa^2)-R(\kappa'^2)\}g\|_X\big\}.\end{aligned}$$

Each term on the right is estimated as follows: By use of Assumption 2 and (1.5),

$$|\kappa-\kappa'|\,\|R(\kappa^2)A^*\phi(\kappa)\|_X\le C_0|\kappa|^{-1}|\kappa-\kappa'|\,\|A^*\phi\|_{X^*}\le C_3^2|\kappa|^{-2}|\kappa-\kappa'|\,\|g\|_{X^*}.$$

Assumption 3 implies

$$\begin{aligned}&|\kappa'|\,\|\{R(\kappa^2)-R(\kappa'^2)\}A^*\phi(\kappa')\|_X\\&\qquad\le|\kappa'|C_1|\kappa-\kappa'|^{\delta_1}\|A^*\phi(\kappa')\|_{X^*}\le C_1C_3\|A^*\|_{\mathcal{B}(\mathcal{H},X^*)}|\kappa-\kappa'|^{\delta_1}\|g\|_{X^*},\\&\|\{R(\kappa^2)-R(\kappa'^2)\}g\|_X\le C_1|\kappa-\kappa'|^{\delta_1}\|g\|_{X^*}.\end{aligned}$$

Estimate (1.6) is thus concluded. □

Lemma 1.3. *Let* $C_5=C_0(C_3\|A^*\|_{\mathcal{B}(\mathcal{H},X^*)}+1)$*. Then*

$$\|v(\kappa)\|_X\le C_5|\kappa|^{-1}\|g\|_{X^*}\quad\text{for }\kappa\in\overline{\mathbf{C}_+}\setminus\{0\}.\tag{1.7}$$

Moreover, let $C_6=C_4(C_3\|A^*\|_{\mathcal{B}(\mathcal{H},X^*)}+1)$*. Then*

$$\|v(\kappa)-v(\kappa')\|_X\le C_6|\kappa-\kappa'|^{\delta_2}\|g\|_{X^*}\quad\text{for }\kappa^2,\kappa'^2\in K_\pm.\tag{1.8}$$

Proof. Apply the inequalities of Assumption 2 to (1.4). Then we have

$$\begin{aligned}\|v\|_X&\le|\kappa|\,\|R(\kappa^2)A^*\phi\|_X+\|R(\kappa^2)g\|_X\\&\le C_0\big\{\|A^*\phi\|_{X^*}+|\kappa|^{-1}\|g\|_{X^*}\big\}\le C_0|\kappa|^{-1}\big\{\|A^*\|_{\mathcal{B}(\mathcal{H},X^*)}C_3+1\big\}\|g\|_{X^*}\end{aligned}$$

showing (1.7). Next, note the identity

$$\begin{aligned}v(\kappa)-v(\kappa')=\;&i\kappa R(\kappa^2)A^*\{\phi(\kappa)-\phi(\kappa')\}\\&+i(\kappa-\kappa')R(\kappa^2)A^*\phi(\kappa')+i\kappa'\{R(\kappa^2)-R(\kappa'^2)\}A^*\phi(\kappa')\\&+\{R(\kappa^2)-R(\kappa'^2)\}g.\end{aligned}$$

Then the estimates given in the proof of Lemma 1.2 are also available in this case to conclude (1.8). □

Lemma 1.4. *There exists* $C_6 > 0$ *such that*

$$\|\kappa v(\kappa) - Pg\|_X \le C_6 |\kappa|^{\delta_1} \|g\|_{X^*} \quad \text{for } |\kappa| \le 1.$$

Proof. By (1.4) we have

$$\kappa v(\kappa) = i\kappa\{\kappa R(\kappa^2) - P_0\}A^*\phi(\kappa) + \{\kappa R(\kappa^2) - P_0\}g + i\kappa P_0 A^*\phi(\kappa) + P_0 g.$$

Here, by Assumption 4 and Lemma 1.2,

$$\begin{aligned}\|\{\kappa R(\kappa^2) - P_0\}(i\kappa A^*\phi + g)\|_X &\le C_2|\kappa|^{\delta_2}\{\|\kappa A^*\phi\|_{X^*} + \|g\|_{X^*}\} \\ &\le |\kappa|^{\delta_2} C_2(C_3\|A^*\|_{\mathcal{B}(\mathcal{H},X^*)} + 1)\|g\|_{X^*}.\end{aligned}$$

Note that

$$\begin{aligned}i\kappa P_0 A^*\phi(\kappa) = {} & iP_0A^*\{1 - i\kappa AR(\kappa^2)A^*\}^{-1}A(\kappa R(\kappa^2) - P_0)g \\ & + iP_0A^*[\{1 - i\kappa AR(\kappa^2)A^*\}^{-1} - \{1 - iAP_0A^*\}^{-1}]AP_0 g \\ & + iP_0A^*\{1 - iAP_0A^*\}^{-1}AP_0 g.\end{aligned}$$

Then since $\|\{1 - i\kappa AR(\kappa^2)A^*\}^{-1}\| \le 1$ and $\|\{1 - iAP_0A^*\}^{-1}\| \le 1$ follow from Lemma 1.1, noting $AP_0 \in \mathcal{B}(X^*, \mathcal{H})$, $P_0A^* \in \mathcal{B}(\mathcal{H}, X)$, we obtain

$$\|i\kappa P_0A^*\phi(\kappa) - iP_0A^*\{1 - iAP_0A^*\}^{-1}AP_0 g\|_X \le C|\kappa|^{\delta_2}\|g\|_{X^*}$$

for a suitable constant $C > 0$. Thus, the desired inequality is concluded. □

Proof of Theorem 1. We start from the expression in X:

$$w(t) = \frac{1}{2\pi i}\lim_{\rho\to\infty}\int_{\rho+i\tau_0}^{-\rho+i\tau_0} \frac{v(\kappa)}{\kappa - \sigma_0} e^{-i\kappa t}\, d\kappa.$$

Note that $v(\kappa)$ is an X-valued analytic function of $\kappa = -\sigma + i\tau$ in $\tau = \operatorname{Im}\kappa > 0$. So, by the use of the Cauchy integral formula,

$$\begin{aligned}\int_{\rho+i\tau_0}^{-\rho+i\tau_0} \frac{v(\kappa)}{\kappa - \sigma_0} e^{-i\kappa t}\, d\kappa = {} & -\lim_{\varepsilon\downarrow 0}\int_{-\rho}^{\rho} \frac{v(\sigma + i\varepsilon)e^{-i(\sigma+i\varepsilon)t}}{\sigma - \sigma_0 + i\varepsilon}\, d\sigma \\ & - \int_0^{\tau_0}\left\{\frac{v(\rho + i\tau)e^{-i(\rho+i\tau)t}}{\tau - i(\rho - \sigma_0)} - \frac{v(-\rho + i\tau)e^{-i(-\rho+i\tau)t}}{\tau + i(\rho + \sigma_0)}\right\} d\tau.\end{aligned}$$

Here, the second term on the right tends to 0 in X as $\rho \to \infty$. Thus, choosing $0 < 2a < \sigma_0$, we have

$$
\begin{aligned}
w(t) = \lim_{\varepsilon \downarrow 0} \frac{-1}{2\pi i} \Bigg[\bigg(\int_{a+\sigma_0}^{\infty} + \int_{a}^{-a+\sigma_0} + \int_{-\infty}^{-a} \bigg) \frac{v(\sigma + i\varepsilon) g e^{-i(\sigma+i\varepsilon)t}}{\sigma - \sigma_0 + i\varepsilon}\, d\sigma \\
+ \int_{-a+\sigma_0}^{a+\sigma_0} \frac{\{v(\sigma+i\varepsilon) - v(\sigma_0 + i\varepsilon)\} e^{(\varepsilon - i\sigma)t}}{\sigma - \sigma_0 + i\varepsilon}\, d\sigma \\
+ v(\sigma_0 + i\varepsilon) e^{(\varepsilon - i\sigma_0)t} \int_{-a}^{a} \frac{e^{-i\sigma t}}{\sigma + i\varepsilon}\, d\sigma \\
+ \int_{-a}^{a} \frac{\{(\sigma + i\varepsilon) v(\sigma + i\varepsilon) - Pg\} e^{(\varepsilon - i\sigma)t}}{(\sigma - \sigma_0 + i\varepsilon)(\sigma + i\varepsilon)}\, d\sigma \\
+ Pg \int_{-a}^{a} \frac{e^{(\varepsilon - i\sigma)t}}{(\sigma - \sigma_0 + i\varepsilon)(\sigma + i\varepsilon)}\, d\sigma \Bigg] \equiv \frac{-1}{2\pi i} \big[I_1 + I_2 + I_3 + I_4 + I_5 \big].
\end{aligned}
$$

By the Hölder continuity and the decay and singularity estimates of $v(\sigma + i\varepsilon)$ in Lemma 1.3, we can use the Riemann–Lebesgue theorem to see $I_1, I_2 \to 0$ strongly in X as $t \to \infty$.

On the other hand, since

$$
\begin{aligned}
\lim_{\varepsilon \downarrow 0} \int_{-a}^{a} \frac{e^{-i\sigma t}}{\sigma + i\varepsilon}\, d\sigma &= -2i \lim_{\varepsilon \downarrow 0} \int_{0}^{a} \frac{\sigma \sin(\sigma t) + \varepsilon \cos(\sigma t)}{\sigma^2 + \varepsilon^2}\, d\sigma \\
&= -2i \bigg\{ \int_0^a \frac{\sin(\sigma t)}{\sigma}\, d\sigma + \int_0^{\infty} \frac{1}{\sigma^2 + 1}\, d\sigma \bigg\} \to -2\pi i \quad \text{as } t \to \infty,
\end{aligned}
$$

it follows that

$$
\frac{-1}{2\pi i} I_3 \to v(\sigma_0 + 0) e^{-i\sigma_0 t} \quad \text{as } t \to \infty.
$$

Moreover, since

$$
\begin{aligned}
&\int_{-a}^{a} \frac{e^{-i\sigma t}}{(\sigma - \sigma_0 + i\varepsilon)(\sigma + i\varepsilon)}\, d\sigma \\
&\quad = \int_0^a \bigg\{ \frac{e^{-i\sigma t}}{(\sigma - \sigma_0 + i\varepsilon)(\sigma + i\varepsilon)} + \frac{e^{i\sigma t}}{(\sigma + \sigma_0 - i\varepsilon)(\sigma - i\varepsilon)} \bigg\}\, d\sigma \\
&\quad = \int_0^a \bigg\{ \frac{2\sigma e^{-i\sigma t}}{(\sigma^2 - (\sigma_0 - i\varepsilon)^2)(\sigma + i\varepsilon)} + \frac{2i\sigma \sin(\sigma t) + 2i\varepsilon \cos(\sigma t)}{(\sigma + \sigma_0 - i\varepsilon)(\sigma^2 + \varepsilon^2)} \bigg\}\, d\sigma,
\end{aligned}
$$

letting $\varepsilon \downarrow 0$ we have

$$
\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \int_{-a}^{a} \frac{e^{(\varepsilon - i\sigma)t}}{(\sigma - \sigma_0 + i\varepsilon)(\sigma + i\varepsilon)}\, d\sigma \\
&\quad = \int_0^a \frac{2e^{-i\sigma t}}{\sigma^2 - \sigma_0^2}\, d\sigma - 2i \int_0^a \bigg\{ \frac{\sin(\sigma t)}{\sigma_0 \sigma} - \frac{\sin(\sigma t)}{\sigma_0(\sigma + \sigma_0)} \bigg\}\, d\sigma - 2i \int_0^{\infty} \frac{1}{\sigma_0(\sigma^2 + 1)}\, d\sigma,
\end{aligned}
$$

and it follows that

$$\frac{-1}{2\pi i} I_5 \to \frac{1}{\sigma_0} Pg \quad \text{as } t \to \infty.$$

Summing up, we obtain the desired conclusion. □

The above theorem is based on [11, Theorem 3], where some dissipative wave equations in the exterior domain in $\mathbf{R}^n$ $(n \geq 2)$ are treated and Assumptions 1 to 5 with $P = 0$ are proved to hold. Note that when we consider the self-adjoint problem without dissipation, the Stone theorem can be replaced for a high-frequency estimate of the resolvent. See, e.g., [14, 15], where Assumptions 3 and 4 with $P = 0$ are proved to establish the principle of limiting amplitude for a wide class of n-dimensional $(n \geq 3)$ wave propagation problems. Wave equations may have resolvent with higher-order singularity at the origin. As we shall see in Sections 2 and 3, the first-order singularity appears in the graph problems. Assumption 4 is formulated to apply to these problems.

§2. The wave equation on the half-line

As the simplest problem we first consider the wave equation

$$\begin{aligned} &\partial_t^2 w - \partial_x^2 w + b(x)\partial_t w + q(x)w = e^{-i\sigma_0 t} g(x), \quad (x,t) \in \mathbf{R}_+ \times (0,\infty), \\ &w|_{x=0} = 0 \quad \text{and} \quad w(x,0) = \partial_x w(x,0) = 0, \end{aligned} \tag{2.1}$$

where $\sigma_0 > 0$, $b(x)$ is a non-negative continuous function satisfying

$$b(x) \leq C(1+x)^{-\alpha} \quad \text{with } \alpha > 1 \tag{2.2}$$

and the potential $q(x)$ is a real function satisfying (0.4).

Let $\mathcal{H} = L^2(\mathbf{R}_+)$ be the Hilbert space with norm $\|f\|^2 = \int_0^\infty |f(x)|^2\,dx < \infty$, and let $Lu = -d^2u/dx^2 + q(x)u$ be the self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(L) = \{u, u' \in \mathcal{H};\ u' \text{ is absolutely continuous, } u(0) = 0 \text{ and } -u'' + q(x)u \in \mathcal{H}\}$.

The essential spectrum of L fills the non-negative half-line $[0,\infty)$. The resolvent of L is an integral operator, whose kernel will be given with the use of two solutions of the equation

$$-\frac{d^2u}{dx^2} + q(x)u = \kappa^2 u \quad \text{in } \mathbf{R}_+. \tag{2.3}$$

Let $\omega(x,\kappa)$ be the solution of (2.3) which satisfies the initial condition $\omega(0,\kappa) = 0$, $\omega'(0,\kappa) = 1$. Let $e(x,\kappa)$ $(\operatorname{Im}\kappa \geq 0)$ be the Jost solution of (2.3) which satisfies the asymptotic condition $e(x,\kappa) \to e^{i\kappa x}$ $(x \to \infty)$. The Wronskian of $\omega(x,\kappa)$ and $e(x,\kappa)$ is

$$\langle \omega(x,\kappa), e(x,\kappa)\rangle = \omega(x,\kappa)e'(x,\kappa) - \omega'(x,\kappa)e(x,\kappa) = -e(0,\kappa).$$

So these solutions are linearly independent when $e(0,\kappa) \neq 0$, and the resolvent $R(\kappa^2) = (L-\kappa^2)^{-1}$ of L forms an integral operator with kernel

$$R(x,y;\kappa) = \begin{cases} \dfrac{\omega(x,\kappa)e(y,\kappa)}{e(0,\kappa)} & (x \le y), \\ \dfrac{e(x,\kappa)\omega(y,\kappa)}{e(0,\kappa)} & (y \le x). \end{cases} \tag{2.4}$$

The unique existence and fundamental properties of solutions $\omega(x,\kappa)$ and $e(x,\kappa)$ are well known ([8]) and follow from the equivalent integral equations

$$\omega(x,\kappa) = \frac{\sin\kappa x}{\kappa} + \int_0^x \frac{\sin\kappa(x-y)}{\kappa} q(y)\omega(y,\kappa)\,dy,$$
$$e(x,\kappa) = e^{i\kappa x} - \int_x^\infty \frac{\sin\kappa(x-y)}{\kappa} q(y)e(y,\kappa)\,dy, \quad \operatorname{Im}\kappa \ge 0.$$

We put

$$\omega(x,\kappa) = e^{-i\kappa x}\xi(x,\kappa), \quad e(x,\kappa) = e^{i\kappa x}z(x,\kappa)$$

and rewrite the equations as those of $\xi(x,\kappa)$ and $z(x,\kappa)$:

$$\xi(x,\kappa) = \frac{e^{2i\kappa x}-1}{2i\kappa} + \int_0^x \frac{e^{2i\kappa(x-y)}-1}{2i\kappa} q(y)\xi(y,\kappa)\,dy, \tag{2.5}$$

$$z(x,\kappa) = 1 - \int_x^\infty \frac{1-e^{2i\kappa(y-x)}}{2i\kappa} q(y)z(y,\kappa)\,dy. \tag{2.6}$$

Lemma 2.1. *For each* κ, $\kappa' \in \overline{\mathbf{C}_+}\backslash\{0\}$ *we have*

$$\left|\frac{e^{2i\kappa x}-1}{2i\kappa}\right| = \left|\int_0^x e^{2i\kappa t}\,dt\right| \le \min\{x, |\kappa|^{-1}\}, \tag{2.7}$$

$$\left|x\frac{e^{2i\kappa x}}{2i\kappa} - \frac{e^{2i\kappa x}-1}{(2i\kappa)^2}\right| = \left|\int_0^x te^{2i\kappa t}\,dt\right| \le \frac{1}{2}\min\{x^2, x|\kappa|^{-1} + |\kappa|^{-2}\}, \tag{2.8}$$

$$\begin{aligned} \left|\frac{e^{2i\kappa x}-1}{2i\kappa} - \frac{e^{2i\kappa' x}-1}{2i\kappa'}\right| &= \left|\int_0^x \{e^{2i\kappa t} - e^{2i\kappa' t}\}\,dt\right| = \left|\int_0^x 2it\int_{\kappa'}^{\kappa} e^{2i\sigma t}\,d\sigma\,dt\right| \\ &\le \min\{|\kappa-\kappa'|x^2, 2x, |\kappa|^{-1}+|\kappa'|^{-1}\}, \end{aligned} \tag{2.9}$$

$$\begin{aligned} \left|\frac{e^{2i\kappa t}-1}{2i\kappa} - x\right| &= \left|\int_0^x \{e^{2i\kappa t}-1\}\,dt\right| = \left|\int_0^x 2it\int_0^\kappa e^{2i\sigma t}\,d\sigma\,dt\right| \\ &\le \min\{|\kappa|x^2, 2x\}. \end{aligned} \tag{2.10}$$

Lemma 2.2. *For each* $\kappa \in \mathbf{C}$, (2.5) *has a unique solution* $\xi(x,\kappa)$, *which is an entire function of* κ, *and if* $\operatorname{Im}\kappa \ge 0$, *we have for any* $x \in \mathbf{R}_+$,

$$|\xi(x,\kappa)| \le \min\{x, |\kappa|^{-1}\}e^{\int_0^x y|q(y)|\,dy}. \tag{2.11}$$

For each $\kappa \in \overline{\mathbf{C}_+}$, (2.6) *has a unique solution* $z(x,\kappa)$, *which is analytic in* $\mathbf{C}_+$ *continuous up to* $\overline{\mathbf{C}_+}$, *and we have*

$$|z(x,\kappa)| \le e^{\int_x^\infty \min\{y,|\kappa|^{-1}\}|q(y)|\,dy} \le e^{\int_x^\infty y|q(y)|\,dy}, \tag{2.12}$$

$$|z(x,\kappa)-1| \le \int_x^\infty \min\{y,|\kappa|^{-1}\}|q(y)|\,dy\, e^{\int_0^\infty y|q(y)|\,dy}, \tag{2.13}$$

$$|z'(x,\kappa)| = |\partial_x z(x,\kappa)| \le \int_x^\infty |q(y)|\,|z(y,\kappa)|\,dy. \tag{2.14}$$

Proof. The usual successive approximation argument can be applied to equations (2.5) and (2.6).

First consider (2.5), restricting ourselves to the case $\operatorname{Im}\kappa \ge 0$. Put $\xi_0(x,\kappa) = \int_0^x e^{2i\kappa t}\,dt$ and determine $\xi_k(x,\kappa)$ successively by

$$\xi_k(x,\kappa) = \int_0^x \frac{e^{2i\kappa(x-y)}-1}{2i\kappa} q(y)\xi_{k-1}(y,\kappa)\,dy.$$

Then (2.7) implies

$$\begin{aligned}
|\xi_1(x,\kappa)| &\le \min\{x,|\kappa|^{-1}\} \int_0^x \min\{y,|\kappa|^{-1}\}|q(y)|\,dy, \\
|\xi_k(x,\kappa)| &\le \min\{x,|\kappa|^{-1}\} \int_0^x |q(y)|\,|\xi_{k-1}(y,\kappa)|\,dy \\
&\le \frac{\min\{x,|\kappa|^{-1}\}}{k!} \left[\int_0^x \min\{y,|\kappa|^{-1}\}|q(y)|\,dy\right]^k.
\end{aligned}$$

Hence $\xi(x,\kappa) = \sum_{k=0}^\infty \xi_k(x,\kappa)$ converges to the unique solution of (2.5) in $x \ge 0$, $\operatorname{Im}\kappa \ge 0$, and satisfies (2.11).

For (2.6) we put $z_0(x,\kappa) = 1$ and determine $z_k(x,\kappa)$ successively by

$$z_k(x,\kappa) = -\int_x^\infty \frac{1-e^{2i\kappa(y-x)}}{2i\kappa} q(y) z_{k-1}(y,\kappa)\,dy.$$

Then (2.7) also shows

$$\begin{aligned}
|z_k(x,\kappa)| &\le \int_x^\infty \min\{y,|\kappa|^{-1}\}|q(y)|\,|z_{k-1}(y,\kappa)|\,dy \\
&\le \frac{1}{k!}\left[\int_x^\infty \min\{y,|\kappa|^{-1}\}|q(y)|\,dy\right]^k,
\end{aligned}$$

and the unique solution of (2.6) is obtained by $z(x,\kappa) = \sum_{k=0}^\infty z_k(x,\kappa)$, and satisfies (2.12) and (2.13). Inequality (2.14) is obvious from the equation

$$z'(x,\kappa) = -\int_x^\infty e^{2i\kappa(y-x)} q(y) z(y,\kappa)\,dy. \qquad \square$$

Lemma 2.3. *For each $\kappa, \kappa' \in \overline{\mathbf{C}_+}\setminus\{0\}$ and $0 \le \delta \le 1$ we have*

$$|\xi(x,\kappa)-\xi(x,\kappa')| \le |\kappa-\kappa'|^\delta x^{2\delta}[\min\{2x, |\kappa|^{-1}+|\kappa'|^{-1}\}]^{1-\delta} \times \left\{1+\int_0^\infty |q(y)|\,|\xi(y,\kappa')|\,dy\right\} e^{\int_0^x y|q(y)|\,dy}, \tag{2.15}$$

$$|z(x,\kappa)-z(x,\kappa')| \le |\kappa-\kappa'|^\delta \int_0^\infty y^{2\delta}[\min\{2y, |\kappa|^{-1}+|\kappa'|^{-1}\}]^{1-\delta} \times |q(y)|\,|z(y,\kappa')|\,dy\, e^{\int_0^\infty y|q(y)|\,dy}. \tag{2.16}$$

Proof. We have from (2.5) and (2.6),

$$\xi(x,\kappa)-\xi(x,\kappa') = a(x,\kappa,\kappa') + \int_0^x \frac{e^{2i\kappa(x-y)}-1}{2i\kappa} q(y)\{\xi(y,\kappa)-\xi(y,\kappa')\}\,dy,$$

$$a(x,\kappa,\kappa') = \int_0^x \{e^{2i\kappa t}-e^{2i\kappa' t}\}\,dt + \int_0^x \left\{\frac{e^{2i\kappa(x-y)}-1}{2i\kappa} - \frac{e^{2i\kappa'(x-y)}-1}{2i\kappa'}\right\} q(y)\xi(y,\kappa')\,dy$$

and

$$z(x,\kappa)-z(x,\kappa') = \tilde a(x,\kappa,\kappa') - \int_x^\infty \frac{1-e^{2i\kappa(y-x)}}{2i\kappa} q(y)\{z(y,\kappa)-z(y,\kappa')\}\,dy,$$

$$\tilde a(x,\kappa,\kappa') = -\int_x^\infty \left\{\frac{1-e^{2i\kappa(y-x)}}{2i\kappa} - \frac{1-e^{2i\kappa'(y-x)}}{2i\kappa'}\right\} q(y)z(y,\kappa')\,dy,$$

respectively. Successive approximation arguments are also applicable to these integral equations. Then since we have from (2.9) that

$$|a(x,\kappa,\kappa')| \le \min\{|\kappa-\kappa'|x^2, 2x, |\kappa|^{-1}+|\kappa'|^{-1}\}\left\{1+\int_0^x |q(y)|\,|\xi(y,\kappa')|\,dy\right\}$$

and

$$|\tilde a(x,\kappa,\kappa')| \le \int_x^\infty \min\{|\kappa-\kappa'|y^2, 2y, |\kappa|^{-1}+|\kappa'|^{-1}\}|q(y)|\,|z(y,\kappa')|\,dy,$$

the desired inequalities (2.15) and (2.16) are concluded, respectively. ☐

Lemma 2.4. *Assume that $\beta > 2$ in (0.4). Then for each $\kappa \in \overline{\mathbf{C}_+}\setminus\{0\}$ we have*

$$|\dot z(x,\kappa)| \le \int_x^\infty \min\{y^2, y|\kappa|^{-1}+|\kappa|^{-2}\}|q(y)|\,|z(y,\kappa)|\,dy\, e^{\int_0^\infty y|q(y)|\,dy}, \tag{2.17}$$

and for each $\kappa \in \overline{\mathbf{C}_+}$, $0 < |\kappa| < 1$,

$$
\begin{aligned}
|\dot{z}(x,\kappa) - \dot{z}(x,0)| \le 2|\kappa|^\delta \Big\{ \int_x^\infty & y^{2+\delta} |q(y)|\,|z(y,0)|\,dy \\
&+ \sup_{x>0,|\kappa|<1} |\dot{z}(x,\kappa)| \int_x^\infty y^2 |q(y)|\,dy \\
&+ \int_x^\infty y^{1+\delta}|q(y)|\,|\dot{z}(y,0)|\,dy \Big\} e^{\int_0^\infty t|g(t)|\,dt},
\end{aligned}
\tag{2.18}
$$

where $\dot{z}(x,\kappa) = \partial_\kappa z(x,\kappa)$ *and* $0 < \delta < \min\{1, \beta - 2\}$.

Proof. Differentiate (2.6) by κ. Then

$$
\begin{aligned}
\dot{z}(x,\kappa) &= a(x,\kappa) + \int_x^\infty \frac{1 - e^{2i\kappa(y-x)}}{2i\kappa} q(y)\dot{z}(y,\kappa)\,dy, \\
a(x,\kappa) &= \int_x^\infty \Big\{ \frac{(y-x)e^{2i\kappa(y-x)}}{\kappa} + \frac{1 - e^{2i\kappa(y-x)}}{2i\kappa^2} \Big\} q(y) z(y,\kappa)\,dy.
\end{aligned}
$$

The use of (2.8) implies

$$
|a(x,\kappa)| \le \int_x^\infty \min\{t^2, t|\kappa|^{-1} + |\kappa|^{-2}\} |q(t)|\,|z(t,\kappa)|\,dt,
$$

which leads us to the desired conclusion (2.17).

Next note

$$
\dot{z}(x,\kappa) - \dot{z}(x,0) = a_1(x,\kappa) + \int_x^\infty \Big[\int_0^{y-x} e^{2i\kappa t}\,dt \Big] q(y) \{\dot{z}(y,\kappa) - \dot{z}(y,0)\}\,dy,
$$

where

$$
\begin{aligned}
a_1(x,\kappa) = \int_x^\infty & \Big[\int_0^{y-x} 2it\{e^{2i\kappa t} - 1\}\,dt \Big] q(y) z(y,0)\,dy \\
&+ \int_x^\infty \Big[\int_0^{y-x} 2ite^{2i\kappa t}\,dt \Big] q(y)\{z(y,\kappa) - z(y,0)\}\,dy \\
&+ \int_x^\infty \Big[\int_0^{y-x} \{e^{2i\kappa t} - 1\}\,dt \Big] q(y)\dot{z}(y,0)\,dy.
\end{aligned}
$$

To estimate the first and the third terms we use (2.10). Then

$$
\begin{aligned}
|a_1(x,\kappa)| \le 4|\kappa|^\delta & \int_x^\infty y^{2+\delta}|q(y)|\,|z(y,0)|\,dy \\
&+ \int_x^\infty y^2 |g(y)| \Big| \int_0^\kappa \dot{z}(y,\sigma)\,d\sigma \Big|\,dy + 2|\kappa|^\delta \int_x^\infty y^{1+\delta} |q(y)|\,|\dot{z}(y,0)|\,dy,
\end{aligned}
$$

and (2.18) follows. □

With these lemmas we return to the expression (2.4) of the resolvent kernel $R(x,y,\kappa)$. We put $K_e = \{\kappa \in \overline{\mathbf{C}_+};\ e(0,\kappa) = 0\}$. It is well known ([8]) that $\{\mu^2;\ \mu \in K_e \cap \mathbf{C}_+\}$ forms the set of negative eigenvalues of L, and we have $K_e \cap \mathbf{R} \subset \{0\}$. Also, in the case $e(0,0) = 0$ there is an identity (cf. [17])

$$\dot{e}(0,0)e'(0,0) = -i.$$

In the following, we require in addition to (0.4) that L is non-negative, i.e., $K_e \subset \{0\}$.

Let $X = L^2_{(1+x)^{-\alpha}}(\mathbf{R}_+)$ be the weighted L^2-space with norm

$$\|f\|_X^2 = \int_0^\infty (1+x)^{-\alpha}|f(x)|^2\,dx < \infty,$$

and let $X^* = L^2_{(1+x)^\alpha}(\mathbf{R}_+)$ be its dual, where we have chosen $\alpha > 1$ as in (2.2). For $b(x)$ of (2.2) let $a(x) = \sqrt{b(x)}$ and define a multiplication operator A by $Af(x) = a(x)f(x)$. Obviously we have $A \in \mathcal{B}(L^2(\mathbf{R}_+), X) \cap \mathcal{B}(X^*, L^2(\mathbf{R}_+))$, and equation (2.1) will be treated in this framework.

Theorem 2. *The following statements hold:*

(1) *Let $\alpha > 1$ in (2.2) and $\beta \geq 1$ in (0.4). Assume that $e(0,0) \neq 0$. If $g(x) \in X^*$ in (2.1), then as $t \to \infty$,*

$$w(x,t) = v(x,\sigma_0)e^{-i\sigma_0 t} + o(1) \quad \text{in } X,$$

where $v(x,\sigma_0)$ is the unique solution in X of the problem

$$Lv - i\kappa Bv - \kappa^2 v = g \quad \text{with } \kappa = \sigma_0.$$

(2) *Let $\alpha > 2$ in (2.2) and $\beta > 2$ in (0.4). Assume that $e(0,0) = 0$. If $g(x) \in X^*$ in (2.1), then as $t \to \infty$,*

$$w(x,t) = v(x,\omega)e^{-i\sigma_0 t} + \frac{1}{\sigma_0}Pg(x) + o(1) \quad \text{in } X,$$

where $v(x,\omega)$ is the unique solution in X of the problem

$$Lv - i\kappa Bv - \kappa^2 v = g \quad \text{with } \kappa = \sigma_0,$$

and the operator $P \in \mathcal{B}(X^*, X)$ is given by

$$Pg = P_0 g + iP_0 A^*\{1 - iAP_0A^*\}^{-1}AP_0 g$$

with

$$[P_0 g](x) = \frac{e(x,0)}{\dot{e}(0,0)}\int_0^x \omega(y,0)g(y)\,dy + \frac{\omega(x,0)}{\dot{e}(0,0)}\int_x^\infty e(y,0)g(y)\,dy.$$

Proof. For the proof it is enough to verify that Assumptions 1–5 of Section 1 hold.

(1) The resolvent kernel of L is given by (2.4):

$$R(x,y;\kappa) = \frac{\xi(x,\kappa)z(y,\kappa)e^{i\kappa(y-x)}}{z(0,\kappa)} \quad \text{if } x \le y,$$

and $R(x,y;\kappa) = R(y,x;\kappa)$ $(x > y)$, where $z(0,\kappa) = e(0,\kappa)$. Since $e(0,0) \ne 0$, it follows from (2.13) that $z(0,\kappa)$ is uniformly positive in $\kappa \in \mathbf{R}$. Combining this with other estimates of Lemma 2.2, we see that $R(x,y,\kappa)$ satisfies

$$|R(x,y;\kappa)| \le C_0|\kappa|^{-1+\delta}x^{\delta/2}y^{\delta/2}$$

for any $0 \le \delta < \alpha - 1$. So $R(\kappa^2)$ is continuously extended in $\mathcal{B}(X,X^*)$ to real $\kappa \in \mathbf{R}\backslash\{0\}$, and the principle of limiting absorption (Assumption 1) holds. Also the high-frequency estimate (Assumption 2) and the low-frequency estimate (Assumption 4 with $P_0 = 0$ and $0 < \delta_2 < \alpha - 1$) hold.

Next, note that for κ, $\kappa' \in \Lambda_\pm$ we have

$$R(x,y,\kappa) - R(x,y,\kappa') = \frac{\xi(x,\kappa)z(y,\kappa)\{e^{i\kappa(y-x)} - e^{i\kappa'(y-x)}\}}{z(0,\kappa)} + \left\{\frac{\xi(x,\kappa)z(y,\kappa)}{z(0,\kappa)} - \frac{\xi(x,\kappa')z(y,\kappa')}{z(0,\kappa')}\right\}e^{i\kappa'(y-x)} \quad \text{if } x < y.$$

Then $|e^{i\kappa(y-x)} - e^{i\kappa'(y-x)}| \le 2^{1-\delta}|\kappa-\kappa'|^\delta y^\delta$ for any $0 \le \delta \le 1$. Thus, choosing $2\delta < \alpha - 1$ and combining the estimates of Lemmas 2.2 and 2.3, we obtain

$$|R(x,y,\kappa) - R(x,y;\kappa')| \le C_1|\kappa-\kappa'|^\delta(1+x)^\delta(1+y)^\delta,$$

where $C_1 > 0$ depends on $\Lambda_\pm$ and $\int_0^\infty (1+y)|q(y)|\,dy$. This implies the local Hölder continuity of the resolvent (Assumption 3 with $0 < \delta_1 < (\alpha-1)/2$):

$$\|R(\kappa^2) - R(\kappa'^2)\|_{\mathcal{B}(X^*,X)} \le C_1|\kappa-\kappa'|^{\delta_1}.$$

Finally, the compactness of $AR(\kappa^2)A^*$ (Assumption 5) is obvious since it is an integral operator of Hilbert–Schmidt type for each $\kappa \in \overline{\mathbf{C}_+}$.

(2) We have only to show the low-frequency estimate of the resolvent.

The kernel of $\kappa R(\kappa^2) - P_0$ is represented as

$$\kappa R(x,y,\kappa) - P_0(x,y) = \frac{\kappa\omega(x,\kappa)e(y.\kappa)}{e(0,\kappa)} - \frac{\omega(x,0)e(y,0)}{\dot{e}(0,0)}$$

$$= \frac{\{\kappa\dot{z}(0,0) - z(0,\kappa)\}\xi(x,\kappa)z(y,\kappa)e^{i\kappa(y-x)}}{z(0,\kappa)\dot{z}(0,0)} + \frac{\xi(x,\kappa)z(y,\kappa) - \xi(x,0)z(y,0)}{\dot{z}(0,0)} + \frac{\xi(x,\kappa)z(y,\kappa)\{e^{i\kappa(y-x)} - 1\}}{\dot{z}(0,0)}$$

when $x \leq y$, and a similar representation also holds when $y < x$. Lemma 2.4 is applied to

$$\frac{\kappa\dot{z}(0,0) - z(0,\kappa)}{z(0,\kappa)} = -\int_0^\kappa \{\dot{z}(0,\sigma) - \dot{z}(0,0)\}\, d\sigma \left[\int_0^\kappa \dot{z}(0,\sigma)\, d\sigma\right]^{-1}$$

of the first term. For the second term we use the inequalities of Lemma 2.3 after letting $\kappa' \to 0$, and for the third term we note $|e^{i\kappa(y-x)} - 1| \leq 2^{1-\delta}|\kappa|^\delta |y-x|^\delta$. Then we finally obtain

$$|\kappa R(x,y;\kappa) - P_0(x,y)| \leq C|\kappa|^\delta (1+x)^{(1+\delta)/2}(1+y)^{(1+\delta)/2},$$

with $C > 0$ independent of $0 < |\kappa| < 1$ and $0 < \delta < \min\{1/2, \alpha - 2, \beta - 2\}$.

This gives the desired low-frequency estimate. □

§3. The wave equation on a star graph

We return to the Schrödinger operator (0.1) defined on the non-compact graph $\Gamma = \gamma_1 \times \cdots \times \gamma_n$ for functions u satisfying the Kirchhoff boundary conditions (0.2), (0.3) on the vertex. The potential $q_j(x)$ $(j = 1, \ldots, n)$ is real and satisfies condition (0.4). Then the operator L restricted to the domain (0.5) forms a lower semi-bounded self-adjoint operator in $\mathcal{H} = L^2(\Gamma) = \prod_{j=1}^n L^2(\gamma_j)$, and its essential spectrum fills the non-negative half-line $[0, \infty)$.

We consider on Γ the generalized eigenvalue problem

$$-\frac{d^2 u_j}{dx^2} + q_j(x)u_j - \kappa^2 u_j = 0 \quad \text{in } \Gamma,\ j \in \{1, \ldots, n\}. \tag{3.1}$$

Let $\omega_j(x,\kappa)$, $x \in \gamma_j$, be the solution of (3.1) which satisfies the initial condition $\omega_j(0,\kappa) = 0$, $\omega_j'(0,\kappa) = 1$. Let $e_j(x,\kappa)$, $j \in \{1, \ldots, n\}$, be the Jost solution of (3.1) which satisfies the asymptotic condition $e_j(x,\kappa) \to e^{i\kappa x}$ $(x \to \infty)$.

When $e_j(0,\kappa) \neq 0$ for all j, the resolvent of the operator $L_D = \mathrm{diag}(-d^2/dx_j^2 + q(x_j))_{j=1}^n$ with 0-Dirichlet boundary conditions forms an integral operator with diagonal kernel

$$R_D(x,y,\kappa) = \mathrm{diag}(R_{D,k}(x_k, y_k; \kappa))_{k=1}^n,$$

where

$$R_{D,k}(x_k, y_k; \kappa) = \begin{cases} \dfrac{\omega_k(x_k, \kappa) e_k(y_k, \kappa)}{e_k(0, \kappa)} & (x_k \le y_k), \\ \dfrac{e_k(x_k, \kappa)\omega_k(y_k, \kappa)}{e_k(0, \kappa)} & (x_k \ge y_k). \end{cases}$$

We modify this to satisfy the Kirchhoff conditions. As is easily seen (cf. [12]), the resolvent kernel $R(x, y; \kappa) = (R_{kj}(x_k, y_j; \kappa))_{k,j=1}^n$ of L is given by

$$R_{kj}(x_k, y_j; \kappa) = \delta_{kj} R_{D,k}(x_k, y_k; \kappa) - \frac{e_k(x_k, \kappa) e_j(y_j, \kappa)}{e_k(0, \kappa) e_j(0, \kappa) G(\kappa)}, \tag{3.2}$$

where δ_{kj} is Kronecker's delta and

$$G(\kappa) = \sum_{\ell=1}^{n} \frac{e'_\ell(0, \kappa)}{e_\ell(0, \kappa)}.$$

Put

$$K_e = \big\{\kappa \in \overline{\mathbf{C}_+};\ \textstyle\prod_{k=1}^n e_k(0, \kappa) = 0\big\}, \quad K_G = \big\{\kappa \in \overline{\mathbf{C}_+};\ G(\kappa) = 0\big\}.$$

Let $\mathbf{N}^0(\kappa) = \{j \in \{1, \ldots, n\};\ e_j(0, \kappa) = 0\}$, and divide K_e into two parts $K_e = K_e(\mathrm{I}) \cup K_e(\mathrm{II})$:

$$K_e(\mathrm{I}) = \big\{\kappa \in K_e;\ \sharp\mathbf{N}^0(\kappa) \ge 2\big\}, \quad K_e(\mathrm{II}) = \big\{\kappa \in K_e;\ \sharp\mathbf{N}^0(\kappa) = 1\big\}.$$

If $\mu \in \mathbf{C}_+$ is in K_e, then for $k \in \mathbf{N}^0(\mu)$,

$$e_k(0, \kappa) G(\kappa) \to \dot{e}_k(0, \mu) \sum_{j \in \mathbf{N}^0(\mu)} \frac{e'_j(0, \mu)}{\dot{e}_j(0, \mu)},$$

and it follows that $G(\kappa) \to \infty$ as $\kappa \to \mu$. So K_e and K_G are disjoint to each other. Repeating the argument of [12] we have more: If $\mu \in [K_e(\mathrm{I}) \cup K_G] \cap \mathbf{C}_+$, μ^2 becomes a negative eigenvalue of L. In fact, when $\mu \in K_e(\mathrm{I}) \cap \mathbf{C}_+$, choosing $\{a_j;\ j \in \mathbf{N}^0(\mu)\}$ to satisfy $\sum_{j \in \mathbf{N}^0(\mu)} a_j e'_j(0, \mu) = 0$, we see that $u = (u_j)_{j=1}^n$ with $u_j = a_j e_j(x, \mu)$ ($j \in \mathbf{N}^0(\mu)$) and $= 0$ (otherwise) becomes an eigenfunction of L, and when $\mu \in K_G \cap \mathbf{C}_+$, μ^2 is an eigenvalue of L with eigenfunction

$$\Big(\frac{e_1(x, \mu)}{e_1(0, \mu)}, \ldots, \frac{e_n(x, \mu)}{e_n(0, \mu)}\Big) \in L^2(\Gamma).$$

Next, $[K_e \cup K_G] \cap \mathbf{R} \subset \{0\}$. In fact, if $\lambda \in \mathbf{R} \backslash \{0\}$, then as in the case of Section 2 we have

$$\mathrm{Im}\Big\{\frac{e'_j(x, \lambda)}{e_j(x, \lambda)}\Big\} |e_j(x, \lambda)|^2 = \lambda. \tag{3.3}$$

Lastly, let $\mu \in K_e(\text{II})$ and $k \in \mathbf{N}^0(\mu)$. Then $e_k(0,\kappa)G(\kappa) \to e_k'(0,\mu)$ and $e_j(0,\kappa)G(\kappa) \to \infty$ $(j \neq k)$ as $\kappa \to \mu$. So $\text{Res}_{\kappa=\mu} R(x,y,\kappa) = 0$ if $\mu \in \mathbf{C}_+$, and this μ becomes a removable singularity.

Now we treat of the principle of limiting amplitude for the solution $w(x,t) = (w_j(x,t))_{j=1}^n$ of the wave equation

$$\partial_t^2 w(x,t) + Lw(x,t) + B\partial_t w(x,t) = g(x)e^{-i\sigma_0 t}, \quad (x,t) \in \Gamma \times (0,\infty),$$
$$w(x,0) = \partial_t w(x,0) = 0, \quad x \in \Gamma.$$

Here L is additionally required to be non-negative, i.e., $\{K_e(\text{I}) \cup K_G\} \subset \{0\}$, and $B = A^*A$ is defined by

$$Af(x) = (a_j(x)f_j(x))_{j=1}^n, \quad a_j(x) = \sqrt{b_j(x)} \le C(1+x)^{-\alpha/2} \quad (\alpha > 1). \tag{3.4}$$

With $\alpha > 1$ in (3.4), let $X = L^2_{(1+x)^\alpha}(\Gamma) = \prod_{j=1}^n L^2_{(1+x_j)^\alpha}(\gamma_j)$ and X^* be its dual.

Theorem 3. *The following statements hold:*

(1) *Let $\alpha > 1$ in (3.4) and $\beta \ge 1$ in (0.4). Assume $K_e \cup K_G = \emptyset$. If $g(x) \in X^*$, then as $t \to \infty$,*

$$w(x,t) = v(x,\sigma_0)e^{-i\sigma_0 t} + o(1) \quad \text{in } X,$$

where $v(x,\sigma_0)$ is the unique solution in X of the problem

$$Lv - i\kappa Bv - \kappa^2 v = g \quad \text{with } \kappa = \sigma_0.$$

(2) *Let $\alpha > 2$ in (3.4) and $\beta > 2$ in (0.4). Assume $K_e(\text{II}) = \{0\}$. If $g(x) \in X^*$, then the same results as (1) hold in this case.*

(3) *Let $\alpha > 2$ in (3.4) and $\beta > 2$ in (0.4). Assume $K_e(\text{I}) \cup K_G = \{0\}$. If $g(x) \in X^*$, then as $t \to \infty$,*

$$w(x,t) = v(x,\sigma_0)e^{-i\sigma_0 t} + \frac{1}{\sigma_0}Pg(x) + o(1) \quad \text{in } X,$$

where the operator $P \in \mathcal{B}(X^*, X)$ is given by

$$P = P_0 + iP_0A^*\{1 - iAP_0A^*\}^{-1}AP_0,$$

where $P_0 = (P_{0kj})_{k,j=1}^n$ is an integral operator with kernel $(P_{0kj}(x,y))_{k,j=1}^n$. When $0 \in K_e(\text{I})$, we have $P_{0kj}(x,y) = 0$ if k or $j \notin \mathbf{N}^0(0)$, and if both $k, j \in \mathbf{N}^0(0)$, then

$$P_{0kj}(x,y) = \delta_{kj}\frac{\omega_k(x,0)e_j(y,0)}{\dot{e}_j(0,0)} - \frac{e_k(x,0)e_j(y,0)}{\dot{e}_k(0,0)\dot{e}_j(0,0)\widehat{G}(0)}, \quad x < y,$$

and $P_{0kj}(x,y) = P_{0kj}(y,x)$, $x > y$, *where*

$$\widehat{G}(0) = \sum_{\ell \in \mathbf{N}^0(0)} \frac{e'_\ell(0,0)}{\dot{e}_\ell(0,0)} = \sum_{\ell \in \mathbf{N}^0(0)} \frac{e'_\ell(0,0)^2}{-i}.$$

On the other hand, when $0 \in K_G$, *we have*

$$P_{0kj}(x,y) = -\frac{e_k(x,0)e_j(y,0)}{e_k(0,0)e_j(0,0)\dot{G}(0)}$$

for all $k, j \in \mathbf{N}$, *where*

$$\dot{G}(0) = \sum_{j=1}^{n} \frac{\dot{e}'_j(0,0)e_j(0,0) - e'_j(0,0)\dot{e}_j(0,0)}{e_j(0,0)^2} = \sum_{j=1}^{n} \frac{i}{e_j(0,0)^2}.$$

To show this theorem we can follow a line similar to that for the single equation.

Lemma 3.1. *Let* $0 \le \delta \le 1$. *Then*

$$|z'_j(0,\kappa) - z'_j(0,\kappa')| \le 2|\kappa-\kappa'|^\delta \left\{ \int_0^\infty y^\delta |q(y)|\,dy + C_\delta(\kappa,\kappa') \right\} \sup_{y\in\gamma_j} |z_j(y,\kappa)|, \tag{3.5}$$

where

$$C(\kappa,\kappa') = \int_0^\infty |q(y)|\,dy \int_0^\infty y^{2\delta}[\min\{y, |\kappa|^{-1}, |\kappa'|^{-1}\}]^{1-\delta}|q(y)|\,dy\, e^{\int_0^\infty y|q(y)|\,dy}.$$

Moreover, in the case $\beta \ge 2$,

$$|\dot{z}'_j(0,\kappa)| \le 4\int_0^\infty (1+y)|q(y)|\,dy \sup_{y\in\gamma_j}\{|z_j(y,\kappa)| + |\dot{z}_j(y,\kappa)|\}, \tag{3.6}$$

$$\begin{aligned} |\dot{z}'_j(0,\kappa) - \dot{z}'_j(0,0)| \le 4|\kappa|^\delta &\int_0^\infty (1+y)^{1+\delta}|q(y)|\,dy \sup_{y\in\gamma_j}\{|z_j(y,\kappa)| + |\dot{z}_j(y,\kappa)\} \\ &+ 2\int_0^\infty (1+y)|q(y)|\,dy \sup_{y\in\gamma_j}\{|z_j(y,\kappa) - z_j(y,0)| \\ &\qquad + |\dot{z}_j(y,\kappa) - \dot{z}_j(y,0)|\}. \end{aligned} \tag{3.7}$$

Proof. Inequality (3.5) is easy from the identity

$$\begin{aligned} z'_j(0,\kappa) - z'_j(0,\kappa') = &-\int_0^\infty \{e^{2i\kappa y} - e^{2i\kappa' y}\}q(y)z_j(y,\kappa)\,dy \\ &-\int_0^\infty e^{2i\kappa' y}q(y)\{z_j(y,\kappa) - z_j(y,\kappa')\}\,dy \end{aligned}$$

if we note (2.16). Inequalities (3.6) and (3.7) are respectively obtained from the identities

$$\dot{z}_j'(0,\kappa) = -\int_0^\infty e^{2i\kappa y} q(y)\{2iyz_j(y,\kappa) + \dot{z}_j(y,\kappa)\}\,dy,$$

$$\begin{aligned}\dot{z}_j'(0,\kappa) - \dot{z}_j'(0,0) = &-\int_0^\infty (e^{2i\kappa y}-1)\{2iyq(y)z_j(y,\kappa) + q(y)\dot{z}_j(y,\kappa)\}\,dy \\ &-\int_0^\infty [2iyq(y)\{z_j(y,\kappa) - z_j(y,0)\} \\ &\qquad + q(y)\{\dot{z}_j(y,\kappa) - \dot{z}_j(y,0)\}]\,dy. \qquad \square\end{aligned}$$

Combining this lemma and Lemmas 2.2 and 2.4, we have the following two lemmas for the function $G(\kappa)$.

Lemma 3.2. *Let $\beta \geq 1$ in (0.4). Then there exists $C(\Lambda_\pm)$ such that for κ, $\kappa' \in \Lambda_\pm$,*

$$|G(\kappa) - G(\kappa')| \leq C(\Lambda_\pm)|\kappa - \kappa'|^{1/2}. \tag{3.8}$$

Proof. By definition we have

$$\begin{aligned}G(\kappa) - G(\kappa') = &\sum_{j=1}^n \frac{z_j'(0,\kappa) - z_j'(0,\kappa')}{z_j(0,\kappa)} \\ &- \sum_{j=1}^n \frac{z_j'(0,\kappa')\{z_j(0,\kappa) - z_j(0,\kappa')\}}{z_j(0,\kappa)z_j(0,\kappa')} + ni(\kappa-\kappa').\end{aligned}$$

So (3.8) follows from (3.5), (2.16), (2.12) and (2.14). $\square$

Lemma 3.3. *Let $\beta > 2$ in (0.4). Assume that $0 \in K_e(\mathrm{I})$ and $j \in \mathbf{N}^0(0)$. Then there exists $C > 0$ such that for $0 < \delta < \min\{1/2, \beta - 2\}$,*

$$|e_j(0,\kappa)G(\kappa) - \dot{e}_j(0,0)\widehat{G}(0)| \leq C|\kappa|^\delta. \tag{3.9}$$

Assume that $G(0) = 0$. Then similarly

$$|\kappa^{-1}G(\kappa) - \dot{G}(0)| = \left|\kappa^{-1}\int_0^\kappa \{\dot{G}(s) - \dot{G}(0)\}\,ds\right| \leq C|\kappa|^\delta. \tag{3.10}$$

Proof. Assume that $0 \in K_e$ and $j \in \mathbf{N}^0(0)$. Then

$$\begin{aligned} & e_j(0,\kappa)G(\kappa) - \dot{e}_j(0,0)\widehat{G}(0) \\ & \quad = i\sharp\mathbf{N}^0(0)\kappa z_j(0,\kappa) + \{\kappa^{-1}z_j(0,\kappa) - \dot{z}_j(0,0)\} \sum_{\ell\in\mathbf{N}^0(0)} \frac{z_\ell'(0,\kappa)}{\kappa^{-1}z_\ell(0,\kappa)} \\ & \qquad + \dot{z}(0,0) \sum_{\ell\in\mathbf{N}^0(0)} \Big\{ \frac{\{\dot{z}_\ell(0,0) - \kappa^{-1}z_\ell(0,\kappa)\}z_\ell'(0,\kappa)}{\kappa^{-1}z_\ell(0,\kappa)\dot{z}_\ell(0,0)} + \frac{z_\ell'(0,\kappa) - z_\ell'(0,0)}{\dot{z}_\ell(0,0)} \Big\}. \end{aligned}$$

Lemmas 2.2–2.4 and 3.1 are applicable to this identity to obtain (3.9). Next, assume $G(0) = 0$. Note the identity

$$\begin{aligned} \dot{G}(\kappa) - \dot{G}(0) = & \sum_{j=1}^{n} \Big\{ \frac{\dot{z}_j'(0,\kappa)}{z_j(0,\kappa)} - \frac{\dot{z}_j'(0,0)}{z_j(0,0)} \Big\} \\ & - \sum_{j=1}^{n} \Big\{ \frac{z_j'(0,\kappa)\dot{z}_j(0,\kappa)}{z_j(0,\kappa)^2} - \frac{z_j'(0,0)\dot{z}_j(0,0)}{z_j(0,0)^2} \Big\} \end{aligned}$$

in this case. Then Lemmas 2.2–2.4 and 3.1 also yield (3.10). □

Proof of Theorem 3. (1) By assumption both $e_k(0,\kappa)$ $(k = 1, \dots, n)$ and $G(\kappa)$ are never zero in $\mathbf{R}$. So Assumption 1 is obvious from the expression (3.2) of the resolvent kernel $R(x, y, \kappa)$.

As is proved in the half-line problem we see

$$\Big| \frac{\omega_k(x,\kappa)e_k(y,\kappa)}{e_k(0,\kappa)} \Big| = \Big| \frac{\xi_k(x,\kappa)z_k(y,\kappa)e^{i\kappa(y-x)}}{z_k(0,\kappa)} \Big| \le C|\kappa|^{-1+\delta}x^{\delta/2}y^{\delta/2}$$

for any $0 \le \delta < \alpha - 1$. On the other hand, the assistance of (3.3) yields

$$\Big| \frac{e_k(x,\kappa)e_j(y,\kappa)}{e_k(0,\kappa)e_j(0,\kappa)G(\kappa)} \Big| = \Big| \frac{z_k(x,\kappa)z_j(y,\kappa)e^{i\kappa x}e^{i\kappa y}}{z_k(0,\kappa)z_j(0,\kappa)G(\kappa)} \Big| \le \frac{C}{|G(\kappa)|} \le \frac{C}{1+|\kappa|}.$$

Applying these estimates to the expression of the resolvent kernel, we obtain both the high-frequency estimate (Assumption 2) and low-frequency estimate (Assumption 4 with $P_0 = 0$).

To verify Assumption 3 we need the Hölder continuity (3.9) of $G(\kappa)$. Then a similar proof to Theorem 2 (1) is possible to the present case. Moreover, we see that $AR(\kappa^2)A^*$ is an integral operator of Hilbert–Schmidt type under (3.4). So we see Assumption 5 holds. Theorem 3 (1) is thus proved.

(2) Let $0 \in K_e(\mathrm{II})$ and $k \in \mathbf{N}^0(0)$. Then

$$R_{kj}(x, y, \kappa) = \Big\{ \delta_{kj}\omega_k(x,\kappa) - \frac{e_j(x,\kappa)}{e_j(0,\kappa)G(\kappa)} \Big\} \frac{e_k(x,\kappa)}{e_k(0,\kappa)}.$$

Applying Lemmas 2.2, 2.3 and the first inequality of Lemma 3.1, we obtain

$$\begin{aligned}\omega_k(x,\kappa)-\frac{e_k(x,\kappa)}{e_k(0,\kappa)G(\kappa)}&=\omega_k(x,\kappa)-\omega_k(x,0)+\frac{e_k(x,0)}{e_k'(0,0)}-\frac{e_k(x,\kappa)}{e_k'(0,\kappa)}\\&\quad+\frac{e_k(x,\kappa)}{e_k'(0,\kappa)}-\frac{e_k(x,\kappa)}{e_k(0,\kappa)G(\kappa)}=O(|\kappa|).\end{aligned}$$

Also,

$$\frac{e_j(x,\kappa)}{e_j(0,\kappa)G(\kappa)}=O(|\kappa|)\quad (j\neq k).$$

So $R_{\ell j}(x,y,\kappa)$ (for all ℓ, j) are bounded near $\kappa=0$, and the proof is concluded.

(3) For the proof it remains only to show the low-frequency estimate (Assumption 4).

When $0\in K_e(\mathrm{I})$ we notice

$$\kappa R_{kj}(x,y,\kappa)=\left\{\delta_{kj}\omega_k(x,\kappa)-\frac{e_k(x,\kappa)}{e_k(0,\kappa)G(\kappa)}\right\}\frac{\kappa e_j(y,\kappa)}{e_j(0,\kappa)}.$$

Here, as $\kappa\to 0$,

$$\frac{\kappa e_j(y,\kappa)}{e_j(0,\kappa)}\to\begin{cases}0 & \text{if } j\notin \mathbf{N}^0(0),\\ \dfrac{e_j(y,0)}{\dot e_j(0,0)} & \text{if } j\in\mathbf{N}^0(0),\end{cases}$$

and

$$\delta_{kj}\omega_k(x,\kappa)-\frac{e_k(x,\kappa)}{e_k(0,\kappa)G(\kappa)}\to\begin{cases}\delta_{kj}\omega_k(x,0) & \text{if } k\notin\mathbf{N}^0(0),\\ \delta_{kj}\omega_k(x,0)-\dfrac{e_k(x,0)}{\dot e_k(0,0)\widehat G(0)} & \text{if } k\in\mathbf{N}^0(0).\end{cases}$$

So we see $\kappa R_{kj}(x,y,\kappa)\to P_{0kj}(x,y)$ as $\kappa\to 0$, where

$$P_{0kj}(x,y)=\begin{cases}0 & \text{if } k \text{ or } j\notin\mathbf{N}^0(0),\\ \left\{\delta_{kj}\omega_k(x,0)-\dfrac{e_k(x,0)}{\dot e_k(0,0)\widehat G(0)}\right\}\dfrac{e_j(y,0)}{\dot e_j(0,0)} & \text{if both } k,j\in\mathbf{N}^0(0).\end{cases}$$

Moreover, with the help of (3.9) we obtain the estimate

$$|\kappa R_{kj}(x,y,\kappa)-P_{0kj}(x,y)|\le C|\kappa|^\delta(1+x)^{(1+\delta)/2}(1+y)^{(1+\delta)/2}.$$

Thus, Assumption 4 holds with

$$[P_0g](x)=\left(\sum_{j=1}^n\int_{\gamma_j}P_{0kj}(x,y)g_j(y)\,dy\right)_{k=1}^n.$$

Next, consider the case $0 \in K_G$. Then $e_k(0,\kappa) \neq 0$ near $\kappa = 0$ for all k and

$$\kappa R_{kj}(x,y,\kappa) \to P_{0kj}(x,y) \quad \text{as } \kappa \to 0,$$

where

$$P_{0kj}(x,y) = -\frac{e_k(x,0)e_j(y,0)}{e_k(0,0)e_j(0,0)\dot{G}(0)}.$$

Moreover, by use of (3.10) we also have

$$|\kappa R_{kj}(x,y,\kappa) - P_{0kj}(x,y)| \leq C|\kappa|^\delta (1+x)^{(1+\delta)/2}(1+y)^{(1+\delta)/2}. \qquad \square$$

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