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Gorenstein Local Rings Whose Cohomological Annihilator Is the Maximal Ideal

by

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Abstract

This work concerns the cohomological annihilators introduced by Iyengar and Takahashi. Let R be a Gorenstein local ring whose cohomological annihilator is the maximal ideal. We prove that R has finite Cohen–Macaulay type if, in addition, R is either of minimal multiplicity, one-dimensional, or an equicharacteristic Artinian algebra over an algebraically closed field of characteristic not two. Under mild assumptions on residue fields, we classify the equicharacteristic complete Gorenstein local rings with finite Cohen–Macaulay type whose cohomological annihilator is the maximal ideal. For an Artinian Gorenstein local ring but not a field, we prove that the nilpotency degree of the cohomological annihilator is at most three.

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§1. Introduction

Let R be a commutative Noetherian ring. In this article, we focus on studying the *cohomological annihilator* of R introduced by Iyengar and Takahashi [24]. By definition, the cohomological annihilator of R, denoted $\operatorname{ca}(R)$, is $\bigcup_{n\geq 0}\operatorname{ca}^n(R)$, where $\operatorname{ca}^n(R)$ consists of elements r in R such that r annihilates $\operatorname{Ext}^n_R(M,N)$ for all finitely generated R-modules M, N; see 3.1. When R is of finite Krull dimension, the cohomological annihilator measures the singularity of R in the sense that R is regular if and only if $\operatorname{ca}(R)=R$; see 3.5.

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The study of the annihilator of Ext groups can be traced back to the 1980s. Dieterich [17] proved that the ideal $\mathrm{ca}^1(R)$ contains some power of the maximal ideal when R is an equicharacteristic isolated Cohen–Macaulay singularity over an algebraically closed field. Furthermore, Dieterich [17] observed that this property implies that the first Brauer–Thrall conjecture holds. This established a close connection between the cohomological annihilator and the Brauer–Thrall conjecture. Buchweitz [8] observed that the ideal $\mathrm{ca}(R)$ contains the Jacobian ideal of R when R is an equicharacteristic complete intersection; see 3.10. Later, Wang [34] extended this result to equicharacteristic complete Cohen–Macaulay local rings.

Cohomological annihilators were systematically studied by Iyengar and Takahashi [24, 25]. In [24], they proved that the cohomological annihilator of R defines the singular locus of R when R is either an equicharacteristic excellent local ring or a localization of a finitely generated algebra over a field. Then, under this assumption, they proved in [24] that the category of finitely generated R-modules contains a strong generator. On the other hand, Wang's result [34] just mentioned was extended by Iyengar and Takahashi [25] to a larger family of rings.

Usually, it is not easy to compute the cohomological annihilator for a given ring. There are only a few classes of rings whose cohomological annihilators are known. When R is a one-dimensional reduced complete Gorenstein local ring, Esentepe [19] observed that the cohomological annihilator of R coincides with the conductor ideal of R, namely the annihilator of $\overline{R}/R$ over R, where $\overline{R}$ is the integral closure of R inside its total ring of fractions.

Recently, the second author [30] proved that the cohomological annihilator of R is equal to the annihilator of the singularity category of R up to radical when R is either an equicharacteristic excellent local ring or a localization of a finitely generated algebra over a field. Moreover, building on works of Dao and Takahashi [14], it is proved in [30] that there is an upper bound of the Rouquier dimension of the singularity category of R by using the minimal number of generators of $\mathrm{ca}(R)$ when R is an equicharacteristic excellent local ring with isolated singularity.

Let $(R, \mathfrak{m})$ be a Noetherian local ring. As mentioned, the cohomological annihilator of R captures the singularity of R. Precisely, R is regular if and only if $\mathrm{ca}(R) = R$. We are motivated by the following natural question: For a Noetherian local ring $(R, \mathfrak{m})$, what can we say if $\mathrm{ca}(R) = \mathfrak{m}$? This is the leading question throughout the article. Note that if the cohomological annihilator of $(R, \mathfrak{m})$ is $\mathfrak{m}$-primary, then R has an isolated singularity; see Proposition 4.3.

A local ring $(R, \mathfrak{m})$ is said to have *finite (resp. countable) Cohen–Macaulay (CM) type* provided that there are finitely (resp. countably) many indecomposable maximal Cohen–Macaulay R-modules up to isomorphism. Our first result is a characterization of Gorenstein local ring $(R, \mathfrak{m})$ with finite CM type and $\mathrm{ca}(R) = \mathfrak{m}$.

Theorem 1.1 (Theorem 4.12). *Let $(R, \mathfrak{m}, k)$ be an equicharacteristic complete Gorenstein local ring of dimension d, where k is an uncountable algebraically closed field of characteristic 0. Then the following conditions are equivalent:*

(1) $\operatorname{ca}(R) = \mathfrak{m}$ *and R has finite CM type.*

(2) $\operatorname{ca}(R) = \mathfrak{m}$ *and R is a hypersurface with countable CM type.*

(3) $R \cong k[\![x_0, \ldots, x_d]\!]/(f)$, *where f is of the form*

$$x_0^2 + x_1^2 + x_2^2 + \cdots + x_d^2 \quad \text{or} \quad x_0^3 + x_1^2 + x_2^2 + \cdots + x_d^2.$$

The above result supports a question of Huneke and Leuschke [22]; see Remark 4.13. The proof of Theorem 1.1 relies on the celebrated characterization of the simple hypersurface singularities due to Buchweitz, Greuel, and Schreyer [9], and Knörrer [28]. It is natural to ask whether we can drop the assumption that R has finite CM type in Theorem 1.1. More generally, we ask: For a Gorenstein local ring $(R, \mathfrak{m})$ with $\operatorname{ca}(R) = \mathfrak{m}$, does R have finite CM type? The question is also motivated by a theorem of Auslander [3] that a complete Cohen–Macaulay local ring with finite CM type has at most an isolated singularity. The following result and Theorem 1.4 support this question.

Theorem 1.2 (Theorems 5.5 and 5.8). *Let $(R, \mathfrak{m})$ be a Gorenstein local ring with* $\operatorname{ca}(R) = \mathfrak{m}$. *Then*

(1) *if R has minimal multiplicity, then R has finite CM type;*

(2) *assume $R = S/(f_1 f_2)$, where $S = (S, \mathfrak{n})$ is a Noetherian local ring and $f_1, f_2 \in \mathfrak{n}$ are S-regular. Then* $\operatorname{embdim}(R) \leq 2$. *If, in addition,* $\dim(R) \geq 1$, *then S is regular,* $\dim R = 1$, *and R has finite CM type.*

Let $(R, \mathfrak{m})$ be a hypersurface with $\operatorname{ca}(R) = \mathfrak{m}$. We observed that the Rouquier dimension of the stable category of maximal Cohen–Macaulay R-modules, denoted $\dim \underline{\mathrm{MCM}}(R)$, is at most one; see Corollary 4.6. Due to Dao and Takahashi [12], complete hypersurface singularity with $\dim \underline{\mathrm{MCM}}(R) = 0$ has finite CM type. Based on the above facts, we conjecture that a hypersurface $(R, \mathfrak{m})$ with $\operatorname{ca}(R) = \mathfrak{m}$ has finite CM type. Theorems 1.2 and 1.4 support this conjecture.

In Section 5, we raise some questions and conjectures including those we mentioned above. We also figure out their connections; see Proposition 5.15 and Proposition 5.20. In Section 6, we study the cohomological annihilators of Artinian Gorenstein local rings. We prove the following.

Theorem 1.3 (Theorem 6.3). *Let $(R, \mathfrak{m}, k)$ be an Artinian Gorenstein local ring. Then*

$$(\operatorname{ca}(R))^2 = 0 \quad \text{or} \quad (\operatorname{ca}(R))^2 = \operatorname{soc}(R) \cong k.$$

In particular, $(\operatorname{ca}(R))^3 = 0$ *if, in addition, R is not a field.*

The above result yields that the nilpotency degree of the cohomological annihilator of R is at most three if R is an Artinian Gorenstein local ring but not a field. In Proposition 6.6, we prove that the cohomological annihilator of the equicharacteristic short Gorenstein local ring is the socle under some mild assumptions. Combining this with Theorem 1.3, we prove the first statement of the following result.

Theorem 1.4 (Theorems 6.7 and 7.3). *The following statements hold:*

(1) *Let $(R, \mathfrak{m}, k)$ be an equicharacteristic Artinian Gorenstein local ring, where k is an algebraically closed field with* $\operatorname{char}(k) \neq 2$. *Then* $\operatorname{ca}(R) = \mathfrak{m}$ *if and only if $R \cong k[\![x]\!]/(x^2)$ or $R \cong k[\![x]\!]/(x^3)$.*

(2) *Let $(R, \mathfrak{m})$ be a one-dimensional Gorenstein local ring. If* $\operatorname{ca}(R) = \mathfrak{m}$, *then R has minimal multiplicity and R has finite CM type.*

The second statement of Theorem 1.4 has an immediate consequence: For a hypersurface $R = (S, \mathfrak{n})/(f)$, where S is a two-dimensional regular local ring, if the cohomological annihilator of R is the maximal ideal, then R has minimal multiplicity; see Corollary 7.4.

§2. Notation and terminology

Throughout this article, R will be a commutative Noetherian ring.

2.1. In this article, we let $\operatorname{mod}(R)$ denote the category of finitely generated R-modules. If R is a local ring with maximal ideal $\mathfrak{m}$ and M is a finitely generated R-module, the depth of M, denoted $\operatorname{depth}_R(M)$, is the length of a maximal M-regular sequence contained in $\mathfrak{m}$. This is well defined as every maximal M-regular sequence contained in $\mathfrak{m}$ has the same length; see [7, Theorem 1.2.8].

We denote by $\operatorname{MCM}(R)$ the full subcategory of $\operatorname{mod}(R)$ consisting of all maximal Cohen–Macaulay R-modules, i.e., R-modules M satisfying

$$\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq \dim(R_\mathfrak{p})$$

for all $\mathfrak{p} \in \operatorname{Spec}(R)$. By convention, $0 \in \operatorname{MCM}(R)$.

2.2 (Syzygy modules). For each $i \geq 0$, $\Omega_R^n(M)$ is an n-th *syzygy module* in some projective resolution of M. That is, there is an exact sequence

$$0 \to \Omega_R^n(M) \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0,$$

where each P_i is finitely generated projective R-module for $0 \leq i \leq n-1$. By Schanuel's lemma, $\Omega_R^n(M)$ is independent of the choice of the projective resolution of M up to projective summands.

If R is local, we always use minimal free resolutions to compute syzygy modules in the article.

2.3 (Annihilators and stable annihilators). For an R-module M, the *annihilator* of M in R is

$$\operatorname{ann}_R M := \{r \in R \mid r \cdot M = 0\}.$$

For R-modules M and N, denote by $\mathcal{P}(M, N)$ the set of R-homomorphisms of M to N which factors through projective modules. This is an R-submodule of $\operatorname{Hom}_R(M, N)$. We put

$$\underline{\operatorname{Hom}}_R(M, N) := \operatorname{Hom}_R(M, N)/\mathcal{P}(M, N).$$

The *stable annihilator* of M, denoted $\underline{\operatorname{ann}}_R M$, is defined to be $\operatorname{ann}_R \underline{\operatorname{Hom}}_R(M, M)$.

We let $\operatorname{Spec}(R)$ denote the prime spectrum of R and write

$$\mathrm{V}(I) := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I \subseteq \mathfrak{p}\}$$

to be the Zariski closed subset of $\operatorname{Spec}(R)$ defined by the ideal I. For each finitely generated R-module M, we denote by $\mu(M)$ the minimal number of generators of M.

2.4 (Regular (local) rings and Gorenstein (local) rings). For a commutative Noetherian local ring $(R, \mathfrak{m}, k)$, the *embedding dimension* of R is defined to be $\mu(\mathfrak{m})$. There is always an inequality $\dim(R) \leq \operatorname{embdim}(R)$. The *codimension* of R, denoted $\operatorname{codim}(R)$, is defined to be their difference. That is,

$$\operatorname{codim}(R) := \operatorname{embdim}(R) - \dim(R).$$

A commutative Noetherian local ring $(R, \mathfrak{m})$ is *regular* if its codimension is zero. Due to Auslander, Buchsbaum, and Serre, a commutative Noetherian local ring is regular if and only if its global dimension is finite; see [7, Theorem 2.2.7]. A commutative Noetherian ring is said to be regular provided that $R_\mathfrak{p}$ is regular for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

A commutative Noetherian local ring $(R, \mathfrak{m})$ is *Gorenstein* if the injective dimension of R is finite. A commutative Noetherian ring is said to be Gorenstein provided that $R_\mathfrak{p}$ is Gorenstein for all $\mathfrak{p} \in \operatorname{Spec}(R)$. For a Gorenstein ring R, if R has finite Krull dimension, then the injective dimension of R is finite; see [8, Theorem 4.1.1].

2.5 (Finite CM type and countable CM type). Recall a local ring $(R, \mathfrak{m})$ is said to have *finite (resp. countable) Cohen–Macaulay (CM) type* provided that there are finitely (resp. countably) many indecomposable maximal Cohen–Macaulay R-modules up to isomorphism. See details in [29] and [35].

§3. Cohomological annihilators

The cohomological annihilator of a ring was introduced by Iyengar and Takahashi [24, Definition 2.1]. We recall the definition below.

3.1. For each integer $n \geq 0$, the n-th *cohomological annihilator* of R is defined to be the ideal

$$\mathrm{ca}^n(R) \coloneqq \bigcap_{M,N \in \mathrm{mod}(R)} \mathrm{ann}_R \operatorname{Ext}^n_R(M,N).$$

By the dimension shifting,

$$\mathrm{ca}^n(R) = \bigcap_{i \geq n, M,N \in \mathrm{mod}(R)} \mathrm{ann}_R \operatorname{Ext}^i_R(M,N).$$

The *cohomological annihilator* of R is defined to be

$$\mathrm{ca}(R) \coloneqq \bigcup_{n \geq 0} \mathrm{ca}^n(R).$$

This is an ideal of R. Indeed, there is an ascending chain

$$0 = \mathrm{ca}^0(R) \subseteq \mathrm{ca}^1(R) \subseteq \mathrm{ca}^2(R) \subseteq \cdots$$

of ideals of R. As R is Noetherian, it stabilizes after enough high terms, i.e.,

$$\mathrm{ca}(R) = \mathrm{ca}^n(R), \quad \text{for } n \gg 0.$$

3.2. It is proved that the socle of R, denoted $\mathrm{soc}(R)$, is always contained in $\mathrm{ca}(R)$. Indeed,

$$\mathrm{soc}(R) \subseteq \mathrm{ca}^1(R);$$

see [24, Example 2.6].

The following lemma is useful for computing $\mathrm{ca}(R)$. We refer the readers to [16, Lemma 3.8] and [24, Lemma 2.14] for its proof.

Lemma 3.3. *Let M be a finitely generated R-module. Then one has equalities of ideals of R:*

$$\begin{aligned} \underline{\mathrm{ann}}_R M &= \mathrm{ann}_R \operatorname{Ext}^1_R(M, \Omega^1_R(M)) \\ &= \bigcap_{i \geq 1, N \in \mathrm{mod}(R)} \mathrm{ann}_R \operatorname{Ext}^i_R(M,N). \end{aligned}$$

For Gorenstein rings with finite Krull dimension, we have the following useful lemma.

Proposition 3.4. *Let R be a Gorenstein ring with finite Krull dimension. Then*

$$\mathrm{ca}(R) = \mathrm{ca}^n(R)$$

for all $n \geq d+1$, where $d = \dim(R)$.

Proof. It is enough to prove that $\mathrm{ca}^n(R) \subseteq \mathrm{ca}^{d+1}(R)$ for all $n \geq d+1$. Consider $x \in \mathrm{ca}^n(R)$. For any $M, N \in \mathrm{mod}(R)$, we have

$$\mathrm{Ext}_R^{d+1}(M,N) \cong \mathrm{Ext}_R^1(\Omega_R^d(M),N).$$

Since $\Omega_R^d(M)$ is maximal Cohen–Macaulay and R is Gorenstein, $\Omega_R^d(M)$ is an i-th syzygy module for any $i \geq 0$. In particular, $\Omega_R^d(M) \cong \Omega_R^n(X)$ for some $X \in \mathrm{mod}\, R$. Thus,

$$\begin{aligned}\mathrm{Ext}_R^{d+1}(M,N) &\cong \mathrm{Ext}_R^1(\Omega_R^d(M),N)\\ &\cong \mathrm{Ext}_R^1(\Omega_R^n(X),N)\\ &\cong \mathrm{Ext}_R^{n+1}(X,N).\end{aligned}$$

We obtain $x \in \mathrm{ca}^n(R) \subseteq \mathrm{ann}_R \mathrm{Ext}_R^{n+1}(X,N) = \mathrm{ann}_R \mathrm{Ext}_R^{d+1}(M,N)$. As M, N are arbitrary, we conclude $x \in \mathrm{ca}^{d+1}(R)$. □

3.5. Let $\mathrm{Sing}(R) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid R_{\mathfrak{p}} \text{ is not regular}\}$ be the *singular locus* of R. It is proved in [24, Lemma 2.10] that

$$\mathrm{Sing}(R) \subseteq \mathrm{V}(\mathrm{ca}(R)).$$

Hence, if $\mathrm{ca}(R) = R$, then R is regular. The converse holds if, in addition, R has finite Krull dimension. Indeed, if R is a regular ring of Krull dimension d, then the global dimension of R is equal to d. This yields that $\mathrm{Ext}_R^i(M,N) = 0$ for all $i \geq d+1$ and $M, N \in \mathrm{mod}(R)$. Thus, $\mathrm{ca}(R) = \mathrm{ca}^{d+1}(R) = R$.

Next, we compute the stable annihilator of cyclic modules.

Lemma 3.6. *Let I be an ideal of R. Then*

$$\underline{\mathrm{ann}}_R(R/I) = I + \mathrm{ann}_R I.$$

Proof. Since $\Omega_R^1(R/I) = I$, one has

$$\mathrm{Ext}_R^1(R/I, \Omega_R^1(R/I)) = \mathrm{Ext}_R^1(R/I, I).$$

This is annihilated by I and $\mathrm{ann}_R I$. It follows from Lemma 3.3 that $I + \mathrm{ann}_R I$ is contained in $\underline{\mathrm{ann}}_R(R/I)$.

Conversely, let $r \in \underline{\mathrm{ann}}_R(R/I)$. We have an exact sequence

$$0 \to I \to R \xrightarrow{g} R/I \to 0,$$

where g is the canonical map. Hence, by the proof of [24, Lemma 2.14], we have a factorization

$$\begin{tikzcd} R/I \arrow[rr, "\cdot r"] \arrow[dr, "f"'] & & R/I. \\ & R \arrow[ur, "g"'] & \end{tikzcd}$$

Since $f(r)$ is annihilated by I, one has $\mathrm{Im}(f) \subseteq \mathrm{ann}_R I$, where $\mathrm{Im}(f)$ denotes the image of f. Then

$$\mathrm{Im}(gf) \subseteq g(\mathrm{ann}_R I) \subseteq (\mathrm{ann}_R I)(R/I) = (\mathrm{ann}_R I + I)/I.$$

As gf is multiplication by r on R/I, we get

$$(rR + I)/I = r \cdot (R/I) = \mathrm{Im}(gf) \subseteq (\mathrm{ann}_R I + I)/I.$$

Lifting the containment to R, we obtain $rR + I \subseteq I + \mathrm{ann}_R I$. Hence, r is contained in $\mathrm{ann}_R I + I$. Thus, $\underline{\mathrm{ann}}_R(R/I) \subseteq I + \mathrm{ann}_R I$. $\square$

Let I be an ideal of R and $r \in R$. We let $\bar{r}$ denote the image of r under the canonical map $R \to R/I$.

Example 3.7. Let $R = S/(x^{2n})$, where $n \geq 0$ and S is a DVR with maximal ideal generated by an element x. Since R is an Artinian Gorenstein local ring, $\mathrm{ca}(R) = \mathrm{ca}^1(R)$ by Proposition 3.4. Note that S is a PID. By the structure theorem of finitely generated modules over PID, we know that the indecomposable modules of R are of the form $S/(x^i) \cong R/(\overline{x^i})$ for $1 \leq i \leq 2n$. Hence, using Lemmas 3.3 and 3.6,

$$\begin{aligned} \mathrm{ca}(R) &= \mathrm{ca}^1(R) \\ &= \bigcap_{1 \leq i \leq 2n} \underline{\mathrm{ann}}_R(R/(\overline{x^i})) \\ &= \bigcap_{1 \leq i \leq 2n} (\overline{x^i}) + \mathrm{ann}_R(\overline{x^i}) \\ &= \bigcap_{1 \leq i \leq 2n} (\overline{x^i}) + (\overline{x^{2n-i}}) \\ &= (\overline{x^n}). \end{aligned}$$

Keeping the same assumption of S, if $R = S/(x^{2n+1})$, where $n \geq 0$, one can compute similarly that $\mathrm{ca}(R) = (\overline{x^n})$.

3.8. Let R be a Gorenstein ring with finite Krull dimension. In this case, the injective dimension of R is finite; see 2.4. By a result of Esentepe [19, Lemma 2.3],

$$\mathrm{ca}(R) = \bigcap_{M \in \mathrm{MCM}(R)} \underline{\mathrm{ann}}_R\, M.$$

Proposition 3.9. *Let $(S, \mathfrak{n})$ be a Gorenstein local ring and $R = S/(f_1 f_2)$, where f_1, f_2 are S-regular elements. Then*

$$\mathrm{ca}(R) \subseteq (\overline{f_1}) + (\overline{f_2}).$$

Proof. As f_1, f_2 are S-regular, hence so is $f_1 f_2$, hence $S/(f_1 f_2)$ is also Gorenstein. Let $d := \dim S$. Then,

$$\operatorname{depth} S/(f_1 f_2) = \dim S/(f_1 f_2) = d - 1 = \operatorname{depth} S/(f_i) = \dim S/(f_i)$$

for $i = 1, 2$. Thus, $S/(f_i) = R/(\overline{f_i})$ are MCM $S/(f_1 f_2)$-modules for $i = 1, 2$.

Combining with Lemma 3.6, one has

$$\begin{aligned}\underline{\mathrm{ann}}_R(R/(\overline{f_1})) &= (\overline{f_1}) + \mathrm{ann}_R(\overline{f_1}) \\ &= (\overline{f_1}) + (\overline{f_2});\end{aligned}$$

here we see that $\mathrm{ann}_R(\overline{f_1}) = (f_1 f_2 :_S f_1)/(f_1 f_2) = (f_2)/(f_1 f_2) = (\overline{f_2})$ because f_1, f_2 are S-regular. Then the desired result follows immediately from 3.8. □

We make use of Proposition 3.9 to compute the cohomological annihilators of Example 3.11. Before that, we recall the following powerful result that will also be used in the computation.

3.10. Let $R = k[\![x_1, \dots, x_n]\!]/(f)$ be a hypersurface, where k is a field. For each $1 \le i \le n$,

$$\overline{\partial f/\partial x_i} \in \mathrm{ca}(R);$$

see [17, Proposition 18] or [19, Remark 3.1].

More generally, if $R = k[\![x_1, \dots, x_n]\!]/(f_1, \dots, f_c)$ is a complete intersection, where k is a field and $f_1, \dots, f_c$ is a regular sequence in $(x_1, \dots, x_n)$, then the Jacobian ideal generated by all maximal minors of the Jacobian matrix $(\partial f_i/\partial x_j)_{i=1,\dots,c}^{j=1,\dots,n}$ is contained in the cohomological annihilator of R; see [8, Corollaries 6.4.1 and 7.8.7].

Example 3.11. Let $R = k[\![x_1, \dots, x_n]\!]/(x_1^2)$, where k is a field. It follows from Proposition 3.9 that

$$\mathrm{ca}(R) \subseteq (\overline{x_1}).$$

Combining with 3.10, if, in addition, the characteristic of k is not equal to 2, then the cohomological annihilator of R is equal to $(\overline{x_1})$.

§4. Gorenstein local rings with $\mathrm{ca}(R) = \mathfrak{m}$

For a Noetherian local ring $(R, \mathfrak{m})$, as mentioned in 3.5, R is regular if and only if $\mathrm{ca}(R) = R$. Thus, the cohomological annihilator can be used to detect whether a local ring is regular. Also, it suggests that a commutative Noetherian ring has better homological properties if $\mathrm{ca}(R)$ is bigger. As mentioned in the introduction, there is a natural question: What can we say about the local ring R if $\mathrm{ca}(R) = \mathfrak{m}$? We consider this question throughout this article.

Let $\mathrm{mod}_0(R)$ denote the full subcategory of $\mathrm{mod}(R)$ consisting of modules that are locally free on the punctured spectrum. Modules with finite length are contained in $\mathrm{mod}_0(R)$. The following is a version of [13, Proposition 4.5 and 4.6 (1)] for rings that are not necessarily Cohen–Macaulay.

Lemma 4.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. For each $n \geq 0$, then one has the following:*

(1) *Let $a \in \bigcap_{M,N \in \mathrm{mod}_0(R)} \mathrm{ann}\,\mathrm{Tor}_n^R(M,N)$. Then $a^{2^{2d}}\,\mathrm{Tor}_i^R(M,N) = 0$ for all $i \geq n + 2d$ and all $M, N \in \mathrm{mod}(R)$.*
(2) *Let $a \in \bigcap_{M,N \in \mathrm{mod}_0(R)} \mathrm{ann}\,\mathrm{Ext}^n_R(M,N)$. Then $a^{2^{2d}}\,\mathrm{Ext}^i_R(M,N) = 0$ for all $i \geq n$ and $M, N \in \mathrm{mod}(R)$.*
(3) *Here is an inclusion:*

$$\mathrm{Sing}(R) \subseteq \mathrm{V}\Bigg(\bigcap_{M,N \in \mathrm{mod}_0(R)} \mathrm{ann}\,\mathrm{Tor}_n^R(M,N)\Bigg) \bigcap \mathrm{V}\Bigg(\bigcap_{M,N \in \mathrm{mod}_0(R)} \mathrm{ann}\,\mathrm{Ext}^n_R(M,N)\Bigg).$$

Proof. (1) If $n = 0$, then $a = 0$ as $R \in \mathrm{mod}_0(R)$ and $\mathrm{Tor}_0^R(R,R) \cong R$. Now, assume $n > 0$. Let M, N be finitely generated R-modules with finite length. For each $i \geq n$, there is an isomorphism

$$\mathrm{Tor}_i^R(M,N) \cong \mathrm{Tor}_n^R(\Omega_R^{i-n}(M), N).$$

We observe that $\mathrm{mod}_0(R)$ is closed under syzygy. Combining with the hypothesis of a, we conclude by the isomorphism above that $a\,\mathrm{Tor}_i^R(M,N) = 0$ for all $i \geq n$ and all finite length modules M, N. Then it follows immediately from [13, Corollary 4.4 (1)] that $a^{2^{2d}}\,\mathrm{Tor}_i^R(M,N) = 0$ for all $i \geq n + 2d$ and all $M, N \in \mathrm{mod}(R)$.

Combining with [13, Corollary 4.4 (2)], the proof of (2) is similar to the proof of (1) and we omit the proof here.

(3) The proof is similar to the proof of [13, Proposition 4.6(1)]. For each $\mathfrak{p} \in \operatorname{Sing}(R)$, one has $\bigcap_{M,N\in \operatorname{mod}_0(R)} \operatorname{ann} \operatorname{Tor}_n^R(M,N) \subseteq \mathfrak{p}$. If not, there exists

$$a \in \bigcap_{M,N\in \operatorname{mod}_0(R)} \operatorname{ann} \operatorname{Tor}_n^R(M,N) \setminus \mathfrak{p}.$$

Then it follows from (1) that $a^{2^{2d}} \operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq n+2d$ and all $M, N \in \operatorname{mod}(R)$. In particular, $a^{2^{2d}} \operatorname{Tor}_i^R(R/\mathfrak{p}, R/\mathfrak{p}) = 0$. Combining with $a \notin \mathfrak{p}$, we get $\operatorname{Tor}_i^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \kappa(\mathfrak{p})) = 0$ for all $i \geq n+2d$, where $\kappa(\mathfrak{p})$ is the residue field of $R_\mathfrak{p}$. This yields that $R_\mathfrak{p}$ is regular, which contradicts $\mathfrak{p} \in \operatorname{Sing}(R)$. Hence, $\bigcap_{M,N\in \operatorname{mod}_0(R)} \operatorname{ann} \operatorname{Tor}_n^R(M,N) \subseteq \mathfrak{p}$. Similarly, one can prove the similar result that the Ext version holds. □

For a Noetherian local ring $(R, \mathfrak{m})$, we let $\widehat{M}$ denote the $\mathfrak{m}$-adic completion of M for each R-module M. We note here that the current authors established Lemma 4.1 and Proposition 4.2 in 2023, and these were also established independently by Kimura [27]. Indeed, a significant part of the arguments are motivated by the arguments in [16, Proposition 4.9 (2)].

Proposition 4.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then for each $n \geq 0$,*

$$\operatorname{Sing}(\widehat{R}) \subseteq \mathrm{V}(\widehat{\operatorname{ca}^n(R)}).$$

In particular, $\operatorname{Sing}(\widehat{R}) \subseteq \mathrm{V}(\widehat{\operatorname{ca}(R)})$.

Proof. The second statement follows immediately from the first one. By Lemma 4.1 (3),

$$\operatorname{Sing}(\widehat{R}) \subseteq \mathrm{V}(I),$$

where $I := \bigcap_{M,N\in \operatorname{mod}_0(\widehat{R})} \operatorname{ann}_{\widehat{R}} \operatorname{Ext}_{\widehat{R}}^n(M,N)$. Then it is enough to show that $\mathrm{V}(I) \subseteq \mathrm{V}(\widehat{\operatorname{ca}^n(R)})$.

For each $M \in \operatorname{mod}_0(\widehat{R})$, it follows from [4, Corollary 4.4] that M is a direct summand of $\widehat{M'}$ for some $M' \in \operatorname{mod}_0(R)$. This yields the second inclusion in the following:

$$\begin{aligned} \operatorname{ca}^n(R) &\subseteq \bigcap_{M,N\in \operatorname{mod}_0(R)} \operatorname{ann}_R \operatorname{Ext}_R^n(M,N) \\ &\subseteq R \bigcap \Bigg(\bigcap_{M,N\in \operatorname{mod}_0(\widehat{R})} \operatorname{ann}_{\widehat{R}} \operatorname{Ext}_{\widehat{R}}^n(M,N) \Bigg). \end{aligned}$$

In particular, $\operatorname{ca}^n(R) \subseteq I$. Hence,

$$\widehat{\operatorname{ca}^n(R)} = \widehat{R} \operatorname{ca}^n(R) \subseteq \widehat{R} I = I,$$

where the first equation follows from [2, Proposition 10.13] and the third one is because I is an ideal of $\widehat{R}$. Then one has $\mathrm{V}(I) \subseteq \mathrm{V}(\widehat{\mathrm{ca}^n(R)})$. This completes the proof. $\square$

Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then R is said to have an *isolated singularity* provided that $R_\mathfrak{p}$ is regular for all prime ideals $\mathfrak{p} \neq \mathfrak{m}$. This is equivalent to saying that $\mathrm{Sing}(R) \subseteq \{\mathfrak{m}\}$. If $\widehat{R}$ has an isolated singularity, then so does R; this can be deduced by [29, Lemma 7.9].

Proposition 4.3 (1) below was established by Kimura [27, Proposition 2.4], and Proposition 4.3 (2) was proved by Bahlekeh, Hakimian, Salarian, and Takahashi [4, Theorem 4.5] under the assumption that $\widehat{R}$ has an isolated singularity.

Proposition 4.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then*

(1) *The following three conditions are equivalent:*

  (a) $\mathrm{ca}(R)$ *contains a power of* $\mathfrak{m}$.

  (b) $\mathrm{ca}(\widehat{R})$ *contains a power of* $\widehat{\mathfrak{m}}$.

  (c) $\widehat{R}$ *has an isolated singularity.*

(2) $\mathrm{ca}(R) = \mathfrak{m}$ *if and only if* $\mathrm{ca}(\widehat{R}) = \widehat{\mathfrak{m}}$.

Proof. (1) The implication (a) $\Rightarrow$ (c) follows from Proposition 4.2 and the fact that completions of $\mathfrak{m}$-primary ideals are $\widehat{\mathfrak{m}}$-primary. The implication (c) $\Rightarrow$ (b) can be proved by combining [24, Theorem 1.1] with [15, Corollary 3.12]. The implication (b) $\Rightarrow$ (a) follows from [4, Theorem 4.5 (1)].

(2) If either $\mathrm{ca}(R) = \mathfrak{m}$ or $\mathrm{ca}(\widehat{R}) = \widehat{\mathfrak{m}}$, then (1) yields that $\widehat{R}$ has an isolated singularity. When $\widehat{R}$ has an isolated singularity, it follows from [4, Theorem 4.5] that $\mathrm{ca}(R) = \mathrm{ca}(\widehat{R}) \cap R$. The statement of (2) follows immediately. $\square$

4.4. Let $\underline{\mathrm{MCM}}(R)$ denote the stable category of the maximal Cohen–Macaulay modules. More precisely, it has the same objects as $\mathrm{MCM}(R)$, and for each $M, N \in \mathrm{MCM}(R)$, the morphism spaces

$$\mathrm{Hom}_{\underline{\mathrm{MCM}}(R)}(M, N) := \underline{\mathrm{Hom}}_R(M, N).$$

The category $\underline{\mathrm{MCM}}(R)$ is a triangulated category if R is Gorenstein; see [8, Chapter 4 and Appendix B].

For a local ring $(R, \mathfrak{m})$ and a finitely generated R-module M, we let $\mu(M)$ denote the minimal number of generators of M. When R is Artinian, we write $\ell\ell(R)$ to be the *Lowey length* of R, namely the minimal integer n such that $\mathfrak{m}^n = 0$.

The following result concerns the dimension of the triangulated category introduced by Rouquier [32, definition 3.2]. For a triangulated category $\mathcal{T}$, we let $\dim \mathcal{T}$ denote dimension in the sense of Rouquier; we refer the reader to [32] for the definition and more details.

Proposition 4.5. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring. Then*

(1) *If $\operatorname{ca}(R)$ is $\mathfrak{m}$-primary, then*

$$\dim \underline{\mathrm{MCM}}(R) \leq \big(\mu(\operatorname{ca}(R)) - \dim(R) + 1\big)\ell\ell(R/\operatorname{ca}(R)) - 1.$$

In particular, if $\operatorname{ca}(R) = \mathfrak{m}$, then

$$\dim \underline{\mathrm{MCM}}(R) \leq \operatorname{codim}(R).$$

(2) *Assume $R = (S, \mathfrak{n})/(f_1, \ldots, f_c)$, where S is regular and $f_1, \ldots, f_c$ is a regular sequence in $\mathfrak{n}^2$. If $\operatorname{ca}(R)$ is the maximal ideal of R, then*

$$\dim \underline{\mathrm{MCM}}(R) = c \quad \textit{or} \quad c - 1.$$

Proof. (1) Due to Buchweitz [8, Theorem 4.4.1], $\underline{\mathrm{MCM}}(R)$ is triangle equivalent to the singularity category of R. Since $\operatorname{ca}(R)$ is $\mathfrak{m}$-primary, R has an isolated singularity by Proposition 4.3. Combining with 3.8, the same proof as [30, Lemma 6.5 (2)] yields the inequality in the first statement. If $\operatorname{ca}(R) = \mathfrak{m}$, then $\dim \underline{\mathrm{MCM}}(R) \leq \operatorname{codim}(R)$ by the first statement.

(2) The assumption means that R is a complete intersection of codimension c, and the desired result follows by combining with the first statement and the following result:

$$\operatorname{codim}(R) - 1 \leq \dim \underline{\mathrm{MCM}}(R);$$

see [5, Corollary 5.10]. □

The above result has an immediate consequence on the hypersurface.

Corollary 4.6. *Let $R = (S, \mathfrak{n})/(f)$, where S is regular and $f \in \mathfrak{n}^2$. If $\operatorname{ca}(R)$ is the maximal ideal of R, then*

$$\dim \underline{\mathrm{MCM}}(R) \leq 1.$$

Remark 4.7. For an isolated complete hypersurface singularity $(R, \mathfrak{m})$, it is worth noting that R has finite CM type if and only if $\dim \underline{\mathrm{MCM}}(R) = 0$; see [12, Propositions 2.4 and 2.5]. Inspired by this and Corollary 4.6, we consider the business about finite CM type under the assumption that $\operatorname{ca}(R) = \mathfrak{m}$; see Proposition 4.11 and Theorem 4.12.

Lemma 4.8. *Let $R = k[\![x_0, \dots, x_d]\!]/(f)$, where $f \in (x_0, \dots, x_{d-1})^2$ is a non-zero element and k is a field. Then* $\operatorname{ca}(R)$ *is not* $\mathfrak{m}$*-primary.*

Proof. Note that $(x_0, \dots, x_{d-1})/(f)$ is a prime ideal of R but not maximal. Denote this prime ideal by $\mathfrak{p}$. We write $S = k[\![x_0, \dots, x_d]\!]$ and $\mathfrak{q} = (x_0, \dots, x_{d-1})$. Then

$$R_{\mathfrak{p}} \cong (S/(f))_{\mathfrak{q}} \cong S_{\mathfrak{q}}/(f)S_{\mathfrak{q}}.$$

Since $f \in \mathfrak{q}^2$, one has $(f)S_{\mathfrak{q}} \subseteq \mathfrak{q}^2 S_{\mathfrak{q}}$. This yields that $R_{\mathfrak{p}}$ is not regular; see [7, Proposition 2.2.4]. Hence, R does not have an isolated singularity. It follows from Proposition 4.3 that $\operatorname{ca}(R)$ is not $\mathfrak{m}$-primary. □

Let $S = k[\![x_0, \dots, x_d]\!]$ and $f \in (x_0, \dots, x_d)^2$, where k is an algebraically closed field of characteristic 0. Consider $R = S/(f)$. For each $i \geq 2$, $R^\sharp := S[\![z]\!]/(f + z^i)$ is called the i-th *branched cover* of R. There is the natural surjective map $\pi\colon R^\sharp \to R$.

Lemma 4.9. *Keep the assumption as above. Let $R^\sharp = S[\![z]\!]/(f + z^i)$ be the i-th branched cover of the hypersurface $R = S/(f)$ as above. If* $\operatorname{ca}(R^\sharp)$ *is the maximal ideal, then so is R. The converse holds if, in addition, $i = 2$.*

Proof. First, assume that $\operatorname{ca}(R^\sharp)$ is the maximal ideal. By [19, Theorem 5.4] one has $\pi(\operatorname{ca}(R^\sharp)) \subseteq \operatorname{ca}(R)$. Thus, $\operatorname{ca}(R)$ contains the maximal ideal of R, denoted $\mathfrak{m}$. We claim that $\operatorname{ca}(R) = \mathfrak{m}$. If not, then $\operatorname{ca}(R) = R$. By 3.5, R is regular. This contradicts that $f \in (x_0, \dots, x_d)^2$; see [7, Proposition 2.2.4]. Hence, $\operatorname{ca}(R) = \mathfrak{m}$.

Next, assume $i = 2$ and $\operatorname{ca}(R)$ is the maximal ideal. When $i = 2$, by [19, Theorem 5.4], $\pi(\operatorname{ca}(R^\sharp)) = \operatorname{ca}(R)$. Combining this with the assumption, we get that $\overline{x_i}$ is in $\operatorname{ca}(R^\sharp)$ for each $0 \leq i \leq d$. It follows from 3.10 that

$$\overline{\partial(f + z^2)/\partial(z)} = \overline{2z} \in \operatorname{ca}(R^\sharp).$$

Thus, $\bar{z} \in \operatorname{ca}(R^\sharp)$ as the characteristic of k is zero. Also, $\operatorname{ca}(R^\sharp) \neq R^\sharp$ as $R^\sharp$ is not regular. We conclude that $\operatorname{ca}(R^\sharp)$ is the maximal ideal of $R^\sharp$. □

The converse of the above lemma does not hold if $i \neq 2$; see [19, Example 5.5] or Example 5.10. For the case $i = 2$, one cannot remove the assumption on the characteristic in the above lemma; see the following example.

Example 4.10. Let $R = k[\![x]\!]/(x^2)$, where k is a field of characteristic 2. Then $\operatorname{ca}(R)$ is the maximal ideal. However, the cohomological annihilator of $R^\sharp = k[\![x, z]\!]/(x^2 + z^2)$ is not the maximal ideal. Indeed, since $x^2 + z^2 = (x + z)^2$, it follows from 3.9 that

$$\operatorname{ca}(R^\sharp) \subseteq (\bar{x} + \bar{z}).$$

Thus, $\operatorname{ca}(R^\sharp)$ is not the maximal ideal.

Recall that a local ring $(R, \mathfrak{m})$ is a *hypersurface* provided that R is isomorphic to a regular local ring modulo an ideal generated by one element.

Proposition 4.11. *Let $(R, \mathfrak{m}, k)$ be an equicharacteristic complete hypersurface of dimension d with countable CM type, where k is an uncountable algebraically closed field of characteristic* 0*. Then*

(1) $\operatorname{ca}(R) = \mathfrak{m}$ *if and only if* $R \cong k[\![x_0, \ldots, x_d]\!]/(f)$*, where f is of the form*

$$x_0^2 + x_1^2 + x_2^2 + \cdots + x_d^2 \quad \text{or} \quad x_0^3 + x_1^2 + x_2^2 + \cdots + x_d^2.$$

(2) *If* $\operatorname{ca}(R) = \mathfrak{m}$*, then R has finite CM type.*

Proof. Combining with the first statement, the second statement follows from a result by Knörrer [28, Theorem] that an isolated simple hypersurface singularity has finite CM type. We prove the first statement in the following.

Consider the direction "$\Leftarrow$". Combining with Lemma 4.9, in order to show that $\operatorname{ca}(R) = \mathfrak{m}$, it is equivalent to show that $\operatorname{ca}(R) = \mathfrak{m}$ when $d = 0$. This follows from Example 3.7.

Consider the direction "$\Rightarrow$". Since R is an equicharacteristic complete hypersurface, R is isomorphic to the ring $k[\![x_0, \ldots, x_d]\!]/(g)$, where g is a non-zero element in $(x_0, \ldots, x_d)$; this can be deduced by combining the Cohen structure theorem with [7, Theorem 2.3.3 (c)]. By $\operatorname{ca}(R) = \mathfrak{m}$, the ring R has an isolated singularity by Proposition 4.3. If $d = 0$, then $R \cong k[\![x_0]\!]/(x_0^i)$. Thus, R is of the form $k[\![x_0]\!]/(x_0^2)$ or $k[\![x_0]\!]/(x_0^3)$ by the computation in Example 3.7.

Next, we assume $d > 0$. First, assume R has finite CM type. Due to Buchweitz, Greuel, and Schreyer [9, Theorem B], $R \cong k[\![x_0, \ldots, x_d]\!]/(f)$, where f is one of the following forms:

$$\begin{aligned}
A_n &\colon f = x_0^{n+1} + x_1^2 + x_2^2 + \cdots + x_d^2, && n \geq 1,\\
D_n &\colon f = x_0^{n-1} + x_0 x_1^2 + x_2^2 + \cdots + x_d^2, && n \geq 4,\\
E_6 &\colon f = x_0^3 + x_1^4 + x_2^2 + \cdots + x_d^2,\\
E_7 &\colon f = x_0^3 + x_0 x_1^3 + x_2^2 + \cdots + x_d^2,\\
E_8 &\colon f = x_0^3 + x_1^5 + x_2^2 + \cdots + x_d^2;
\end{aligned}$$

see also [29, Theorem 9.8]. Let f be of the form A_n. Since $\operatorname{ca}(R) = \mathfrak{m}$, Lemma 4.9 yields that this is equivalent to that the cohomological annihilator of $k[\![x_0]\!]/(x_0^{n+1})$ is the maximal ideal. By Example 3.7, we get that $n = 1$ or 2. Next, we prove that f cannot be of the forms D_n, E_6, E_7, E_8. Let f be of the form D_n; if the cohomological annihilator of $k[\![x_0, \ldots, x_d]\!]/(f)$ is the maximal ideal, then the cohomological annihilator of $R' := k[\![x_0, x_1]\!]/(x_0^{n-1} + x_0 x_1^2)$ is the maximal ideal by Lemma 4.9. It follows from Proposition 3.9 that

$$\operatorname{ca}(R') \subseteq (\overline{x_0}) + (\overline{x_0^{n-2}+x_1^2}) = (\overline{x_0}, \overline{x_1^2}),$$

where the equation holds as $n \geq 4$ in the form D_n. This contradicts that $\operatorname{ca}(R')$ is the maximal ideal. Thus, f cannot be of the form D_n. Similarly, f cannot be of the form E_7. Let f be of the form E_6; if the cohomological annihilator of $k[\![x_0,\dots,x_d]\!]/(f)$ is the maximal ideal, we conclude by Lemma 4.9 that the cohomological annihilator of $k[\![x_1]\!]/(x_1^4)$ is the maximal ideal. This is a contradiction; see Example 3.7. Similarly, f cannot be of the form E_8.

We now assume that R has countable but not finite CM type. Buchweitz, Greuel, and Schreyer [9, Theorem B] proved that $R \cong k[\![x_0,\dots,x_d]\!]/(f)$, where f is one of the following forms:

$$A_\infty\colon f = x_1^2 + x_2^2 + \cdots + x_d^2,$$
$$D_\infty\colon f = x_0x_1^2 + x_2^2 + \cdots + x_d^2;$$

see also [29, Theorem 14.16]. In the above two cases, $f \in (x_1,\dots,x_d)^2$, it follows from Lemma 4.8 that $\operatorname{ca}(R)$ is not $\mathfrak{m}$-primary. In particular, $\operatorname{ca}(R) \neq \mathfrak{m}$.

By the above, we prove that if $\operatorname{ca}(R) = \mathfrak{m}$, then R must have the form in the proposition. □

Theorem 4.12. *Let $(R,\mathfrak{m},k)$ be an equicharacteristic complete Gorenstein local ring of dimension d, where k is an algebraically closed field of characteristic 0. Consider the following conditions:*

(1) $\operatorname{ca}(R) = \mathfrak{m}$ *and R has finite CM type.*

(2) $\operatorname{ca}(R) = \mathfrak{m}$ *and R is a hypersurface with countable CM type.*

(3) $R \cong k[\![x_0,\dots,x_d]\!]/(f)$*, where f is of the form*

$$x_0^2 + x_1^2 + x_2^2 + \cdots + x_d^2 \quad \textit{or} \quad x_0^3 + x_1^2 + x_2^2 + \cdots + x_d^2.$$

Then (1) ⇔ (3) ⇒ (2). All these conditions are equivalent if, in addition, k is uncountable.

Proof. (1) ⇒ (3). Assume $\operatorname{ca}(R) = \mathfrak{m}$ and R has finite CM type. The desired result follows from the same argument as in Proposition 4.11; when R has finite CM type, the assumption that k is uncountable is not needed (see [29, Theorem 9.8]).

(3) ⇒ (1). Assume R is of the form in the statement. Combining Example 3.7 with Lemma 4.9, $\operatorname{ca}(R) = \mathfrak{m}$. On the other hand, it follows from a result by Knörrer [28, Theorem] that R has finite CM type.

(1) $\Rightarrow$ (2). If R is a complete Gorenstein local ring with finite CM type, then it is trivial that R has countable CM type. On the other hand, it follows from a result by Herzog that R is a hypersurface; see [29, Theorem 9.15].

If, in addition, k is uncountable, then the implication (2) $\Rightarrow$ (3) follows from Proposition 4.11. □

Remark 4.13. In [22], Huneke and Leuschke raised a question: Let R be a complete Cohen–Macaulay local ring with countable CM type. If R has an isolated singularity, does R have finite CM type?

Theorem 4.12 gives some sufficient conditions which support the above question. Assume $(R, \mathfrak{m}, k)$ is a complete hypersurface with countable CM type, where k is an uncountable algebraically closed field with characteristic 0. If $\mathrm{ca}(R) = \mathfrak{m}$, then R has finite CM type by Theorem 4.12.

The results in the next three sections will also support the above question.

§5. Minimal multiplicity case and some questions

Throughout this section, $(R, \mathfrak{m})$ will be a Cohen–Macaulay local ring of dimension d. The main result of this section is Theorem 1.2 from the introduction. We prove that the condition $\mathrm{ca}(R) = \mathfrak{m}$ implies that R has finite CM type if $(R, \mathfrak{m})$ is a Gorenstein local ring with minimal multiplicity. Motivated by this and some evidence, we propose two conjectures and two questions at the end of this section. Also, in the next two sections, we prove, under some mild assumptions, that these conjectures and questions are true if $\dim(R) \leq 1$.

5.1. We recall here the definition of Ulrich modules; see [6] and [33] for more details.

A finitely generated R-module M is called *Ulrich* provided that $M \in \mathrm{MCM}(R)$ and $\mathrm{e}(M) = \mu(M)$, where $\mathrm{e}(M)$ is the multiplicity of M with respect to $\mathfrak{m}$ (see [7, Section 4.6]), and $\mu(M)$ is the minimal number of generators of M.

Assume k is infinite. Then $\mathfrak{m}$ contains a system of parameters $x_1, \dots, x_d$ such that $Q := (x_1, \dots, x_d)$ is a minimal reduction of $\mathfrak{m}$ in the sense of Northcott and Rees [31, Definition 2]. For each maximal Cohen–Macaulay R-module M, note that Q is a reduction of $\mathfrak{m}$ with respect to M. One has

$$\mathrm{e}(M) = \ell(M/QM) \geq \ell(M/\mathfrak{m}M) = \mu(M),$$

where $\ell(-)$ represents the length of the R-module. Thus, M is Ulrich if and only if $\mathfrak{m}M = QM$.

Example 5.2. Let $(R, \mathfrak{m}, k)$ be an Artinian local ring. Then an R-module is Ulrich if and only if it is a direct sum of finite copies of k.

5.3. Abhyankar [1] observed that the multiplicity of R has a lower bound. More precisely,

$$\mathrm{e}(R) \geq \operatorname{embdim}(R) - \dim(R) + 1,$$

and R is said to have *minimal multiplicity* provided that the equality holds.

When k is infinite, R has minimal multiplicity if and only if $\mathfrak{m}^2 = Q\mathfrak{m}$ for some minimal reduction Q of $\mathfrak{m}$ as in 5.1; see [7, Exercise 4.6.14].

Example 5.4. For a local ring $(R, \mathfrak{m})$, the *order* of an element $f \in R$, denoted $\operatorname{ord}_{\mathfrak{m}}(f)$, is defined as follows: $\operatorname{ord}_{\mathfrak{m}}(f) := \sup\{n \in \mathbb{Z}_{\geq 0} \mid f \in \mathfrak{m}^n\}$; if this supremum does not exist, then we set $\operatorname{ord}_{\mathfrak{m}}(f) = \infty$. By the Krull-intersection theorem, $\operatorname{ord}_{\mathfrak{m}}(f) = \infty$ if and only if $f = 0$.

Let $(S, \mathfrak{n})$ be a regular local ring and consider $R = S/(f)$, where $f \in \mathfrak{n}^2$. By definition, R has minimal multiplicity if and only if $\mathrm{e}(R) = 2$. It is known that the multiplicity of R is equal to $\operatorname{ord}_{\mathfrak{n}}(f)$; see [23, Example 11.2.8]. Thus, R has minimal multiplicity if and only if $f \in \mathfrak{n}^2 \setminus \mathfrak{n}^3$.

Theorem 5.5. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring with minimal multiplicity. If $\operatorname{ca}(R) = \mathfrak{m}$, then R has finite CM type.*

Proof. Let M be a non-free indecomposable maximal Cohen–Macaulay $\widehat{R}$-module. Since R is Gorenstein with minimal multiplicity, then so is $\widehat{R}$. It follows from [10, Lemma 4.3] that M is an Ulrich $\widehat{R}$-module.

By Proposition 4.3, $\operatorname{ca}(\widehat{R}) = \widehat{\mathfrak{m}}$. Thus,

$$\widehat{\mathfrak{m}} \subseteq \underline{\operatorname{ann}}_{\widehat{R}}(M) = \operatorname{ann}_{\widehat{R}} \operatorname{Ext}^1_{\widehat{R}}(M, \Omega^1_{\widehat{R}}(M)),$$

where the first inclusion is from 3.8 and the equality is from Lemma 3.3. As mentioned, M is an Ulrich $\widehat{R}$-module. Combining with the fact that M is indecomposable, it follows from [10, Proposition 3.3] that M is a direct summand of $\Omega^d_{\widehat{R}}(k)$, where k is the residue field of R.

Since $\widehat{R}$ is Henselian, the Krull–Schmidt theorem holds in $\operatorname{mod}(\widehat{R})$. Thus, there is a decomposition

$$\Omega^d_{\widehat{R}}(k) \cong M_1 \oplus M_2 \oplus \cdots \oplus M_n,$$

where each M_i is an indecomposable maximal Cohen–Macaulay $\widehat{R}$-module. It follows from this that $M \cong M_i$ for some i. We conclude that $\widehat{R}$ has finite CM type. Hence, R has finite CM type by [29, Theorem 10.1]. This completes the proof. $\square$

Remark 5.6. Let $(R,\mathfrak{m})$ be a local ring. It is known that R is Gorenstein with minimal multiplicity if and only if its completion $\widehat{R}$ is a hypersurface with minimal multiplicity. If R is not regular, these conditions are further equivalent to $\widehat{R}$ being a hypersurface of multiplicity 2.

Example 5.7. Let $(R,\mathfrak{m}) = k[\![x_1,\dots,x_n,y_1,\dots,y_n]\!]/(x_1y_1+\cdots+x_ny_n)$, where k is a field and $n \geq 1$. It follows from 3.5 and 3.10 that the cohomological annihilator of R is $\mathfrak{m}$. By Theorem 5.5, R has finite CM type.

For the case $n=1$, consider $R = k[\![x,y]\!]/(xy)$, where k is a field. By the proof of Theorem 5.5, one can conclude that each indecomposable maximal Cohen–Macaulay R-module is isomorphic to either R, $R/(\bar{x})$, or $R/(\bar{y})$.

A local ring $(R,\mathfrak{m})$ is said to be an *abstract hypersurface* provided that the $\mathfrak{m}$-adic completion of R is isomorphic to a regular local ring modulo an ideal generated by one element, namely $\widehat{R}$ is a hypersurface.

Theorem 5.8. *Let $(S,\mathfrak{n})$ be a Gorenstein local ring and set $(R,\mathfrak{m}) = S/(f_1f_2)$, where $f_1, f_2 \in \mathfrak{n}$ are S-regular elements. Assume $\operatorname{ca}(R) = \mathfrak{m}$; then*

(1) $\operatorname{embdim}(R) \leq 2$ *and S is either regular or an abstract hypersurface;*
(2) $\max\{\operatorname{ord}_{\mathfrak{n}}(f_1), \operatorname{ord}_{\mathfrak{n}}(f_2)\} \leq 2$, *and if equality holds, then* $\min\{\operatorname{ord}_{\mathfrak{n}}(f_1), \operatorname{ord}_{\mathfrak{n}}(f_2)\} = 1$ *and S is a DVR;*
(3) *if* $\dim R \geq 1$, *then* $\operatorname{embdim}(R) = 2 = 1 + \dim R$ *and S is regular;*
(4) *if* $\dim R \geq 1$, *then* $\mathrm{e}(R) = 2$ *and R has finite CM type.*

Proof. (1) Combining with the assumption, by Proposition 3.9,

$$(\overline{f_1}) + (\overline{f_2}) = \mathfrak{m}.$$

This implies that $\operatorname{embdim}(R) \leq 2$.

Since $f_1f_2 \in \mathfrak{n}^2$, wet get that $\operatorname{embdim}(S) = \operatorname{embdim}(R)$. This yields that $\operatorname{embdim}(S) \leq 2$. Note that $\dim(S) \geq 1$ as f_1f_2 is S-regular. Thus, $\operatorname{codim} S \leq 1$. If $\operatorname{codim}(S) = 0$, then S is regular. It remains to prove that S is an abstract hypersurface if $\operatorname{codim}(S) = 1$. Assume now $\operatorname{codim}(S) = 1$. This implies that $\operatorname{codim}(\widehat{S}) = 1$. By Cohen's structure theorem, $\widehat{S}$ is isomorphic to a regular local ring modulo an ideal. We can choose a minimal presentation of $\widehat{S}$ and write $\widehat{S} = Q/I$, where Q is a regular local ring and I is in the square of the maximal ideal of Q. Combining with $\operatorname{embdim}(\widehat{S}) = 1$, it follows from a theorem of Serre that I is a principal ideal; see [18, Corollary 21.20]. Thus, S is an abstract hypersurface if $\operatorname{codim}(S) = 1$.

(2) First, we prove that if $\max\{\mathrm{ord}_{\mathfrak{n}}(f_1), \mathrm{ord}_{\mathfrak{n}}(f_2)\} \geq 2$, then S is a DVR. Indeed, assume $f_1 \in \mathfrak{n}^2$. It follows from this and the proof of (1) that

$$\mathfrak{m}^2 + (\overline{f_2}) = \mathfrak{m}.$$

By Nakayama's lemma, $(\overline{f_2}) = \mathfrak{m}$. This yields that $(f_2) = \mathfrak{n}$. As mentioned in the proof of (1), $\dim(S) \geq 1$. Hence, $\dim(S) = \mathrm{embdim}(S) = 1$. This yields that S is a DVR.

Next, we prove that $\max\{\mathrm{ord}_{\mathfrak{n}}(f_1), \mathrm{ord}_{\mathfrak{n}}(f_2)\} \leq 2$. If not, one has f_1 or f_2 is in $\mathfrak{n}^3$. We may assume $f_1 \in \mathfrak{n}^3$. By the above argument, S is a DVR with $\mathfrak{n} = (f_2)$. Hence, $f_1 f_2 \in (f_2^4)$. It follows from Example 3.7 that $\mathrm{ca}(R) \subseteq \mathfrak{m}^2$. This yields that $\mathfrak{m} = \mathfrak{m}^2$ as $\mathrm{ca}(R) = \mathfrak{m}$. By Nakayama's lemma, $\mathfrak{m} = 0$. This contradicts $\mathfrak{m} \neq 0$; here, $\mathfrak{m} \neq 0$ is because S is a DVR and $f_1 f_2$ is S-regular. Thus, $\max\{\mathrm{ord}_{\mathfrak{n}}(f_1), \mathrm{ord}_{\mathfrak{n}}(f_2)\} \leq 2$.

Now assume $\max\{\mathrm{ord}_{\mathfrak{n}}(f_1), \mathrm{ord}_{\mathfrak{n}}(f_2)\} = 2$. This yields that S is a DVR with $\mathfrak{n} = (f_2)$. If $\min\{\mathrm{ord}_{\mathfrak{n}}(f_1), \mathrm{ord}_{\mathfrak{n}}(f_2)\} \geq 2$, then $f_1 f_2 \in (f_2^4)$. The same argument as above yields that $\mathrm{ca}(R) \subseteq \mathfrak{m}^2$ and $\mathfrak{m} = 0$. This again contradicts $\mathfrak{m} \neq 0$. Thus, $\min\{\mathrm{ord}_{\mathfrak{n}}(f_1), \mathrm{ord}_{\mathfrak{n}}(f_2)\} = 1$.

(3) Note that $1 \leq \dim R \leq \mathrm{embdim}(R)$. We claim that $\mathrm{embdim}(R) \neq 1$. If not, assume $\mathrm{embdim}(R) = 1$. By $\dim(R) \geq 1$, we conclude that R is regular. This contradicts $\mathrm{ca}(R) = \mathfrak{m}$; see 3.5. Combining with (1), $\mathrm{embdim}(R) = 2$.

By 3.5 and $\mathrm{ca}(R) = \mathfrak{m}$, R is not regular. Hence, $\dim R < \mathrm{embdim}(R) = 2$. Combining this with $\dim R \geq 1$, we conclude $\dim R = 1$. Thus, $\dim S = \dim R + 1 = 2$. Since $f_1 f_2 \in \mathfrak{n}^2$, we also know $\mathrm{embdim}(S) = \mathrm{embdim}(R) = 2$. Thus, S is regular.

(4) By (3), S is regular. We claim that $f_1 \notin \mathfrak{n}^2$. If not, assume $f_1 \in \mathfrak{n}^2$. It follows from this and the proof of (1) that $(f_2) = \mathfrak{n}$. Hence, S is a DVR. In particular, $\dim R = 0$, contradicting our hypothesis. Thus, $f_1 \in \mathfrak{n} \setminus \mathfrak{n}^2$. Similarly, $f_2 \in \mathfrak{n} \setminus \mathfrak{n}^2$. It follows from Example 5.4 that $\mathrm{e}(R) = 2$ and R has minimal multiplicity. Combining this with Theorem 5.5, R has finite CM type. □

Remark 5.9. Regarding the type of rings that appear in Theorem 5.8, we mention that if S is a Cohen–Macaulay local ring that admits a canonical module and is factorial, then S is Gorenstein ([7, Corollary 3.3.19]). Hence, one can apply Theorem 5.8 to get structural results on rings of the form $S/(f)$, where $f \in \mathfrak{m}$ is not a prime element, whose cohomology annihilator is the maximal ideal.

Example 5.10. Let $(R, \mathfrak{m}) = k[\![x, y]\!]/(x^3 + y^3)$, where k is a field. By 3.5, $\mathrm{ca}(R)$ is contained in $\mathfrak{m}$. Note that R does not have minimal multiplicity; see Example

5.4. Combining this with $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$, it follows immediately from Example 5.4 and Theorem 5.8 that

$$\mathrm{ca}(R) \subsetneqq \mathfrak{m} = (\bar{x}, \bar{y}).$$

Note that R is the 3rd branched cover of $k[\![x]\!]/(x^3)$. The computation of Example 3.7 implies that the cohomological annihilator of $k[\![x]\!]/(x^3)$ is the maximal ideal. Thus, the converse of Lemma 4.9 does not hold if we do not consider the double branched covers.

Inspired by Corollary 4.6 and Theorem 5.8, we propose the following conjectures.

Conjecture 5.11. *Let $(S, \mathfrak{n})$ be a regular local ring and $(R, \mathfrak{m}) = S/(f)$, where $f \in \mathfrak{n}$. If $\mathrm{ca}(R) = \mathfrak{m}$, then R has finite CM type.*

Conjecture 5.12. *Let $(S, \mathfrak{n})$ be a regular local ring and $(R, \mathfrak{m}) = S/(f)$, where $f \in \mathfrak{n}$. If $\dim(R) \geq 1$ and $\mathrm{ca}(R) = \mathfrak{m}$, then R has minimal multiplicity.*

Remark 5.13. In regard to the above conjectures, we note the following:

(1) In the above conjectures, if $\mathrm{ca}(R) = \mathfrak{m}$, then R is not regular; see 3.5. Thus, $f \in \mathfrak{n}^2$.
(2) Let R be a hypersurface as Conjecture 5.12. Then R has minimal multiplicity if and only if $f \in \mathfrak{n}^2 \setminus \mathfrak{n}^3$; see Example 5.4.
(3) We cannot remove the assumption that $\dim(R) \geq 1$ in Conjecture 5.12. For example, the cohomological annihilator of $k[\![x]\!]/(x^3)$ is the maximal ideal, but $k[\![x]\!]/(x^3)$ does not have minimal multiplicity.

5.14. Keep the assumption in Conjecture 5.11. If, in addition, R is Artinian, then S is a DVR. By the structure theorem of finitely generated modules over PID, one sees easily that R has finite CM type in this case. Thus, Conjecture 5.11 is true when R is Artinian.

The next result concerns the relations between the above two conjectures.

Proposition 5.15. *Let $(S, \mathfrak{n})$ be a regular local ring and $(R, \mathfrak{m}, k) = S/(f)$, where $f \in \mathfrak{n}$. Assume $\dim(R) \geq 1$ and $\mathrm{ca}(R) = \mathfrak{m}$. If Conjecture 5.12 is true, then so is Conjecture 5.11. The converse holds if, in addition, k is an algebraically closed field of characteristic 0 and R is equicharacteristic complete.*

Proof. The first statement holds by applying Theorem 5.5. Now, assume $\mathrm{ca}(R) = \mathfrak{m}$, k is an algebraically closed field of characteristic 0, and R is equicharacteristic complete. If Conjecture 5.11 is true, then R has finite CM type. Combining the assumption $\dim(R) \geq 1$ with Example 5.4, R is isomorphic to a hypersurface with minimal multiplicity by Theorem 4.12. Thus, R has minimal multiplicity. $\square$

5.16. In [3, Section 10], Auslander proved that a complete Cohen–Macaulay local ring with finite CM type has at most an isolated singularity. Due to Huneke and Leuschke [21, Corollary 2], this theorem still holds if the complete assumption is removed. Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring which is not regular. Combining with Proposition 4.3, there are the following implications:

$$
\begin{tikzcd}
\boxed{\begin{array}{c} R \text{ has} \\ \text{finite CM type} \end{array}} \arrow[rr, Rightarrow, shift left] \arrow[dr, Rightarrow, "/" description] & & \boxed{\begin{array}{c} R \text{ has an} \\ \text{isolated singularity} \end{array}} \arrow[ll, Rightarrow, shift left, "/" description] \arrow[dl, Rightarrow, shift left, "/" description] \\
& \boxed{\operatorname{ca}(R) = \mathfrak{m}} \arrow[ur, Rightarrow, shift left] &
\end{tikzcd}
.
$$

Note that the ring $k[\![x, y]\!]/(x^2, y^2)$ has an isolated singularity with $\operatorname{ca}(R) \neq \mathfrak{m}$; see Example 6.4. Moreover, it does not have finite CM type.

Motivated by the above diagram, Theorem 5.5, and the examples we computed, we put the following question.

Question 5.17. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring. If $\operatorname{ca}(R) = \mathfrak{m}$, does R have finite CM type?*

Let $(R, \mathfrak{m})$ be a Gorenstein local ring with finite CM type. Due to Herzog, R is an abstract hypersurface; see [29, Theorem 9.15]. Based on Question 5.17, we put another question.

Question 5.18. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring. If $\operatorname{ca}(R) = \mathfrak{m}$, is R an abstract hypersurface?*

Remark 5.19. In regard to the above questions, we note the following:

(1) Theorem 5.5 yields that Questions 5.17 and 5.18 have a positive answer if, in addition, R has minimal multiplicity.

(2) The assumption that R is Gorenstein is needed in Questions 5.17 and 5.18. For example, the cohomological annihilator of $k[\![x, y]\!]/(x, y)^2$ is the maximal ideal; see 3.2 and 3.5. Obviously, $k[\![x, y]\!]/(x, y)^2$ is not a hypersurface. It is known that this algebra does not have finite CM type.

We prove in the next two sections that, under some mild assumptions, all these conjectures and questions are true if $\dim(R) \leq 1$. Before ending this section, we give the following relations between these conjectures and questions.

Proposition 5.20. *Let $(R, \mathfrak{m}, k)$ be a complete Gorenstein local ring with $\mathrm{ca}(R) = \mathfrak{m}$, where k is an algebraically closed field of characteristic 0. Then we have the following:*

(1) *If Question 5.17 is true, then so is Question 5.18.*

(2) *Assume furthermore $\dim R \geq 1$ and R is equicharacteristic. If Question 5.17 is true, then Conjecture 5.12 holds.*

(3) *If Conjecture 5.12 holds and Question 5.18 is true, then Question 5.17 is true.*

Proof. (1) As mentioned above, this is due to Herzog; see [29, Theorem 9.15].

(2) Assume Question 5.17 is true. Then R has finite CM type. With the assumption that R is equicharacteristic complete, Theorem 4.12 yields that R has two possible forms as in Theorem 4.12. Since $\dim(R) \geq 1$, we conclude that R has minimal multiplicity. That is, Conjecture 5.12 holds.

(3) Assume Conjecture 5.12 holds and Question 5.18 is true. Then R is a hypersurface with minimal multiplicity. It follows from Theorem 5.5 that R has finite CM type. Hence, Question 5.17 is true. □

§6. Zero-dimensional case

The first main result of this section is Theorem 1.3 from the introduction. It turns out that the nilpotency degree of the cohomological annihilator $\mathrm{ca}(R)$ is at most three if R is an Artinian Gorenstein local ring but not a field. By making use of this and Proposition 6.6, we prove the first statement of Theorem 1.4 from the introduction.

Lemma 6.1. *For ideals I and J of R such that $J \subseteq I \subseteq \mathrm{ca}^1(R)$, we have*

$$\mathrm{ca}^1(R)J = IJ.$$

Proof. By Lemma 3.6, we have $\mathrm{ca}^1(R) \subseteq I + \mathrm{ann}_R I$ for every I. If $J \subseteq I$, then $J(\mathrm{ann}_R I) = 0$. Hence, $\mathrm{ca}^1(R)J \subseteq IJ$ for all $J \subseteq I$. As $I \subseteq \mathrm{ca}^1(R)$, we also have $IJ \subseteq \mathrm{ca}^1(R)J$. Thus, $\mathrm{ca}^1(R)J = IJ$ for all ideals $J \subseteq I \subseteq \mathrm{ca}^1(R)$. □

Proposition 6.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then*

$$(\mathrm{ca}^1(R))^2 \subseteq \mathrm{soc}(R).$$

In particular, if R is not a field, then $(\mathrm{ca}^1(R))^3 = 0$.

Proof. Take $I = J = \mathrm{ca}^1(R)\mathfrak{m}$. By Lemma 6.1,

$$(\mathrm{ca}^1(R))^2\mathfrak{m} = \big((\mathrm{ca}^1(R))^2\mathfrak{m}\big)\mathfrak{m}.$$

This yields $(\mathrm{ca}^1(R))^2\mathfrak{m} = 0$ by Nakayama's lemma. Hence,

$$(\mathrm{ca}^1(R))^2 \subseteq \mathrm{ann}_R(\mathfrak{m}) = \mathrm{soc}(R).$$

Assume R is not a field. Since $\mathrm{ca}^1(R) \subseteq \mathrm{ann}_R \mathrm{Ext}^1_R(R/\mathfrak{m}, R/\mathfrak{m})$, one has $\mathrm{ca}^1(R) \neq R$. Then $\mathrm{ca}^1(R) \subseteq \mathfrak{m}$. Thus, $(\mathrm{ca}^1(R))^3 \subseteq (\mathrm{ca}^1(R))^2\mathfrak{m} = 0$. □

Applying the above result to Artinian Gorenstein local rings, we get the following result.

Theorem 6.3. *Let $(R, \mathfrak{m}, k)$ be an Artinian Gorenstein local ring. Then*

$$(\mathrm{ca}(R))^2 = 0 \quad \textit{or} \quad (\mathrm{ca}(R))^2 = \mathrm{soc}(R) \cong k.$$

In particular, $(\mathrm{ca}(R))^3 = 0$ if, in addition, R is not a field.

Proof. As R is Artinian Gorenstein, one has

$$\mathrm{ca}(R) = \mathrm{ca}^1(R)$$

by Proposition 3.4. In an Artinian Gorenstein local ring, every non-zero ideal contains the socle ([7, Exercise 3.2.15 (d)]). It follows from Proposition 6.2 that either $(\mathrm{ca}(R))^2 = 0$ or $(\mathrm{ca}(R))^2 = \mathrm{soc}(R)$. Then the second statement is a direct consequence; see also Proposition 6.2. □

Example 6.4. Let k be a field and consider $R = k[\![x, y]\!]/(x^2, y^2)$. Then one has $\mathrm{ca}(R) = \mathrm{soc}(R)$.

Indeed, it follows from 3.2 that $\mathrm{ca}(R) \supseteq \mathrm{soc}(R)$. On the other hand, by 3.8,

$$\mathrm{ca}(R) \subseteq \underline{\mathrm{ann}}_R(R/(x)) \bigcap \underline{\mathrm{ann}}_R(R/(y)).$$

By Lemma 3.6, $\underline{\mathrm{ann}}_R(R/(x)) = (\bar{x}) + \mathrm{ann}_R(\bar{x}) = (\bar{x})$. Similarly, $\underline{\mathrm{ann}}_R(R/(y)) = (\bar{y})$. Then

$$\mathrm{ca}(R) \subseteq (\bar{x}) \bigcap (\bar{y}) = (\overline{xy}) = \mathrm{soc}(R).$$

Thus, $\mathrm{ca}(R) = \mathrm{soc}(R)$.

Lemma 6.5. *Let R be an Artinian Gorenstein ring. Assume there exist elements a_1, a_2, a_3 in R such that $a_1a_2 = a_1a_3 = a_2a_3 = 0$ and $(a_1)^2 = (a_2)^2 = (a_3)^2$. Then for each $r \in \mathrm{ca}(R)$,*

$$ra_1 = ra_2 = ra_3.$$

Proof. By 3.8, one has

$$\operatorname{ca}(R) = \bigcap_{M \in \operatorname{mod}(R)} \underline{\operatorname{ann}}_R M;$$

this can also be deduced by combining Lemma 3.3 with Lemma 3.6. Thus, the assumption yields that $r \in \underline{\operatorname{ann}}_R M$ for each $M \in \operatorname{mod}(R)$.

Denote $\{i, j, l\} = \{1, 2, 3\}$. Let M be the image of the following map:

$$R \oplus R \xrightarrow{\left(\begin{smallmatrix} a_i & a_j \\ a_l & a_i \end{smallmatrix}\right)} R \oplus R.$$

Denote by $\pi\colon R \oplus R \to M$ (resp. $\iota\colon M \to R \oplus R$) the canonical surjection (resp. the canonical injection) of the above map. Since $r \in \underline{\operatorname{ann}}_R M$, there is a factorization

$$\begin{tikzcd} M \arrow[rr, "\cdot r"] \arrow[dr, "f"'] & & M, \\ & P \arrow[ur, "g"'] & \end{tikzcd}$$

where P is a projective module. Combining with the property that π is surjective, the morphism g factors through π. In particular, the multiplication $r\colon M \to M$ factors through π. That is, there is a morphism $\varphi\colon M \to R \oplus R$ such that $\pi\varphi = r$. Since $\iota\colon M \to R \oplus R$ is an injection and $R \oplus R$ is an injective module, there exists a morphism $\varphi'\colon R \oplus R \to R \oplus R$ such that $\varphi'\iota = \varphi$. Thus, there exists a commutative diagram

$$\begin{tikzcd} & R \oplus R \arrow[dl, "\varphi'"'] \arrow[dr, "\left(\begin{smallmatrix} r & 0 \\ 0 & r \end{smallmatrix}\right)"] & \\ & M \arrow[u, "\iota"] \arrow[dl, "\varphi"] \arrow[d, "\cdot r"] & \\ R \oplus R \arrow[r, "\pi"] & M \arrow[r, "\iota"] & R \oplus R. \end{tikzcd}$$

This yields that $\left(\begin{smallmatrix} r & 0 \\ 0 & r \end{smallmatrix}\right)\iota = \iota\pi\varphi'\iota$, and hence,

$$\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \iota\pi = \iota\pi\varphi'\iota\pi.$$

We write φ' to be $\left(\begin{smallmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{smallmatrix}\right)$, where $b_{ij} \in R$. Since $\iota\pi = \left(\begin{smallmatrix} a_i & a_j \\ a_l & a_i \end{smallmatrix}\right)$, one has

$$\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} a_i & a_j \\ a_l & a_i \end{pmatrix} = \begin{pmatrix} a_i & a_j \\ a_l & a_i \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} a_i & a_j \\ a_l & a_i \end{pmatrix}.$$

Combining with $a_1a_2 = a_1a_3 = a_2a_3 = 0$, a direct calculation of the above equation shows that

$$\begin{pmatrix} ra_i & ra_j \\ ra_l & ra_i \end{pmatrix} = \begin{pmatrix} b_{11}(a_i)^2 & b_{21}(a_j)^2 + b_{12}(a_i)^2 \\ b_{21}(a_i)^2 + b_{12}(a_l)^2 & b_{22}(a_i)^2 \end{pmatrix}.$$

Since $(a_i)^2 = (a_j)^2 = (a_l)^2$, we conclude from the above equation that $ra_j = ra_l$. Since this is true for each $\{i, j, l\} = \{1, 2, 3\}$, one has $ra_1 = ra_2 = ra_3$. This completes the proof. □

A local ring $(R, \mathfrak{m})$ is called *short* provided that $\mathfrak{m}^3 = 0$. For a local ring $(R, \mathfrak{m})$, the embedding dimension of R, denoted $\operatorname{embdim}(R)$, is the minimal number of generators of the maximal ideal $\mathfrak{m}$.

Proposition 6.6. *Let $(R, \mathfrak{m}, k)$ be an equicharacteristic short Gorenstein local ring, where k is an algebraically closed field with $\operatorname{char}(k) \neq 2$. If the embedding dimension satisfies that $\operatorname{embdim}(R) \geq 2$, then*

$$\operatorname{ca}(R) = \operatorname{soc}(R).$$

Proof. By Cohen's structure theorem, $R \cong k[\![x_1, \ldots, x_n]\!]/I$, where $n = \operatorname{embdim}(R)$ and $I \subseteq (x_1, \ldots, x_n)^2$. Since R is a short Gorenstein ring, the structure of R is determined by the multiplication

$$\mathfrak{m}/\mathfrak{m}^2 \times \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}^2 \cong k.$$

This is a symmetric bilinear form. Classifying the structures of R is equivalent to classifying the symmetric bilinear forms [26, Section 6.3]. It follows from [20, Chapter 2] that R is isomorphic to the algebra

$$k[\![x_1, \ldots, x_n]\!]/(x_i^2 - x_j^2, x_ix_j \mid 1 \leq i < j \leq n).$$

If $e = 2$, then $R \cong k[\![x_1, x_2]\!]/(x_1^2 - x_2^2, x_1x_2)$. We observe that the maps

$$s \mapsto x_1 - \sqrt{-1}x_2 \quad \text{and} \quad t \mapsto x_1 + \sqrt{-1}x_2$$

induce an isomorphism of algebras

$$k[\![s, t]\!]/(s^2, t^2) \cong k[\![x_1, x_2]\!]/(x_1^2 - x_2^2, x_1x_2).$$

Hence, $R \cong k[\![s, t]\!]/(s^2, t^2)$. It follows from Example 6.4 that $\operatorname{ca}(R) = \operatorname{soc}(R)$.

If $n \geq 3$, then $R \cong k[\![x_1, \ldots, x_n]\!]/(x_i^2 - x_j^2, x_ix_j \mid 1 \leq i < j \leq n)$. It follows from 3.2 that $\operatorname{soc}(R) = \mathfrak{m}^2$ is contained in $\operatorname{ca}(R)$. Next, we prove $\operatorname{ca}(R) = \operatorname{soc}(R)$. If not, there exists an element in $\operatorname{ca}(R) \setminus \operatorname{soc}(R)$. Hence, there exists an element $r = b_1\overline{x_1} + b_2\overline{x_2} + \cdots + b_n\overline{x_n}$ in $\operatorname{ca}(R)$, where $b_1, b_2, \ldots, b_n \in k$ are not all zero.

Choose i, j, l such that $1 \leq i < j < l \leq n$ and consider $a_1 = \overline{x_i}, a_2 = \overline{x_j}, a_3 = \overline{x_l}$. It follows from Lemma 6.5 that $r\overline{x_i} = r\overline{x_j} = r\overline{x_l}$. This implies that

$$b_i\overline{x_i^2} = b_j\overline{x_j^2} = b_l\overline{x_l^2}$$

in R. Combining with $x_i^2 = x_j^2 = x_k^2 \neq 0$ in R, one has $b_i = b_j = b_l$. Since i, j, l are arbitrary, we get that $b_i = b_j \neq 0$ for all $1 \leq i, j \leq n$. In particular, $b_1\overline{x_1^2} = b_2\overline{x_2^2}$. Now, choose $a_1 = -\overline{x_1}, a_2 = \overline{x_2}, a_3 = \overline{x_3}$. It follows from Lemma 6.5 that $r(-\overline{x_1}) = r\overline{x_2} = r\overline{x_3}$. This yields that

$$-b_1\overline{x_1^2} = b_2\overline{x_2^2} = b_3\overline{x_3^2}.$$

Combining with $b_1\overline{x_1^2} = b_2\overline{x_2^2}$, we get that $2b_1\overline{x_1^2} = 0$. Since $\operatorname{char}(k) \neq 2$ and $b_1 \neq 0$, $\overline{x_1^2} = 0$. This contradicts $\overline{x_1^2} \neq 0$. Hence, $\operatorname{ca}(R) = \operatorname{soc}(R)$. $\square$

Theorem 6.7. *Let $(R, \mathfrak{m}, k)$ be an equicharacteristic Artinian Gorenstein local ring, where k is an algebraically closed field with $\operatorname{char}(k) \neq 2$. Then $\operatorname{ca}(R) = \mathfrak{m}$ if and only if $R \cong k[\![x]\!]/(x^2)$ or $R \cong k[\![x]\!]/(x^3)$.*

Proof. The backward direction follows from Example 3.7.

For the forward direction, assume $\operatorname{ca}(R) = \mathfrak{m}$. Then R is not regular; see 3.5. Combining with R being an Artinian Gorenstein local ring, it follows from Theorem 6.3 that $\mathfrak{m}^3 = 0$.

We claim $\operatorname{embdim}(R) = 1$. If not, assume $\operatorname{embdim}(R) \geq 2$. Combining with Proposition 6.6, we conclude that $\mathfrak{m} = \operatorname{soc}(R)$. Thus, $\dim_k \mathfrak{m} = 1$; see [7, Proposition 3.3.13]. This contradicts $\operatorname{embdim}(R) \geq 2$. Thus, $\operatorname{embdim}(R) = 1$.

By above, $\operatorname{embdim}(R) = 1$, $\mathfrak{m}^3 = 0$, and R is an equicharacteristic Gorenstein local ring, we conclude that either $R \cong k[\![x]\!]/(x^2)$ or $R \cong k[\![x]\!]/(x^3)$. $\square$

§7. One-dimensional case

The main result of this section is the second statement of Theorem 1.4 from the introduction.

7.1. Let $Q(R)$ be the total ring of fractions of R. Denote by $\overline{R}$ the integral closure of R inside $Q(R)$. The *conductor ideal* of R, denoted $\mathcal{C}(R)$, is the ideal $\operatorname{ann}_R(\overline{R}/R)$.

Note that the integral closure $\overline{R}$ is finitely generated as an R-module if and only if the conductor ideal $\mathcal{C}(R)$ contains a non-zero divisor.

There are only a few classes of rings whose cohomological annihilators are known. When R is a one-dimensional reduced complete Gorenstein local ring, this is described by Esentepe [19].

7.2. Let R be a one-dimensional Gorenstein local ring with reduced completion. Esentepe [19, Theorem 5.10] observed that

$$\mathrm{ca}(R) = \mathcal{C}(R).$$

Theorem 7.3. *Let $(R, \mathfrak{m})$ be a one-dimensional Gorenstein local ring. If $\mathrm{ca}(R) = \mathfrak{m}$, then R has minimal multiplicity and R has finite CM type.*

Proof. Since $\mathrm{ca}(R) = \mathfrak{m}$, we get $\mathrm{ca}(\widehat{R}) = \widehat{\mathfrak{m}}$ by Proposition 4.3. By assumption, $\widehat{R}$ is also a one-dimensional Gorenstein local ring. This implies that $\widehat{\mathfrak{m}}$ contains a non-zero divisor of $\widehat{R}$. Hence, $\mathrm{ca}(\widehat{R})$ contains a non-zero divisor of $\widehat{R}$. It follows from [24, Lemma 6.6] that $\widehat{R}$ is reduced. Then we conclude by 7.2 that

$$\mathcal{C}(R) = \mathrm{ca}(R) = \mathfrak{m}.$$

In particular, $\mathcal{C}(R)$ contains a non-zero divisor of R. This yields that the integral closure $\overline{R}$ is finitely generated as an R-module. It follows immediately from [11, Corollary 4.10] that $\mathcal{C}(R)$ is Ulrich. Hence, $\mathfrak{m}$ is Ulrich. This yields that R has minimal multiplicity. Combining with Theorem 5.5, we get that R has finite CM type. $\square$

Corollary 7.4. *Let $(R, \mathfrak{m}) = (S, \mathfrak{n})/(f)$, where S is a regular local ring of dimension 2. If $\mathrm{ca}(R) = \mathfrak{m}$, then $f \in \mathfrak{n}^2 \setminus \mathfrak{n}^3$.*

Proof. Since $\mathrm{ca}(R) \neq R$, R is not regular; see 3.5. Hence, $f \in \mathfrak{n}^2$. It follows from Theorem 7.3 that R has minimal multiplicity. This yields that $f \notin \mathfrak{n}^3$; see Example 5.4. This finishes the proof. $\square$

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