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Levi Problem Under the Negativity of the Canonical Bundle near the Boundary

In memory of J. J. Kohn

by

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Abstract

A theorem asserting the existence of proper holomorphic maps with connected fibers to an open subset of $\mathbb{C}^N$ from a locally pseudoconvex bounded domain in a complex manifold will be proved under the negativity of the canonical bundle on the boundary. Related results by Takayama on the holomorphic embeddability and holomorphic convexity of pseudoconvex manifolds will be extended under similar curvature conditions.

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§1. Introduction

This is a continuation of [Oh-4] where the following was proved, among other things.

Theorem 1.1. *Let M be a complex manifold and let Ω be a proper bounded domain in M with C^2-smooth pseudoconvex boundary $\partial\Omega$. Assume that M admits a Kähler metric and the canonical bundle K_M of M admits a fiber metric whose curvature form is negative on a neighborhood of $\partial\Omega$. Then there exists a holomorphic map with connected fibers from Ω to $\mathbb{C}^N$ for some $N \in \mathbb{N}$ which is proper onto the image.*

The main purpose of the present article is to strengthen it by removing the Kählerness assumption (see Section 3). For that, the proof of Theorem 1.1 given in [Oh-4] by an application of the L^2 vanishing theorem on complete Kähler manifolds

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will be replaced by an argument which is more involved but also seems to be basic (see Section 2).

More precisely, the proof is an application of the finite-dimensionality of L^2 $\bar{\partial}$-cohomology groups on M with coefficients in line bundles whose curvature form is positive at infinity. Recall that the idea of exploiting the finite-dimensionality for producing holomorphic sections originates in a celebrated paper [G] by Grauert. In short, it amounts to finding infinitely many linearly independent C^∞ sections $s_1, s_2, \ldots$ of the bundle in such a way that some nontrivial linear combination of $\bar{\partial}s_1, \bar{\partial}s_2, \ldots$, say $\sum_{k=1}^N c_k \bar{\partial} s_k$ ($c_k \in \mathbb{C}$), is equal to $\bar{\partial}u$ for some u which is more regular than $\sum_{k=1}^N c_k s_k$. This works if one can attach mutually different orders of singularities to s_k, for instance as in [G] where the holomorphic convexity of strongly pseudoconvex domains was proved. Although such a method does not directly work for the weakly pseudoconvex cases, the method of solving the $\bar{\partial}$-equation with L^2 estimates is available to produce a nontrivial holomorphic section of the form $\sum c_k s_k - u$ by appropriately estimating u. More precisely, instead of specifying singularities of s_k, one finds a solution u which has more zeros than $\sum_{k=1}^N c_k s_k$. For that, finite-dimensionality of the L^2 cohomology with respect to singular fiber metrics would be useful. However, this part of the analysis does not seem to have been explored much. For instance, the author does not know whether or not Nadel's vanishing theorem as in [Na] can be extended as a finiteness theorem with coefficients in the multiplier ideal sheaves of singular fiber metrics under an appropriate positivity assumption of the curvature current near infinity.

So, instead of analyzing the L^2 cohomology with respect to singular fiber metrics, we shall avoid the singularities by simply removing them from the manifold and consider the L^2 cohomology of the complement, which turns out to have a similar finite-dimensionality property because of the L^2 estimate on complete Hermitian manifolds. Such an argument is restricted to the cases where the singularities of the fiber metric are isolated. As a technique, it was first introduced in [D-Oh-3] to estimate the Bergman distances. It is useful for other purposes and applied also in [Oh-3, Oh-4, Oh-5, Oh-6], but will be repeated here for the reader's convenience.

Once one has infinitely many linearly independent holomorphic sections of a line bundle $L \to M$, one can find singular fiber metrics of L by taking the reciprocal of the sum of squares of the moduli of local trivializations of the sections. Very roughly speaking, this is the main trick to derive the conclusion of Theorem 1.1 from $K_M|_{\partial\Omega} < 0$. In fact, for the bundles L with $L|_{\partial\Omega} > 0$, the proof of $\dim H^{n,0}(\Omega, , L^m) = \infty$ for $m \gg 1$ will be given in detail here (see Theorems 2.4, 2.5 and 2.6). The rest is actually similar to the case $K_M < 0$. We shall also generalize the following theorems of Takayama.

Theorem 1.2 (Cf. [T-1]). *Weakly* 1*-complete manifolds with positive line bundles are embeddable into* $\mathbb{CP}^N$ *(*$N \gg 1$*).*

Theorem 1.3 (Cf. [T-2]). *Pseudoconvex manifolds with negative canonical bundles are holomorphically convex.*

Let M be a complex manifold. We shall say that M is a C^k pseudoconvex manifold if M is equipped with a C^k plurisubharmonic exhaustion function, say φ. Note, C^∞ (resp. C^0) pseudoconvex manifolds are also called weakly 1-complete (resp. pseudoconvex) manifolds. The sublevel sets $\{x;\ \varphi(x) < c\}$ will be denoted by M_c.

Theorems 1.2 and 1.3 are respectively a generalization of Kodaira's embedding theorem and that of Grauert's characterization of Stein manifolds. Our intention here is to draw similar conclusions by assuming the curvature conditions only on the complement of a compact subset of the manifold in question.

Theorem 1.2 will be generalized as follows.

Theorem 1.4. *Let* (M, φ) *be a connected and noncompact* C^2 *pseudoconvex manifold which admits a holomorphic Hermitian line bundle whose curvature form is positive on* $M \setminus M_c$*. Then there exists a holomorphic embedding of* $M \setminus M_c$ *into* $\mathbb{CP}^N$ *which extends to* M *meromorphically.*

Theorem 1.3 will be extended to the following.

Theorem 1.5. *A* C^2 *pseudoconvex manifold* (M, φ) *is holomorphically convex if the canonical bundle is negative outside a compact set.*

This extends Grauert's theorem asserting that strongly 1-convex manifolds are holomorphically convex.

The proofs will be done by combining the method of Takayama with an L^2 variant of the Andreotti–Grauert theory [A-G] on complete Hermitian manifolds whose special form needed here will be recalled in Section 3.

In Section 4 we shall extend Theorem 1.4 for the domains Ω as in Theorem 1.1. Whether or not Ω in Theorem 1.1 is holomorphically convex is still open.

§2. Preliminaries

The proof of the desired improvement of Theorem 1.1 will rely on the following.

Theorem 2.1 (Cf. [Oh-6, Theorems 0.3 and 4.1]). *Let* M *be a complex manifold, let* $\Omega \subsetneq M$ *be a relatively compact pseudoconvex domain with a* C^2*-smooth boundary and let* B *be a holomorphic line bundle over* M *with a fiber metric* h *whose* *curvature form is positive on a neighborhood of* $\partial\Omega$*. Then there exists a positive integer* m_0 *such that for all* $m \geq m_0$*,* $\dim H^{0,0}(\Omega, B^m) = \infty$ *and that, for any compact set* $K \subset \Omega$ *and for any positive number* R*, one can find a compact set* $\widehat{K} \subset \Omega$ *such that for any point* $x \in \Omega \setminus \widehat{K}$ *there exists an element* s *of* $H^{0,0}(\Omega, B^m)$ *satisfying*

$$\sup_K |s|_{h^m} < 1 \quad \text{and} \quad |s(x)|_{h^m} > R. \tag{2.1}$$

We shall give the proof of Theorem 2.1 in this section for the convenience of the reader, after recalling the basic L^2 estimates in a general setting.

Let (M, g) be a complete Hermitian manifold of dimension n and let (E, h) be a holomorphic Hermitian vector bundle over M.

Let $C^{p,q}(M, E)$ denote the space of E-valued C^∞ (p, q)-forms on M and let

$$C_0^{p,q}(M, E) = \{u \in C^{p,q}(M, E);\ \operatorname{supp} u \text{ is compact}\}.$$

Given a C^2 function $\varphi\colon M \to \mathbb{R}$, let $L^{p,q}_{(2),\varphi}(M, E)$ $(= L^{p,q}_{(2),g,\varphi}(M, E))$ be the space of E-valued square integrable measurable (p, q)-forms on M with respect to g and $he^{-\varphi}$. The definition of $L^{p,q}_{(2),\varphi}(M, E)$ will be naturally extended for continuous metrics and continuous weights.

Recall that $L^{p,q}_{(2),\varphi}(M, E)$ is identified with the completion of $C_0^{p,q}(M, E)$ with respect to the L^2 norm

$$\|u\|_\varphi := \left(\int_M e^{-\varphi} |u|^2_{g,h}\, dV_g \right)^{1/2}.$$

Here we put $dV_g := \frac{1}{n!}\omega^n$ for the fundamental form $\omega = \omega_g$ of g. More explicitly, when E is given by a system of transition functions $e_{\alpha\beta}$ with respect to a trivializing covering $\{U_\alpha\}$ of M and h is given as a system of C^∞ positive definite Hermitian matrix-valued functions $h\alpha$ on U_α satisfying $h_\alpha = {}^t e_{\beta\alpha} h_\beta \overline{e_{\beta\alpha}}$ on $U_\alpha \cap U_\beta$, $|u|^2_{g,h}\, dVg$ is defined by ${}^t u_\alpha h_\alpha \wedge \overline{* u_\alpha}$, where $u = \{u_\alpha\}$ with $u_\alpha = e_{\alpha\beta} u_\beta$ on $U_\alpha \cap U_\beta$ and $*$ stands for the Hodge star operator with respect to g. We put $\bar{*} u = \overline{* u}$ so that ${}^t u_\alpha h_\alpha \wedge \bar{*} u_\alpha = {}^t u_\alpha h_\alpha \wedge \bar{*} u_\alpha$.

Let us denote by $\bar\partial$ (resp. ∂) the complex exterior derivative of type (0,1) (resp. (1,0)). Then the correspondence $u_\alpha \to \bar\partial u_\alpha$ defines a linear differential operator $\bar\partial\colon C^{p,q}(M, E) \to C^{p,q+1}(M, E)$. The Chern connection D_h is defined to be $\bar\partial + \partial_h$, where ∂_h is defined by $u_\alpha \to h_\alpha^{-1} \partial(h_\alpha u_\alpha)$. Since $\bar\partial^2 = \partial_h^2 = \bar\partial\partial + \partial\bar\partial = 0$, there exists an $(E^* \otimes E)$-valued (1,1)-form Θ_h such that $D_h^2 u = \Theta_h \wedge u$ holds for all $u \in C^{p,q}(M, E)$, where Θ_h is called the curvature form of h. Note that $\Theta_h e^{-\varphi} = \Theta_h + \mathrm{Id}_E \otimes \bar\partial\partial\varphi$.

The form Θ_h is said to be positive (resp. semipositive) at $x \in M$ if $\Theta_h = \sum_{j,k=1}^n \Theta_{j\bar k}\, dz_j \wedge d\bar z_k$ in terms of a local coordinate $(z_1, \ldots, z_n)$ around x and

$(\Theta_{j\bar{k}}(x))_{j,k} = (\Theta^{\mu}_{\nu j\bar{k}}(x))_{j,k,\mu,\nu}$ is positive (semipositive) in the sense (of Nakano) that the quadratic form

$$\sum \Big(\sum_{\mu} h_{\mu\bar{\kappa}} \Theta^{\mu}_{\nu j\bar{k}}\Big)(x)\xi^{\nu j}\overline{\xi^{\kappa k}}$$

is positive definite (resp. positive semidefinite). Thus, $\Theta > 0$ (resp. ≥ 0) for an $(E^* \otimes E)$-valued (1,1)-form Θ will mean the positivity (resp. semipositivity) in this sense.

Whenever there is no fear of confusion, as well as the Levi form $\partial\bar{\partial}\varphi$ of φ, Θ_h will be identified with a Hermitian form along the fibers of $E \otimes TM$, where TM stands for the holomorphic tangent bundle of M. By an abuse of notation, $\bar{\partial}$ (resp. $\partial_{he^{-\varphi}}$) will also stand for the maximal closed extension of $\bar{\partial}|_{C_0^{p,q}(M,E)}$ (resp. $\partial_{he^{-\varphi}}|_{C_0^{p,q}(M,E)}$) as a closed operator from $L^{p,q}_{(2),\varphi}(M,E)$ to $L^{p,q+1}_{(2),\varphi}(M,E)$ (resp. $L^{p+1,q}_{(2),\varphi}(M,E)$). The adjoint of $\bar{\partial}$ (resp. $\partial_{he^{-\varphi}}$) will be denoted by $\bar{\partial}^* = \bar{\partial}^*_{g,he^{-\varphi}}$ (resp. $\partial^*_{he^{-\varphi}}$).

We recall that $\partial^*_{he^{-\varphi}} = -\bar{*}\bar{\partial}\bar{*}$ holds as a differential operator acting on $C^{p,q}(M,E)$, so that $\partial^*_{he^{-\varphi}}$ will be also denoted by ∂^*. By $\operatorname{Dom}\bar{\partial}$ (resp. $\operatorname{Dom}\bar{\partial}^*$) we shall denote the domain of $\bar{\partial}$ (resp. $\bar{\partial}^*$).

We put

$$H^{p,q}_{(2),g,\varphi}(M,E) = \frac{\operatorname{Ker}(\bar{\partial}\colon L^{p,q}_{(2),\varphi}(M,E) \to L^{p,q+1}_{(2),\varphi}(M,E))}{\operatorname{Im}(\bar{\partial}\colon L^{p,q-1}_{(2),\varphi}(M,E) \to L^{p,q}_{(2),\varphi}(M,E))}$$

and

$$\mathscr{H}^{p,q}_{\varphi}(M,E) = \operatorname{Ker}\bar{\partial} \cap \operatorname{Ker}\bar{\partial}^* \cap L^{p,q}_{(2),\varphi}(M,E).$$

Let $\Lambda = \Lambda_g$ denote the adjoint of the exterior multiplication by ω. Then the Nakano formula

$$\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} - \partial_h\partial^* - \partial^*\partial_h = \sqrt{-1}(\Theta_h\Lambda - \Lambda\Theta_h) \tag{2.2}$$

holds if $d\omega = 0$. Here, Θ_h also stands for the exterior multiplication by Θ_h from the left-hand side. Hence, for any open set $\Omega \subset M$ such that $d\omega|_\Omega = 0$ and for any $u \in C_0^{n,q}(\Omega, E)$, one has

$$\|\bar{\partial}u\|^2_{\varphi} + \|\bar{\partial}^*u\|^2_{\varphi} \geq (\sqrt{-1}(\Theta_h + \mathrm{Id}_E \otimes \partial\bar{\partial}\varphi)\Lambda u, u)_{\varphi}. \tag{2.3}$$

Here, $(u,w)_\varphi$ stands for the inner product of u and v with respect to $(g, he^{-\varphi})$. The following direct consequence of (2.3) is important for our purpose.

Proposition 2.1. *Let M, E, g, h and φ be as above. Assume that there exists a compact set $K \subset M$ such that $d\omega_g = 0$ holds on $M \setminus K$. Then there exist a compact* *set K' containing K and a constant C such that K' and C do not depend on the choice of φ and*

$$\sqrt{-1}((\Theta_h + \mathrm{Id}_E \otimes \partial\bar{\partial}\varphi)\Lambda u, u)_\varphi \leq C\left(\|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2 + \int_{K'} e^{-\varphi}|u|^2_{g,h}\, dV_g\right)$$

holds for any $u \in C_0^{n,q}(M, E)$ $(q \geq 1)$.

From Proposition 2.1 one infers the following.

Proposition 2.2. *Let (M, E, g, h, φ, K) and (K', C) be as above. Assume moreover that one can find a constant $C_0 > 0$ such that $C_0(\Theta_h + \mathrm{Id}_E \otimes \partial\bar{\partial}\varphi) - \mathrm{Id}_E \otimes g \geq 0$ holds on $M \setminus K$. Then there exist a constant C' depending only on C, K' and C_0 such that*

$$\|u\|^2_\varphi \leq C'\left(\|\bar{\partial} u\|^2_\varphi + \|\bar{\partial}^* u\|^2_\varphi + \int_{K'} e^{-\varphi}|u|^2_{g,h}\, dV_g\right)$$

holds for any $u \in C_0^{n,q}(M, E)$ $(q \geq 1)$.

By a theorem of Gaffney, the estimate in Proposition 2.2 implies the following.

Proposition 2.3. *In the situation of Proposition 2.2,*

$$\|u\|^2_\varphi \leq C'\left(\|\bar{\partial} u\|^2_\varphi + \|\bar{\partial}^* u\|^2 + \int_{K'} |u|^2_{g,h}\, dV_g\right)$$

holds for all $u \in L^{n,q}_{(2),\varphi}(M, E) \cap \mathrm{Dom}\,\bar{\partial} \cap \mathrm{Dom}\,\bar{\partial}^*$ $(q \geq 1)$.

Recall that the following was proved in [H] by a basic argument of functional analysis.

Theorem 2.2 ([H, Theorems 1.1.2 and 1.1.3]). *Let H_1 and H_2 be Hilbert spaces and let $T\colon H_1 \to H_2$ be a densely defined closed operator. Let H_3 be another Hilbert space and let $S\colon H_2 \to H_3$ be a densely defined closed operator such that $ST = 0$. Then a necessary and sufficient condition for the ranges R_T, R_S of T, S to both be closed is that there exists a constant C such that*

$$\|g\|_{H_2} \leq C(\|T^* g\|_{H_1} + \|Sg\|_{H_3});\ g \in D_{T^*} \cap D_S, g \perp (N_{T^*} \cap N_S), \tag{2.4}$$

where D_{T^*} and D_S denote the domains of T^* and S, respectively, and $N_{T^*} = \mathrm{Ker}\, T^*$ and $N_S = \mathrm{Ker}\, S$. Moreover, if one can select a strongly convergent subsequence from every sequence $g_k \in D_{T^*} \cap D_S$ with $\|g_k\|_{H_2}$ bounded and $T^* g_k \to 0$ in H_1, $Sg_k \to 0$ in H_3, then $N_S/R_T \cong N_{T^*} \cap N_S$ holds and $N_{T^*} \cap N_S$ is finite-dimensional.

Hence we obtain the following theorem.

Theorem 2.3. *In the situation of Proposition 2.2,* $\dim H^{n,q}_{(2),\varphi}(M,E) < \infty$ *and* $\mathscr{H}^{n,q}_{\varphi} \cong H^{n,q}_{(2),\varphi}(M,E)$ *hold for all* $q \geq 1$.

It is an easy exercise to deduce from Theorem 2.3 that every strongly pseudoconvex manifold is holomorphically convex (cf. [G] or [H]). We are going to extend this application to the domains with weaker pseudoconvexity.

For any Hermitian metric g on M, a C^∞ function $\psi\colon M \to \mathbb{R}$ is called g-psh (g-plurisubharmonic) if $g + \partial\bar{\partial} g \geq 0$ holds everywhere.

Then Theorem 2.3 can be restated as follows.

Theorem 2.4. *Let* (M,g) *be an* n*-dimensional complete Hermitian manifold and let* (E,h) *be a Hermitian holomorphic vector bundle over* M*. Assume that there exists a compact set* $K \subset M$ *such that* $\Theta_h - \mathrm{Id}_E \otimes g \geq 0$ *and* $d\omega_g = 0$ *hold on* $M \backslash K$*. Then, for any* g*-psh function* φ *on* M *and for any* $\varepsilon \in (0,1)$*,* $\dim H^{n,q}_{(2),\varepsilon\varphi}(M,E) < \infty$ *and* $\mathscr{H}^{n,q}_{\varepsilon\varphi}(M,E) \cong H^{n,q}_{(2),\varepsilon\varphi}(M,E)$ *for* $q \geq 1$.

By applying Theorem 2.4, we shall first show the following.

Theorem 2.5. *Let* (M,E,g,h) *be as in Theorem 1.4 and let* x_μ $(\mu = 1,2,\ldots)$ *be a sequence of points in* M *without accumulation points. Assume that there exists a* $(1-\varepsilon)g$*-psh function* φ *on* $M \setminus \{x_\mu\}_{\mu=1}^\infty$ *for some* $\varepsilon \in (0,1)$ *such that* $e^{-\varphi}$ *is not integrable on any neighborhood of* x_μ *for any* μ*. Then* $\dim H^{n,0}(M,E) = \infty$.

Proof. We put $M' = M \setminus \{x_\mu\}_{\mu=1}^\infty$ and let ψ be a bounded $\frac{\varepsilon}{2}g$-psh C^∞ function on M' such that $g' := g + \bar{\partial}\partial\psi$ is a complete metric on M'. Take $s_\mu \in C^{n,0}(M,E)$ $(\mu \in \mathbb{N})$ in such a way that $|s_\mu(x_\nu)|_{g,h} = \delta_{\mu\nu}$ and $\int_{M'} |\bar{\partial} s_\mu|^2_{g,h}\, dV_g < \infty$. Since $\int_{M'} e^{-\varphi-\psi}|\bar{\partial} s_\mu|^2_{g',h}\, dV_{g'} \leq \int_{M'} e^{-\varphi-\psi}|\bar{\partial} s_\mu|^2_{g,h}\, dV_g$ and $H^{n,1}_{(2),g',\varphi}(M',E) < \infty$ by Theorem 2.4, one can find a nontrivial linear combination of $\bar{\partial} s_\mu$, say $v = \sum c_\mu \bar{\partial} s_\mu$, which is in the range of $L^{n,0}_{(2),\varphi}(M,E) \xrightarrow{\bar{\partial}} L^{n,1}_{(2),g',\varphi}(M',E)$. Then take $u \in L^{n,0}_{(2),\varphi(M',E)}$ satisfying $\bar{\partial} u = v$ and put $s = \sum c_\mu s_\mu - u$. Clearly, s extends to a nonzero element of $H^{n,0}(M,E)$ which is zero at x_μ except for finitely many μ. Hence, one can find mutually disjoint finite subsets $\Sigma_\nu \neq \emptyset$ $(\nu = 1,2,\ldots)$ of $\mathbb{N}$ and nonzero holomorphic sections s_μ of E such that $s_\nu(x_\mu) = 0$ if $\mu \notin \Sigma_\nu$, so that $\dim H^{n,0}(M,E) = \infty$. ☐

This observation will be basic for the proofs of Theorems 1.4 and 1.5. Let us introduce some terminology related to g-plurisubharmonicity.

Definition 2.1. A Hermitian manifold (M,g) is said to be g-convex if M is equipped with a g-psh exhaustion function.

Definition 2.2. An open set Ω in a complex manifold M is said to be locally pseudoconvex if, for every point $x \in M$, one can find a neighborhood $U \ni x$ such that $U \cap \Omega$ is Stein.

We note that “pseudoconvex” in Theorems 1.1 and 2.1 means “locally pseudoconvex” in the above sense. Since Steinness of domains in $\mathbb{C}^n$ can be characterized by local pseudoconvexity, one can see immediately the following.

Proposition 2.4. *Let (M, g) be a Hermitian manifold and let $\Omega \subset M$ be a relatively compact domain. If Ω is g-convex, then Ω is locally pseudoconvex.*

It is easy to see that a relatively compact locally pseudoconvex domain $\Omega \subset M$ is g-convex with respect to some metric g on M if every connected component of $\partial\Omega$ is either a real hypersurface of class C^2 or the support of an effective divisor on M.

Definition 2.3. A g-psh function $\varphi \colon M \to \mathbb{R}$ is said to be hyper-g-psh if one can find a positive number C such that $\partial\bar{\partial}\varphi + g \geq C\partial\varphi\bar{\partial}\varphi$ holds everywhere. A Hermitian manifold (M, g) is said to be hyperconvex if M admits an exhaustion function φ which is hyper-g-psh.

Proposition 2.5. *Let (M, g) be a Hermitian manifold and let Ω be a relatively compact locally pseudoconvex domain. Assume that each connected component of $\partial\Omega$ is either a C^2 real hypersurface of M or the support of an effective divisor on M whose associated line bundle admits a fiber metric whose curvature form is semipositive on the Zariski tangent spaces. Then (Ω, g) is hyperconvex.*

Proof. Under the above condition, Ω admits a complete Hermitian metric of the form $g + \partial\bar{\partial}\Psi$ for some hyper-g-psh exhaustion function $\Psi \colon \Omega \to \mathbb{R}$. For the detail, see [Oh-6, Propositions 4.1 and 4.2]. □

In what follows, we shall refer to a domain Ω satisfying the assumption of Proposition 2.2 as a *model domain* for short, by an abuse of language.

Remark 2.1. Describing geometric properties by using the potentials as above has become common in complex geometry. See [C-L] for instance.

Definition 2.4. Given a Hermitian manifold (M, g), a sequence $x_\mu \in M$ ($\mu = 1, 2, \ldots$) is said to be *sparsely discrete* with respect to g if one can find a C^∞ function $\Phi \colon M \setminus \{x_\mu;\ \mu \in \mathbb{N}\} \to (-\infty, 0)$ such that $e^{-\Phi}$ is not integrable on any neighborhood of any point x_μ and $\partial\bar{\partial}\Phi + Ag \geq 0$ holds everywhere on $M \setminus \{x_\mu\}_{\mu=1}^{\infty}$ for some $A > 0$. The term Φ is called a g-polarization function associated to the sequence x_μ.

Lemma 2.1. *Let (M, g, φ) be a g-convex Hermitian manifold with a hyper-g-convex exhaustion function φ. Assume that for any point $x \in M$ there exists a negative g-psh function Φ_x on $M \setminus \{x\}$ such that, for any neighborhood $U \ni x$, $e^{-\Phi_x}$ is not integrable on U and $|d\Phi_x|_g$ is bounded on $M \setminus U$. Then, for any $\varepsilon \in (0,1)$ and for any sequence $\{x_\mu\}_{\mu=1}^\infty \subset M$ without accumulation points, there exist a subsequence $\{x_{\mu_k}\}_{k=1}^\infty$ and a complete metric g^* on M of the form $g* = g + (1-\varepsilon)\partial\bar\partial\varphi$ such that $\{x_{\mu_k}\}_{k=1}^\infty$ is sparsely discrete with respect to g^*.*

Proof. Let g^* be as above and let χ be a $C\infty$ function $\chi\colon \mathbb{R} \to [0,1]$ satisfying $\chi|_{(-\infty,\frac{1}{2})} \equiv 1$ and $\chi|_{[1,\infty)} \equiv 0$. Then, since φ is assumed to be of hyperconvex type, one can find a constant C' such that, for any $c \in \mathbb{R}$ and $R > 1$, there exists $c' > c$ such that, for any $x_0 \in M$ with $\varphi(x_0) > c'$ the function $\chi_{x_0,R} = \chi(\frac{|\varphi-\varphi(x_0)|}{R})$ satisfies $\operatorname{supp}\chi_{x_0,R} \subset M \setminus M_c$, $|d\chi_{x_0,R}|_{g^*} < \frac{C'}{R}$ and $\partial\bar\partial\chi_{X_0,R}|_{g^*} < \frac{C'}{R}$. Hence, for any $\{x_\mu\}$ one can find a subsequence $\{x_{\mu_k}\}$ and $R_k > 0$ such that the supports of $\chi_{x_{\mu_k},R_k}$ are mutually disjoint and $\sum \chi_{x_{\mu_k},R_k}\Phi_{x_{\mu_k}}$ is $(1+\varepsilon)g^*$-plurisubharmonic. □

Combining Proposition 2.5 with Lemma 2.1 and its proof, we obtain the following.

Proposition 2.6. *Let (M, g) be a Hermitian manifold and let $\Omega \subset M$ be a model domain. Then there exist a C^∞ g-psh exhaustion function $\Psi\colon \Omega \to \mathbb{R}$ such that $g^* := g + \partial\bar\partial\Psi$ is a complete metric on Ω and a C^∞ exhaustion function φ on Ω which is hyper-g^*-psh, such that any sequence of points x_μ in Ω accumulating to no points in the interior admits a sparsely discrete subsequence x_{μ_k} with respect to $\hat g := g^* + \partial\bar\partial\varphi$. Moreover, for any Hermitian holomorphic vector bundle (E, h) over M, the sequence x_{μ_k} and the associated polarization function Φ can be chosen in such a way that one can find C^∞ sections s_k of $K_M \otimes E$ on Ω with mutually disjoint supports such that $\lim_{k\to\infty} |s_k(x_{\mu_k})|_{g,h} = \infty$ and*

$$\sum_{k=1}^\infty \int_M e^{-\phi}|\bar\partial s_k|^2_{\hat g,h}\, dV_{\hat g} < \infty.$$

Note that, once one has s_k as above, one can also find sequences $s_{k,\nu}$, $\nu = 1, 2, \ldots$ of C^∞ sections of $K_M \otimes E$ on Ω with $\overline{\{s_{j,\nu} = 0\}} \cap \overline{\{s_{k,\nu} = 0\}} = \emptyset$ if $j \neq k$ and

$$\sum_{k=1}^\infty \int_M e^{-\phi}|\bar\partial s_\nu k|^2_{\hat g,h}\, dV_{\hat g} < \infty,$$

such that the conditions

$$\lim_{k-to\infty} \frac{|s_{k,\nu+1}(x_{\nu_k})|}{|s_{k,\nu}(x_{\nu_k})|} = \infty$$

are fulfilled for all ν.

Hence, similarly to the proof of Theorem 2.5, one has the following.

Theorem 2.6. *Let (M, g) be a connected Hermitian manifold of dimension n and let $\Omega \subset M$ be a model domain. Assume that there exists a Hermitian holomorphic line bundle (B, a) over M whose curvature form is positive on a neighborhood of $\partial\Omega$. Then, for any Hermitian holomorphic vector bundle (E, h) over M, there exists a positive integer m_0 such that, for any compact subset K of Ω and $C > 0$, the set $\{x \in \Omega;\ |s(x)|_{g,ha^m} \le C \sup_K |s|_{g,ha^m}$ for all $s \in H^{n,0}(\Omega, E \otimes B^m)\}$ is relatively compact in Ω for all $m \ge m_0$.*

Clearly, Theorem 1.1 is a corollary of Theorem 2.6.

Remark 2.2. It is clear that Theorem 2.6 is only a slight improvement of Theorem 1.1. However, it seems that the proof has acquired more flexibility for applications. For instance, once one has a C^∞ function ρ on Ω such that $g + \partial\bar\partial\rho$ is complete on Ω, it is an easy exercise to see that one can use $H^{n,0}_{(2),\psi+\rho}(\Omega, E \otimes B^m)$ instead of $H^{n,0}(\Omega, E \otimes B^m)$, where ψ is any hyper-$(g + \partial\bar\partial\rho)$-psh exhaustion function on Ω.

§3. Proof of the main result

Now we are going to prove the following.

Theorem 3.1. *Let M be a complex manifold and let $\Omega \subsetneq M$ be a model domain. Assume that the canonical bundle of M admits a fiber metric whose curvature form is negative on a neighborhood of $\partial\Omega$. Then there exists a holomorphic map with connected fibers from Ω to $\mathbb{C}^N$ for some $N \in \mathbb{N}$ which is proper onto its image.*

Proof. Similarly to the proof of Theorem 2.2, by the negativity of K_M on $\partial\Omega$, there exist holomorphic sections $s_0, \dots, s_N$ of K_M^{-m} over Ω for sufficiently large m, such that the set A of common zeros of k $(0 \le k \le N)$ has compact connected components. Then $\varpi := (s_0 : s_1 : \cdots : s_N)$ defines a holomorphic map from $\Omega \setminus A$ to $\mathbb{CP}^N$. Since K_M is negative on $\partial\Omega$, one can find a compact set $K \subset \Omega$ such that $\varpi|_{\Omega\setminus(A\cup K)}$ has discrete fibers. Therefore, by virtue of Hironaka's desingularization theorem, one can find a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_M$ with $B := \operatorname{supp}(\mathcal{O}_M/\mathcal{I}) \subset K$, a C^∞ function ψ on Ω vanishing along the critical set of $\varpi|_{\Omega\setminus B}$ and a compactly supported C^∞ function χ on Ω such that, after a monoidal transformation $f\colon \widehat\Omega \to \Omega$, one has a Kähler metric on $\widehat\Omega \setminus f^{-1}(A)$ of the form $f^*(-\Theta_\kappa + \partial\bar\partial(\chi\psi)) + \Theta_b + Cf^*\varpi^* g_{\mathrm{FS}}$ for sufficiently large constant C, where κ and b denote a fiber metric of K_M and that of the invertible sheaf $f^*\mathcal{I}$, respectively, and g_{FS} denotes the Fubini–Study metric. Hence $\Omega \setminus A$ has a complete Kähler metric.

On the other hand, since $H^{0,0}(\Omega, K_M^{-m})$ is infinite-dimensional for sufficiently large m, we may assume that, for any finitely many points $(y_1, \dots, y_\ell)$ given in advance, $y_j \in A$ for all j and $\exp(-\frac{\psi}{m})$ is not integrable on any neighborhood of y_j. Under this condition, let A_j be the connected component of A containing y_j and set $\mathcal{A} = \{A_j;\ 1 \leq j \leq \ell\}$. Then, by applying the L^2 vanishing theorem in [Dm] or [Oh-2] (see also [Oh-1]) to the bundle K_M^{-1} on $\Omega \setminus A$ with respect to the fiber metric $\kappa^{-1} \exp(-\frac{\psi}{m})$, and taking the negligibility of analytic sets with respect to L^2 holomorphic functions into account, one has the surjectivity of the restriction map $H^{0,0}(\Omega) \to \mathbb{C}^{\mathcal{A}}$. From this, the conclusion of Theorem 1.1 is straightforward. □

Remark 3.1. Although the conclusion of Theorem 3.1 is far from the holomorphic convexity of Ω, which has been anticipated for a long time (cf. [N]), it seems that one may still expect it here because of the regularity of $\partial\Omega$. As for related affirmative results, see [D-Oh-1, D-Oh-2]. Furthermore, in view of Theorems 1.1 and 3.1, it seems plausible to conjecture that every model domain in Fano manifolds (compact complex manifolds with negative anticanonical bundle) are holomorphically convex. Note that not all locally pseudoconvex domains in Fano manifolds are holomorphically convex, since the one point blow-up of $\mathbb{CP}^2$ is Fano.

§4. Generalization of Takayama's theorems

For the proofs of Theorems 1.4 and 1.5, the following is crucial.

Theorem 4.1. *Let (M, g) be a complete Hermitian manifold of dimension n, let (E, h) be a Hermitian holomorphic vector bundle over M and let φ be a C^2 plurisubharmonic function on M. Assume that $M_0 := \{x \in M;\ \varphi(x) < 0\}$ is relatively compact and $\Theta_h - \mathrm{Id}_E \otimes g \geq 0$ holds on M_0. Then for any $q \geq 1$, $\dim H^{n,q}_{(2),\varphi}(M, E) < \infty$ and the restriction homomorphisms*

$$H^{n,q}_{(2),\mu\varphi}(M, E) \to H^{n,q}_{(2),\varphi}(M_0, E)$$

are bijective for sufficiently large μ. Moreover, the restriction homomorphism

$$\varinjlim_{\mu} H^{n,0}_{(2),\mu\varphi}(M, E) \left(= \bigcup_{\mu=1}^{\infty} H^{n,0}_{(2),\mu\varphi} \right) \to H^{n,0}(M_0, E)$$

has a dense image.

For the proof of Theorem 4.1, the reader is referred to [H, Proposition 3.4.5] or [Oh-3]. We note that some assumptions in Theorem 4.1 are slightly weaker than those in [H] and [Oh-3] but the proof stays essentially the same.

The point is that, in the above situation, the $\bar{\partial}$-equation $\bar{\partial}u = v$ has a solution whenever $\bar{\partial}v = 0$ and $v|_{M_0} = 0$ and that every E-valued holomorphic n-form on M_0 can be approximated by those on M. Therefore, if Ω is a sublevel set of some C^2 plurisubharmonic function in Theorem 4.1, there remains no difficulty in strengthening the conclusion of Theorem 4.1 to the assertion that Ω is holomorphically convex. The rest of the detail of the proof of Theorem 1.5 may well be left to the reader.

Thus, Theorem 1.5 is essentially a small supplement to Takayama's theorem for the negative canonical bundle case. The situation of Theorem 1.4 is similar.

Proof of Theorem 1.4. By the approximation theorem contained in Theorem 4.1 it suffices to show that, one can find $m_1, m_2 \in \mathbb{N}$ depending only on the dimension n such that, for any $c' > c$, one can find a proper modification $f\colon \widehat{M} \to M$ along a set lying in M_c in such a way that $K_M \otimes B^{m_1}|_{M_{c'} \setminus M_c}$ is positive and $H^{n,0}(f^{-1}(M_{c'}), (f^*(K_M \otimes B^{m_1}))^{m_2})$ embeds $M_{c'} \setminus M_c$ into some $\mathbb{CP}^N$. On the other hand, similarly to the proof of Theorem 3.1, one can find a complex manifold $\hat{}\ M$, a proper holomorphic map $f\colon \widehat{M} \to M$ and a compactly supported divisor D on $\widehat{M}$ such that$f|_{M \setminus f^{-1}(M_c)}$ is a biholomorphic map onto its image and the bundle $f^*B \otimes [D]$ is positive. Hence the required assertion follows from [T-1, Theorem 1.2]. □

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