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Spectral properties of symmetrized AMV operators

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Abstract. The symmetrized Asymptotic Mean Value Laplacian $\widetilde{\Delta}$, obtained as limit of approximating operators $\widetilde{\Delta}_r$, is an extension of the classical Euclidean Laplace operator to the realm of metric measure spaces. We show that, as $r \downarrow 0$, the operators $\widetilde{\Delta}_r$ eventually admit isolated eigenvalues defined via min-max procedure on any compact uniformly locally doubling metric measure space. Then we prove L^2 and spectral convergence of $\widetilde{\Delta}_r$ to the Laplace–Beltrami operator of a compact Riemannian manifold, imposing Neumann conditions when the manifold has a non-empty boundary.

1. Introduction

In the past thirty years, much research has been carried out to extend the classical Euclidean Laplace operator to metric measure spaces: see e.g., [4, 8, 9, 13]. This paper deals with such an extension, namely the symmetrized Asymptotic Mean Value (AMV) Laplacian, proposed in [15], see also [1, 2, 12, 14]. The symmetrized AMV Laplacian is set as

$$\widetilde{\Delta} := \lim_{r \downarrow 0} \widetilde{\Delta}_r, \tag{1}$$

where for μ-a.e. $x \in X$,

$$\widetilde{\Delta}_r f(x) := \frac{1}{2r^2} \fint_{B_r(x)} \Big(1 + \frac{V(x,r)}{V(y,r)}\Big)(f(y) - f(x)) \,\mathrm{d}\mu(y).$$

Here f is a locally integrable function defined on a metric measure space (X, d, μ). Throughout the paper, $B_r(z)$ denotes the metric open ball centered at $z \in X$ with radius $r > 0$, the notation $V(z,r)$ stands for $\mu(B_r(z))$, and $\fint_{B_r(z)}$ is shorthand for $V(z,r)^{-1} \int_{B_r(z)}$.

Part of the study on the symmetrized AMV Laplacian consists in finding a relevant meaning to the limit in (1). If this is intended in the L^2 sense, then the associated

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spectral convergence can be investigated. This is the goal of the present paper. For any $k \in \mathbb{N}$, set

$$\tilde{\lambda}_{k,r} := \inf_{V \in \mathcal{G}_{k+1}(L^2(X,\mu))} \sup_{f \in V} \frac{\widetilde{E}_r(f)}{\|f\|_2},$$

where $\mathcal{G}_{k+1}(L^2(X,\mu))$ is the $(k+1)$-th Grassmannian of $L^2(X,\mu)$, and $\widetilde{E}_r(f)$ is the energy functional naturally associated with $\widetilde{\Delta}_r$ (Definition 3.10). These form a non-decreasing sequence of non-negative numbers. Our first main result states that these numbers eventually correspond to isolated eigenvalues of $-\widetilde{\Delta}_r$ when (X, d, μ) is compact and uniformly locally doubling (Definition 2.8).

Theorem 1. *Let (X, d, μ) be a compact uniformly locally doubling metric measure space. For any integer $k \geq 2$, there exists $r_k > 0$ such that for any $r \in (0, r_k)$, the operator $-\widetilde{\Delta}_r$ admits $k+1$ eigenvalues*

$$0 = \lambda_0(-\widetilde{\Delta}_r) < \lambda_1(-\widetilde{\Delta}_r) \leq \cdots \leq \lambda_k(-\widetilde{\Delta}_r)$$

such that $\lambda_i(-\widetilde{\Delta}_r) = \tilde{\lambda}_{i,r}$ for any $i \in \{0, \ldots, k\}$.

Our second main result deals with a smooth manifold M endowed with a smooth Riemannian metric g. We write Δ_g for the (negative) Laplace–Beltrami operator of (M, g). We let $m \geq 2$ be the dimension of M, and we set

$$C_m := \frac{1}{2} \fint_{\mathbb{B}_1^m(0)} \xi_1^2 \, \mathrm{d}\xi = \frac{1}{2(m+2)}, \tag{2}$$

where $\mathbb{B}_1^m(0)$ is the unit Euclidean ball of $\mathbb{R}^m$. In this context, it follows from the equality between symmetrized and non-symmetrized AMV Laplacian and a simple calculation in normal coordinates that

$$\widetilde{\Delta}_r f(x) \xrightarrow{r \downarrow 0} C_m \Delta_g f(x) \tag{3}$$

for any $f \in \mathcal{C}^2(M)$ and any interior point $x \in M$, see [14, 15] – the convergence is even locally uniform in the interior of M, see [1]. We refer to [1, 2, 14, 15] for related pointwise results in various settings like Carnot groups or Alexandrov spaces.

In this paper, we are interested in the L^2 version of (3) with a particular interest in the case where M admits a non-empty boundary $\partial M \neq 0$. In this case, we write $\partial_\nu f \in \mathcal{C}^\infty(\partial M)$ for the normal derivative of a smooth function $f : M \to \mathbb{R}$, and we define

$$\mathcal{C}_\nu^\infty(M) := \{f \in \mathcal{C}^\infty(M) : \partial_\nu f = 0\}. \tag{4}$$

We see (M, g) as a metric measure space $(M, \mathsf{d}_g, \mathrm{vol}_g)$ where d_g and vol_g are the Riemannian distance and volume measure on M associated with g. Then our statement reads as follows.

Theorem 2. *Let (M^m, g) be a compact, connected, smooth Riemannian manifold with a non-empty (resp. empty) boundary ∂M. Then for any $f \in \mathcal{C}^\infty_\nu(M)$ (resp. $\mathcal{C}^\infty(M)$), as $r \downarrow 0$,*

$$\widetilde{\Delta}_r f \xrightarrow{L^2} C_m \Delta_g f.$$

We point out that the boundaryless version of this result is rather easy to obtain, while a non-empty boundary is quite tricky to handle. The Neumann condition in the latter case is crucial to ensure convergence: indeed, the sequence $\widetilde{\Delta}_r f$ may blow-up if this is not imposed.

After the previous L^2-convergence result, we address the question of spectral convergence, that is to say, the convergence of the associated eigenvalues and eigenfunctions. In this regard we show that, for any $k \in \mathbb{N}$, the function $r \mapsto \tilde{\lambda}_{k,r}$ is bounded in a neighborhood of 0, as proved in the course of Theorem 1. This ensures that the k-th lowest eigenvalue of the operator $-\widetilde{\Delta}_r$, which we denote $\lambda_k(-\widetilde{\Delta}_r)$, exists for small enough r, and that it coincides with $\tilde{\lambda}_{k,r}$. Let $f_{k,r}$ be an L^2-normalized eigenfunction of $-\widetilde{\Delta}_r$ associated with $\lambda_k(-\widetilde{\Delta}_r)$. Recall that if $\partial M = \emptyset$ (resp. $\partial M \neq \emptyset$), a Laplace (resp. Neumann) eigenvalue of (M, g) is a number $\mu \geq 0$ for which there exists an associated eigenfuction $f \in \mathcal{C}^\infty(M)$ (resp. $\mathcal{C}^\infty_\nu(M)$) of $-\Delta_g$, i.e., $-\mu f = \Delta_g f$.

Theorem 3. *Let (M^m, g) be a compact, connected, smooth Riemannian manifold. Assume that $\partial M = \emptyset$ (resp. $\partial M \neq \emptyset$). For $k \in \mathbb{N}$, let μ_k be the k-th lowest Laplace (resp. Neumann) eigenvalue of Δ_g. For any $(r_n) \subset (0, +\infty)$ such that $r_n \to 0$, there exists an L^2-normalized Laplace (resp. Neumann) eigenfunction $f \in \mathcal{C}^\infty(M)$ (resp. $\mathcal{C}^\infty_\nu(M)$) associated with μ_k such that, up to a subsequence,*

$$\begin{cases} \lambda_k(-\widetilde{\Delta}_{r_n}) \to C_m \mu_k, \\ f_{k,r_n} \xrightarrow{L^2} f. \end{cases}$$

We point out that the question of spectral convergence for the Gaussian approximation of the Laplace–Beltrami operator of a compact Euclidean submanifold with boundary was raised in [5]. This has been one motivation for the present work: to study this convergence with the intrinsic approximation provided by the symmetrized AMV operators $\widetilde{\Delta}_r$ instead of the extrinsic Gaussian one.

2. Averaging-like operators

In this section, we consider a fixed metric measure space, that is to say, a triple (X, d, μ) where (X, d) is a metric space and μ is a fully supported regular Borel measure on (X, d) such that

$$V(x,r) := \mu(B_r(x)) < +\infty$$

for any $x \in X$ and $r > 0$, where $B_r(x)$ denotes the open ball $\{y \in X : \mathsf{d}(x,y) < r\}$. Notice that for any $x \in X$ and $r > 0$,

$$V(x,r) > 0,$$

because μ is fully supported. Moreover, if X is compact, then

$$\mu(X) < +\infty$$

since μ is finite on any ball of radius the diameter of X. We set

$$0 \le m(r) := \inf_{x\in X} V(x,r) \le M(r) := \sup_{x \in X} V(x,r) \le +\infty.$$

Note that our assumptions yield the following preliminary result.

Lemma 2.1. *$L^2(X,\mu)$ is separable.*

Proof. We start by proving that (X, d) is a second countable space. Fix $o \in X$. Given $\varepsilon > 0$ and $N \in \mathbb{N}$ positive, consider the value given by

$$\alpha_{\varepsilon,N} = \sup\left\{\mu|_{B_N(o)}\left(\bigcup_n B_\varepsilon(x_n)\right) : \{x_n\}_{n\in\mathbb{N}} \subset X\right\}, \tag{5}$$

where $\mu|_{B_N(o)}(\cdot) := \mu(\cdot \cap B_N(o))$. First we show this supremum is attained. Consider $\delta_k \to 0$ and let $\{x_n^k\}_{n\in\mathbb{N}} \subset X$ such that

$$\mu|_{B_N(o)}\left(\bigcup_n B_\varepsilon(x_n^k)\right) > \alpha - \delta_k.$$

Taking

$$\{y_n\}_{n\in\mathbb{N}} = \bigcup_k \{x_n^k\}_{n\in\mathbb{N}}$$

we have that

$$\mu|_{B_N(o)}\left(\bigcup_n B_\varepsilon(y_n)\right) = \alpha_{\varepsilon,N}.$$

Now, we prove $\alpha_{\varepsilon,N} = \mu(B_N(z)) < \infty$. If $\alpha_{\varepsilon,N} < \mu(B_N(z))$, then

$$\mu\left(B_N(z) \backslash \bigcup_n B_\varepsilon(y_n)\right) > 0,$$

where $\{y_n\}_{n\in\mathbb{N}}$ is a maximizer of (5). Since the measure is inner regular, there must exist some compact $K \subset B_N(z)\backslash \bigcup_n B_\varepsilon(y_n)$ such that

$$\mu(K) > 0.$$

Since we can cover K by a finite number of balls $B_\varepsilon(z_k)$, there must exist some $z = z_k$ such that

$$\mu(B_\varepsilon(z) \cap K) > 0.$$

We then have

$$\begin{aligned}\mu|_{B_N(o)}\big(\textstyle\bigcup_n B_\varepsilon(y_n) \cup B_\varepsilon(z)\big) &\geq \mu|_{B_N(o)}\big(\textstyle\bigcup_n B_\varepsilon(y_n) \cup (B_\varepsilon(z) \cap K)\big) \\ &= \mu\big(\textstyle\bigcup_n B_\varepsilon(y_n)\big) + \mu(B_\varepsilon(z) \cap K) > \alpha_{N,\varepsilon}.\end{aligned}$$

And so $\{y_n\}_{n\in\mathbb{N}} \cup \{z\}$ contradicts the maximality of $\{y_n\}_{n\in\mathbb{N}}$. This shows that $\alpha_{\varepsilon,N} = \mu(B_N(o))$. Since $\bigcup_n B_\varepsilon(y_n) \cap B_N(o)$ has full measure in the support $B_N(o)$ of $\mu|_{B_N(o)}$, it is a dense subset of $B_N(o)$. This implies that $\bigcup_n B_{2\varepsilon}(y_n)$ is a countable cover of $B_N(o)$. To build a countable basis of X, consider a sequence $\delta_k \to 0$. For any k, take $\bigcup_{n\in\mathbb{N}} B_{\delta_k}(y_n^{k,N})$ a countable cover of $B_N(z)$. Then the set given by

$$\mathcal{B} = \bigcup_{k,N\in\mathbb{N}} \{B_{\delta_k}(y_n^{k,N})\}_{n\in\mathbb{N}}$$

is a countable basis of X. Given this basis $\mathcal{B}$ we can create a new basis given by the finite union of elements of $\mathcal{B}$, and we call this new basis $\mathcal{B}'$, which will also be countable. In particular, we have that given some open set $V \subset X$ we can find a sequence V_n such that

$$V_n \subset V_{n+1}, \quad \textstyle\bigcup_n V_n = V.$$

This is the case since $\mathcal{B}$ is a countable basis, we can find elements $B_k \in \mathcal{B}$ such that

$$\textstyle\bigcup_k B_k = V.$$

We conclude by taking $V_n = \bigcup_{k=1}^n B_k \in \mathcal{B}'$. To construct our dense subset of $L^2(X, \mu)$ we take finite sums with rational coefficients of the characteristic functions χ_V, with $V \in \mathcal{B}'$. To show that this is dense in $L^2(X, \mu)$, we only need to show that we can approximate arbitrarily well simple functions χ_U where $U \subset X$ is open and $\mu(U) < \infty$ since the measure μ is outer regular. Given such an open set U, take χ_{U_n} where $U_n \in \mathcal{B}'$ and $U_n \subset U_{n+1}$ and $\bigcup_n U_n = U$. Then we have by dominated convergence

$$\chi_{U_n} \to_{L^2(X,\mu)} \chi_U,$$

concluding the proof. ■

2.1. Averaging operator

For any $x, y \in X$ and $r > 0$, set

$$a_r(x, y) := \frac{1_{B_r(x)}(y)}{V(x, r)}.$$

Consider $u \in L^1_{\mathrm{loc}}(X,\mu)$. For any $x \in X$ and $r > 0$ such that u is μ-integrable on $B_r(x)$, set

$$A_r u(x) := \fint_{B_r(x)} u \,\mathrm{d}\mu = \int_X a_r(x,y) u(y) \,\mathrm{d}\mu(y).$$

Notice that, since u is locally integrable, for any $x \in X$ there exists $r_x > 0$ such that $A_{r_x} u(x)$ is well defined. However, there may be no uniform $r > 0$ for which the integral $A_r u(x)$ is well defined for every $x \in X$.

Let us also set

$$a_r^*(x,y) := a_r(y,x) = \frac{1_{B_r(x)}(y)}{V(y,r)}$$

for any $x, y \in X$ and $r > 0$. Consider $u \in L^0(X,\mu)$ such that $v(\cdot) := u(\cdot)/V(\cdot,r) \in L^1_{\mathrm{loc}}(X,\mu)$. For any $x \in X$ and $r > 0$ such that v is μ-integrable on $B_r(x)$, set

$$A_r^* u(x) := \int_{B_r(x)} \frac{u(y)\,\mathrm{d}\mu(y)}{V(y,r)} = \int_X a_r^*(x,y) u(y) \,\mathrm{d}\mu(y).$$

Notice that, just like $A_r u(x)$, $A_r^* u(x)$ may not make sense uniformly with respect to $x \in X$.

For any $r > 0$, we introduce the following conditions:

$$\|A_r^* 1\|_\infty < +\infty, \tag{I$_r$}$$

$$V(\cdot,r)^{-1} \in L^1(X,\mu). \tag{II$_r$}$$

Note that (II$_r$) implies (I$_r$) since

$$\|A_r^* 1\|_\infty = \sup_{x \in X} |A_r^* 1(x)| = \sup_{x \in X} \int_{B_r(x)} \frac{\mathrm{d}\mu(y)}{V(y,r)} \le \int_X \frac{\mathrm{d}\mu(y)}{V(y,r)}.$$

In the next lemma, we discuss the boundedness and the compactness of the averaging operator A_r acting on Lebesgue spaces.

Lemma 2.2. *Assume that there exists $r > 0$ such that (I$_r$) holds. Then for any $p \in [1,+\infty]$ the linear operator $A_r \colon L^p(X,\mu) \to L^p(X,\mu)$ is well defined and bounded with*

$$\|A_r\|_{p \to p} \le \|A_r^* 1\|_\infty^{1/p}.$$

Moreover, if (II$_r$) holds, then $A_r \colon L^2(X,\mu) \to L^2(X,\mu)$ is compact.

Proof. The case $p = +\infty$ is obvious and holds regardless of (I$_r$). Let us assume that $p < +\infty$. Let $u \in L^p(X,\mu)$. By Jensen's inequality, for any $x \in X$,

$$|A_r u(x)|^p \le \Bigg(\fint_{B_r(x)} |u|^p \,\mathrm{d}\mu \Bigg).$$

Thus,

$$\begin{aligned}\|A_r u\|_p^p &\le \int_X \frac{1}{V(x,r)} \int_{B_r(x)} |u(y)|^p \,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\\ &= \int_X \int_X \frac{1}{V(x,r)} \underbrace{1_{B_r(x)}(y)}_{=1_{B_r(y)}(x)} |u(y)|^p \,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\\ &= \int_X |u(y)|^p \underbrace{\int_{B_r(y)} \frac{\mathrm{d}\mu(x)}{V(x,r)}}_{=A_r^* 1(y)} \mathrm{d}\mu(y) \le \|A_r^* 1\|_\infty \|u\|_p^p,\end{aligned}$$

where we have used the Fubini–Tonelli theorem to get the second equality and (I_r) for the last inequality.

Let us now assume that (II_r) holds. Since

$$\int_X \int_X a_r^2(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x) = \int_X \frac{1}{V(x,r)} \fint_{B_r(x)} \mathrm{d}\mu(y)\,\mathrm{d}\mu(x) = \int_X \frac{\mathrm{d}\mu(x)}{V(x,r)}$$

we obtain that A_r is a Hilbert–Schmidt integral operator acting on the separable space $L^2(X,\mu)$ (recall Lemma 2.1); in particular, A_r is compact [16, Section IV.6]. ■

In the next statement, we provide an alternative way to prove the compactness of A_r from $L^2(X,\mu)$ to itself. This goes through the compactness of A_r from $L^2(X,\mu)$ to the space of continuous functions $\mathcal{C}(X)$ which we obtain for compact spaces X satisfying the following condition:

$$\sup_{x\in X} \mu(S_r(x)) = 0, \tag{S_r}$$

where $S_r(x) := \{y \in X : \mathsf{d}(x,y) = r\}$.

Lemma 2.3. *Assume that (X,d,μ) is compact and satisfies (S_r) for some $r > 0$. Then $A_r\colon L^2(X,\mu) \to \mathcal{C}(X)$ is compact and satisfies*

$$\|A_r\|_{2\to\infty} \le \frac{1}{m(r)^{1/2}}. \tag{6}$$

Proof. We start by noticing that if $u \in L^2(X,\mu)$, then $A_r(u)$ is continuous. This follows from $V(\cdot,r)^{-1}$ and $\int_{B_r(x)} u(y)d\mu(y)$ being continuous. The former holds by assumption. To prove the latter, assume that $\|u\|_{L^2(X)} = 1$. Then for any $x,z \in X$,

$$\begin{aligned}\left| \int_{B^r(x)} u(y)d\mu(y) - \int_{B^r(z)} u(y)d\mu(y) \right| &\le \|1_{B_r(x)} - 1_{B_r(z)}\|_{L^2(X)} \|u\|_{L^2(X)}\\ &\le \mu(B_{r+d(x,z)}(x) - B_{r-d(x,z)}(x))^{1/2}, \end{aligned} \tag{7}$$

and $\mu(B_{r+d(x,z)}(x) - B_{r-d(x,z)}(x)) \to 0$ as $\mathsf{d}(x,z) \to 0$ due to (S_r). Moreover, the bound (6) is obtained via Hölder's inequality: for any $x \in X$,

$$|A_r u(x)| \leq \Big(\fint_{B_r(x)} u^2 \, \mathrm{d}\mu \Big)^{1/2} \leq \frac{1}{m(r)^{1/2}}.$$

To prove compactness, consider $\{f_n\} \subset L^2(X,\mu)$ such that $\sup_n \|f_n\|_2 \leq 1$. Uniform boundedness of $\{A_r(f_n)\}$ follows from (6), and equicontinuity can be obtained by using the inequality (7) applied to the sequence. By the Ascoli–Arzelà theorem, we can extract from $\{A_r(f_n)\}$ a subsequence which converges in $\mathscr{C}(X)$, concluding the proof. ∎

2.2. Adjoint

Let us focus now on the boundedness and the compactness of the adjoint operator A_r^*. We begin with a simple observation.

Lemma 2.4. *The operator $A_r^* \colon L^1(X,\mu) \to L^1(X,\mu)$ is a contraction for any $r > 0$.*

Proof. For any $u \in L^1(X,\mu)$,

$$\begin{aligned}
\int_X |A_r^* u(x)| \, \mathrm{d}\mu(x) &\leq \int_X \int_{B_r(x)} \frac{|u(y)|}{V(y,r)} \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(x) \\
&= \int_X \int_X 1_{B_r(x)}(y) \frac{|u(y)|}{V(y,r)} \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(x) \\
&= \int_X \Big(\int_X 1_{B_r(y)}(x) \, \mathrm{d}\mu(x) \Big) \frac{|u(y)|}{V(y,r)} \, \mathrm{d}\mu(y) = \int_X |u(y)| \, \mathrm{d}\mu(y),
\end{aligned}$$

where we used the Fubini–Tonelli theorem to get the penultimate equality. ∎

We continue with the next lemma which covers the case $p > 1$.

Lemma 2.5. *Assume that there exists $r > 0$ such that (I_r) holds. Then for any $p \in [1,+\infty]$, the linear operator $A_r^* \colon L^p(X,\mu) \to L^p(X,\mu)$ is well defined and bounded with*

$$\|A_r^*\|_{p\to p} \leq \|A_r^* 1\|_\infty^{(p-1)/p}$$

Moreover, this operator is the adjoint of $A_r \colon L^q(X,\mu) \to L^q(X,\mu)$ for $q \in [1,+\infty]$ such that $1/p + 1/q = 1$. Lastly, if (II_r) holds, then the operator $A_r^* \colon L^2(X,\mu) \to L^2(X,\mu)$ is compact.

Proof. For the proof of the first assertion, consider $u \in L^\infty(X,\mu)$. Thanks to (I_r), for μ-a.e. $x \in X$,

$$|A_r^* u|(x) \le \int_{B_r(x)} \frac{|u(y)|}{V(y,r)} \,\mathrm{d}\mu(y) \le \|u\|_\infty \int_{B_r(x)} \frac{\mathrm{d}\mu(y)}{V(y,r)} \le \|A_r^* 1\|_\infty \|u\|_\infty.$$

Thus, $A_r^* \colon L^\infty(X,\mu) \to L^\infty(X,\mu)$ is bounded with $\|A_r^*\|_{\infty\to\infty} \le \|A_r^* 1\|_\infty$. The conclusion for $p \in (1,+\infty)$ follows from the Riesz-Thorin theorem and Lemma 2.4.

Let us prove that A_r and A_r^* are adjoint of each other. Consider $u \in L^p(X,\mu)$ and $v \in L^q(X,\mu)$. Then

$$\begin{aligned}\int_X A_r^* u(x) v(x)\,\mathrm{d}\mu(x) &= \int_X \int_X 1_{B_r(x)}(y) \frac{u(y)}{V(y,r)} v(x)\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\\ &= \int_X \frac{u(y)}{V(y,r)} \int_X 1_{B_r(x)}(y) v(x)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\\ &= \int_X u(y) A_r v(y)\,\mathrm{d}\mu(y),\end{aligned}$$

where we used the Fubini–Tonelli to get the second equality, and the equality

$$1_{B_r(x)}(y) = 1_{B_r(y)}(x)$$

to get the last one. As for the compactness of A_r^* under (II_r), this result is a direct consequence of the Schauder theorem for compact operators which can be applied thanks to Lemma 2.2. ∎

2.3. Discussion on the assumptions

Let us discuss the validity of (I_r) and (II_r). Recall first that (II_r) $\implies$ (I_r). Both properties hold on totally bounded spaces, as seen in the next lemma.

Lemma 2.6. *Assume that* (X,d) *is totally bounded. Then* (II_r) *(and then* (I_r)*) holds for any* $r > 0$.

Proof. Consider $r > 0$ and a finite cover $\{B_{r/2}(x_i)\}$ of X. For any $x \in X$, there exists i such that $x \in B_{r/2}(x_i)$. Then $B_r(x)$ contains $B_{r/2}(x_i)$ so $V(x,r) \ge V(x_i,r/2) \ge \min_j V(x_j,r/2) > 0$. Thus,

$$\int_X \frac{\mathrm{d}\mu(x)}{V(x,r)} \le \frac{\mu(X)}{\min_j V(x_j,r/2)} < +\infty. \quad ∎$$

Remark 2.7. If $\mu(X) = +\infty$ and $M(r) < +\infty$ then (II_r) cannot hold:

$$\int_X \frac{\mathrm{d}\mu(x)}{V(x,r)} \geq \frac{\mu(X)}{M(r)} = +\infty.$$

This happens, for instance, on $\mathbb{R}^n$ endowed with the Euclidean distance and the Lebesgue measure. More generally, this property cannot hold on a locally compact topological group endowed with a left-invariant metric compatible with the Haar measure and with infinite volume (see [17, Lemma 1] for more details about these spaces).

If $\mu(X) < \infty$ and $m(r) > 0$, then (II_r) always holds:

$$\int_X \frac{\mathrm{d}\mu(x)}{V(x,r)} \leq \frac{\mu(X)}{m(r)} < +\infty.$$

In this regard, observe that if X is not totally bounded, then there exist $r > 0$ small enough and a countable family of disjoint balls $\{B_r(x_i)\}$ in X, so that $\mu(X) < \infty$ and $m(r) > 0$ cannot hold simultaneously:

$$\mu(X) \geq \sum_i V(x_i, r).$$

Let us now focus on (I_r). We show below that this condition holds on so-called *doubling spaces*. Let us recall this classical property and its uniform local variant, see e.g., [10] for more details.

Definition 2.8. The space (X, d, μ) is called *globally doubling* if there exists $C > 0$ such that for any $x \in X$ and $r > 0$,

$$V(x, 2r) \leq C\, V(x, r). \tag{8}$$

It is called *uniformly locally doubling* if there exist $C, r_0 > 0$ such that (8) holds for any $x \in X$ and $r \in (0, r_0)$.

The celebrated Bishop–Gromov theorem (see e.g., [7, Theorem III.4.5]) implies that any complete Riemannian manifold with a uniform lower bound on the Ricci curvature is uniformly locally doubling, and globally doubling if the uniform bound is non-negative. This is also true for metric spaces with generalized sectional curvature bounded from below in the sense of Alexandrov [6, Theorem 10.6.6] and $\mathrm{CD}(K, N)$ metric measure space [18, Corollary 30.14].

The next lemma relates the uniformly local doubling condition with (I_r).

Lemma 2.9. *Let (X, d, μ) be uniformly locally doubling with parameters C, r_0. Then (I_r) holds with $\|A_r^* 1\|_\infty \leq C$ for any $r \in (0, r_0)$.*

Proof. For any $x \in X$ and $r \in (0, r_0)$, the triangle inequality yields that $B_r(x) \subset B_{2r}(y)$ for any $y \in B_r(x)$. Then

$$A_r^* 1(x) = \fint_{B_r(x)} \frac{V(x,r)}{V(y,r)} \, \mathrm{d}\mu(y) \le \fint_{B_r(x)} \frac{V(y,2r)}{V(y,r)} \, \mathrm{d}\mu(y) \le C. \qquad \blacksquare$$

Remark 2.10. Of course, if (X, d, μ) is globally doubling with constant C, then (I_r) holds with $\|A_r^* 1\|_\infty \le C$ for any $r > 0$.

Remark 2.11. The previous result notably implies that (I_r) may hold in situations where (II_r) does not. This happens e.g., on a non-compact Riemannian manifold (M, g) with non-negative Ricci curvature endowed with its canonical Riemannian distance d and volume measure μ. Indeed, such a space has infinite volume, and the Bishop–Gromov theorem implies that $M(r) \le \mathbb{V}^n(r)$ for any $r > 0$, where $\mathbb{V}^n(r)$ is the Lebesgue measure of an Euclidean ball of radius r in $\mathbb{R}^n$. From Remark 2.7, we get that M cannot satisfy (II_r) for any $r > 0$, while it does satisfies (I_r) thanks to the global doubling condition.

We conclude this discussion with two final remarks. First, if $m(r) > 0$ and $M(r) < +\infty$, then (I_r) always holds with

$$\|A_r^* 1\|_\infty \le \frac{M(r)}{m(r)}.$$

This happens on locally compact topological groups endowed with a left-invariant distance and their Haar measure, compare with Remark 2.7. Secondly, (I_r) can easily be seen as a weak variant of the comparability conditions introduced in [3, 15].

2.4. Symmetrization

For any $x, y \in X$ and $r > 0$, set

$$\begin{aligned} \tilde{a}_r(x,y) &= \frac{1}{2}(a_r(x,y) + a_r^*(x,y)) \\ &= \frac{1}{2}\Big(\frac{1}{V(x,r)} + \frac{1}{V(y,r)}\Big) 1_{B_r(x)}(y) \end{aligned}$$

Consider $u \in L^1_{\mathrm{loc}}(X, \mu)$ such that $v(\cdot) := u(\cdot)/V(\cdot, r) \in L^1_{\mathrm{loc}}(X, \mu)$. For any $x \in X$ and $r > 0$ such that u and v are μ-integrable on $B_r(x)$, set

$$\tilde{A}_r u(x) := \frac{1}{2}(A_r u(x) + A_r^* u(x)) = \int_X \tilde{a}_r(x,y) u(y) \, \mathrm{d}\mu(y). \tag{9}$$

Then the next lemma is an obvious consequence of Lemma 2.2 and Lemma 2.5.

Corollary 2.12. *Assume that there exists $r > 0$ such that (I_r) holds. Then the map $\widetilde{A}_r\colon L^2(X,\mu) \to L^2(X,\mu)$ is a self-adjoint operator such that*

$$\|\widetilde{A}_r\|_{2\to 2} \le \|A_r^* 1\|_\infty^{1/2}.$$

Moreover, if (II_r) holds, then $\widetilde{A}_r\colon L^2(X,\mu) \to L^2(X,\mu)$ is compact.

3. Symmetrized AMV operators

In this section, we provide our working definition of the symmetrized AMV r-Laplace operator $\widetilde{\Delta}_r$ and we derive several spectral properties in a general setting.

3.1. Definitions

For this subsection, we consider a metric measure space (X, d, μ) satisfying (I_r) for some fixed $r > 0$.

Definition 3.1. The symmetrized AMV r-Laplace operator of (X, d, μ) is

$$\widetilde{\Delta}_r := \frac{1}{r^2}(\widetilde{A}_r - [\widetilde{A}_r 1]\mathrm{I}),$$

where we recall that $\widetilde{A}_r$ is defined in (9).

Remark 3.2. We may use the notation $\widetilde{\Delta}_{r,\mathfrak{X}}$ to specify that we work on the metric measure space $\mathfrak{X} = (X, \mathsf{d}, \mu)$.

Lemma 3.3. *$\widetilde{\Delta}_r$ is a bounded, self-adjoint operator acting on $L^2(X,\mu)$ with*

$$\|\widetilde{\Delta}_r\|_{2\to 2} \le \frac{1}{2r^2}(2\|A_r^* 1\|_\infty^{1/2} + \|A_r^* 1\|_\infty + 1). \tag{10}$$

Proof. The self-adjointness of $\widetilde{\Delta}_r$ is obvious because $\widetilde{A}_r$ and $[\widetilde{A}_r 1]\mathrm{I}$ are self-adjoint too. The boundedness is a consequence of Lemma 2.12. Indeed,

$$\begin{aligned}
\|\widetilde{\Delta}_r\|_{2\to 2} &\le \frac{1}{r^2}(\|\widetilde{A}_r\|_{2\to 2} + \|[\widetilde{A}_r 1]I\|_{2\to 2}) \\
&\le \frac{1}{r^2}(\|A_r^* 1\|_\infty^{1/2} + \|\widetilde{A}_r 1\|_\infty \underbrace{\|I\|_{2\to 2}}_{=1}) \\
&\le \frac{1}{r^2}\Big(\|A_r^* 1\|_\infty^{1/2} + \frac{\|A_r 1\|_\infty + \|A_r^* 1\|_\infty}{2}\Big) \\
&= \frac{1}{r^2}\Big(\|A_r^* 1\|_\infty^{1/2} + \frac{1 + \|A_r^* 1\|_\infty}{2}\Big),
\end{aligned}$$

hence $\widetilde{\Delta}_r\colon L^2(X,\mu) \to L^2(X,\mu)$ is bounded and (10) holds. ■

Definition 3.4. The energy functional $\widetilde{E}_r$ of (X, d, μ) is the quadratic form on the space $L^2(X, \mu)$ defined by

$$\widetilde{E}_r(f) := \frac{1}{4} \int_X \int_X 1_{B_r(x)}(y) \Big(\frac{1}{V(x,r)} + \frac{1}{V(y,r)} \Big) \Big(\frac{f(x) - f(y)}{r} \Big)^2 \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(x).$$

The associated bilinear form, which we still denote by $\widetilde{E}_r$, is given by

$$\widetilde{E}_r(f, \psi) := \frac{1}{4} \int_X \int_X 1_{B_r(x)}(y) \Big(\frac{1}{V(x,r)} + \frac{1}{V(y,r)} \Big) \times \frac{(f(x) - f(y))(\psi(x) - \psi(y))}{r^2} \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(x).$$

Remark 3.5. A suitable use of the Fubini–Tonelli theorem shows that the energy functional $\widetilde{E}_r(f)$ equals the approximate Korevaar–Schoen energy [11]

$$\frac{1}{2} \int_X \fint_{B_r(x)} \frac{|f(y) - f(x)|^2}{r^2} \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(x).$$

Remark 3.6. We may also use the notation $\widetilde{E}_{r,\mathfrak{X}}$ to specify the metric measure space $\mathfrak{X} = (X, \mathsf{d}, \mu)$.

The next lemma goes back to [2, Lemma 3.1]. We provide a quick proof for completeness.

Lemma 3.7. *For any $f, \psi \in L^2(X, \mu)$,*

$$\widetilde{E}_r(f, \psi) = \langle -\widetilde{\Delta}_r f, \psi \rangle_{L^2}. \tag{11}$$

Proof. Note that

$$\begin{aligned} \widetilde{E}_r(f, \psi) &= \frac{1}{4} \int_X \int_{B_r(x)} \Big(\frac{1}{V(x,r)} + \frac{1}{V(y,r)} \Big) \frac{(f(x) - f(y))\psi(x)}{r^2} \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(x) \\ &\quad - \frac{1}{4} \int_X \int_X 1_{B_r(x)}(y) \Big(\frac{1}{V(x,r)} + \frac{1}{V(y,r)} \Big) \frac{(f(x) - f(y))\psi(y)}{r^2} \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(x). \end{aligned}$$

Using $1_{B_r(x)}(y) = 1_{B_r(y)}(x)$ and then the Fubini theorem, we can rewrite the second term as the opposite of the first one, so that we eventually get (11). ∎

Remark 3.8. Observe that (11) implies that $-\widetilde{\Delta}_r$ is a non-negative operator, since for any $f \in L^2(X, \mu)$,

$$\langle -\widetilde{\Delta}_r f, f \rangle_{L^2} = \widetilde{E}_r(f) \geq 0$$

Let us recall the definition of spectrum.

Definition 3.9. We let $\sigma(-\widetilde{\Delta}_r)$ denote the spectrum of $-\widetilde{\Delta}_r$, that is to say, the set of elements $\lambda \in \mathbb{C}^*$ such that $-\widetilde{\Delta}_r - \lambda \mathrm{I}\colon L^2(X,\mu) \to L^2(X,\mu)$ is not a bijection.

It is well known from classical functional analysis that the spectrum $\sigma(T)$ of a bounded operator T acting on a Banach space E can be decomposed as

$$\sigma(T) = \sigma_{\mathrm{p}}(T) \cup \sigma_{\mathrm{c}}(T) \cup \sigma_{\mathrm{a}}(T),$$

where

- $\sigma_{\mathrm{p}}(T)$ is the point spectrum, that is to say, the set of $\lambda \in \mathbb{C}^+$ such that $(T - \lambda \mathrm{I}) f = 0$ for some non-zero $f \in E$, in which case λ is called an *eigenvalue* and f an *eigenvector* of T;
- $\sigma_{\mathrm{c}}(T)$ is the compression spectrum, that is to say, the set of $\lambda \in \mathbb{C}^+$ whose conjugate $\bar{\lambda}$ is an eigenvalue of the adjoint T^*;
- $\sigma_{\mathrm{a}}(T)$ is the approximate point spectrum, that is to say, the set of $\lambda \in \mathbb{C}^+$ for which there exists $(f_n) \subset E$ with $\|f_n\| = 1$ for any n such that $\|(T - \lambda \mathrm{I}) f_n\| \to 0$.

Since $-\widetilde{\Delta}_r$ is self-adjoint and non-negative, we know that

$$\sigma(-\widetilde{\Delta}_r) \subset [0, +\infty].$$

This implies that $\sigma_{\mathrm{p}}(-\widetilde{\Delta}_r) = \sigma_{\mathrm{c}}(-\widetilde{\Delta}_r)$, so that

$$\sigma(-\widetilde{\Delta}_r) = \sigma_{\mathrm{p}}(-\widetilde{\Delta}_r) \cup \sigma_{\mathrm{a}}(-\widetilde{\Delta}_r). \tag{12}$$

Definition 3.10. For any $k \in \mathbb{N}$, we define

$$\tilde{\lambda}_{k,r} := \inf_{V \in \mathcal{G}_{k+1}(L^2(X,\mu))} \sup_{f \in V} \frac{\widetilde{E}_r(f)}{\|f\|_2},$$

where $\mathcal{G}_{k+1}(L^2(X,\mu))$ is the $(k+1)$-th Grassmannian of $L^2(X,\mu)$.

Remark 3.11. Let $\sigma_{\mathrm{ess}}(-\widetilde{\Delta}_r)$ denote the essential spectrum of $-\widetilde{\Delta}_r$, i.e., the closed subset of $\sigma(-\widetilde{\Delta}_r)$ made of those λ such that $-\widetilde{\Delta}_r - \lambda \mathrm{I}$ is not a Fredholm operator. Since $-\widetilde{\Delta}_r$ is self-adjoint, the Fischer–Polyà minimum-maximum principle (see e.g., [19, p. 12]) asserts that if there exists a positive integer N such that $\tilde{\lambda}_{N,r} < \min \sigma_{\mathrm{ess}}(-\widetilde{\Delta}_r)$, then $-\widetilde{\Delta}_r$ admits $N+1$ isolated eigenvalues

$$\lambda_0(-\widetilde{\Delta}_r) \le \cdots \le \lambda_N(-\widetilde{\Delta}_r) < \min \sigma_{\mathrm{ess}}(-\widetilde{\Delta}_r)$$

such that for any $k \in \{0, \dots, N\}$,

$$\tilde{\lambda}_{k,r} = \lambda_k(-\widetilde{\Delta}_r).$$

3.2. Spectral properties

Our first spectral result on $\widetilde{\Delta}_r$ is the following. Note that we need the compactness of $\widetilde{A}_r$ here, thus we assume (II_r).

Proposition 3.12. *Let (X, d, μ) be a metric measure space satisfying (II_r) for some fixed $r > 0$. Assume that $\lambda \in \sigma(-\widetilde{\Delta}_r)$ satisfies*

$$\lambda < \inf_{y \in X} \frac{[\widetilde{A}_r 1](y)}{r^2}.$$

Then λ is an isolated eigenvalue of $-\widetilde{\Delta}_r$ which does not belong to $\sigma_{\mathrm{ess}}(-\widetilde{\Delta}_r)$.

Proof. Let us first show that λ is an eigenvalue, that is to say, that λ belongs to the point spectrum. According to (12), it is enough to show that if λ is in the approximate point spectrum, then it is in the point spectrum. If this is the case, then there exists a sequence $(f_n) \in L^2(X, \mu)$ such that $\|f_n\|_{L^2(X)} = 1$ and

$$\| -\widetilde{\Delta}_r f_n - \lambda f_n\|_{L^2(X)} = \Big\| \Big(\frac{[\widetilde{A}_r 1]}{r^2} - \lambda \Big) f_n - \frac{\widetilde{A}_r f_n}{r^2} \Big\|_{L^2(X)} \to 0. \tag{13}$$

Since $\widetilde{A}_r$ is compact, we have that $\widetilde{A}_r f_n$ converges up to a subsequence, and as such by equation (13) so does $([\widetilde{A}_r 1] - r^2\lambda) f_n$, with limit $g \in L^2(X)$. Consider $\delta > 0$ such that $0 \le \lambda + \delta/r^2 < \inf_{y \in X} [\widetilde{A}_r 1](y)/r^2$ and define

$$b_r(x) := [\widetilde{A}_r 1] - r^2 \lambda \ge \delta.$$

Thus, we have that $f := g/b_r \in L^2(X, \mu)$, and so

$$\delta \Big\| f_n - \frac{g}{b_r} \Big\|_{L^2(X)} \le \|b_r f_n - g\|_{L^2(X)} \to 0.$$

Thus, f_n converges in $L^2(X, \mu)$ to the limit function f. Using continuity of $-\widetilde{\Delta}_r$ we conclude that

$$-\widetilde{\Delta}_r f = \lambda f,$$

and thus λ is in the point spectrum.

To prove that λ is an isolated point, we suppose by contradiction that there exists an infinite sequence $(\lambda_n) \subset \sigma(-\widetilde{\Delta}_r)$ of distinct values such that $\lambda_n \to \lambda$. Then there exists $\delta > 0$ such that for any high enough n,

$$\lambda_n + \delta/r^2 < \inf_{y \in X} \frac{[\widetilde{A}_r 1](y)}{r^2}, \quad \lambda_n \to \lambda.$$

From the previous paragraph, we know that λ_n is in the point spectrum, thus there exists $f_n \in L^2(X, \mu)$ satisfying $\|f_n\|_{L^2(X)} = 1$, such that

$$-\widetilde{\Delta}_r f_n = \lambda_n f_n.$$

This can be written as

$$-\frac{\widetilde{A}_r}{r^2} f_n = \Big(\lambda_n - \frac{[\widetilde{A}_r 1]}{r^2}\Big) f_n.$$

Using compactness of $\widetilde{A}_r$, we know that $\widetilde{A}_r f_n$ converges up to a subsequence. This implies that $(r^2\lambda_n - [\widetilde{A}_r 1]) f_n$ converges up to a subsequence to some $g \in L^2(X, \mu)$. Define

$$b_{r,n}(x) := [\widetilde{A}_r 1] - r^2\lambda_n.$$

With this we have that

$$\begin{aligned}\delta\Big\| f_n - \frac{g}{b_r}\Big\|_{L^2(X)} &\leq \delta\Big(\Big\| f_n - \frac{g}{b_{r,n}}\Big\|_{L^2(X)} + \Big\|\frac{g}{b_{r,n}} - \frac{g}{b_r}\Big\|_{L^2(X)}\Big)\\ &\leq \|b_{r,n} f_n - g\|_{L^2(X)} + \delta\Big\|\frac{g}{b_{r,n}} - \frac{g}{b_r}\Big\|_{L^2(X)}.\end{aligned}$$

We have that $\|b_{r,n} f_n - g\|_{L^2(X)} \to 0$ and also $\|\frac{g}{b_{r,n}} - \frac{g}{b_r}\|_{L^2(X)} \to 0$ since $0 < \delta \leq b_{r,n}, b_r$ and $\lambda_n \to \lambda$. Thus, f_n converges. However, since all the eigenvalues are different, we know that $\langle f_n, f_j\rangle = \delta_{n,j}$, and so the sequence cannot converge up to a subsequence, achieving contradiction. This shows that λ is an isolated point of $\sigma(-\widetilde{\Delta}_r)$ finishing the first part of the proof.

Let us now prove that $-\widetilde{\Delta}_r - \lambda I$ is a Fredholm operator.

To show that $\ker(-\widetilde{\Delta}_r - \lambda I)$ is finite dimensional we proceed by contradiction. Assume that there exists an infinite sequence $(f_n) \subset \ker(-\widetilde{\Delta}_r - \lambda I)$ such that

$$\langle f_n, f_j\rangle = \delta_{n,j}.$$

Thus,

$$-\frac{\widetilde{A}_r f_n}{r^2} = \Big(\lambda - \frac{[\widetilde{A}_r 1]}{r^2}\Big) f_n.$$

Similar to before we can use compactness of $\widetilde{A}_r$ and the condition on λ to conclude that f_n converges in $L^2(X, \mu)$ up to a subsequence. However, this is prevented by $\langle f_n, f_j\rangle = \delta_{n,j}$.

To show that the image of $-\widetilde{\Delta}_r - \lambda I$ is closed, consider a sequence

$$g_n := (-\widetilde{\Delta}_r - \lambda I)(f_n)$$

such that $g_n \to g$. Similarly to before, we can conclude that since g_n converges, then f_n converges to some f, and thus $g = (-\widetilde{\Delta}_r - \lambda I)(f)$. ■

Corollary 3.13. *Let* (X, d, μ) *be a metric measure space such that for some* $r_0 > 0$ *the assumption* (II_r) *holds for any* $r \in (0, r_0)$. *Then*

$$\lim_{r\downarrow 0}(\min \sigma_{\mathrm{ess}}(-\widetilde{\Delta}_r)) = +\infty.$$

Proof. Proposition 3.12 implies that

$$\inf_{y\in X} \frac{[\widetilde{A}_r 1](y)}{r^2} \leq \min \sigma_{\mathrm{ess}}(-\widetilde{\Delta}_r). \tag{14}$$

But for any $r > 0$,

$$\widetilde{A}_r 1 = \frac{1}{2}(A_r 1 + A_r^* 1) \geq \frac{1}{2} A_r 1 = \frac{1}{2}$$

hence (14) implies that

$$\min \sigma_{\mathrm{ess}}(-\widetilde{\Delta}_r) \geq \frac{1}{2r^2} \xrightarrow{r\downarrow 0} +\infty. \qquad \blacksquare$$

Let us provide our second spectral result on $\widetilde{\Delta}_r$.

Proposition 3.14. *Let* (X, d, μ) *be a connected metric measure space satisfying* (II_r) *for some fixed* $r > 0$. *Then the kernel of* $\widetilde{\Delta}_r$ *contains constant functions only, and* $\widetilde{E}_r$ *defines a scalar product on*

$$\Pi(X, \mu) := \left\{ f \in L^2(X, \mu) : \int_X f \,\mathrm{d}\mu = 0 \right\}. \tag{15}$$

Proof. Consider $f \in L^2(X, \mu)\backslash\{0\}$ such that $\widetilde{\Delta}_r f = 0$. Then we have $\widetilde{E}_r(f) = 0$. This implies that for μ-a.e. $x \in X$,

$$\int_X 1_{B_r(x)}(y)\Big(\frac{1}{V(x,r)} + \frac{1}{V(y,r)}\Big)\Big(\frac{f(x) - f(y)}{r}\Big)^2 d\mu(y) = 0$$

which implies, in turn,

$$\mu(\{y \in B_r(x) : f(y) = f(x)\}) = \mu(B_r(x)).$$

Consider $F := \{x \in X : f(x) \text{ is a well-defined real number}\}$ and

$$A := \{x \in F : \mu(\{y \in B_r(x) : f(y) = f(x)\}) = \mu(B_r(x))\}.$$

Then

$$\mu(X\backslash A) = 0.$$

Take $z \in A$ and let $c = f(z)$. Consider

$$I = \{x \in A : f(x) = c\}, \quad I' = \{x \in A : f(x) \neq c\},$$

and notice that $I \cup I' = A$. Suppose by contradiction that $I' \neq \emptyset$. Set

$$W := \bigcup_{x \in I} B_r(x), \quad V := \bigcup_{y \in I'} B_r(y).$$

Since V and W are open sets whose union contains A which has full measure in X, we must have

$$W \cup V = X,$$

otherwise $X \backslash A$ would contain an open ball with positive measure. Since W and V form an open cover of X, and X is connected, if both V and W are different from the empty set, then there exist $x \in I$ and $y \in I'$ such that

$$B_r(x) \cap B_r(y) \neq \emptyset.$$

However,

$$f|_{B_r(x) \cap B_r(y)}(w) = f(x) \quad \mu\text{-a.e. } w \in X,$$
$$f|_{B_r(x) \cap B_r(y)}(w) = f(y) \quad \mu\text{-a.e. } w \in X.$$

This is not possible since $f(x) \neq f(y)$ and $\mu(B_r(x) \cap B_r(y)) > 0$. This implies that $I = A$ and that $f(w) = c$ for μ-a.e. $w \in X$. Then $-\widetilde{\Delta}_r$ has a non-trivial kernel consisting of the constant functions only. Moreover, since $-\widetilde{\Delta}_r$ is non-negative (Remark 3.8), we get the desired property on (15). ■

We are now in a position to prove Theorem 1. We recall that the context of this statement is a compact uniformly locally doubling metric measure space (X, d, μ). The compactness of the space ensures that (II_r) holds for any $r > 0$, see Lemma 2.6.

Proof. For $k \geq 1$ integer, let $x_0, \ldots, x_k \in X$ be distinct points. Set

$$\bar{r}_k := \min_{0 \leq i \neq j \leq k} \frac{\mathsf{d}(x_i, x_j)}{4}.$$

For any $i \in \{0, \ldots, k\}$ and $y \in X$, define

$$\tilde{f}_i(y) := \left(1 - \frac{\mathsf{d}(x_i, y)}{\bar{r}_k}\right)^+ \quad \text{and} \quad f_i(y) := \frac{\tilde{f}_i(y)}{\|\tilde{f}_i\|_2}.$$

Note that each f_i is an L^2-normalized Lipschitz function supported in $B_{\bar{r}_k}(x_i)$, and that $(f_0, \ldots, f_k)$ is an orthonormal family of $L^2(X, \mu)$. Set

$$V := \mathrm{Span}(f_0, \ldots, f_k) \in \mathcal{G}_{k+1}(L^2(X, \mu))$$

and observe that for any $r > 0$,

$$\tilde{\lambda}_{k,r} \leq \max_{\substack{f \in V \\ \|f\|_2 = 1}} \widetilde{E}_r(f).$$

Consider $r \in (0, \bar{r}_k)$ and $i \neq j$ in $\{0, \dots, k\}$. If $x \in B_{2\bar{r}_k}(x_i)$, then $f_j(y) = 0$ for any $y \in B_r(x)$, while if $x \notin B_{2\bar{r}_k}(x_i)$, then $f_i(y) = 0$ for any $y \in B_r(x)$. In both cases,

$$(f_i(x) - f_i(y))(f_j(x) - f_j(y)) = 0$$

for any $y \in B_r(x)$. Thus,

$$\widetilde{E}_r(f_i, f_j) = 0. \tag{16}$$

Consider $f \in V$ such that $\|f\|_2 = 1$. Then $f = \sum_{i=0}^k a_i f_i$ for some $a_0, \dots, a_k \in \mathbb{R}$ such that $\sum_{i=0}^k a_i^2 = 1$. By (16), we get

$$\widetilde{E}_r(f) = \sum_{i=0}^k a_i^2 \widetilde{E}_r(f_i) \leq \max_{0 \leq i \leq k} \widetilde{E}_r(f_i).$$

Let r_0, C be the parameters of the uniform local doubling property of (X, d, μ), see Definition 2.8. Then for any $i \in \{0, \dots, k\}$ and $r \in (0, r_0)$,

$$\begin{aligned}
\widetilde{E}_r(f_i) &\leq \frac{\mathrm{Lip}^2(f_i)}{4} \int_X \fint_{B_r(x)} \underbrace{\left(1 + \frac{V(x,r)}{V(y,r)}\right)}_{\leq 1 + C^2} \underbrace{\frac{\mathsf{d}^2(x,y)}{r^2}}_{\leq 1} \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(x) \\
&\leq \frac{\mathrm{Lip}^2(f_i)(1 + C^2)\mu(X)}{4}.
\end{aligned}$$

Therefore, for any $r < \tilde{r}_k := \min(\bar{r}_k, R)$, we get

$$\tilde{\lambda}_{k,r} \leq \widetilde{C} := \frac{(1 + C^2)\mu(X)}{4} \max_{0 \leq i \leq k} \mathrm{Lip}^2(f_i)\cdot \tag{17}$$

By Corollary 3.13, there exists $r_k \in (0, \tilde{r}_k)$ such that $\widetilde{C} < \min \sigma_{\mathrm{ess}}(-\widetilde{\Delta}_r)$ for any $r \in (0, r_k)$. Then Remark 3.11 implies that for such an r the operator $-\widetilde{\Delta}_r$ admits $k + 1$ eigenvalues $\lambda_0(-\widetilde{\Delta}_r) \leq \lambda_1(-\widetilde{\Delta}_r) \leq \dots \leq \lambda_k(-\widetilde{\Delta}_r)$ such that $\lambda_i(-\widetilde{\Delta}_r) = \tilde{\lambda}_{i,r}$ for any $i \in \{0, \dots, k\}$. That $\lambda_0(-\widetilde{\Delta}_r) = 0$ follows from Proposition 3.14. Moreover, by Remark 3.11 and Proposition 3.14, we know that

$$\tilde{\lambda}_{1,r} = \min_{f \in \Pi(X,\mu)} \frac{\widetilde{E}_r(f)}{\|f\|_2},$$

where $\Pi(X, \mu)$ is as in (15). Since $-\Delta_r$ has a kernel which is L^2-orthogonal to $\Pi(X, \mu)$ (Proposition 3.14), we have $\widetilde{E}_r(f) > 0$ for any $f \in \Pi(X, \mu)$, hence we get

$$\tilde{\lambda}_{1,r} > 0.$$

■

4. First eigenvalue of torus and hypercubes

In this section, we derive some results which will be applied in Section 6. We let m be a positive integer kept fixed throughout the section.

4.1. A preliminary lemma

We begin with a result where we use the normalized sinc function, namely

$$\operatorname{sinc}(\rho) := \begin{cases} \dfrac{\sin(\pi\rho)}{\pi\rho} & \text{if } \rho \in \mathbb{R}\backslash\{0\}, \\ 1 & \text{if } \rho = 0, \end{cases}$$

and the following notation: for any $p = (p_1, \dots, p_m) \in \mathbb{Z}^m$,

$$J(p) := \{i \in \{1, \dots, m\} : p_i \neq 0\}, \quad j(p) := \# J(p).$$

Lemma 4.1. *We have*

$$\liminf_{r\to 0} \inf_{0\neq p\in\mathbb{Z}^m} \left| \frac{1}{r^2}\Big(1 - \prod_{i\in J(p)} \operatorname{sinc}(p_i r)\Big)\right| > 0. \tag{18}$$

Proof. We start by pointing out that for any $r > 0$ and $p \in \mathbb{Z}^m\backslash\{0\}$,

$$\prod_{i\in J(p)} \operatorname{sinc}(p_i r) \neq 1,$$

so that

$$\left| \frac{1}{r^2}\Big(1 - \prod_{i\in J(p)} \operatorname{sinc}(p_i r)\Big)\right| > 0.$$

Moreover, for any $r > 0$, if $|p|_\infty := \max_{1\leq i\leq m} |p_i| \to +\infty$, then

$$\left| \frac{1}{r^2}\Big(1 - \prod_{i\in J(p)} \operatorname{sinc}(p_i r)\Big)\right| \to \frac{1}{r^2} > 0,$$

hence there exists $R > 0$ such that

$$\inf_{0\neq p\in\mathbb{Z}^m} \left| \frac{1}{r^2}\Big(1 - \prod_{i\in J(p)} \operatorname{sinc}(p_i r)\Big)\right| = \min_{\substack{0\neq p\in\mathbb{Z}^m \\ |p|_\infty < R}} \left| \frac{1}{r^2}\Big(1 - \prod_{i\in J(p)} \operatorname{sinc}(p_i r)\Big)\right| > 0.$$

If (18) were to fail, due to the previous line, there would exist sequences $(r_n) \subset (0, +\infty)$ and $(p^{(n)}) \subset \mathbb{Z}^m\backslash\{0\}$ such that $r_n \to 0$ and

$$\lim_n \left| \frac{1}{r_n^2}\Big(1 - \prod_{i\in J(p)} \operatorname{sinc}(p_i^{(n)} r)\Big)\right| = 0. \tag{19}$$

We have two cases:

- there exists $\alpha > 0$ and $j \in \{1, \dots, m\}$ such that $\liminf_n p_j^{(n)} r_n > \alpha$;
- for any $j \in \{1, \dots, m\}$, one has $\liminf_n p_j^{(n)} r_n = 0$.

If the first one were true, then we would have

$$\liminf_n \Big| 1 - \prod_{i \in J(p)} \operatorname{sinc}(p_i^{(n)} r) \Big| > |1 - \alpha|,$$

and so (19) could not hold. On the contrary, if the second case were true, up to extracting a subsequence we would have $p_j^{(n)} r_n \to 0$ for any $j \in \{1, \dots, m\}$. For any $y = (y_1, \dots, y_m) \in \mathbb{R}^m$, set

$$G(y) := 1 - \prod_{j=1}^{m} \operatorname{sinc}(y_j).$$

Then G is smooth on $\mathbb{R}^m$ and satisfies

$$G(y) = \frac{1}{2}|y|^2 + o(|y|^3), \quad |y| \to 0.$$

As a consequence, for $y \in \mathbb{R}^m$ such that $|y|$ is small enough,

$$|G(y)| \geq \frac{1}{4}|y|^2.$$

Then, for large enough n,

$$\Big| 1 - \prod_{i \in I(p^{(n)})} \operatorname{sinc}(p_i^{(n)} r_n) \Big| = G(r_n p^{(n)}) \geq \frac{1}{4}|r_n p^{(n)}|^2 \geq \frac{1}{4} r_n^2,$$

because $p^{(n)} \neq 0$ implies that there exists at least one i such that $|p_i^{(n)}| \geq 1$. Thus, we obtain

$$\lim_n \Big| \frac{1}{r_n^2} \Big(1 - \prod_{i \in J(p)} \operatorname{sinc}(\pi p_i^{(n)} r_n) \Big) \Big| \geq \frac{1}{4}$$

and so (19) could not hold. This concludes the proof. ∎

4.2. Torus

Consider the torus

$$\mathbb{T}^m := \mathbb{R}^m / (-1 + 2\mathbb{Z})^m$$

with its natural quotient map $\pi : \mathbb{R}^m \to \mathbb{T}^m$. Let π^{-1} be the inverse of the bijective map

$$\pi : [-1, 1)^m \to \mathbb{T}^m.$$

Let d_∞ be the distance in $\mathbb{R}^m$ associated with the infinity norm, given by

$$\mathsf{d}_\infty(x,y) := \max\{|x_i - y_i| : i \in \{1,\dots,m\}\}$$

for any $x = (x_1,\dots,x_m)$ and $y = (y_1,\dots,y_m)$ in $\mathbb{R}^m$. With respect to this distance, the open ball of radius $r > 0$ centered at $x \in \mathbb{R}^m$ is

$$Q_r(x) := \prod_{i=1}^{m} (-r + x_i, x_i + r). \tag{20}$$

For any $x, y \in \mathbb{T}^m$, set

$$\tilde{\mathsf{d}}_\infty(x,y) = \inf_{\substack{z\in\pi^{-1}(x)\\ w\in\pi^{-1}(y)}} \mathsf{d}_\infty(z,w).$$

Then $\tilde{\mathsf{d}}_\infty$ defines a distance on $\mathbb{T}^m$, and we denote by $\widetilde{Q}_r(x)$ the open ball of radius $r > 0$ centered at $x \in \mathbb{T}^m$ with respect to this distance. We also introduce the probability measure

$$\mathbb{L}^m := \pi_\#\Big(\frac{\mathcal{L}^m}{2^m}\Big)$$

on $\mathbb{T}^m$. Note that this is also the normalized Haar measure of $\mathbb{T}^m$ seen as a Lie group.

It is obvious that the metric measure space $\mathfrak{T}^m := (\mathbb{T}^m, \tilde{\mathsf{d}}_\infty, \mathbb{L}^m)$ satisfies the assumptions of Theorem 1, hence we know that for any small enough r,

$$\tilde{\lambda}_{1,r} = \lambda_1(-\widetilde{\Delta}_r) > 0.$$

Then the following holds.

Proposition 4.2. *We have*

$$\liminf_{r\to 0} \lambda_1(-\widetilde{\Delta}_r) > 0.$$

Proof. For any $x \in \mathbb{T}^m$ and $r > 0$ small enough, the ball $\widetilde{Q}_r(x) \subset \mathbb{T}^m$ is given by

$$\widetilde{Q}_r(x) = \pi(Q_r(\tilde{x})),$$

where $\tilde{x}$ is any element in $\pi^{-1}(x)$, and $Q_r(\tilde{x})$ is as in (20). Then

$$\mathbb{L}^m(\widetilde{Q}_r(x)) = \frac{\mathcal{L}^m(Q_r(\tilde{x}))}{2^m} = r^m \tag{21}$$

so that the r-energy functional of $\mathfrak{T}^m$ writes as

$$\widetilde{E}_{r,\mathfrak{T}^m}(f) = \frac{1}{r^m} \int_{\mathbb{T}^m} \int_{\widetilde{Q}_r(x)} \Big(\frac{f(x)-f(y)}{r}\Big)^2 \,\mathrm{d}\mathbb{L}^m(y)\,\mathrm{d}\mathbb{L}^m(x) \tag{22}$$

for any $f \in L^2(\mathbb{T}^m, \mathbb{L}^m)$.

We act by contradiction. Assume that there exist $r_n \to 0$ and $\{f_n\} \subset L^2(\mathbb{T}^m, \mathbb{L}^m)$ satisfying $\int_{\mathbb{T}^m} f_n \, d\mathbb{L}^m = 0$ and $\|f_n\|_{L^2(\mathbb{T}^m)} = 1$, such that

$$-\widetilde{\Delta}_{r_n} f_n = \lambda_1(-\widetilde{\Delta}_{r_n}) f_n, \quad \lambda_1(-\widetilde{\Delta}_{r_n}) \to 0. \tag{23}$$

With no loss of generality, we assume that each r_n is small enough to ensure that $\widetilde{E}_{r_n,\mathfrak{T}^m}$ writes as in (22).

From (21) we can write, for any n and $\mathbb{L}^m$-a.e. $x \in \mathbb{T}^m$,

$$\begin{aligned} -\widetilde{\Delta}_{r_n} f_n(x) &= \frac{1}{r_n^2}\Big(f_n(x) - \frac{1}{r_n^m} \int\limits_{\mathbb{T}^m} 1_{\widetilde{Q}_{r_n}(x)}(y) f_n(y) \, d\mathbb{L}^m(y) \Big) \\ &= \frac{1}{r_n^2}\Big(f_n(x) - \frac{1}{r_n^m} (1_{\widetilde{Q}_{r_n}(0)} * f_n)(x) \Big). \end{aligned} \tag{24}$$

We consider the Fourier decomposition of f_n, $1_{\widetilde{Q}_{r_n}(0)}$, and $-\widetilde{\Delta}_{r_n} f_n$, namely

$$f_n = \sum_{p \in \mathbb{Z}^m} a_{p,n} e_p, \quad 1_{\widetilde{Q}_{r_n}(0)} = \sum_{p \in \mathbb{Z}^m} b_{p,n} e_p, \quad -\widetilde{\Delta}_{r_n} f_n = \sum_{p \in \mathbb{Z}^m} c_{p,n} e_p,$$

where $\{e_p\}_{p \in \mathbb{Z}^m}$ is the orthonormal basis of $L^2(\mathbb{T}^m, \mathbb{L}^m)$ given by

$$e_p \colon \mathbb{T}^m \ni x \mapsto e^{i\pi p \cdot \pi^{-1}(x)} \quad \text{for all } p \in \mathbb{Z}^m.$$

Since the Fourier coefficients of a convolution are the product of the coefficients, we obtain from (24) that

$$c_{p,n} = \frac{1}{r_n^2}\Big(a_{p,n} - \frac{b_{p,n}}{r_n^m} a_{p,n} \Big) = \frac{a_{p,n}}{r_n^2}\Big(1 - \frac{b_{k,n}}{r_n^m} \Big).$$

We can compute each coefficient $b_{p,n}$ by means of Fubini's theorem; we obtain

$$b_{p,n} = \int\limits_{Q_{r_n}(0)} e^{i\pi p \cdot x} \frac{d\mathcal{L}^m(x)}{2^m} = r_n^m \prod_{i \in J(p)} \operatorname{sinc}(p_i r_n).$$

Thus,

$$c_{p,n} = \frac{a_{p,n}}{r_n^2}\Big(1 - \prod_{i \in J(p)} \operatorname{sinc}(p_i r_n) \Big).$$

Using Lemma 4.1, for $p \in \mathbb{Z} \backslash \{0\}$ we conclude that there exists $\alpha > 0$ such that

$$\frac{1}{r_n^2}\Big| 1 - \prod_{i \in J(p)} \operatorname{sinc}(p_i r_n) \Big| \geq \alpha.$$

This implies that

$$|c_{p,n}|^2 \geq |a_{p,n}|^2 \alpha^2.$$

By Parseval's identity, and since $f_n \in \Pi(\mathbb{T}^m, \mathbb{L}^m)$ we have $a_{0,n} = 0$,

$$\| -\widetilde{\Delta}_{r_n} f_n\|^2_{L^2(\mathbb{T}^m)} = \sum_{p\in\mathbb{Z}^m} |c_{p,n}|^2 \geq \sum_{p\in\mathbb{Z}^m\setminus\{0\}} |a_{p,n}|^2\alpha^2 = \alpha^2\|f_n\|^2_{L^2(\mathbb{T}^m)},$$

in contradiction with $\| -\widetilde{\Delta}_{r_n} f_n\|_{L^2(\mathbb{T}^m)} \to 0$ provided by (23). ■

4.3. Shrinking hypercubes

For any $b > 0$, consider the metric measure space $\mathfrak{Q}^m(b) := ([0, b]^m, \mathsf{d}_\infty, \mathcal{L}^m)$. It trivially satisfies the assumptions of Theorem 1, so that for any small enough r,

$$\tilde{\lambda}_{1,r} = \lambda_1(-\widetilde{\Delta}_{r,\mathfrak{Q}^m(b)}) > 0. \tag{25}$$

Then the following holds.

Lemma 4.3. *We have*

$$\liminf_{b\to 0} \lim_{r\to 0} \lambda_1(-\widetilde{\Delta}_{r,\mathfrak{Q}^m(b)}) = +\infty. \tag{26}$$

Proof. We suppose by contradiction that (26) fails. Then there exist $b_n \to 0$ and $r_n \to 0$ such that

$$\lim_n \lambda_1(-\widetilde{\Delta}_{r_n,\mathfrak{Q}^m(b)}) < +\infty.$$

Since we are first taking the limit in r and then in b, we can assume $\bar{r}_n := r_n/b_n \to 0$. For any n, by a simple scaling argument we have

$$\lambda_1(-\widetilde{\Delta}_{r_n,\mathfrak{Q}^m(b_n)}) = \frac{1}{b_n^2}\lambda_1(-\widetilde{\Delta}_{\bar{r}_n,\mathfrak{Q}^m(1)})$$

thus

$$\lambda_1(-\widetilde{\Delta}_{\bar{r}_n,\mathfrak{Q}^m(1)}) \to 0.$$

From (25), assuming that each r_n is small enough, we know that there exists $f_n \in \Pi([0, 1]^m, \mathcal{L}^m)$ such that $\|f_n\|_{L^2([0,1]^m,\mathcal{L}^m)} = 1$ and

$$\widetilde{E}_{\bar{r}_n,\mathfrak{Q}^m(1)}(f_n) = \lambda_1(-\widetilde{\Delta}_{\bar{r}_n,\mathfrak{Q}^m(1)}).$$

Consider the continuous function

$$T : \mathbb{T}^m \to [0, 1]^m, \quad x \mapsto (|\bar{x}_1|, \dots, |\bar{x}_m|), ,$$

where $\bar{x} = \pi^{-1}(x) \in [-1, 1)^m$. For any n, set

$$\tilde{f}_n = f_n \circ T \in L^2(\mathbb{T}^m, \mathbb{L}^m).$$

From this we have that $f_n \in \Pi(\mathbb{T}^m, \mathbb{L}^m)$ Let us prove that

$$\|\tilde{f}_n\|_{L^2(\mathbb{T}^m,\mathbb{L}^m)} = 1.$$

Let $C_1, \ldots, C_{2^m}$ denote the 2^m sets of the form $I_1 \times \cdots \times I_m$ where each I_i is either $[-1, 0]$ or $[0, 1]$. For any $j \in \{1, \ldots, 2^m\}$, set

$$\begin{aligned} N_j \colon [0,1]^m &\to C_j, \\ (\xi_1, \ldots, \xi_m) &\mapsto (\varepsilon_i \xi_1, \ldots, \varepsilon_m \xi_m), \end{aligned}$$

where ε_i is 1 if $I_i = [0, 1]$ and -1 otherwise. Note that N_j is an isometry which preserves the Lebesgue measure, and that $T \circ \pi \circ N_j$ is equal to the identity. Then

$$\begin{aligned} \|\tilde{f}_n\|^2_{L^2(\mathbb{T}^m,\mathbb{L}^m)} = \int_{\mathbb{T}^m} (f_n \circ T)^2 \, \mathrm{d}\mathbb{L}^m &= \frac{1}{2^m} \int_{[-1,1]^m} (f_n \circ T \circ \pi)^2 \, \mathrm{d}\mathcal{L}^m \\ &= \frac{1}{2^m} \sum_{j=1}^{2^m} \int_{C_j} (f_n \circ T \circ \pi)^2 \, \mathrm{d}\mathcal{L}^m \\ &= \frac{1}{2^m} \sum_{j=1}^{2^m} \int_{[0,1]^m} (f_n \circ T \circ \pi \circ N_j)^2 \, \mathrm{d}\mathcal{L}^m \\ &= \|f_n\|^2_{L^2([0,1]^m,\mathcal{L}^m)} = 1. \end{aligned}$$

We claim that

$$\widetilde{E}_{\bar{r}_n,\mathfrak{T}^m}(\tilde{f}_n) \leq 2^m \widetilde{E}_{\bar{r}_n,\mathfrak{Q}^m(1)}(f_n). \tag{27}$$

Since $\tilde{f}_n \in \Pi(\mathbb{T}^m, \mathbb{L}^m)$, the latter provides a contradiction with Proposition 4.2, namely

$$0 < \lambda_1(-\widetilde{\Delta}_{\bar{r}_n,\mathfrak{T}^m}) \leq 2^m \lambda_1(-\widetilde{\Delta}_{\bar{r}_n,\mathfrak{Q}^m(1)}) \to 0.$$

Given $x \in \mathbb{T}^m$, define

$$\widetilde{G}_n(x) := \frac{1}{\bar{r}_n^m} \int_{\widetilde{Q}_{\bar{r}_n}(x)} \left(\frac{\tilde{f}_n(x) - \tilde{f}_n(y)}{\bar{r}_n} \right)^2 \mathrm{d}\mathbb{L}^m(y).$$

For any $x \in [0, 1]^m$, set

$$G_n(x) := \int_{\widetilde{Q}_{\bar{r}_n}(x)} \left(\frac{1}{V(x, \bar{r}_n)} + \frac{1}{V(y, \bar{r}_n)} \right) \left(\frac{f_n(x) - f_n(y)}{\bar{r}_n} \right)^2 \mathrm{d}\mathcal{L}^m(y),$$

where

$$V(z, r) := \mathcal{L}^m(Q_r(z) \cap [0, 1]^m)$$

for any $z \in [0,1]^m$. Then

$$4\widetilde{E}_{\bar{r}_n,\mathfrak{T}^m}(\tilde{f}_n) = \int_{\mathbb{T}^m} \widetilde{G}_n(x)\,\mathrm{d}\mathbb{L}^m(x)$$

$$4\widetilde{E}_{\bar{r}_n,\mathfrak{Q}^m(1)}(f_n) = \int_{[0,1]^m} G_n(x)\,\mathrm{d}\mathcal{L}^m(x).$$

Moreover, for all $x \in \mathbb{T}^m$,

$$\frac{1}{\bar{r}_n^m} \leq \frac{2^m}{\mathcal{L}^m(Q_{\bar{r}_n}(T(x)) \cap [0,1]^m)} = \frac{2^m}{V(T(x),\bar{r}_n)}. \tag{28}$$

For any $x \in \mathbb{T}^m$, there exists some $k \in \{1,\dots,2^m\}$ and $\bar{x} \in C_k$ such that $\pi(\bar{x}) = x$. We will now consider for each $j \in \{1,\dots,2^m\}$ the rectangle given by

$$R_{n,j}(x) := T(\pi(C_j) \cap \widetilde{Q}_{\bar{r}_n}(x)) \subset [0,1]^m.$$

and we point out that

$$R_{n,j}(x) \subset R_{n,k}(x) = [0,1]^m \cap Q_{\bar{r}_n}(T(x)) \tag{29}$$

for any $j \in \{1,\dots,2^m\}$. This follows since for each $i \in \{1,\dots,m\}$ we have 4 possibilities

(1) $|\bar{x}_i| < \bar{r}_n$

(2) $|1-\bar{x}_i| < \bar{r}_n$

(3) $|-1-\bar{x}_i| < \bar{r}_n$

(4) $\neg((1) \vee (2) \vee (3))$.

If $\pi_i : \mathbb{R}^m \to \mathbb{R}$ is the projection in the i-th coordinate, we conclude

$$\pi_i(R_{n,k}(x)) = \begin{cases} [0, \bar{r}_n + |\bar{x}_i|] & \text{if (1)}, \\ [-\bar{r}_n + |\bar{x}_i|, 1] & \text{if (2)}, \\ [-\bar{r}_n + |\bar{x}_i|, 1] & \text{if (3)}, \\ [-\bar{r}_n, \bar{r}_n] & \text{if (4)}, \end{cases}$$

and

$$\pi_l(R_{n,k}(x)) = \begin{cases} \pi_i(R_{n,l}(x)) = \pi_i(R_{n,k}(x)) & \text{if } \pi_i(C_j) = \pi_i(C_k), \\ [0, \bar{r}_n - |\bar{x}_i|] & \text{if (1) and } \neg(\pi_i(C_j) = \pi_i(C_k)), \\ [2 - \bar{r}_n - |\bar{x}_i|, 1] & \text{if (2) and } \neg(\pi_i(C_j) = \pi_i(C_k)), \\ [2 - \bar{r}_n - |\bar{x}_i|, 1] & \text{if (3) and } \neg(\pi_i(C_j) = \pi_i(C_k)), \\ \emptyset & \text{if (4) and } \neg(\pi_i(C_j) = \pi_i(C_k)). \end{cases}$$

Thus, we conclude that for all $i \in \{1, \ldots, m\}$ and $l \in \{1, \ldots, 2^m\}$ we have

$$\pi_i(R_{n,l}(x)) \subset \pi_i(R_{n,k}(x)),$$

and since these sets are rectangles, we conclude equation (29).

From (28), we can deduce

$$\begin{aligned}
\widetilde{G}_n(x) &= \frac{1}{\bar{r}_n^m} \int_{\widetilde{Q}_{\bar{r}_n}(x)} \Big(\frac{\tilde{f}_n(x) - \tilde{f}_n(y)}{\bar{r}_n}\Big)^2 \, \mathrm{d}\mathbb{L}^m(y) \\
&\leq \int_{\widetilde{Q}_{\bar{r}_n}(x)} \Big(\frac{2^m}{V(T(x), \bar{r}_n)} + \frac{2^m}{V(T(y), \bar{r}_n)}\Big)\Big(\frac{f_n(T(x)) - f_n(T(y))}{\bar{r}_n}\Big)^2 \, \mathrm{d}\mathbb{L}^m(y) \\
&= \sum_{j=1}^{2^m} \int_{\widetilde{Q}_{\bar{r}_n}(x) \cap \pi(C_j)} \Big(\frac{2^m}{V(T(x), \bar{r}_n)} + \frac{2^m}{V(T(y), \bar{r}_n)}\Big) \\
&\qquad\qquad \times \Big(\frac{f_n(T(x)) - f_n(T(y))}{\bar{r}_n}\Big)^2 \, \mathrm{d}\mathbb{L}^m(y) \\
&= \frac{1}{2^m} \sum_{j=1}^{2^m} \int_{\pi^{-1}(\widetilde{Q}_{\bar{r}_n}(x)) \cap C_j} \Big(\frac{2^m}{V(T(x), \bar{r}_n)} + \frac{2^m}{V(T(\pi(y)), \bar{r}_n)}\Big) \\
&\qquad\qquad \times \Big(\frac{f_n(T(x)) - f_n(T(\pi(y)))}{\bar{r}_n}\Big)^2 \, \mathrm{d}\mathcal{L}^m(y).
\end{aligned}$$

For each integral, change coordinates by N_j to conclude

$$\begin{aligned}
\widetilde{G}_n(x) \leq \frac{1}{2^m} \sum_{j=1}^{2^m} \int_{N_j^{-1}(\pi^{-1}(\widetilde{Q}_{\bar{r}_n}(x)) \cap C_j)} &\Big(\frac{2^m}{V(T(x), \bar{r}_n)} + \frac{2^m}{V(T(\pi(N_j(y))), \bar{r}_n)}\Big) \\
&\times \Big(\frac{f_n(T(x)) - f_n(T(\pi(N_j(y))))}{\bar{r}_n}\Big)^2 \, \mathrm{d}\mathcal{L}^m(y).
\end{aligned}$$

We have that

$$N_j^{-1}(\pi^{-1}(\widetilde{Q}_{\bar{r}_n}(x)) \cap C_j) = R_{n,j}(x),$$

so by equation (29) and the fact that $T \circ \pi \circ N_j = id$, we conclude

$$\begin{aligned}
\widetilde{G}_n(x) \leq \frac{1}{2^m} \sum_{j=1}^{2^m} \int_{R_{n,j}(x)} &\Big(\frac{2^m}{V(T(x), \bar{r}_n)} + \frac{2^m}{V(T(y), \bar{r}_n)}\Big) \\
&\times \Big(\frac{f_n(T(x)) - f_n(y)}{\bar{r}_n}\Big)^2 \, \mathrm{d}\mathcal{L}^m(y)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2^m}\sum_{j=1}^{2^m}\int_{R_{n,k}(x)}\Big(\frac{2^m}{V(T(x),\bar r_n)}+\frac{2^m}{V(y,\bar r_n)}\Big)\\
&\qquad\qquad\qquad\qquad\times\Big(\frac{f_n(T(x))-f_n(y)}{\bar r_n}\Big)^2\,\mathrm{d}\mathcal{L}^m(y)\\
&=\int_{[0,1]^m\cap Q_{\bar r_n}(T(x))}\Big(\frac{2^m}{V(T(x),\bar r_n)}+\frac{2^m}{V(y,\bar r_n)}\Big)\Big(\frac{f_n(T(x))-f_n(y)}{\bar r_n}\Big)^2\,\mathrm{d}\mathcal{L}^m(y)\\
&=2^m G_n(T(x)).
\end{aligned}
$$

Now, we integrate both sides in $\mathbb{T}^m$ and change variables by π and N_j to conclude

$$
\begin{aligned}
4\widetilde{E}_{\bar r_n,\mathfrak{T}^m}(\tilde f_n) &= \int_{\mathbb{T}^m}\widetilde{G}_n(x)\,\mathrm{d}\mathbb{L}^m(x)\leq 2^m\int_{\mathbb{T}^m}G_n(T(x))\,\mathrm{d}\mathbb{L}^m(x)\\
&=\int_{[-1,1)^m}G_n(T(\pi(x)))\,\mathrm{d}\mathcal{L}^m(x)=\sum_{j=1}^{2^m}\int_{C_j}G_n(T(\pi(x)))\,\mathrm{d}\mathcal{L}^m(x)\\
&=\sum_{j=1}^{2^m}\int_{[0,1]^m}G_n(T(\pi(N_j(x))))\,\mathrm{d}\mathcal{L}^m(x)=\sum_{j=1}^{2^m}\int_{[0,1]^m}G_n(x)\,\mathrm{d}\mathcal{L}^m(x)\\
&=2^m 4\widetilde{E}_{\bar r_n,[0,1]^m}(f_n)=2^m4\lambda_1(-\widetilde{\Delta}_{\bar r_n,\mathfrak{Q}^m(1)}).
\end{aligned}
$$

With this we obtain (27). ∎

5. L^2 convergence

In this section, we prove Theorem 2. Let M be a smooth, compact, connected manifold of dimension $m\geq 2$. Assume that M is endowed with a smooth Riemannian metric g and let d_g and vol_g be the associated Riemannian distance and volume measure on M.

In this smooth context, the function $V(\cdot,r)$ is obviously continuous for any $r>0$. Since M is compact, this implies that the metric measure space $(M,\mathsf{d}_g,\mathrm{vol}_g)$ satisfies (II_r) and (I_r). Then Lemma 3.3 applies and ensures that $\widetilde{\Delta}_r$ is a bounded self-adjoint operator acting on $L^2(M,\mathrm{vol}_g)$. The compactness of M also ensures that $(M,\mathsf{d}_g,\mathrm{vol}_g)$ is locally Ahlfors regular: there exists a constant $C>1$ such that for any $x\in M$ and $r\in(0,\mathrm{diam}(M)]$,

$$
C^{-1}r^m\leq V(x,r)\leq Cr^m. \tag{30}
$$

Note that this condition trivially implies a uniform local doubling property for $(M,\mathsf{d}_g,\mathrm{vol}_g)$.

5.1. Convergence in the sense of distributions

Recall that C_m is defined in (2). For any $x \in M \backslash \partial M$, we let $\exp_x$ be the exponential map centered at x. We identify $T_x M$ with $\mathbb{R}^m$ and write $\mathbb{B}_r^m(v)$ for the Euclidean ball in $\mathbb{R}^m$ centered at v with radius $r > 0$. Then there exists $\delta > 0$ such that the restriction of $\exp_x$ to $\mathbb{B}_\delta^m(0)$ is a diffeomorphism onto its image; recall that the injectivity radius $\mathrm{i}_M(x)$ of M at x is the supremum of the set of such numbers δ. We let J_x be the Radon-Nikodym derivative of the measure $(\exp_x^{-1})_\# \mathrm{vol}_g$ with respect to the Lebesgue measure $\mathscr{L}^m$. It is well known that for any $\xi \in \mathbb{B}_{\mathrm{i}_M}^m(0)$,

$$J_x(\xi) = 1 + \underline{O}_K(|\xi|^2),$$

where for any $h > 0$, the notation $\underline{O}_K(h)$ stands for a quantity independent on $x \in K$ whose absolute value divided by h is bounded. Here K is a compact subset of M. We write $\underline{O}$ instead of $\underline{O}_M$. Then the following holds.

Proposition 5.1. *Consider $f, \psi \in C^2(M)$. Then*

$$\lim_{r \to 0} \langle -\widetilde{\Delta}_r f, \psi \rangle_2 = C_m \int_M \langle \mathrm{d} f, \mathrm{d}\psi \rangle_g \, \mathrm{dvol}_g. \tag{31}$$

Proof. For any $x \in M$ and $r > 0$, set

$$\tilde{e}_r(f, \psi; x) := \frac{1}{4} \fint_{B_r(x)} \left(1 + \frac{V(x,r)}{V(y,r)}\right) \frac{(f(x) - f(y))(\psi(x) - \psi(y))}{r^2} \, \mathrm{dvol}_g(y)$$

so that

$$\langle -\widetilde{\Delta}_r f, \psi \rangle_2 = \int_M \tilde{e}_r(f, \psi; y) \, \mathrm{dvol}_g(y).$$

On one hand,

$$\begin{aligned} |\tilde{e}_r(f, \psi; x)| &\leq \frac{1}{4} \fint_{B_r(x)} \left(1 + \frac{V(x,r)}{V(y,r)}\right) \frac{|f(x) - f(y)||\psi(x) - \psi(y)|}{r^2} \, \mathrm{dvol}_g(y) \\ &\leq \frac{1}{4} \fint_{B_r(x)} \left(1 + \frac{V(x,r)}{V(y,r)}\right) \frac{\mathrm{Lip}(f) \, \mathrm{Lip}(\psi) \mathrm{d}_g^2(x,y)}{r^2} \, \mathrm{dvol}_g(y) \\ &\leq \frac{\mathrm{Lip}(f) \, \mathrm{Lip}(\psi)}{4} \fint_{B_r(x)} \left(1 + \frac{V(x,r)}{V(y,r)}\right) \mathrm{dvol}_g(y). \end{aligned}$$

By (30), we obtain

$$|\tilde{e}_r(f, \psi; x)| \leq \frac{\mathrm{Lip}(f) \, \mathrm{Lip}(\psi)(1 + C^2)}{4}. \tag{32}$$

On the other hand, assume that r is smaller than $\mathrm{i}_M(x)$, and consider $\tilde{f} := f \circ \exp_x$ and $\tilde{\psi} := \psi \circ \exp_x$ on $\mathbb{B}^m_r(0)$. The first-order Taylor expansion of $\tilde{f}$ and $\tilde{\psi}$ yields

$$\tilde{f}(\xi) = \tilde{f}(0) + (\mathrm{d}\tilde{f})_0(\xi) + \underline{O}_{\{0\}}(|\xi|^2),$$
$$\tilde{\psi}(\xi) = \tilde{\psi}(0) + (\mathrm{d}\tilde{\psi})_0(\xi) + \underline{O}_{\{0\}}(|\xi|^2).$$

Then

$$\begin{aligned}
&\int_{B_r(x)} (f(x) - f(y))(\psi(x) - \psi(y))\,\mathrm{dvol}_g(y) \\
&\quad = \int_{\mathbb{B}^m_r(0)} (\tilde{f}(0) - \tilde{f}(\xi))(\tilde{\psi}(0) - \tilde{\psi}(\xi))\,J(\xi)\,\mathrm{d}\mathcal{L}^m(\xi) \\
&\quad = \int_{\mathbb{B}^m_r(0)} ((\mathrm{d}\tilde{f})_0(\xi) + \underline{O}_{\{0\}}(r^2))((\mathrm{d}\tilde{\psi})_0(\xi) + \underline{O}_{\{0\}}(r^2))(1 + \underline{O}_{\{0\}})\,\mathrm{d}\mathcal{L}^m(\xi) \\
&\quad = \sum_{i,j=1}^m [(\mathrm{d}\tilde{f})_0]_i [(\mathrm{d}\tilde{\psi})_0]_j \int_{\mathbb{B}^m_r(0)} \xi_j \xi_i \,\mathrm{d}\mathcal{L}^m(\xi) + \underline{O}_{\{0\}}(r^{m+3}) \\
&\quad = (\mathrm{d}\tilde{f})_0 \cdot (\mathrm{d}\tilde{\psi})_0 \int_{\mathbb{B}^m_r(0)} \xi_1^2 \,\mathrm{d}\mathcal{L}^m(\xi) + \underline{O}_{\{0\}}(r^{m+3}).
\end{aligned}$$

Moreover, it is known that (see e.g., [15, Remark 2.11])

$$1 + \frac{V(x,r)}{V(y,r)} = 2 + \underline{O}_{\{x\}}(r^2)$$

and since $V(x,r)/\mathcal{L}^m(\mathbb{B}^m_r(0))) \to 1$ as $r \downarrow 0$, we obtain

$$\begin{aligned}
&\fint_{B_r(x)} \Big(1 + \frac{V(x,r)}{V(y,r)}\Big)(f(x) - f(y))(\psi(x) - \psi(y))\,\mathrm{dvol}_g(y) \\
&\quad = \frac{\mathcal{L}^m(\mathbb{B}^m_r(0))(2 + \underline{O}_{\{x\}}(r^2))}{V(x,r)} \fint_{\mathbb{B}^m_r(0)} (\tilde{f}(0) - \tilde{f}(\xi))(\tilde{\psi}(0) - \tilde{\psi}(\xi))\,J(\xi)\,\mathrm{d}\mathcal{L}^m(\xi) \\
&\quad = (2 + \underline{O}_{\{x\}}(r^2))\bigg((\mathrm{d}\tilde{f})_0 \cdot (\mathrm{d}\tilde{\psi})_0 \fint_{\mathbb{B}^m_r(0)} \xi_1^2 \,\mathrm{d}\mathcal{L}^m(\xi) + \underline{O}_{\{0\}}(r^3)\bigg).
\end{aligned}$$

Since $(\mathrm{d}\tilde{f})_0 \cdot (\mathrm{d}\tilde{\psi})_0 = \langle \mathrm{d}f, \mathrm{d}\psi \rangle_g(x)$ and

$$\fint_{\mathbb{B}^m_r(0)} \xi_1^2 \,\mathrm{d}\mathcal{L}^m(\xi) = 2r^2 C_m$$

by change of variable $\xi \leftrightarrow \eta/r^2$, we eventually obtain that

$$\tilde{e}_r(f,\psi;x) = C_m\langle \mathrm{d}f, \mathrm{d}\psi\rangle_g(x) + \underline{O}_{\{x\}}(r) \quad \text{as } r \to 0.$$

By (32) and the compactness of M, we can apply the dominated convergence theorem to the functions $\tilde{e}_r(f,\psi;\cdot)$. Then we get (31). ■

Using integration by parts in (31), we immediately obtain the following.

Corollary 5.2. *Let ∂g be the Riemannian metric induced by g on ∂M. For any $f \in \mathcal{C}^\infty(M)$, the following convergence holds in the sense of distributions as $r \downarrow 0$:*

$$(\widetilde{\Delta}_r f)\mathrm{vol}_g \to C_m((\Delta_g f)\mathrm{vol}_g + (\partial_\nu^g f)\mathrm{vol}_{\partial g}).$$

5.2. Pointwise convergence

We aim to prove Theorem 2 in a similar way as Proposition 5.1, that is to say, by means of the dominated convergence theorem. To this aim, we first establish that pointwise convergence holds vol_g-a.e. on M. We recall that ∂M is a vol_g-negligible subset of M.

Proposition 5.3. *Let $f \in C^\infty(M)$. Then for any $x \in M - \partial M$,*

$$\lim_{r\to 0} \widetilde{\Delta}_r f(x) = C_m \Delta_g f(x).$$

Moreover, the convergence is uniform on any compact subset of $M - \partial M$.

Proof. Let K be a compact subset of $M - \partial M$. Consider $x \in K$ and $r \in (0, \mathrm{i}_M(x))$. Set $\tilde{f}_x := f \circ \exp_x$. Acting like in the proof of Proposition 5.1, we get

$$\widetilde{\Delta}_r f(x) = \frac{(2 + \underline{O}_K(r^2))}{2r^2} \fint_{\mathbb{B}_r^m(0)} (\tilde{f}_x(\xi) - \tilde{f}_x(0))\, \mathrm{d}\mathcal{L}^m(\xi).$$

The second-order Taylor expansion of $\tilde{f}_x$ yields

$$\tilde{f}_x(\xi) = \tilde{f}_x(0) + (\mathrm{d}\tilde{f}_x)_0(\xi) + \frac{1}{2}(\mathrm{d}^{(2)}\tilde{f}_x)_0(\xi,\xi) + \underline{O}_K(|\xi|^3)$$

hence we get

$$\fint_{\mathbb{B}_r^m(0)} \tilde{f}_x(\xi) - \tilde{f}_x(0)\, \mathrm{d}\mathcal{L}^m(\xi)$$
$$= \fint_{\mathbb{B}_r^m(0)} (\mathrm{d}\tilde{f}_x)_0(\xi)\, \mathrm{d}\mathcal{L}^m(\xi) + \frac{1}{2}\fint_{\mathbb{B}_r^m(0)} (\mathrm{d}^{(2)}\tilde{f}_x)_0(\xi,\xi)\, \mathrm{d}\mathcal{L}^m(\xi) + \underline{O}_K(r^3).$$

The first term vanishes by symmetry. The second term is equal to

$$\frac{1}{2}\Delta \tilde{f}_x(0) \fint_{\mathbb{B}^m_r(0)} \xi_1^2 \,\mathrm{d}\mathcal{L}^m(\xi) = \Delta_g f(x) r^2 C_m.$$

In the end we get

$$\begin{aligned}\widetilde{\Delta}_r f(x) &= \frac{2 + \underline{O}_K(r^2)}{2r^2}(\Delta_g f(x) r^2 C_m + \underline{O}_K(r^3)) \\ &= (1 + \underline{O}_K(r^2))(C_m \Delta_g f(x) + \underline{O}_K(r))\end{aligned}$$

from which follows the desired result, by letting $r \downarrow 0$. ∎

5.3. Uniform bound

We wish now to provide a uniform L^∞ bound for the functions $-\widetilde{\Delta}_r \psi$, where r is in a neighborhood of zero.

Let us first consider the case $\partial M = \emptyset$. From Proposition 5.3, we have uniform convergence

$$\|\widetilde{\Delta}_r f - C_m \Delta_g f\|_\infty \to 0$$

so that

$$\|\widetilde{\Delta}_r f - C_m \Delta_g f\|_2 \leq \|\widetilde{\Delta}_r f - C_m \Delta_g f\|_\infty \mathrm{vol}_g(M) \to 0.$$

Let us now deal with the case $\partial M \neq \emptyset$.

Proposition 5.4. *Assume that $\partial M \neq \emptyset$. Consider $f \in C^\infty_\nu(M)$. Then there exists $r_0 > 0$ such that*

$$\sup_{0<r<r_0} \|\widetilde{\Delta}_r f\|_{L^\infty(M)} < +\infty.$$

Proof. Since ∂M is compact, we can find a finite collection of smooth parameterizations $\psi_i \colon (-4, 4)^{m-1} \to \partial M$ such that

$$\partial M = \bigcup_i \psi_i([-1, 1]^{m-1}). \tag{33}$$

Step 1. We work with any of the previous ψ_i which we denote by ψ. For any $x \in \partial M$, let $\nu_x \in T_x M$ be the unit inner normal vector of ∂M at x. Since ∂M is smooth, there exists $\varepsilon > 0$ such that the map $E \colon \partial M \times [0, \varepsilon] \to M$ given by

$$E(x, t) = \exp^M_x(t\nu_x)$$

is an embedding, and there exists a smooth family of metrics $\{g_t\}_{t\in[0,\varepsilon]}$ on ∂M such that for any $(x, t) \in \partial M \times [0, \varepsilon]$,

$$(E^* g)_{(x,t)} = (g_t \oplus d\tau^2)_{(x,t)}. \tag{34}$$

Pulling back each metric g_t by ψ we have, for any $\xi \in (-4,4)^{m-1}$ and $v, w \in \mathbb{R}^{m-1}$,

$$(\psi^* g_t)_\xi(v,w) = v^T \cdot A_{(\xi,t)} \cdot w, \tag{35}$$

for some positive definite, symmetric $(m-1)$-square matrix $A_{(\xi,t)}$. From the non-degeneracy of the metric and a Lipschitz bound, we have that there exist $C, \tilde{c} > 0$ such that for all $t, s \in [0,\varepsilon]$, $\xi \in [-3,3]^{m-1}$, $v \in \mathbb{R}^m$ we have

$$[(\psi^* g_t) \oplus d\tau^2]_{(\xi,t)}(v,v) \geq \tilde{c}|v|^2, \tag{36}$$

$$|[(\psi^* g_t) \oplus d\tau^2]_{(\xi,t)}(v,v) - [(\psi^* g_s) \oplus d\tau^2]_{(\xi,t)}(v,v)| \leq C|v|^2|t-s|. \tag{37}$$

Claim 1. *There exists $K > 0$ such that for any $t, s \in [0,\varepsilon]$, $\xi \in [-3,3]^{m-1}$, $v \in \mathbb{R}^m$ such that*

$$|O(\xi,t,s,v)| \leq K|v| \cdot |t-s|, \tag{38}$$

where

$$O(\xi,t,s,v) = [(\psi^* g_t) \oplus d\tau^2]^{1/2}_{(\xi,t)}(v,v) - [(\psi^* g_s) \oplus d\tau^2]^{1/2}_{(\xi,t)}(v,v). \tag{39}$$

Proof. Consider $\tilde{c} > 0$ given by (36). We know that there exists $M > 0$ such that the map $\sqrt{\cdot}\colon [\tilde{c}, +\infty) \to \mathbb{R}$ is M-Lipschitz. By homogeneity in $|v|$ of (38), we can assume that $|v| = 1$. Then the Lipschitz condition and (37) yield

$$\begin{aligned}
|O(\xi,t,s,v)| &= |[(\psi^* g_t) \oplus d\tau^2]^{1/2}_{(\xi,t)}(v,v) - [(\psi^* g_s) \oplus d\tau^2]^{1/2}_{(\xi,t)}(v,v)| \\
&\leq M|[(\psi^* g_t) \oplus d\tau^2]_{(\xi,t)}(v,v) - [(\psi^* g_s) \oplus d\tau^2]_{(\xi,t)}(v,v)| \\
&\leq MC|v|^2|t-s| = MC|t-s|. \qquad \blacksquare
\end{aligned}$$

Step 2. Let $T(\varepsilon) := E(\partial M \times [0,\varepsilon])$ be the ε tubular neighborhood of the boundary. Then E is a diffeomorphism between $\partial M \times [0,\varepsilon]$ and $T(\varepsilon)$. For fixed $s \in [0,\varepsilon]$, consider the product metric $g_s \oplus d\tau^2$ in $\partial M \times [0,\varepsilon]$ and define the metric in $T(\varepsilon)$

$$\eta_s := (E^{-1})^*(g_s \oplus d\tau^2).$$

Let d_s be the distance induced from this metric. Consider the map $\Phi\colon (-4,4)^{m-1} \times [0,\varepsilon] \to M$ given by

$$\Phi(\xi,t) = E(\psi(\xi),t) \tag{40}$$

and note that (34) implies that for any $(\xi,t) \in (-4,4)^{m-1} \times [0,\varepsilon]$,

$$(\Phi^* g)_{(\xi,t)} = (\psi^* g_t \oplus \mathrm{d}\tau^2)_{(\xi,t)}. \tag{41}$$

Set $\mathbb{H}^m := \{v \in \mathbb{R}^m : v_m \geq 0\}$. Let $L(\tilde{\gamma})$ be the Euclidean length of a curve $\tilde{\gamma}\colon [0,1] \to \mathbb{R}^m$.

Claim 2. *Let $\tilde{c} > 0$ be given by (36). For any $r > 0$ and $s \in [0, \varepsilon]$, if a couple $(\xi, t) \in [-2, 2]^{m-1} \times [0, \varepsilon]$ is such that $\mathbb{B}^m_{r/\sqrt{\tilde{c}}}(\xi, t) \cap \mathbb{H}^m \subset [-3, 3]^{m-1} \times [0, \varepsilon]$, then the following holds for any $y \in M$.*

(1) *If $\mathsf{d}(\Phi(\xi, t), y) < r$, then the image of any d-minimizing geodesic $\gamma\colon [0, 1] \to M$ is contained in $\Phi(\mathbb{B}^m_{r/\sqrt{\tilde{c}}}(\xi, t) \cap \mathbb{H}^m)$ and $\tilde{\gamma} = \Phi^{-1} \circ \gamma$ satisfies $L(\tilde{\gamma}) < r/\sqrt{\tilde{c}}$.*

(2) *If $\mathsf{d}_s(\Phi(\xi,t), y) < r$, then the image of any d_s-minimizing geodesic $\gamma\colon [0,1] \to M$ is contained in $\Phi(\mathbb{B}^m_{r/\sqrt{\tilde{c}}}(\xi, t) \cap \mathbb{H}^m)$ and $\tilde{\gamma} = \Phi^{-1} \circ \gamma$ satisfies $L(\tilde{\gamma}) < r/\sqrt{\tilde{c}}$.*

Proof. We prove the first result only since the proof of the second one follows from similar lines. Consider a d-minimizing geodesic $\gamma\colon [0, 1] \to M$ from $\Phi(\xi, t)$ to y. Set

$$\delta := \sup\{t \in [0, 1] : \gamma(s) \in \Phi(\mathbb{B}^m_{r/\sqrt{c}}(\xi, t) \cap \mathbb{H}^m) \text{ for any } s \in [0, t)\},$$

$$\tilde{\gamma} := \Phi^{-1} \circ \gamma|_{[0,\delta]},$$

and observe that $\gamma([0, 1]) \subset \Phi(\mathbb{B}^m_{r/\sqrt{c}}(\xi, t) \cap \mathbb{H}^m)$ if and only if $\delta = 1$. We claim that

$$L(\tilde{\gamma}) \leq \frac{\mathsf{d}_g(\Phi(\xi, t), y)}{\sqrt{\tilde{c}}}. \tag{42}$$

Indeed, setting $(\tilde{\alpha}, \tilde{\gamma}_m) := \tilde{\gamma}$, where $\tilde{\alpha}\colon [0, \delta] \to [-3, 3]^{m-1}$ and $\tilde{\gamma}_m\colon [0, \delta] \to [0, \varepsilon]$, we have

$$\begin{aligned}
\mathsf{d}(\Phi(\xi, t), y) &= \int_0^1 g^{1/2}_{\gamma(w)}(\dot{\gamma}(w), \dot{\gamma}(w))\,\mathrm{d}w \geq \int_0^\delta g^{1/2}_{\gamma(w)}(\dot{\gamma}(w), \dot{\gamma}(w))\,\mathrm{d}w \\
&= \int_0^\delta (\Phi^* g)^{1/2}_{\tilde{\gamma}(w)}(\dot{\tilde{\gamma}}(w), \dot{\tilde{\gamma}}(w))\,\mathrm{d}w \\
&= \int_0^\delta (\psi^* g_{\tilde{\gamma}_m(w)} \oplus d\tau^2)^{1/2}_{\tilde{\gamma}(w)}(\dot{\tilde{\gamma}}(w), \dot{\tilde{\gamma}}(w))\,\mathrm{d}w && \text{(by (41))} \\
&\geq \int_0^\delta \sqrt{\tilde{c}}|\dot{\tilde{\gamma}}(w)|\,\mathrm{d}w && \text{(by (36))} \\
&= \sqrt{\tilde{c}}L(\tilde{\gamma}).
\end{aligned}$$

Now, we claim that

$$\delta < 1 \implies \mathsf{d}_g(\Phi(\xi, t), y) \geq r. \tag{43}$$

Indeed, if $\delta < 1$, since $\tilde{\gamma}(0) = (\xi, t)$ and $\tilde{\gamma}(\delta) \in \partial\mathbb{B}^m_{r/\sqrt{\tilde{c}}}(\xi, t) \cap \mathbb{H}^m$, then

$$L(\tilde{\gamma}) \geq \frac{r}{\sqrt{\tilde{c}}}$$

and we get $\mathsf{d}_g(\Phi(\xi,t),y) \geq r$ from (42). Therefore, if $\mathsf{d}_g(\Phi(\xi,t),y) < r$, then (43) implies that $\delta = 1$ which means $\gamma([0,1])$ is included in $\Phi(\mathbb{B}^m_{r/\sqrt{c}}(\xi,t) \cap \mathbb{H}^m)$, and (42) yields $L(\tilde{\gamma}) < r/\sqrt{\tilde{c}}$ as desired. ∎

Claim 3. *There exists $K > 0$ such that for all $(\xi,t),(\eta,s) \in [-2,2]^{m-1} \times [0,\varepsilon]$ and $r > 0$ such that $\mathbb{B}^m_{r/\sqrt{\tilde{c}}}(\xi,t) \cap \mathbb{H}^m \subset [-3,3]^{m-1} \times [0,\varepsilon]$:*

$$\mathsf{d}(\Phi(\xi,t),\Phi(\eta,s)) < r \implies \mathsf{d}_t(\Phi(\xi,t),\Phi(\eta,s)) < r + Kr^2, \tag{44}$$

$$\mathsf{d}(\Phi(\xi,t),\Phi(\eta,s)) \geq r \implies \mathsf{d}_t(\Phi(\xi,t),\Phi(\eta,s)) \geq r - Kr^2. \tag{45}$$

Proof. Suppose that $\mathsf{d}(\Phi(\xi,t),\Phi(\eta,s)) < r$. Let $\gamma\colon [0,1] \to M$ be a d-minimizing geodesic between $\Phi(\xi,t)$ and $\Phi(\eta,s)$. Then by Claim 2, we have that $\gamma([0,1]) \subset \Phi(\mathbb{B}^m_{r/\sqrt{\tilde{c}}}(\xi,t) \cap \mathbb{H}^m)$, and by defining $(\tilde{\alpha},\tilde{\gamma}_m) = \tilde{\gamma} := \Phi^{-1} \circ \gamma$, we have that $L(\tilde{\gamma}) < r/\sqrt{\tilde{c}}$. Thus, we obtain

$$\begin{aligned}
\mathsf{d}(\Phi(\xi,t),\Phi(\eta,s)) &= \int_0^1 g^{1/2}_{\gamma(w)}(\dot{\gamma}(w),\dot{\gamma}(w))\,\mathrm{d}w \\
&= \int_0^1 g^{1/2}_{\gamma(w)}(\dot{\gamma}(w),\dot{\gamma}(w))\,\mathrm{d}w - \int_0^1 (\eta_t)^{1/2}_{\gamma(w)}(\dot{\gamma}(w),\dot{\gamma}(w))\,\mathrm{d}w \\
&\quad + \int_0^1 (\eta_t)^{1/2}_{\gamma(w)}(\dot{\gamma}(w),\dot{\gamma}(w))\,\mathrm{d}w \\
&= \int_0^1 (\psi^* g_{\tilde{\gamma}_m(w)} \oplus d\tau^2)^{1/2}_{(\tilde{\alpha}(w),\tilde{\gamma}_m(w))}(\dot{\tilde{\gamma}}(w),\dot{\tilde{\gamma}}(w))\,\mathrm{d}w \\
&\quad - \int_0^1 (\psi^* g_t \oplus d\tau^2)^{1/2}_{(\tilde{\alpha}(w),\tilde{\gamma}_m(w))}(\dot{\tilde{\gamma}}(w),\dot{\tilde{\gamma}}(w))\,\mathrm{d}w \\
&\quad + \int_0^1 (\eta_t)^{1/2}_{\gamma(w)}(\dot{\gamma}(w),\dot{\gamma}(w))\,\mathrm{d}w \\
&= \int_0^1 O(\tilde{\alpha}(w),\tilde{\gamma}_m(w),t,\dot{\gamma}(w))\,\mathrm{d}w \\
&\quad + \int_0^1 (\eta_t)^{1/2}_{\gamma(w)}(\dot{\gamma}(w),\dot{\gamma}(w))\,\mathrm{d}w,
\end{aligned} \tag{46}$$

where we use (39) to get the last equality. By Claim 1, we have that

$$\left|\int_0^1 O(\tilde{\alpha}(w), \tilde{\gamma}_m(w), t, \dot{\gamma}(w))\,\mathrm{d}w\right| \leq \int_0^1 |O(\tilde{\alpha}(w), \tilde{\gamma}_m(w), t, \dot{\gamma}(w))|\,\mathrm{d}w$$
$$\leq \int_0^1 K|\dot{\tilde{\gamma}}(w)||t - \tilde{\gamma}_m(w)|\,\mathrm{d}w.$$

By Claim 2, we have that $|t - \tilde{\gamma}_m(s)| < r/\sqrt{\tilde{c}}$ and $L(\tilde{\gamma}) < r/\sqrt{\tilde{c}}$, hence we get

$$\left|\int_0^1 O(\tilde{\alpha}(w), \tilde{\gamma}_m(w), t, \dot{\gamma}(w))\,\mathrm{d}w\right| \leq K\frac{r}{\sqrt{\tilde{c}}}\int_0^1 |\dot{\tilde{\gamma}}(w)|\,\mathrm{d}w < K\frac{r^2}{\tilde{c}}. \tag{47}$$

Thus,

$$\begin{aligned}
\mathsf{d}_t(\Phi(\xi,t), \Phi(\eta,s)) &\leq \int_0^1 (\eta_t)^{1/2}_{\gamma(w)}(\dot{\gamma}(w), \dot{\gamma}(w))\,\mathrm{d}w \\
&\leq \mathsf{d}(\Phi(\xi,t), \Phi(\eta,s)) + \left|\int_0^1 O(\tilde{\alpha}(w), \tilde{\gamma}_m(w), t, \dot{\gamma}(w))\,\mathrm{d}w\right| \\
&\leq \mathsf{d}(\Phi(\xi,t), \Phi(\eta,s)) + K\frac{r^2}{\tilde{c}} \\
&< r + K\frac{r^2}{\tilde{c}},
\end{aligned}$$

where we use (46) to get the second inequality and (47) to get the third one. This proves (44).

To prove (45), we may assume $\mathsf{d}(\Phi(\xi,t), \Phi(\eta,s)) \geq r$ and $\mathsf{d}_t(\Phi(\xi,t), \Phi(\eta,s)) < r$. Let $\gamma\colon [0,1] \to M$ be a geodesic in the metric $g_t \oplus d\tau^2$. By Claim 2, we have $\gamma([0,1]) \subset \Phi(\mathbb{B}^m_{r/\sqrt{\tilde{c}}}(\xi,t) \cap \mathbb{H}^m)$ with $(\tilde{\alpha}, \tilde{\gamma}_m) = \tilde{\gamma} = \Phi^{-1} \circ \gamma$ satisfying $L(\tilde{\gamma}) < r/\sqrt{\tilde{c}}$. The same estimates as before are satisfied, hence we conclude

$$\begin{aligned}
r < \mathsf{d}(\Phi(\xi,t), \Phi(\eta,s)) &\leq \int_0^1 g^{1/2}_{\gamma(w)}(\dot{\gamma}(w), \dot{\gamma}(w))\,\mathrm{d}w \\
&\leq \int_0^1 (\eta_t)^{1/2}_{\gamma(w)}(\dot{\gamma}(w), \dot{\gamma}(w))\,\mathrm{d}w + \left|\int_0^1 O(\tilde{\alpha}(w), \tilde{\gamma}_m(w), t, \dot{\tilde{\gamma}}(w))\,\mathrm{d}w\right| \\
&\leq \mathsf{d}_t(\Phi(\xi,t), \Phi(\eta,s)) + \frac{K}{\tilde{c}}r^2.
\end{aligned}$$

∎

We omit the proof of the next elementary claim.

Claim 4. *Let* $(x,t),(y,\tau) \in \partial M \times [0,\varepsilon]$ *and* $s \in [0,\varepsilon]$. *Let* $\gamma\colon [0,1] \to \partial M$ *be the geodesic between* x *and* y *in the metric* g_s. *Then the curve*

$$w \in [0,1] \mapsto (\gamma(w),(1-w)t + w\tau)$$

is a geodesic in the metric $g_s \oplus d\tau^2$.

Step 3. For every $t \in [0,\varepsilon]$, consider the exponential map $\exp^{g_t}$ given by the metric g_t. Since g_t varies smoothly with respect to t, and $\partial M \times [0,\varepsilon]$ is compact, we know that there exists $\delta > 0$ lower than the injectivity radius of each $\exp^{g_t}$. For any (ξ,t,ζ,s) in

$$D_\delta := [-1,1]^{m-1} \times [0,\varepsilon] \times \mathbb{B}_\delta^{m-1}(0) \times [0,\varepsilon]$$

define

$$\Psi(\xi,t,\zeta,s) := E(\exp^{g_t}_{\psi(\xi)}((d\psi)_\xi A^{-1/2}_{(\xi,t)}\zeta),s).$$

We may write

$$\Psi_{(\xi,t)}(\zeta,s) = \Psi(\xi,t,\zeta,s)$$

to see Ψ as a function of the two last variables only, the two first being frozen. Observe that for any (ξ,t,ζ) in $[-1,1]^{m-1} \times [0,\varepsilon] \times \mathbb{B}_\delta^{m-1}(0)$,

$$\begin{aligned}(g_t)^{1/2}_{\psi(\xi)}((d\psi)_\xi A^{-1/2}_{(\xi,t)}\zeta,(d\psi)_\xi A^{-1/2}_{(\xi,t)}\zeta) &= (\psi^* g_t)^{1/2}_\xi (A^{-1/2}_{(\xi,t)}\zeta, A^{-1/2}_{(\xi,t)}\zeta)\\ &= |\zeta^T A^{-1/2}_{(\xi,t)} A_{(\xi,t)} A^{-1/2}_{(\xi,t)}\zeta|^{1/2} \qquad \text{(by (35))}\\ &= |\zeta| < \delta.\end{aligned}$$

Since δ is lower than the injectivity radius of the exponentials, the map

$$\zeta \in \mathbb{B}_\delta^{m-1}(0) \mapsto \exp^{g_t}_{\psi(\xi)}((d\psi)_\xi A^{-1/2}_{(\xi,t)}\zeta)$$

is injective. Thus, for every $(\xi,t) \in [-1,1]^{m-1} \times [0,\varepsilon]$ the map $\Psi_{(\xi,t)}$ defined on $\mathbb{B}_\delta^{m-1}(0) \times [0,\varepsilon]$ is a local parametrization of M. Moreover,

$$\Psi_{(\xi,t)}(0,t) = E(\psi(\xi),t) = \Phi(\xi,t), \tag{48}$$

and

$$\det([\Psi^*_{(\xi,t)} g]_{(0,t)}) = 1.$$

Claim 5. *Consider* $(\xi,t) \in [-1,1]^{m-1} \times [0,\varepsilon]$, *and* $(\zeta,s) \in \mathbb{B}_\delta^{m-1}(0) \times [0,\varepsilon]$. *Then*

$$\mathsf{d}_t(\Psi_{(\xi,t)}(\zeta,s),\Psi_{(\xi,t)}(0,t)) = \sqrt{|\zeta|^2 + (t-s)^2}.$$

Proof. By Claim 4, the geodesic between $E^{-1}(\Psi_{(\xi,t)}(0,t))$ and $E^{-1}(\Psi_{(\xi,t)}(\zeta,s))$ in the metric $g_s \oplus d\tau^2$ is

$$\begin{aligned}\gamma\colon w \in [0,1] \mapsto (\exp^{g_t}_{\psi(\xi)}((d\psi)_\xi A^{-1/2}_{(\xi,t)} w\zeta), (1-w)t + ws)\\ =: (\tilde{\gamma}(w), \gamma_m(w)).\end{aligned}$$

Then we also know that, for any $w \in [0,1]$,

$$(g_t)^{1/2}_{\tilde{\gamma}(w)}(\dot{\tilde{\gamma}}(w), \dot{\tilde{\gamma}}(w)) = |\zeta|.$$

Thus,

$$\begin{aligned}\mathsf{d}_t(\Psi_{(\xi,t)}(\zeta,s), \Psi_{(\xi,t)}(0,t)) &= \int_0^1 \sqrt{(g_t)^2_{\tilde{\gamma}(w)}(\dot{\tilde{\gamma}}(w), \dot{\tilde{\gamma}}(w)) + (s-t)^2}\, \mathrm{d}w\\ &= \sqrt{|\zeta|^2 + (s-t)^2}.\end{aligned}$$

∎

Claim 6. *There exist $r_0, \kappa > 0$ such that for all $(\xi,t) \in [-2,2]^{m-1} \times [0, \varepsilon/2]$ and $r \in (0, r_0)$ such that $\mathbb{B}^m_{r/\sqrt{c}}(\xi,t) \subset [-3,3]^{m-1} \times [0,\varepsilon]$, we have*

$$\mathcal{L}^m(\mathbb{B}^m_r(0,t) \triangle (\Psi^{-1}_{(\xi,t)}(B_r(\Psi_{(\xi,t)}(0,t))))) \le \kappa r^{m+1}.$$

Proof. For any $(\xi,t) \in [-2,2]^{m-1} \times [0,\varepsilon/2]$, there exists $r_0(\xi,t) > 0$ small enough such that

$$B_{r_0}(\Psi_{(\xi,t)}(0,t)) = B_{r_0}(\Phi(\xi,t)) \subset \Psi_{(\xi,t)}(\mathbb{B}^{m-1}_\delta(0) \times [0,\varepsilon]). \tag{49}$$

By compactness of $[-2,2]^{m-1} \times [0,\varepsilon/2]$ and continuity of the maps $\Psi^{-1}_{(\xi,t)}$, we get that there exists a common $r_0 > 0$ such that the previous holds for any $(\xi,t) \in [-2,2]^{m-1} \times [0,\varepsilon/2]$. Consider $r < r_0$ and $(\xi,t) \in [-2,2]^{m-1} \times [0,\varepsilon/2]$, then

$$B_r(\Psi_{(\xi,t)}(0,t)) \subset \mathrm{Im}(\Psi_{(\xi,t)}).$$

Set $A_1 := \Psi^{-1}_{(\xi,t)}(B_r(\Psi_{(\xi,t)}(0,t)))$ and $A_2 := \mathbb{B}^m_r(0,t)$. We will show that there exists $K > 0$ such that

$$A_1 \backslash A_2 \subset \mathbb{B}^m_{r+Kr^2}(0,t) \backslash \mathbb{B}^m_r(0,t). \tag{50}$$

For $(\zeta,s) \in A_1 \backslash A_2$, we know that

$$\mathsf{d}(\Psi_{(\xi,t)}(\zeta,s), \Psi_{(\xi,t)}(0,t)) < r.$$

Therefore, from (48) and Claim 3, we conclude that there exists $K > 0$ such that

$$\mathsf{d}_t(\Psi_{(\xi,t)}(\zeta,s), \Psi_{(\xi,t)}(0,t)) < r + Kr^2.$$

Then we get from Claim 5 that

$$\sqrt{|\zeta|^2 + (t-s)^2} < r + Kr^2$$

hence (50) is proved. A similar proof shows that

$$A_2 \backslash A_1 \subset \mathbb{B}^m_r(0,t) \backslash \mathbb{B}^m_{r-Kr^2}(0,t).$$

From the latter and (50), we conclude that

$$A_1 \triangle A_2 \subset \mathbb{B}^m_{r+Kr^2}(0,t) \backslash \mathbb{B}^m_{r-Kr^2}(0,t).$$

Thus,

$$\begin{aligned}
\mathcal{L}^m(A_1 \triangle A_2) &\le \mathcal{L}^m(\mathbb{B}^m_{r+Kr^2}(0,t) \backslash \mathbb{B}^m_{r-Kr^2}(0,t)) \\
&\le r^m \mathcal{L}^m(\mathbb{B}^m_{1+Kr}(0) \backslash \mathbb{B}^m_{1-Kr}(0)) \\
&= r^m \omega_m((1+Kr)^m - (1-Kr)^m) \\
&\le r^m C(Kr) \qquad \text{(for some } C > 0) \\
&\le (CK) r^{m+1}.
\end{aligned}$$

■

Claim 7. *There exist $C, r_0 > 0$ such that for all $r \in (0, r_0)$ and*

$$(\xi, t, \zeta, s) \in [-1,1]^{m-1} \times [0, \varepsilon/4] \times \mathbb{B}^{m-1}_\delta(0) \times [0, \varepsilon]$$

such that $\mathbb{B}^m_{2r/\sqrt{\tilde{c}}}(\xi,t) \cap \mathbb{H}^m \subset [-2,2]^{m-1} \times [0, \varepsilon/2]$ and $\mathsf{d}(\Psi_{(\xi,t)}(\zeta,s), \Phi(\xi,t)) < r$, then

$$\left| \frac{1}{V(\Psi_{(\xi,t)}(\zeta,s), r)} - \frac{1}{\mathcal{L}^m(\mathbb{B}^m_r(0,s) \cap \mathbb{H}^m)} \right| \le \frac{C}{r^{m-1}}.$$

Proof. Let us first consider the map $G \colon D_\delta \to \mathbb{R}$ given by

$$G(\xi, t, \zeta, s) := \det([\Psi^*_{(\xi,t)} g]_{(\zeta,s)})^{1/2}.$$

This map is C^∞, and its value at any $(\xi, t) \times (0, t)$ is 1, thus by a Taylor expansion in the variable (ζ, s) centered at $(0, t)$, and compactness of D_δ, we obtain that there exists $k > 0$ such that for all $(\xi, t, \zeta, s) \in D_\delta$ we have

$$|\det([\Psi^*_{(\xi,t)} g]_{(\zeta,s)})^{1/2} - 1| \le k|(\zeta,s) - (0,t)|. \tag{51}$$

In particular, there exists $C > 0$ such that for all $(\xi, t, \zeta, s) \in D_\delta$,

$$|\det([\Psi^*_{(\xi,t)} g]_{(\zeta,s)})^{1/2}| \le C.$$

Let us now consider r, ξ, t, ζ, s as in the statement of the claim. Set $y := \Psi_{(\xi,t)}(\zeta, s)$. Since $\mathsf{d}(y, \Phi(\xi,t)) < r$, and $\mathbb{B}^m_{2r/\sqrt{\tilde{c}}}(\xi,t) \subset [-2,2]^{m-1} \times [0, \varepsilon/2]$, we know by Claim 2 that $y \in \Phi(\mathbb{B}^m_{r/\sqrt{\tilde{c}}}(\xi,t) \cap \mathbb{H}^m)$. Thus, if we set

$$(\eta, \tau) := \Phi^{-1}(y) \in [-2,2]^{m-1} \times [0, \varepsilon/2], \tag{52}$$

we obtain that

$$\mathbb{B}^m_{r/\sqrt{\tilde{c}}}(\eta,\tau) \cap \mathbb{H}^m \subset \mathbb{B}^m_{2r/\sqrt{\tilde{c}}}(\xi,t) \cap \mathbb{H}^m \subset [-2,2]^{m-1} \times [0,\varepsilon/2].$$

Thus, by Claim 2, we conclude that $B_r(y) = B_r(\Phi(\eta,\tau)) \subset \Phi(\mathbb{B}^m_{r/\sqrt{\tilde{c}}}(\eta,\tau))$.

By (48) and (40), we easily see that

$$s = \tau. \tag{53}$$

Moreover, by (52) and (53), we also have that

$$\Psi_{(\eta,s)}(0,s) = E(\psi(\eta), s) = \Phi(\eta, s) = y.$$

Choose r_0 such that $r_0/\sqrt{\tilde{c}} < \varepsilon/4$. Since $t \le \varepsilon/4$ and $(\eta, s) \in \mathbb{B}^m_{r/\sqrt{\tilde{c}}}(\xi,t)$, this implies that $s \le \varepsilon/2$. Thus, we can use Claim 6 to ensure that

$$\mathcal{L}^m(\mathbb{B}^m_r(0,s) \triangle (\Psi_{(\eta,s)})^{-1}(B_r(\Psi_{(\eta,s)}(0,s)))) \le \kappa r^{m+1}. \tag{54}$$

Then

$$\begin{aligned} V(y,r) &= \int_{\Psi^{-1}_{(\eta,z)}(B_r(y))} |\det[\Psi^*_{(\eta,s)} g]_w|^{1/2} \, \mathrm{d}\mathcal{L}^m(w) \\ &= \int_{\Psi^{-1}_{(\eta,s)}(B_r(y))} 1 \cdot \mathrm{d}\mathcal{L}^m(w) + O(r^{m+1}) && \text{(by (51))} \\ &= \mathcal{L}^m(\mathbb{B}^m_r(0,s) \cap \mathbb{H}^m) + O(r^{m+1}) && \text{(by (54))} \end{aligned}$$

that is, there exists $\widetilde{C} > 0$ such that

$$|\mathrm{vol}_g(B_r(y)) - \mathcal{L}^m(\mathbb{B}^m_r(0,z) \cap \mathbb{H}^m)| \le \widetilde{C} r^{m+1}$$

Thus, using the local Ahlfors regularity of (M, g) and Claim 6, we obtain

$$\begin{aligned} \left| \frac{1}{\mathrm{vol}_g(B_r(\bar{y}))} - \frac{1}{\mathcal{L}^m(\mathbb{B}^m_r(0,z) \cap \mathbb{H}^m)} \right| &= \left| \frac{\mathcal{L}^m(\mathbb{B}^m_r(0,z) \cap \mathbb{H}^m) - \mathrm{vol}_g(B_r(\bar{y}))}{\mathrm{vol}_g(B_r(\bar{y}))\mathcal{L}^m(\mathbb{B}^m_r(0,z) \cap \mathbb{H}^m)} \right| \\ &\le \frac{C\widetilde{C} r^{m+1}}{r^m \mathcal{L}^m(\mathbb{B}^m_r(0,z) \cap \mathbb{H}^m)} \\ &\le \frac{C\widetilde{C}}{\mathcal{L}^m(\mathbb{B}^m_1(0) \cap \mathbb{H}^m) r^{m-1}} \end{aligned}$$

concluding the proof. ■

Step 4. We start with the following claim.

Claim 8. *For all* $f \in C^\infty(M)$*, there exists* $C > 0$ *such that for all* $(\xi, t, \zeta, s) \in D_\delta$ *we have*

$$|f \circ \Psi_{(\xi,t)}(\zeta, s) - f \circ \Psi_{(\xi,t)}(0, t)| \leq C\|(\zeta, s) - (0, t)\|_2$$

and

$$\begin{aligned}|f \circ \Psi_{(\xi,t)}(\zeta, s) - f \circ \Psi_{(\xi,t)}(0, t) - \nabla(f \circ \Psi_{(\xi,t)})_{(0,t)} \cdot ((\zeta, s) - (0, t))| \\ \leq C\|(\zeta, s) - (0, t)\|_2^2.\end{aligned}$$

Also if $\partial_\nu f|_{\partial M} = 0$*, then there exists* $C > 0$ *such that*

$$|\partial_m f \circ \Psi_{(\xi,t)}(0, t)| \leq Ct \tag{55}$$

Proof. The map $\tilde{f}((\xi, t), (\zeta, s)) = f \circ \Psi_{(\xi,t)}(\zeta, s)$ is C^∞, thus by a Taylor expansion of order 1 and 2 respectively, and compactness of D_δ, we conclude the first two inequalities. For the last, we notice that

$$\partial_m f \circ \Psi_{(\xi,0)}(0, 0) = (\partial_\nu f)(\psi(\xi, 0)) = 0.$$

Thus, by a Taylor expansion and compactness of D_δ we conclude (55). ∎

Now, we will fix a function $f \in C^\infty(M)$ such that $\partial_\nu f|_{\partial M} = 0$, and show that for $x \in M$ such that $d(x, \partial M) < \varepsilon/4$ then $\Delta_r f(x)$ is uniformly bounded. The proof for points x with $d(x, \partial M) \geq \varepsilon/4$, follows from the uniform convergence obtained in Proposition 5.3. We will study the following term of the AMV:

$$G_r(x) := \frac{1}{r^2} \int_{B_r(x)} \frac{1}{\mathrm{vol}_g(B_r(y))}(f(y) - f(x))\, \mathrm{dvol}_g(y)$$

since the bound for the remainder follows similarly.

We notice that by equation (33) we have that there exists some $i \in \{1, \ldots, l\}$ and $(\xi, t) \in [-1, 1]^{m-1} \times [0, \varepsilon/4]$ such that $x = \Phi_i(\xi, t)$. We let $\Phi = \Phi_i$. Also by (49), for $0 < r < r_0$ in the conditions of the claim, we have that

$$B_r(x) = B_r(\Psi_{(\xi,t)}(0, t)) \subset \Psi_{(\xi,t)}(\mathbb{B}_\delta^{m-1}(0) \times [0, \varepsilon]).$$

Also we can choose r_0 small enough so that for all $(\xi, t) \in [-1, 1]^{m-1} \times [0, \varepsilon/4]$ we have $\mathbb{B}_{2r_0/\sqrt{\tilde{c}}}(x, t) \cap \mathbb{H}^m \subset [-2, 2]^{m-1} \times [0, \varepsilon/2]$. With this we can apply Claim 2 to conclude that $B_r(\Psi_{(\xi,t)(0,t)}) \subset \Phi(\mathbb{B}^m_{r/\sqrt{\tilde{c}}}(\xi, t))$ and we can also apply Claim 7 for points $\Psi_{(\xi,t)}(\zeta, s) \in B_r(\Psi_{(\xi,t)}(0, t))$.

Thus, we can change variables of the integral to obtain

$$G_r(x) = \frac{1}{r^2} \int\limits_{\Psi^{-1}_{(\xi,t)}(B_r(\Psi_{(\xi,t)}(0,t)))} \frac{1}{\mathrm{vol}_g(B_r(\Psi_{(\xi,t)}(\zeta,s)))} \times (f(\Psi_{(\xi,t)}(\zeta,s)) - f(\Psi_{(\xi,t)}(0,t)))\det([\Psi^*_{(\xi,t)}g]_{\zeta,s})^{1/2}\,\mathrm{d}\mathcal{L}^m(\zeta,s)$$

$$= \frac{1}{r^2} \int\limits_{\Psi^{-1}_{(\xi,t)}(B_r(\Psi_{(\xi,t)}(0,t)))} \frac{f(\Psi_{(\xi,t)}(\zeta,s)) - f(\Psi_{(\xi,t)}(0,t))}{\mathcal{L}^m(\mathbb{B}^m_r(0,s)\cap\mathbb{H}^m)} \times \det([\Psi^*_{(\xi,t)}g]_{\zeta,s})^{1/2}\;\mathrm{d}\mathcal{L}^m(\zeta,s) + O(1) \qquad \text{(by Claims 7 and 8)}$$

$$= \frac{1}{r^2} \int\limits_{\mathbb{B}^m_r(0,t)\cap\mathbb{H}^m} \frac{f(\Psi_{(\xi,t)}(\zeta,s)) - f(\Psi_{(\xi,t)}(0,t))}{\mathcal{L}^m(\mathbb{B}^m_r(0,s)\cap\mathbb{H}^m)} \times \det([\Psi^*_{(\xi,t)}g]_{\zeta,s})^{1/2}\,\mathrm{d}\mathcal{L}^m(\zeta,s) + O(1) \qquad \text{(by Claims 6 and 8)}$$

$$= \frac{1}{r^2} \int\limits_{\mathbb{B}^m_r(0,t)\cap\mathbb{H}^m} \frac{f(\Psi_{(\xi,t)}(\zeta,s)) - f(\Psi_{(\xi,t)}(0,t))}{\mathcal{L}^m(\mathbb{B}^m_r(0,s)\cap\mathbb{H}^m)} \times \mathrm{d}\mathcal{L}^m(\zeta,s) + O(1) \qquad \text{(by (51) and Claim 8)}$$

$$= \frac{1}{r^2} \int\limits_{\mathbb{B}^m_r(0,t)\cap\mathbb{H}^m} \frac{\sum_{i=1}^{m-1}\partial_j(f\circ\Psi_{(\xi,t)})(0,t)\zeta_i + \partial_m(f\circ\Psi_{(\xi,t)})(0,t)(s-t)}{\mathcal{L}^m(\mathbb{B}^m_r(0,s)\cap\mathbb{H}^m)} \times \mathrm{d}\mathcal{L}^m(\zeta,s) + O(1) \qquad \text{(by Claim 8)}$$

$$= \frac{1}{r^2} \int\limits_{\mathbb{B}^m_r(0,t)\cap\mathbb{H}^m} \frac{\sum_{i=1}^{m-1}\partial_j(f\circ\Psi_{(\xi,t)})(0,t)\zeta_i + \partial_m(f\circ\Psi_{(\xi,t)})(0,t)(s-t)}{\mathcal{L}^m(\mathbb{B}^m_r(0,s)\cap\mathbb{H}^m)} \times \mathrm{d}\mathcal{L}^{m-1}(\zeta)\,\mathrm{d}\mathcal{L}(s) + O(1) \qquad \text{(by Claim 8)}$$

$$= \frac{1}{r^2} \int\limits_{\mathbb{B}^m_r(0,t)\cap\mathbb{H}^m} \frac{\partial_m(f\circ\Psi_{(\xi,t)})(0,t)(s-t)}{\mathcal{L}^m(\mathbb{B}^m_r(0,s)\cap\mathbb{H}^m)} \times \mathrm{d}\mathcal{L}^{m-1}(\zeta)\,\mathrm{d}\mathcal{L}(s) + O(1). \qquad \text{(by symmetry)}$$

Now, we separate further in two cases. First, if $\mathsf{d}(x,\partial M) > 2r$, then $t > 2r$ and so for all $(\zeta,s)\in\mathbb{B}^m_r(0,t)\cap\mathbb{H}^m$ we have

$$\mathcal{L}^m(\mathbb{B}^m_r(0,s)\cap\mathbb{H}^m) = \mathcal{L}^m(\mathbb{B}^m_r(0)),$$

and so we conclude that

$$\frac{1}{r^2} \int\limits_{\mathbb{B}_r^m(0,t)\cap\mathbb{H}^m} \frac{\partial_m(f\circ\Psi_{(\xi,t)})(0,t)(s-t)}{\mathcal{L}^m(\mathbb{B}_r^m(0,s)\cap\mathbb{H}^m)} \, \mathrm{d}\mathcal{L}^{m-1}(\zeta)\,\mathrm{d}\mathcal{L}(s)$$
$$= \frac{1}{r^2\mathcal{L}^m(\mathbb{B}_r^m(0))} \int\limits_{\mathbb{B}_r^m(0,t)\cap\mathbb{H}^m} \partial_m(f\circ\Psi_{(\xi,t)})(0,t)(s-t) \, \mathrm{d}\mathcal{L}^{m-1}(\zeta)\,\mathrm{d}\mathcal{L}(s) = 0.$$

This shows that $G_r(x) = O(1)$ for x such that $2r < d(x,\partial M) < \varepsilon/4$ since there are only a finite number of parametrizations. On the other hand, if $d(x,\partial M) \le 2r$, then we have by Lemma 8 that for $t \le 2r$ then $|\partial_m(f\circ\Psi_{(x,t)})(0,t)| \le 2Cr$, and so

$$\frac{1}{r^2} \int\limits_{\mathbb{B}_r^m(0,t)\cap\mathbb{H}^m} \frac{\partial_m(f\circ\Psi_{(x,t)})(0,t)(s-t)}{\mathcal{L}^m(\mathbb{B}_r^m(0,s)\cap\mathbb{H}^m)} \, \mathrm{d}\mathcal{L}^{m-1}(y)\,\mathrm{d}\mathcal{L}(s) = O(1),$$

which shows that $G_r(x) = O(1)$ for x such that $d(x,\partial M) < 2r$. ∎

6. Spectral convergence

In this section, we prove Theorem 3. We consider a smooth, compact, connected manifold M^m endowed with a smooth Riemannian metric g. We let d_g and vol_g be the associated Riemannian distance and volume measure on M, respectively. If $\partial M = \emptyset$ (resp. $\partial M \neq \emptyset$), we let $\{\mu_k\}_{k\in\mathbb{N}}$ be the sequence of Laplace (resp. Neumann) eigenvalues of (M,g).

6.1. Existence of limit eigenfunctions

Recalling that $C_\nu^\infty(M)$ is defined in (4), we define the Hilbert space

$$H := \overline{C_\nu^\infty(M)}^{\|\cdot\|_{W^{2,2}}}.$$

We let $\Pi(M,\mathrm{vol}_g)$ be defined as in (15) and we consider the operator

$$T\colon \Pi(M,\mathrm{vol}_g) \to H^*$$

which maps any $f \in \Pi(M,\mathrm{vol}_g)$ to

$$T(f) := \Big(v \in H \mapsto -\int\limits_M f\Delta_g v \,\mathrm{dvol}_g\Big).$$

Lemma 6.1. *The operator T is injective.*

Proof. For $f \in \Pi(M, \mathrm{vol}_g)$ such that $f \neq 0$, let $v \in H$ be the solution of

$$\begin{cases} -\Delta_g v = f & \text{in } M, \\ \partial_\nu v = 0 & \text{in } \partial M. \end{cases}$$

The solution to this problem exists since $\int_M f = 0 = \int_{\partial M} \partial_\nu v$. In fact, by regularity theory we can conclude that $v \in H$, and so we conclude that

$$T(f)(v) = -\int_M f \Delta_g v \,\mathrm{dvol}_g = \int_M f^2 \,\mathrm{dvol}_g \neq 0.$$

Thus, $T(f) \neq 0$, concluding that T is injective. ∎

Let us now prove the existence of L^2-weak limit eigenfunctions.

Proposition 6.2. *Let (r_n) be a sequence of positive numbers such that $r_n \to 0$. For any n, let (λ_{k,r_n}) be the eigenvalues of the operator $\widetilde{\Delta}_{r_n}$ and let (f_{k,r_n}) be corresponding eigenfunctions. Then for any k, there exists a Laplace (resp. Neumann) eigenfunction f of (M, g) with associated eigenvalue μ such that, up to extracting subsequences, satisfy*

$$\begin{aligned} f_{k,r_n} &\overset{L^2}{\rightharpoonup} f, \\ \lambda_{k,r_n} &\to C_m \mu, \\ \sup_n E_{r_n}(f_{k,r_n} - f) &< +\infty. \end{aligned} \tag{56}$$

Proof. By the proof of Theorem 1, in particular (17), there exists $\lambda \geq 0$ and a subsequence such that $\lambda_k(-\widetilde{\Delta}_{r_n}) \to \lambda$ up to subsequence. Since $\|f_{k,r_n}\|_{L^2(M)} = 1$ for any n, there exists $f \in L^2(M, \mathrm{vol}_g)$ such that the weak convergence (56) holds up to subsequence. Therefore, by Theorem 2, we get that for any $\psi \in C^\infty(M)$ (resp. $C^\infty_{\partial_\nu}(M)$),

$$\begin{aligned} \int_M f \Delta_g \psi \,\mathrm{dvol}_g &= \lim_n \frac{1}{C_m} \int_M f_{k,r_n} \widetilde{\Delta}_{r_n} \psi \,\mathrm{dvol}_g = \frac{1}{C_m} \lim_n \int_M (\widetilde{\Delta}_{r_n} f_{k,r_n}) \psi \,\mathrm{dvol}_g \\ &= -\frac{1}{C_m} \lim_n \lambda_k(-\widetilde{\Delta}_{r_n}) \int_M f_{k,r_n} \psi \,\mathrm{dvol}_g = -\frac{1}{C_m} \lambda \int_M f \psi \,\mathrm{dvol}_g. \end{aligned}$$

Moreover, since for any n it holds that $\lambda_{k,r_n} > 0$ and

$$0 = \int_M -\widetilde{\Delta}_{r_n} f_{k,r_n} \,\mathrm{dvol}_g = \int_M \tilde{\lambda}_{k,r_n} f_{k,r_n} \,\mathrm{dvol}_g,$$

we get that $\int_M f_{k,r_n} = 0$. Thus, $\int_M f \,\mathrm{dvol}_g = 0$ by weak convergence.

Now, let $v \in W^{2,2}(M)$ be the solution of

$$\begin{cases} -\Delta_g v = \frac{\lambda}{C_m} f & \text{in } M, \\ \partial_\nu v = 0 & \text{in } \partial M \end{cases}$$

satisfying $\int_M v = 0$. Then we have that for $\psi \in H \cap C^\infty(M)$,

$$\int_M v \Delta_g \psi \,\mathrm{dvol}_g = -\frac{1}{C_m}\lambda \int_M f\psi \,\mathrm{dvol}_g = \int_M f \Delta_g \psi \,\mathrm{dvol}_g.$$

Since this is a dense subspace of H and the functionals are continuous with respect to $W^{2,2}(M)$ in ψ, the equality holds for all H. Thus, by Lemma 6.1 we conclude that $v = f$, and so f satisfies

$$\begin{cases} -\Delta_g f = \frac{\lambda}{C_m} f & \text{in } M, \\ \partial_\nu f = 0 & \text{in } \partial M, \end{cases}$$

thus f is a Neumann eigenfunction, and so it must be $C^\infty(M)$. Also since both $f_{k,r_n}, f \in \Pi(M, \mathrm{vol}_g)$, we know by Proposition 3.14 using triangle inequality of the inner product,

$$E_{r_n}(f_{k,r_n} - f)^{1/2} \leq E_{r_n}(f_{k,r_n})^{1/2} + E_{r_n}(f)^{1/2}.$$

We know that $E_{r_n}(f_{k,r_n}) = \lambda_{k,r_n} \| f_{k,r_n} \|_{L^2(M)} = \lambda_{k,r_n}$ which is uniformly bounded. Also since $f \in C^\infty(M)$, by Lemma 5.1, we know that $E_{r_n}(f)$ is also uniformly bounded, concluding the proof. ∎

6.2. Energy comparison

Let us now compare the energy of a map defined on M with the energy of the image of the map through a local chart parametrizing a neighborhood of an open subset of ∂M. To this aim, up to scaling, we consider a map $\Phi\colon (-1,1)^{m-1} \times [0,1) \to M$ which is a bi-Lipschitz homeomorphism onto its image. We set

$$\mathcal{Q} := (-1/2, 1/2)^{m-1} \times [0, 1/2). \tag{57}$$

Lemma 6.3. *There exist constants $\widetilde{C} = \widetilde{C}(\Phi) > 0$ and $\tilde{c} = \tilde{c}(\Phi) > 0$ such that for any $f \in L^2(M)$, for any $r \in (0, 1/2)$,*

$$\widetilde{E}_{\tilde{c}r,\mathfrak{Q}}(f \circ \Phi) \leq \widetilde{C} E_{r,\mathfrak{M}}(f),$$

where $\mathfrak{Q} := (\mathcal{Q}, d_\infty, \mathcal{L}^m)$ and $\mathfrak{M} := (M, d, \mu)$.

Proof. We start by pointing out that there exist constants $c = c(\Phi) > 0$ and $C = C(\Phi) > 0$ such that for all $x \in \mathcal{Q}$ and $r \in (0, 1/2)$,

$$\Phi(Q_{cr}(x)) \subset B_r(\Phi(x)) \subset \Phi(Q_{Cr}(x)),$$

$$V(\Phi(x), r) \leq C\mathcal{L}^m(Q_{cr}(x) \cap \mathcal{Q}),$$

$$\det(g_x) \geq 0,$$

where g_x is the metric in the coordinates given by Φ. Then for any $x, y \in \mathcal{Q}$,

$$\begin{aligned}\tilde{a}_{r,\mathfrak{M}}(\Phi(x), \Phi(y)) &= 1_{B_r(\Phi(x))}(\Phi(y))\Big(\frac{1}{V(\Phi(x), r)} + \frac{1}{V(\Phi(y), r)}\Big) \\ &\geq 1_{Q_{cr}(x)}(y)\Big(\frac{1}{C\mathcal{L}^m(Q_{cr}(x) \cap \mathcal{Q})} + \frac{1}{C\mathcal{L}^m(Q_{cr}(y) \cap \mathcal{Q})}\Big) \\ &= \frac{\tilde{a}_{cr,\mathfrak{Q}}(x, y)}{C}.\end{aligned}$$

Thus,

$$\begin{aligned}\widetilde{E}_{r,\mathfrak{M}}(f) &\geq \iint\limits_{\Phi(\mathcal{Q})^2} \tilde{a}_{r,\mathfrak{M}}(p, q)\Big(\frac{f(p) - f(q)}{r}\Big)^2 \,\mathrm{dvol}_g(q)\,\mathrm{dvol}_g(p) \\ &= \iint\limits_{\mathcal{Q}^2} \tilde{a}_{r,\mathfrak{M}}(\Phi(x), \Phi(y))\Big(\frac{f(\Phi(x)) - f(\Phi(y))}{r}\Big)^2 \\ &\qquad\qquad \times \sqrt{\det(g_x)\det(g_y)}\,\mathrm{d}\mathcal{L}^m(y)\,\mathrm{d}\mathcal{L}^m(x) \\ &\geq \iint\limits_{\mathcal{Q}^2} \frac{c}{C}\tilde{a}_{cr,\mathfrak{Q}}(x, y)\Big(\frac{f(\Phi(x)) - f(\Phi(y))}{r}\Big)^2 \,\mathrm{d}\mathcal{L}^m(y)\,\mathrm{d}\mathcal{L}^m(x) \\ &= \frac{c}{C}E_{cr,\mathfrak{Q}}(f \circ \Phi).\end{aligned}$$

Taking $\tilde{c} = c$ and $\widetilde{C} = C/c$, we obtain the result. ∎

6.3. Proof of Theorem 3

We are now in a position to prove Theorem 3. Recall the context of this result: $(r_n) \subset (0, +\infty)$ is a sequence such that $r_n \to 0$, (M^m, g) is a compact, connected, smooth Riemannian manifold with $\partial M = \emptyset$ (resp. $\partial M \neq \emptyset$), k is a positive integer, μ_k is the k-th lowest Laplace (resp. Neumann) eigenvalue of Δ_g, and f_{k,r_n} is an eigenfunction of $-\widetilde{\Delta}_{r_n}$ associated with the k-th eigenvalue $\lambda_k(-\widetilde{\Delta}_{r_n})$ of this operator.

Proof. We proceed it two steps.

Step 1. First we show strong L^2-convergence of the sequence (f_{k,r_n}). We proceed by contradiction. By Proposition 6.2, we can assume that there exist $\alpha > 0$, $f \in L^2(M,\mu)$ which is a Neumann eigenfunction, and $(r_n) \subset (0,+\infty)$ such that $r_n \to 0$ and

$$f_{k,r_n} \overset{L^2}{\rightharpoonup} f, \quad \|f_{k,r_n} - f\|^2_{L^2(M)} \geq \alpha.$$

Since M is a compact manifold with boundary, up to scaling there exist finitely many bi-Lipschitz homeomorphisms $\{\Phi_j : (-1,1)^{m-1} \times [0,1) \to M\}_{j\in\{1,\dots,\ell\}}$ such that

$$\bigcup_j \Phi_j(\mathcal{Q}) = M,$$

where $\mathcal{Q}$ is as in (57), and

$$\bigcup_j \Phi_j((-1/2,1/2)^{m-1} \times \{0\}) = \partial M.$$

Then there exists $j \in \{1,\dots,\ell\}$ such that, up to a subsequence,

$$\inf_n \int_{\Phi_j(\mathcal{Q})} |f_{k,r_n} - f|^2 \,\mathrm{dvol}_g \geq \frac{\alpha}{\ell} > 0.$$

From this, we conclude that there exists $\tilde{\alpha} > 0$ such that

$$\inf_n \int_{\mathcal{Q}} |f_{k,r_n} - f|^2 \circ \Phi_j \,\mathrm{d}\mathcal{L}^m \geq \tilde{\alpha} > 0.$$

Let us set $\Phi := \Phi_j$. Then there exist $C, \widetilde{C}, \tilde{c} > 0$ such that for any n,

$$\begin{aligned} C &\geq \widetilde{E}_{r_n}(f_{k,r_n} - f) && \text{(by Proposition 6.2)}\\ &\geq \widetilde{C}^{-1}\widetilde{E}_{\tilde{c}r_n,\mathfrak{Q}}((f_{k,r_n} - f)\circ\Phi). && \text{(by Lemma 6.3)} \end{aligned}$$

By the weak convergence, we also have

$$h_n := (f_{k,r_n} - f)\circ\Phi \overset{L^2}{\rightharpoonup} 0. \tag{58}$$

Let us set $\bar{r}_n = \tilde{c}r_n$. For an integer N to be chosen later, consider a decomposition of $\mathcal{Q}$ into L_N disjoint subcubes $\{\widetilde{Q}_i\}$ of size $1/N$. For any $x, y \in \mathcal{Q}$, we set

$$a_r(x,y) := \chi_{Q_r(x)\cap\mathcal{Q}}(y)\Big(\frac{1}{\mathcal{L}^m(Q_r(x)\cap\mathcal{Q})} + \frac{1}{\mathcal{L}^m(Q_r(y)\cap\mathcal{Q})}\Big),$$

$$a_{r,i}(x,y) := \chi_{Q_r(x)\cap\mathcal{Q}_i}(y)\Big(\frac{1}{\mathcal{L}^m(Q_r(x)\cap\mathcal{Q}_i)} + \frac{1}{\mathcal{L}^m(Q_r(y)\cap\mathcal{Q}_i)}\Big),$$

and we point out that for $x, y \in \mathcal{Q}_i$

$$a_r(x,y) \geq \frac{1}{2^m} a_{r,i}(x,y).$$

We also set for any n,

$$\varepsilon_{i,n,N} := \int_{\mathcal{Q}_i} h_n \, \mathrm{d}\mathcal{L}^m, \quad \delta_{n,N} := \max_i |\varepsilon_{i,n,N}|.$$

We obtain that for any n,

$$\begin{aligned}
\widetilde{E}_{\bar{r}_n,\mathfrak{Q}}(h_n) &= \int_{\mathcal{Q}} \left(\int_{\mathcal{Q}} a_{\bar{r}_n}(x,y) \frac{(h_n(x)-h_n(y))^2}{\bar{r}_n^2} \, \mathrm{d}\mathcal{L}^m(y) \right) \mathrm{d}\mathcal{L}^m(x) \\
&= \sum_i \int_{\mathcal{Q}_i} \left(\int_{\mathcal{Q}} a_{\bar{r}_n}(x,y) \frac{(h_n(x)-h_n(y))^2}{\bar{r}_n^2} \, \mathrm{d}\mathcal{L}^m(y) \right) \mathrm{d}\mathcal{L}^m(x) \\
&\geq \frac{1}{2^m} \sum_i \int_{\mathcal{Q}_i} \left(\int_{\mathcal{Q}_i} a_{\bar{r}_n,i}(x,y) \frac{(h_n(x)-h_n(y))^2}{\bar{r}_n^2} \, \mathrm{d}\mathcal{L}^m(y) \right) \mathrm{d}\mathcal{L}^m(x) \\
&= \frac{1}{2^m} \sum_i \widetilde{E}_{\bar{r}_n,\mathcal{Q}_i}(h_n) = \frac{1}{2^m} \sum_i \widetilde{E}_{\bar{r}_n,\mathcal{Q}_i}(h_n - \varepsilon_{i,n,N}) \\
&\geq \frac{1}{2^m} \sum_i \|h_n - \varepsilon_{i,n,N}\|^2_{L^2(\mathcal{Q}_i)} \lambda_1(-\widetilde{\Delta}_{\bar{r}_n,\mathcal{Q}_i}) \\
&\geq \frac{\lambda_1(-\widetilde{\Delta}_{\bar{r}_n,\mathfrak{Q}^m(1/N)})}{2^m} \sum_i \|h_n - \varepsilon_{i,n,N}\|^2_{L^2(\mathcal{Q}_i)} \\
&= \frac{\lambda_1(-\widetilde{\Delta}_{\bar{r}_n,\mathfrak{Q}^m(1/N)})}{2^m} \\
&\qquad \times \sum_i \left(\|h_n\|^2_{L^2(\mathcal{Q}_i)} - 2\varepsilon_{i,n,N} \int_{\mathcal{Q}_i} h_n \, \mathrm{d}\mathcal{L}^m + \mathcal{L}^m(\mathcal{Q}_i) \varepsilon^2_{i,n,N} \right) \\
&\geq \frac{\lambda_1(-\widetilde{\Delta}_{\bar{r}_n,\mathfrak{Q}^m(1/N)})}{2^m} (\|h_n\|^2_{L^2(\mathcal{Q})} - 3L_n \delta_{n,N}) \\
&\geq \frac{\lambda_1(-\widetilde{\Delta}_{\bar{r}_n,\mathfrak{Q}^m(1/N)})}{2^m} (\tilde{\alpha} - 3L_n \delta_{n,N}).
\end{aligned}$$

By Lemma 4.3, we choose N big enough to ensure that for any n,

$$\lambda_1(-\widetilde{\Delta}_{\bar{r}_n,\mathfrak{Q}(1/N)}) > C\widetilde{C} \frac{2^{m+2}}{\tilde{\alpha}}.$$

By the weak convergence (58), we know that $\delta_{n,N} \to 0$, and so we can choose n big enough to guarantee

$$\delta_{n,N} < \frac{\tilde{\alpha}}{6L_N}.$$

With these choices we eventually get

$$\widetilde{E}_{r_n}(f_{k,r_n} - f) > C,$$

which is a contradiction.

Step 2. Now, we show that $\tilde{\lambda}_{k,r_n} \to \mu_k$, where μ_k is the k-th Neumann eigenvalue. Let $r_n \to 0$. We know by Proposition 6.2 that there exist eigenfunctions $f_0, \dots, f_k$ with Neumann eigenvalue $\lambda_0, \dots, \lambda_k$ such that

$$\begin{aligned} f_{i,r_n} &\overset{L^2}{\to} f_i, \quad \text{for all } i \in \{0, \dots, k\}, \\ \tilde{\lambda}_{k,r_n} &\to C_m \lambda_k, \end{aligned}$$

and

$$\lambda_i \leq \lambda_k \quad \text{for all } i \in \{0, \dots, k\}. \tag{59}$$

Since $\langle f_{i,r_n}, f_{j,r_n} \rangle = \delta_{i,j}$, we also have by strong convergence that $\langle f_i, f_j \rangle = \delta_{i,j}$. Thus, we have that

$$V_{k+1} := \operatorname{Span}(f_0, \dots, f_k) \in \mathcal{G}_{k+1}(L^2(M, \mathrm{vol}_g)),$$

and so by equation (59), we conclude

$$C_m \mu_k \leq \max_{f \in V_{k+1}} \frac{\langle \nabla f, \nabla f \rangle}{\| f \|_{L^2}} = C_m \lambda_k = \lim_n \tilde{\lambda}_{k,r_n}.$$

This shows that $\liminf_{r \to 0} \tilde{\lambda}_{k,r_n} \geq C_m \mu_k$

To prove $\limsup_{r \to 0} \tilde{\lambda}_{k,r} \leq C_m \mu_k$, let $\{f_0, \dots, f_k\}$ be an $\langle \cdot, \cdot \rangle_2$-orthonormal family of Laplace (resp. Neumann) eigenfunctions associated with the eigenvalues $\{\mu_0, \dots, \mu_k\}$ respectively satisfying $\mu_0 \leq \dots \leq \mu_k$. By elliptic regularity, we know that these functions belong to $C^\infty(M)$. Then Proposition 5.1 implies that given $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that for $r \in (0, r_\varepsilon)$,

$$|\langle -\widetilde{\Delta}_r f_i, f_j \rangle_2 - \delta_{i,j} C_m \mu_j| < \varepsilon,$$

where $\delta_{i,j}$ is the usual Kronecker delta. Set $U := \operatorname{Span}(f_0, \dots, f_k)$ and

$$v := \sum_{i=1}^{k} a_i \psi_i$$

for some $a = (a_1, \ldots, a_k) \in \mathbb{S}^{k-1}$. Then

$$\Big|\langle -\widetilde{\Delta}_r v, v\rangle - \sum_{i=1}^k a_i^2 C_m \mu_i\Big| = \Big|\sum_{i,j=1}^k a_i a_j \langle -\widetilde{\Delta}_r f_i, f_j\rangle - \sum_{i=1}^k a_i^2 C_m \mu_i\Big| \le k^2\varepsilon.$$

Since U is a $k+1$-dimensional subspace, we conclude that

$$\tilde{\lambda}_{k,r} \le \max_{v \in U} \frac{\langle -\widetilde{\Delta}_r v, v\rangle}{\|v\|_2^2} \le \max_{a \in \mathbb{S}^k} \sum_{i=1}^k a_i^2 \mu_i + k^2\varepsilon \le \mu_k + k^2\varepsilon.$$

Take the limit superior as $r \to 0$ and then let $\varepsilon \to 0$ to obtain $\limsup_{r\to 0} \tilde{\lambda}_{k,r} \le C_m \mu_k$. Combined with Corollary 3.13, the latter implies the existence of $r_k > 0$ such that $\min \sigma_{\mathrm{ess}}(-\widetilde{\Delta}_r) \ge \mu_k + 1 \ge \tilde{\lambda}_{k,r}$ for any $r \in (0, r_k)$, so that $\tilde{\lambda}_{k,r}$ indeed coincides with $\lambda_k(-\widetilde{\Delta}_r)$. ∎

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