J. Spectr. Theory 16 (2026), 1–49
DOI 10.4171/JST/590

The Duistermaat index and eigenvalue interlacing for self-adjoint extensions of a symmetric operator

Gregory Berkolaiko, Graham Cox, Yuri Latushkin, and Selim Sukhtaiev

Abstract. Eigenvalue interlacing is a useful tool in linear algebra and spectral analysis. In its simplest form, the interlacing inequality states that a rank-one positive perturbation shifts each eigenvalue up, but not further than the next unperturbed eigenvalue. For different types of perturbations, this idea is known as Weyl interlacing, Cauchy interlacing, Dirichlet–Neumann bracketing, and so on.

We prove a sharp version of the interlacing inequalities for "finite-dimensional perturbations in boundary conditions," expressed as bounds on the spectral shift between two self-adjoint extensions of a fixed semibounded symmetric operator with finite and equal defect numbers. The bounds are given in terms of the Duistermaat index, a topological invariant describing the relative position of three Lagrangian planes in a symplectic space. Two of the Lagrangian planes describe the self-adjoint extensions being compared, while the third corresponds to the Friedrichs extension, which acts as a reference point.

Along the way, numerous auxiliary results are established, including one-sided continuity properties of the Duistermaat index, smoothness of the Cauchy data space without unique continuation-type assumptions, and a formula for the Morse index of an extension of a non-negative symmetric operator.

1. Introduction

1.1. Background and motivation

Let H_1 and H_2 be two $N \times N$ Hermitian matrices, with eigenvalues $\lambda_1(H_j) \leq \lambda_2(H_j) \leq \cdots \leq \lambda_N(H_j)$. The following inequalities [43, Corollary 4.3.3] are often referred to as "Weyl interlacing":

$$\lambda_{k-\sigma_-}(H_1) \leq \lambda_k(H_2) \leq \lambda_{k+\sigma_+}(H_1), \tag{1.1}$$

where σ_+ and σ_- are given by the number of positive and negative eigenvalues of the perturbation $H_2 - H_1$. The fact that $\sigma_- + \sigma_+$ gives the rank of the perturbation

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Mathematics Subject Classification 2020: 15A42 (primary); 47B25, 53D12 (secondary).
Keywords: eigenvalue interlacing, self-adjoint extensions, Duistermaat index, Maslov index.

from H_1 to H_2 suggests that the bounds in (1.1) are optimal. Cauchy interlacing [43, Corollary 4.3.17], where H_2 is obtained from H_1 by removing rows and columns, can also be put in the form (1.1) by adjusting σ_+ to include the number of removed rows and columns.

The principal aim of this work is to establish inequality (1.1) for any two self-adjoint extensions of a bounded from below symmetric operator S with finite and equal defect numbers. The results are directly applicable to differential operators with finite-dimensional changes in boundary conditions, in settings such as linear Hamiltonian systems [45,46], quantum graphs [4,14,15], Šeba billiards [21,48,57,58,68,72], and manifolds with conical singularities [39]; some concrete examples are discussed in Section 7. Unlike the matrix case discussed above, the perturbations here are not additive (since one cannot take the difference of two unbounded operators with different domains), so it is not immediately clear how to define such quantities as the signature (σ_-, σ_+) of the perturbation.

Our results characterize the shifts in the interlacing, σ_- and σ_+, in terms of the relative topological position of three pieces of data: the two Lagrangian planes that describe the self-adjoint extensions of interest and a third Lagrangian plane describing the Friedrichs extension. This topological position is expressed via the Duistermaat triple index [5, 32, 44, 75], an integer-valued symplectic invariant whose definition is recalled and supplemented with several new computational tools in Section 3; see also Section 7 for examples of computation.

For illustrative purposes, we present two proofs of our main result: via Maslov-type index theory [3, 25, 27, 69] and via the Kreĭn resolvent formula [7, 38, 49, 54, 55, 70], in Sections 5 and 6, correspondingly. Along the way, we sharpen existing techniques, in particular to avoid relying on unique continuation-type conditions; see Section 4.

1.2. Main results

Let $\mathcal{H}$ be a separable Hilbert space and let S be a closed, bounded from below, densely defined symmetric operator with finite and equal defect numbers (n, n). Under these assumptions, all self-adjoint extensions H of S have the same essential spectrum,[1]

$$\operatorname{Spec}_{\mathrm{ess}}(H) = \operatorname{Spec}_{\mathrm{ess}}(S) := \{z \in \mathbb{C} : S - z \text{ is not Fredholm}\}. \tag{1.2}$$

[1] This coincides with $\sigma_{e3}(S)$ in [33, Chapter IX], so the first equality in (1.2) follows from [33, Corollary IX.4.2.]

We will be describing the self-adjoint extensions of S in terms of a boundary triplet $(\mathcal{K}, \Gamma_0, \Gamma_1)$, see [70, Section 14.2]. Here $\mathcal{K}$ is an n-dimensional[2] complex Hilbert space and the linear mappings $\Gamma_0, \Gamma_1 \colon \mathrm{dom}(S^*) \to \mathcal{K}$ are such that the operator

$$\boldsymbol{\Gamma} \colon \mathrm{dom}(S^*) \to \mathcal{K} \oplus \mathcal{K}, \quad \boldsymbol{\Gamma} f := (\Gamma_0 f, \Gamma_1 f)$$

is surjective and the *abstract Green's identity*

$$\langle f, S^* g\rangle_{\mathcal{H}} - \langle S^* f, g\rangle_{\mathcal{H}} = \langle \Gamma_0 f, \Gamma_1 g\rangle_{\mathcal{K}} - \langle \Gamma_1 f, \Gamma_0 g\rangle_{\mathcal{K}} \tag{1.3}$$

holds for all $f, g \in \mathrm{dom}(S^*)$.

We will view $\mathcal{K} \oplus \mathcal{K}$ as a complex symplectic space with the symplectic form

$$\omega(u, v) := \langle u_0, v_1\rangle_{\mathcal{K}} - \langle u_1, v_0\rangle_{\mathcal{K}}, \quad u = (u_0, u_1) \text{ and } v = (v_0, v_1) \in \mathcal{K} \oplus \mathcal{K}, \tag{1.4}$$

in terms of which the right-hand side of Green's identity (1.3) is $\omega(\boldsymbol{\Gamma} f, \boldsymbol{\Gamma} g)$. Self-adjoint extensions H of S are in one-to-one correspondence with Lagrangian planes[3] $\mathcal{L}$ in $\mathcal{K} \oplus \mathcal{K}$ via

$$\mathrm{dom}(H) := \{f \in \mathrm{dom}(S^*) : (\Gamma_0 f, \Gamma_1 f) \in \mathcal{L}\}. \tag{1.5}$$

Heuristically, this says that one must impose $\dim \mathcal{L} = n$ "boundary conditions" on S^* to obtain a self-adjoint operator.

To state an analogue of inequalities (1.1) for two self-adjoint extensions H_1 and H_2 of S, in terms of the corresponding Lagrangian planes $\mathcal{L}_1$ and $\mathcal{L}_2$, we will need a third Lagrangian plane $\mathcal{F}$, which corresponds to the Friedrichs extension H_F of S, namely

$$\mathcal{F} := \{(\Gamma_0 f, \Gamma_1 f) : f \in \mathrm{dom}(H_F)\}.$$

It is common in applications to choose the triple $(\mathcal{K}, \Gamma_0, \Gamma_1)$ so that the domain of the Friedrichs extension is $\ker \Gamma_0$ and thus $\mathcal{F}$ coincides with the *vertical subspace* $\mathcal{V} := 0 \oplus \mathcal{K}$. Some of the results below take a simplified form under the assumption that $\mathcal{F} = \mathcal{V}$.

Since H is a self-adjoint extension, $\mathrm{Spec}(H) \setminus \mathrm{Spec}_{\mathrm{ess}}(H)$ consists of isolated eigenvalues of finite multiplicity. For an interval I whose closure is disjoint from $\mathrm{Spec}_{\mathrm{ess}}(H)$, we define[4] the *counting function*

$$N(H; I) := \sum_{\lambda \in I} \dim \ker(H - \lambda), \tag{1.6}$$

[2] A boundary triplet exists if and only if the defect numbers of S are equal, in which case $\dim \mathcal{K}$ equals this common value, see [70, Proposition 14.5].

[3] We refer to Lagrangian subspaces as "planes" regardless of their dimension. For a review of symplectic linear algebra in complex vector spaces, we refer the reader to [34, 40] or [16, Appendix D].

[4] The requirement that $\bar{I}$ lies outside the essential spectrum guarantees $N(H; I)$ is finite.

i.e., the number of eigenvalues of H in I counted with multiplicity. We will abbreviate

$$n_-(H) := N(H; (-\infty, 0)), \quad n_0(H) := \dim \ker H \tag{1.7}$$

when these are well defined. The number $n_-(H)$ is called the *Morse index* of H. If $\lambda \in \mathbb{R}$ is below the essential spectrum, we also define the *spectral shift*

$$\sigma(H_1, H_2; \lambda) := N(H_1; (-\infty, \lambda]) - N(H_2; (-\infty, \lambda])$$

between self-adjoint extensions H_1 and H_2. Finally, assuming S is semibounded from below, we will label the eigenvalues of H below the essential spectrum by $\lambda_1(H) \leq \lambda_2(H) \leq \cdots$, i.e., in increasing order, repeated according to their multiplicity.

We are now ready to formulate our main result.

Theorem 1.1. *Suppose S is a closed, bounded from below, densely defined symmetric operator with finite and equal defect numbers and a boundary triplet $(\mathcal{K}, \Gamma_0, \Gamma_1)$. Let $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{F}$ be Lagrangian planes in $(\mathcal{K} \oplus \mathcal{K}, \omega)$ corresponding to self-adjoint extensions H_1, H_2 and the Friedrichs extension H_F of S. Define*

$$\sigma_- := \iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{F}), \quad \sigma_+ := \iota(\mathcal{L}_2, \mathcal{L}_1, \mathcal{F}), \tag{1.8}$$

where ι is the Duistermaat index (see Section 3). Then the bounds

$$-\sigma_- \leq \sigma(H_1, H_2; \lambda) \leq \sigma_+ \tag{1.9}$$

hold for all $\lambda \in \mathbb{R}$ below $\mathrm{Spec}_{\mathrm{ess}}(S)$. *Equivalently, each of the inequalities*

$$\lambda_{k-\sigma_-}(H_1) \leq \lambda_k(H_2) \leq \lambda_{k+\sigma_+}(H_1) \tag{1.10}$$

holds for all k such that the eigenvalues in question are below $\mathrm{Spec}_{\mathrm{ess}}(S)$.

It follows from elementary properties of the Duistermaat index that

$$\sigma_- + \sigma_+ = n - \dim(\mathcal{L}_1 \cap \mathcal{L}_2), \tag{1.11}$$

where $n = \dim \mathcal{L}_1 = \dim \mathcal{L}_2 = \dim \mathcal{K}$ is the defect number of the operator S. Relation (1.11) gives a shortcut for computing one index in (1.8) from the other. The fact that (1.11) is the rank of the perturbation from H_1 to H_2 (in the sense of (1.18) below) suggests that the bounds in (1.9) are optimal. Proposition 7.5 puts this on a rigorous footing: for any σ_- and σ_+, there are extensions H_1 and H_2 for which both estimates in (1.9) are sharp.

The intuition behind needing $\mathcal{F}$ is as follows. The (complex) Lagrangian Grassmannian Λ – the set of all Lagrangian planes in $\mathcal{K} \oplus \mathcal{K}$ – is diffeomorphic to the unitary group $U(n)$, a compact manifold without boundary. The plane $\mathcal{F}$ provides a point of reference in Λ, which allows us to establish facts on the ordering of the eigenvalues in $\mathbb{R}$, such as (1.10). Heuristically, H_F is the extension of S with the largest number of "eigenvalues at $\pm\infty$."[5]

One approach to Theorem 1.1 proceeds via the Maslov index of a special path of Lagrangian planes, the *Cauchy data space*, defined here with the parameter $z \in \mathbb{C}$ by

$$\mathcal{M}(z) := \{(\Gamma_0 f, \Gamma_1 f) : f \in \ker(S^* - z)\} \subset \mathcal{K} \oplus \mathcal{K}. \tag{1.12}$$

Relevant properties of $\mathcal{M}(z)$ are given in Proposition 4.5. Note that $\mathcal{M}(z)$ is the graph of the Dirichlet-to-Neumann map (Weyl–Titschmarsh function) $M(z)$ when the latter is defined.

Theorem 1.2. *Under the assumptions of Theorem 1.1, one has*

$$N(H_1; (a,b]) - N(H_2; (a,b]) = \iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{M}(b)) - \iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{M}(a)) \tag{1.13}$$

for any $[a,b] \subset \mathbb{R} \setminus \mathrm{Spec}_{\mathrm{ess}}(S)$. *In particular, the spectral shift is given by*

$$\sigma(H_1, H_2; \lambda) = \iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{M}(\lambda)) - \iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{F}) \tag{1.14}$$

for any $\lambda \in \mathbb{R}$ *below the essential spectrum.*

We remark that it is common to express eigenvalue counting functions as Maslov indices (see, for instance, the survey [6] and references therein), but only under the assumption of the unique continuation property (UCP) for the symmetric operator S: the mapping $f \mapsto (\Gamma_0 f, \Gamma_1 f)$ is injective on $\ker(S^* - z)$ for all $z \in \mathbb{C}$. Without the UCP, the Maslov index may miss some eigenvalues, as demonstrated in Section 7.5. A novel feature of Theorem 1.2 is that *the UCP is not necessary when evaluating the spectral shift*.

Another remarkable feature of Theorem 1.2 is that using a Maslov-type index (or a spectral flow, or the Kreĭn shift function) invariably involves integration or evaluation over a path (for example, continuously tracking a branch of the logarithm). In contrast, equation (1.13) involves only the data collected at the endpoints of the interval!

An easy corollary of Theorem 1.2 is the following elegant formula for the Morse index of an extension $H_{\mathcal{L}}$ of a non-negative symmetric operator S:

$$n_-(H_{\mathcal{L}}) = \iota(\mathcal{M}(0), \mathcal{L}, \mathcal{F}).$$

We refer the reader to Corollary 7.1 for a precise formulation and references to related results.

[5] See, for instance, [66, Theorem 1.4], which says a Lagrangian plane $\mathcal{L}$ intersects $\mathcal{F}$ nontrivially if and only if every neighborhood of $\mathcal{L}$ contains an $\mathcal{L}'$ whose self-adjoint extension $H_{\mathcal{L}'}$ has eigenvalues close to $-\infty$.

Another approach to Theorem 1.1 is via the resolvents of H_1 and H_2, which can be compared using the Kreĭn resolvent formula. Throughout, we use the notation $\mathcal{N}_z := \ker(S^* - z)$, and denote the number of zero, positive and negative eigenvalues of a self-adjoint operator (whenever these quantities are finite) by $n_\bullet(\cdot)$ with $\bullet \in \{0, +, -\}$; cf. (1.7).

Theorem 1.3. *Assume the setting of Theorem 1.1. For $\lambda \in \rho(H_1) \cap \rho(H_2) \cap \mathbb{R}$, we define the operator*

$$D(\lambda) := (H_1 - \lambda)^{-1} - (H_2 - \lambda)^{-1}. \tag{1.15}$$

Then one has

$$\begin{aligned} n_-(D(\lambda)) &= \iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{M}(\lambda)), \\ n_+(D(\lambda)) &= \iota(\mathcal{L}_2, \mathcal{L}_1, \mathcal{M}(\lambda)), \end{aligned} \tag{1.16}$$

$$n_0(D(\lambda)|_{\mathcal{N}_\lambda}) = \dim \mathcal{L}_1 \cap \mathcal{L}_2. \tag{1.17}$$

In addition, $D(\lambda)$ has constant rank given by

$$\operatorname{rank}(D(\lambda)) = n - \dim \mathcal{L}_1 \cap \mathcal{L}_2, \tag{1.18}$$

while the functions $\lambda \mapsto n_\pm(D(\lambda))$ are locally constant on $\rho(H_1) \cap \rho(H_2) \cap \mathbb{R}$.

In Section 6 we will explain how Theorem 1.1 can be obtained from Theorem 1.3. In particular, we will use the index formulas in (1.16) to derive the interlacing inequality (1.10).

Remark 1.4. Assume the setting of Theorem 1.3.

(1) Since H_1 and H_2 are both extensions of S, we have $D(\lambda)f = 0$ provided $f \in \mathcal{N}_\lambda^\perp = \overline{\operatorname{ran}(S - \lambda)}$, thus $\mathcal{N}_\lambda^\perp \subset \ker D(\lambda)$. In this context, the identity in (1.17) provides new information about the part of $\ker D(\lambda)$ contained in $\mathcal{N}_\lambda$.

(2) The fact that the rank of $D(\lambda)$ is constant on $\rho(H_1) \cap \rho(H_2) \cap \mathbb{R}$ was shown in [7, Theorem 2.8.1]; however, the explicit value $n - \dim \mathcal{L}_1 \cap \mathcal{L}_2$ appears to be new.

(3) The indices $n_-(D(\lambda))$ and $n_+(D(\lambda))$ may change value when λ passes through $\operatorname{Spec}(H_1) \cup \operatorname{Spec}(H_2)$, but their sum remains constant, as seen from (1.18), cf. (1.11).

1.3. Related results and possible extensions

The most general prior result on eigenvalue interlacing known to us in the context of self-adjoint extension is [19, Theorem 10.2.5]. This theorem only applies when the operator H_2 is obtained from H_1 by the restriction of its form domain. For instance, no pair of extensions from the example in Section 7.1 satisfies this condition. In contrast, our Theorem 1.1 allows one to compare *any* pair of self-adjoint extensions of S.

Relations between the Morse index of self-adjoint extensions of S and the boundary operators in $\mathcal{K} \oplus \mathcal{K}$ constructed by means of the Weyl function (as in Corollary 7.1, for example) are classical, and known at least since M. G. Kreĭn [50], Birman [17], and Derkach and Malamud [31]; see also the recent treatise [7] and the literature cited therein. However, the geometric approach via the Duistermaat index offered in the current paper allow us, on the one hand, to drop inessential restrictions (such as the form domain inclusion condition in [31, Theorems 5 and 6]) and, on the other hand, to compute the spectral shift using the Duistermaat index calculus that we review and extend in Section 3.

The inequalities (1.9) and (1.10) give different but equivalent points of view on the same result; we include both of them for completeness and ease of use in applications. However, the discrete spectral shift we consider is extended by a more flexible concept of *Krein spectral shift* [18, 20, 52, 53, 63, 70], valid also on the continuous spectrum and in the gaps. With this extension, we conjecture that (1.9) holds for all real λ. The case of a rank-one perturbation has been thoroughly investigated in [8, 11], in the more general setting of self-adjoint operators on a Kreĭn space. The latter results are formulated in the gaps of the essential spectrum (see also [10]), which gives further support to our conjecture about the universal validity of (1.9). Note, additionally, that (most of) the conclusions in Theorems 1.2 and 1.3 are already valid in the gaps.

Outline of paper

In Section 2 we review the crossing form and the computation of the Maslov index for monotone paths. In Section 3 we recall the definition of the Duistermaat index, in addition to obtaining new results on its one-sided limits and explicit formulas for its calculation via Lagrangian frames. Section 4 derives fundamental properties of the Cauchy data space, which we use in Section 5 to calculate its Maslov index and hence prove our main theorems. In Section 6 we give a second proof using the Kreĭn resolvent formula, and in Section 7 we present some applications of our results. Appendix A summarizes basic properties of the Lagrangian Grassmannian that are used throughout.

Notation and conventions

We use $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ to denote the scalar products on $\mathcal{H}$ on $\mathcal{K}$. These are taken to be linear in the *second* argument, as is the symplectic form ω in (1.4). We use $\oplus$ to denote the direct (not necessarily orthogonal) sum, and the set of all Lagrangian planes in $(\mathcal{K} \oplus \mathcal{K}, \omega)$ is denoted Λ. The zero subspace will be denoted by 0 when the ambient space is clear from the context. The space of bounded linear operators between Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ is denoted by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. We denote by $\operatorname{Spec}(\cdot)$ and $\rho(\cdot)$ the spectrum and the resolvent set.

2. Crossing forms and monotonicity

We first review crossing forms and the computation of the Maslov index for monotone paths. This suffices for the purposes of this paper; see Appendix A for a general discussion of the Maslov index and different parameterizations of Lagrangian subspaces. We work in a finite-dimensional complex symplectic space, $\mathcal{K} \oplus \mathcal{K}$, with Lagrangian Grassmannian Λ.

Let $\mathcal{L}(\cdot)\colon [0,1] \to \Lambda$ be a differentiable path of Lagrangian planes. For $t_0 \in [0,1]$ and $u, v \in \mathcal{L}(t_0)$, we define the crossing form $\mathfrak{q}$ by

$$\mathfrak{q}(u,v) := \frac{d}{dt}\omega(u, \tilde{v}(t))\Big|_{t=t_0}, \tag{2.1}$$

where $\tilde{v}(t)$ is any differentiable path in $\mathcal{K} \oplus \mathcal{K}$ such that $\tilde{v}(t_0) = v$ and $\tilde{v}(t) \in \mathcal{L}(t)$ for all t near t_0. For this to be a valid definition, we must show that it does not depend on the choice of the path $\tilde{v}$.

Suppose $\tilde{u}(t)$ is a differentiable path in $\mathcal{K} \oplus \mathcal{K}$ with $\tilde{u}(t_0) = u$ and $\tilde{u}(t) \in \mathcal{L}(t)$. The fact that $\tilde{u}(t)$ and $\tilde{v}(t)$ are both in $\mathcal{L}(t)$ implies $\omega(\tilde{u}(t), \tilde{v}(t)) = 0$. Differentiating at t_0, we get

$$\omega(u, \tilde{v}'(t_0)) = -\omega(\tilde{u}'(t_0), v) = \overline{\omega(v, \tilde{u}'(t_0))}.$$

Since the left-hand side does not depend on $\tilde{u}(t)$ and the right-hand side does not depend on $\tilde{v}(t)$, both sides are path independent. This proves that $\mathfrak{q}(u,v)$ is well defined, and is equal to $\overline{\mathfrak{q}(v,u)}$.

We now give some equivalent expressions for the quadratic form $\mathfrak{q}[v] := \mathfrak{q}(v,v)$. The analogous expressions for the sesquilinear form $\mathfrak{q}(u,v)$ can be obtained by polarization.

Theorem 2.1. *Let $\mathcal{L}(\cdot)\colon [0,1] \to \Lambda$ be a differentiable path of Lagrangian planes. Fix $v \in \mathcal{L}(t_0)$ and let $\widehat{\mathcal{L}}$ be a Lagrangian subspace transversal to $\mathcal{L}(t_0)$.*

(1) *If $w(t)$ is the unique path in $\widehat{\mathcal{L}}$ for which $v + w(t) \in \mathcal{L}(t)$, then*

$$\mathfrak{q}[v] = \frac{d}{dt}\omega(v, w(t))\Big|_{t=t_0}. \tag{2.2}$$

(2) *If $\mathcal{L}^\sharp$ is transversal to $\widehat{\mathcal{L}}$, $L_t\colon \mathcal{L}^\sharp \to \widehat{\mathcal{L}}$ is a differentiable family of operators such that $\mathcal{L}(t) = \{u + L_t u \colon u \in \mathcal{L}^\sharp\}$ (i.e., $\mathcal{L}(t)$ is the graph of L_t) and $u \in \mathcal{L}^\sharp$*

is such that $v = u + L_{t_0} u$, *then*

$$\mathfrak{q}[v] = \frac{d}{dt}\omega(u, L_t u)\Big|_{t=t_0}. \tag{2.3}$$

(3) *If* $Z(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$ *is a differentiable frame for* $\mathcal{L}(t)$ *and* $\kappa \in \mathcal{K}$ *is such that* $v = Z(t_0)\kappa$, *then*

$$\mathfrak{q}[v] = \langle \kappa, (X^*(t_0)Y'(t_0) - Y^*(t_0)X'(t_0))\kappa \rangle_{\mathbb{C}^n}.$$

(4) *If* G_t *is a differentiable family of operators on* $\mathcal{K} \oplus \mathcal{K}$ *such that* $G_t \mathcal{L}(t_0) = \mathcal{L}(t)$ *and* $G_{t_0} = I_{\mathcal{K}\oplus\mathcal{K}}$, *then*

$$\mathfrak{q}[v] = \frac{d}{dt}\omega(v, G_t v)\Big|_{t=t_0}. \tag{2.4}$$

Note that (2.2) is the definition of $\mathfrak{q}$ given by Robbin and Salamon in [69, Theorem 1.1], so this theorem shows that our definition in (2.1) is equivalent to theirs. Similarly, choosing $\mathcal{L}^\sharp = \mathcal{L}(t_0)$ in (2.3) recovers the definition in [22, eq. (2.1)].

Proof. The given expressions for $\mathfrak{q}[v]$ follow from (2.1) with appropriate choices of the path $\tilde{v}(t)$, namely $v + w(t)$, $u + L_t u$, $Z(t)\kappa$ and $G_t v$. ■

The crossing form allows us to define a notion of monotonicity for differentiable paths.

Definition 2.2. A differentiable path $\mathcal{L}(\cdot)\colon (0,1) \to \Lambda$ is *non-decreasing* (corresp. *increasing*) if the crossing form $\mathfrak{q} = \mathfrak{q}_{t_0}$ on $\mathcal{L}(t_0)$ is non-negative (corresp. positive) at every t_0.

The Maslov index of an increasing C^1 path $\mathcal{M}(\cdot)$, with reference plane $\mathcal{L}$, is given by

$$\operatorname{Mas}_{[a,b]}(\mathcal{M}(\cdot), \mathcal{L}) = \sum_{t\in[a,b)} \dim(\mathcal{M}(t) \cap \mathcal{L}). \tag{2.5}$$

The fact that $\mathfrak{q}$ is positive (and hence non-degenerate) guarantees that the crossings are isolated, therefore the above sum is finite. This is a special case of [25, Proposition 3.27]; see (A.7) for the general formula. In practice, we will use the equivalent formula

$$\operatorname{Mas}_{[a,b]}(\mathcal{L}, \mathcal{M}(\cdot)) = -\sum_{t\in(a,b]} \dim(\mathcal{M}(t) \cap \mathcal{L}). \tag{2.6}$$

which is obtained from (2.5) using the identity (A.6). Note that the sum in (2.5) is over $t \in [a,b)$, whereas the sum in (2.6) is over $t \in (a,b]$.

Remark 2.3. Our notion of path monotonicity is the standard one for the Lagrangian Grassmannian [3] and is local in nature. If $\mathcal{L}(t)$ never intersects the vertical subspace $\mathcal{V} = 0 \oplus \mathcal{K}$, then this coincides with the partial ordering defined in [7, Section 5.2] for families of self-adjoint linear relations. Namely, $\mathcal{L}(\cdot)$ is non-decreasing in the sense of Definition 2.2 if and only if $\mathcal{L}(t_1) \leq \mathcal{L}(t_2)$ for all $t_1 \leq t_2$ in the sense of [7, Definition 5.2.3].

This is no longer the case if $\mathcal{L}(t)$ intersects $\mathcal{V}$ at some time. A simple example is $\mathcal{L}(t) = \{(z \cos t, z \sin t) : z \in \mathbb{C}\}$, which is Lagrangian in $\mathbb{C}^2$ for real values of t. At an arbitrary point $v = (z_0 \cos t_0, z_0 \sin t_0) \in \mathcal{L}(t_0)$, we compute $\mathfrak{q}[v] = |z_0|^2$, and conclude that $\mathcal{L}(\cdot)$ is increasing in the sense of Definition 2.2. On the other hand, for small positive ε, $\mathcal{L}(\pi/2 + \varepsilon) \leq \mathcal{L}(\pi/2 - \varepsilon)$ in the sense of [7, Definition 5.2.3].

3. The Duistermaat triple index

Again, assuming that $\mathcal{K} \oplus \mathcal{K}$ is a finite-dimensional complex symplectic space, we now recall the definition of the *Duistermaat index* ι, which first appeared in [32, eq. (2.16)]. It is closely related to other symplectically invariant triple indices, such as Kashiwara–Wall index [27, 64, 73] – also known as the "triple signature" [62, Appendix 6.2] – and Leray–de Gosson index of inertia defined in [30, Definition 148] (generalizing [61, Section I.2.4] to remove the assumption of transversality). In fact, there is only one non-trivial symplectic invariant of a triple of Lagrangian planes [1, Proposition 4.4], and the above indices are just different incarnations of it.

When describing the Duistermaat index, we follow the original definition [32, eq. (2.16)], but use the notational conventions of [75], whose results we will use below. An alternative axiomatic approach to the Duistermaat index is presented in our follow-up work [13].

3.1. Definition and basic properties

For Lagrangian planes α, β, and γ such that $\alpha \cap \beta = 0 = \beta \cap \gamma$, we can view γ as the graph of a linear mapping $L: \alpha \to \beta$, i.e., $\gamma = \{u + Lu : u \in \alpha\}$. This gives rise to a bilinear form $Q(\alpha, \beta; \gamma): \alpha \times \alpha \to \mathbb{C}$ acting by

$$Q(\alpha, \beta; \gamma): (u_1, u_2) \mapsto \omega(u_1, Lu_2). \tag{3.1}$$

It is easily shown that

$$\ker Q(\alpha, \beta; \gamma) = \alpha \cap \gamma. \tag{3.2}$$

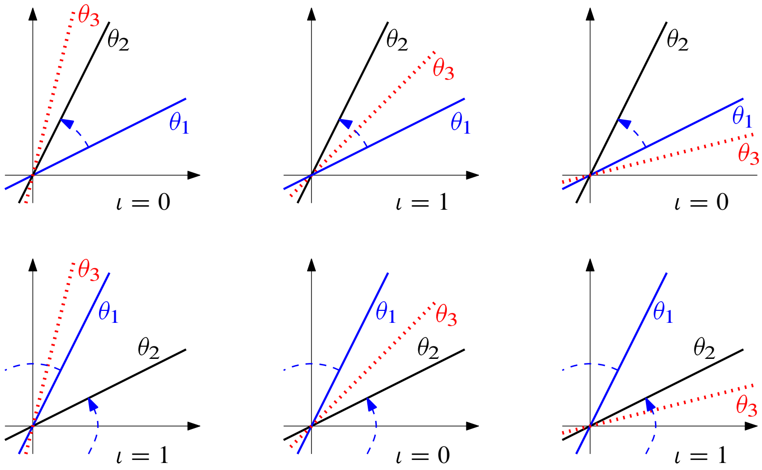

Figure 1. Possible transversal configurations of three Lagrangian planes (illustrated in $\mathbb{R}^2$) and the corresponding values of the Duistermaat index.

For arbitrary $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \in \Lambda$, we choose an $\widehat{\mathcal{L}}$ that is transversal to all three and define the *Duistermaat index* of $(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$ to be[6]

$$\begin{aligned}\iota(&\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \\ &:= n_-(Q(\mathcal{L}_2, \widehat{\mathcal{L}}; \mathcal{L}_3)) - n_-(Q(\mathcal{L}_1, \widehat{\mathcal{L}}; \mathcal{L}_3)) + n_-(Q(\mathcal{L}_1, \widehat{\mathcal{L}}; \mathcal{L}_2)).\end{aligned} \tag{3.3}$$

One can geometrically describe the integer $\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$ as the maximal dimension of a subspace $\widehat{\mathcal{L}}_3 \subset \mathcal{L}_3$ which lies between $\mathcal{L}_1$ and $\mathcal{L}_2$, where "between" is defined in terms of the positive direction of rotation in the Lagrangian Grassmannian, as introduced in Section 2. This geometric interpretation, which is not immediately obvious from the definition, is illustrated by the following example (see also Corollary 3.7).

Example 3.1. Consider the Lagrangian planes $\mathcal{L}_j = \{(z, \theta_j z) : z \in \mathbb{C}\}$ in the symplectic space $\mathbb{C}^2$, with $\theta_j \in \mathbb{R}$ and $j = 1, 2, 3$. Simple calculations (for example, using Theorem 3.5 where we can take $\varepsilon = 0$) show that

$$\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) = \begin{cases} 0, & \theta_1 \le \theta_2 \le \theta_3 \text{ or } \theta_3 < \theta_1 \le \theta_2 \text{ or } \theta_2 \le \theta_3 < \theta_1, \\ 1, & \theta_1 \le \theta_3 < \theta_2 \text{ or } \theta_2 < \theta_1 \le \theta_3 \text{ or } \theta_3 < \theta_2 < \theta_1. \end{cases}$$

These results are illustrated in Figure 1.

[6]The negative index of a bilinear form is defined as the maximal dimension of a subspace on which the form is negative definite. Alternatively, one may count the number of negative eigenvalues of the corresponding Hermitian matrix, as in (1.7).

We next recall some basic properties of ι that will be useful later. The Duistermaat index ι is a symplectic invariant: for any symplectic automorphism g of $(\mathcal{K} \oplus \mathcal{K}, \omega)$, we have

$$\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) = \iota(g(\mathcal{L}_1), g(\mathcal{L}_2), g(\mathcal{L}_3)). \tag{3.4}$$

For any Lagrangian $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, and $\mathcal{L}_4$, ι satisfies the *cocycle property*:

$$\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) - \iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_4) + \iota(\mathcal{L}_1, \mathcal{L}_3, \mathcal{L}_4) - \iota(\mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4) = 0.$$

This follows from [75, Theorem 1.1], cf. [64, Proposition 1.5.8], [29, eq. (I.2.13)], [27, Section 8, Proposition VI]. In fact, the definition in (3.3) can be interpreted as letting

$$\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) = n_-(Q(\mathcal{L}_1, \mathcal{L}_3; \mathcal{L}_2)), \quad \text{assuming } \mathcal{L}_1 \cap \mathcal{L}_3 = 0 = \mathcal{L}_2 \cap \mathcal{L}_3, \tag{3.5}$$

then using the cocycle property to extend to Lagrangian planes with no transversality assumptions. Equation (3.5) will be derived from the definition (3.3) of ι in Corollary 3.3 below (see also [32, Lemma 2.4] and [75, Lemma 3.13]).

From [32, Lemma 2.4], we immediately get an estimate

$$0 \le \iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \le n - \dim((\mathcal{L}_1 \cap \mathcal{L}_2) + (\mathcal{L}_2 \cap \mathcal{L}_3)) \le n - \dim \mathcal{L}_1 \cap \mathcal{L}_2. \tag{3.6}$$

Under permutation of the first two arguments, we have

$$\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) + \iota(\mathcal{L}_2, \mathcal{L}_1, \mathcal{L}_3) = n - \dim \mathcal{L}_1 \cap \mathcal{L}_2. \tag{3.7}$$

This follows, for instance, from Theorem 3.5 below. We also have the cyclic identity

$$\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) - \dim \mathcal{L}_1 \cap \mathcal{L}_3 = \iota(\mathcal{L}_3, \mathcal{L}_1, \mathcal{L}_2) - \dim \mathcal{L}_2 \cap \mathcal{L}_3, \tag{3.8}$$

which follows from [75, Lemma 3.2 and Lemma 3.13]. Combining (3.7) and (3.8), we obtain identities for other permutations,

$$\begin{aligned} \iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) + \iota(\mathcal{L}_1, \mathcal{L}_3, \mathcal{L}_2) &= n - \dim \mathcal{L}_2 \cap \mathcal{L}_3, \\ \iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) + \iota(\mathcal{L}_3, \mathcal{L}_2, \mathcal{L}_1) &= n - \dim \mathcal{L}_1 \cap \mathcal{L}_2 \\ &\qquad - \dim \mathcal{L}_2 \cap \mathcal{L}_3 + \dim \mathcal{L}_1 \cap \mathcal{L}_3, \end{aligned} \tag{3.9}$$

and the important special cases

$$\iota(\mathcal{L}, \mathcal{L}, \mathcal{L}_3) = 0, \quad \iota(\mathcal{L}_1, \mathcal{L}, \mathcal{L}) = 0, \quad \iota(\mathcal{L}, \mathcal{L}_2, \mathcal{L}) = n - \dim \mathcal{L} \cap \mathcal{L}_2. \tag{3.10}$$

We finally recall an identity of Zhou, Wu, and Zhu that relates the difference of Maslov indices with different reference planes to the difference of Duistermaat indices:

$$\begin{aligned} &\mathrm{Mas}_{[a,b]}(\mathcal{L}_2, \mathcal{M}(\cdot)) - \mathrm{Mas}_{[a,b]}(\mathcal{L}_1, \mathcal{M}(\cdot)) \\ &\quad = \iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{M}(b)) - \iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{M}(a)) \end{aligned} \tag{3.11}$$

for any continuous path $\mathcal{M}(\cdot)$; see [75, Theorem 1.1]. The difference of Maslov indices is also known as the *Hörmander index* [42].

3.2. One-sided limits

The notion of monotonicity in Definition 2.2 allows us to compute one-sided limits of the Duistermaat index.

Theorem 3.2. *Suppose* $\mathcal{L}\colon(-1,1)\to\Lambda$ *is a continuous path that is differentiable and increasing on* $(-1,0)\cup(0,1)$*, and set* $\mathcal{L}_0 := \mathcal{L}(0)$*. For any* $\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3\in\Lambda$ *and* $0<|t|\ll 1$*, we have*

$$\iota(\mathcal{L}(t),\mathcal{L}_2,\mathcal{L}_3) = \iota(\mathcal{L}_0,\mathcal{L}_2,\mathcal{L}_3) + \begin{cases} 0, & t<0,\\ \dim\mathcal{L}_2\cap\mathcal{L}_0 - \dim\mathcal{L}_3\cap\mathcal{L}_0, & t>0, \end{cases} \tag{3.12}$$

$$= \begin{cases} \iota(\mathcal{L}_0,\mathcal{L}_2,\mathcal{L}_3), & t<0,\\ \iota(\mathcal{L}_2,\mathcal{L}_3,\mathcal{L}_0), & t>0, \end{cases}$$

$$\iota(\mathcal{L}_1,\mathcal{L}(t),\mathcal{L}_3) = \iota(\mathcal{L}_1,\mathcal{L}_0,\mathcal{L}_3) + \begin{cases} \dim\mathcal{L}_1\cap\mathcal{L}_0, & t<0,\\ \dim\mathcal{L}_3\cap\mathcal{L}_0, & t>0, \end{cases} \tag{3.13}$$

$$\iota(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}(t)) = \iota(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_0) + \begin{cases} \dim\mathcal{L}_2\cap\mathcal{L}_0 - \dim\mathcal{L}_1\cap\mathcal{L}_0, & t<0,\\ 0, & t>0, \end{cases} \tag{3.14}$$

$$= \begin{cases} \iota(\mathcal{L}_0,\mathcal{L}_1,\mathcal{L}_2), & t<0,\\ \iota(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_0), & t>0. \end{cases} \tag{3.15}$$

In particular, we see that the index $\iota(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3)$ is left-continuous in $\mathcal{L}_1$ and right-continuous in $\mathcal{L}_3$, but in general is neither right- nor left-continuous in $\mathcal{L}_2$.

Proof. We first prove (3.14), as it follows most directly from the definition of ι. Choose $\widehat{\mathcal{L}}$ that is transversal to $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_0$ (and hence to $\mathcal{L}(t)$ for small t). Recalling the definition in (3.3), we have

$$\begin{aligned}\iota(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}(t)) \\ = n_-(Q(\mathcal{L}_2,\widehat{\mathcal{L}};\mathcal{L}(t))) - n_-(Q(\mathcal{L}_1,\widehat{\mathcal{L}};\mathcal{L}(t))) + n_-(Q(\mathcal{L}_1,\widehat{\mathcal{L}};\mathcal{L}_2)).\end{aligned} \tag{3.16}$$

Starting with the first term on the right-hand side, we abbreviate

$$Q_t := Q(\mathcal{L}_2,\widehat{\mathcal{L}};\mathcal{L}(t)).$$

This acts by $Q_t[u] = \omega(u, L_t u)$, where $L_t \colon \mathcal{L}_2 \to \widehat{\mathcal{L}}$ is such that $\mathcal{L}(t) = \{u + L_t u \colon u \in \mathcal{L}_2\}$. From Theorem 2.1, namely (2.3) with $\mathcal{L}^\sharp = \mathcal{L}_2$, we see that $Q_t'[u] = \mathfrak{q}[u + L_t u]$ for any $u \in \mathcal{L}_2$, therefore Q_t' is positive for $t \in (-1, 0) \cup (0, 1)$. It follows from the mean value theorem that Q_t is increasing on $(-1, 1)$, therefore $n_-(Q_t) = n_-(Q_0)$ for small $t > 0$, and

$$n_-(Q_t) = n_-(Q_0) + \dim\ker Q_0 = n_-(Q_0) + \dim \mathcal{L}_2 \cap \mathcal{L}_0$$

for small $t < 0$, where we have used (3.2). An analogous formula holds for the term $n_-(Q(\mathcal{L}_1, \widehat{\mathcal{L}}; \mathcal{L}(t)))$. Using this in (3.16) completes the proof of (3.14).

To prove (3.12), we combine (3.14) with the identities

$$\begin{aligned}
&\iota(\mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_0) = \iota(\mathcal{L}_0, \mathcal{L}_2, \mathcal{L}_3) + \dim \mathcal{L}_2 \cap \mathcal{L}_0 - \dim \mathcal{L}_3 \cap \mathcal{L}_0,\\
&\iota(\mathcal{L}_2, \mathcal{L}_3, \mathcal{L}(t)) = \iota(\mathcal{L}(t), \mathcal{L}_2, \mathcal{L}_3), \quad 0 < |t| \ll 1,
\end{aligned}$$

which follow from (3.8) and the observation that $\mathcal{L}(t)$ is transversal to $\mathcal{L}_2$ and $\mathcal{L}_3$ except at isolated values of t. The proof of (3.13) is analogous. ■

Corollary 3.3. *Under the assumption $\mathcal{L}_1 \cap \mathcal{L}_3 = 0 = \mathcal{L}_2 \cap \mathcal{L}_3$, we have*

$$\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) = n_-(Q(\mathcal{L}_1, \mathcal{L}_3; \mathcal{L}_2)).$$

Proof. Let $\mathcal{L}(t)$ be an increasing path with $\mathcal{L}(0) = \mathcal{L}_3$. Since $\mathcal{L}_3$ is transversal to $\mathcal{L}_1$ and $\mathcal{L}_2$, Theorem 3.2 implies $\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) = \iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}(t))$ for $|t| \ll 1$. For small non-zero t, we can choose $\widehat{\mathcal{L}} = \mathcal{L}_3$ in the definition (3.3) of $\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}(t))$ to get

$$\begin{aligned}
&\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}(t))\\
&\quad = n_-(Q(\mathcal{L}_2, \mathcal{L}_3; \mathcal{L}(t))) - n_-(Q(\mathcal{L}_1, \mathcal{L}_3; \mathcal{L}(t))) + n_-(Q(\mathcal{L}_1, \mathcal{L}_3; \mathcal{L}_2)).
\end{aligned}$$

Using the identity $n_-(Q(\alpha, \beta; \gamma)) + n_-(Q(\beta, \alpha; \gamma)) = n$, valid when α, β, and γ are pairwise transversal, we can rewrite this as

$$\begin{aligned}
&\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}(t))\\
&\quad = n_-(Q(\mathcal{L}_3, \mathcal{L}_1; \mathcal{L}(t))) - n_-(Q(\mathcal{L}_3, \mathcal{L}_2; \mathcal{L}(t))) + n_-(Q(\mathcal{L}_1, \mathcal{L}_3; \mathcal{L}_2)).
\end{aligned}$$

The quadratic form $Q(\mathcal{L}_3, \mathcal{L}_1; \mathcal{L}(t))$ is increasing in t and is identically zero when $t = 0$, so $n_-(Q(\mathcal{L}_3, \mathcal{L}_1; \mathcal{L}(t))) = 0$ for small positive t and similarly for $n_-(Q(\mathcal{L}_3, \mathcal{L}_2; \mathcal{L}(t)))$. ■

3.3. Computing with frames

We now give a simple formula for computing the Duistermaat index using linear algebra. First, we recall that any n-dimensional subspace $\mathcal{M} \subset \mathcal{K} \oplus \mathcal{K}$ can be described by a *frame*, which is an injective linear operator

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix} : \mathcal{K} \to \mathcal{K} \oplus \mathcal{K},$$

whose range is $\mathcal{M}$. Moreover, $\mathcal{M}$ is Lagrangian if and only if $X^*Y = Y^*X$ (a review of this and other parametrizations of Lagrangian planes is given in Appendix A.2).

This description is not unique, but it is easy to see that frames Z and $\widetilde{Z}$ describe the same subspace if and only if $\widetilde{Z} = ZC$ for some invertible $C : \mathcal{K} \to \mathcal{K}$. Therefore, the set

$$E(\mathcal{M}) := \{\varepsilon \in \mathbb{R} : X + \varepsilon Y \text{ is not invertible}\}$$

and the operator

$$R^\varepsilon := Y(X + \varepsilon Y)^{-1}, \quad \varepsilon \in \mathbb{R} \setminus E(\mathcal{M}), \tag{3.17}$$

are independent of the choice of frame. The set $E(\mathcal{M})$ is finite, since $\det(X + \varepsilon Y)$ is a polynomial in ε that is not identically zero because Z has rank n. When $\mathcal{M}$ is Lagrangian, the condition $X^*Y = Y^*X$ implies that

$$(X + \varepsilon Y)^* R^\varepsilon (X + \varepsilon Y) = X^*Y + \varepsilon Y^*Y$$

is Hermitian, therefore R^ε is Hermitian.

Remark 3.4. For the Cauchy data space $\mathcal{M}(z)$ defined in (1.12), the corresponding $R^\varepsilon(z)$ acts by $u \mapsto \Gamma_1 f$, where $f \in \ker(S^* - z)$ satisfies $\Gamma_0 f + \varepsilon \Gamma_1 f = u$. (The condition $\varepsilon \in \mathbb{R} \setminus E(\mathcal{M}(z))$ guarantees there is a unique such f for each $u \in \mathcal{K}$.) In particular, $R^0(z)$ is the Dirichlet-to-Neumann map, whenever it is defined. We thus refer to the operator R^ε in (3.17) as the *ε-Robin map*, whether or not the corresponding subspace $\mathcal{M}$ is the Cauchy data space.

An intuitive description of R^ε is the "regularized slope" of $\mathcal{M}$, as drawn in $\mathcal{K} \oplus \mathcal{K}$. Referring to Figure 1, it is therefore natural that the Duistermaat index can be computed by comparing slopes of pairs of planes, in the following sense.

Theorem 3.5. *Let R_1^ε, R_2^ε, and R_3^ε be the ε-Robin maps for the Lagrangian planes $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$. Then the Duistermaat index $\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$ is given by*

$$\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) = n_-(R_2^\varepsilon - R_1^\varepsilon) + n_-(R_3^\varepsilon - R_2^\varepsilon) - n_-(R_3^\varepsilon - R_1^\varepsilon) \tag{3.18}$$

for any $\varepsilon \in \mathbb{R} \setminus E_{123}$, where

$$E_{123} := E(\mathcal{L}_1) \cup E(\mathcal{L}_2) \cup E(\mathcal{L}_3)$$

is a finite set. In particular, for the vertical plane $\mathcal{V} = 0 \oplus \mathcal{K}$ *we have*

$$\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{V}) = n_-(R_2^\varepsilon - R_1^\varepsilon) \tag{3.19}$$

for any $0 < \varepsilon \ll 1$.

Proof. We first assume that $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ are transversal to $\mathcal{V}$ and establish (3.18) with $\varepsilon = 0$, using the definition (3.3) with $\widehat{\mathcal{L}} = \mathcal{V}$. In this case, each $\mathcal{L}_j$ is represented by the frame (I, R_j^0). We will use this to compute the index of $Q(\mathcal{L}_2, \mathcal{V}; \mathcal{L}_3)$ to be used in (3.3). To represent $\mathcal{L}_3$ as the graph of an operator $L\colon \mathcal{L}_2 \to \mathcal{V}$, we write

$$\mathcal{L}_3 \ni \begin{pmatrix} \kappa \\ R_3^0 \kappa \end{pmatrix} = \begin{pmatrix} \kappa \\ R_2^0 \kappa \end{pmatrix} + \begin{pmatrix} 0 \\ R_3^0 \kappa - R_2^0 \kappa \end{pmatrix} = u + Lu \in \mathcal{L}_2 + \mathcal{V}.$$

For any $u_1 = (\kappa_1, R_2^0 \kappa_1)^\intercal$ and $u_2 = (\kappa_2, R_2^0 \kappa_2)^\intercal$ in $\mathcal{L}_2$, we thus obtain

$$Q(\mathcal{L}_2, \mathcal{V}; \mathcal{L}_3)\colon (u_1, u_2) \mapsto \omega(u_1, Lu_2) = \langle \kappa_1, (R_3^0 - R_2^0)\kappa_2 \rangle_{\mathcal{K}}.$$

Since the index is invariant under isomorphism, the form $Q(\mathcal{L}_2, \mathcal{V}; \mathcal{L}_3)$ on $\mathcal{L}_2$ has the same index as the form $\langle \cdot, (R_3^0 - R_2^0)\cdot \rangle_{\mathcal{K}}$ on $\mathcal{K}$, that is

$$n_-(Q(\mathcal{L}_2, \mathcal{V}; \mathcal{L}_3)) = n_-(R_3^0 - R_2^0).$$

Evaluating the other two terms in (3.3) similarly, we obtain (3.18) with $\varepsilon = 0$, i.e.,

$$\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) = n_-(R_2^0 - R_1^0) + n_-(R_3^0 - R_2^0) - n_-(R_3^0 - R_1^0). \tag{3.20}$$

In the general case, we can use the symplectic transformation

$$g_\varepsilon = \begin{pmatrix} I & \varepsilon I \\ 0 & I \end{pmatrix}$$

to make $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ transversal to $\mathcal{V}$. In other words, we consider Lagrangian planes

$$\mathcal{L}_j^\varepsilon := g^\varepsilon(\mathcal{L}_j) = \operatorname{ran} \begin{pmatrix} X_j + \varepsilon Y_j \\ Y_j \end{pmatrix},$$

with ε chosen so that all $X_j + \varepsilon Y_j$ are invertible. From (3.4), we have $\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) = \iota(\mathcal{L}_1^\varepsilon, \mathcal{L}_2^\varepsilon, \mathcal{L}_3^\varepsilon)$, so we can compute $\iota(\mathcal{L}_1^\varepsilon, \mathcal{L}_2^\varepsilon, \mathcal{L}_3^\varepsilon)$ according to (3.20) and thus obtain (3.18).

Finally, we consider the case $\mathcal{L}_3 = \mathcal{V}$ and prove (3.19). We have $R_3^\varepsilon = \varepsilon^{-1} I$, so the result follows once we establish that for any Lagrangian frame $(X, Y)^\intercal$, the operator $\varepsilon^{-1} I - Y(X + \varepsilon Y)^{-1}$ is non-negative definite. It is equivalent to consider

$$\varepsilon(X + \varepsilon Y)^*(\varepsilon^{-1} I - Y(X + \varepsilon Y)^{-1})(X + \varepsilon Y) = X^* X + \varepsilon Y^* X.$$

The right-hand side is a Hermitian perturbation of the non-negative operator X^*X. As functions of ε, zero eigenvalues of the unperturbed problem remain identically zero, since the perturbation Y^*X vanishes on $\ker(X^*X) = \ker X$. On the other hand, the non-zero eigenvalues are positive at $\varepsilon = 0$ and thus remain bounded away from zero for small ε. ■

Using Theorem 3.5, we obtain a formula for the Duistermaat index in the case when $\mathcal{L}_3 = \mathcal{V} = 0 \oplus \mathcal{K}$ and $\mathcal{L}_1 \cap \mathcal{V} = 0$. We remark here that any Lagrangian plane can be described in terms of a frame $(X, Y) = (P, P\Theta P + P - I)$, where $P : \mathcal{K} \to \mathcal{K}$ is an orthogonal projector and Θ is a Hermitian operator acting on ran P; see Section A.2.

Proposition 3.6. *Suppose the planes $\mathcal{L}_1$ and $\mathcal{L}_2$ are described by the frames (I, M) and $(P, P\Theta P + P - I)$, respectively. Then*

$$\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{V}) = n_-(\Theta - PMP).$$

This result is inspired by a counting formula in [9] for the eigenvalues of the Laplacian on a metric graph, which we will rederive in Section 7.2.

Proof. Let $\mathcal{L}(t)$ denote the path given by the frames $(I, M + tI)$. Since $\mathcal{L}(t)$ is increasing and $\mathcal{L}(0) = \mathcal{L}_1$, Theorem 3.2 gives

$$\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) = \lim_{t \to 0-} \iota(\mathcal{L}(t), \mathcal{L}_2, \mathcal{L}_3).$$

We now use (3.19) to compute $\iota(\mathcal{L}(t), \mathcal{L}_2, \mathcal{L}_3)$. Writing $R_j^\varepsilon = Y_j(X_j + \varepsilon Y_j)^{-1}$ in block form corresponding to the decomposition $\mathcal{K} = \ker P \oplus \operatorname{ran} P$, we obtain

$$R_2^\varepsilon - R_1^\varepsilon = \begin{pmatrix} \varepsilon^{-1} I - M_{11} - tI & -M_{12} \\ -M_{21} & \Theta - M_{22} - tI \end{pmatrix} + O(\varepsilon),$$

where $M_{11} = (I - P)M(I - P)$, $M_{22} = PMP$ and so on. The top-left block is strictly positive (and in particular invertible) for small $\varepsilon > 0$, so the Haynsworth formula [41] implies

$$\begin{aligned} n_-(R_2^\varepsilon - R_1^\varepsilon) &= n_-(\Theta - M_{22} - tI + O(\varepsilon) + M_{21}(\varepsilon^{-1} I + O(1))^{-1} M_{12}) \\ &= n_-(\Theta - M_{22} - tI + O(\varepsilon)) \end{aligned}$$

as $\varepsilon \to 0$. The operator $\Theta - M_{22} - tI_2$ is invertible for all t in some interval $(t_*, 0)$, therefore

$$\begin{aligned} \iota(\mathcal{L}(t), \mathcal{L}_2, \mathcal{L}_3) &= \lim_{\varepsilon \to 0+} n_-(R_2^\varepsilon - R_1^\varepsilon) \\ &= n_-(\Theta - M_{22} - tI_2) = n_-(\Theta - M_{22} + |t| I_2). \end{aligned}$$

On the other hand, since negative eigenvalues cannot be produced by a small positive perturbation, we have, for sufficiently small t,

$$\iota(\mathcal{L}(t), \mathcal{L}_2, \mathcal{L}_3) = n_-(\Theta - M_{22}) = n_-(\Theta - PMP),$$

completing the proof. ∎

Finally, we give a corollary that will be useful in applications, and also clearly illustrates the idea that the index $\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$ quantifies how much of $\mathcal{L}_3$ lies "between" $\mathcal{L}_1$ and $\mathcal{L}_2$ (in the special case that $\mathcal{L}_1$ and $\mathcal{L}_2$ are horizontal and vertical, respectively).

Corollary 3.7. *Let $\widehat{\mathcal{K}} \subseteq \mathcal{K}$ be a subspace and $\Theta\colon \widehat{\mathcal{K}} \to \widehat{\mathcal{K}}$ a self-adjoint operator. Denote its number of non-negative eigenvalues by $n_{0+}(\Theta)$ and consider the Lagrangian plane $\mathcal{L}_\Theta := \{(\kappa, \kappa' + \Theta\kappa)\colon \kappa \in \widehat{\mathcal{K}},\ \kappa' \in \widehat{\mathcal{K}}^\perp\}$. Then*

$$\iota(\mathcal{K} \oplus 0, 0 \oplus \mathcal{K}, \mathcal{L}_\Theta) = n_{0+}(\Theta).$$

Remark 3.8. If we view $\mathcal{L}_\Theta$ as a self-adjoint linear relation from $\mathcal{K}$ to $\mathcal{K}$, then Θ is the "operator part" of $\mathcal{L}_\Theta$, as in [70, Proposition 14.2].

Proof. Using Proposition 3.6 with $\mathcal{L}_1 = \mathcal{K} \oplus 0$, $\mathcal{L}_2 = \mathcal{L}_\Theta$, $M = 0$ and P being the orthogonal projector onto $\widehat{\mathcal{K}}$, we get $\iota(\mathcal{K} \oplus 0, \mathcal{L}_\Theta, 0 \oplus \mathcal{K}) = n_-(\Theta)$. Since

$$n - \dim(\mathcal{L}_\Theta \cap (0 \oplus \mathcal{K})) = n - \dim \widehat{\mathcal{K}}^\perp = \dim \widehat{\mathcal{K}} = n_-(\Theta) + n_{0+}(\Theta),$$

the result follows from (3.9). ∎

4. The Cauchy data space

The main object in the proof of Theorems 1.1 and 1.2 is the Cauchy data space

$$\mathcal{M}(z) = \mathbf{\Gamma}(\ker(S^* - z)) \subset \mathcal{K} \oplus \mathcal{K} \tag{4.1}$$

introduced in (1.12), where $\mathbf{\Gamma} = (\Gamma_0, \Gamma_1)$. We now establish its fundamental properties, which will be needed below, in particular in the proofs of Proposition 5.3 and Theorem 6.2. In this section we allow $\mathcal{K}$ to be infinite dimensional, as the results presented herein are of independent interest.

Recall that the *deficiency* of a closed operator $T\colon X \to Y$ is the codimension of $\operatorname{ran} T$ in Y, i.e., $\operatorname{def} T := \dim(Y/\operatorname{ran} T)$, see [33, Section I.3]. We define

$$\Phi_-(T) := \{z \in \mathbb{C}\colon \operatorname{def}(T - z) < \infty\}$$

and observe that $z \in \Phi_-(T)$ implies $\operatorname{ran}(T - z)$ is closed, by [33, Theorem I.3.2]. If S is closed and symmetric, then

$$\mathbb{C} \setminus \operatorname{Spec}_{\mathrm{ess}}(H) \subseteq \Phi_-(S^*) \tag{4.2}$$

for any self-adjoint extension H of S, with equality when the defect numbers of S are finite; see [33, Corollary IX.4.2].

We now show that $\mathcal{M}(z)$ depends on z analytically as long as $z \in \Phi_-(S^*)$.

Theorem 4.1. *Let S be a closed, densely defined symmetric operator on a Hilbert space $\mathcal{H}$ with equal (possibly infinite) defect numbers, and let $(\mathcal{K}, \Gamma_0, \Gamma_1)$ be a boundary triplet. In a neighborhood of any $z_0 \in \Phi_-(S^*)$, there exists an analytic family of invertible operators $G_z \in \mathcal{B}(\mathcal{K} \oplus \mathcal{K})$ such that $G_{z_0} = I_{\mathcal{K}\oplus\mathcal{K}}$ and $G_z \mathcal{M}(z_0) = \mathcal{M}(z)$.*

An equivalent formulation of the theorem is that $\mathcal{M}(z)$ is an *analytic Banach bundle* over $\Phi_-(S^*)$; see [74] for definitions. To compare this to previous results in the literature, we first recall that S has the *unique continuation property* (or, equivalently, has no *inner solutions*) if $\ker(S^* - z) \cap \ker(\Gamma_0) \cap \ker(\Gamma_1) = 0$ for all $z \in \mathbb{C}$. Since $S \subset S^*$, the identity

$$\ker(\boldsymbol{\Gamma}) = \ker(\Gamma_0) \cap \ker(\Gamma_1) = \operatorname{dom}(S), \tag{4.3}$$

see, e.g., [70, Lemma 14.6 (iv)], implies that this is equivalent to

$$\ker(S - z) = 0. \tag{4.4}$$

Non-trivial elements of $\ker(S - z)$, if they exist, are called *inner solutions*.

In general, $\ker(S^* - z)$ is not analytic or even continuous on $\Phi_-(S^*)$, since the dimension of $\ker(S^* - z)$ will jump at points z which are eigenvalues of S. For this reason, similar results in the literature have assumed one of the following:

(1) S has no inner solutions (see [22, Theorem 3.8] and [37, Section 6]);

(2) $z \in \rho(H_0)$, where H_0 is the Dirichlet-type extension of S (see [7, Theorem 5.5.1] and [70, Proposition 14.15]). Note that this is stronger than assuming $\ker(S - z) = 0$.

Theorem 4.1, on the other hand, requires no such assumptions on S or z. The reason is that when passing to the Cauchy data $\mathcal{M}(z)$, the jump in the dimension disappears, leaving only the "analytic component" of $\ker(S^* - z)$. To make this intuition rigorous, we use the observation of M. G. Kreĭn [51] (see also [7, Remark 2.3.10]) that one can split off a maximal self-adjoint part of S – which is responsible for the inner solutions – and hence consider only simple symmetric operators.

We recall from [7, Section 3.4] that a closed, symmetric operator is *simple* if 0 is the largest reducing subspace on which it is self-adjoint. Simple symmetric operators have no eigenvalues, by [7, Lemma 3.4.7], and hence satisfy (4.4) for all $z \in \mathbb{C}$. Since S is closed and symmetric, there is a splitting $\mathcal{H} = \mathcal{H}_{\rm sim} \oplus \mathcal{H}_{\rm sa}$ with respect to which S is diagonal and

$$S_{\rm sim} := S|_{\mathcal{H}_{\rm sim}}, \quad S_{\rm sa} := S|_{\mathcal{H}_{\rm sa}} \tag{4.5}$$

are simple symmetric and self-adjoint in $\mathcal{H}_{\rm sim}$ and $\mathcal{H}_{\rm sa}$, respectively; see [7, Section 3.4] for details.

Proof of Theorem 4.1. We proceed in six steps.

Step one: Reducing to the simple symmetric case. Decomposing $\mathcal{H}$ as in (4.5), we have $S = S_{\rm sim} \oplus S_{\rm sa}$ and hence

$$S^* - z = (S^*_{\rm sim} - z) \oplus (S_{\rm sa} - z) \tag{4.6}$$

for all $z \in \mathbb{C}$. Since $\mathrm{dom}(S_{\rm sa}) \subset \mathrm{dom}(S) = \ker \boldsymbol{\Gamma}$, we have $\boldsymbol{\Gamma}(\ker(S_{\rm sa} - z)) = 0$ and thus

$$\mathcal{M}(z) = \boldsymbol{\Gamma}(\ker(S^* - z)) = \boldsymbol{\Gamma}(\ker(S^*_{\rm sim} - z)).$$

It therefore suffices to prove the result for $S_{\rm sim}$, so we will assume for the rest of the proof that S is simple.

Step two: $S^* - z$ is onto for $z \in \Phi_-(S^*)$. By the definition of Φ_-, the range of $S^* - z$ has finite codimension and hence is closed. On the other hand, S being simple implies $\ker(S - z) = 0$, therefore $\mathrm{ran}(S^* - z)$ is dense.

Step three: $\ker(S^* - z)$ *is analytic in z when $S^* - z$ is onto.* Let $\mathcal{H}_+ := \mathrm{dom}(S^*)$, equipped with the graph scalar product of S^*, so $\mathcal{H}_+$ is a Hilbert space and $S^* \in \mathcal{B}(\mathcal{H}_+, \mathcal{H})$. Since $S^* - z$ is surjective and its kernel (a closed subspace of a Hilbert space) is complemented, it has a bounded right inverse [26, Theorem 2.12], i.e., $B_z \in \mathcal{B}(\mathcal{H}, \mathcal{H}_+)$ such that $(S^* - z)B_z = I_{\mathcal{H}}$. In a neighborhood of any $z_0 \in \Phi_-(S^*)$, B_z can be chosen to be analytic using the formula

$$B_z := B_{z_0}((S^* - z)B_{z_0})^{-1} = B_{z_0}(I_{\mathcal{H}} + (z_0 - z)B_{z_0})^{-1}.$$

Now,

$$P_z \colon \mathcal{H}_+ \to \mathcal{H}_+, \quad P_z := I_{\mathcal{H}_+} - B_z(S^* - z),$$

defines an analytic family of projectors (in general not orthogonal) onto

$$\mathcal{N}_z = \ker(S^* - z).$$

Step four: Transformation functions (see [47, Section II.4.2] *and* [28, Section IV.1.1]*) for* $\mathcal{N}_z$*.* Fixing $z_0 \in \Phi_-(S^*)$, define the operator family $F_z : \mathcal{H}_+ \to \mathcal{H}_+$ by

$$F_z := (I - P_z)(I - P_{z_0}) + P_z P_{z_0} = I + (2P_z - I)(P_{z_0} - P_z).$$

The latter expression and $F_{z_0} = I$ show that F_z is invertible for z close to z_0; from the former expression, we immediately get $F_z P_{z_0} = P_z F_z$ and $F_z^{-1} P_z = P_{z_0} F_z^{-1}$, therefore

$$F_z \mathcal{N}_{z_0} = \mathcal{N}_z. \tag{4.7}$$

Step five: A right inverse for the boundary trace. The boundary trace operator $\mathbf{\Gamma} : u \mapsto (\Gamma_0 u, \Gamma_1 u)$ is a bounded surjection from $\mathcal{H}_+$ onto $\mathcal{K} \oplus \mathcal{K}$ with $\ker \mathbf{\Gamma} = \operatorname{dom}(S)$, as in (4.3), so it has a bounded right inverse $\mathbf{\Gamma}^R$, which can be chosen to satisfy

$$\mathbf{\Gamma}^R \mathbf{\Gamma}|_{\mathcal{N}_{z_0}} = I_{\mathcal{N}_{z_0}}. \tag{4.8}$$

More explicitly, since $\mathcal{N}_{z_0} \cap \ker \mathbf{\Gamma} = 0$, the operator $\mathbf{\Gamma} : (\operatorname{dom}(S) \oplus \mathcal{N}_{z_0})^\perp \oplus \mathcal{N}_{z_0} \to \mathcal{K} \oplus \mathcal{K}$ is an isomorphism and $\mathbf{\Gamma}^R$ is the corresponding inverse.

Step six: Transformation functions for $\mathcal{M}(z)$*.* Finally, the analytic family of operators

$$G_z : \mathcal{K} \oplus \mathcal{K} \to \mathcal{K} \oplus \mathcal{K}, \quad G_z := \mathbf{\Gamma} F_z \mathbf{\Gamma}^R \tag{4.9}$$

satisfies

$$G_z \mathcal{M}(z_0) = \mathbf{\Gamma} F_z \mathbf{\Gamma}^R \mathcal{M}(z_0) = \mathbf{\Gamma} F_z \mathbf{\Gamma}^R \mathbf{\Gamma} \mathcal{N}_{z_0} = \mathbf{\Gamma} F_z \mathcal{N}_{z_0} = \mathbf{\Gamma} \mathcal{N}_z = \mathcal{M}(z)$$

and is invertible for z close to z_0 because $G_{z_0} = \mathbf{\Gamma}\mathbf{\Gamma}^R = I_{\mathcal{K}\oplus\mathcal{K}}$. ∎

Remark 4.2. From G_z, one can define an analytic family of oblique projectors onto $\mathcal{M}(z)$ by

$$P_z = \begin{pmatrix} I & 0 \\ G_z^{21}(G_z^{11})^{-1} & 0 \end{pmatrix},$$

where

$$G_z = \begin{pmatrix} G_z^{11} & G_z^{12} \\ G_z^{21} & G_z^{22} \end{pmatrix}$$

is the block decomposition of G_z in the direct sum decomposition $\mathcal{M}(z_0) \oplus \mathcal{M}(z_0)^\perp$. Using [23, Lemma 12.8], we see that the corresponding family of *orthogonal* projections, $P_z P_z^*(P_z P_z^* + (I - P_z^*)(I - P_z))^{-1}$, is smooth. It is not analytic, however, since a family of orthogonal projections is analytic only if it is constant.

The existence of the family G_z allows us to define the crossing form as in (2.4) and thereby extend the notion of monotonicity, Definition 2.2, to our present setting of possibly infinite-dimensional $\mathcal{K}$. In fact, the crossing form for $\mathcal{M}(z)$ has a beautiful explicit form.

Corollary 4.3. *Fix $z_0 \in \Phi_-(S^*)$ and let G_z be the operator family from Theorem 4.1. Then, for any $v \in \mathcal{M}(z_0)$,*

$$\mathfrak{q}[v] := \frac{d}{dz}\omega(v, G_z v)\Big|_{z=z_0} = \min\{\|g\|^2_{\mathcal{H}} : (S^* - z_0)g = 0,\ \mathbf{\Gamma} g = v\} = \|f\|^2_{\mathcal{H}_{\text{sim}}}, \tag{4.10}$$

where ω is the symplectic form (1.4), $\mathcal{H}_{\text{sim}}$ and S_{sim} are defined in (4.5) and f is the unique vector in $\mathcal{H}_{\text{sim}}$ with $S^*_{\text{sim}} f = z_0 f$ and $\mathbf{\Gamma} f = v$.

Remark 4.4. Equation (4.10) generalizes known formulas: for the derivative of the Dirichlet-to-Neumann map in the resolvent set of the “Dirichlet” extension, as in [70, Proposition 14.15 (iv)]; and for the crossing form when S satisfies the unique continuation condition, for instance [22, Theorem 5.1] or [59, Theorem 5.10]. Under such conditions, the solution f to $S^* f = z_0 f$, $\mathbf{\Gamma} f = v$ is unique and the operator that maps the first component of the vector $v \in \mathcal{M}(z) \subset \mathcal{K} \oplus \mathcal{K}$ into f is known as the *γ-field.*

Proof of Corollary 4.3. In view of the decomposition (4.6) and its properties, the general solution of $S^* g = z_0 g$, $\mathbf{\Gamma} g = v$ has the form

$$g = f + \ker(S_{\text{sa}} - z_0),$$

with $\|g\|^2 \geq \|f\|^2$, where $f \in \ker(S^*_{\text{sim}} - z_0) \subset \mathcal{H}_{\text{sim}}$ is unique by (4.4). Therefore, we can restrict ourselves to the case when S is simple symmetric. In this case, there exists a unique $f \in \ker(S^* - z_0)$ with $v = \mathbf{\Gamma} f$. From (4.7), (4.8), and (4.9) we have

$$G_z v = \mathbf{\Gamma} F_z \mathbf{\Gamma}^R v = \mathbf{\Gamma} F_z \mathbf{\Gamma}^R \mathbf{\Gamma} f = \mathbf{\Gamma} F_z f = \mathbf{\Gamma} f_z,$$

where $f_z := F_z f \in \ker(S^* - z)$. By Green’s identity (1.3), we get

$$\begin{aligned}\omega(v, G_z v) = \omega(\mathbf{\Gamma} f, \mathbf{\Gamma} f_z) &= \langle f, (S^* - z_0) f_z\rangle_{\mathcal{H}} - \langle (S^* - z_0) f, f_z\rangle_{\mathcal{H}} \\ &= \langle f, (z - z_0) f_z\rangle_{\mathcal{H}} = \langle f, (z - z_0) F_z f\rangle_{\mathcal{H}}.\end{aligned}$$

Equation (4.10) follows from $F_{z_0} = I$ and the continuity of F_z. ■

We now state some further properties of the Cauchy data space, recalling $\mathcal{F} = \mathbf{\Gamma}(\operatorname{dom}(H_F))$ for the Friedrichs extension H_F of S.

Proposition 4.5. *Under the assumptions in Theorem 4.1 the Cauchy data space $\mathcal{M}(\cdot)$ has the following properties.*

(1) *For all $z \in \Phi_-(S^*)$,*

$$\mathcal{M}(\bar{z}) = \mathcal{M}(z)^{\omega} := \{u \in \mathcal{K} \oplus \mathcal{K} \colon \omega(u, v) = 0 \text{ for all } v \in \mathcal{M}(z)\}. \tag{4.11}$$

In particular, $\mathcal{M}(s)$ is Lagrangian and increasing for $s \in \Phi_-(S^*) \cap \mathbb{R}$.

(2) *If S is bounded from below, with lower bound γ, then $\mathcal{M}(s)$ has limits (over real s)*

$$\lim_{s\to-\infty}\mathcal{M}(s)=\mathcal{F},\qquad \lim_{s\to\gamma-}\mathcal{M}(s)=:\mathcal{M}(\gamma-),$$

in the strong graph sense, and the limiting subspace $\mathcal{M}(\gamma-)$ is Lagrangian.

We recall from [7, Definition 1.9.1] that the strong graph limit of $\mathcal{M}(s)$ consists of all $u\in\mathcal{K}\oplus\mathcal{K}$ for which there exists a sequence $u_s\in\mathcal{M}(s)$ with $u_s\to u$.

Remark 4.6. In general, $\mathcal{M}(\gamma-)\neq\mathcal{M}(\gamma)$. For the example of $S=-d^2/dx^2$ on the half-line (with the standard Dirichlet and Neumann traces), we have $\gamma=0\in\mathrm{Spec}_{\mathrm{ess}}(S)=\mathbb{C}\setminus\Phi_-(S^*)$ and $\mathcal{M}(0)$, when computed from the definition (4.1), is equal to the zero subspace of $\mathbb{C}\oplus\mathbb{C}$ (and, in particular, is not Lagrangian). On the other hand, the limit $\mathcal{M}(0-)$ is the Lagrangian plane $\mathbb{C}\oplus 0$.

Proof of Proposition 4.5. (1) It follows from Green's identity (1.3) that $\omega(u,v)=0$ for all $u\in\mathcal{M}(\bar z)$ and $v\in\mathcal{M}(z)$, therefore $\mathcal{M}(\bar z)\subset\mathcal{M}(z)^\omega$. To prove the other inclusion, suppose $u\in\mathcal{M}(z)^\omega$, so $\omega(u,\boldsymbol{\Gamma} f)=0$ for all $f\in\ker(S^*-z)$. Since $\boldsymbol{\Gamma}$ is surjective, there exists $g\in\mathrm{dom}(S^*)$ such that $\boldsymbol{\Gamma} g=u$. Using Green's identity and $(S^*-z)f=0$, we get

$$\begin{aligned}0=\omega(\boldsymbol{\Gamma} g,\boldsymbol{\Gamma} f)&=\langle S^*g,f\rangle_{\mathcal{H}}-\langle g,S^*f\rangle_{\mathcal{H}}\\&=\langle (S^*-\bar z)g,f\rangle_{\mathcal{H}}-\langle g,(S^*-z)f\rangle_{\mathcal{H}}=\langle (S^*-\bar z)g,f\rangle_{\mathcal{H}}.\end{aligned}$$

This means $(S^*-\bar z)g\in\ker(S^*-z)^\perp=\mathrm{ran}(S-\bar z)$, where the last equality holds because $z\in\Phi_-(S^*)$ implies $\mathrm{ran}(S-\bar z)$ is closed, thus $(S^*-\bar z)g=(S-\bar z)h$ for some $h\in\mathrm{dom}(S)=\ker\boldsymbol{\Gamma}$. Since S^* is an extension of S, this implies $g-h\in\ker(S^*-\bar z)$ and so $u=\boldsymbol{\Gamma} g=\boldsymbol{\Gamma}(g-h)\in\mathcal{M}(\bar z)$, as required.

When s is real, (4.11) gives $\mathcal{M}(s)^\omega=\mathcal{M}(s)$, so $\mathcal{M}(s)$ is Lagrangian. It is increasing because the crossing form $\mathfrak{q}[v]$ in (4.10) is positive definite ($f\neq 0$ for non-zero v in Corollary 4.3).

(2) Because the lower bounds of S and its Friedrichs extension H_F coincide, we have $(-\infty,\gamma)\subset\rho(H_F)\subset\Phi_-(S^*)$ by (4.2), therefore $\mathcal{M}(s)$ is continuous on $(-\infty,\gamma)$ by Theorem 4.1. From [7, Corollary 5.2.14], we have that the $s\downarrow-\infty$ limit of $\mathcal{M}(s)$ exists in the strong resolvent sense, and [7, Theorem 5.5.1] gives $\mathcal{M}(-\infty)=\mathcal{F}$.

For the limit $s\uparrow\gamma$, we first use [7, Corollary 5.5.5] to find a boundary triplet $(\mathcal{K}',\Gamma_0',\Gamma_1')$ such that $\mathrm{dom}(H_F)=\ker\Gamma_0'$. It then follows from [7, Corollary 5.2.14] that the corresponding Cauchy data space $\mathcal{M}'(z)$ has a left-hand limit $\mathcal{M}'(\gamma-)$ in the strong resolvent sense, and this limit is Lagrangian. By [7, Theorem 2.5.1], the triplets $(\mathcal{K},\Gamma_0,\Gamma_1)$ and $(\mathcal{K}',\Gamma_0',\Gamma_1')$ are related by a bounded symplectic transformation, thus the Cauchy data spaces $\mathcal{M}(z)$ and $\mathcal{M}'(z)$ are related by a Möbius transform [7, eq. (2.5.4)], which preserves convergence and the Lagrangian property. To complete the proof, we note that for Lagrangian subspaces, strong resolvent convergence is equivalent to strong graph convergence, by [7, Corollary 1.9.6]. ∎

5. First proof of main theorems

We are now ready to prove our main results, namely Theorems 1.1 and 1.2, using the Maslov index. We are thus back to the assumption that $\mathcal{K}$ is finite dimensional. There are three key ingredients in the proof, two of which have already been established:

(1) a formula for the difference of counting functions in terms of the difference of Maslov indices (Proposition 5.3),

(2) the identity of Zhou–Wu–Zhu relating the difference of Maslov indices to the difference of Duistermaat indices (formula (3.11)),

(3) a one-sided continuity result for the Duistermaat index (Theorem 3.2).

Working towards the counting formula in Proposition 5.3, we first relate the eigenvalues of a self-adjoint extension to the intersections of the corresponding Lagrangian plane with the Cauchy data space $\mathcal{M}(\cdot)$ defined in (1.12).

Lemma 5.1. *Under the assumptions of Theorem 1.1, let H_1 and H_2 be self-adjoint extensions of S corresponding to Lagrangian planes $\mathcal{L}_1$ and $\mathcal{L}_2$. If $z \in \mathbb{C} \setminus \operatorname{Spec}_{\mathrm{ess}}(S)$, then*

$$\dim\ker(H_j - z) = \dim(\mathcal{M}(z) \cap \mathcal{L}_j) + \dim\ker(S - z) \tag{5.1}$$

for $j = 1, 2$, therefore

$$\dim\ker(H_1 - z) - \dim\ker(H_2 - z) = \dim(\mathcal{M}(z) \cap \mathcal{L}_1) - \dim(\mathcal{M}(z) \cap \mathcal{L}_2). \tag{5.2}$$

Remark 5.2. If S has the unique continuation property, so that (4.4) holds for all $z \in \mathbb{C}$, then (5.1) implies $\dim\ker(H_j - z) = \dim(\mathcal{M}(z) \cap \mathcal{L}_j)$ for all $z \in \mathbb{C} \setminus \operatorname{Spec}_{\mathrm{ess}}(S)$. This is no longer true if S does not have the unique continuation property (see Section 7.5 for an elementary example), but (5.2) holds regardless.

Proof. From the definition of the extension H_j, using the Lagrangian plane $\mathcal{L}_j$ in (1.5), and the definition of $\mathcal{M}(z)$ in (4.1), we have

$$\mathbf{\Gamma}(\ker(H_j - z)) = \mathcal{M}(z) \cap \mathcal{L}_j.$$

By the rank-nullity theorem,

$$\dim\ker(H_j - z) = \dim(\mathcal{M}(z) \cap \mathcal{L}_j) + \dim\ker(\mathbf{\Gamma}|_{\ker(H_j - z)}). \tag{5.3}$$

In turn,

$$\ker(\mathbf{\Gamma}|_{\ker(H_j - z)}) = \ker(H_j - z) \cap \ker(\mathbf{\Gamma}) = \ker(H_j - z) \cap \operatorname{dom}(S) = \ker(S - z).$$

Substituting this into (5.3) gives (5.1). ∎

We now relate the eigenvalue counting functions and Maslov indices, recalling the definition of $N(H;I)$ from (1.6).

Proposition 5.3. *With the hypotheses and notation of Theorem 1.1, for any interval* $[a,b] \subset \mathbb{R} \setminus \operatorname{Spec}_{\mathrm{ess}}(S)$*, we have*

$$N(H_1;(a,b]) - N(H_2;(a,b]) = \operatorname{Mas}_{[a,b]}(\mathcal{L}_2, \mathcal{M}(\cdot)) - \operatorname{Mas}_{[a,b]}(\mathcal{L}_1, \mathcal{M}(\cdot)). \quad (5.4)$$

As mentioned in the introduction, it is only possible to express the counting functions $N(H_j;(a,b])$ in terms of Maslov indices if one assumes the unique continuation property of S, but formula (5.4) for their *difference* does not require this assumption; see Remark 5.2.

Proof. It follows from Proposition 4.5 (1) and Corollary 4.3 that $\mathcal{M}(s)$ is an increasing path of Lagrangian planes; so we can use (2.6), (5.2), and (1.6) to obtain

$$\begin{aligned}
&\operatorname{Mas}_{[a,b]}(\mathcal{L}_2, \mathcal{M}(\cdot)) - \operatorname{Mas}_{[a,b]}(\mathcal{L}_1, \mathcal{M}(\cdot)) \\
&\qquad = \sum_{t \in (a,b]} [\dim(\mathcal{M}(t) \cap \mathcal{L}_1) - \dim(\mathcal{M}(t) \cap \mathcal{L}_2)] \\
&\qquad = \sum_{\lambda \in (a,b]} [\dim\ker(H_1 - \lambda) - \dim\ker(H_2 - \lambda)] \\
&\qquad = N(H_1;(a,b]) - N(H_2;(a,b])
\end{aligned}$$

as claimed. ∎

We are now ready to prove our main results.

Proof of Theorem 1.2. For $[a,b] \subset \mathbb{R} \setminus \operatorname{Spec}_{\mathrm{ess}}(S)$, we combine (5.4) and the Zhou–Wu–Zhu identity (3.11) to obtain

$$N(H_1;(a,b]) - N(H_2;(a,b]) = \iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{M}(b)) - \iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{M}(a)), \quad (5.5)$$

which is exactly (1.13).

For λ below the essential spectrum, we use (5.5) with $b = \lambda$ and $a < b$. Corollary 4.3 and Proposition 4.5 (2) say that $\mathcal{M}(a)$ converges to $\mathcal{F}$ from above as $a \to -\infty$, so we can use (3.14) from Theorem 3.2 to compute

$$\lim_{a \to -\infty} \iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{M}(a)) = \iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{F}) \quad (5.6)$$

and hence arrive at (1.14). ∎

Proof of Theorem 1.1. We start with (1.14) from Theorem 1.2,

$$\sigma(H_1, H_2; \lambda) = \iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{M}(\lambda)) - \iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{F}),$$

and use the bound (3.6) to estimate $\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{M}(\lambda))$, arriving at

$$-\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{F}) \le \sigma(H_1, H_2; \lambda) \le n - \dim \mathcal{L}_1 \cap \mathcal{L}_2 - \iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{F}). \tag{5.7}$$

We then use (3.7) to simplify the right-hand side to $\iota(\mathcal{L}_2, \mathcal{L}_1, \mathcal{F})$. ∎

6. The Duistermaat index of a self-adjoint linear relation

A Lagrangian plane in $(\mathcal{K} \oplus \mathcal{K}, \omega)$ can be viewed as a self-adjoint linear relation; see [25, Section 4.2]. In this context, the difference of two Lagrangian planes, $\mathcal{L}$ and $\mathcal{M}$, is the Lagrangian plane

$$\mathcal{L} - \mathcal{M} := \left\{ \begin{pmatrix} u \\ v_{\mathcal{L}} - v_{\mathcal{M}} \end{pmatrix} \in \mathcal{K} \oplus \mathcal{K} : \begin{pmatrix} u \\ v_{\mathcal{L}} \end{pmatrix} \in \mathcal{L}, \begin{pmatrix} u \\ v_{\mathcal{M}} \end{pmatrix} \in \mathcal{M} \right\}, \tag{6.1}$$

the *inverse* of $\mathcal{L}$ is

$$\mathcal{L}^{-1} := \left\{ \begin{pmatrix} v \\ u \end{pmatrix} \in \mathcal{K} \oplus \mathcal{K} : \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{L} \right\}, \tag{6.2}$$

and the *kernel* and *multivalued part* of $\mathcal{L}$ are

$$\ker \mathcal{L} := \left\{ u \in \mathcal{K} : \begin{pmatrix} u \\ 0 \end{pmatrix} \in \mathcal{L} \right\}, \quad \operatorname{mul} \mathcal{L} := \left\{ v \in \mathcal{K} : \begin{pmatrix} 0 \\ v \end{pmatrix} \in \mathcal{L} \right\}.$$

The dimension of $\ker \mathcal{L}$ is called the *nullity* and denoted $n_0(\mathcal{L})$. If $\operatorname{mul} \mathcal{L} = 0$, then $\mathcal{L}$ is the graph of a self-adjoint linear operator on $\mathcal{K}$. In this case, we define the *index* $n_-(\mathcal{L})$ to be the number of negative eigenvalues of the corresponding operator.

Proposition 6.1. *Assume $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ are Lagrangian planes such that $\mathcal{L}_3$ is transversal to $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{V} = 0 \oplus \mathcal{K}$. Then the Lagrangian plane*

$$\Delta := (\mathcal{L}_1 - \mathcal{L}_3)^{-1} - (\mathcal{L}_2 - \mathcal{L}_3)^{-1} \tag{6.3}$$

is a graph of an operator on $\mathcal{K}$, with nullity and index

$$n_0(\Delta) = \dim \mathcal{L}_1 \cap \mathcal{L}_2, \quad n_-(\Delta) = \iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3).$$

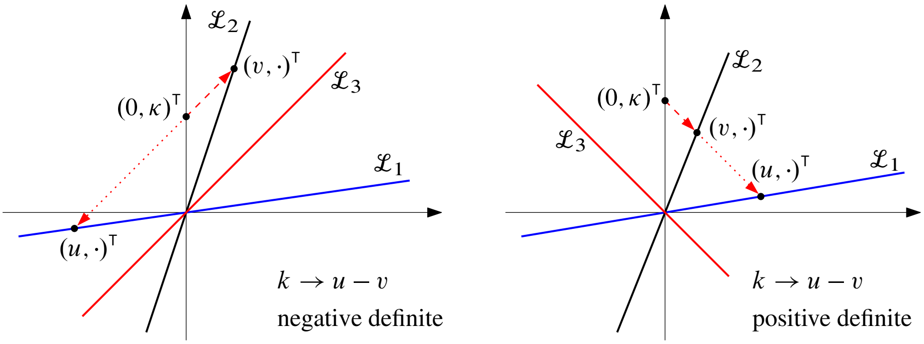

Figure 2. Examples of the action $\Delta\colon \kappa \mapsto u(\kappa) - v(\kappa)$ with $n_-(\Delta) = 1$ (left) and $n_-(\Delta) = 0$ (right). This should be compared with Figure 1, top row. Dotted and dashed lines illustrate the action of the projectors P_1 and P_2, respectively.

Proof. Let P_1 and P_2 be the projections onto $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, parallel to $\mathcal{L}_3$. The transversality conditions on $\mathcal{L}_3$ imply that their restrictions $P_1\colon \mathcal{V} \to \mathcal{L}_1$ and $P_2\colon \mathcal{V} \to \mathcal{L}_2$ are isomorphisms. Written explicitly, they are

$$P_1\colon \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \in \mathcal{V} \;\mapsto\; \begin{pmatrix} u \\ \alpha \end{pmatrix} \in \mathcal{L}_1 \quad \text{such that} \begin{pmatrix} u \\ \alpha \end{pmatrix} - \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \in \mathcal{L}_3, \tag{6.4}$$

and

$$P_2\colon \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \in \mathcal{V} \;\mapsto\; \begin{pmatrix} v \\ \beta \end{pmatrix} \in \mathcal{L}_2 \quad \text{such that} \begin{pmatrix} v \\ \beta \end{pmatrix} - \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \in \mathcal{L}_3. \tag{6.5}$$

For future use, we observe that

$$P_1 \begin{pmatrix} 0 \\ \kappa \end{pmatrix} - P_2 \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \in \mathcal{L}_3, \tag{6.6}$$

and

$$P_1 \begin{pmatrix} 0 \\ \kappa \end{pmatrix} = P_2 \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \iff P_1 \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \in \mathcal{L}_1 \cap \mathcal{L}_2. \tag{6.7}$$

The non-trivial $\Longrightarrow$ direction in (6.7) follows from the observation that if $(u,\alpha)^\mathsf{T}$ in (6.4) belongs to both $\mathcal{L}_1$ and $\mathcal{L}_2$, then it satisfies the condition in (6.5) and therefore coincides with the unique value of $(v,\beta)^\mathsf{T}$.

We now compute the Lagrangian plane Δ from (6.3). Using definition (6.1), we write

$$\begin{aligned} \mathcal{L}_1 - \mathcal{L}_3 &= \left\{ \begin{pmatrix} u \\ \alpha - \gamma \end{pmatrix} \in \mathcal{K} \oplus \mathcal{K}\colon \begin{pmatrix} u \\ \alpha \end{pmatrix} \in \mathcal{L}_1,\ \begin{pmatrix} u \\ \gamma \end{pmatrix} \in \mathcal{L}_3 \right\} \\ &= \left\{ \begin{pmatrix} u \\ \kappa \end{pmatrix} \colon \exists \alpha \in \mathcal{K} \text{ such that} \begin{pmatrix} u \\ \alpha \end{pmatrix} \in \mathcal{L}_1, \begin{pmatrix} u \\ \alpha - \kappa \end{pmatrix} \in \mathcal{L}_3 \right\}, \end{aligned}$$

where the vector $(u, \alpha)^{\intercal}$ is exactly the image of $(0, \kappa)^{\intercal}$ under P_1. Defining $\pi\colon \mathcal{K} \oplus \mathcal{K} \to \mathcal{K}$ to be the projection onto the first component and using (6.2), we arrive at

$$(\mathcal{L}_1 - \mathcal{L}_3)^{-1} = \left\{ \begin{pmatrix} \kappa \\ u(\kappa) \end{pmatrix} \in \mathcal{K} \oplus \mathcal{K} \colon \kappa \in \mathcal{K}, \right\}, \quad \text{where } u(\kappa) := \pi P_1 \begin{pmatrix} 0 \\ \kappa \end{pmatrix}.$$

Analogous considerations for $\mathcal{L}_2$ result in

$$\Delta = \left\{ \begin{pmatrix} \kappa \\ u(\kappa) - v(\kappa) \end{pmatrix} \in \mathcal{K} \oplus \mathcal{K} \colon \kappa \in \mathcal{K} \right\}, \quad \text{where } v(\kappa) := \pi P_2 \begin{pmatrix} 0 \\ \kappa \end{pmatrix}. \tag{6.8}$$

In other words, Δ is the graph of an operator mapping $\kappa \in \mathcal{K}$ to $u(\kappa) - v(\kappa) \in \mathcal{K}$, see Figure 2 for an example. The kernel of Δ is the space of κ such that the corresponding $u(\kappa)$ and $v(\kappa)$ coincide. More precisely,

$$\begin{aligned} \begin{pmatrix} \kappa \\ 0 \end{pmatrix} \in \ker \Delta \iff & \pi P_1 \begin{pmatrix} 0 \\ \kappa \end{pmatrix} - \pi P_2 \begin{pmatrix} 0 \\ \kappa \end{pmatrix} = 0 \\ \iff & P_1 \begin{pmatrix} 0 \\ \kappa \end{pmatrix} - P_2 \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \in \mathcal{V}. \end{aligned}$$

Combining this with (6.6) and the condition $\mathcal{V} \cap \mathcal{L}_3 = 0$, we obtain

$$\begin{pmatrix} \kappa \\ 0 \end{pmatrix} \in \ker \Delta \iff P_1 \begin{pmatrix} 0 \\ \kappa \end{pmatrix} = P_2 \begin{pmatrix} 0 \\ \kappa \end{pmatrix}.$$

Finally, (6.7) yields that $\ker \Delta$ is isomorphic to $\mathcal{L}_1 \cap \mathcal{L}_2$.

We now calculate the Duistermaat index using Corollary 3.3, namely by evaluating the Morse index of $Q(\mathcal{L}_1, \mathcal{L}_3; \mathcal{L}_2)$. The mapping $L\colon \mathcal{L}_1 \to \mathcal{L}_3$ appearing in the definition (3.1) of $Q(\mathcal{L}_1, \mathcal{L}_3; \mathcal{L}_2)$ acts as

$$\mathcal{L}_1 \ni \begin{pmatrix} u \\ \alpha \end{pmatrix} = P_1 \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \mapsto P_2 \begin{pmatrix} 0 \\ \kappa \end{pmatrix} - P_1 \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \in \mathcal{L}_3,$$

cf. (3.1), (6.6), and the fact that P_2 acts into $\mathcal{L}_2$. Consequently, for any two vectors from $\mathcal{L}_1$,

$$\begin{pmatrix} u_1 \\ \alpha_1 \end{pmatrix} = P_1 \begin{pmatrix} 0 \\ \kappa_1 \end{pmatrix}, \quad \begin{pmatrix} u_2 \\ \alpha_2 \end{pmatrix} = P_1 \begin{pmatrix} 0 \\ \kappa_2 \end{pmatrix},$$

the form Q acts as

$$\left(\begin{pmatrix} u_1 \\ \alpha_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ \alpha_2 \end{pmatrix} \right) \mapsto \omega \left(P_1 \begin{pmatrix} 0 \\ \kappa_1 \end{pmatrix}, P_2 \begin{pmatrix} 0 \\ \kappa_2 \end{pmatrix} - P_1 \begin{pmatrix} 0 \\ \kappa_2 \end{pmatrix} \right).$$

We now note that for the purpose of computing the index, the form Q can be viewed as a Hermitian form on $\mathcal{V}$. Observing that

$$\omega\left(P_1\begin{pmatrix}0\\ \kappa_1\end{pmatrix}-\begin{pmatrix}0\\ \kappa_1\end{pmatrix},\ P_2\begin{pmatrix}0\\ \kappa_2\end{pmatrix}-P_1\begin{pmatrix}0\\ \kappa_2\end{pmatrix}\right)=0,$$

which holds because both arguments are in the Lagrangian plane $\mathcal{L}_3$, we have for the value of Q,

$$\begin{aligned}\omega\left(P_1\begin{pmatrix}0\\ \kappa_1\end{pmatrix}, P_2\begin{pmatrix}0\\ \kappa_2\end{pmatrix}-P_1\begin{pmatrix}0\\ \kappa_2\end{pmatrix}\right)&=\omega\left(\begin{pmatrix}0\\ \kappa_1\end{pmatrix},\ P_2\begin{pmatrix}0\\ \kappa_2\end{pmatrix}-P_1\begin{pmatrix}0\\ \kappa_2\end{pmatrix}\right)\\ &=\langle\kappa_1, u(\kappa_2)-v(\kappa_2)\rangle_{\mathcal{K}},\end{aligned}$$

which is exactly the sesquilinear form corresponding to $\Delta\colon\kappa\mapsto u(\kappa)-v(\kappa)$, see (6.8). By Corollary 3.3, we conclude that

$$\iota(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3)=n_-(\Delta),$$

which establishes the desired result. ∎

Next we provide a proof of Theorem 1.3 by virtue of a slightly more general result.

Theorem 6.2. *Assume the setting of Theorem 1.1. For $\lambda\in\rho(H_1)\cap\rho(H_2)\cap\mathbb{R}$, the operator $D(\lambda)$, defined in (1.15), is reduced by the decomposition $\mathcal{H}=\mathcal{N}_\lambda\oplus\mathcal{N}_\lambda^\perp$, where $\mathcal{N}_\lambda:=\ker(S^*-\lambda)$. The block form of $D(\lambda)$ with respect to this decomposition is*

$$D(\lambda)=\begin{pmatrix}K(\lambda) & 0\\ 0 & 0\end{pmatrix}, \tag{6.9}$$

where $K(\lambda)\colon\mathcal{N}_\lambda\to\mathcal{N}_\lambda$ is the restriction of $D(\lambda)$ to $\mathcal{N}_\lambda$. For an arbitrary interval $I\subset\rho(H_1)\cap\rho(H_2)\cap\mathbb{R}$, the eigenvalues of $K(\lambda)$ depend continuously on $\lambda\in I$ and one has

$$n_0(K(\lambda))=\dim\mathcal{L}_1\cap\mathcal{L}_2, \tag{6.10}$$

$$n_-(K(\lambda))=n_-(D(\lambda))=\iota(\mathcal{L}_1,\mathcal{L}_2,\mathcal{M}(\lambda)), \tag{6.11}$$

$$n_+(K(\lambda))=n_+(D(\lambda))=\iota(\mathcal{L}_2,\mathcal{L}_1,\mathcal{M}(\lambda)). \tag{6.12}$$

In particular, $K(\lambda)$ has constant rank

$$r=n-\dim\mathcal{L}_1\cap\mathcal{L}_2=\iota(\mathcal{L}_1,\mathcal{L}_2,\mathcal{M}(\lambda))+\iota(\mathcal{L}_2,\mathcal{L}_1,\mathcal{M}(\lambda)) \tag{6.13}$$

and the functions $\lambda\mapsto n_\pm(K(\lambda))$ are constant on I.

Proof. Since $H_j \subset S^*$, we have $(S^* - \lambda)(H_j - \lambda)^{-1} = I_{\mathcal{H}}$ for $j = 1, 2$, therefore $(S^* - \lambda)D(\lambda) = 0$, i.e., $\operatorname{ran} D(\lambda) \subset \mathcal{N}_\lambda$. This establishes the second row of (6.9) and, by self-adjointness, the whole of (6.9).

To prove the continuity of the eigenvalues of $K(\lambda)$ (acting on the λ-dependent spaces $\mathcal{N}_\lambda$), we first note that $S^* - \lambda$ is onto in $\mathcal{H}$ as an extension of a surjective operator $H_1 - \lambda$ with $\lambda \in \rho(H_1)$. Therefore, by steps three and four in the proof of Theorem 4.1, there exists a continuous family of bijections F_λ mapping $\mathcal{N}_{\lambda_0}$ onto $\mathcal{N}_\lambda$. The family of operators $\lambda \mapsto F_\lambda^{-1} K(\lambda) F_\lambda$ acting on the λ-independent Hilbert space $\mathcal{N}_{\lambda_0}$ is continuous and for each $\lambda \in I$ the eigenvalues of $K(\lambda)$ and $F_\lambda K(\lambda) F_\lambda^{-1}$ coincide. These facts yield the desired continuity assertion.

We next show that (6.10) holds for all $\lambda \in \rho(H_1) \cap \rho(H_2) \cap \mathbb{R}$. For this, we use the resolvent difference formula from [60, Theorem 2.5],

$$K(\lambda) = (\mathbf{\Gamma}(H_1 - \lambda)^{-1})^* P_1 J P_2 \mathbf{\Gamma}(H_2 - \lambda)^{-1}|_{\mathcal{N}_\lambda}, \tag{6.14}$$

where P_j denotes the orthogonal projection onto $\mathcal{L}_j$ in $\mathcal{K} \oplus \mathcal{K}$. By Lemma 6.3, the maps

$$\mathbf{\Gamma}(H_2 - \lambda)^{-1} \colon \mathcal{N}_\lambda \to \mathcal{L}_2, \quad (\mathbf{\Gamma}(H_1 - \lambda)^{-1})^* \colon \mathcal{L}_1 \to \mathcal{N}_\lambda,$$

are bijective. Using (6.14) and the identity $P_1 J = J(I - P_1)$ from (A.1), we get

$$\dim\ker K(\lambda) = \dim\ker(P_1 J|_{\mathcal{L}_2}) = \ker((I - P_1)|_{\mathcal{L}_2}) = \mathcal{L}_1 \cap \mathcal{L}_2,$$

proving (6.10).

It follows that $n_0(K(\lambda))$ is constant on the interval I, therefore the same is true of $n_\pm(K(\lambda))$, since the eigenvalues are continuous. It thus suffices to prove (6.11) and (6.12) for a single $\lambda \in I$. We will choose $\lambda \in I \setminus \operatorname{Spec}(H_0)$, where H_0 is the extension of S with $\mathbf{\Gamma}(\operatorname{dom}(H_0)) = \ker \Gamma_0$. Such a λ always exists because $I \subset \rho(H_1) \cap \rho(H_2)$ does not intersect $\operatorname{Spec}_{\mathrm{ess}}(H_j) = \operatorname{Spec}_{\mathrm{ess}}(S)$, and therefore I can only contain isolated eigenvalues of H_0.

For $\lambda \in I \setminus \operatorname{Spec}(H_0)$, we may apply the classical Kreĭn–Naimark formula,

$$(H_j - \lambda)^{-1} - (H_0 - \lambda)^{-1} = \gamma(\lambda)(\mathcal{L}_j - \mathcal{M}(\lambda))^{-1}\gamma(\lambda)^*,$$

where $\gamma(\lambda) := (\Gamma_0|_{\mathcal{N}_\lambda})^{-1} \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is the γ-field; see, for example, [70, Theorem 14.18]. Applying this to $D(\lambda) = (H_1 - \lambda)^{-1} - (H_2 - \lambda)^{-1}$ yields

$$D(\lambda) = \gamma(\lambda)((\mathcal{L}_1 - \mathcal{M}(\lambda))^{-1} - (\mathcal{L}_2 - \mathcal{M}(\lambda))^{-1})\gamma(\lambda)^*.$$

Employing Proposition 6.1 with $\mathcal{L}_3 = \mathcal{M}(\lambda)$ (the transversality conditions are satisfied since λ is not an eigenvalue of H_1, H_2 or H_0) together with the fact that $\gamma(\lambda) \colon \mathcal{K} \to \mathcal{N}_\lambda$ is a bijection (see [70, Lemma 14.13 (ii)]), we get

$$\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{M}(\lambda)) = n_-((\mathcal{L}_1 - \mathcal{M}(\lambda))^{-1} - (\mathcal{L}_2 - \mathcal{M}(\lambda))^{-1}) = n_-(D(\lambda)),$$

which is exactly (6.11). To complete the proof, we note that (6.13) follows from (6.10) and the identity (3.7) for the Duistermaat index. Combining this with (6.11) yields (6.12). ∎

Lemma 6.3. *Assume that H is a self-adjoint extension of S corresponding to the Lagrangian plane $\mathcal{L} \subset \mathcal{K} \oplus \mathcal{K}$. For $\lambda \in \rho(H)$, we let $R_\lambda := (H-\lambda)^{-1}$ and consider $\mathbf{\Gamma} R_\lambda \colon \mathcal{H} \to \mathcal{K} \oplus \mathcal{K}$. Then*

$$\ker(\mathbf{\Gamma} R_\lambda) = \overline{\operatorname{ran}(S-\lambda)}, \quad \operatorname{ran}(\mathbf{\Gamma} R_\lambda) = \mathcal{L}.$$

In particular, $\mathbf{\Gamma} R_\lambda$ is a bijection between $\ker(S^* - \bar{\lambda})$ *and $\mathcal{L}$.*

Proof. To show the inclusion $\ker(\mathbf{\Gamma} R_\lambda) \subset \operatorname{ran}(S-\lambda)$, suppose that $\mathbf{\Gamma} R_\lambda u = 0$. Then by (4.3) one has $R_\lambda u = v$ for some $v \in \operatorname{dom}(S)$, hence $u = (S-\lambda)v$ as required. To prove $\ker(\mathbf{\Gamma} R_\lambda) \supset \operatorname{ran}(S-\lambda)$, we note that $R_\lambda(S-\lambda)v = v$ and use (4.3). For the second identity we note that $\operatorname{ran}(\mathbf{\Gamma} R_\lambda) = \mathbf{\Gamma}(\operatorname{dom}(H)) = \mathcal{L}$. ∎

Finally, we explain how this implies the interlacing formulas in Theorem 1.1.

Proof of Theorem 1.1 using Theorem 1.3. Using (5.6), we can choose a large negative number λ_* that satisfies

$$\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{M}(\lambda_*)) = \iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{F}) = \sigma_-, \quad \iota(\mathcal{L}_2, \mathcal{L}_1, \mathcal{M}(\lambda_*)) = \iota(\mathcal{L}_2, \mathcal{L}_1, \mathcal{F}) = \sigma_+$$

and is below the spectra of H_1 and H_2. For such λ_*, the eigenvalues of $R_1 := (\lambda_* - H_1)^{-1}$ are negative and bounded; we label them in increasing order as $\mu_k(R_1)$, and likewise for $R_2 := (\lambda_* - H_2)^{-1}$. Recalling the definition of $D(\lambda)$ in (1.15), we write $R_2 = R_1 + D(\lambda_*)$ and note that $n_-(D(\lambda_*)) = \sigma_-$ and $n_+(D(\lambda_*)) = \sigma_+$ by Theorem 1.3. Applying Weyl interlacing for additive finite-rank perturbations, we get

$$\mu_{k-\sigma_-}(R_1) \le \mu_k(R_2) \le \mu_{k+\sigma_+}(R_1).$$

The corresponding eigenvalues of H_j are computed by the monotone increasing trasformation $\lambda_k(H_j) = \lambda_* - 1/\mu_k(R_j)$, yielding (1.10). ∎

7. Examples and applications

Having proved Theorem 1.1, we now discuss some of its consequences. In particular, we compare a variety of boundary conditions for a Schrödinger operator on the interval (0, 1) and derive a counting formula of Behrndt and Luger for the Laplacian on quantum graphs. We also demonstrate that the upper and lower bounds in (1.9) are optimal. Finally, we give an example of an operator not satisfying the UCP, where we see that the Maslov index undercounts the eigenvalues even though our theorem remains valid.

Name	Conditions	Frame	Notation
Periodic	$f(0) = f(1)$, $f'(0) = f'(1)$	$X = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$	$\mathcal{L}_{\text{per}}$
δ-type	$f(0) = f(1)$, $f'(0) - f'(1) = sf(0)$	$X = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} s & 1 \\ 0 & -1 \end{pmatrix}$	$\mathcal{L}_{\delta}(s)$
Antiperiodic	$f(0) = -f(1)$, $f'(0) = -f'(1)$	$X = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	$\mathcal{L}_{\text{aper}}$
δ'-type	$f(0) + f(1) = sf'(0)$, $f'(0) = -f'(1)$	$X = \begin{pmatrix} 1 & s \\ -1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	$\mathcal{L}_{\delta'}(s)$

Table 1. Some commonly encountered boundary conditions and the corresponding Lagrangian frames. Note that periodic conditions are also called *Neumann–Kirchhoff* or *standard* conditions in the context of quantum graphs.

7.1. Examples with $n = 2$

The prototypical example with $n = 2$ is an interval $(0, 1)$ with the Schrödinger operator $S = -\frac{d^2}{dx^2} + q(x)$. However, the results below apply equally well to a compact quantum graph [16] with self-adjoint conditions imposed everywhere except for two vertices of degree one. More sophisticated contexts include manifolds with conical singularities [39] and Šeba billiards [21, 48, 57, 58, 68, 72] with two or more delta potentials.

For $S = -\frac{d^2}{dx^2} + q(x)$, with potential $q \in L^\infty(0, 1)$, we have $\mathcal{H} = L^2(0, 1)$, $\operatorname{dom}(S) = H_0^2(0, 1)$ and $\operatorname{dom}(S^*) = H^2(0, 1)$. Both defect numbers are 2, so the self-adjoint extensions are parameterized by Lagrangian planes in $\mathbb{C}^4$, see [70, Examples 14.2 and 14.10]. The traditional choice of traces is

$$\Gamma_0 f = \begin{pmatrix} f(0) \\ f(1) \end{pmatrix}, \quad \Gamma_1 f = \begin{pmatrix} f'(0) \\ -f'(1) \end{pmatrix}.$$

With this choice, the Friedrichs extension corresponds to the vertical plane, $\mathcal{F} = \mathcal{V}$.

We now use Theorem 1.1 to compare the boundary conditions listed in Table 1.

7.1.1. Periodic vs δ-type conditions. At $s = 0$, the δ-type condition reduces to the periodic condition, $\mathcal{L}_\delta(0) = \mathcal{L}_{\text{per}}$, so it suffices to consider $s \neq 0$. In this case the rank of the perturbation is $2 - \dim(\mathcal{L}_{\text{per}} \cap \mathcal{L}_\delta(s)) = 1$. The corresponding ε-Robin maps are

$$R^\varepsilon_{\text{per}} = \frac{1}{2\varepsilon}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad R^\varepsilon_\delta = \frac{1}{\varepsilon(2 + \varepsilon s)}\begin{pmatrix} 1 + \varepsilon s & -1 \\ -1 & 1 + \varepsilon s \end{pmatrix};$$

therefore,

$$R^\varepsilon_\delta - R^\varepsilon_{\text{per}} = \frac{s}{2(2 + \varepsilon s)}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Using Theorem 3.5, we conclude that

$$\iota(\mathscr{L}_{\mathrm{per}}, \mathscr{L}_\delta(s), \mathscr{F}) = \begin{cases} 0, & s > 0, \\ 1, & s < 0, \end{cases} \quad \text{and} \quad (\sigma_-, \sigma_+) = \begin{cases} (0,1), & s > 0, \\ (1,0), & s < 0, \end{cases}$$

recovering a well-known result [14, 15].

7.1.2. Periodic vs antiperiodic. This important case arises in the spectral analysis of Hill's operator and certain other $\mathbb{Z}$-periodic quantum graphs [35] (namely, those with one edge crossing the boundary of the fundamental domain).

For the antiperiodic conditions, we have

$$R^\varepsilon_{\mathrm{aper}} = \frac{1}{2\varepsilon}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad R^\varepsilon_{\mathrm{aper}} - R^\varepsilon_{\mathrm{per}} = \frac{1}{\varepsilon}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We therefore get

$$(\sigma_-, \sigma_+) = (1, 1),$$

which agrees, for instance, with the interlacing in Hill's equation [65, eq. (2.4) in Theorem 2.1].

7.1.3. Antiperiodic vs δ'-type conditions. The antiperiodic conditions are a special case of the δ'-type conditions with $s = 0$. For $s \neq 0$, we have $\dim(\mathscr{L}_{\mathrm{aper}} \cap \mathscr{L}_{\delta'}(s)) = 1$, so the rank of the perturbation is 1. Computing the ε-Robin map, we obtain

$$R^\varepsilon_{\delta'} = \frac{1}{s + 2\varepsilon}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad R^\varepsilon_{\mathrm{aper}} - R^\varepsilon_{\delta'} = -\frac{s}{2\varepsilon(s + 2\varepsilon)}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

For $s \neq 0$, we see that $-\frac{s}{2\varepsilon(s+2\varepsilon)} \approx -\frac{1}{2\varepsilon}$ is negative for $0 < \varepsilon \ll 1$, and hence $\sigma_- = 1$. Taking the rank of the perturbation into account, we get

$$(\sigma_-, \sigma_+) = (1, 0),$$

or, in terms of the eigenvalues,

$$\lambda_{k-1}(H_{\mathrm{aper}}) \leq \lambda_k(H_{\delta'}(s)) \leq \lambda_k(H_{\mathrm{aper}}).$$

This may be somewhat unexpected, since it implies that $\lambda_k(H_{\delta'}(s))$ achieves its maximum at $s = 0$, while the eigenvalue variation formulas of [60] can be used to show that

$$\frac{d\lambda}{ds} = -|f_s'(0)|^2,$$

where f_s is the normalized eigenfunction corresponding to the eigenvalue $\lambda(s)$ (assuming it is simple). These two facts are reconciled by the observation that $\lambda_1(s) \to -\infty$ as $s \to 0-$, therefore the *ordered* eigenvalue curves $s \mapsto \lambda_k(H_{\delta'}(s))$ are discontinuous at $s = 0$, as shown in Figure 3.

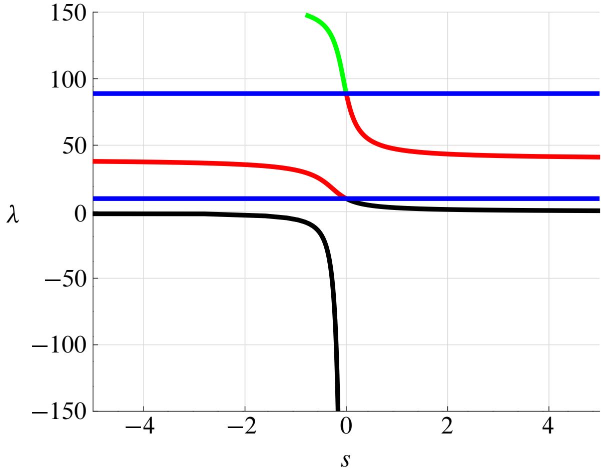

Figure 3. The first five eigenvalues of $H_{\delta'}(s)$, plotted as functions of s. The curves are colored according to the index of the eigenvalue: λ_1 is black, λ_2 is blue, λ_3 is red etc. In particular, the blue lines (from bottom to top) are $\lambda_2(H_{\delta'}(s)) = \lambda_{1,2}(H_{\text{aper}})$ and $\lambda_4(H_{\delta'}(s)) = \lambda_{3,4}(H_{\text{aper}})$.

7.2. Counting negative eigenvalues: the Behrndt–Luger formula

In [9], Behrndt and Luger derived a convenient formula for the number of negative eigenvalues of the Laplacian on a metric graph. Here we show how their formula can be obtained from our results.

We recall that any Lagrangian plane $\mathcal{L}$ can be described by a frame of the form $(P, P\Theta P + P - I)$, where $P: \mathcal{K} \to \mathcal{K}$ is an orthogonal projector and $\Theta: \operatorname{ran} P \to \operatorname{ran} P$ is a self-adjoint operator; see Section A.2. This corresponds to imposing the Dirichlet-type condition $(I - P)\Gamma_0 f = 0$ and the Robin-type condition $P\Gamma_1 f = \Theta P \Gamma_0 f$.

Corollary 7.1. *Within the setting of Theorem 1.1, assume S is non-negative*[7] *and let H be an extension of S specified by a Lagrangian plane $\mathcal{L}$. Then the Morse index of H is*

$$n_-(H) = \iota(\mathcal{M}(0-), \mathcal{L}, \mathcal{F}), \tag{7.1}$$

where $\mathcal{M}(0-)$ is the left-hand limit of $\mathcal{M}(s)$, as in Proposition 4.5.

Furthermore, if $\operatorname{dom}(H_F) = \ker \Gamma_0$ *and $\mathcal{M}(0-)$ is a graph of a self-adjoint operator $M_0: \mathcal{K} \to \mathcal{K}$, then*

$$n_-(H) = n_+(PM_0P - \Theta), \tag{7.2}$$

where $(P, P\Theta P + P - I)$ is a Lagrangian frame for $\mathcal{L}$.

[7] Equivalently, the Friedrichs extension H_F is non-negative.

Remark 7.2. Equation (7.1) generalizes results of Derkach and Malamud [31] in the setting of finite defect numbers. In particular, [31, eq. (0.4)] equates the Morse indices of H and the operator Δ that appears in Proposition 6.1; thus, under more restrictive assumptions, one can see that [31, eq. (0.4)] and (7.1) give the same expressions for $n_-(H)$. However, the basis-independent nature of the Duistermaat index (manifested as symplectic invariance (3.4)) saves one from superfluous restrictions such as the form domain inclusion condition in [31, Theorems 5 and 6].

Remark 7.3. In the setting of [9], where S^* is the Laplacian on a metric graph and Γ_0 and Γ_1 are the standard Dirichlet and Neumann traces, the limit $\mathcal{M}(0-)$ is the graph of an operator M_0, by [9, Lemma 3], thus (7.2) holds. To relate this to [9, Theorem 1] we simply take $\Theta = -L$ and observe that the boundary condition in [9] is written in terms of a co-frame rather than a frame (see Section A.2). In this setting, M_0 is the Dirichlet-to-Neumann map at $\lambda = 0$, which is explicitly computable; see also [16, Section 3.5].

Proof. Since H_F is non-negative, we have $\operatorname{Spec}_{\mathrm{ess}}(H_F) \subset [0, \infty)$. Therefore, we can apply Theorem 1.2 with $\mathcal{L}_2 = \mathcal{F}$ and negative λ to obtain

$$N(H_1; (-\infty, \lambda]) = \iota(\mathcal{L}_1, \mathcal{F}, \mathcal{M}(\lambda)) - \iota(\mathcal{L}_1, \mathcal{F}, \mathcal{F}) = \iota(\mathcal{L}_1, \mathcal{F}, \mathcal{M}(\lambda)), \quad (7.3)$$

using (3.10) to eliminate $\iota(\mathcal{L}, \mathcal{F}, \mathcal{F})$. By Proposition 4.5, the path $\mathcal{M}(\lambda)$ converges to a Lagrangian plane $\mathcal{M}(0-)$ as $\lambda \to 0-$. Since the convergence is monotone, (3.15) gives $\iota(\mathcal{L}_1, \mathcal{F}, \mathcal{M}(\lambda)) = \iota(\mathcal{M}(0-), \mathcal{L}_1, \mathcal{F})$ for sufficiently small λ. It follows from (7.3) that $N(H_1; (-\infty, \lambda])$ is constant for small negative λ and hence is equal to $n_-(H_1)$, proving equation (7.1).

Combining (7.1) with Proposition 3.6, we obtain

$$n_-(H) = \iota(\mathcal{M}(0-), \mathcal{L}, \mathcal{F}) = n_-(\Theta - PM_0P) = n_+(PM_0P - \Theta),$$

since $\mathcal{M}(0-)$ is the graph of the operator M_0 and $\mathcal{L}$ is described by the frame $(P, P\Theta P + P - I)$. ∎

7.3. Comparing Dirichlet and Neumann eigenvalues

Another easy consequence of our results is a version of Friedlander's well-known interlacing formula [36], which, in our case of finite defect indices, takes the geometric form of a Duistermaat index. Defining the *Dirichlet and Neumann extensions* of S to be the self-adjoint extensions H_D and H_N with

$$\operatorname{dom}(H_D) = \{f \in \operatorname{dom}(S^*) : \Gamma_0 f = 0\}, \quad \operatorname{dom}(H_N) = \{f \in \operatorname{dom}(S^*) : \Gamma_1 f = 0\},$$

respectively, we get the following.

Corollary 7.4. *Assume, in addition to the hypotheses of Theorem 1.1, that $H_F = H_D$. For any $\lambda \in \mathbb{R}$ below the essential spectrum, we have*

$$\sigma(H_N, H_D; \lambda) = \iota(\mathcal{K} \oplus 0, 0 \oplus \mathcal{K}, \mathcal{M}(\lambda)).$$

If λ is not a Dirichlet eigenvalue, then $\mathcal{M}(\lambda)$ is the graph of an operator $M(\lambda)$ on $\mathcal{K}$, and

$$\sigma(H_N, H_D; \lambda) = n_{0+}(M(\lambda)). \tag{7.4}$$

Proof. Since H_D and H_N correspond to the Lagrangian planes $\mathcal{L}_D = 0 \oplus \mathcal{K}$ and $\mathcal{L}_N = \mathcal{K} \oplus 0$, the first equation follows from Theorem 1.2, using (3.10) to eliminate $\iota(\mathcal{L}_N, \mathcal{L}_D, \mathcal{F}) = \iota(\mathcal{L}_N, \mathcal{F}, \mathcal{F}) = 0$, and the second follows from Corollary 3.7 . ■

When λ is also not a Neumann eigenvalue (and hence $\ker M(\lambda) = 0$), this is exactly the formula of Friedlander [36, Lemma 1] (see also [2] and references therein) in the context of finite defect numbers. Note that the Dirichlet-to-Neumann map in [36] is $-M(\lambda)$ here, due to the choice of normal derivative – the abstract Green's identity (1.3) requires Γ_1 to be the *inward* normal derivative.

Corollary 3.7 can also be used when λ is a Dirichlet eigenvalue, leading to a more general version of (7.4) in terms of a "reduced" Dirichlet-to-Neumann map, as in [12].

7.4. Sharpness of the bounds

We now prove that the bounds in (1.9) are sharp for any symmetric operator S.

Proposition 7.5. *Let S satisfy the assumptions of Theorem 1.1. For any numbers $\tilde{\sigma}_\pm \geq 0$ with $\tilde{\sigma}_- + \tilde{\sigma}_+ \leq n$, there exist Lagrangian planes $\mathcal{L}_1$ and $\mathcal{L}_2$ such that*

$$\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{F}) = \tilde{\sigma}_-, \quad \iota(\mathcal{L}_2, \mathcal{L}_1, \mathcal{F}) = \tilde{\sigma}_+, \tag{7.5}$$

and the bounds in (1.9) are sharp for the corresponding extensions H_1 and H_2 of S. To be precise, there exists $\lambda_0 \in \mathbb{R}$ such that

$$\sigma(H_1, H_2; \lambda_0 - 0) = \tilde{\sigma}_-, \quad \sigma(H_1, H_2; \lambda_0 + 0) = \tilde{\sigma}_+.$$

Proof. Without loss of generality, we assume that our boundary triplet is chosen such that $\mathcal{F} = \mathcal{V} = 0 \oplus \mathcal{K}$. Now, choose $\lambda_0 \in \mathbb{R}$ below the essential spectrum such that $\mathcal{M}(\lambda_0)$ is transversal to $\mathcal{V}$. This is always possible because $\mathcal{M}$ is increasing, therefore its intersections with any Lagrangian plane are isolated. Letting $M_0 \colon \mathcal{K} \to \mathcal{K}$ be the operator whose graph is $\mathcal{M}(\lambda_0)$, we define $\mathcal{L}_1$ and $\mathcal{L}_2$ via the frames (I, M_0) and $(I, M_0 - P_- + P_+)$, respectively, where $P_-, P_+ \colon \mathcal{K} \to \mathcal{K}$ are arbitrary mutually orthogonal projectors of rank $\tilde{\sigma}_-$ and $\tilde{\sigma}_+$. Note that $\mathcal{L}_1 = \mathcal{M}(\lambda_0)$.

We first show that this choice of $\mathcal{L}_1$ and $\mathcal{L}_2$ gives the desired Duistermaat indices. Using Proposition 3.6 with $P = I$, we get

$$\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{F}) = n_-((M_0 - P_- + P_+) - M_0) = \operatorname{rank} P_- = \tilde{\sigma}_-,$$
$$\iota(\mathcal{L}_2, \mathcal{L}_1, \mathcal{F}) = n_-(P_- - P_+) = \operatorname{rank} P_+ = \tilde{\sigma}_+,$$

which is exactly (7.5).

Now, we recall from the proof of Theorem 1.1, in particular (5.7), that the lower bound in (1.9) is attained at some λ if $\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{M}(\lambda)) = 0$, and the upper bound is attained if $\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{M}(\lambda)) = n - \dim \mathcal{L}_1 \cap \mathcal{L}_2$. We then use (3.10) and Theorem 3.2 to obtain

$$\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{M}(\lambda_0 - 0)) = \iota(\mathcal{M}(\lambda_0), \mathcal{L}_1, \mathcal{L}_2) = \iota(\mathcal{L}_1, \mathcal{L}_1, \mathcal{L}_2) = 0$$

and

$$\iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{M}(\lambda_0 + 0)) = \iota(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_1) = n - \dim \mathcal{L}_1 \cap \mathcal{L}_2,$$

completing the proof. ∎

Example 7.6. It is not true that for any $\mathcal{L}_1$ and $\mathcal{L}_2$ the bounds of Theorem 1.1 are achieved at some λ. A simple example is the Neumann versus the Dirichlet Laplacian on the interval $(0, \pi)$. The spectra are $\{0, 1, 4, 9, \ldots\}$ and $\{1, 4, 9, \ldots\}$, respectively, so the spectral shift takes the values 0 and 1 while $\sigma_- = 0$ and $\sigma_+ = 2$.

7.5. An operator with inner solutions

Defining the space

$$\mathcal{D} := \{f = (f_1, f_2) \in H^2(-\pi, 0) \oplus H^2(0, \pi) :$$
$$f_1(-\pi) = f_2(\pi) = 0,\ f_1(0) = f_2(0)\},$$

we consider the symmetric operator S acting as $-\frac{d^2}{dx^2}$ on $L^2(-\pi, 0) \oplus L^2(0, \pi)$, with

$$\operatorname{dom}(S) := \{(f_1, f_2) \in \mathcal{D} : f_1(0) = f_2(0) = 0,\ f_1'(0) = f_2'(0)\}.$$

It easily follows that $\operatorname{dom}(S^*) = \mathcal{D}$. As a boundary triple, we can take $\mathcal{K} = \mathbb{C}$ and

$$\Gamma_0 f = \frac{1}{2}(f_1(0) + f_2(0)), \quad \Gamma_1 f = f_2'(0) - f_1'(0).$$

For every $z \in \mathbb{C} \setminus \{1, 4, 9, \ldots\}$, the kernel

$$\ker(S^* - zI) = \operatorname{span}\Big\{\Big(\frac{1}{\sqrt{z}} \sin \sqrt{z}(\pi + x), \frac{1}{\sqrt{z}} \sin \sqrt{z}(\pi - x)\Big)\Big\}$$

is one-dimensional (with $\ker(S^*) = \operatorname{span}\{(\pi + x, \pi - x)\}$ understood as the $z \to 0$ limit). However, when $z = k^2$ for some $k \in \mathbb{N}$, the kernel is two-dimensional,

$$\ker(S^* - k^2 I) = \operatorname{span}\{(\sin k(\pi + x), 0), (0, \sin k(\pi - x))\}. \tag{7.6}$$

On the other hand, the Cauchy data space from (1.12) is one-dimensional for all $z \in \mathbb{C}$,

$$\mathcal{M}(z) = \operatorname{span}\Big\{\Big(\frac{1}{\sqrt{z}} \sin \pi\sqrt{z},\ -2\cos \pi\sqrt{z}\Big)\Big\},$$

since the trace $\boldsymbol{\Gamma}$ vanishes on the difference of the two basis vectors in (7.6) when $z \in \{1, 4, 9, \dots\}$. In other words, for any $k \in \mathbb{N}$, the function

$$g_k = (\sin k(\pi + x), -\sin k(\pi - x))$$

is an inner solution in the sense of (4.4).

Consider, for example, the Lagrangian planes

$$\mathcal{F} = \operatorname{span}\Big\{\begin{pmatrix}0\\1\end{pmatrix}\Big\}, \quad \mathcal{L}_1 = \operatorname{span}\Big\{\begin{pmatrix}1\\0\end{pmatrix}\Big\},$$

and the corresponding extensions of S,

$$\begin{aligned} \operatorname{dom}(H_F) &:= \{f \in \mathcal{D} \colon f_1(0) = f_2(0) = 0\}, \\ \operatorname{dom}(H_1) &:= \{f \in \mathcal{D} \colon f_1'(0) = f_2'(0)\}. \end{aligned}$$

As the notation suggests, H_F is the Friedrichs extension of S. The corresponding spectra are easily computed to be

$$\operatorname{Spec}(H_F) = \{k^2 \text{ with multiplicity } 2 : k \in \mathbb{N}\}, \quad \operatorname{Spec}(H_1) = \{(k/2)^2 \colon k \in \mathbb{N}\}.$$

We now see that if one tries to use the intersections $\mathcal{M}(z) \cap \mathcal{F}$ and $\mathcal{M}(z) \cap \mathcal{L}_1$ to search for the eigenvalues of H_F and H_1, every second eigenvalue will be missed, because these are the eigenvalues that correspond to inner solutions.

A. Lagrangian preliminaries

A.1. The Lagrangian Grassmannian

Given a complex symplectic space $(\mathcal{K} \oplus \mathcal{K}, \omega)$, we say that a subspace $\mathcal{L} \subset \mathcal{K} \oplus \mathcal{K}$ is *Lagrangian* if it equals its *symplectic complement*

$$\mathcal{L}^\omega := \{u \in \mathcal{K} \oplus \mathcal{K} \colon \omega(u, v) = 0 \text{ for all } v \in \mathcal{L}\}.$$

There are several convenient reformulations of this definition. These involve the operator

$$J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

on $\mathcal{K} \oplus \mathcal{K}$, defined so that $\omega(u, v) = \langle u, Jv \rangle_{\mathcal{K} \oplus \mathcal{K}}$. It follows that $\mathcal{L}^\omega = (J\mathcal{L})^\perp$, thus $\mathcal{L}$ is Lagrangian if and only if $J\mathcal{L} = \mathcal{L}^\perp$, where $(\cdot)^\perp$ is the usual orthogonal complement. A useful consequence of this is the identity

$$(\mathcal{L}_1 + \mathcal{L}_2)^\perp = \mathcal{L}_1^\perp \cap \mathcal{L}_2^\perp = J\mathcal{L}_1 \cap J\mathcal{L}_2 = J(\mathcal{L}_1 \cap \mathcal{L}_2),$$

valid for any Lagrangian planes $\mathcal{L}_1$ and $\mathcal{L}_2$. This implies $\dim \mathcal{L}_1 \cap \mathcal{L}_2 = \operatorname{codim}(\mathcal{L}_1 + \mathcal{L}_2)$. In particular, $\mathcal{L}_1 + \mathcal{L}_2 = \mathcal{K} \oplus \mathcal{K}$ if and only if $\mathcal{L}_1 \cap \mathcal{L}_2 = 0$. Another useful fact is that a subspace $\mathcal{L}$ is Lagrangian if and only if

$$J = JP_{\mathcal{L}} + P_{\mathcal{L}}J, \tag{A.1}$$

where $P_{\mathcal{L}}$ is the orthogonal projection onto $\mathcal{L}$; see [37, Proposition 2.11].

The set of all Lagrangian subspaces in $(\mathcal{K} \oplus \mathcal{K}, \omega)$ is called the *Lagrangian Grassmannian* and is denoted by Λ.

A.2. Parameterizations of Lagrangian subspaces

A Lagrangian subspace $\mathcal{L} \subset \mathcal{K} \oplus \mathcal{K}$ can be described in many different ways. For convenience, we summarize the most useful ones here.

A *frame* is an injective linear map $Z\colon \mathcal{K} \to \mathcal{K} \oplus \mathcal{K}$ whose range is $\mathcal{L}$. A frame is typically written in block form

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix},$$

with $X, Y\colon \mathcal{K} \to \mathcal{K}$. We often abbreviate this as (X, Y). The range of Z is Lagrangian if and only if the condition $X^*Y = Y^*X$ holds. This frame is not uniquely determined by $\mathcal{L}$, since for any $C \in \mathrm{GL}(\mathcal{K})$, ZC is also a frame for $\mathcal{L}$. Any frame for $\mathcal{L}$ is of this form for a suitable choice of C.

A closely related notion is that of a *co-frame*, which is a surjective linear map $\mathcal{K} \oplus \mathcal{K} \to \mathcal{K}$ whose kernel is $\mathcal{L}$; see [16, Theorem 1.4.4]. This can be written in block form as $(A \;\; B)$, with $A, B\colon \mathcal{K} \to \mathcal{K}$. The Lagrangian condition is equivalent to $AB^* = BA^*$, and it is easily seen that the frame (X, Y) corresponds to the co-frame $(Y^* \;\; -X^*)$.

It was shown in [56, Corollary 5] (see also [9, Lemma 2]) that any Lagrangian plane $\mathcal{L}$ can be represented by a co-frame with $A = I - P - P\Theta P$ and $B = P$, or equivalently by the frame

$$\begin{pmatrix} B^* \\ -A^* \end{pmatrix} = \begin{pmatrix} P \\ P\Theta P + P - I \end{pmatrix},$$

where $P: \mathcal{K} \to \mathcal{K}$ is an orthogonal projection and Θ is a self-adjoint operator on $\operatorname{ran} P$. This parameterization arises naturally when describing boundary conditions: for $f \in \operatorname{dom}(S^*)$, we have $\mathbf{\Gamma} f = (\Gamma_0 f, \Gamma_1 f) \in \mathcal{L}$ precisely when

$$(I - P)\Gamma_0 f = 0, \quad P\Gamma_1 f = \Theta P \Gamma_0 f,$$

so we interpret P and $I - P$ as projections onto the Robin and the Dirichlet parts of $\mathcal{K}$, respectively.

Another possibility is to write $\mathcal{L}$ as the graph of an operator defined on a reference Lagrangian subspace. Let $\mathcal{L}^\sharp$ and $\widehat{\mathcal{L}}$ be transversal Lagrangian subspaces. If $\mathcal{L}$ is transversal to $\widehat{\mathcal{L}}$, then there exists an operator $L: \mathcal{L}^\sharp \to \widehat{\mathcal{L}}$ whose graph is $\mathcal{L}$, in the sense that

$$\mathcal{L} = \{v + Lv : v \in \mathcal{L}^\sharp\}.$$

In particular, if $\widehat{\mathcal{L}} = (\mathcal{L}^\sharp)^\perp$, then we can write $L = JT$ for some $T \in \mathcal{B}(\mathcal{L}^\sharp)$, and the Lagrangian condition is equivalent to $T^* = T$. For instance, if $\mathcal{L}^\sharp = \mathcal{K} \oplus 0$ and $\widehat{\mathcal{L}} = 0 \oplus \mathcal{K}$, then $\mathcal{L}$ is transversal to $\widehat{\mathcal{L}}$ if and only if it has a frame (X, Y) for which X is invertible, and hence an equivalent frame

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \begin{pmatrix} I_{\mathcal{K}} \\ YX^{-1} \end{pmatrix}.$$

It therefore corresponds to the graph of $L = JT$ with $T = YX^{-1} \in \mathcal{B}(\mathcal{L}^\sharp)$.

Another possibility is to fix a Lagrangian subspace $\mathcal{L}_0$ and write $\mathcal{L}$ as the image of an invertible operator $G_{\mathcal{L}} \in \mathcal{B}(\mathcal{K} \oplus \mathcal{K})$. Such an operator can be explicitly constructed as follows. If (X, Y) is a frame for $\mathcal{L}$, then

$$G^{X,Y} = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \tag{A.2}$$

is invertible and maps the horizontal subspace $\mathcal{K} \oplus 0$ to $\mathcal{L}$; cf. [67, Proposition 1]. Choosing a frame (X_0, Y_0) for $\mathcal{L}_0$, we see that $G^{X,Y}(G^{X_0,Y_0})^{-1}$ is a valid choice of $G_{\mathcal{L}}$.

Finally, we recall that $\mathcal{L}$ can be written as the graph of a unitary operator from $\ker(J - i)$ to $\ker(J + i)$. This can be related to the frame description of $\mathcal{L}$ by decomposing

$$\begin{pmatrix} Xu \\ Yu \end{pmatrix} = \underbrace{\frac{1}{2}\begin{pmatrix} (X - iY)u \\ (Y + iX)u \end{pmatrix}}_{\in \ker(J-i)} + \underbrace{\frac{1}{2}\begin{pmatrix} (X + iY)u \\ (Y - iX)u \end{pmatrix}}_{\in \ker(J+i)}$$

for arbitrary $u \in \mathcal{K}$. The desired unitary map is

$$\begin{pmatrix} a \\ ia \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (X - iY)u \\ (Y + iX)u \end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} (X + iY)u \\ (Y - iX)u \end{pmatrix} = \begin{pmatrix} b \\ -ib \end{pmatrix}, \tag{A.3}$$

therefore $b = (X + iY)(X - iY)^{-1}a$. (This agree with the formula in [71, Proposition 1], since the $\mathcal{J}$ used there coincides with $-J$ in this paper.) This parameterization will be used in Section A.4 to define the Maslov index.

A.3. Smooth structure on the Lagrangian Grassmannian

We now recall the smooth structure on Λ and give some equivalent formulations for the differentiability of a path $\mathcal{L}(\cdot)\colon (0,1) \to \Lambda$, in terms of the different parameterizations given above.

To prove that Λ is a smooth manifold, one first shows that it is a topological manifold, and then equips it with a smooth atlas of coordinate charts. The topology is given by the *gap metric*, $d(\mathcal{L}_1, \mathcal{L}_2) := \|P_{\mathcal{L}_1} - P_{\mathcal{L}_2}\|$, where $\mathcal{L}_j \in \Lambda$ and $P_{\mathcal{L}_j} \in \mathcal{B}(\mathcal{K} \oplus \mathcal{K})$ are the corresponding orthogonal projections.

Next, given a Lagrangian subspace $\mathcal{L}$, we let $U_{\mathcal{L}^\perp}$ denote the set of Lagrangian subspaces that are transversal to $\mathcal{L}^\perp = J\mathcal{L}$. This is an open neighborhood of $\mathcal{L}$ in Λ, and is homeomorphic to the set $\mathcal{B}_{\mathrm{sa}}(\mathcal{L})$ of self-adjoint operators on $\mathcal{L}$, via the map

$$A \mapsto \Psi_{\mathcal{L}}(A) := \{v + JAx : v \in \mathcal{L}\}. \tag{A.4}$$

See [37, Proposition 2.21] for details. It follows that Λ is a topological manifold. Moreover, it can be shown that the "transition functions" are smooth on any intersection $U_{\mathcal{L}_1^\perp} \cap U_{\mathcal{L}_2^\perp}$ of coordinate charts. That is, referring to the map $\Psi_{\mathcal{L}}\colon \mathcal{B}_{\mathrm{sa}}(\mathcal{L}) \to \Lambda$ defined in (A.4), we have that the composition

$$\Psi_{\mathcal{L}_2}^{-1} \circ \Psi_{\mathcal{L}_1}\colon \underbrace{\Psi_{\mathcal{L}_1}^{-1}(U_{\mathcal{L}_1^\perp} \cap U_{\mathcal{L}_2^\perp})}_{\subseteq \mathcal{B}_{\mathrm{sa}}(\mathcal{L}_1)} \to \underbrace{\Psi_{\mathcal{L}_2}^{-1}(U_{\mathcal{L}_1^\perp} \cap U_{\mathcal{L}_2^\perp})}_{\subseteq \mathcal{B}_{\mathrm{sa}}(\mathcal{L}_2)}$$

is C^∞; see, for instance, [37, Corollary 2.25]. This gives Λ the structure of a smooth manifold.

Differentiability of a path $\mathcal{L}(t)$ of Lagrangian subspaces is defined with respect to this manifold structure. However, it is often easier to work with the following equivalent conditions, which are in fact taken as definitions in some papers.

Theorem A.1. *Given a path $\mathcal{L}(\cdot)\colon (0,1) \to \Lambda$, the following are equivalent:*

(1) *$\mathcal{L}(t)$ is differentiable;*

(2) *the family of orthogonal projections $P_{\mathcal{L}(t)}$ is differentiable;*

(3) *there exists a differentiable family of invertible operators* G_t *on* $\mathcal{K} \oplus \mathcal{K}$ *such that, for some* $\mathcal{L}_0$, $G_t \mathcal{L}_0 = \mathcal{L}(t)$*;*

(4) *there exists a differentiable frame* $Z(t)$ *for* $\mathcal{L}(t)$*;*

(5) *there exists a differentiable family of unitary operators* $U_t \colon \ker(J - i) \to \ker(J + i)$ *such that* $\mathcal{L}(t) = \{v + U_t v : v \in \ker(J - i)\}$.

Proof. (1) $\Longrightarrow$ (2). Suppose $\mathcal{L}(t)$ is differentiable. In a neighborhood of an arbitrary point t_0, $\mathcal{L}(t)$ is given by $\mathcal{L}(t) = \{x + JA(t)x : x \in \mathcal{L}(t_0)\}$ for some differentiable family $A(t)$. Using [37, eq. (2.16)], we see that the orthogonal projection $P_{\mathcal{L}(t)}$ is differentiable.

(2) $\Longrightarrow$ (3). We fix an arbitrary $t_0 \in (0, 1)$ and then, following [28, Theorem IV.1.1], define G_t as the solution to the differential equation

$$\frac{d}{dt} G_t = P'_{\mathcal{L}(t)}(2P_{\mathcal{L}(t)} - I)G_t, \quad G_{t_0} = I.$$

Choosing $\mathcal{L}_0 = \mathcal{L}(t_0)$, we find that G_t has the desired property (and in addition is unitary).

(3) $\Longrightarrow$ (4). Defining G^{X_0,Y_0} as in (A.2), we see that $G_t G^{X_0,Y_0}$ is differentiable and maps $\mathcal{K} \oplus 0$ onto $\mathcal{L}(t)$. Writing this in block form

$$G_t G^{X_0,Y_0} = \begin{pmatrix} X(t) & * \\ Y(t) & * \end{pmatrix}$$

in the decomposition $\mathcal{K} \oplus \mathcal{K}$, it follows that $(X(t), Y(t))$ is a differentiable frame for $\mathcal{L}(t)$.

(4) $\Longrightarrow$ (1). Let $t_0 \in (0, 1)$. Using the frame $Z(t)$, any point $v \in \mathcal{L}(t)$ can be written as

$$v = Z(t)u = P_0 Z(t)u + (I - P_0)Z(t)u$$

for some $u \in \mathcal{K}$, where $P_0 = P_{\mathcal{L}(t_0)}$. Since the frame $Z(t_0) \colon \mathcal{K} \to \mathcal{K} \oplus \mathcal{K}$ is injective and has range $\mathcal{L}(t_0)$, the map $C(t) := P_0 Z(t) \colon \mathcal{K} \to \mathcal{L}(t_0)$ is invertible at $t = t_0$, and therefore is invertible for t sufficiently close to t_0. It follows that any $v \in \mathcal{L}(t)$ can be written as

$$v = w + (I - P_0)Z(t)C(t)^{-1}w$$

for some $w \in \mathcal{L}(t_0)$. This show that $\mathcal{L}(t)$ is the graph of the differentiable family $A(t) = -J(I - P_0)Z(t)C(t)^{-1} \colon \mathcal{L}(t_0) \to \mathcal{L}(t_0)$, and hence is differentiable at t_0.

(4) $\Longleftrightarrow$ (5) This is an immediate consequence of (A.3). ∎

A.4. Maslov indices for Lagrangian paths

For a continuous family $\mathcal{M}(\cdot) : [a,b] \to \Lambda$ of Lagrangian planes, we can represent each $\mathcal{M}(t)$ as the graph of a unitary operator $U_{\mathcal{M}}(t)\colon \ker(J-i) \to \ker(J+i)$. Doing the same for $\mathcal{L}(t)$, we obtain a unitary family

$$W(t) := U_{\mathcal{M}}(t)(U_{\mathcal{L}}(t))^{-1}$$

on $\ker(J+i)$ such that $\dim\ker(W(t)-1) = \dim(\mathcal{M}(t) \cap \mathcal{L}(t))$; see [24, Lemma 2]. The *Maslov index of* $\mathcal{M}(\cdot)$ *with respect to* $\mathcal{L}(\cdot)$ is defined to be the spectral flow of $W(t)$ through the point 1 on the unit circle, in the counterclockwise direction. More precisely,

$$\mathrm{Mas}_{[a,b]}(\mathcal{M}(\cdot), \mathcal{L}(\cdot)) := \sum_{j=1}^{n} \Big(\Big\lceil \frac{\theta_j(b)}{2\pi} \Big\rceil - \Big\lceil \frac{\theta_j(a)}{2\pi} \Big\rceil \Big),$$

where $\lceil\cdot\rceil$ denotes the ceiling and $\theta_1, \dots, \theta_n\colon [a,b] \to \mathbb{R}$ are continuous functions such that $e^{i\theta_1(t)}, \dots, e^{i\theta_n(t)}$ are the eigenvalues of $W(t)$. See [25, Section 2.2] or [75, Section 2] for details.

Remark A.2. We follow the conventions and notation of [75] so we can directly use their formula (3.11) relating the Duistermaat and Maslov indices. Compared to the Maslov index defined by Cappell, Lee, and Miller in [27], we have

$$\mathrm{Mas}^{\mathrm{CLM}}_{[a,b]}(\mathcal{M}(\cdot), \mathcal{L}(\cdot)) = \mathrm{Mas}_{[a,b]}(\mathcal{L}(\cdot), \mathcal{M}(\cdot)), \tag{A.5}$$

see the remark in [5, Definition A.9]. On the other hand, this is related to the Maslov index defined by Robbin and Salamon in [69] by

$$\mathrm{Mas}^{\mathrm{RS}}_{[a,b]}(\mathcal{M}(\cdot), \mathcal{L}(\cdot)) = \mathrm{Mas}_{[a,b]}(\mathcal{M}(\cdot), \mathcal{L}(\cdot)) + \frac{1}{2}h(b) - \frac{1}{2}h(a),$$

where we have abbreviated $h(t) := \dim(\mathcal{L}(t) \cap \mathcal{M}(t))$; see [5, eq. (A.7)].

Comparing (A.5) with [27, eq. (1.12)], we obtain

$$\mathrm{Mas}_{[a,b]}(\mathcal{L}(\cdot), \mathcal{M}(\cdot)) = -\mathrm{Mas}_{[a,b]}(\mathcal{M}(\cdot), \mathcal{L}(\cdot)) + h(a) - h(b). \tag{A.6}$$

That is, the Maslov index is antisymmetric up to boundary terms.

We now explain how to compute the Maslov index, assuming for the rest of the section that $\mathcal{L}(t) = \mathcal{L}$ is constant and $\mathcal{M}(t)$ is differentiable. We say t_0 is a *crossing* if $\mathcal{M}(t_0) \cap \mathcal{L} \neq 0$. The associated *crossing form* is $\mathfrak{m}_{t_0} := \mathfrak{q}|_{\mathcal{M}(t_0)\cap\mathcal{L}}$, where $\mathfrak{q}$ is the form on $\mathcal{M}(t_0)$ defined by (2.1). A crossing t_0 is *regular* if the form $\mathfrak{m}_{t_0}$ is non-degenerate. For a C^1 path $\mathcal{M}(\cdot)$ with only regular crossings on $[a,b]$, the Maslov index with respect to $\mathcal{L}$ is then given by

$$\mathrm{Mas}_{[a,b]}(\mathcal{M}(\cdot), \mathcal{L}) = n_+(\mathfrak{m}_a) + \sum_{t_0\in(a,b)} (n_+(\mathfrak{m}_{t_0}) - n_-(\mathfrak{m}_{t_0})) - n_-(\mathfrak{m}_b). \tag{A.7}$$

Regular crossings of a C^1 path are isolated, so the sum over t_0 is finite. For complex symplectic spaces (A.7), was proved in [25, Proposition 3.27]. For real spaces, this method of computing the Maslov index first appeared in [69]. In this paper we only require the special case when $\mathcal{M}(\cdot)$ is an increasing path, hence all crossing forms are positive definite; see (2.5).

Acknowledgments. The authors are grateful to Ram Band, Dean Baskin, Maurice de Gosson, Peter Howard, Alexander V. Kiselev, Peter Kuchment, Marina Prokhorova, Gilad Sofer, and Igor Zelenko for their help, encouragement, and suggestions of references. The authors also thank the anonymous referees for suggesting several important corrections to the manuscript.

Funding. G.B. was partially supported by NSF grant DMS-2247473. G.C. acknowledges the support of NSERC Discovery Grant 2017-04259. Y.L. was supported by the NSF grant DMS-2106157, and would like to thank the Courant Institute of Mathematical Sciences at NYU and especially Prof. Lai-Sang Young for their hospitality. S.S. was supported in part by NSF grants DMS–2510344, DMS–2418900, Simons Foundation grant MP–TSM–00002897, grant no. 2024154 from the U.S.-Israel Binational Science Foundation Jerusalem, Israel, and by the Office of the Vice President for Research & Economic Development (OVPRED) at Auburn University through the Research Support Program grant.

References

[1] A. Agrachev and R. Gamkrelidze, Symplectic methods for optimization and control. In *Geometry of feedback and optimal control*, pp. 19–77, Monogr. Textbooks Pure Appl. Math. 207, Dekker, New York, 1998 Zbl 0965.93034 MR 1493010

[2] W. Arendt and R. Mazzeo, Friedlander's eigenvalue inequalities and the Dirichlet-to-Neumann semigroup. *Commun. Pure Appl. Anal.* **11** (2012), no. 6, 2201–2212 Zbl 1267.35139 MR 2912743

[3] V. I. Arnol'd, Characteristic class entering in quantization conditions. *Funkcional. Anal. i Priložen.* **1** (1967), 1–14. English translation: *Funct. Anal. Its Appl.* (1967) **1**, 1–13 Zbl 0175.20303 MR 0211415

[4] R. Band, G. Berkolaiko, H. Raz, and U. Smilansky, The number of nodal domains on quantum graphs as a stability index of graph partitions. *Comm. Math. Phys.* **311** (2012), no. 3, 815–838 Zbl 1244.81026 MR 2909765

[5] V. L. Barutello, D. Offin, A. Portaluri, and L. Wu, Sturm theory with applications in geometry and classical mechanics. *Math. Z.* **299** (2021), no. 1-2, 257–297 Zbl 1521.37053 MR 4311604

[6] M. Beck, Spectral stability and spatial dynamics in partial differential equations. *Notices Amer. Math. Soc.* **67** (2020), no. 4, 500–507 Zbl 1448.35037 MR 4186268

[7] J. Behrndt, S. Hassi, and H. de Snoo, *Boundary value problems, Weyl functions, and differential operators*. Monogr. Math. 108, Birkhäuser/Springer, Cham, 2020 Zbl 1457.47001 MR 3971207

[8] J. Behrndt, L. Leben, F. Martínez Pería, R. Möws, and C. Trunk, Sharp eigenvalue estimates for rank one perturbations of nonnegative operators in Krein spaces. *J. Math. Anal. Appl.* **439** (2016), no. 2, 864–895 Zbl 1341.47045 MR 3475956

[9] J. Behrndt and A. Luger, On the number of negative eigenvalues of the Laplacian on a metric graph. *J. Phys. A* **43** (2010), no. 47, article no. 474006 Zbl 1204.81076 MR 2738101

[10] J. Behrndt, R. Möws, and C. Trunk, On finite rank perturbations of selfadjoint operators in Krein spaces and eigenvalues in spectral gaps. *Complex Anal. Oper. Theory* **8** (2014), no. 4, 925–936 Zbl 1348.47009 MR 3178038

[11] J. Behrndt, R. Möws, and C. Trunk, Eigenvalue estimates for operators with finitely many negative squares. *Opuscula Math.* **36** (2016), no. 6, 717–734 Zbl 1370.47045 MR 3571418

[12] G. Berkolaiko, G. Cox, B. Helffer, and M. P. Sundqvist, Computing nodal deficiency with a refined Dirichlet-to-Neumann map. *J. Geom. Anal.* **32** (2022), no. 10, article no. 246 Zbl 1495.35130 MR 4456213

[13] G. Berkolaiko, G. Cox, Y. Latushkin, and S. Sukhtaiev, The calculus of Duistermaat's triple index. [v1] 2024, [v2] 2025, arXiv:2412.20825v2

[14] G. Berkolaiko, J. B. Kennedy, P. Kurasov, and D. Mugnolo, Surgery principles for the spectral analysis of quantum graphs. *Trans. Amer. Math. Soc.* **372** (2019), no. 7, 5153–5197 Zbl 1451.34029 MR 4009401

[15] G. Berkolaiko and P. Kuchment, Dependence of the spectrum of a quantum graph on vertex conditions and edge lengths. In *Spectral geometry*, pp. 117–137, Proc. Sympos. Pure Math. 84, Amer. Math. Soc., Providence, RI, 2012 Zbl 1317.81127 MR 2985312

[16] G. Berkolaiko and P. Kuchment, *Introduction to quantum graphs*. Math. Surveys Monogr. 186, American Mathematical Society, Providence, RI, 2013 Zbl 1318.81005 MR 3013208

[17] M. Š. Birman, On the theory of self-adjoint extensions of positive definite operators. *Mat. Sb. (N.S.)* **38(80)** (1956), 431–450 Zbl 0070.12102 MR 0080271

[18] M. S. Birman and A. B. Pushnitski, Spectral shift function, amazing and multifaceted. *Integral Equations Operator Theory* 30 (1998), no. 2, 191—199 Zbl 0907.47014 MR 1607900

[19] M. S. Birman and M. Z. Solomjak, *Spectral theory of selfadjoint operators in Hilbert space*. Math. Appl. (Soviet Ser.), D. Reidel, Dordrecht, 1987 Zbl 0744.47017 MR 1192782

[20] M. S. Birman and D. R. Yafaev, The spectral shift function. The papers of M. G. Kreĭn and their further development (in Russian). *Algebra i Analiz* **4** (1992), no. 5, 1–44. English translation: *St. Petersburg Math. J.* **4** (1993), no. 5, 833—870 Zbl 0791.47013 MR 1202723

[21] E. Bogomolny, O. Giraud, and C. Schmit, Nearest-neighbor distribution for singular billiards. *Phys. Rev. E (3)* **65** (2002), no. 5, article no. 056214 MR 1920077

[22] B. Booss-Bavnbek and K. Furutani, The Maslov index: a functional analytical definition and the spectral flow formula. *Tokyo J. Math.* **21** (1998), no. 1, 1–34 Zbl 0932.37063 MR 1630119

[23] B. Booß-Bavnbek and K. P. Wojciechowski, *Elliptic boundary problems for Dirac operators*. Math. Theory Appl., Birkhäuser, Boston, MA, 1993 Zbl 0797.58004 MR 1233386

[24] B. Booß-Bavnbek and C. Zhu, The Maslov index in weak symplectic functional analysis. *Ann. Global Anal. Geom.* **44** (2013), no. 3, 283–318 Zbl 1307.53063 MR 3101860

[25] B. Booß-Bavnbek and C. Zhu, The Maslov index in symplectic Banach spaces. *Mem. Amer. Math. Soc.* **252** (2018), no. 1201 Zbl 07000096 MR 3778556

[26] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*. Universitext, Springer, New York, 2011 Zbl 1220.46002 MR 2759829

[27] S. E. Cappell, R. Lee, and E. Y. Miller, On the Maslov index. *Comm. Pure Appl. Math.* **47** (1994), no. 2, 121–186 Zbl 0805.58022 MR 1263126

[28] J. L. Dalec'kiĭ and M. G. Kreĭn, *Stability of solutions of differential equations in Banach space*. Transl. Math. Monogr. 43, American Mathematical Society, Providence, RI, 1974 Zbl 0286.34094 MR 0352639

[29] M. de Gosson, The structure of q-symplectic geometry. *J. Math. Pures Appl. (9)* **71** (1992), no. 5, 429–453 Zbl 0829.58015 MR 1191584

[30] M. A. de Gosson, *The principles of Newtonian and quantum mechanics*. Imperial College Press, London, 2001 Zbl 0991.70001 MR 1897416

[31] V. A. Derkach and M. M. Malamud, Generalized resolvents and the boundary value problems for Hermitian operators with gaps. *J. Funct. Anal.* **95** (1991), no. 1, 1–95 Zbl 0748.47004 MR 1087947

[32] J. J. Duistermaat, On the Morse index in variational calculus. *Advances in Math.* **21** (1976), no. 2, 173–195 Zbl 0361.49026 MR 0649277

[33] D. E. Edmunds and W. D. Evans, *Spectral theory and differential operators*. Oxford Math. Monogr., The Clarendon Press, Oxford University Press, New York, 1987 Zbl 0628.47017 MR 0929030

[34] W. N. Everitt and L. Markus, Complex symplectic geometry with applications to ordinary differential operators. *Trans. Amer. Math. Soc.* **351** (1999), no. 12, 4905–4945 Zbl 0936.34005 MR 1637066

[35] P. Exner, P. Kuchment, and B. Winn, On the location of spectral edges in $\mathbb{Z}$-periodic media. *J. Phys. A* **43** (2010), no. 47, article no. 474022 Zbl 1204.81063 MR 2738117

[36] L. Friedlander, Some inequalities between Dirichlet and Neumann eigenvalues. *Arch. Rational Mech. Anal.* **116** (1991), no. 2, 153–160 Zbl 0789.35124 MR 1143438

[37] K. Furutani, Fredholm-Lagrangian-Grassmannian and the Maslov index. *J. Geom. Phys.* **51** (2004), no. 3, 269–331 Zbl 1076.53100 MR 2079414

[38] F. Gesztesy, K. A. Makarov, and E. Tsekanovskii, An addendum to Krein's formula. *J. Math. Anal. Appl.* **222** (1998), no. 2, 594–606 Zbl 0922.47006 MR 1628437

[39] J. B. Gil, T. Krainer, and G. A. Mendoza, Geometry and spectra of closed extensions of elliptic cone operators. *Canad. J. Math.* **59** (2007), no. 4, 742–794 Zbl 1126.58014 MR 2338233

[40] M. Harmer, Hermitian symplectic geometry and extension theory. *J. Phys. A* **33** (2000), no. 50, 9193–9203 Zbl 0983.53051 MR 1804888

[41] E. V. Haynsworth, Determination of the inertia of a partitioned Hermitian matrix. *Linear Algebra Appl.* **1** (1968), no. 1, 73–81 Zbl 0155.06304 MR 0223392

[42] L. Hörmander, Fourier integral operators. I. *Acta Math.* **127** (1971), no. 1-2, 79–183 Zbl 0212.46601 MR 0388463

[43] R. A. Horn and C. R. Johnson, *Matrix analysis*. Second edn., Cambridge University Press, Cambridge, 2013 Zbl 1267.15001 MR 2978290

[44] P. Howard, Hörmander's index and oscillation theory. *J. Math. Anal. Appl.* **500** (2021), no. 1, article no. 125076 Zbl 1465.37072 MR 4224840

[45] P. Howard, S. Jung, and B. Kwon, The Maslov index and spectral counts for linear Hamiltonian systems on $[0, 1]$. *J. Dynam. Differential Equations* **30** (2018), no. 4, 1703–1729 Zbl 1404.37064 MR 3871619

[46] P. Howard and A. Sukhtayev, The Maslov and Morse indices for Sturm-Liouville systems on the half-line. *Discrete Contin. Dyn. Syst.* **40** (2020), no. 2, 983–1012 Zbl 1448.34065 MR 4043663

[47] T. Kato, *Perturbation theory for linear operators*. Classics in Mathematics, Springer, Berlin, 1995 Zbl 0836.47009 MR 1335452

[48] J. P. Keating, J. Marklof, and B. Winn, Localized eigenfunctions in šeba billiards. *J. Math. Phys.* **51** (2010), no. 6, article no. 062101 Zbl 1311.81115 MR 2675179

[49] M. G. Kreĭn, Concerning the resolvents of an Hermitian operator with the deficiency-index (m, m). *C. R. (Doklady) Acad. Sci. URSS (N.S.)* **52** (1946), 651–654 Zbl 0063.03358 MR 0018341

[50] M. G. Kreĭn, The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications. I. *Rec. Math. [Mat. Sbornik] N.S.* **20(62)** (1947), 431–495 Zbl 0029.14103 MR 0024574

[51] M. G. Kreĭn, The fundamental propositions of the theory of representations of Hermitian operators with deficiency index (m, m) (in Russian). *Ukrain. Mat. Žurnal* **1** (1949), no. 2, 3–66. English translation: in *Eleven papers on logic, algebra, analysis and topology,* pp. 75-143, Trans. Amer. Math. Soc. Ser. 2, 97: American Mathematical Society, Providence, RI, 1971 Zbl 0258.47025 MR 0048704 MR 0270854

[52] M. G. Kreĭn, On the trace formula in perturbation theory (in Russian). *Mat. Sbornik N.S.* **33(75)** (1953), 597–626 Zbl 0052.12303 MR 0060742

[53] M. G. Kreĭn, On perturbation determinants and a trace formula for unitary and self-adjoint operators (in Russian). *Dokl. Akad. Nauk SSSR* **144** (1962), 268–271. English translation: *Sov. Math., Dokl.* (1962) **3** (1962), 707–710 Zbl 0191.15201 MR 0139006

[54] M. G. Kreĭn and I. E. Ovčarenko, Q-functions and sc-resolvents of nondensely defined Hermitian contractions (in Russian). *Sibirsk. Mat. Ž.* **18** (1977), no. 5, 1032–1056, 1206 Zbl 0384.47006 MR 0482343

[55] M. G. Kreĭn and Š. N. Saakjan, Certain new results in the theory of resolvents of Hermitian operators (in Russian). *Dokl. Akad. Nauk SSSR* **169** (1966), 1269–1272. English translation: *Sov. Math., Dokl.* **7** (1966), 1086–1089 Zbl 0193.10002 MR 0203474

[56] P. Kuchment, Quantum graphs. I. Some basic structures. *Waves Random Media* **14** (2004), no. 1, pp. S107–S128, 14, 2004 Zbl 1063.81058 MR 2042548

[57] P. Kurlberg and H. Ueberschär, Quantum ergodicity for point scatterers on arithmetic tori. *Geom. Funct. Anal.* **24** (2014), no. 5, 1565–1590 Zbl 1366.35023 MR 3261634

[58] P. Kurlberg and H. Ueberschär, Non-Gaussian waves in šeba's billiard. *Int. Math. Res. Not. IMRN* (2023), no. 2, 932–955 Zbl 1514.37044 MR 4537316

[59] Y. Latushkin and S. Sukhtaiev, The Maslov index and the spectra of second order elliptic operators. *Adv. Math.* **329** (2018), 422–486 Zbl 06863449 MR 3783419

[60] Y. Latushkin and S. Sukhtaiev, First-order asymptotic perturbation theory for extensions of symmetric operators. *J. Lond. Math. Soc. (2)* **110** (2024), no. 5, article no. e13005 MR 4810573

[61] J. Leray, *Lagrangian analysis and quantum mechanics*. MIT Press, Cambridge, Mass.-London, 1981 Zbl 0483.35002 MR 0644633

[62] P. Libermann and C.-M. Marle, *Symplectic geometry and analytical mechanics*. Mathematics and its Applications 35, D. Reidel Publishing Co., Dordrecht, 1987 Zbl 0643.53002 MR 0882548

[63] I. M. Lifšic, On a problem of the theory of perturbations connected with quantum statistics (in Russian). *Uspehi Matem. Nauk (N.S.)* **7** (1952), no. 1(47), 171–180 Zbl 0046.21203 MR 0049490

[64] G. Lion and M. Vergne, *The Weil representation, Maslov index and theta series*. Progr. Math. 6, Birkhäuser, Boston, MA, 1980 Zbl 0444.22005 MR 0573448

[65] W. Magnus and S. Winkler, Hill's equation. Interscience Tracts in Pure and Applied Mathematics 20, Interscience Publishers John Wiley & Sons, New York-London-Sydney, 1966 Zbl 0158.09604 MR 0197830

[66] G. A. Mendoza, Spectrally unstable domains. *Illinois J. Math.* **59** (2015), no. 4, 979–997 Zbl 1370.47024 MR 3628297

[67] K. Pankrashkin, Resolvents of self-adjoint extensions with mixed boundary conditions. *Rep. Math. Phys.* **58** (2006), no. 2, 207–221 Zbl 1143.47017 MR 2281536

[68] S. Rahav and S. Fishman, Spectral statistics of rectangular billiards with localized perturbations. *Nonlinearity* **15** (2002), no. 5, 1541–1594 Zbl 1082.81508 MR 1925428

[69] J. Robbin and D. Salamon, The Maslov index for paths. *Topology* **32** (1993), no. 4, 827–844 Zbl 0798.58018 MR 1241874

[70] K. Schmüdgen, *Unbounded self-adjoint operators on Hilbert space*. Grad. Texts in Math. 265, Springer, Dordrecht, 2012 Zbl 1257.47001 MR 2953553

[71] H. Schulz-Baldes, Sturm intersection theory for periodic Jacobi matrices and linear Hamiltonian systems. *Linear Algebra Appl.* **436** (2012), no. 3, 498–515 Zbl 1232.15010 MR 2854887

[72] P. Šeba, Wave chaos in singular quantum billiard. *Phys. Rev. Lett.* **64** (1990), no. 16, 1855–1858 Zbl 1050.81535 MR 1046366

[73] C. T. C. Wall, Non-additivity of the signature. *Invent. Math.* **7** (1969), 269–274 Zbl 0176.21501 MR 0246311

[74] M. G. Zaĭdenberg, S. G. Kreĭn, P. A. Kučment, and A. A. Pankov, Banach bundles and linear operators (in Russian). *Uspehi Mat. Nauk* **30** (1975), no. 5(185), 101–157 Zbl 0317.47018 MR 0415661

[75] Y. Zhou, L. Wu, and C. Zhu, Hörmander index in finite-dimensional case. *Front. Math. China* **13** (2018), no. 3, 725–761 Zbl 1405.53110 MR 3813014

Received 24 June 2024; revised 25 August 2025.

Gregory Berkolaiko
Department of Mathematics, Texas A&M University, College Station, TX, 77843, USA; gberkolaiko@tamu.edu

Graham Cox
Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NL, A1C 5S7, Canada; gcox@mun.ca

Yuri Latushkin
Department of Mathematics, The University of Missouri, Columbia, MO, 65211, USA; latushkiny@missouri.edu

Selim Sukhtaiev
Department of Mathematics and Statistics, Auburn University, Auburn, AL, 36849, USA; szs0266@auburn.edu