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On the spectra of periodic elastic beam lattices: Single-layer graph

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Abstract. We consider planar elastic beam Hamiltonians defined on hexagonal lattices. These quantum graphs are constructed from Euler–Bernoulli beams, each governed by the fourth-order Schrödinger operator with a real periodic symmetric potential function. In contrast to the second-order Schrödinger operator commonly studied in the quantum graph literature, here vertex matching conditions encode the geometry of the underlying graph by their dependence on angles at which the edges meet.

We show that on the hexagonal lattice, the dispersion relation has a structure similar to that reported for the periodic second-order Schrödinger operator, known as the "graphene Hamiltonian." This property is then utilized to prove the existence of Dirac points (conical singularities). We further discuss the (ir)reducibility of Fermi surfaces. Moreover, we obtain the point spectrum, the absolutely continuous spectrum, and the singular continuous spectrum.

Applying perturbation analysis, we derive the dispersion relation for the planar elastic beam Hamiltonians on angle-perturbed irregular hexagonal lattices, defined in a geometric neighborhood of the hexagonal lattice. On these graphs, we find that, unlike the hexagonal lattice, the dispersion relation is not split into purely energy- and quasimomentum-dependent terms; however, Dirac points exist similar to the hexagonal-lattice case.

1. Introduction

Lattice materials are cellular structures obtained by tessellating a unit cell comprising a few beams. Such lattice materials exhibit the characteristics of pass and stop bands, determining frequency intervals over which wave motion can or cannot occur, respectively [21, 28, 35]. This unique directional behavior complements the stop–pass band pattern and makes the application of 2D periodic structures as directional mechanical filters possible [35]. For models on special lattices, for example, the hexagonal lattice, interesting physical and spectral properties have been observed, such as the presence

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of special conical points in the dispersion relation, where its different sheets touch to form a two-sided conical singularity [5, 25, 36, 37, 39].

The analysis of wave motion in periodic systems, such as lattice materials and vibrations in harmonic atomic lattices, goes back to the early studies of string vibration and later to Brillouin [10]. Under certain assumptions, the modeling of natural and engineered tessellated lattices is studied under beam theories.[1] Under the Euler–Bernoulli beam model, each beam is described by an energy functional which involves four degrees of freedom for every infinitesimal element along the beam: axial, lateral (2 degrees of freedom) and angular displacements. At a joint (vertex), these four functions, supported on adjacent beams (edges), are related through matching conditions that take into account the physics of a joint; see [7, 9, 16] for more details. In the special case of planar frames, the operator is decomposed into a direct sum of two operators, one coupling out-of-plane displacement to angular (torsional) displacement and the other coupling in-plane displacement to axial displacement [7].

From a more theoretical point of view, the analysis of Hamiltonians corresponding to these symplectic structures has recently gained the interest of mathematicians working on differential operators on metric graphs (also known as "quantum graphs") [7, 14, 19]. Elastic beam Hamiltonians of the corresponding wave equations are given by fourth-order operators; see [7, 19] and references therein. This makes understanding the spectral properties of fourth-order quantum graphs important in the study of elasticity models.

Early studies on the derivation of the dispersion relation (or variety) of second-order Schrödinger operators defined on a periodic graph split the Hamiltonian into two essentially unrelated parts: the analysis on a single edge and the spectral analysis on the combinatorial graph; the former being independent of the graph structure and the latter independent of the potential function [25]. However, contrary to second-order Schrödinger-type operators on graphs, vertex conditions for elastic beam Hamiltonians encode geometry by their dependence on the angles at which the edges meet. As a result, an extension of the existing theory to the latter operator on periodic lattices is not trivially accessible.

Studying elastic beams through quantum graph models, Kiik, Kurasov, and Usman provided the vertex conditions that make the operator d^4/dx^4 self-adjoint on the Y-graph (also known as the 3-edge star graph or the claw graph) [19]. Berkolaiko and the first named author constructed three-dimensional elastic frames from Euler–Bernoulli beams as quantum graphs and described vertex conditions from the geometric description of the frame (or the combinatorial graph) [7]. The vertex conditions

[1] Most inclusive classical beam models are the Euler–Bernoulli and Timoshenko beam theories.

have physical meaning (continuity of displacement, continuity of rotation, equilibrium of forces, and equilibrium of moments) and make the corresponding differential operator self-adjoint [7]. These results open the way for studying the spectral properties of elastic beam Hamiltonians.

In this paper, we study the spectral properties of planar elastic beam Hamiltonians on the hexagonal lattice and its geometric perturbations defined in (3.1)–(3.4). We note that the fundamental domains of these Hamiltonians are 3-edge star graphs; see Figure 1. To keep the Hamiltonians we study more general as quantum graph models, we consider the operator $d^4/dx^4 + q(x)$. Therefore, the current work focuses on extensions of second-order results obtained by Kuchment and Post [25] to the fourth-order operator $\mathscr{H} = d^4/dx^4 + q(x)$ with self-adjoint vertex conditions and a real periodic symmetric potential on the hexagonal lattice and the lattices in the geometric neighborhood of it; see Figures 1 and 4. Compared to the Kuchment–Post theory of the second-order model (graphene Hamiltonian), the description of the spectrum (Theorem 4.11) follows their results. However, our study on the dispersion relation (Proposition 3.3 and Theorem 4.5), Dirac points (Theorem 4.12), and (ir)reducibility of Fermi surfaces (Theorem 4.14) includes important differences. Moreover, we extend our discussions to irregular hexagonal lattices in the geometric neighborhood of the hexagonal lattice in Section 5.

We study hexagonal elastic lattice Hamiltonians by considering the analysis of the operator $\mathscr{H}$ on a single edge, and then the spectral analysis of $\mathscr{H}$ on the combinatorial graph. The spectrum of the self-adjoint operator $\mathscr{H}^{\text{per}} = d^4/dx^4 + q_0(x)$ on the real line with a real periodic potential (known as the *Hill operator* for the second-order operator) has a band gap structure and is bounded below. In contrast to the Hill operator, the edges of the spectral bands may belong to not only the periodic or antiperiodic spectra of $\mathscr{H}$ on (0, 1), but also the set of resonances [3]. However, the latter case occurs at most for finitely many bands [3]. Resonances are the branch points of the Lyapunov function, which is an analytic function on a two-sheeted Riemann surface and depends on the monodromy matrix of $\mathscr{H}$. The Lyapunov function characterizes the spectrum $\sigma(\mathscr{H})$ and the multiplicities of its points. We refer the interested reader to Section 2.2 of this work and [1–3, 31] for detailed discussions.

Before stating the structure of this paper, we briefly summarize our main results.

- Floquet–Bloch theory allows us to study spectral properties of a periodic operator through spectral properties of the same operator defined on a compact set (fundamental domain) with additional vertex conditions (cyclic or Floquet–Bloch conditions) depending on quasimomentum parameters θ_1, θ_2 [23,24]; see Figure 1 and Section 3. The graph of the multivalued function mapping quasimomenta $\Theta := \{\theta_1, \theta_2\}$ to the spectrum of the corresponding Bloch Hamiltonian $\mathscr{H}^{\Theta}$ is called the *dispersion relation,* or the *Bloch variety* of $\mathscr{H}$; see Section 3 for a detailed discussion of $\mathcal{H}^{\Theta}$ and its connection with $\mathcal{H}$. In Theorem 4.5, we obtain the dispersion relation of the Hamiltonian $\mathcal{H}$ on the hexagonal lattice.

- The hexagonal elastic lattice Hamiltonian $\mathcal{H}$ is a self-adjoint operator, so its spectrum consists of the absolutely continuous spectrum, the pure point spectrum, and the singular continuous spectrum. Theorem 4.11 describes the absolutely continuous, pure point, and the singular continuous spectra of $\mathcal{H}$.
- If two sheets of the dispersion relation (surface) of an operator touch at a point and form a conical singularity, such a point is called a *Dirac point,* see Figure 5 and Figure 6. In Theorem 4.12, we prove a representation of the set of Dirac points of $\mathcal{H}$ in terms of the two branches of the Lyapunov function.
- The level surfaces of the dispersion relation for fixed energy values are called *Fermi surfaces* at the corresponding energy. At any energy level, the Fermi surface for $\mathcal{H}$ is a Laurent polynomial in $z_1 := e^{i\theta_1}$ and $z_2 := e^{i\theta_2}$ for the quasimomenta θ_1 and θ_2. A Fermi surface is called *(ir)reducible* if the corresponding Laurent polynomial is (ir)reducible. Theorem 4.14 characterizes reducible and irreducible Fermi surfaces for $\mathcal{H}$.
- In Corollary 5.4 and Theorem 5.6, we investigate the role of angle-dependent vertex conditions. Under perturbed angles, see Figure 4, we show the existence of Dirac points and describe the spectrum, which becomes purely absolutely continuous.

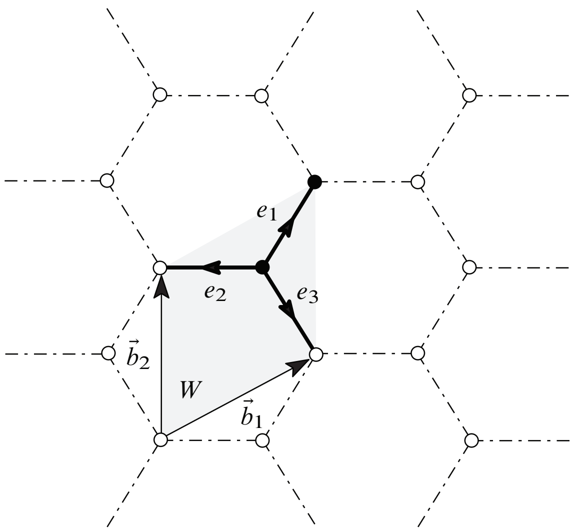

Figure 1. The hexagonal lattice Γ and a fundamental domain W together with its set of vertices $V(W) = \{v_1, v_2\}$ and set of edges $E(W) = \{e_1, e_2, e_3\}$.

The paper is structured as follows.

- In Section 2, we summarize background material starting with a discussion of the parameterization of the beam deformation, energy functional, quadratic form, and Hamiltonian on planar frames. This discussion is continued with a brief review of the spectral properties of the fourth-order periodic operator $\mathcal{H}^{\text{per}}$ on the real line.
- In Section 3, we give a characterization of the spectrum of elastic lattice Hamiltonians on the hexagonal lattice and its perturbations.
- Section 4 is devoted to the derivation of the dispersion relation, Dirac points, and the spectral structure of the hexagonal lattice.
- Section 5 discusses extensions of our results for perturbed angles.
- Section 6 contains additional remarks and potential future extensions.

2. Preliminaries

In this section we briefly review the literature to build the necessary background for understanding the forthcoming material. Elastic beams are modeled by quantum graphs [7]. A quantum graph is a metric graph equipped with a differential operator and vertex conditions [8].

In the first part of this section, the self-adjoint beam operator $\mathcal{H}$ is defined on the graph Γ along with the corresponding vertex conditions. Next, we briefly discuss the spectral properties of the second-order periodic operator defined on the real line, $\mathcal{H}^{\text{per}}$, known as the Hill operator. We summarize these results from [1,3,9] in Theorem 2.4, which will be referred to in the following sections.

2.1. Elastic planar graphs

Under the Euler–Bernoulli beam model, each beam is described by an energy functional which involves four degrees of freedom for every infinitesimal element along the beam: axial, lateral (2 degrees of freedom) and angular displacements [7]. A critical step here is how to derive vertex matching conditions that are both mathematically general and physically sound for application purposes.

By restricting to one degree of freedom, that is, lateral displacement, vertex conditions for planar graphs have been derived by assuming that the deformed lattice will remain locally planar at a vertex, that is, the tangent plane exists at that vertex [19]. The resulting scalar-valued operator is shown to be self-adjoint [19]. The extension of these results to general three-dimensional graphs is developed in [7]. This has been done by introducing the notion of rigidity at the vertex, so that the corresponding matching conditions are derived. Interestingly, the remaining vertex conditions, which make the vector-valued operator self-adjoint, have a connection to the engineering world, namely satisfying equilibrium of forces and moments at the vertices. Further extensions of these results to semi-rigid joints have recently been proposed in [4] where the discontinuity of the displacement and rotation fields is admissible at a vertex.

In the special case of planar frames, the operator decomposes into a direct sum of two operators, one coupling out-of-plane displacement to angular (torsional) displacement and the other coupling in-plane with axial displacements. However, achieving this level of physically sound models means that the operator is no longer scalar-valued and contains at least two degrees of freedom (for planar graphs) coupled at the joints [7].

In this work, we follow the results of the scalar-valued operator from [19] with the benefit of revealing some solid theoretical results regarding the spectra of the corresponding Hamiltonian on the hexagonal lattice and its geometric perturbations.

2.1.1. Energy functional on planar lattice. A beam frame is a collection of beams connected at joints. We describe a beam frame as a combinatorial graph $\Gamma = (V, E)$, where V denotes the set of vertices and E the set of edges. Let us note that in this paper we consider the hexagonal lattice and its geometric perturbations (see Figure 4), so in this general planar lattice framework, we assume Γ to be an equilateral planar graph with countably many vertices of finite degree (identical for each vertex) and without loops. Now, letting the distance between two points, not necessarily vertices, on Γ be the minimal length of the path connecting them over Γ, we can introduce a metric. This makes Γ a metric graph.

The vertices $\mathfrak{v} \in V$ correspond to joints and the edges $e \in E$ are the beams. Each edge e is a collection of the following information:

- origin and terminus vertices $\mathfrak{v}_e^o, \mathfrak{v}_e^t \in V$,
- length ℓ_e and
- the local basis $\{\vec{\imath}_e, \vec{\jmath}_e, \vec{k}_e\}$.

For special planar graphs, which we focus on in this paper, the local basis $\vec{k}_e = \vec{k}$ is constant for all edges $e \in E$, so the graph Γ can be embedded in $\mathbb{R}^2$, see Figure 2. Describing the vertices V as points in $\mathbb{R}^2$ also fixes the axial direction $\vec{\imath}_e$ (from origin to terminus) and the length ℓ_e, which without loss of generality will be assumed to be 1. However, the choice of $\vec{\jmath}_e$ in the plane orthogonal to $\vec{\imath}_e$ still needs to be specified externally. The distinction between origin and terminus, and thus the direction of $\vec{\imath}_e$, is not important in analysis.

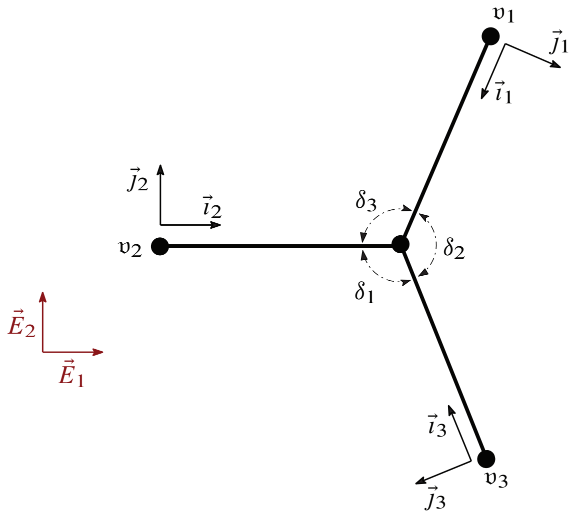

Figure 2. Planar local and global coordinate bases.

In the context of the kinematic Bernoulli assumptions for beam frames without prestress and external force, the total strain energy of the beam frame is expressed as

$$\mathcal{U} := \frac{1}{2} \sum_{e \in E} \int_e (a_e |u_e''(x)|^2 + q(x)|u_e(x)|^2) dx, \tag{2.1}$$

where the parameter a_e is positive and fixed on the edge e representing the bending stiffness about the local axis $\vec{j}_e$ and $q \in L^2(e)$ is real-valued.

Assumption 1. The potential function $q \in L^2(e)$ satisfies the evenness (symmetry) property, i.e.,

$$q(x) = q(1-x) \tag{2.2}$$

after parameterizing the corresponding edge e as $[0, 1]$. The evenness assumption (2.2) is made not just for mathematical convenience, but this condition is required if one considers operators invariant with respect to all symmetries of the periodic lattice.

2.1.2. Quadratic and operator forms. We now give a formal mathematical description of the Euler–Bernoulli strain energy form.

Theorem 2.1 (Sesquilinear form [7, 19]). *Energy functional* (2.1) *of the planar beam lattice with free rigid joints is the quadratic form corresponding to the positive closed* *sesquilinear form*

$$\mathcal{Q}[u,\tilde{u}] := \sum_{e\in E}\int_e (a_e u_e''(x)\overline{\tilde{u}_e''(x)} + q(x)u_e(x)\overline{\tilde{u}_e(x)})dx \tag{2.3}$$

densely defined on the Hilbert space

$$L^2(\Gamma) := \bigoplus_{e\in E} L^2(e)$$

with the domain of $\mathcal{Q}$ *consisting of the vectors from*

$$\bigoplus_{e\in E} H^2(e)$$

that satisfy at every vertex $\mathfrak{v}\in V$ *rigid joint conditions, namely for any vertex* $\mathfrak{v}$ *and edges* e *and* e' *adjacent to* $\mathfrak{v}$

$$u_e(\mathfrak{v}) = u_{e'}(\mathfrak{v}) \tag{2.4a}$$

and

$$(\vec{\jmath}_2\cdot\vec{\imath}_e)u_1'(\mathfrak{v}) + (\vec{\jmath}_e\cdot\vec{\imath}_1)u_2'(\mathfrak{v}) + (\vec{\jmath}_1\cdot\vec{\imath}_2)u_e'(\mathfrak{v}) = 0 \tag{2.4b}$$

for fixed edges $1,2$ *adjacent to* $\mathfrak{v}$ *and any edge* e *adjacent to* $\mathfrak{v}$*, where the local basis functions* $\vec{\imath}_e$*,* $\vec{\jmath}_e$ *are defined above.*

In Theorem 2.1, all functions are evaluated at the vertex $\mathfrak{v}$ and all derivatives are taken in the direction $\vec{\imath}_e$. One may notice that condition (2.4a) guarantees the continuity of u at a vertex $\mathfrak{v}$ while (2.4b) preserves the local tangent plane in which the edges $e\in\mathfrak{v}$ reside initially; see [9, 19] for details.

Obviously, for the cases $e = 1,2$ in (2.4b), trivial conditions appear, for example, for $e = 1$, equation (2.4b) has the form

$$(\vec{\jmath}_2\cdot\vec{\imath}_1)u_1'(\mathfrak{v}) + (\vec{\jmath}_1\cdot\vec{\imath}_1)u_2'(\mathfrak{v}) + (\vec{\jmath}_1\cdot\vec{\imath}_2)u_1'(\mathfrak{v}) = 0.$$

But here $\vec{\jmath}_1\cdot\vec{\imath}_1 = 0$ since the two vectors are orthonormal and, moreover, $(\vec{\jmath}_2\cdot\vec{\imath}_1) = -(\vec{\jmath}_1\cdot\vec{\imath}_2)$, so we get $0=0$. Then, for the non-trivial cases, i.e., $e = 3,4,\ldots,n_{\mathfrak{v}}$, we get non-trivial conditions, one for each e. Therefore, for the case of the hexagonal periodic graph, see Figure 1, we get only one non-trivial identity from condition (2.4b). The following theorem characterizes the Hamiltonian of the frame as a self-adjoint differential operator on the metric graph. We state it for vertices of degree 3, since in this paper we focus on hexagonal lattices.

Theorem 2.2 (Operator form [19]). *The energy form* (2.3) *on a hexagonal beam frame with free rigid joints corresponds to the self-adjoint operator* $\mathcal{H}\colon L^2(\Gamma) \to L^2(\Gamma)$ *acting as*

$$u_e \mapsto a_e u_e'''' + q u_e \tag{2.5}$$

on every edge $e \in E$ *of the graph. The domain of the operator* $\mathcal{H}$ *consists of the functions from*

$$\bigoplus_{e \in E} H^4(e)$$

that satisfy at each vertex $\mathfrak{v} \in V$ *with degree* $n_{\mathfrak{v}}$ *the following*

(i) *primary conditions*

$$u_1(\mathfrak{v}) = u_2(\mathfrak{v}) = u_3(\mathfrak{v}),$$
$$(\vec{j}_2 \cdot \vec{\imath}_3)u_1'(\mathfrak{v}) + (\vec{j}_3 \cdot \vec{\imath}_1)u_2'(\mathfrak{v}) + (\vec{j}_1 \cdot \vec{\imath}_2)u_3'(\mathfrak{v}) = 0;$$

(ii) *conjugate conditions*

$$\sum_{n=1}^{3} s_{\mathfrak{v}}^n a_n u_n'''(\mathfrak{v}) = 0$$

and

$$\sum_{n=1}^{3} s_{\mathfrak{v}}^n a_n (\vec{\imath}_1 \cdot \vec{j}_n) u_n''(\mathfrak{v}) = 0,$$
$$\sum_{n=1}^{3} s_{\mathfrak{v}}^n a_n (\vec{\imath}_2 \cdot \vec{j}_n) u_n''(\mathfrak{v}) = 0,$$
$$\sum_{n=1}^{3} s_{\mathfrak{v}}^n a_n (\vec{\imath}_3 \cdot \vec{j}_n) u_n''(\mathfrak{v}) = 0.$$

The defined operator $\mathcal{H}$ is unbounded and self-adjoint in the Hilbert space $L^2(\Gamma)$. Due to the condition (2.2) on the potential, the Hamiltonian $\mathcal{H}$ is invariant with respect to all symmetries of the hexagonal lattice Γ, in particular, with respect to the $\mathbb{Z}^2$-shifts, which will play a crucial role in our consideration; see [25] for a detailed discussion on the role of symmetry of the potential.

2.2. Periodic fourth-order operator on the real line

Next, we summarize existing results on the spectral theory of the periodic fourth-order operator on the real line. There are key differences compared to the second-order (Hill) operator, which are essential for us to develop our results. The reader familiar with the aforementioned discussions can skip this subsection and jump to Theorem 2.4.

Consider the self-adjoint operator

$$\mathcal{H}^{\mathrm{per}} := d^4/dx^4 + q_0(x)$$

acting on $L^2(\mathbb{R})$, where the real 1-periodic potential $q_0(x)$ belongs to the real space

$$L_0^2(\mathbb{T}) := \left\{ q_0 \in L^2(\mathbb{T}) : \int_0^1 q_0(x)dx = 0 \right\},$$

where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

We introduce the fundamental solutions $\{g_k(x)\}_{k=1}^4$ of the eigenvalue problem

$$\mathcal{H}^{\mathrm{per}} u(x) = \lambda u(x), \quad (x, \lambda) \in \mathbb{R} \times \mathbb{C} \tag{2.8}$$

satisfying for $j, k \in \{1, 2, 3, 4\}$, the conditions

$$g_k^{(j-1)}(0) = \delta_{jk},$$

where δ_{jk} is the Kronecker delta function and $g^{(k)} = d^k g/dx^k$. The monodromy matrix is defined as

$$M(\lambda) := \mathcal{M}(1, \lambda)$$

for

$$\mathcal{M}(x, \lambda) := \begin{pmatrix} g_1(x) & g_2(x) & g_3(x) & g_4(x) \\ g_1'(x) & g_2'(x) & g_3'(x) & g_4'(x) \\ g_1''(x) & g_2''(x) & g_3''(x) & g_4''(x) \\ g_1'''(x) & g_2'''(x) & g_3'''(x) & g_4'''(x) \end{pmatrix} \tag{2.9}$$

and it shifts by the period along the solutions of (2.8). It is well known that the monodromy matrix $M(\lambda)$ is entire as a complex function of λ. Its eigenvalue $\tau \in \mathbb{C}$, i.e., the root of the algebraic polynomial

$$D(\tau, \lambda) := \det(M(\lambda) - \tau \mathbb{I}_4),$$

is called a *multiplier.*

According to the Lyapunov theorem, if for some $\lambda \in \mathbb{C}$, $\tau(\lambda)$ is a multiplier, then $1/\tau(\lambda)$ is another multiplier of the same multiplicity. Moreover, each $M(\lambda)$ has exactly four multipliers $\tau_1(\lambda)$, $\tau_2(\lambda)$, $1/\tau_1(\lambda)$ and $1/\tau_2(\lambda)$, see [1].

If we let

$$D_+(\lambda) := D(1, \lambda)/4 \qquad \text{and} \qquad D_-(\lambda) := D(-1, \lambda)/4,$$

then the zeros of $D_+(\lambda)$ and $D_-(\lambda)$ are the eigenvalues of the periodic and antiperiodic problem, respectively, for (2.8). Denote by λ_0^+, $\lambda_{2n}^\pm$, and $\lambda_{2n-1}^\pm$ with $n \in \mathbb{N}$, the sequence of zeros of D_+ and D_- (counted with multiplicity) respectively such that

$$\lambda_0^+ \le \lambda_2^- \le \lambda_2^+ \le \lambda_4^- \le \lambda_4^+ \le \cdots$$

and

$$\lambda_1^- \le \lambda_1^+ \le \lambda_3^- \le \lambda_3^+ \le \lambda_5^- \le \cdots .$$

It is well known that the spectrum of $\mathcal{H}^{\mathrm{per}}$ is purely absolutely continuous and consists of non-degenerate intervals [1, 3]. These intervals are separated by the gaps $G_n = (\lambda_n^-, \lambda_n^+)$, $n \in \mathbb{N}$.

We introduce the functions

$$T_1(\lambda) := \frac{1}{4}\mathrm{tr}(M(\lambda)), \quad T_2(\lambda) := \frac{1}{2}[\mathrm{tr}(M^2(\lambda)) + 1] - [\mathrm{tr}(M(\lambda))]^2.$$

The complex functions T_1 and T_2 are entire, real on $\mathbb{R}$, and determine the discriminant as

$$D(\tau, \cdot) = \big(\tau^2 - 2(T_1 - T_2^{1/2})\tau + 1\big)\big(\tau^2 - 2(T_1 + T_2^{1/2})\tau + 1\big).$$

For the special case of the free operator, that is, $q_0 \equiv 0$, the corresponding functions have the form

$$T_1^0(\lambda) = \frac{1}{2}(\cosh(\lambda^{1/4}) + \cos(\lambda^{1/4})),$$

$$T_2^0(\lambda) = \frac{1}{4}(\cosh(\lambda^{1/4}) - \cos(\lambda^{1/4}))^2$$

with $\arg(\lambda^{1/4}) \in (-\pi/4, \pi/4]$.

Let $\{r_0^-, r_n^\pm\}_{n\in\mathbb{N}}$ be the sequence of zeros of T_2 in $\mathbb{C}$ (counted with multiplicity) such that r_0^- is the maximal real zero, and

$$\cdots \le \mathrm{Re}\, r_{n+1}^+ \le \mathrm{Re}\, r_n^+ \le \cdots \le \mathrm{Re}\, r_1^+.$$

If $r_n^+ \in \mathbb{C}_+$, then

$$r_n^- = \overline{r_n^+} \in \mathbb{C}_-,$$

and if $r_n^+ \in \mathbb{R}$, then

$$r_n^- \le r_n^+ \le \mathrm{Re}\, r_m^-$$

for $m = 1, \ldots, n-1$.

Under extra mild conditions, it was shown that

$$r_n^\pm = -4(n\pi)^4 + \mathcal{O}(n^2)$$

as $n \to \infty$. Let

$$\cdots \le r_{n_j}^- \le r_{n_j}^+ \le \cdots \le r_{n_1}^- \le r_{n_1}^+ \le r_0^-$$

be the subsequence of the real zeros of T_2. Then $T_2(\lambda) < 0$ for any $\lambda \in R_j^0 := (r_{n_{j+1}}^+, r_{n_j}^-)$ for $j \in \mathbb{N}$. A zero of the function $T_2(\lambda)$ is called a *resonance* of the operator $\mathcal{H}^{\text{per}}$ and the interval $R_j^0 \subset \mathbb{R}$ is called a *resonance gap*.

Let us introduce $R^0 := \bigcup R_j^0$ and η^0 which joins the points $r_n^+, \overline{r_n^+}$ and does not cross R^0. To deal with the roots of the function $T_2(\lambda)$, the Riemann surface $\mathcal{R}$ is constructed by taking two replicas of the λ-plane cut along R^0 and $\bigcup \eta_n$. They are called *sheets* $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. As a result, there exists a unique analytic continuation of the function $T_2^{1/2}(\lambda)$. We denote the Lyapunov function by

$$\Delta(\xi) = T_1(\xi) + T_2^{1/2}(\xi)$$

with $\xi \in \mathcal{R}$. Letting $\Delta(\xi) = \Delta_1(\lambda)$ on $\mathcal{R}_1$ and $\Delta(\xi) = \Delta_2(\lambda)$ on the second sheet $\mathcal{R}_2$, we get

$$\Delta_1(\lambda) = T_1(\lambda) + T_2^{1/2}(\lambda), \tag{2.10a}$$

$$\Delta_2(\lambda) = T_1(\lambda) - T_2^{1/2}(\lambda). \tag{2.10b}$$

For $q_0 \in L_0^2(\mathbb{T})$, the function $\Delta(\lambda) = T_1(\lambda) + T_2^{1/2}(\lambda)$ is analytic on the two sheeted Riemann surface $\mathcal{R}$ and the branches Δ_k of Δ have the forms

$$\Delta_k(\lambda) = \frac{1}{2}(\tau_k(\lambda) + \tau_k^{-1}(\lambda))$$

for $\lambda \in \mathcal{R}_k$ with $k = 1, 2$. For the special case $q_0 \equiv 0$, the corresponding functions are given by

$$\Delta_1^0(\lambda) = \cosh(\lambda^{1/2}) \quad \text{and} \quad \Delta_2^0(\lambda) = \cos(\lambda^{1/2}).$$

For the operator $\mathcal{H}^{\text{per}}$, the Lyapunov function Δ_1 is increasing and Δ_2 is bounded on the real line at high energy levels (large λ values). The Lyapunov function for the operator $\mathcal{H}^{\text{per}}$ defines the band structure of the spectrum and is an analytic function on a 2-sheeted Riemann surface. The qualitative behavior of the Lyapunov function for identically vanishing and small potentials is shown in Figure 3.

Remark 2.3. In the case of the Hill operator, the monodromy matrix has exactly 2 eigenvalues τ and τ^{-1}. The Lyapunov function $\frac{1}{2}(\tau + \tau^{-1})$ is an entire function of the spectral parameter. It defines the band structure of the spectrum; see [24] for a detailed discussion.

Theorem 2.4 (Spectral properties of $\mathcal{H}^{\text{per}}$ [1, 3, 9]). *Let $\Delta_1(\lambda)$ and $\Delta_2(\lambda)$ be as defined in* (2.10)*. Then, for the eigenvalue problem* (2.8)*, the following results hold.*

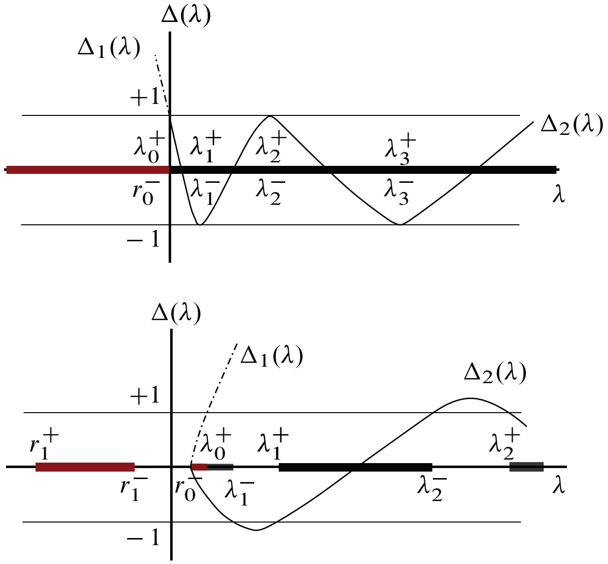

Figure 3. The function Δ for the zero potential and a small potential q_0.

(i) *The spectrum of $\mathscr{H}^{\text{per}}$, $\sigma(\mathscr{H}^{\text{per}})$, is purely absolutely continuous.*

(ii) *$\lambda \in \sigma(\mathscr{H}^{\text{per}})$ if and only if $\Delta_k(\lambda) \in [-1, 1]$ for some $k = 1, 2$. If $\lambda \in \sigma(\mathscr{H}^{\text{per}})$, then $T_2(\lambda) \geq 0$.*

(iii) *There exists an integer $n_0 \in \mathbb{N}_0$ such that for all $n \geq n_0$,*

$$\lambda_n^- \leq \lambda_n^+ \leq \lambda_{n+1}^- \leq \lambda_{n+1}^+ \leq \lambda_{n+2}^- \leq \lambda_{n+2}^+ \leq \cdots,$$

where the intervals $[\lambda_n^+, \lambda_{n+1}^-]$ are spectral bands of multiplicity 2 and the intervals $(\lambda_n^-, \lambda_n^+)$ are gaps.

(iv) *Each gap $G_n = (\lambda_n^-, \lambda_n^+)$ for $n \in \mathbb{N}$ is a bounded interval and $\lambda_n^\pm$ are either periodic (anti-periodic) eigenvalues or resonance point, namely, real branch point of Δ_k for some $k = 1, 2$ which is a zero of T_2.*

(v) *Any $\lambda \in \sigma(\mathscr{H}^{\text{per}})$ on an interval $S \subset \mathbb{R}$ has multiplicity 4 if and only if $-1 < \Delta_k(\lambda) < 1$ for all $k = 1, 2$ and $\lambda \in S$, except for finite number of points.*

(vi) *Any $\lambda \in \sigma(\mathscr{H}^{\text{per}})$ on an interval $S \subset \mathbb{R}$ has multiplicity 2 if and only if $-1 < \Delta_1(\lambda) < 1$, $\Delta_2(\lambda) \in \mathbb{R} \setminus [-1, 1]$ or $-1 < \Delta_2(\lambda) < 1$, $\Delta_1(\lambda) \in \mathbb{R} \setminus [-1, 1]$ for all $\lambda \in S$, except for finite number of points.*

(vii) *Let Δ_k be real analytic on some interval $I \subset \mathbb{R}$ and $-1 < \Delta_k(\lambda) < 1$ for any $\lambda \in I$ for some $k = 1, 2$. Then $\Delta_k'(\lambda) \neq 0$ for $\lambda \in I$ (monotonicity).*

(viii) *The dispersion relation for* $\mathcal{H}^{\text{per}}$ *is given by* $\Delta_{1,2}(\lambda) = \cos(\theta)$ *or equivalently*

$$T_1(\lambda) \pm T_2^{1/2}(\lambda) = \cos(\theta),$$

where θ *is the one-dimensional quasimomentum.*

3. Spectra of the perturbed hexagonal elastic lattice Hamiltonians

In this section, our aim is to adapt and characterize the spectrum $\sigma(\mathcal{H})$ of the operator $\mathcal{H}$ defined in Theorem 2.2 on the hexagonal lattice and its perturbations. We construct such lattices by considering a 3-edge star graph as the fundamental domain and extending it to the plane, see Figure 1. Our focus will be on the hexagonal lattice (Section 4) and its perturbations (Section 5). In order to cover both cases in Proposition 3.3, we define angles $\delta_0 := 2\pi/3$ and

$$\delta_c^\varepsilon := \delta_0 + c\varepsilon. \tag{3.1}$$

Then by letting the three angles between the edges of the fundamental domain be δ_1^ε, δ_c^ε and $\delta_{-1-c}^\varepsilon$ for $c \in [-1, 1]$ and $0 < \varepsilon \ll 1$, we construct ε-perturbations of the hexagonal lattice. The hexagonal lattice corresponds to the case $\varepsilon = 0$, see Figure 4.

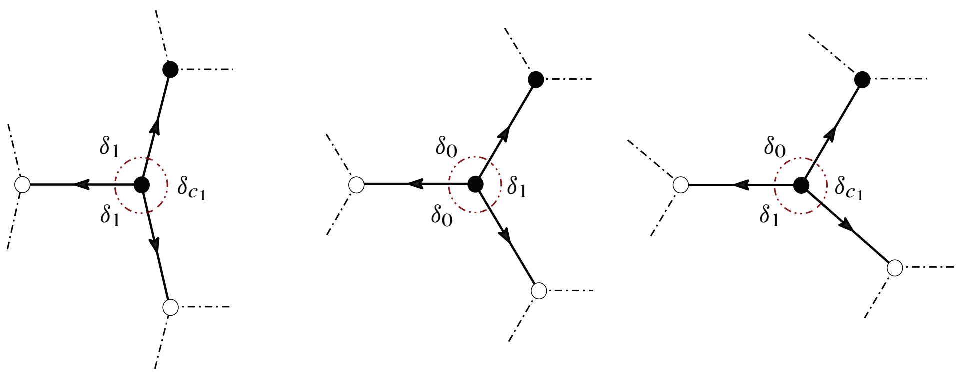

Figure 4. Fundamental domains for angle perturbed hexagonal lattices, see (3.1). The middle picture shows the hexagonal lattice in which the edges meet with equal angles at each vertex. Recall that $\delta_0 := 2\pi/3$ and $\delta_c := \delta_0 + c\varepsilon$. In the left picture, the angles are $2\pi/3 + \varepsilon$, $2\pi/3 + \varepsilon$ and $2\pi/3 - 2\varepsilon$. In the right picture, the angles are $2\pi/3$, $2\pi/3 + \varepsilon$ and $2\pi/3 - \varepsilon$.

Due to the positivity and self-adjointness of the operator $\mathcal{H}$, its spectrum is real and positive. Let $\lambda \in \sigma(\mathcal{H})$ with $\lambda > 0$ be an eigenvalue of $\mathcal{H}$ with the associated eigenfunction $(u_e)_{e\in E} \in \mathcal{D}(\mathcal{H})$. Then $u_e(x)$ satisfies on each edge $e \in E$

$$\mathcal{H}u_e(x) := u_e''''(x) + q(x)u_e(x) = \lambda u_e(x). \tag{3.2}$$

Remark 3.1. Note that since a_e – the bending stiffness about the local axis $\vec{\jmath}_e$ in (2.5) – is a positive constant and identical on the hexagonal lattice (because of the periodicity), we assume that it is identically one.

Recalling the definition (3.1) and letting $c_1 \in [-1, 1]$ be an arbitrary parameter and $c_2 := -(1 + c_1)$, the eigenfunction $(u_e)_{e\in E} \in \mathcal{D}(\mathcal{H})$ corresponding to the ε-perturbed lattice at each vertex $\mathfrak{v}$ satisfies (see Theorem 2.2) the primary vertex conditions

$$u_1(\mathfrak{v}) = u_2(\mathfrak{v}) = u_3(\mathfrak{v}), \tag{3.3a}$$
$$\sin(\delta_1^\varepsilon)u_1'(\mathfrak{v}) + \sin(\delta_{c_1}^\varepsilon)u_2'(\mathfrak{v}) + \sin(\delta_{c_2}^\varepsilon)u_3'(\mathfrak{v}) = 0, \tag{3.3b}$$

along with their conjugate vertex conditions

$$u_1''(\mathfrak{v})/\sin(\delta_1^\varepsilon) = u_2''(\mathfrak{v})/\sin(\delta_{c_1}^\varepsilon) = u_3''(\mathfrak{v})/\sin(\delta_{c_2}^\varepsilon), \tag{3.4a}$$
$$u_1'''(\mathfrak{v}) + u_2'''(\mathfrak{v}) + u_3'''(\mathfrak{v}) = 0. \tag{3.4b}$$

We note that the result in (3.4a) is obtained using the fact that $\vec{\imath}_3 \cdot \vec{\jmath}_2 = -\vec{\imath}_2 \cdot \vec{\jmath}_3$. Moreover, for the hexagonal lattice and its ε-perturbed angles, the conditions above are well defined; we refer the reader to [19] for a discussion of special cases, e.g., when $\delta_1^\varepsilon = \pi$.

Using Floquet–Bloch theory with respect to the $\mathbb{Z}^2$-action, we can reduce the study of the Hamiltonian $\mathcal{H}$ to the study of the Bloch Hamiltonians $\mathcal{H}^\Theta$ acting in $L^2(W)$ for the values of the *quasimomentum* Θ in the (first) Brillouin zone $[-\pi, \pi]^2$ (see, e.g., [24]). The Bloch Hamiltonian $\mathcal{H}^\Theta$ and Hamiltonian $\mathcal{H}$ correspond to the same differential operator, but act on different spaces of functions. Any function

$$u = \bigoplus_{e\in E} u_e$$

in the domain of $\mathcal{H}^\Theta$ belongs to the Sobolev space $H^4(e)$ on each edge e and satisfies the vertex conditions (3.3)–(3.4). Additionally, it satisfies the cyclic conditions (Floquet–Bloch conditions)

$$u(x + n_1\vec{b}_1 + n_2\vec{b}_2) = e^{i\vec{n}\cdot\Theta}u(x) = e^{i(n_1\theta_1+n_2\theta_2)}u(x) \tag{3.5}$$

for any x in the fundamental domain W, vector $\vec{n} = (n_1, n_2) \in \mathbb{Z}^2$, and quasimomentum $\Theta = (\theta_1, \theta_2) \in [-\pi, \pi]^2$, see Figure 1. Cyclic conditions (3.5) provide unique determination of the function u from its restriction to the fundamental domain W.

Here, without loss of generality in the fundamental domain, we assume the edge e_1 is $[0, 1]$, so the vertices $\mathfrak{v}_1$ and $\mathfrak{v}_2$ are 0 and 1. Then conditions (3.3)–(3.4) at the central vertex, i.e., $x = 0$, become

$$u_1(0) = u_2(0) = u_3(0) =: A, \tag{3.6a}$$
$$\sin(\delta_1^\varepsilon)u_1'(0) + \sin(\delta_{c_1}^\varepsilon)u_2'(0) + \sin(\delta_{c_2}^\varepsilon)u_3'(0) = 0, \tag{3.6b}$$
$$u_1''(0)/\sin(\delta_1^\varepsilon) = u_2''(0)/\sin(\delta_{c_1}^\varepsilon) = u_3''(0)/\sin(\delta_{c_2}^\varepsilon) =: B, \tag{3.6c}$$
$$u_1'''(0) + u_2'''(0) + u_3'''(0) = 0. \tag{3.6d}$$

Similarly, at the other endpoint of edge e_1, i.e., $x = 1$, we have

$$u_1(1) = u_2(1)e^{i\theta_1} = u_3(1)e^{i\theta_2} =: C \tag{3.7a}$$
$$\sin(\delta_1^\varepsilon)u_1'(1) + \sin(\delta_{c_1}^\varepsilon)u_2'(1)e^{i\theta_1} + \sin(\delta_{c_2}^\varepsilon)u_3'(1)e^{i\theta_2} = 0 \tag{3.7b}$$
$$u_1''(0)/\sin(\delta_1^\varepsilon) = u_2''(0)e^{i\theta_1}/\sin(\delta_{c_1}^\varepsilon) = u_3''(0)e^{i\theta_2}/\sin(\delta_{c_2}^\varepsilon) =: D \tag{3.7c}$$
$$u_1'''(1) + u_2'''(1)e^{i\theta_1} + u_3'''(1)e^{i\theta_2} = 0. \tag{3.7d}$$

It is well known that $\mathcal{H}^\Theta$ has a purely discrete spectrum $\sigma(\mathcal{H}^\Theta) = \{\lambda_k(\Theta)\}_{k\in\mathbb{N}}$, see [24]. The graph of the multi-valued function

$$\Theta \mapsto \{\lambda_k(\Theta)\}$$

is called *the dispersion relation*, or *the Bloch variety* of the operator $\mathcal{H}$. The range of the dispersion relation is the spectrum of $\mathcal{H}$ (see [24]):

$$\sigma(\mathcal{H}) = \bigcup_{\Theta\in[-\pi,\pi]^2} \sigma(\mathcal{H}^\Theta). \tag{3.8}$$

Taking into account (3.8), the dispersion relation of $\mathcal{H}$ can be determined from the spectrum of $\mathcal{H}^\Theta$. Therefore, we need to solve the eigenvalue problem

$$\mathcal{H}^\Theta u(x) = \lambda u(x) \tag{3.9}$$

for $\lambda \in \mathbb{R}$ and $u_e(x) \in L_e^2(W)$ with the boundary conditions (3.6) and (3.7). Let Σ^D denote the spectrum of the operator

$$\mathcal{H}u(x) = u''''(x) + q(x)u(x) \tag{3.10}$$

on the interval $(0, 1)$ with boundary conditions

$$u(0) = 0, \quad u''(0) = 0, \quad u(1) = 0, \quad u''(1) = 0. \tag{3.11}$$

If $\lambda \notin \Sigma^D$, then there exist four linearly independent solutions ϕ_1, ϕ_2, ϕ_3, and ϕ_4 (depending on λ) of (3.10) on (0, 1) such that

$$\begin{aligned} &\phi_1(0) = 1, \quad \phi_1''(0) = 0, \quad \phi_1(1) = 0, \quad \phi_1''(1) = 0, \\ &\phi_2(0) = 0, \quad \phi_2''(0) = 1, \quad \phi_2(1) = 0, \quad \phi_2''(1) = 0, \\ &\phi_3(0) = 0, \quad \phi_3''(0) = 0, \quad \phi_3(1) = 1, \quad \phi_3''(1) = 0, \\ &\phi_4(0) = 0, \quad \phi_4''(0) = 0, \quad \phi_4(1) = 0, \quad \phi_4''(1) = 1. \end{aligned} \tag{3.12}$$

For example, if $q \equiv 0$ and $\lambda > 0$, then we have $\lambda \notin \Sigma^D$ if and only if $\lambda^{1/4} \notin \pi\mathbb{Z}$. If $\lambda \notin \Sigma^D$, then

$$\begin{aligned} \phi_1(x) &= \frac{1}{2}\big(\cos(\lambda^{1/4}x) + \cosh(\lambda^{1/4}x) - \cot(\lambda^{1/4})\sin(\lambda^{1/4}x) \\ &\qquad - \coth(\lambda^{1/4})\sinh(\lambda^{1/4}x)\big), \\ \phi_2(x) &= \frac{1}{2}\big(-\cos(\lambda^{1/4}x) - \cosh(\lambda^{1/4}x) + \cot(\lambda^{1/4})\sin(\lambda^{1/4}x) \\ &\qquad - \coth(\lambda^{1/4})\sinh(\lambda^{1/4}x)\big), \\ \phi_3(x) &= \frac{1}{2}\big(\sin(\lambda^{1/4}x)/\sin(\lambda^{1/4}) + \sinh(\lambda^{1/4}x)/\sinh(\lambda^{1/4})\big), \\ \phi_4(x) &= \frac{1}{2}\big(-\sin(\lambda^{1/4}x)/\sin(\lambda^{1/4}) + \sinh(\lambda^{1/4}x)/\sinh(\lambda^{1/4})\big). \end{aligned}$$

Next, we assume that the functions ϕ_k, defined on [0, 1] with the conditions (3.12), are lifted to each edge in the fundamental domain W. Moreover, with abuse of notation, ϕ_k denotes the lifted functions.

For $\lambda \notin \Sigma^D$, we use (3.12) to represent any solution u of (3.9) from the domain of $\mathcal{H}^\Theta$ on each edge in W as follows:

$$\begin{aligned} u_1(x) = A\phi_1(x) &+ B\sin(\delta_1^\varepsilon)\phi_2(x) + C\phi_3(x)e^{-i\theta_0} \\ &+ D\sin(\delta_1^\varepsilon)\phi_4(x)e^{-i\theta_0}, \end{aligned} \tag{3.13a}$$

$$\begin{aligned} u_2(x) = A\phi_1(x) &+ B\sin(\delta_{c_1}^\varepsilon)\phi_2(x) + C\phi_3(x)e^{-i\theta_1} \\ &+ D\sin(\delta_{c_1}^\varepsilon)\phi_4(x)e^{-i\theta_1}, \end{aligned} \tag{3.13b}$$

$$\begin{aligned} u_3(x) = A\phi_1(x) &+ B\sin(\delta_{c_2}^\varepsilon)\phi_2(x) + C\phi_3(x)e^{-i\theta_2} \\ &+ D\sin(\delta_{c_2}^\varepsilon)\phi_4(x)e^{-i\theta_2}, \end{aligned} \tag{3.13c}$$

with $\theta_0 = 0$. Next, let us introduce the (Wronskian) operator

$$\mathcal{W}\colon L^2[0,1] \times L^2[0,1] \to \mathbb{C},$$

defined as

$$\mathcal{W}_x(u_1, u_2) := u_1'''(x)u_2(x) - u_1''(x)u_2'(x) + u_1'(x)u_2''(x) - u_1(x)u_2'''(x)$$

for $x \in [0, 1]$. Then for the fourth-order Hamiltonian $\mathcal{H}$, we get

$$u_2(x)\mathcal{H}u_1(x) - u_1(x)\mathcal{H}u_2(x) = (\mathcal{W}_1(u_1, u_2) - \mathcal{W}_0(u_1, u_2))'. \tag{3.14}$$

If u_1 and u_2 solve $\mathcal{H}u = \lambda u$, then $\mathcal{W}_1(u_1, u_2) - \mathcal{W}_0(u_1, u_2)$ is a constant. In the next lemma, we use $\mathcal{W}$ to show some identities of ϕ_k at the endpoints.

Lemma 3.2. *Applying the symmetry property* (2.2) *of the operator* $\mathcal{H}$ *acting on the interval* $(0, 1)$*, we get*

$$\begin{aligned}
&\phi_3'(1) = -\phi_1'(0), && \phi_3'(0) = -\phi_1'(1), && \\
&\phi_3'''(1) = -\phi_1'''(0), && \phi_3'''(0) = -\phi_1'''(1), && \phi_2'''(0) = \phi_1'(0), \\
&\phi_4'(1) = -\phi_2'(0), && \phi_4'(0) = -\phi_2'(1), && \\
&\phi_4'''(1) = -\phi_2'''(0), && \phi_4'''(0) = -\phi_2'''(1), && \phi_2'''(1) = \phi_1'(1).
\end{aligned}$$

Proof of Lemma 3.2. The proof of this lemma is based on (3.14). In fact, for $n, m \in \{1, 2, 3, 4\}$ and $n \neq m$, let $\phi_n(x)$ and $\phi_m(x)$ be two independent solutions of eigenvalue problem

$$\mathcal{H}u(x) = u''''(x) + q(x)u(x) = \lambda u(x)$$

on $(0, 1)$ satisfying the boundary conditions (3.12). Now, observe that

$$\phi_m(x)\mathcal{H}\phi_n(x) - \phi_n(x)\mathcal{H}\phi_m(x) = \phi_m(x)\lambda\phi_n(x) - \phi_n(x)\lambda\phi_m(x) = 0. \tag{3.15}$$

However, by (3.14),

$$\phi_m(x)\mathcal{H}\phi_n(x) - \phi_n(x)\mathcal{H}\phi_m(x) = (\mathcal{W}_1(\phi_n, \phi_m) - \mathcal{W}_0(\phi_n, \phi_m))'. \tag{3.16}$$

For a constant c, (3.15) and (3.16) then imply that $\mathcal{W}_1(\phi_n, \phi_m) - \mathcal{W}_0(\phi_n, \phi_m) = \mathrm{c}$. For any choice of $n \neq m$, observe that the boundary conditions in (3.12) implies $\mathrm{c} = 0$, i.e.,

$$\mathcal{W}_1(\phi_n, \phi_m) = \mathcal{W}_0(\phi_n, \phi_m).$$

Finally, applying the properties of ϕ_n from (3.12), we get the result we look for.

As an example, setting $(n, m) = (1, 3)$ and using the property that the only non-zero terms are $\phi_1(0)$ and $\phi_3(1)$, one gets

$$\phi_3'''(0) = -\phi_1'''(0).$$

Similar conclusions can be made to derive the desired relations stated in the lemma.

∎

Next, for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and an arbitrary $\varepsilon > 0$, let

$$S_k^\varepsilon(\Theta) := \frac{1}{\sin^k(\delta_0)}(\sin^k(\delta_1^\varepsilon) + \sin^k(\delta_{c_1}^\varepsilon)e^{-i\theta_1} + \sin^k(\delta_{c_2}^\varepsilon)e^{-i\theta_2}),$$

where $\Theta \in [-\pi, \pi]^2$, $\delta_0 = 2\pi/3$, $c_1 \in [-1, 1]$ and $c_2 = -(1 + c_1)$. We introduce the scaled versions of B and D from (3.6c) and (3.7c) as $\widetilde{B} := \sin(\delta_0)B$ and $\widetilde{D} := \sin(\delta_0)D$, respectively. Using the functions u_i defined in (3.13) with the vertex conditions (3.6b), (3.6d), (3.7b), and (3.7d) reduces the problem to finding the vector $\vec{\xi} := (A\ \widetilde{B}\ C\ \widetilde{D})^T$ that satisfies

$$\mathbb{M}_\varepsilon \vec{\xi} = \begin{pmatrix} A_0(\varepsilon) & -A_1(\varepsilon) \\ -\widetilde{A}_1(\varepsilon) & A_0(\varepsilon) \end{pmatrix} \vec{\xi} = 0. \tag{3.17}$$

The block components of the matrix $\mathbb{M}_\varepsilon$ are written in terms of the quasimomentum Θ and the solutions ϕ_k as

$$A_0(\varepsilon) := \begin{pmatrix} S_1^\varepsilon(0)\phi_1'(0) & S_2^\varepsilon(0)\phi_2'(0) \\ S_0^\varepsilon(0)\phi_1'''(0) & S_1^\varepsilon(0)\phi_2'''(0) \end{pmatrix}, \quad A_1(\varepsilon) := \begin{pmatrix} S_1^\varepsilon(\Theta)\phi_1'(1) & S_2^\varepsilon(\Theta)\phi_2'(1) \\ S_0^\varepsilon(\Theta)\phi_1'''(1) & S_1^\varepsilon(\Theta)\phi_2'''(1) \end{pmatrix},$$

and

$$\widetilde{A}_1(\varepsilon) := -\begin{pmatrix} \overline{S_1^\varepsilon(\Theta)}\phi_1'(1) & \overline{S_2^\varepsilon(\Theta)}\phi_2'(1) \\ \overline{S_0^\varepsilon(\Theta)}\phi_1'''(1) & \overline{S_1^\varepsilon(\Theta)}\phi_2'''(1) \end{pmatrix}.$$

Clearly, a non-trivial solution exists, if the matrix $\mathbb{M}_\varepsilon(\lambda)$ is singular. This is formally stated in the following proposition.

Proposition 3.3. *If $\lambda \notin \Sigma^D$, then λ is in spectrum of the hexagonal elastic lattice Hamiltonian $\mathscr{H}$ if and only if there exists $\Theta \in [-\pi, \pi]^2$ such that*

$$\det(\mathbb{M}_\varepsilon(\lambda)) = 0. \tag{3.18}$$

The result in Proposition 3.3 can be directly (numerically) investigated, but we will split the discussion into two parts. In the following section, we state the theoretical results for case $\varepsilon = 0$, namely, the hexagonal lattice. In Section 5, extensions of these results are presented for the perturbed angles, applying tools from perturbation analysis.

4. Spectral properties of the hexagonal elastic lattice Hamiltonian

In this section, we discuss the outcome of the results from the previous section for the special case of $\varepsilon = 0$, i.e., the hexagonal lattice, and obtain the spectral properties of the hexagonal elastic lattice Hamiltonian. In this case, for any $k \in \mathbb{N}_0$, we let

$$s_0(\Theta) := S_k^0(\Theta) = 1 + e^{-i\theta_1} + e^{-i\theta_2}. \tag{4.1}$$

From (4.1), the components of the block matrix defined in (3.17) split into

$$A_0(\lambda) = s_0(0)\Phi_0(0), \quad A_1(\lambda) = -s_0(\Theta)\Phi_0(1), \quad \widetilde{A}_1(\lambda) = -\overline{s_0(\Theta)}\Phi_0(1),$$

where the matrices $\Phi_0(0)$ and $\Phi_0(1)$ are of the form

$$\Phi_0(0) := \begin{pmatrix} \phi_1'(0) & \phi_2'(0) \\ \phi_1'''(0) & \phi_2'''(0) \end{pmatrix} \quad \text{and} \quad \Phi_0(1) := \begin{pmatrix} \phi_1'(1) & \phi_2'(1) \\ \phi_1'''(1) & \phi_2'''(1) \end{pmatrix}. \tag{4.2}$$

Lemma 4.1. *The matrix $\Phi_0(1)$ defined in (4.2) is non-singular.*

Proof of Lemma 4.1. By contradiction, let us assume $\Phi_0(1)$ is singular, which by the identities in Lemma 3.2 reduces to the condition

$$\det(\Phi_0(1)) = \phi_1'(1)\phi_2'''(1) - \phi_2'(1)\phi_1'''(1) = \phi_3'(0)\phi_4'''(0) - \phi_4'(0)\phi_3'''(0) = 0. \tag{4.3}$$

Using the fact that $\phi_4'''(0) = \phi_3'(0)$, equation (4.3) implies (at least) one of the following conditions is true:

(i) $\phi_3'(0) = 0$ and $\phi_3'''(0) = 0$;

(ii) $\phi_3'(0) = 0$ and $\phi_4'(0) = 0$;

(iii) $\phi_3'(0) \neq 0$, $\phi_3'''(0) \neq 0$, and $\phi_4'(0) \neq 0$.

Recalling the fundamental solutions g_k from Section 2.2, we can represent ϕ_3 as a linear combination of them in the form

$$\phi_3(x) = b_1 g_1(x) + b_2 g_2(x) + b_3 g_3(x) + b_4 g_4(x). \tag{4.4}$$

Then, using (4.4) along with the conditions $\phi_3(0) = 0$, $\phi_3''(0) = 0$, item (i) implies that $\phi_3 \equiv 0$, which is a contradiction.

A similar discussion holds to show that item (ii) above implies $\phi_4 \equiv 0$.

Considering the last case above, let us introduce

$$r := \frac{\phi_3'(0)}{\phi_3'''(0)} = \frac{\phi_4'(0)}{\phi_3'(0)}.$$

Then, obviously, by our assumption, $r \neq 0$. Utilizing the representation (4.4) and a similar representation for ϕ_4, we get

$$\phi_3(x) = \phi_3'(0) g_2(x) + \frac{\phi_3'(0)}{r} g_4(x) \quad \text{and} \quad \phi_4(x) = r\phi_3'(0) g_2(x) + \phi_3'(0) g_4(x).$$

Comparing these two representations implies that $\phi_4 = r\phi_3$, which is a contradiction, since by our assumption ϕ_3 and ϕ_4 are linearly independent solutions. This proves the non-singularity of the matrix $\Phi_0(1)$. ∎

Applying the fact that $s_0(0) = 3$ along with Lemma 4.1 reduces condition (3.18) in Proposition 3.3 to

$$\det\Big(\Lambda_0^2(\lambda) - \frac{|s_0(\Theta)|^2}{9}\mathbb{I}_2\Big) = 0, \tag{4.5}$$

where $\Lambda_0(\lambda) := \Phi_0^{-1}(1)\Phi_0(0)$. This means that for the hexagonal lattice we proved the following result.

Proposition 4.2. *If $\lambda \notin \Sigma^D$, then λ is in the spectrum of the hexagonal elastic lattice Hamiltonian $\mathcal{H}$ if and only if there is $\Theta \in [-\pi, \pi]^2$ such that*

$$\det\Big(\Lambda_0(\lambda) - \frac{|s_0(\Theta)|}{3}\mathbb{I}_2\Big)\det\Big(\Lambda_0(\lambda) + \frac{|s_0(\Theta)|}{3}\mathbb{I}_2\Big) = 0.$$

In other words, $|s_0(\Theta)|/3$ is a root of the characteristic polynomial for $\Lambda_0(\lambda)$ or $-\Lambda_0(\lambda)$ matrices, i.e., a root of

$$\mathcal{P}(z;\lambda) = (z^2 - \operatorname{tr}(\Lambda_0(\lambda))z + \det(\Lambda_0(\lambda)))(z^2 + \operatorname{tr}(\Lambda_0(\lambda))z + \det(\Lambda_0(\lambda))). \tag{4.6}$$

Proposition 4.2, in particular, says that in order to find the spectrum of $\mathcal{H}$, we need to calculate the range of $|s_0(\Theta)|$ on $[-\pi, \pi]^2$. This function is identical to the one reported for the second-order Schrödinger operator on the hexagonal lattice (the graphene Hamiltonian), see [25].

In summary, $|s_0(\Theta)|$ has range $[0, 3]$, its maximum is at $(0, 0)$, and minimum at $(2\pi/3, -2\pi/3)$ and $(-2\pi/3, 2\pi/3)$. These properties are based on the simple observation that

$$|s_0(\Theta)|^2 = |1 + e^{i\theta_1} + e^{i\theta_2}|^2$$

with range $[0, 9]$. See Figure 8 (left) for a plot of the level curves of this function.

4.1. The dispersion relation via fundamental solutions

Our next goal is to obtain the dispersion relation by representing the functions ϕ_k, and hence the matrix Λ_0, in terms of the potential q_0 on $[0, 1]$. To this end, let us recall the operator $\mathcal{H}^{\text{per}}$ on $\mathbb{R}$, defined in Section 2 as

$$\mathcal{H}^{\text{per}}u(x) = u''''(x) + q_0(x)u(x) \tag{4.7}$$

with the periodic potential extended from q_0. Note that we maintain the notation q_0 for the extended potential. The fundamental solutions $\{g_k(x)\}_{k=1}^4$ of $\mathcal{H}^{\text{per}}$ satisfy for $j, k \in \{1, 2, 3, 4\}$, the conditions

$$g_k^{(j-1)}(0) = \delta_{jk}.$$

Therefore, the monodromy matrix $M(\lambda)$ defined through (2.9), shifts by the period along the solutions of (4.7), that is,

$$\begin{pmatrix} u(1) \\ u'(1) \\ u''(1) \\ u'''(1) \end{pmatrix} = \begin{pmatrix} g_1(1) & g_2(1) & g_3(1) & g_4(1) \\ g_1'(1) & g_2'(1) & g_3'(1) & g_4'(1) \\ g_1''(1) & g_2''(1) & g_3''(1) & g_4''(1) \\ g_1'''(1) & g_2'''(1) & g_3'''(1) & g_4'''(1) \end{pmatrix} \begin{pmatrix} u(0) \\ u'(0) \\ u''(0) \\ u'''(0) \end{pmatrix}.$$

The 4×4 matrix-valued function $\lambda \mapsto M(\lambda)$ is entire, see Section 2 and the references therein for more detailed discussions. Since our goal is to obtain the dispersion relation of the operator $\mathcal{H}$, next we derive relations among g_k and ϕ_k. For simplicity, let us introduce the following notation:

$$\mathcal{D}(f, g) := f'(0)g'''(1) - g'(1)f'''(0). \tag{4.8}$$

Lemma 4.3. *The fundamental solutions $\{g_k(x)\}_{k=1}^4$ of $\mathcal{H}^{\mathrm{per}}$ can be represented in terms of the functions ϕ_1 and ϕ_2 introduced in (3.12) as*

$$g_1(x) = \phi_1(x) + \frac{1}{\det(\Phi_0(1))}(\mathcal{D}(\phi_1, \phi_2)\phi_3(x) - \mathcal{D}(\phi_1, \phi_1)\phi_4(x)),$$
$$g_3(x) = \phi_2(x) + \frac{1}{\det(\Phi_0(1))}(\mathcal{D}(\phi_2, \phi_2)\phi_3(x) + \mathcal{D}(\phi_1, \phi_2)\phi_4(x)),$$

and

$$g_2(x) = \frac{-1}{\det(\Phi_0(1))}(\phi_1'(1)\phi_3(x) - \phi_1'''(1)\phi_4(x)),$$
$$g_4(x) = \frac{1}{\det(\Phi_0(1))}(\phi_2'(1)\phi_3(x) - \phi_1'(1)\phi_4(x)).$$

Proof of Lemma 4.3. Starting with the observation that $\{\phi_k\}_{k=1}^4$ and $\{g_k\}_{k=1}^4$ solve the eigenvalue problem

$$u''''(x) + q(x)u(x) = \lambda u(x)$$

and the fact that these are linearly independent sets of solutions, we represent each g_k as

$$g_k(x) = a_k\phi_1(x) + b_k\phi_2(x) + c_k\phi_3(x) + d_k\phi_4(x).$$

Applying the properties of ϕ_k given in (3.12), we observe that coefficients corresponding to g_1 satisfy

$$g_1(0) = 0 \implies a_1 = 1, \quad g_1''(0) = 0 \implies b_1 = 0.$$

Moreover, the remaining conditions result in

$$\begin{aligned} g_1'(0) = 0 \implies g_1'(0) &= \phi_1'(0) + c_1\phi_3'(0) + d_1\phi_4'(0) \\ &= \phi_1'(0) - c_1\phi_1'(1) - d_1\phi_2'(1) = 0 \end{aligned}$$

and

$$\begin{aligned} g_1'''(0) = 0 \implies g_1'''(0) &= \phi_1'''(0) + c_1\phi_3'''(0) + d_1\phi_4'''(0) \\ &= \phi_1'(0) - c_1\phi_1'''(1) - d_1\phi_2'''(1) = 0. \end{aligned}$$

Solving for c_1 and d_1, we get

$$c_1 = \frac{\mathcal{D}(\phi_1, \phi_2)}{\det(\Phi_0(1))}, \quad d_1 = -\frac{\mathcal{D}(\phi_1, \phi_1)}{\det(\Phi_0(1))}.$$

Similar discussions can be followed to obtain the coefficients corresponding to remaining g_k. ■

The symmetry of the potential q_0 brings additional properties on the fundamental solutions which are summarized in the following lemma.

Lemma 4.4. *Under the symmetry property of the potential q_0, the fundamental solutions satisfy*

$$\begin{aligned} g_1''(1) = g_2'''(1), \quad g_1'(1) = g_3'''(1), \quad g_1(1) = g_4'''(1) \\ g_2'(1) = g_4'''(1), \quad g_2(1) = g_4''(1), \quad g_3(1) = g_4'(1) \end{aligned}$$

Proof of Lemma 4.4. Establishing this result is similar to the proof of Lemma 3.2 along with an application of the symmetry of the potential. ■

Next, let us introduce the matrix $\mathbb{G}_0(\lambda)$ as

$$\mathbb{G}_0(\lambda) := \begin{pmatrix} g_1(1) & g_3(1) \\ g_1''(1) & g_3''(1) \end{pmatrix}. \tag{4.9}$$

This matrix can be interpreted as an extension of the (scalar-valued) discriminant function $D(\lambda) = g_1(1) + g_2'(1) = 2g_1(1)$ for the eigenvalue problem corresponding to the second-order Schrödinger operator [25]. Putting all the observations above together allows us to derive the dispersion relation of $\mathcal{H}$, stated formally in the following theorem.

Theorem 4.5 (Dispersion relation). *The dispersion relation of the hexagonal elastic lattice Hamiltonian $\mathcal{H}$ consists of the variety*

$$\det\Big(\mathbb{G}_0^2(\lambda) - \frac{|s_0(\Theta)|^2}{9}\mathbb{I}_2\Big) = 0 \tag{4.10}$$

and the collection of flat branches $\lambda \in \Sigma^D$.

Proof of Theorem 4.5. Recalling the notation (4.8), and then applying Lemma 4.3 and (3.12), we get the following identities:

$$g_1(1) + g_4'''(1) = +2\frac{\mathcal{D}(\phi_1,\phi_2)}{\det(\Phi_0(1))}, \quad g_3(1) + g_4'(1) = +2\frac{\mathcal{D}(\phi_2,\phi_2)}{\det(\Phi_0(1))},$$

$$g_1''(1) + g_2'''(1) = -2\frac{\mathcal{D}(\phi_1,\phi_1)}{\det(\Phi_0(1))}, \quad g_3''(1) + g_2'(1) = -2\frac{\mathcal{D}(\phi_2,\phi_1)}{\det(\Phi_0(1))}.$$

Since $\phi_2'''(0) = \phi_1'(0)$ and $\phi_2'''(1) = \phi_1'(1)$, observe that the right-hand sides of the above equations are the entries of $2\Lambda_0(\lambda)$ introduced in (4.5). Therefore, using Lemma 4.4, one gets

$$2\Lambda_0(\lambda) = \begin{pmatrix} g_1(1) + g_4'''(1) & g_3(1) + g_4'(1) \\ g_1''(1) + g_2'''(1) & g_3''(1) + g_2'(1) \end{pmatrix} = 2\begin{pmatrix} g_1(1) & g_3(1) \\ g_1''(1) & g_3''(1) \end{pmatrix} = 2\mathbb{G}_0(\lambda),$$

Then, combining the results from Proposition 4.2 and Lemma 4.7 establishes the claimed result. ■

For specific purposes, e.g., the reducibility of the Fermi surface, it may be desirable to rephrase (4.10) in terms of the characteristic polynomials.

Remark 4.6. The eigen-parameter λ is in the spectrum of the hexagonal elastic lattice Hamiltonian $\mathcal{H}$ if and only if $\lambda \in \Sigma^D$, or $|s_0(\Theta)|/3$ is a root of the characteristic polynomial for $\mathbb{G}_0(\lambda)$ or $-\mathbb{G}_0(\lambda)$, i.e., $\lambda \in \Sigma^D$ or is a root of

$$\mathcal{P}(z;\lambda) = (z^2 - \operatorname{tr}(\mathbb{G}_0(\lambda))z + \det(\mathbb{G}_0(\lambda)))(z^2 + \operatorname{tr}(\mathbb{G}_0(\lambda))z + \det(\mathbb{G}_0(\lambda)))$$

by equation (4.6).

Noting that $\phi_2'''(1) = \phi_1'(1)$, we can also write the dispersion relation as follows: the eigen-parameter λ is in the spectrum of $\mathcal{H}$ if and only if

$$\left(\Delta_1(\lambda) \pm \frac{|s_0(\Theta)|}{3}\right)\left(\Delta_2(\lambda) \pm \frac{|s_0(\Theta)|}{3}\right) = 0$$

or $\lambda \in \Sigma^D$, where $\Delta_{1,2}(\lambda)$ were defined in (2.10) and

$$T_1 = \frac{\operatorname{tr}(\mathbb{G}_0)}{2}, \quad T_2 = \frac{\operatorname{tr}^2(\mathbb{G}_0)}{4} - \det(\mathbb{G}_0). \tag{4.11}$$

We have not yet discussed the points in the Dirichlet spectrum Σ^D of a single edge. In order to deal with them, we will explicitly construct their corresponding eigenfunctions. We refer the reader to [25] for a similar type of construction for the second-order operator case.

Lemma 4.7. *Each point $\lambda \in \Sigma^D$ is an infinite multiplicity eigenvalue of the hexagonal elastic lattice Hamiltonian $\mathcal{H}$. The corresponding eigen-space is generated by simple loop states, i.e., by eigen-functions which are supported on a single hexagon and vanish at the vertices.*

Proof of Lemma 4.7. Let us first show that each $\lambda \in \Sigma^D$ is an eigenvalue. Let u be an eigenfunction of the operator $d^4/dx^4 + q_0(x)$ with the eigenvalue λ and (Dirichlet-type) boundary conditions on [0, 1] as stated in (3.11). Note that $u(1-x)$ is also an eigenfunction with the same eigenvalue, since $q_0(x)$ is even. If $u(x)$ is neither even nor odd, then $u(x) - u(1-x)$ is an odd eigenfunction. For an odd eigenfunction, repeating it on each of the six edges of a hexagon and letting the eigenfunction be zero on any other hexagon, we get an eigenfunction of the operator $\mathcal{H}$. If u is an even eigenfunction, then repeating it around the hexagon with an alternating sign and letting the eigenfunction be zero on any other hexagon, we get an eigenfunction of the operator $\mathcal{H}$. Therefore, $\lambda \in \sigma_{pp}(\mathcal{H})$. We get the rest of the proof following the arguments of [25, Lemma 3.5]. ∎

Remark 4.8. Compared to Schrödinger, the Dirichlet-type boundary conditions for the fourth-order operator may be a place to be cautious. Naturally, one may select vanishing boundary conditions in quadratic form as Dirichlet ones (this choice holds for the second-order operator). However, here we defined Σ^D as (3.11) to accommodate Floquet vertex conditions in (3.6a), (3.6c), etc. We refer the interested reader to Section 6 for a further discussion along this line.

Example 1. Consider the free operator, i.e., $q \equiv 0$. Setting $\mu := \sqrt[4]{\lambda}$ and using the convention

$$C_\mu^\pm(x) := \cosh(\mu x) \pm \cos(\mu x) \quad \text{and} \quad S_\mu^\pm(x) := \sinh(\mu x) \pm \sin(\mu x),$$

the fundamental solutions take the forms

$$g_1(x) = \frac{1}{2} C_\mu^+(x), \qquad g_2(x) = \frac{1}{2\mu} S_\mu^+(x),$$

$$g_3(x) = \frac{1}{2\mu^2} C_\mu^-(x), \quad g_4(x) = \frac{1}{2\mu^3} S_\mu^-(x),$$

and hence

$$\mathbb{G}_0(\lambda) = \begin{pmatrix} g_1(1) & g_3(1) \\ g_1''(1) & g_3''(1) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} C_\mu^+(1) & \mu^{-2} C_\mu^-(1) \\ \mu^2 C_\mu^-(1) & C_\mu^+(1) \end{pmatrix}.$$

This then implies

$$\det\Big(\mathbb{G}_0(\lambda) \pm \frac{|s_0(\Theta)|}{3} \mathbb{I}_2\Big) = \Big(\frac{|s_0(\Theta)|}{3}\Big)^2 \pm \operatorname{tr}(\mathbb{G}_0)\Big(\frac{|s_0(\Theta)|}{3}\Big) + \det(\mathbb{G}_0).$$

Therefore, the dispersion relation is equivalent to

$$\left(\cos(\lambda^{1/4}) \pm \frac{|s_0(\Theta)|}{3}\right)\left(\cosh(\lambda^{1/4}) \pm \frac{|s_0(\Theta)|}{3}\right) = 0.$$

Since $\cosh(x) \geq 1$, and taking into account $|s_0(\Theta)| \leq 3$, the only root of the second factor is at $\Theta = (0,0)$ and $\lambda = 0$ which also solve the first phrase. Therefore, the dispersion relation for $q_0 \equiv 0$ reduces to

$$\cos(\lambda^{1/4}) = \pm\frac{|s_0(\Theta)|}{3}. \tag{4.12}$$

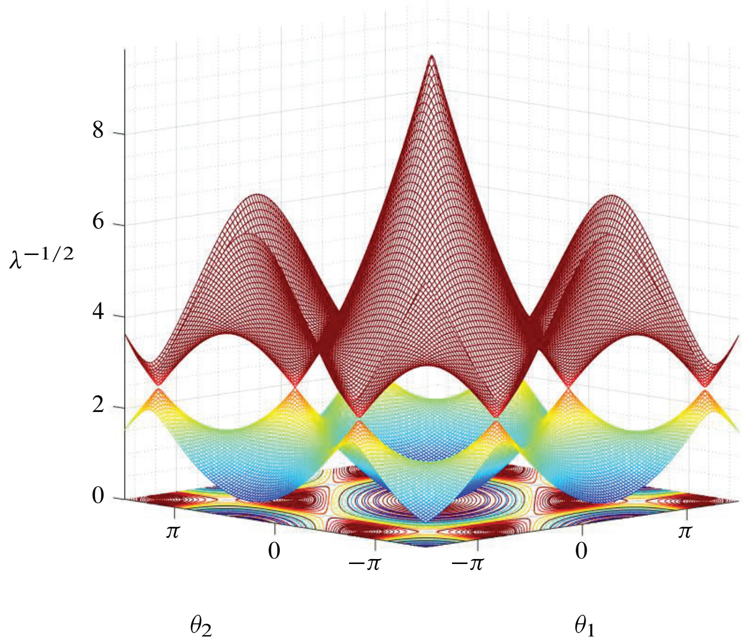

Figure 5. Dispersion relation for the zero potential case, see (4.12).

Remark 4.9. The dispersion relation of the second-order Schrödinger operator with the vanishing potential, that is, $\mathcal{H}^s u(x) = -u''(x)$ on graphene has the form

$$\cos(\lambda^{1/2}) = \pm\frac{|s_0(\Theta)|}{3},$$

which is interestingly very similar to (4.12). Therefore, Example 1 shows that the dispersion relation of the hexagonal elastic lattice Hamiltonian $\mathcal{H}$ coincides with that for the second-order Schrödinger operator (graphene Hamiltonian) $\mathcal{H}^s$, if the eigenvalue problems $\mathcal{H}^s u = \lambda u$ and $\mathcal{H} u = \lambda^{1/2} u$ are considered. Figure 5 shows the plot of the first two spectral sheets of the dispersion relation.

4.2. The spectrum of the hexagonal elastic lattice Hamiltonian

This section is devoted to the full description of the spectra, conical singularities, and Fermi surfaces corresponding to $\mathcal{H}$ defined on the hexagonal lattice.

Lemma 4.10. *As a set, Σ^D belongs to the union of the periodic and the antiperiodic spectra of $\mathcal{H}^{\mathrm{per}}$.*

Proof of Lemma 4.10. Let $\lambda \in \Sigma^D$. Since the potential q_0 is even, if $u(x)$ is an eigenfunction, then $u(1-x)$ is also an eigenfunction. Therefore, we consider the two cases: u even or u odd. If u is odd, it satisfies the periodic boundary conditions, i.e.,

$$u(0) = u(1), \quad u'(0) = u'(1), \quad u''(0) = u''(1), \quad u'''(0) = u'''(1). \tag{4.13}$$

On the other-hand, if u is even, it satisfies the anti-periodic boundary conditions

$$u(0) = -u(1), \quad u'(0) = -u'(1), \quad u''(0) = -u''(1), \quad u'''(0) = -u'''(1). \tag{4.14}$$

■

We can now fully describe the spectral structure of the hexagonal elastic lattice operator $\mathcal{H}$.

Theorem 4.11 (Spectral description). (i) *The singular continuous spectrum $\sigma_{\mathrm{sc}}(\mathcal{H})$ is empty.*

(ii) *The absolutely continuous spectrum $\sigma_{\mathrm{ac}}(\mathcal{H})$ has a band-gap structure and coincides as a set with the spectrum $\sigma(\mathcal{H}^{\mathrm{per}})$ of the fourth-order operator $\mathcal{H}^{\mathrm{per}}$ with potential q_0 periodically extended from* $[0, 1]$. *Moreover, the absolutely continuous spectrum $\sigma_{\mathrm{ac}}(\mathcal{H})$ has the representation*

$$\sigma_{\mathrm{ac}}(\mathcal{H}) = \{\lambda \in \mathbb{R} \mid \Delta_k(\lambda) = [-1, 1] \text{ for some } k = 1, 2\}, \tag{4.15}$$

where

$$\Delta_{1,2}(\lambda) := \frac{1}{2}(\mathrm{tr}(\mathbb{G}_0(\lambda)) \pm (\mathrm{tr}^2(\mathbb{G}_0(\lambda)) - 4\det(\mathbb{G}_0(\lambda)))^{1/2}),$$

and $\mathbb{G}_0$ is defined in (4.9).

(iii) *The pure point spectrum $\sigma_{\mathrm{pp}}(\mathcal{H})$ coincides with Σ^D as a set, and for large energy values, it belongs to the union of the edges of the spectral bands of $\sigma_{\mathrm{ac}}(\mathcal{H})$.*

Proof of Theorem 4.11. The proof is based on the tools developed in this paper, along with the results already established in the relevant references.

For (i), observe that the singular continuous spectrum is empty, since $\mathcal{H}$ is a self-adjoint elliptic operator (see e.g., [24, Corollary 6.11]).

The proof of (ii) is based on Theorem 4.5, as we know that any $\lambda \notin \Sigma^D$ belongs to $\sigma(\mathcal{H})$ if and only if $|s_0(\Theta)|/3$ is a root of the characteristic polynomial for $D(\lambda)$ or $-D(\lambda)$, i.e., a root of

$$\mathcal{P}(z;\lambda) := (z^2 - \operatorname{tr}(\mathbb{G}_0(\lambda))z + \det(\mathbb{G}_0(\lambda)))(z^2 + \operatorname{tr}(\mathbb{G}_0 D(\lambda))z + \det(\mathbb{G}_0(\lambda))).$$

Since the range of $|s_0(\Theta)|$ is $[0,3]$, then $\mathcal{P}(|s_0(\Theta)|/3;\lambda) = 0$ if and only if $\Delta_1 \in [-1,1]$ or $\Delta_2 \in [-1,1]$. This observation along with Proposition 4.2 provide the desired representation (4.15). According to Thomas' analytic continuation argument, eigenvalues correspond to the constant branches of the dispersion relation [25,34,38]. Since the dispersion surfaces

$$\{(\Theta,\lambda) \in \mathbb{R}^3 \mid \Delta_k(\lambda) = \pm|s_0(\Theta)|/3 \text{ for some } k = 1,2\}$$

have no constant branches outside Σ^D, we get $\sigma_{\mathrm{pp}}(\mathcal{H}) \subseteq \Sigma^D$ and hence

$$\sigma_{\mathrm{ac}}(\mathcal{H}) = \{\lambda \in \mathbb{R} \mid \Delta_k(\lambda) \in [-1,1] \text{ for some } k = 1,2\}.$$

Note that (4.15) also represents $\sigma(\mathcal{H}^{\mathrm{per}}) = \sigma_{\mathrm{ac}}(\mathcal{H}^{\mathrm{per}})$ by Theorem 2.4 (ii). So, the absolutely continuous spectrum $\sigma_{\mathrm{ac}}(\mathcal{H})$ has band-gap structure and coincides as a set with the spectrum $\sigma(\mathcal{H}^{\mathrm{per}})$ of the operator $\mathcal{H}^{\mathrm{per}}$ with potential q_0 periodically extended from $[0,1]$.

Finally, for (iii), we observed that $\sigma_{\mathrm{pp}}(\mathcal{H}) \subseteq \Sigma^D$ and in Lemma 4.7 we showed that $\Sigma^D \subseteq \sigma_{\mathrm{pp}}(\mathcal{H})$. Then Lemma 4.10 implies that $\sigma_{\mathrm{pp}}(\mathcal{H}) \subset \Sigma^{\mathrm{p}} \cup \Sigma^{\mathrm{ap}}$, where Σ^{p} and Σ^{ap} denote the periodic and anti-periodic spectra of (3.10), i.e., with the boundary conditions (4.13) and (4.14) respectively. However, from Theorem 2.4 (iii), there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the edges of the n-th spectral band are the n-th periodic and anti-periodic eigenvalues. This concludes the proof. ■

The next theorem proves the existence of Dirac points, also called *diabilical points,* in the dispersion relation of $\mathcal{H}$, where its different sheets touch to form a conical singularity.

Theorem 4.12 (Dirac points). *The set of Dirac points of $\mathcal{H}$ in the (first) Brillouin zone is*

$$\{(\Theta,\lambda) \in \mathbb{R}^3 \mid \Theta = \pm(2\pi/3, -2\pi/3),\ T_2(\lambda-\varepsilon,\lambda+\varepsilon) \subset [0,\infty) \text{ and}$$
$$\Delta_k(\lambda) = 0 \text{ for some } \varepsilon > 0, k \in \{1,2\}\}.$$

Proof of Theorem 4.12. If $T_2(\lambda-\varepsilon,\lambda+\varepsilon) \not\subset [0,\infty)$, then λ cannot belong to the interior of a spectral band. If it is a band edge, it cannot be a Dirac point, since the dispersion relation may exhibit only a one-sided conical singularity. Observe that the function $|s_0(\Theta)|$ on $[-\pi,\pi]^2$ has vanishing conical singularities at the points

$\pm(2\pi/3, -2\pi/3)$. From Theorem 2.4 (vii), we know for $k = 1, 2$ and λ that $\Delta_k(\lambda) \in [-1, 1]$, Δ_k is analytic, and has non-zero derivative in the neighborhood of λ restricted to the interior of the corresponding band. Therefore, Δ_k is monotonic in any spectral band around any λ satisfying $\Delta_k(\lambda) = 0$; so using the dispersion relation of $\mathcal{H}$, we get the set of Dirac points. ∎

Remark 4.13. One can classify the Dirac points $(\pm\Theta^*, \lambda^*)$ with $\Theta^* := (2\pi/3, -2\pi/3)$ of the dispersion relation as follows.

- If λ^* is not a resonance point (i.e., $T_2(\lambda^*) \neq 0$) and $\Delta_k(\lambda^*) = 0$ for some $k \in \{1, 2\}$, then the dispersion relation around each of the singularities $(\pm\Theta^*, \lambda^*)$ consists of two cones located in opposite directions in λ^*-axis with the common vertex singularity $(\pm\Theta^*, \lambda^*)$. See Figure 6 (left). This is the case for large λ^*, i.e., a high energy level scheme.
- If $\Delta_1(\lambda^*) = \Delta_2(\lambda^*) = 0$ and there exists $\delta > 0$ such that $|T_2(\lambda)| < 1$ for all $\lambda \in [\lambda^* - \delta, \lambda^* + \delta]$, and $T_1(\lambda^* - \lambda) \neq T_1(\lambda^* + \lambda)$ for $\lambda \in (0, \delta)$, then the dispersion relation around each of the singularities $(\pm\Theta^*, \lambda^*)$ consists of four cones, two of them located in directions opposite to the other two on the λ-axis with the common vertex singularity at $(\pm\Theta^*, \lambda^*)$. See Figure 6 (right). Note that if $T_1(\lambda^* - \lambda) = T_1(\lambda^* + \lambda)$ for $\lambda \in (0, \delta)$, then the pairs of cones that are in the same direction coincide, so we get the first item above.

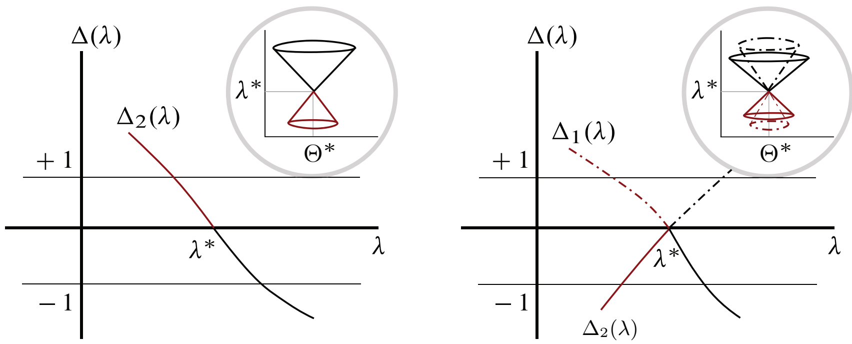

Figure 6. Behavior of functions Δ_1 and Δ_2 near Dirac point λ^*. The circular windows schematically show the dispersion relation in a neighborhood of $(\pm\Theta^*, \lambda^*)$, see Remark 4.13 for details.

The next result of this section is on the irreducibility of the Fermi surfaces corresponding to the Hamiltonian $\mathcal{H}$ at high-energy levels. Depending on the potential, the reducibility of this surface may occur for uncountably many (low) energies. Reducibility is required for the existence of embedded eigenvalues engendered by local defects, except for anomalous situations, where an eigenvalue has compact support [26]. In summary, the Fermi surface of a 2-periodic operator at an energy λ is the set of wave vectors (θ_1, θ_2) admissible by the operator at that energy.

For the periodic graph Hamiltonian, the dispersion function is a Laurent polynomial in the Floquet variables $(z_1, z_2) = (e^{i\theta_1}, e^{i\theta_2})$. When the dispersion function is factored, for each fixed energy, into a product of two or more Laurent polynomials in (θ_1, θ_2), each irreducible component contributes to a sequence of special bands and gaps. We refer the reader to [12] and references there for a detailed discussion.

From Theorem 4.5, the dispersion relation of $\mathscr{H}$ is equivalent to the fact that $|s_0(\Theta)|^2/9$ is an eigenvalue of $\mathbb{G}_0^2(\lambda)$, that is, it is a root of the polynomial

$$z^2 - \operatorname{tr}(\mathbb{G}_0^2(\lambda))z + \det(\mathbb{G}_0^2(\lambda)).$$

The roots of this quadratic polynomial have the following forms:

$$\frac{|s_0(\Theta)|^2}{9} = \frac{\operatorname{tr}(\mathbb{G}_0^2(\lambda))}{2} \pm \frac{1}{2}(\operatorname{tr}^2(\mathbb{G}_0^2(\lambda)) - 4\det(\mathbb{G}_0^2(\lambda)))^{1/2}.$$

Now, observe that

$$\begin{aligned}\frac{|s_0(\Theta)|^2}{9} &= \frac{\operatorname{tr}^2(\mathbb{G}_0(\lambda))}{2} - \det(\mathbb{G}_0(\lambda)) \\ &\quad \pm \frac{1}{2}(\operatorname{tr}(\mathbb{G}_0(\lambda))(\operatorname{tr}^2(\mathbb{G}_0(\lambda)) - 4\det(\mathbb{G}_0(\lambda)))^{1/2}) \\ &= -\det(\mathbb{G}_0(\lambda)) + \frac{1}{2}\operatorname{tr}(\mathbb{G}_0(\lambda))(\operatorname{tr}(\mathbb{G}_0(\lambda)) \\ &\quad \pm (\operatorname{tr}^2(\mathbb{G}_0(\lambda)) - 4\det(\mathbb{G}_0(\lambda)))^{1/2}).\end{aligned}$$

Then, using $T_1(\lambda)$ and $T_2(\lambda)$ from (4.11) in Δ_k implies that

$$\frac{|s_0(\Theta)|^2}{9} = T_1^2(\lambda) + T_2(\lambda) \pm 2T_1(\lambda)T_2^{1/2}(\lambda) = \Delta_{1,2}^2(\lambda).$$

So we proved the following result on the reducibility of the Fermi surface of $\mathscr{H}$.

Theorem 4.14 (Fermi surfaces). *The relation* (4.10) *has representation*

$$\big(P(z_1, z_2)P(z_1^{-1}, z_2^{-1}) - 9\Delta_1^2(\lambda)\big)\big(P(z_1, z_2)P(z_1^{-1}, z_2^{-1}) - 9\Delta_2^2(\lambda)\big) = 0,$$

where $P(z_1, z_2) := 1 + z_1 + z_2$ *and* $z_1 = e^{i\theta_1}$ *and* $z_2 = e^{i\theta_2}$*. Moreover, if*

$$\mathcal{S}_1 := \{\lambda \in \mathbb{R} \mid \Delta_1(\lambda) \in [-1, 1]\} \quad \textit{and} \quad \mathcal{S}_2 := \{\lambda \in \mathbb{R} \mid \Delta_2(\lambda) \in [-1, 1]\},$$

then the Fermi surface with the energy level $\lambda \notin \Sigma^D$ *is*

- *reducible if* $\lambda \in (\mathcal{S}_1 \cap \mathcal{S}_2)$,
- *irreducible if* $\lambda \in (\mathcal{S}_1 \setminus \mathcal{S}_2) \cup (\mathcal{S}_2 \setminus \mathcal{S}_1)$,
- *absent if* $\lambda \in \mathbb{R} \setminus (\mathcal{S}_1 \cup \mathcal{S}_2)$.

A remark on choices of the Brillouin zone. There is some room for the choice of the fundamental domain W for the hexagonal lattice. For the one selected in Figure 1, the space and the quasimomentum (conjugate) basis with respect to the global coordinate system are of the form

$$\vec{b}_1 = \frac{1}{2}\begin{pmatrix} 3 \\ \sqrt{3} \end{pmatrix}, \quad \vec{b}_2 = \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}, \quad \vec{b}_1^* = \frac{2}{3}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{b}_2^* = \frac{1}{3}\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}.$$

The dual basis then satisfies

$$\vec{b}_n^* \cdot \vec{b}_m = \delta_{nm} \tag{4.16}$$

and vectors $2\pi\vec{b}_1^*$ and $2\pi\vec{b}_2^*$ also span the hexagonal lattice, denoted by Γ^*. Now, the orthonormality condition (4.16) implies

$$n_1\theta_1 + n_2\theta_2 = (\theta_1\vec{b}_1^* + \theta_2\vec{b}_2^*) \cdot (n_1\vec{b}_1 + n_2\vec{b}_2).$$

The two choices of the Brillouin zone using coordinates $\Theta = (\theta_1, \theta_2)$ with respect to the dual basis vectors $\vec{b}_1^*, \vec{b}_2^*$ are shown in Figure 7. In the literature, it is common to represent these Brillouin zones in the corresponding Cartesian coordinates $\vec{\kappa} = (k_1, k_2)^T$ given by $\vec{\kappa} = B^*\Theta$, where B^* is the transformation matrix with columns formed by the dual basis vectors, i.e.,

$$B^* = (\vec{b}_1^* \quad \vec{b}_2^*) = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 0 & \sqrt{3} \end{pmatrix}.$$

As shown in Figure 7 (right), the resulting Brillouin zones will be symmetric in the new coordinate system $\vec{\kappa}$. We arrive at the first picture using θ_1 and θ_2 as parameters for the dispersion relation ranging within $[-\pi, \pi]^2$, and then plot the result using k_1 and k_2 as Cartesian coordinates. Although these two representations are equivalent, for the symmetry discussion, it may be preferable to work with the $\vec{\kappa}$ coordinate system, while for our case we followed the Brillouin zone in the Θ coordinates due to simpler presentation of the vertex conditions; see (3.7a)–(3.7d). Interested readers are encouraged to look at the work [6] for detailed discussions.

5. Spectral properties of the perturbed hexagonal elastic lattice Hamiltonian

In this section we will apply tools from perturbation theory to characterize the dispersion relation for the case in which edges meet at generally different angles; see Figure 4 for schematic fundamental domains. Restricted to the fundamental domain W,

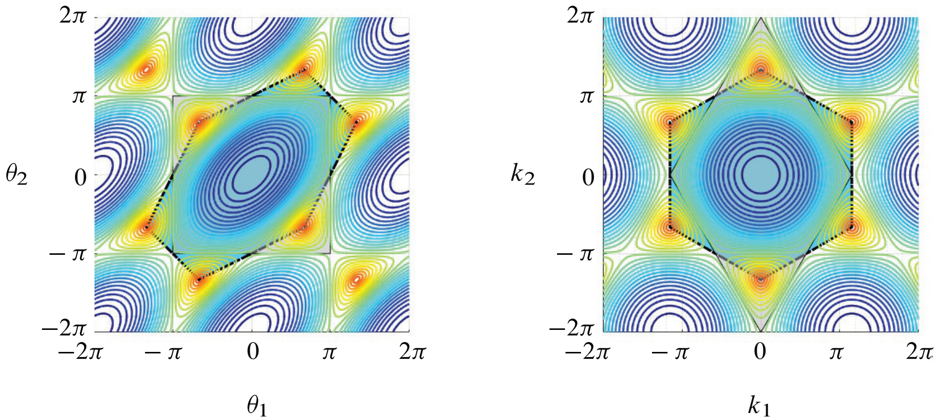

Figure 7. Choices of Brillouin zone, contour plot of second sheet of the dispersion surface in Left: coordinates θ_1, θ_2 (drawn as if they were Cartesian) and Right: coordinates k_1, k_2 (which are Cartesian).

this is equivalent to finding $(\lambda, \Theta) \in \mathbb{R} \times [-\pi, \pi]^2$ so that $\det(\mathbb{M}_\varepsilon(\lambda)) = 0$ as stated in Proposition 3.3.

First, observe that for the angle $\delta_c^{(\varepsilon)}$, see (3.1), an expansion of sine function has the form

$$\sin(\delta_c^{(\varepsilon)}) = \sin(\delta_0) + \varepsilon c \cos(\delta_0) + \mathcal{O}(\varepsilon^2)$$

as ε goes to zero. A similar result holds as

$$\sin^2(\delta_c^{(\varepsilon)}) = \sin^2(\delta_0) + 2\varepsilon c \cos^2(\delta_0) + \mathcal{O}(\varepsilon^2).$$

Let us introduce

$$s_1(\Theta) := \cot(\delta_0)(1 + c_1 e^{-i\theta_1} + c_2 e^{-i\theta_2}). \tag{5.1}$$

Then, up to order $\mathcal{O}(\varepsilon^2)$ accuracy, $\mathbb{M}_\varepsilon(\lambda)$ has an expansion of the form

$$\mathbb{M}_\varepsilon := \mathbb{M}_0 + \varepsilon \mathbb{M}_1 + O(\varepsilon^2),$$

in which the two 4×4 matrices have block structures as

$$\mathbb{M}_0 := \begin{pmatrix} s_0(0)\Phi_0(0) & -s_0(\Theta)\Phi_0(1) \\ -\overline{s_0(\Theta)}\Phi_0(1) & s_0(0)\Phi_0(0) \end{pmatrix},$$

$$\mathbb{M}_1 := \begin{pmatrix} s_1(0)\Phi_1(0) & -s_1(\Theta)\Phi_1(1) \\ -\overline{s_1(\Theta)}\Phi_1(1) & s_1(0)\Phi_1(0) \end{pmatrix}$$

with 2×2 blocks

$$\Phi_0(0) := \begin{pmatrix} \phi_1'(0) & \phi_2'(0) \\ \phi_1'''(0) & \phi_2'''(0) \end{pmatrix}, \quad \Phi_0(1) := \begin{pmatrix} \phi_1'(1) & \phi_2'(1) \\ \phi_1'''(1) & \phi_2'''(1) \end{pmatrix},$$

$$\Phi_1(0) := \begin{pmatrix} \phi_1'(0) & 2\phi_2'(0) \\ 0 & \phi_2'''(0) \end{pmatrix}, \quad \Phi_1(1) := \begin{pmatrix} \phi_1'(1) & 2\phi_2'(1) \\ 0 & \phi_2'''(1) \end{pmatrix}.$$

Using the fact that $\Phi_0(1)$ is non-singular, see Lemma 4.1, we introduce

$$\Lambda_0(0) := \Phi_0^{-1}(1)\Phi_0(0), \quad \Lambda_1(1) := \Phi_0^{-1}(1)\Phi_1(1).$$

Then up to $\mathcal{O}(\varepsilon^2)$ error, the perturbed matrix $\mathbb{M}_\varepsilon$ can be explicitly written as

$$\mathbb{M}_\varepsilon(\lambda) = \begin{pmatrix} 3\Lambda_0(0) & -s_0(\Theta) \\ -\overline{s_0(\Theta)} & 3\Lambda_0(0) \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & -s_1(\Theta)\Lambda_1(1) \\ -\overline{s_1(\Theta)}\Lambda_1(1) & 0 \end{pmatrix}. \tag{5.2}$$

As stated in Theorem 4.5, the equality $\mathbb{G}_0(\lambda) = \Lambda_0(0)$ is maintained with the components of $\mathbb{G}_0(\lambda)$ in terms of fundamental solutions

$$\mathbb{G}_0(\lambda) = \begin{pmatrix} g_1(1) & g_3(1) \\ g_1''(1) & g_3''(1) \end{pmatrix}.$$

Introducing $\widetilde{\mathcal{D}}(f,g) := f(1)g''(1) - g(1)f''(1)$, we represent the functions ϕ_1 and ϕ_2 in terms of fundamental solutions.

Lemma 5.1. *The functions ϕ_1 and ϕ_2 have representations*

$$\phi_1(x) = g_1(x) + \widetilde{\mathcal{D}}^{-1}(g_2, g_4)(\widetilde{\mathcal{D}}(g_4, g_1)g_2(x) + \widetilde{\mathcal{D}}(g_1, g_2)g_4(x)),$$
$$\phi_2(x) = g_3(x) + \widetilde{\mathcal{D}}^{-1}(g_2, g_4)(\widetilde{\mathcal{D}}(g_4, g_3)g_2(x) + \widetilde{\mathcal{D}}(g_3, g_2)g_4(x)),$$

where $\widetilde{\mathcal{D}}(g_k, g_n) := g_k(1)g_n''(1) - g_n(1)g_k''(1)$.

Using Lemma 5.1 along with the identity $\phi_2'''(1) = \phi_1'(1)$ yields the representation of $\Lambda_1(1)$ in (5.2) in terms of fundamental solutions. From now on, we call this representation $\mathbb{G}_1(\lambda)$ *matrix*. One way to calculate the determinant of $\mathbb{M}_\varepsilon(\lambda)$ is to apply results on the analysis of perturbed matrices, e.g., see [20] and references there. However, we calculate this quantity directly up to the $\mathcal{O}(\varepsilon^2)$ order, which under heavy simplification of the terms turns out to be

$$\det(\mathbb{M}_\varepsilon) = d_0 + d_1\varepsilon + \mathcal{O}(\varepsilon^2). \tag{5.3}$$

The quantity d_0 is equal to the determinant of the matrix $\mathbb{M}_0$ and

$$d_0 = \det(\mathbb{M}_0) = \frac{|s_0(\Theta)|^4}{81} - \frac{|s_0(\Theta)|^2}{9}\mathrm{tr}(\mathbb{G}_0^2) + \det(\mathbb{G}_0^2).$$

Moreover, the ε term is

$$d_1 = -4\frac{|s_0(\Theta)|^2}{9}\,\mathrm{Re}(s_0(\Theta)\overline{s_1(\Theta)})G(\lambda)$$

with the purely λ-dependent function

$$\begin{aligned} G(\lambda) = -\frac{1}{2}\{&(1-(\mathbb{G}_0^2)_{22})(\mathbb{G}_1)_{11} + (1-(\mathbb{G}_0^2)_{11})(\mathbb{G}_1)_{22} \\ &+ (\mathbb{G}_0^2)_{21}(\mathbb{G}_1)_{12} + (\mathbb{G}_0^2)_{12}(\mathbb{G}_1)_{21}\}. \end{aligned}$$

Therefore, up to ε^2 accuracy, the zeros of the perturbed determinant (5.3) are equivalent to the fact that $|s_0(\Theta)|^2/9$ is a root of the polynomial

$$\mathscr{P}(z) = z^4 - (\mathrm{tr}(\mathbb{G}_0^2) + 4\varepsilon\,\mathrm{Re}(s_0(\Theta)\overline{s_1(\Theta)})\xi(\lambda))z^2 + \det(\mathbb{G}_0^2). \tag{5.4}$$

Notice that a fourth-order polynomial of form $z^4 - az^2 + b$ can be factorized as

$$z^4 - az^2 + b = (z^2 + \tilde{a}z + \tilde{b})(z^2 - \tilde{a}z + \tilde{b}),$$

where $\tilde{a} = (a + 2b^{1/2})^{1/2}$ and $\tilde{b} = b^{1/2}$. This realization along with the form (5.4) implies that $\pm|s_0(\Theta)|/3$ are roots of $\mathscr{P}(z) = \mathscr{P}_1(z)\mathscr{P}_2(z)$, where

$$\begin{aligned} \mathscr{P}_{1,2}(z) = z^2 &\pm \big(\mathrm{tr}(\mathbb{G}_0^2) + 2\mathrm{det}^{1/2}(\mathbb{G}_0^2) + 4\varepsilon\,\mathrm{Re}(s_0(\Theta)\overline{s_1(\Theta)})G(\lambda)\big)^{1/2} z \\ &+ \big(\mathrm{tr}(\mathbb{G}_0^2) + 4\varepsilon\,\mathrm{Re}(s_0(\Theta)\overline{s_1(\Theta)})G(\lambda)\big)^{1/2} \qquad . \end{aligned}$$

Without loss of generality, assume that $|s_0(\Theta)|/3$ is a root of $\mathscr{P}_2$, that is,

$$\begin{aligned} \frac{2}{3}|s_0(\Theta)| &= \big(\mathrm{tr}(\mathbb{G}_0^2) + 2\mathrm{det}^{1/2}(\mathbb{G}_0^2) + 4\varepsilon\,\mathrm{Re}(s_0(\Theta)\overline{s_1(\Theta)})G(\lambda)\big)^{1/2} \\ &\pm \big(\mathrm{tr}(\mathbb{G}_0^2) - 2\mathrm{det}^{1/2}(\mathbb{G}_0^2) + 4\varepsilon\,\mathrm{Re}(s_0(\Theta)\overline{s_1(\Theta)})G(\lambda)\big)^{1/2}. \end{aligned}$$

Now, applying the fact that

$$\mathrm{tr}(\mathbb{G}_0^2) = \mathrm{tr}^2(\mathbb{G}_0) - 2\det(\mathbb{G}_0)$$

along with the equality $\mathrm{det}^{1/2}(\mathbb{G}_0^2) = \det(\mathbb{G}_0)$ implies that

$$\begin{aligned} \frac{|s_0(\Theta)|}{3} &= \Big(\frac{1}{4}\mathrm{tr}^2(\mathbb{G}_0) + \varepsilon\,\mathrm{Re}(s_0(\Theta)\overline{s_1(\Theta)})G(\lambda)\Big)^{1/2} \\ &\pm \Big(\frac{1}{4}\mathrm{tr}^2(\mathbb{G}_0) - \det(\mathbb{G}_0) + \varepsilon\,\mathrm{Re}(s_0(\Theta)\overline{s_1(\Theta)})G(\lambda)\Big)^{1/2}. \end{aligned}$$

Now, using the definitions of T_1 and T_2 from (4.11), we introduce the ε-extensions of these functions as

$$T_1^{(\varepsilon)} := (T_1^2(\lambda) + \varepsilon\,\mathrm{Re}(s_0(\Theta)\overline{s_1(\Theta)})G(\lambda))^{1/2}$$

and

$$T_2^{(\varepsilon)} := T_2(\lambda) + \varepsilon\,\mathrm{Re}(s_0(\Theta)\overline{s_1(\Theta)})G(\lambda).$$

Finding the roots of the quadratic polynomials $\mathscr{P}_{1,2}$ then reduces to the condition that the equations

$$\pm\frac{|s_0(\Theta)|}{3} = T_1^{(\varepsilon)} + (T_2^{(\varepsilon)})^{1/2} \quad \text{or} \quad \pm\frac{|s_0(\Theta)|}{3} = T_1^{(\varepsilon)} - (T_2^{(\varepsilon)})^{1/2} \tag{5.5}$$

are satisfied. Thus, we proved an ε-extended dispersion relation for perturbed Hamiltonian as stated below.

Theorem 5.2 (Perturbed dispersion). *The dispersion relation for the perturbed elastic lattice Hamiltonian satisfies*

$$\Big(\Delta_1^{(\varepsilon)}(\lambda,\Theta) \pm \frac{|s_0(\Theta)|}{3}\Big)\Big(\Delta_2^{(\varepsilon)}(\lambda,\Theta) \pm \frac{|s_0(\Theta)|}{3}\Big) + \mathcal{O}(\varepsilon^2) = 0, \tag{5.6}$$

as ε goes to zero, where $\Delta_{1,2}^{(\varepsilon)} := T_1^{(\varepsilon)} \pm (T_2^{(\varepsilon)})^{1/2}$.

We note here that for the case $\varepsilon = 0$, the above results are consistent with those stated for the hexagonal elastic lattice Hamiltonian. One of the interesting aspects of Theorem 5.2 is to answer whether singular Dirac points will be preserved under ε-perturbed geometry. To answer this, we first characterize the behavior of the Θ-dependent function $\mathrm{Re}(s_0(\Theta)\overline{s_1(\Theta)})$ in the perturbed part.

Lemma 5.3. *The function* $\mathrm{Re}(s_0(\Theta)\overline{s_1(\Theta)})$ *is* $2\pi\mathbb{Z}^2$ *periodic, its magnitude is bounded by* $2(1+|c_1|)$ *and the zeros are at* $(0,0)$ *and* $\pm(2\pi/3, -2\pi/3)$.

Proof of Lemma 5.3. Recalling the definitions of $s_0(\Theta)$ and $s_1(\Theta)$ from (4.1) and (5.1), respectively, we get

$$s_0(\theta)\overline{s_1(\Theta)} = -\cot(\delta_0)(1 + e^{-i\theta_1} + e^{-i\theta_2})(1 + c_1 e^{i\theta_1} + c_2 e^{i\theta_2}).$$

Next, by representing the exponential terms using Euler's formula, we get

$$\begin{aligned}\mathrm{Re}(s_0(\Theta)\overline{s_1(\Theta)}) = -\cot(\delta_0)\big((1+c_1)\cos(\theta_1) + (1+c_2)\cos(\theta_2) \\ + (c_1+c_2)\cos(\theta_2-\theta_1)\big),\end{aligned}$$

which, after further simplification and application of the identity $1 + c_1 + c_2 = 0$, reduces to

$$\mathrm{Re}(s_0(\Theta)\overline{s_1(\Theta)}) = -\cot(\delta_0)\big(\cos(\theta_2-\theta_1) + c_1\cos(\theta_2) + c_2\cos(\theta_1)\big).$$

Applying the identity $\cos(\delta_0) = \cos(2\delta_0)$, we find that $(0,0)$ and $\pm(2\pi/3, -2\pi/3)$ are zeros of these functions. Finally, setting $c_2 = -1 - c_1$ above, we get

$$\begin{aligned}|\operatorname{Re}(s_0(\Theta)\overline{s_1(\Theta)})| &\le |\cos(\theta_2 - \theta_1) - \cos(\theta_1) + c_1(\cos(\theta_2) - \cos(\theta_1))| \\ &\le 2(1 + |c_1|)\end{aligned}$$

as desired. ∎

Figure 8 (right) shows the level curves of the function $\operatorname{Re}(s_0(\Theta)\overline{s_1(\Theta)})$ for a fixed value of the parameter c_1.

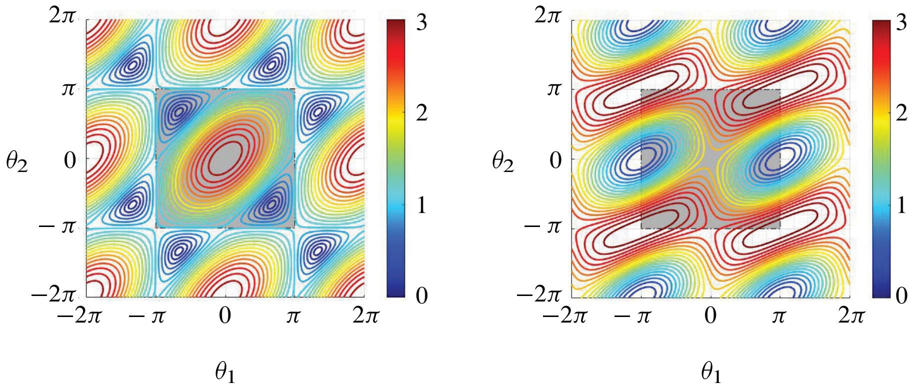

Figure 8. Level curves of $|s_0(\Theta)|$ and $\operatorname{Re}(s_0(\Theta)\overline{s_1(\Theta)})$. The highlighted rectangle shows the first Brillouin zone.

The observations we made in Lemma 5.3 allow us to show the existence of Dirac points.

Corollary 5.4 (Dirac points). *The Dirac points persist under the angle-perturbed hexagonal elastic lattice Hamiltonian.*

Proof of Corollary 5.4. Let (Θ^*, λ^*) with $\Theta^* := \pm(2\pi/3, -2\pi/3)$ be a Dirac point for the hexagonal elastic lattice Hamiltonian. Then, Lemma 5.3 implies that the function $\operatorname{Re}(s_0(\Theta)\overline{s_1(\Theta)})$ vanishes at the quasimomentum Θ^*. Therefore, there is no spectral gap at the energy λ^* for the perturbed Hamiltonian too. Regarding the singularity at this point, let us define the ε-dependent function

$$D_\varepsilon(\lambda, \Theta) := \pm\frac{|s_0(\Theta)|}{3} - T_1^{(\varepsilon)} - (T_2^{(\varepsilon)})^{1/2}$$

and similarly for $\Delta_2^{(\varepsilon)}$. By the continuity of $D_\varepsilon(\lambda, \Theta)$ with respect to Θ, there exists an ε-dependent neighborhood $\mathcal{N}_{\lambda,\Theta}^{(\varepsilon)} := \mathcal{N}_{\lambda^*}(\lambda) \times \mathcal{N}_{\Theta^*}^{(\varepsilon)}(\Theta)$ containing (λ^*, Θ^*) so that

$D_\varepsilon(\lambda, \Theta)$ is well defined for all $(\lambda, \Theta) \in \mathcal{N}^{(\varepsilon)}_{\lambda,\Theta}$. For $\lambda \in \mathcal{N}^{(\varepsilon)}_{\lambda,\Theta} \setminus \{\lambda^*\}$ and the case $T_2(\lambda) > 0$, the inverse function theorem implies that the solution set for $D_\varepsilon(\lambda, \Theta) = 0$ is a simple closed loop (distorted ellipse) in the quasimomentum neighborhood $\mathcal{N}^{(\varepsilon)}_{\Theta^*}(\Theta)$. Moreover, observe that the singularity of the function $D_\varepsilon(\lambda, \Theta)$ only occurs at Θ^* due to $|s_0(\Theta)|$. For the case $T_2(\lambda) = 0$, the function $D_\varepsilon(\lambda, \Theta)$ is only well defined for $\mathcal{N}^{(\varepsilon)}_{\Theta^*}(\Theta) \cap \{\Theta : \mathrm{Re}(s_0(\Theta)\overline{s_1(\Theta)})G(\lambda) \geq 0\}$. A similar discussion implies that the solution set for $D_\varepsilon(\lambda, \Theta) = 0$ is a simply-connected curve (not closed) in quasimomentum $\mathcal{N}^{(\varepsilon)}_{\Theta^*}(\Theta)$. In this case, the dispersion relation is lost locally for Θ such that $\mathrm{Re}(s_0(\Theta)\overline{s_1(\Theta)})G(\lambda) < 0$. In both cases, the gap remains closed at Dirac point, however only one-sided differentiability exists for the latter case. ∎

Remark 5.5. Here we note that for the case $T_2(\lambda) = 0$ explained in the proof of Corollary 5.4, the issue only concerns $\lambda \neq \lambda^*$ since at Θ^* the ε-term vanishes. Moreover, Corollary 5.4 guarantees that Dirac points appear at the quasimomenta $\Theta^* = \pm(2\pi/3, -2\pi/3)$, but possibly with a shift in the energy.

Investigation of the presence of a pure point spectrum has been an active research area. Changing the geometry of medium (e.g., working with 2D periodic graph instead of a real line), imposing perturbation through potential, and applying a different Hamiltonian model are among few ways to guarantee the presence of a pure point spectrum; e.g., see [18,22,25,29]. As stated in Theorem 4.11 the pure point spectrum for the hexagonal elastic lattice Hamiltonian is non-empty. This has been proved by explicit construction of even (or odd) eigenfunctions supported on a single hexagon. However, the existence of a pure point spectrum fails for the perturbed Hamiltonian.

Theorem 5.6 (Spectral description). *The spectrum of the perturbed Hamiltonian is purely absolutely continuous.*

Proof of Theorem 5.6. The singular continuous spectrum is empty, since the Hamiltonian is a self-adjoint elliptic operator, as in the unperturbed case (see the proof of Theorem 4.11). Next, let us show the absence of the pure point spectrum, unlike the hexagonal lattice case. Using the dispersion relation, we obtain $\sigma_{\mathrm{pp}}(\mathcal{H}) \subset \Sigma^D$ as we did in the unperturbed case. Now, let us assume $\sigma_{\mathrm{pp}}(\mathcal{H}) \neq \emptyset$. Then, the corresponding eigenfunction u restricted to any edge must be identically zero or solve $d^4u(x)/dx^4 + q(x)u(x) = \lambda u(x)$ with the boundary conditions $u(0) = u(1) = u''(0) = u''(1) = 0$ on that edge. Therefore, the restriction of an eigenfunction to any edge on its support must be an eigenfunction of the operator $d^4/dx^4 + q(x)$ for the same eigenvalue λ on the interval $[0, 1]$ with the boundary conditions $u(0) = u(1) = u''(0) = u''(1) = 0$. Note that u must also satisfy the vertex conditions.

If u is compactly supported, then the vertex conditions at the vertices of the boundary of the support of u imply $\varepsilon = c_1\varepsilon = c_2\varepsilon$. Recall that $1 + c_1 + c_2 = 0$, so $\varepsilon = 0$ is the only solution, which is the unperturbed case. For a non-compactly supported

$u \in H^4(\Gamma)$, the same argument shows that the vertex conditions cannot be satisfied at any vertex. Therefore, the pure point spectrum is also empty, and we obtain the desired conclusion that the spectrum is purely absolutely continuous. ■

Remark 5.7. Applying the result in Lemma 5.3 and the proof of Corollary 5.4, one can make arguments to quantify the shift of the dispersion relation (5.6) for the perturbed Hamiltonian compared to the hexagonal lattice case at any λ. More precisely, for $T_2 > 0$, the expansions of $T_1^{(\varepsilon)}$ and $T_2^{(\varepsilon)}$ in (5.5) imply that

$$\pm\frac{|s_0(\Theta)|}{3} = \Delta_1(\lambda)\{1 + \varepsilon\,\mathrm{Re}(s_0(\Theta)\overline{s_1(\Theta)})G(\lambda)T_1^{-1}(\lambda)T_2^{-1/2}(\lambda)\} + \mathcal{O}(\varepsilon^2) \quad (5.7)$$

and similarly for Δ_2 with sign changes. Now, for a fixed value of λ, the shift with respect to the hexagonal lattice, i.e., the case $\varepsilon = 0$, in quasimomentum can be found by solving (5.7).

Finally, in the next section we give a partial list of topics that may be interesting for future extensions of the present work.

6. Outlook

The viability of the frame model as a structure composed of one-dimensional segments needs to be verified mathematically, as a limit of a three-dimensional structure when beam widths go to zero. There is a significant mathematical literature on this question for second-order operators (see, for example, [15, 33, 40]), with a variety of operators arising in the limit. This variety will increase in the case of fourth-order equations, and may be expected to incorporate masses concentrating at joints and other cases of applied interest. Moreover, the validity of Euler–Bernoulli beam theory, especially at the high-energy level, is questionable. In contrast, the richer Timoshenko model no longer assumes that the cross-sections remain orthogonal to the deformed axis and therefore incorporates more degrees of freedom [13, 30, 32]. It would be of applied interest to extend the current results to the latter model.

In this work, we focus on Euler–Bernoulli beam theory and its restriction to the scalar-valued lateral displacement $u(x)$. In the work [7], it is shown that for planar graphs, a more accurate presentation of the operator is to include the angular displacement field $\eta(x)$ as well. This shifts our problem to a vector-valued operator and more complicated vertex conditions. We refer to the recent work [4] for an analysis in this direction and potential future developments for three-dimensional periodic graphs.

An interesting problem is to employ two-scale analysis to understand the homogenized behavior and spectra of Hamiltonians on periodic lattices with more complex fundamental domains; see, e.g., [11, 17, 24, 27, 41] and references therein. Of similar interest is the generalization of our result to multi-layer quantum graph models equipped with a beam Hamiltonian. In [12], it is shown that for the multi-layer Schrödinger operator, the dispersion relation between wave vector and energy is a polynomial in the dispersion relation of the single layer. This leads to the reducibility of the algebraic Fermi surface, at any energy, into several components. For the beam Hamiltonian, it has been shown that, in the special case of planar frames, the operator decomposes into a direct sum of two operators: one coupling out-of-plane displacement to angular displacement, and the other coupling in-plane displacement to axial displacement [7]. Understanding the interaction between these decoupled systems on multilayer graphs may be of interest from both theoretical and applied perspectives.

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