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On multiplicity bounds for eigenvalues of the clamped round plate

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Abstract. We ask whether the only multiplicities in the spectrum of the clamped round plate are trivial, i.e., whether all existing multiplicities are due to the isometries of the sphere, or, equivalently, whether any eigenfunction is separated. We prove that any eigenfunction can be expressed as a sum of at most two separated ones, by showing that otherwise the corresponding eigenvalue is algebraic, contradicting the Siegel–Shidlovskii theory. In two dimensions, it follows that no eigenvalue is of multiplicity greater than four. The proof exploits a linear recursion of order two for cross-product Bessel functions with coefficients which are not even algebraic functions, though they do satisfy a non-linear algebraic recursion.

1. Introduction and background

1.1. The vibrating clamped round plate

In this paper we are concerned with the vibrating clamped round plate ([2, Chapter V, Section 6] and [9, Chapter X]), that is, the following fourth order eigenvalue problem in the unit ball $B \subset \mathbb{R}^d$:

$$\begin{cases} \Delta^2 u = \lambda u & \text{in } B, \\ u = 0 & \text{on } \partial B, \\ \partial_n u = 0 & \text{on } \partial B, \end{cases} \tag{VP}$$

where $\Delta = \operatorname{div} \circ \operatorname{grad}$ is the Laplacian. A standard separation of variables argument [9, Section 218] shows that an orthogonal basis of eigenfunctions is given in spherical coordinates by the family

$$u_{l,k,j}(r,\phi) = (I_l^{\text{sp}}(w_{l,k}^{\text{sp}}) J_l^{\text{sp}}(w_{l,k}^{\text{sp}} r) - J_l^{\text{sp}}(w_{l,k}^{\text{sp}}) I_l^{\text{sp}}(w_{l,k}^{\text{sp}} r)) Y_{l,j}(\phi), \tag{1.1}$$

where

$$J_l^{\text{sp}}(\rho) = \rho^{1-d/2} J_{l+d/2-1}(\rho), \quad I_l^{\text{sp}}(\rho) = \rho^{1-d/2} I_{l+d/2-1}(\rho)$$

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denote the spherical Bessel function of the first kind and the modified spherical Bessel function of the first kind respectively, and where $(Y_{l,j})_{j=1}^{N_l}$ with $N_l = \binom{l+d-1}{d-1} - \binom{l+d-3}{d-1}$ form an orthonormal basis for spherical harmonics of degree l on the sphere $\mathbb{S}^{d-1}$. The value $w_{l,k}^{\mathrm{sp}}$ is the k-th positive root of the Wronskian

$$W_l^{\mathrm{sp}} := (I_l^{\mathrm{sp}})' J_l^{\mathrm{sp}} - I_l^{\mathrm{sp}} (J_l^{\mathrm{sp}})' = I_{l+1}^{\mathrm{sp}} J_l^{\mathrm{sp}} + I_l^{\mathrm{sp}} J_{l+1}^{\mathrm{sp}} = I_{l-1}^{\mathrm{sp}} J_l^{\mathrm{sp}} - I_l^{\mathrm{sp}} J_{l-1}^{\mathrm{sp}}.$$

The eigenvalue corresponding to an eigenfunction of the form (1.1) is $\lambda = (w_{l,k}^{\mathrm{sp}})^4$. The multiplicities in the spectrum which we call "trivial" are those due to the multiplicities appearing in the spectrum of the sphere $\mathbb{S}^{d-1}$. We are interested to know whether non-trivial multiplicities occur in the spectrum, i.e., whether an eigenvalue $\lambda = w_{l,k}^{\mathrm{sp}}$ can have multiplicity bigger than N_l, or, equivalently, whether any eigenfunction u is of the separated form

$$u(r, \phi) = R(r) Y(\phi), \tag{1.2}$$

where Y is a spherical harmonic. The results in this paper continue the study initiated in [7]. In the next section we describe our main result.

The analogous problem for the vibrating round *membrane* with Dirichlet boundary conditions was solved by Siegel in 1929 [12], proving that any eigenfunction is separated. To that end, he proved the deep fact that all non-zero roots of Bessel functions J_m where $m \in \mathbb{N}$ are transcendental; the case of odd dimension d, corresponding to $m \in \frac{1}{2} + \mathbb{N}$ follows from the Hermite–Lindemann–Weierstrass theorem [4, 6, 14] as shown by Porter in [8] (see also [13, Section 15.28], [11], and [10, p. 217]). On the other hand, any eigenvalue of non-trivial multiplicity must be algebraic due to the algebraic recursion formula satisfied by Bessel functions. Ruling out non-trivial multiplicities in the case of the free vibrating round membrane problem was achieved in [1, 3].

1.2. Common roots of cross-product of Bessel functions

One can write

$$W_l^{\mathrm{sp}}(\rho) = \rho^{2-d} W_{l+d/2-1}(\rho)$$

where

$$W_m := I_m' J_m - I_m J_m' = I_{m+1} J_m + I_m J_{m+1} = I_{m-1} J_m - I_m J_{m-1} \tag{1.3}$$

In particular, $w_{l,k}^{\mathrm{sp}} = w_{l+d/2-1,k}$, where $w_{m,k}$ denotes the k-th positive root of W_m. Hence, the problem treated in this paper amounts to the question whether there exist $x_0 > 0$ and $m_1, m_2 \in \frac{1}{2}\mathbb{N}_0$ differing by a non-zero integer such that $W_{m_1}(x_0) = W_{m_2}(x_0) = 0$, as it would imply that $\lambda = x_0^4$ is an eigenvalue of non-trivial multiplicity (at least $N_{l_1} + N_{l_2}$ where $m_j = l_j + d/2 - 1$). This seems to be a difficult open problem. In [7] it was shown that there do not exist $x_0 > 0$ and $m_1, m_2, m_3, m_4 \in \mathbb{N}_0$ pairwise distinct for which $W_{m_1}(x_0) = W_{m_2}(x_0) = W_{m_3}(x_0) = W_{m_4}(x_0) = 0$. In two dimensions, this implied a uniform bound, namely, six, on the multiplicity. The proof was based on a *fourth* order recursion formula for the sequence W_m with rational functions as coefficients. One of the main goals of this paper is to eliminate the possibility that three functions W_m vanish simultaneously. More precisely, we prove the following result.

Theorem 1.1. *There do not exist* $x_0 > 0$ *and* $m_1, m_2, m_3 \in \mathbb{Q}_{\geq 0}$ *with* $|m_i - m_j| \in \mathbb{N}$ *for all* $1 \leq i < j \leq 3$ *such that* $W_{m_1}(x_0) = W_{m_2}(x_0) = W_{m_3}(x_0) = 0$. *Equivalently, any eigenfunction of the clamped round plate is a sum of at most two ones of the separated form* (1.2).

One corollary is the following.

Corollary 1.2. *Any eigenvalue of the two-dimensional clamped round plate is of multiplicity at most four. In particular, the multiplicities are uniformly bounded.*

1.3. Schanuel's conjecture and the trivial multiplicity conjecture

A natural conjecture, due to Rayleigh [9], is the following one.

Conjecture 1.3 (Trivial multiplicity). *There do not exist* $x_0 > 0$ *and* $m_1, m_2 \in \frac{1}{2}\mathbb{N}_0$ *with* $|m_1 - m_2| \in \mathbb{N}$ *such that* $W_{m_1}(x_0) = W_{m_2}(x_0) = 0$. *Equivalently, there are no non-trivial multiplicities in the spectrum of the clamped round plate, or, alternatively, any eigenfunction of the clamped round plate is separated.*

In a similar vein to the work of Porter [8], we explain in Section 6 that in odd dimensions a special case of the classical Schanuel conjecture of transcendental number theory implies Conjecture 1.3. For even dimensions, a Schanuel-type conjecture for Bessel functions which would imply Conjecture 1.3 is formulated.

1.4. Idea of proof of main Theorem 1.1

To prove Theorem 1.1, we show that if there exist $x_0 > 0$ and $m_1, m_2, m_3 \in \mathbb{Q}_{\geq 0}$ pairwise differing by a non-zero integer, with $W_{m_1}(x_0) = W_{m_2}(x_0) = W_{m_3}(x_0) = 0$, then x_0 must be algebraic. However, an immediate application of the Siegel–Shidlovskii theory shows that any positive root of the equation $W_m(x_0) = 0$ is transcendental. The main new idea in our proof with respect to [7] is that it is possible to exploit a *second* order linear recursion formula for the sequence W_m which has *non-algebraic* coefficients. We make use of the fact that these coefficients satisfy an *algebraic non-linear* recursion formula of degree two. At a first step, we show that each joint root x_0 of

W_m and $W_{m'}$ leads to an equation of the form

$$P_{m,m'}(x_0, F_m(x_0)) = 0, \tag{1.4}$$

where $P_{m,m'}(x, y)$ is a polynomial of degree two with respect to y, and F_m is a quotient of successive modified Bessel functions (see Def. 2.4). At a second step, we prove that it is possible to eliminate F_m from a system of *any* two such equations, leading to a non-trivial polynomial equation for x_0.

2. Bessel functions and their quotients

Let $m \in \mathbb{Q}$. The Bessel function J_m is defined by the series

$$J_m(x) = \left(\frac{x}{2}\right)^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(m+k+1)} \left(\frac{x}{2}\right)^{2k}.$$

For the purpose of this paper, we consider the above series as a holomorphic function in the domain $\mathbb{C} \setminus (-\infty, 0]$ which is real on the positive real axis. Similarly, the modified Bessel function I_m is given by

$$I_m(x) = \left(\frac{x}{2}\right)^m \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(m+k+1)} \left(\frac{x}{2}\right)^{2k}.$$

Observe that $I_m(\mathrm{i}x) = \mathrm{i}^m J_m(x)$. If m is a negative integer, we ignore the terms for which $m + k + 1$ is a pole of the Γ-function. In view of this, $I_{-m} = I_m$ when $m \in \mathbb{Z}$.

Lemma 2.1. *If $m \in \mathbb{Z}$ or $m \in \mathbb{Q}_{\geq 0}$ and $x > 0$, then $I_m(x) > 0$.*

Proof. For $m \geq 0$, all the terms of the power series are positive for $x > 0$. If $m < 0$ is an integer, we have $I_{-m} = I_m$. ∎

We record the following formulae for a few special cases which follow directly from the definitions.

Lemma 2.2. *We have*

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x, \qquad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x,$$

$$I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cosh(x), \quad I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh(x).$$

Proposition 2.3 ([13, Sections 12 and 3.71]). *Let $m \in \mathbb{Q}$. The following recursions are satisfied:*

$$J_{m+1}(x) = \frac{2m}{x} J_m(x) - J_{m-1}(x),$$
$$I_{m+1}(x) = -\frac{2m}{x} I_m(x) + I_{m-1}(x).$$

We will consider quotients of successive modified Bessel functions.

Definition 2.4. For $m \in \mathbb{Q}$, let F_m be the following meromorphic function on $\mathbb{C} \setminus (-\infty, 0]$:

$$F_m(x) := \frac{I_m(x)}{x I_{m-1}(x)}.$$

The following identity, which can be viewed as a discrete Riccati equation will be important in the sequel.

Key identity 2.5. *For $m \in \mathbb{Q}$, we have*

$$x^2 F_{m+1}(x) F_m(x) = 1 - 2m F_m(x).$$

Proof. From the definition of F_m and Proposition 2.3, we have

$$x^2 F_{m+1} F_m = x^2 \cdot \frac{I_{m+1}}{x I_m} \cdot \frac{I_m}{x I_{m-1}} = \frac{1}{I_{m-1}} \left(I_{m-1} - \frac{2m}{x} I_m \right) = 1 - 2m F_m. \qquad \blacksquare$$

3. Second order recursion for cross products of Bessel functions

Fix $m \in \mathbb{Q}$. The sequence $(W_{m+n})_{n=0}^{\infty}$ satisfies a fourth order linear recurrence with non-constant coefficients in $\mathbb{Q}(x)$ (see [7]). However, it also satisfies a second order linear recurrence whose coefficients, while not even algebraic, satisfy themselves a quadratic recursion. We prove the following result.

Theorem 3.1. *Let $m \in \mathbb{Q}$. The following recursions formulae hold:*

$$W_{m+1} = 2m F_m W_m - (1 - 2m F_m) W_{m-1}$$

and

$$W_{m-1} = 2m G_m W_m - (1 + 2m G_m) W_{m+1},$$

where $G_m = 1/(x^2 F_{m+1}) = I_m/(x I_{m+1})$.

Proof. On the one hand, we have by (1.3) and Proposition 2.3,

$$\begin{aligned} W_{m-1} + W_{m+1} &= (I_m J_{m-1} + I_{m-1} J_m) + (I_m J_{m+1} - I_{m+1} J_m) \\ &= I_m \frac{2m}{x} J_m + \frac{2m}{x} I_m J_m = \frac{4m}{x} I_m J_m. \end{aligned}$$

On the other hand,

$$\begin{aligned} 2m F_m (W_m + W_{m-1}) &= 2m \frac{I_m}{x I_{m-1}} (I_{m-1} J_m - I_m J_{m-1} + I_m J_{m-1} + I_{m-1} J_m) \\ &= \frac{4m}{x} I_m J_m. \end{aligned}$$

Comparing the preceding expressions gives the forward recursion formula. The backward recursion formula follows immediately from the forward one once we take into account Key identity 2.5. ∎

4. Rolling out the recursion

In this section we use the recursion formula for W_m (Theorem 3.1) in order to express any element in the sequence in terms of two initial consecutive terms.

Proposition 4.1. *Let $m \in \mathbb{Q}$, and $n \in \mathbb{N}_0$. There exist polynomials $A_{m,n}$, $B_{m,n}$, $\tilde{B}_{m,n}$ and $C_{m,n} \in \mathbb{Q}[x]$ such that*

$$\begin{aligned} x^{2n} W_{m+n+1} = {} & (A_{m,n} F_m + x^2 B_{m,n} + C_{m,n} F_m^{-1}) W_m \\ & + (A_{m,n} F_m + \tilde{B}_{m,n} - C_{m,n} F_m^{-1}) W_{m-1}. \end{aligned}$$

Remark. Note that the coefficients in the preceding formula are of degree one in F_m and F_m^{-1}.

Proof. We prove the claim by induction on n. The case $n = 0$ follows from Theorem 3.1. For $n \geq 1$,

$$\begin{aligned} x^{2n} W_{m+n+1} &= x^2 x^{2n-2} W_{(m+1)+(n-1)+1} \\ &= (x^2 A_{m+1,n-1} F_{m+1} + x^4 B_{m+1,n-1} + x^2 C_{m+1,n-1} F_{m+1}^{-1}) W_{m+1} \\ &\quad + (x^2 A_{m+1,n-1} F_{m+1} + x^2 \tilde{B}_{m+1,n-1} - x^2 C_{m+1,n-1} F_{m+1}^{-1}) W_m. \end{aligned}$$

We substitute W_{m+1} using Theorem 3.1 and Key identity 2.5:

$$\begin{aligned} & x^{2n} W_{m+n+1} \\ &= (2mx^2 A_{m+1,n-1} F_m F_{m+1} + 2mx^4 B_{m+1,n-1} F_m + 2mx^2 C_{m+1,n-1} F_m F_{m+1}^{-1}) W_m \\ &\quad - (x^2 A_{m+1,n-1} F_{m+1} + x^4 B_{m+1,n-1} + x^2 C_{m+1,n-1} F_{m+1}^{-1}) x^2 F_m F_{m+1} W_{m-1} \\ &\quad + (x^2 A_{m+1,n-1} F_{m+1} + x^2 \tilde{B}_{m+1,n-1} - x^2 C_{m+1,n-1} F_{m+1}^{-1}) W_m. \end{aligned}$$

Applying Key identity 2.5 and collecting terms gives

$$\begin{aligned}
&x^{2n} W_{m+n+1} \\
&= \big(2m A_{m+1,n-1}(1-2m F_m)+2m x^4 B_{m+1,n-1} F_m - x^2 C_{m+1,n-1}(1-2m F_m) F_{m+1}^{-1} \\
&\quad + A_{m+1,n-1}(F_m^{-1}-2m)+x^2 \tilde{B}_{m+1,n-1}\big) W_m \\
&\quad - \big(x^2 A_{m+1,n-1} F_{m+1}(1-2m F_m)+x^4 B_{m+1,n-1}(1-2m F_m) \\
&\qquad + x^4 C_{m+1,n-1} F_m\big) W_{m-1}.
\end{aligned}$$

Applying once more Key identity 2.5 and collecting terms gives

$$\begin{aligned}
x^{2n} W_{m+n+1} = {} & \big((-4m^2 A_{m+1,n-1} + 2m x^4 B_{m+1,n-1} - x^4 C_{m+1,n-1}) F_m \\
& + x^2 \tilde{B}_{m+1,n-1} + A_{m+1,n-1} F_m^{-1}\big) W_m \\
& + \big(2m A_{m+1,n-1}(1-2m F_m) - A_{m+1,n-1}(F_m^{-1}-2m) \\
& \quad + (2m x^4 B_{m+1,n-1} - x^4 C_{m+1,n-1}) F_m - x^4 B_{m+1,n-1}\big) W_{m-1}
\end{aligned}$$

and finally,

$$\begin{aligned}
x^{2n} W_{m+n+1} = {} & \big((-4m^2 A_{m+1,n-1} + 2m x^4 B_{m+1,n-1} - x^4 C_{m+1,n-1}) F_m \\
& + x^2 \tilde{B}_{m+1,n-1} + A_{m+1,n-1} F_m^{-1}\big) W_m \\
& + \big((-4m^2 A_{m+1,n-1} + 2m x^4 B_{m+1,n-1} - x^4 C_{m+1,n-1}) F_m \\
& \quad + 4m A_{m+1,n-1} - x^4 B_{m+1,n-1} - A_{m+1,n-1} F_m^{-1}\big) W_{m-1},
\end{aligned}$$

which is of the desired form. ■

As an immediate consequence of the above computation, we obtain the following lemma.

Lemma 4.2. *Let $A_{m,n}$, $B_{m,n}$, $\tilde{B}_{m,n}$, and $C_{m,n}$ be as in Proposition 4.1. Then, the following recurrence relations hold:*

(i) $A_{m,0} = 2m$; $A_{m,n} = -4m^2 A_{m+1,n-1} + 2m x^4 B_{m+1,n-1} - x^4 C_{m+1,n-1}$;

(ii) $B_{m,0} = 0$; $B_{m,n} = \tilde{B}_{m+1,n-1}$;

(iii) $\tilde{B}_{m,0} = -1$; $\tilde{B}_{m,n} = 4m A_{m+1,n-1} - x^4 B_{m+1,n-1}$;

(iv) $C_{m,0} = 0$; $C_{m,n} = A_{m+1,n-1}$.

As a corollary, we have the following result.

Lemma 4.3. *Let $m \in \mathbb{Q}$ and $n \in \mathbb{N}$. Then,*

$$A_{m,n} \equiv 2(-4)^n (m+n) \prod_{k=0}^{n-1} (m+k)^2 \mod x^4.$$

In particular, if $m \notin \mathbb{Z}$ or $m > 0$ or $m < -n$, then $A_{m,n} \not\equiv 0 \mod x$.

Proof. The proof follows immediately from Lemma 4.2 (i). ∎

The next proposition rolls the recursion backward and shows the connection to the forward recursion.

Proposition 4.4. *Let $m \in \mathbb{Q}$, and $n \in \mathbb{N}_0$. With the same notations as in Proposition 4.1,*

$$(-1)^{n+1}x^{2n}W_{m-n-1} = (A_{-m,n}G_m + x^2 B_{-m,n} + C_{-m,n}G_m^{-1})W_m - (A_{-m,n}G_m + \tilde{B}_{-m,n} - C_{-m,n}G_m^{-1})W_{m+1}.$$

Sketch of Proof. In case $m \in \mathbb{Z}$, the proposition follows immediately from Proposition 4.1, since $W_{-m} = (-1)^m W_m$ and $F_{-m} = G_m$. For any $m \in \mathbb{Q}$, one follows the same pattern of proof of Proposition 4.1. The Key identity 2.5 is replaced by the identity

$$x^2 G_m G_{m-1} = 1 + 2mG_m.$$

We need also to use the recursion relations for the coefficients, given in Lemma 4.2. ∎

5. Proof of Theorem 1.1

We recall the following fact.

Proposition 5.1 ([7]). *Let $m \in \mathbb{Q}_{\geq 0}$ or $m \in \mathbb{Z}$. The functions W_m and W_{m+1} have no joint positive roots.*

Proof. Assume $W_m(x_0) = W_{m+1}(x_0) = 0$ for some $x_0 > 0$. Observe that

$$W_{m+1} + W_m = I_m J_{m+1} - I_{m+1}J_m + I_{m+1}J_m + I_m J_{m+1} = 2I_m J_{m+1}$$

and

$$W_{m+1} - W_m = I_m J_{m+1} - I_{m+1}J_m - I_{m+1}J_m - I_m J_{m+1} = -2I_{m+1}J_m.$$

If $m \in \mathbb{Z}$ or $m \geq 0$, then $I_m(x_0) > 0$ (see Lemma 2.1). In view of this observation and the above formulae, the assumption implies that $J_m(x_0) = J_{m+1}(x_0) = 0$. However, this is impossible since it would imply that $J'_m(x_0) = (m/x_0)J_m(x_0) - J_{m+1}(x_0)$ is also zero, while J_m satisfies a second order linear ODE. ∎

A direct consequence of the preceding proposition and Propositions 4.1 and 4.4 is the following.

Corollary 5.2. *Let $n \in \mathbb{N}_0$ and $x_0 > 0$.*

(a) *If x_0 is a joint root of W_m, and W_{m+n+2} where $m \in \mathbb{Q}_{\geq 0}$, then*

$$A_{m+1,n}(x_0)F_{m+1}(x_0)^2 + x_0^2 B_{m+1,n}(x_0)F_{m+1}(x_0) + C_{m+1,n}(x_0) = 0.$$

(b) *If x_0 is a joint root of W_m, and W_{m-n-2} where $m-1 \in \mathbb{Q}_{\geq 0}$, then*

$$\begin{aligned} &A_{-m+1,n}(x_0)(x_0^2 F_{m+1}(x_0) + 2m)^2 \\ &\quad + x_0^4 B_{-m+1,n}(x_0)(x_0^2 F_{m+1}(x_0) + 2m) + x_0^4 C_{-m+1,n}(x_0) = 0. \end{aligned}$$

Proof. Part (a) follows from Proposition 4.1 with m replaced by $m+1$, taking into account Proposition 5.1. To prove part (b) observe first that from Proposition 4.4 with m replaced by $m-1$ and Proposition 5.1 we get

$$A_{-m+1,n}(x_0)G_{m-1}(x_0) + x_0^2 B_{-m+1,n}(x_0) + C_{-m+1,n}(x_0)G_{m-1}^{-1}(x_0) = 0.$$

Multiply this equation by $x_0^4 G_{m-1}$, while noting that $x^2 G_{m-1} = x^2 F_{m+1} + 2m$. ∎

Proof of Theorem 1.1. Assume $x_0 > 0$ and $0 \leq m_1 < m_2 < m_3$ are such that one has $W_{m_1}(x_0) = W_{m_2}(x_0) = W_{m_3}(x_0) = 0$. By Proposition 5.1, we can write $m_1 = m_2 - l - 2$, $m_2 = m$ and $m_3 = m_2 + n + 2$ with $l, m, n \in \mathbb{N}_0$. By Corollary 5.2, setting $x = x_0$ solves a system

$$\begin{cases} A_{m+1,n}(x)F_{m+1}(x)^2 + x^2 B_{m+1,n}(x)F_{m+1}(x) + C_{m+1,n}(x) = 0, \\ A_{-m+1,l}(x)(x^2 F_{m+1}(x) + 2m)^2 \\ \quad + x^4 B_{-m+1,l}(x)(x^2 F_{m+1}(x) + 2m) + x^4 C_{-m+1,l}(x) = 0. \end{cases} \tag{5.1}$$

Eliminating F_{m+1}^2 from the preceding system, we obtain that x_0 is a root of an equation of the form

$$\begin{aligned} &\big(4m A_{m+1,n}(x)A_{-m+1,l}(x) + x^4 P_1(x)\big)x^2 F_{m+1}(x) \\ &\quad + 4m^2 A_{m+1,n}(x)A_{-m+1,l}(x) + x^4 P_2(x) = 0 \end{aligned}$$

for some polynomials $P_1, P_2 \in \mathbb{Q}[x]$, depending on l, m, n.

By Lemma 4.3 and the fact that $m > l + 1$, the polynomial $4m A_{m+1,n} A_{-m+1,l} + x^4 P_1$ is not zero. Hence, if it vanishes at the point x_0 we get that x_0 is algebraic. Otherwise, using the preceding equation to eliminate F_{m+1} from the first equation in (5.1) leads to an equation of the form

$$16m^4 A_{m+1,n}(x_0)^3 A_{-m+1,l}(x_0)^2 + x_0^4 P_3(x_0) = 0$$

with $P_3 \in \mathbb{Q}[x]$ depending on l, m, n. From Lemma 4.3, it follows that x_0 is also algebraic in this case. We have shown that x_0 is algebraic. However, this contradicts Proposition 5.3. ∎

Proposition 5.3 ([7, Corollary 6.4]). *Let $x_0 > 0$ be algebraic, and let $m \in \mathbb{Q}_{\geq 0}$. Then,*

$$W_m(x_0) \neq 0.$$

Proof. In case $2m$ is not an odd integer, it was proved in [12] that J_m, J'_m, I_m, I'_m are algebraically independent functions over $\mathbb{C}(x)$. Observe that the following ODE is satisfied:

$$\begin{pmatrix} J_m \\ J'_m \\ I_m \\ I'_m \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1+\frac{m^2}{x^2} & -\frac{1}{x} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1+\frac{m^2}{x^2} & -\frac{1}{x} \end{pmatrix} \cdot \begin{pmatrix} J_m \\ J'_m \\ I_m \\ I'_m \end{pmatrix}.$$

Hence, if x_0 is algebraic, it follows from the Siegel–Shidlovskii theory, that $J_m(x_0)$, $J'_m(x_0)$, $I_m(x_0)$, $I'_m(x_0)$ are algebraically independent and in particular $W_m(x_0) \neq 0$.

In case $2m$ is odd, the statement in the proposition is simpler, since it is essentially a special case of the Lindemann–Weierstrass Theorem 5.6. Indeed, suppose that $W_m(x_0) = 0$. Then, $\frac{I_{m+1}}{I_m}(x_0) = -\frac{J_{m+1}}{J_m}(x_0)$. Hence, combining this with Lemma 5.5 iterated we get (see Definition 5.4)

$$\begin{aligned} 1 &\geq \operatorname{tr.deg}\Big(\frac{I_{m+1}}{I_m}(x_0)\Big) = \operatorname{tr.deg}\Big(x_0, \frac{I_{m+1}}{I_m}(x_0), \frac{J_{m+1}}{J_m}(x_0)\Big) \\ &= \operatorname{tr.deg}\Big(x_0, \frac{I_{1/2}}{I_{-1/2}}(x_0), \frac{J_{1/2}}{J_{-1/2}}(x_0)\Big) \overset{\text{Lemma 2.2}}{=} \operatorname{tr.deg}(\tanh(x_0), \tan(x_0)) \\ &= \operatorname{tr.deg}(e^{x_0}, e^{ix_0}) \overset{\text{L.-W.}}{=} 2. \end{aligned}$$

which is absurd. ∎

For the lemma below recall the following.

Definition 5.4. The *transcendental degree over* $\mathbb{Q}$ of a set $A \subset \mathbb{C}$, denoted by $\operatorname{tr.deg} A$, is the cardinality of a maximal subset $A' \subset A$ such that the numbers in A' are algebraically independent over $\mathbb{Q}$.

Lemma 5.5. *Let $m \in \mathbb{Q}_{\geq 0}$ and $x > 0$. Then,*

$$\operatorname{tr.deg}\Big(x, \frac{I_{m+1}}{I_m}(x), \frac{J_{m+1}}{J_m}(x)\Big) = \operatorname{tr.deg}\Big(x, \frac{I_m}{I_{m-1}}(x), \frac{J_m}{J_{m-1}}(x)\Big).$$

Here, if a denominator vanishes we regard the corresponding quotient as algebraic.

Proof. The lemma follows immediately from the identities

$$\frac{I_{m+1}}{I_m}(x) = -\frac{2m}{x} + \frac{I_{m-1}}{I_m}(x), \qquad \frac{J_{m+1}}{J_m}(x) = \frac{2m}{x} - \frac{J_{m-1}}{J_m}(x). \qquad \blacksquare$$

Theorem 5.6 (Lindemann–Weierstrass [6, 14]). *Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ be algebraic numbers that are linearly independent over $\mathbb{Q}$. Then, the numbers $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$.*

6. Schanuel's conjecture and the trivial multiplicities conjecture

We observe that a special case of the classical Schanuel's conjecture from transcendental number theory implies that in odd dimensions non-trivial multiplicities in the spectrum of the clamped round plate do not exist. This leads us to a formulation of a Schanuel-type conjecture which would eliminate non-trivial multiplicities in even dimensions. First, let us record the following special case of Schanuel's conjecture (see [5, Chapter III, Historical note, p. 30])

Conjecture 6.1 (Schanuel's conjecture – special case). *Let $x \in \mathbb{R}$ be non-zero. Then,*

$$\operatorname{tr.deg}(x, \mathrm{e}^x, \mathrm{e}^{\mathrm{i}x}) \geq 2.$$

Proposition 6.2. *Conjecture 6.1 implies Conjecture 1.3 in odd dimensions.*

Proof. If $W_m(x_0) = W_{m+n}(x_0) = 0$ for some $m \in \mathbb{Q}_{\geq 0}$ and $n \in \mathbb{N}$, then the first equation in (5.1) shows that x_0 and $I_{m+1}(x_0)/I_m(x_0)$ are algebraically dependent. Moreover, $\frac{I_{m+1}}{I_m}(x_0) + \frac{J_{m+1}}{J_m}(x_0) = 0$. It follows that

$$\operatorname{tr.deg}\Big(x_0, \frac{I_{m+1}}{I_m}(x_0), \frac{J_{m+1}}{J_m}(x_0)\Big) \leq 1.$$

From Lemma 5.5, we conclude that

$$\operatorname{tr.deg}\Big(x_0, \frac{I_{1/2}}{I_{-1/2}}(x_0), \frac{J_{1/2}}{J_{-1/2}}(x_0)\Big) \leq 1.$$

Hence, by Lemma 2.2, we obtain that

$$\operatorname{tr.deg}\big(x, \tanh(x), \tan(x)\big) \leq 1.$$

On the other hand, Conjecture 6.1 shows that

$$\operatorname{tr.deg}\big(x, \tanh(x), \tan(x)\big) \geq 2,$$

and we get a contradiction. ■

The argument in the preceding proof shows that the following conjecture implies the non-trivial multiplicity conjecture in even dimensions.

Conjecture 6.3 (Bessel–Schanuel-type conjecture). *Let* $x > 0$. *Then,*

$$\operatorname{tr.deg}\Big(x, \frac{I_1}{I_0}(x), \frac{I_1}{I_0}(\mathrm{i}x)\Big) \geq 2.$$

Remark. Here, as in Lemma 5.5, if $I_0(\mathrm{i}x) = 0$ we interpret the quotient $(I_1/I_0)(\mathrm{i}x)$ as an algebraic number.

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