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Spectral and dynamical results related to certain non-integer base expansions on the unit interval

Horia D. Cornean, Ira W. Herbst, and Giovanna Marcelli

Abstract. We consider certain non-integer base β-expansions of Parry's type and we study various properties of the transfer (Perron–Frobenius) operator $\mathcal{P}: L^p([0,1]) \to L^p([0,1])$ with $p \geq 1$ and its associated composition (Koopman) operator, which are induced by a discrete dynamical system on the unit interval related to these β-expansions.

We show that if f is Lipschitz, then the iterated sequence $\{\mathcal{P}^N f\}_{N\geq 1}$ converges exponentially fast (in the L^1 norm) to an invariant state corresponding to the eigenvalue 1 of $\mathcal{P}$. This "attracting" eigenvalue is not isolated: for $1 \leq p \leq 2$ we show that the point spectrum of $\mathcal{P}$ also contains the whole open complex unit disk and we explicitly construct an eigenfunction for every z with $|z| < 1$.

1. Introduction and main results

Let us fix two integers $n \geq 2$ and $q \geq 1$. There exists a unique positive number (see Lemma B.1)

$$\beta_{n,q} \equiv \beta \in (q, q+1) \tag{1.1}$$

which obeys the following equation:

$$1 = \frac{q}{\beta} + \frac{q}{\beta^2} + \cdots + \frac{q}{\beta^n}. \tag{1.2}$$

We consider representations of real numbers in non-integer base β of the type (1.2), which are called *β-expansions.* Expansions in non-integer bases were firstly introduced by the seminal work of Rényi [16], as a generalization of the standard integer base expansions. The original method to determine the "digits" is the greedy algorithm [14–16], which is tightly connected to the study of the map

$$T_\beta : [0,1) \mapsto [0,1), \quad T_\beta(x) = \beta x - \lfloor \beta x \rfloor, \tag{1.3}$$

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see Appendix A for some of its basic properties. Without putting certain restrictions on the coefficients, such expansions are far from being unique (see [11] and references therein). Such expansions are also related to symbolic dynamics [3, 14, 16], which is not the main focus of the current paper.

We are mostly interested in the investigation of certain spectral and dynamical properties of the transfer (or Perron–Frobenius) operator $\mathscr{P}: L^p([0,1]) \mapsto L^p([0,1])$ with $p \geq 1$, and its associated composition (or Koopman) operator $\mathfrak{K}$, which are induced by the above map T_β, see [12, 17, 18].

In general, the transfer operator $\mathscr{P}$ describes the discrete time evolution of certain probability densities associated to some stochastic variables, evolution related to the iteration of a certain map, in our case T_β, see [4, 6, 8]. More specific details about these objects will be given in the subsequent part of the introduction, where we will also formulate our main results: Theorems 1.2 and 1.4. There, it is stated that if f is Lipschitz, then the iterates $\mathscr{P}^N f$ converge exponentially fast (in the L^1 norm and $N \to \infty$) to an invariant state corresponding to the eigenvalue 1 of $\mathscr{P}$. On the other hand, the eigenvalue 1 is far from being isolated: if $1 \leq p \leq 2$ we show that the point spectrum of $\mathscr{P}$ also contains the open complex unit disk; namely, for every $|z| < 1$ and we explicitly construct a corresponding ψ_z such that $\mathscr{P}\psi_z = z\psi_z$.

1.1. The transfer operator

Let us assume that $X: \Omega \mapsto [0,1]$ is an absolutely continuous stochastic variable with a probability density function (PDF) denoted by $f \in L^1([0,1])$. More precisely: for every $x \geq 0$,

$$\mathrm{Prob}(X \leq x) := \int_0^x f(t)\,dt.$$

Any number $X(\omega) \in (0,1)$ has a well-defined *"greedy" decomposition* of the type (see Lemma A.1)

$$X(\omega) = \sum_{k\geq 1} X_k(\omega)\beta^{-k}, \quad X_k(\omega) \in \{0, 1, \dots, q\}.$$

The first coefficient X_1 defines a discrete stochastic variable $X_1: \Omega \mapsto \{0, \dots, q\}$, where (remember that $q < \beta < q+1$)

$$X_1(\omega) := j \in \{0, \dots, q\} \quad \text{whenever } j/\beta \leq X(\omega) < (j+1)/\beta$$

which implies

$$\mathrm{Prob}(X_1 = j) = \mathrm{Prob}\Big(\frac{j}{\beta} \leq X < \frac{j+1}{\beta}\Big), \quad 0 \leq j \leq q.$$

Assuming that $f(x) = 0$ if $x \notin [0, 1]$, then the new stochastic variable $\widetilde{X} = \beta(X - X_1/\beta)$ is also absolutely continuous and has a PDF (denoted by $\mathcal{P} f$) which equals

$$(\mathcal{P} f)(x) = \beta^{-1} \sum_{j=0}^{q} f\Big(\frac{j + x}{\beta}\Big). \tag{1.4}$$

Formula (1.4) is due to the fact that for $x \geq 0$ we have

$$\mathrm{Prob}\Big(\beta\Big(\frac{X - X_1}{\beta}\Big) \leq x\Big) = \mathrm{Prob}\Big(X \leq \frac{X_1 + x}{\beta}\Big) = \sum_{j=0}^{q} \mathrm{Prob}\Big(\frac{j}{\beta} \leq X \leq \frac{j + x}{\beta}\Big),$$

which we then differentiate with respect to x.

In order to formulate our first theorem, we need to state the following result, which goes back to [14].

Proposition 1.1. *There exists a piecewise constant function u_1 which is positive a.e. with $\int_0^1 u_1(x)\, dx = 1$ such that $\mathcal{P} u_1 = u_1$.*

Our first main theorem is as follows.

Theorem 1.2. *Let $n \geq 2$ and $q \geq 1$ be two integers. Let $\mathcal{P} \equiv \mathcal{P}_\beta \colon L^1([0, 1]) \to L^1([0, 1])$ be defined as in (1.4), where $\beta \equiv \beta_{n,q}$ is introduced in (1.1). Then there exist two constants $K_1(n, q) \geq 0$ and $K_2(n, q) \geq 1/2$ such that for every Lipschitz function f with $|f(x) - f(y)| \leq L_f |x - y|$ we have*

$$\Big\| \mathcal{P}^N f - u_1 \int_0^1 f(t)\, dt \Big\|_{L^1} \leq K_1(L_f + \|f\|_{L^\infty})\beta^{-K_2 N} \quad \textit{for all } N \geq 1.$$

If $n = 2$, we have

$$\beta = \frac{q + \sqrt{q^2 + 4q}}{2}, \quad K_2 = \frac{2 - \ln(q)/\ln(\beta)}{3 - \ln(q)/\ln(\beta)}. \tag{1.5}$$

Remark 1.3. We have a few extra comments.

(i) By using that the map $\mathcal{P}$ is non-expansive on L^1 (see (2.2)), a density argument implies that if $f \in L^1([0, 1])$, then

$$\lim_{N\to\infty} \Big\| \mathcal{P}^N f - u_1 \int_0^1 f(t)\, dt \Big\|_{L^1} = 0.$$

(ii) Point (i) implies that the function u_1 constructed in Proposition 1.1 is, up to a constant factor, the unique L^1 eigenfunction of $\mathcal{P}$ corresponding to the eigenvalue 1. We note that Parry [14] also obtained an explicit formula for u_1 in an even more general case. For $q = 1$ (see (1.2)), an exponential decay in sup norm with the same exponent as ours has been previously obtained in [9, 10], but using a slightly different approach (we will explain it in a moment) and with a very different method concerning the convergence. Namely, let

$$X = \sum_{k=1}^{\infty} X_k \beta^{-k}$$

be the β-expansion (with $q = 1$) of an absolutely continuous random variable X on the unit interval. Then [10] analyzes the convergence rate of the PDF of the scaled remainder $\sum_{k=1}^{\infty} X_{m+k}\beta^{-k}$ when m tends to infinity to the asymptotic distribution u_1. If the density of X is f, then $\mathcal{P}^m f$ is nothing but the density associated with the above scaled remainder.

(iii) In [12] it is shown the existence of a Césaro limit $\frac{1}{N}\sum_{k=1}^{N} \mathcal{P}^k f$ in the L^1-norm for the more general case of piecewise monotonic and expanding maps.

(iv) We now briefly outline some consequences for the ergodicity properties [7] of the map T_β in (1.3). It is measure preserving on [0, 1] equipped with the measure density u_1. We consider stochastic variables of the type $F\colon [0, 1] \mapsto \mathbb{R}$ with

$$\mathrm{Prob}(F \in (c, d)) := \int_{F^{-1}((c,d))} u_1(x)\, dx, \quad \text{for all } c < d.$$

For every integer $k \geq 0$, we define $\mathcal{X}_k\colon [0, 1] \mapsto \mathbb{R}$ given by

$$\mathcal{X}_k(x) := g(T_\beta^k(x)),$$

for some $g \in L^p([0, 1])$ with $1 \leq p \leq \infty$. If g is Lipschitz, by using Theorem 1.2 one can prove that these random variables have the same mean value and exponentially decaying correlations, which in turn implies [1, Theorem 1] the strong law of large numbers.

The proof of Theorem 1.2 is given in Section 2.

1.2. The composition (Koopman) operator

Let us recall the definition of $T_\beta\colon [0, 1) \mapsto [0, 1)$ given by

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor = \beta x - j, \quad j/\beta \leq x < (j+1)/\beta,\ x \in [0, 1),\ j \in \{0, 1, \dots, q\}.$$

We define the operator

$$\mathcal{K}\colon L^p([0, 1]) \mapsto L^p([0, 1]), \quad (\mathcal{K}g)(x) := g(T_\beta(x)), \quad 1 \leq p \leq \infty. \tag{1.6}$$

We may also consider the operator $\mathcal{P}$ from (1.4) acting on $L^{p'}([0,1])$ to itself with $1/p + 1/p' = 1$ and $1 \le p' \le \infty$. Then if $f \in L^{p'}([0,1])$ and $g \in L^p([0,1])$, we have

$$\begin{aligned} \int_0^1 \overline{f(t)}(\mathfrak{K}g)(t)\,dt &= \sum_{j=0}^{q-1} \int_{j/\beta}^{(j+1)/\beta} \overline{f(t)} g(\beta t - j)\,dt + \int_{q/\beta}^{1} \overline{f(t)} g(\beta t - q)\,dt \\ &= \int_0^1 \overline{[\mathcal{P} f](x)} g(x)\,dx, \end{aligned} \tag{1.7}$$

where in the last equality we used that $f(x) = 0$ when $x > 1$.

The main spectral results of this paper are contained in the next theorem.

Theorem 1.4. *The following properties hold.*

(i) *Define the numbers*

$$x_j := q\beta^{-2} + \cdots + q\beta^{-n} + j/\beta, \quad 0 \le j \le q.$$

They obey $j/\beta < x_j < (j+1)/\beta$ *when* $0 \le j \le q-1$, *and* $x_q = 1$. *If* $q = 1$, *we define*

$$\psi_0(t) = \begin{cases} e^{\pi i \beta t} & \text{if } j/\beta \le t < x_j,\ 0 \le j \le 1, \\ 0 & \text{if } x_0 \le t < 1/\beta. \end{cases}$$

If $q > 1$, *we define*

$$\psi_0(t) = \begin{cases} e^{2\pi i \beta t/(q+1)} & \text{if } j/\beta \le t < x_j,\ 0 \le j \le q, \\ e^{2\pi i \beta t/q} & \text{if } x_j \le t < (j+1)/\beta,\ 0 \le j \le q-1. \end{cases}$$

Then $\psi_0 \in L^\infty$ *and* $\mathcal{P}\psi_0 = 0$ *almost everywhere. Note that when* $n \equiv \infty$, *then* $\beta \equiv q + 1$ *and* $\psi_0(t) \equiv e^{2\pi i t}$. *See Figure 1 for an illustration of the function* ψ_0 *for the cases* $q = 1$ *and* $q = 3$.

(ii) *The operator* $\widetilde{\mathfrak{K}} := u_1^{1/p} \mathfrak{K} u_1^{-1/p}$ *is a non-surjective isometry on* $L^p([0,1])$ *for* $1 \le p \le \infty$.

(iii) *The spectrum of* $\widetilde{\mathfrak{K}}$ *and* $\mathfrak{K}$ *equals* $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \le 1\}$ *for* $1 \le p \le \infty$.

(iv) *Let* $|z| < 1$. *Then the function*

$$\psi_z = u_1^{1/2}(\mathrm{Id} - z u_1^{1/2} \mathfrak{K} u_1^{-1/2})^{-1} u_1^{-1/2} \psi_0 \in L^2([0,1]) \subset L^{p'}([0,1]),$$

$1 \le p' \le 2$, *is an eigenfunction of* $\mathcal{P}$ *which obeys* $\mathcal{P}\psi_z = z\psi_z$.

The proof of this theorem is given in Section 3. We note that when $\mathcal{P}$ is restricted to functions of bounded variations, its spectrum is quite different [17].

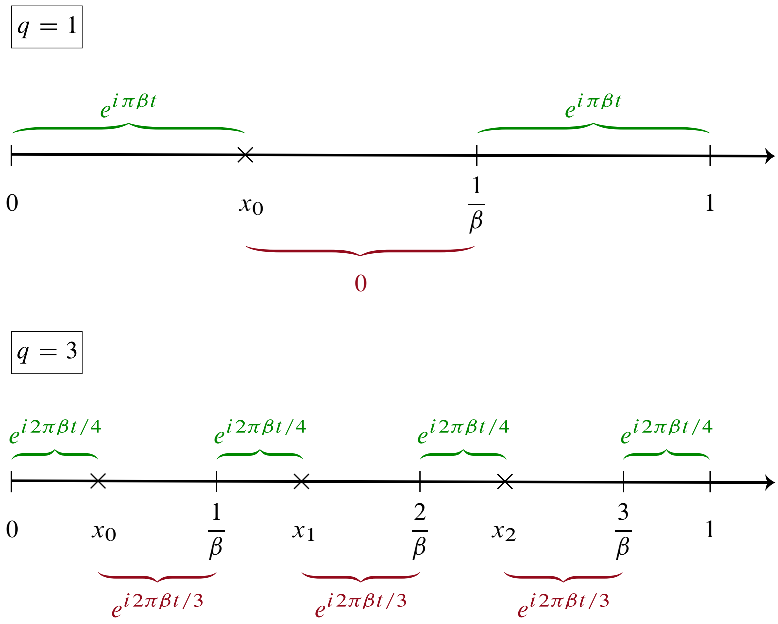

Figure 1. Illustration of the map ψ_0.

2. Proof of Theorem 1.2

2.1. Preliminaries

Notice that $\mathcal{P}$ maps non-negative functions into non-negative functions and for any function $f \in L^1([0,1])$ we have

$$\int_0^1 (\mathcal{P} f)(x)dx = \int_0^1 f(x)\, dx. \tag{2.1}$$

Indeed, if $0 \le j \le q-1$, we have

$$[0,1] \ni x \mapsto \frac{j+x}{\beta} \in \left[\frac{j}{\beta}, \frac{j+1}{\beta}\right],$$

hence these intervals cover the interval $[0, q/\beta]$. Also, due to (1.2) we have

$$\left[0, \frac{q}{\beta} + \cdots + \frac{q}{\beta^{n-1}}\right] \ni x \mapsto \frac{q+x}{\beta} \in \left[\frac{q}{\beta}, 1\right].$$

Equality 2.1 follows after a change of variable on each interval. Moreover, this together with $|\mathcal{P} f| \le \mathcal{P}|f|$ imply that the linear map $\mathcal{P}$ is non-expansive on L^1, i.e.,

$$\|\mathcal{P} f\|_{L^1} \le \|f\|_{L^1} \quad \text{for all } f \in L^1([0,1]). \tag{2.2}$$

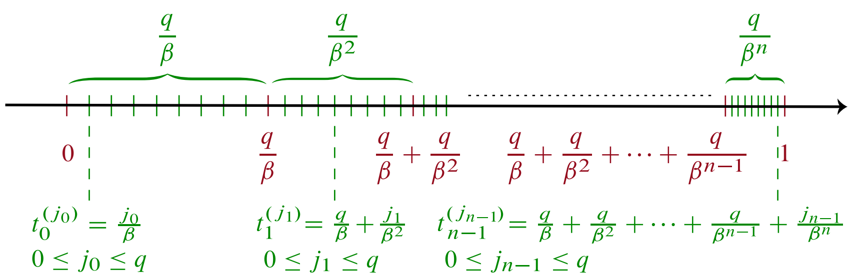

Figure 2. The first layer.

2.2. Subdividing the interval [0, 1]

In Figure 2 we introduce a decomposition of the interval [0, 1], which we will explain in what follows. The characteristic functions of the intervals between two consecutive red points will form a generating system, and it is important to know how $\mathcal{P}$ acts on them. This will be done in Lemma 2.1.

First, we have the numbers in red given by $0, q/\beta, q/\beta + q/\beta^2, \ldots$, and $q/\beta + q/\beta^2 + \cdots q/\beta^{n-1}, 1$.

Second, we want to define the green numbers, which include the red ones, see Figure 2. Let us start with those between 0 and q/β. For $j_0 \in \{0, \ldots, q\}$, we define the first set of green numbers: $t_0^{(j_0)} = j_0/\beta$, with $t_0^{(q)} = q/\beta$. The distance between two consecutive such numbers is $1/\beta$.

The green numbers between q/β and $q/\beta + q/\beta^2$ are indexed by $t_1^{(j_1)} = q/\beta + j_1/\beta^2$ where $j_1 \in \{0, \ldots, q\}$. The distance between two such consecutive numbers is $1/\beta^2$.

For the interval between $q/\beta + \cdots + q/\beta^{n-1}$ and 1, we let $j_{n-1} \in \{0, \ldots, q\}$ and define $t_{n-1}^{(j_{n-1})} := q/\beta + \cdots + q/\beta^{n-1} + j_{n-1}/\beta^n$. We also have the identities $t_k^{(q)} = t_{k+1}^{(0)}$ when $0 \le k \le n-1$, and $t_{n-1}^{(q)} = 1$.

The distance between two consecutive points depends on which "red" interval they are situated and is given by

$$t_{k_1}^{(j_{k_1}+1)} - t_{k_1}^{(j_{k_1})} = \beta^{-(k_1+1)}, \quad 0 \le k_1 \le n-1.$$

By definition, the *first layer* means the set of all numbers $t_{k_1}^{(j_{k_1})}$ where $k_1 \in \{0, \ldots, n-1\}$ and $j_{k_1} \in \{0, \ldots, q\}$.

At this point, we are able to further refine any interval between two consecutive elements of the first layer, where the endpoints 0 and 1 are replaced by $t_{k_1}^{(j_{k_1})}$ and $t_{k_1}^{(j_{k_1}+1)}$, and the width 1 is replaced by β^{-k_1-1}. More precisely, the points of the *second layer* are defined for $0 \le k_1, k_2 \le n-1$:

$$t_{k_1,k_2}^{(j_{k_1},j_{k_2})} = t_{k_1}^{(j_{k_1})} + \beta^{-(k_1+1)} t_{k_2}^{(j_{k_2})}.$$

Thus, in particular, we have that

$$t_{k_1}^{(j_{k_1})} \le t_{k_1,k_2}^{(j_{k_1},j_{k_2})} \le t_{k_1}^{(j_{k_1}+1)}, \quad t_{k_1,q}^{(j_{k_1},n-1)} = t_{k_1}^{(j_{k_1}+1)}.$$

In general, the *m-th layer* consists of the points for $0 \le k_1, k_2, \dots, k_m \le n-1$:

$$t_{k_1,k_2,\dots,k_m}^{(j_{k_1},j_{k_2},\dots,j_{k_m})} = t_{k_1}^{(j_{k_1})} + \beta^{-(k_1+1)} t_{k_2}^{(j_{k_2})} + \dots + \beta^{-(k_1+1)} \dots \beta^{-(k_{m-1}+1)} t_{k_m}^{(j_{k_m})}.$$

We now introduce the L^1 normalized indicator functions of intervals between two "consecutive points" of layer m denoted by

$$F_{k_1,k_2,\dots,k_m}^{(j_{k_1},j_{k_2},\dots,j_{k_m})}(x) = \beta^{k_1+1} \dots \beta^{k_m+1} \chi_{[t_{k_1,k_2,\dots,k_m}^{(j_{k_1},j_{k_2},\dots,j_{k_m})},\, t_{k_1,k_2,\dots,k_m}^{(j_{k_1},j_{k_2},\dots,j_{k_m}+1)}]}(x). \tag{2.3}$$

Finally, let us introduce a special notation for the red numbers including the end-points 0 and 1. They are

$$\begin{aligned} t_0 &:= t_0^{(0)} = 0, \\ t_1 &:= t_0^{(q)} = t_1^{(0)} = \frac{q}{\beta}, \\ t_2 &:= t_1^{(q)} = t_2^{(0)} = \frac{q}{\beta} + \frac{q}{\beta^2}, \\ &\vdots \\ t_{n-1} &:= t_{n-2}^{(q)} = t_{n-1}^{(0)} = \frac{q}{\beta} + \dots + \frac{q}{\beta^{n-1}}, \\ t_n &:= t_{n-1}^{(q)} = 1. \end{aligned}$$

The two very last notations give the L^1 normalized indicator functions of the intervals between two such consecutive points:

$$F_r(x) := q^{-1} \sum_{j=0}^{q-1} F_r^{(j)}(x) = q^{-1} \beta^{r+1} \chi_{[t_r,t_{r+1}]}(x), \quad 0 \le r \le n-1. \tag{2.4}$$

Lemma 2.1. *We have*

$$\mathcal{P} F_0 = \chi_{[0,1]} = q \sum_{j=0}^{n-1} \beta^{-(j+1)} F_j, \quad \text{and} \quad \mathcal{P} F_r = F_{r-1}, \quad \text{where } 1 \le r \le n-1.$$

In particular, the subspace generated by these functions is invariant under the action of $\mathcal{P}$, namely $\mathcal{P}(\operatorname{span}\{F_0, \dots, F_{n-1}\}) \subseteq \operatorname{span}\{F_0, \dots, F_{n-1}\}$.

Moreover, for all $m \geq 2$ and all possible tuples $(j_{k_1}, j_{k_2}, \ldots, j_{k_m}) \in \{0, \ldots, q\}^m$ we have

$$\mathcal{P} F_{k_1,k_2,\ldots,k_m}^{(j_{k_1},j_{k_2},\ldots,j_{k_m})} = F_{k_2,\ldots,k_m}^{(j_{k_2},\ldots,j_{k_m})} \qquad \text{if } k_1 = 0, \tag{2.5}$$

$$\mathcal{P} F_{k_1,k_2,\ldots,k_m}^{(j_{k_1},j_{k_2},\ldots,j_{k_m})} = F_{k_1-1,k_2,\ldots,k_m}^{(j_{k_1},j_{k_2},\ldots,j_{k_m})} \qquad \text{if } k_1 \geq 1, \tag{2.6}$$

and

$$\mathcal{P}^{m-1+k_1+k_2+\cdots+k_{m-1}} F_{k_1,\ldots,k_m}^{(j_{k_1},j_{k_2},\ldots,j_{k_m})} \in \operatorname{span}\{F_0, F_1, \ldots, F_{n-1}\}. \tag{2.7}$$

Proof. For $x \in [0,1]$, we have

$$\chi_{[t_{k_1,\ldots,k_m}^{(j_{k_1},j_{k_2},\ldots,j_{k_m})}, t_{k_1,\ldots,k_m}^{(j_{k_1},j_{k_2},\ldots,j_{k_m}+1)}]}\left(\frac{x+j}{\beta}\right)$$
$$= \chi_{[0,1]}(x)\chi_{[\beta t_{k_1,\ldots,k_m}^{(j_{k_1},j_{k_2},\ldots,j_{k_m})}-j, \beta t_{k_1,\ldots,k_m}^{(j_{k_1},j_{k_2},\ldots,j_{k_m}+1)}-j]}(x),$$

which introduced in (1.4) gives for the functions $F_{k_1,k_2,\ldots,k_m}^{(j_{k_1},j_{k_2},\ldots,j_{k_m})}$ defined in (2.3):

$$(\mathcal{P} F_{k_1,k_2,\ldots,k_m}^{(j_{k_1},j_{k_2},\ldots,j_{k_m})})(x)$$
$$= \beta^{-1}\beta^{k_1+1}\ldots\beta^{k_m+1}\chi_{[0,1]}(x)\sum_{j=0}^{q}\chi_{[\beta t_{k_1,\ldots,k_m}^{(j_{k_1},j_{k_2},\ldots,j_{k_m})}-j, \beta t_{k_1,\ldots,k_m}^{(j_{k_1},j_{k_2},\ldots,j_{k_m}+1)}-j]}(x). \tag{2.8}$$

First, let us consider $m = 1$. We start by computing $\mathcal{P} F_0^{(j_0)}$, thus we put $m = 1$ and $k_1 = 0$. Then $\beta t_0^{(j_0)} = j_0 \in \{0, \ldots, q-1\}$ and

$$\chi_{[\beta t_0^{(j_0)}-j, \beta t_0^{(j_0+1)}-j]}(x) = \chi_{[j_0-j, j_0-j+1]}(x).$$

By summing over j in (2.8), we get

$$\mathcal{P} F_0^{(j_0)} = \chi_{[0,1]}, \quad 0 \leq j_0 \leq q-1.$$

Since the above formula is independent of j_0, it also implies that $\mathcal{P} F_0 = \chi_{[0,1]}$, see (2.4) for the definition of F_0.

We now want to compute $\mathcal{P} F_{k_1}^{(j_{k_1})}$ with $0 < k_1 \leq n-1$. Since $k_1 \geq 1$, then $\beta t_{k_1}^{(j_{k_1})} \geq q$, and so the interval $[\beta t_{k_1}^{(j_{k_1})} - j, \beta t_{k_1}^{(j_{k_1}+1)} - j]$ is disjoint from $[0,1]$ if $j \leq q-1$. On the other hand, since

$$t_{k_1}^{(j_{k_1})} = q/\beta + \cdots + q/\beta^{k_1} + j_{k_1}/\beta^{k_1+1}$$

we have

$$0 \le \beta t_{k_1}^{(j_{k_1})} - q = t_{k_1-1}^{(j_{k_1})} < t_{k_1-1}^{(j_{k_1}+1)} = \beta t_{k_1}^{(j_{k_1}+1)} - q \le 1.$$

This implies that

$$\mathcal{P} F_{k_1}^{(j_{k_1})} = F_{k_1-1}^{(j_{k_1})}, \quad 1 \le k_1 \le n-1, \, 0 \le j_{k_1} \le q-1.$$

This shows that $\mathcal{P}^{1+k_1} F_{k_1}^{(j_{k_1})} = \mathcal{P} F_0^{(j_{k_1})} = \chi_{[0,1]}$ belongs to the subspace spanned by $F_0, \ldots, F_{n-1}$ (see (2.4)). Applying $\mathcal{P}$ to (2.4) we obtain

$$\mathcal{P} F_r = F_{r-1}, \quad 1 \le r \le n-1.$$

This ends the proof of the first part of the lemma.

Now, let us consider $m > 1$, i.e., more than just one layer. We have the following cases.

- If $k_1 = 0$, then

$$\begin{aligned}
&\beta t_{0,\ldots,k_m}^{(j_0, j_{k_2}, \ldots, j_{k_m})} - j \\
&\quad = \beta(j_0/\beta + \beta^{-1} t_{k_2}^{(j_{k_2})} + \cdots + \beta^{-1} \cdots \beta^{-(k_{m-1}+1)} t_{k_m}^{(j_{k_m})}) - j \\
&\quad = j_0 - j + t_{k_2,\ldots,k_m}^{(j_{k_2},\ldots,j_{k_m})}
\end{aligned}$$

  which introduced in (2.8) gives

$$\mathcal{P} F_{0,k_2,\ldots,k_m}^{(j_0, j_{k_2}, \ldots, j_{k_m})} = F_{k_2,\ldots,k_m}^{(j_{k_2},\ldots,j_{k_m})}.$$

  This shows that if we apply $\mathcal{P}$ on a function with $k_1 = 0$, then we go down to a lower layer where m is replaced by $m - 1$ and j_0 is "erased." This proves (2.5).

- If $1 \le k_1 \le n-1$, then $\beta t_{k_1,\ldots,k_m}^{(j_{k_1}, j_{k_2}, \ldots, j_{k_m})} \ge q$ and so the sum over $j \le q-1$ in (2.8) equals zero. On the other hand,

$$\begin{aligned}
0 \le \beta t_{k_1,k_2,\ldots,k_m}^{(j_{k_1}, j_{k_2}, \ldots, j_{k_m})} - q = t_{k_1-1,k_2,\ldots,k_m}^{(j_{k_1}, j_{k_2}, \ldots, j_{k_m})} &< t_{k_1-1,k_2,\ldots,k_m}^{(j_{k_1}, j_{k_2}, \ldots, j_{k_m}+1)} \\
&= \beta t_{k_1,k_2,\ldots,k_m}^{(j_{k_1}, j_{k_2}, \ldots, j_{k_m}+1)} - q \le 1,
\end{aligned}$$

  hence

$$\mathcal{P} F_{k_1,k_2,\ldots,k_m}^{(j_{k_1}, j_{k_2}, \ldots, j_{k_m})} = F_{k_1-1,k_2,\ldots,k_m}^{(j_{k_1}, j_{k_2}, \ldots, j_{k_m})}.$$

  This shows that when we apply $\mathcal{P}$ on a function of the type (2.3) with $k_1 > 0$, then k_1 is reduced with one unit. This proves (2.6).

Conclusion: it takes $k_1 + 1$ applications of $\mathcal{P}$ in order to go down from layer m to layer $m - 1$, then $k_2 + 1$ applications in order to get from layer $m - 1$ to layer $m - 2$, so $\mathcal{P}^{k_1+k_2+\cdots k_{m-1}+m-1}$ gets us to the lowest layer with $m = 1$. ∎

2.2.1. Proof of Proposition 1.1

Lemma 2.2. *Denote by $\mathcal{T}$ the $n \times n$ matrix obtained by restricting $\mathcal{P}$ to the subspace generated by $\{F_0, \ldots, F_{n-1}\}$. Then $\mathcal{T}$ is a left-stochastic matrix. If λ is an eigenvalue, then it obeys the equation $P_{n,q}(\lambda\beta) = 0$ with $P_{n,q}$ from Lemma B.1. For $\lambda_1 = 1$, we can construct a positive eigenvector. If λ_2 is the second largest eigenvalue in absolute value, then*

$$q^{1/(n-1)}\beta^{-n/(n-1)} \leq |\lambda_2| < \beta^{-1}. \tag{2.9}$$

There exists an explicitly computable piecewise constant function u_1 which is positive a.e. such that

$$\mathcal{P}u_1 = u_1, \quad u_1 \in \operatorname{span}\{F_0, \ldots, F_{n-1}\}, \quad \int_0^1 u_1(x)\,dx = 1. \tag{2.10}$$

Moreover, there exists $C < \infty$ such that for every $r \in \mathbb{N}$ and any $g \in \operatorname{span}\{F_0, \ldots, F_{n-1}\}$ we have

$$\left\| (\mathcal{P}^r g)(\cdot) - u_1(\cdot) \int_0^1 g(t)\,dt \right\|_{L^\infty} \leq C|\lambda_2|^r \|g\|_{L^1}. \tag{2.11}$$

Proof. We have

$$\mathcal{P}F_{j-1} = \sum_{i=1}^{n} \mathcal{T}_{ij} F_{i-1}, \quad 1 \leq j \leq n,\ \mathcal{T} = \begin{bmatrix} q\beta^{-1} & 1 & 0 & \ldots & 0 & 0 \\ q\beta^{-2} & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ q\beta^{-(n-1)} & 0 & 0 & \ldots & 0 & 1 \\ q\beta^{-n} & 0 & 0 & \ldots & 0 & 0 \end{bmatrix},$$

then $\mathcal{T}$ is left-stochastic by (1.2). Observe that

$$z\,\mathrm{Id}_n - \mathcal{T} = \begin{bmatrix} z - q\beta^{-1} & -1 & 0 & \ldots & 0 & 0 \\ -q\beta^{-2} & z & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -q\beta^{-(n-1)} & 0 & 0 & \ldots & z & -1 \\ -q\beta^{-n} & 0 & 0 & \ldots & 0 & z \end{bmatrix}.$$

Expanding the determinant with respect to the first row, we get

$$\det(z\,\mathrm{Id}_n - \mathcal{T}) = (z - q\beta^{-1})z^{n-1} + \det(\mathcal{T}_{n-1})$$

where

$$\mathcal{T}_{n-1} = \begin{bmatrix} -q\beta^{-2} & -1 & \dots & 0 & 0 \\ -q\beta^{-3} & z & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -q\beta^{-(n-1)} & 0 & \dots & z & -1 \\ -q\beta^{-n} & 0 & \dots & 0 & z \end{bmatrix}.$$

By recursion, we get

$$\begin{aligned} \det(z \operatorname{Id}_n - \mathcal{T}) &= (z - q\beta^{-1})z^{n-1} - q\beta^{-2}z^{n-2} - \cdots - q\beta^{-(n-1)}z - q\beta^{-n} \\ &= \beta^{-n} P_{n,q}(z\beta). \end{aligned}$$

Thus, λ is an eigenvalue if and only if $\lambda\beta$ is a zero of $P_{n,q}$, hence all eigenvalues are simple due to Lemma B.1 (i) and (iii). While $\lambda_1 = 1$ (notice that $\lambda_1 = 1$ is an eigenvalue due to (1.2)), all other eigenvalues are in absolute value less than $\beta^{-1} < 1$ due to Lemma B.1(iii). Since the product of all roots of $P_{n,q}$ must equal $(-1)^{n-1}q$, we have

$$\beta|\beta\lambda_2|\cdots|\beta\lambda_n| = q.$$

If λ_2 has the second largest modulus, we have $q \leq \beta^n|\lambda_2|^{n-1}$, which proves the lower bound in (2.9).

Now, let us compute an eigenfunction corresponding to the eigenvalue 1. We solve the system

$$\begin{bmatrix} 1 - q\beta^{-1} & -1 & 0 & \dots & 0 & 0 \\ -q\beta^{-2} & 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -q\beta^{-(n-1)} & 0 & 0 & \dots & 1 & -1 \\ -q\beta^{-n} & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_{n-1} \\ s_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

We may choose s_1 as a free variable. In that case, we may choose

$$\begin{aligned} s_1 &= 1, \\ s_2 &= 1 - q\beta^{-1}, \\ s_3 &= 1 - q\beta^{-1} - q\beta^{-2}, \\ &\vdots \\ s_n &= 1 - q\beta^{-1} - \cdots - q\beta^{n-1} = q\beta^{-n}. \end{aligned}$$

Now, let us define (see (2.4)) $\widetilde{F}_k(x) = \sqrt{q}\beta^{-(k+1)/2}F_k(x)$ for $0 \leq k \leq n-1$. They form an L^2-orthonormal basis in the span of $\{F_0, \dots, F_{n-1}\}$. The restriction of $\mathcal{P}$ to this subspace, in the new basis, will have a matrix (here $1 \le i, j \le n$)

$$\widetilde{\mathcal{T}}_{ij} := \langle \widetilde{F}_{i-1}, \mathcal{P}\widetilde{F}_{j-1}\rangle = \sqrt{q}\beta^{-j/2}\langle \widetilde{F}_{i-1}, \mathcal{P}F_{j-1}\rangle = \sqrt{q}\beta^{-j/2}\sum_{r=1}^{n}\mathcal{T}_{rj}\langle \widetilde{F}_{i-1}, F_{r-1}\rangle$$
$$= \beta^{i/2}\mathcal{T}_{ij}\beta^{-j/2}.$$

Since $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ are similar, $\widetilde{\mathcal{T}}$ has the same spectrum as $\mathcal{T}$. Moreover, the vector $\tilde{s}$ with coordinates $\tilde{s}_j = \beta^{j/2}s_j$, where $1 \le j \le n$, is a not-normalized eigenvector of $\widetilde{\mathcal{T}}$ corresponding to the eigenvalue 1. The adjoint matrix $\widetilde{\mathcal{T}}^*$ has the matrix elements

$$(\widetilde{\mathcal{T}}^*)_{ij} = \widetilde{\mathcal{T}}_{ji} = \beta^{j/2}\mathcal{T}_{ji}\beta^{-i/2}.$$

By direct computation, using that $\sum_{j=1}^{n}\mathcal{T}_{ji} = 1$ for all i, we can check that the vector $\tilde{t}$ with entries $\tilde{t}_j = \beta^{-j/2}$ is an eigenvector of $\widetilde{\mathcal{T}}^*$ corresponding to the same eigenvalue 1.

Getting back to functions, the operator $\mathcal{P}$ has an eigenfunction $u(x)$ corresponding to eigenvalue 1 given a.e. by

$$u(x) = \sum_{j=1}^{n}\tilde{s}_j\widetilde{F}_{j-1}(x) = \sqrt{q}\sum_{j=1}^{n}s_jF_{j-1}(x) > 0,$$

and we denote by

$$u_1(x) := \frac{u(x)}{\int_0^1 u(t)\,dt}, \qquad \int_0^1 u_1(x)\,dx = 1,$$

which satisfies (2.10).

Using the information we have about the eigenvector $\tilde{t}$ of $\widetilde{\mathcal{T}}^*$, the adjoint $\mathcal{P}^*$ of $\mathcal{P}$ seen as an operator on the span of $\{F_0, \dots, F_{n-1}\}$ has an eigenfunction

$$w(x) = \sum_{j=1}^{n}\tilde{t}_j\widetilde{F}_{j-1}(x) = \sqrt{q}\sum_{j=1}^{n}\beta^{-j}F_{j-1}(x) = q^{-1/2}\chi_{[0,1]}(x), \quad \mathcal{P}^*\chi_{[0,1]} = \chi_{[0,1]}.$$

Then the rank-one Riesz projection corresponding to the eigenvalue 1 can be written as

$$\Pi_1 = |u_1\rangle\langle\chi_{[0,1]}|, \quad \Pi_1^2 = \Pi_1.$$

Moreover, we may write

$$\mathcal{P}|_{\mathrm{span}\{F_0,\dots,F_{n-1}\}} = \Pi_1 + \sum_{j=2}^{n}\lambda_j\Pi_j$$

where each projection has rank one and $\Pi_j \Pi_k = \delta_{jk} \Pi_k$. Now, if g is in the span of $\{F_0, \dots, F_{n-1}\}$, we have

$$\mathcal{P}^r g = u_1 \int_0^1 g(t)dt + \sum_{j=2}^{n} \lambda_j^r \Pi_j g.$$

Since each Π_j is a rank one operator of the form

$$\left(\frac{1}{\langle v_j, u_j \rangle_{L^2}}\right) |u_j\rangle\langle v_j|$$

with u_j and v_j bounded functions in the span of $\{F_0, \dots, F_{n-1}\}$, we have

$$\|\Pi_j g\|_{L^\infty} \le C \|g\|_{L^1}, \quad 2 \le j \le n. \qquad \blacksquare$$

2.2.2. Finalizing the proof of Theorem 1.2. The first step is to approximate f with piecewise constant functions using its Lipschitz property. For example, using the first layer in Figure 2 we have (in the sup-norm)

$$f - \sum_{k_1=0}^{n-1} \sum_{j_{k_1}=0}^{q-1} f(t_{k_1}^{(j_{k_1})}) \beta^{-1-k_1} F_{k_1}^{(j_{k_1})} = \mathcal{O}(L_f \beta^{-1}),$$

where $F_{k_1}^{(j_{k_1})}$ is defined in (2.3). The error is largest on the interval between 0 and q/β, because the distance between two consecutive points is only β^{-1}. On the other intervals, where $k_1 \ge 1$, the distance between two consecutive points is at least β^{-2} and the error is of order β^{-2} or better.

It is possible to improve the above estimate and get a global error of order β^{-2}. To achieve this, we have to refine the interval $[0, q/\beta]$ by going to the second layer, while keeping unchanged the other intervals where $k_1 \ge 1$. This leads to

$$f - \sum_{j_0=0}^{q-1} \sum_{k_2=0}^{n-1} \sum_{j_{k_2}=0}^{q-1} f(t_{0,k_2}^{(j_0,j_{k_2})}) \beta^{-2-k_2} F_{0,k_2}^{(j_0,j_{k_2})}$$
$$- \sum_{k_1=1}^{n-1} \sum_{j_{k_1}=0}^{q-1} f(t_{k_1}^{(j_{k_1})}) \beta^{-1-k_1} F_{k_1}^{(j_{k_1})} = \mathcal{O}(L_f \beta^{-2}).$$

If we want a global error of order β^{-3}, we need to go up to the third layer on the subintervals where $k_2 = 0$ in the triple sum, and to the second layer on the subintervals where $k_1 = 1$ in the double sum.

If we want a global error of order β^{-n-1}, even the old subinterval $[1 - q/\beta^n, 1]$ corresponding to $k_1 = n - 1$ in the first layer has now to be refined with a second layer.

In the general case, let us fix some integer $M \geq n+1$ and let us investigate in which way we should split the interval [0, 1] so that the error we make is not bigger than β^{-M+n}. From the above discussion, this amounts to adjust the length of the subintervals obtained by picking points from different layers.

For a given layer of order $m \geq 1$, the support of $F_{k_1,\dots,k_m}^{(j_{k_1},\dots,j_{k_m})}$ has a width of $\beta^{-m-k_1-\cdots-k_m}$. We have the following double inequality:

$$\begin{aligned} k_1 + k_2 + \cdots + k_m + m &< k_1 + k_2 + \cdots + k_m + k_{m+1} + (m+1) \\ &\leq k_1 + k_2 + \cdots + k_m + m + n, \end{aligned} \tag{2.12}$$

where the first one is trivial while the second one is due to $k_{m+1} \leq n-1$.

Remember that $M \geq n+1$. The first layer has $m = 1$ with $k_1 + 1 < M$ because $k_1 \leq n-1$. By refining each subinterval of layer 1 by adding points of higher layers, we have two alternatives:

- either

$$k_1 + k_2 + \cdots + k_m + k_{m+1} + (m+1) < M$$

- or

$$k_1 + k_2 + \cdots + k_m + m < M \leq k_1 + k_2 + \cdots + k_m + k_{m+1} + (m+1).$$

If the first alternative is realized, then we perform another refinement. If the second alternative is realized (this must happen at some point), then by coupling it with (2.12), we obtain

$$k_1 + k_2 + \cdots + k_m + m < M \leq n + k_1 + k_2 + \cdots + k_m + m. \tag{2.13}$$

No further refinement is performed on a subinterval where (2.13) holds. Also, when (2.13) is satisfied, we write

$$m + k_1 + \cdots + k_m \approx M.$$

Replacing f on the support of $\chi_{[t_{k_1,\dots,k_m}^{(j_{k_1},\dots,j_{k_m})}, t_{k_1,\dots,k_m}^{(j_{k_1},\dots,j_{k_m}+1)}]}$ with $f(t_{k_1,\dots,k_m}^{(j_{k_1},\dots,j_{k_m})})$ and using the Lipschitz property of f, the error is of order $\beta^{-m-k_1-\cdots-k_m}$. Thus, we have (even in the sup-norm)

$$f - \sum_{\substack{m+k_1+\cdots+k_m \approx M \\ j_{k_1},\dots,j_{k_m}}} f(t_{k_1,\dots,k_m}^{(j_{k_1},\dots,j_{k_m})}) \beta^{-m-k_1-\cdots-k_m} F_{k_1,\dots,k_m}^{(j_{k_1},\dots,j_{k_m})} = \mathcal{O}(L_f \beta^{-M}).$$

According to (2.2), $\mathcal{P}$ is a non-expansive map on L^1, hence there exists a constant $C < \infty$ such that for all $N \geq 1$ we have

$$\Bigg\| \mathcal{P}^N f - \sum_{\substack{m+k_1+\cdots+k_m \approx M \\ j_{k_1},\dots,j_{k_m}}} f(t_{k_1,\dots,k_m}^{(j_{k_1},\dots,j_{k_m})}) \beta^{-m-k_1-\cdots-k_m} \mathcal{P}^N F_{k_1,\dots,k_m}^{(j_{k_1},\dots,j_{k_m})} \Bigg\|_{L^1} \leq C L_f \beta^{-M}.$$

If N is larger than M, which is already larger than $m + k_1 + \cdots + k_m$ (due to (2.13)), then according to (2.7) in Lemma 2.1, we have that both $\mathcal{P}^N F_{k_1,\ldots,k_m}^{(j_{k_1},\ldots,j_{k_m})}$ and $\mathcal{P}^M F_{k_1,\ldots,k_m}^{(j_{k_1},\ldots,j_{k_m})}$ belong to the invariant subspace, are non-negative, and their L^1 norm is constant equal to 1 due to (2.1). Using (2.11) with $r = N - M$, we have that in the L^1 sense,

$$(\mathcal{P}^N f)(\cdot) - \sum_{\substack{m+k_1+\cdots+k_m \approx M \\ j_{k_1},\ldots,j_{k_m}}} f(t_{k_1,\ldots,k_m}^{(j_{k_1},\ldots,j_{k_m})})\beta^{-m-k_1-\cdots-k_m}(u_1(\cdot) + \mathcal{O}(|\lambda_2|^{N-M}))$$
$$= \mathcal{O}(L_f \beta^{-M}),$$

where the bounding constants appearing in the two errors are independent of N and M. Up to another error of order $\mathcal{O}(\beta^{-M})$, we may replace the Riemann sum with $\int_0^1 f(t)dt$. Hence, we have

$$\mathcal{P}^N f - u_1 \int_0^1 f(t)dt = \mathcal{O}(\|f\|_{L^\infty} |\lambda_2|^{N-M}) + \mathcal{O}(L_f \beta^{-M}), \quad N > M.$$

Given $N \gg 1$, we may choose an "optimal" M as a function of N such that

$$|\lambda_2|^{N-M} \sim \beta^{-M},$$

where $\sim$ means that they may differ by a numerical factor which is independent on N. If $n = 2$, then $|\lambda_2| = q\beta^{-2}$, hence we may choose M to be the integer part of x where x solves the equation

$$x \ln(\beta) = (N - x) \ln(\beta^2/q),$$

which gives $x = K_2 N$ with K_2 in (1.5).

Also, since $|\lambda_2| < 1/\beta$ for all n (see (2.9)), by choosing M to be the integer part of $N/2$, we see that the decay is always faster than $\beta^{-N/2}$.

3. Proof of Theorem 1.4

3.1. Proof of (i)

We only prove the result for $q > 1$. Let us first show that $j/\beta < x_j < (j+1)/\beta$ for all $0 \le j \le q-1$. The first inequality follows directly from the definition of x_j, while the second one is equivalent with

$$q\beta^{-2} + \cdots + q\beta^{-n} < \beta^{-1} \quad \text{or} \quad q\beta^{-1} + \cdots + q\beta^{-(n-1)} < 1,$$

the latter holds true by (1.2). This shows that ψ_0 is well defined on [0, 1] and by convention, it equals zero outside this interval.

In view of (1.4), if $q\beta^{-1} + \cdots + q\beta^{-(n-1)} < x < 1$, we have

$$(\mathcal{P}\psi_0)(x) = \beta^{-1} \sum_{j=0}^{q-1} \psi_0\Big(\frac{x+j}{\beta}\Big).$$

For x in that interval, we also have

$$q\beta^{-2} + \cdots + q\beta^{-n} + \frac{j}{\beta} = x_j < \frac{x+j}{\beta} < \frac{j+1}{\beta}, \quad 0 \le j \le q-1,$$

which from the definition of ψ_0 it implies

$$(\mathcal{P}\psi_0)(x) = \beta^{-1} \sum_{j=0}^{q-1} e^{2\pi i(x+j)/q} = \beta^{-1} e^{2\pi i x/q} \sum_{j=0}^{q-1} (e^{2\pi i/q})^j = 0.$$

If $0 < x < q\beta^{-1} + \cdots + q\beta^{-(n-1)}$, we have

$$(\mathcal{P}\psi_0)(x) = \beta^{-1} \sum_{k=0}^{q} \psi_0\Big(\frac{x+k}{\beta}\Big).$$

For x in the above interval, we also have

$$\frac{k}{\beta} < \frac{x+k}{\beta} < \frac{q}{\beta^2} + \cdots + \frac{q}{\beta^n} + \frac{k}{\beta} = x_k, \quad 0 \le k \le q,$$

which from the definition of ψ_0 it implies

$$(\mathcal{P}\psi_0)(x) = \beta^{-1} \sum_{k=0}^{q} e^{2\pi i(x+k)/(q+1)} = \beta^{-1} e^{2\pi i x/(q+1)} \sum_{k=0}^{q} (e^{2\pi i/(q+1)})^k = 0.$$

3.2. Proof of (ii) and (iii)

Let us show that $\widetilde{\mathfrak{K}} = u_1^{1/p} \mathfrak{K}(1/u_1^{1/p})$ is an isometry on $L^p([0, 1])$. If $p = \infty$ then this follows directly from the definition in (1.6). If $1 \le p < \infty$, we have (using (1.7) in the third equality)

$$\begin{aligned}\|\widetilde{\mathfrak{K}}(f)\|_{L^p}^p &= \int_0^1 |\widetilde{\mathfrak{K}}(f)|^p\, dx = \int_0^1 u_1 \mathfrak{K}\Big(\frac{|f|^p}{u_1}\Big)\, dx \\ &= \int_0^1 (\mathcal{P}u_1)\frac{|f|^p}{u_1}\, dx = \|f\|_{L^p}^p.\end{aligned}$$

The operator $\widetilde{\mathfrak{K}} - z\,\mathrm{Id} = u_1^{1/p}(\mathfrak{K} - z\,\mathrm{Id})u_1^{-1/p}$ is invertible if and only if $\mathfrak{K} - z\,\mathrm{Id}$ is invertible, hence $\widetilde{\mathfrak{K}}$ and $\mathfrak{K}$ have the same spectrum. Since $\widetilde{\mathfrak{K}}$ is an isometry, it is also injective, hence $\mathfrak{K}$ is injective, too.

Now, let us show that $\mathfrak{K}$ (thus also $\widetilde{\mathfrak{K}}$) is not surjective. Using (1.7) and the eigenvector ψ_0 of $\mathscr{P}$ constructed at point (i) (ψ_0 belongs to any $L^{p'}$ with $1 \le p' \le \infty$), we have

$$\int_0^1 \overline{\psi_0(x)}(\mathfrak{K}g)(x)\,dx = 0 \quad \text{for all } g \in L^p,\ 1 \le p \le \infty,$$

which implies that ψ_0 does not belong to the range of $\mathfrak{K}$. This also implies that $u_1^{1/p}\psi_0$ does not belong to the range of $\widetilde{\mathfrak{K}}$.

Thus, $\widetilde{\mathfrak{K}}$ is a non-surjective isometry and its spectrum must equal the closed unit disk due to the following result which may be found in [2, Proposition 5.2], but we also prove it here (in a more self-contained way) for the convenience of the reader.

Lemma 3.1. *Assume that U defined on some Banach space is a linear isometry. If U is surjective, then $\sigma(U) \subset \mathbb{S}^1$. If U is not surjective, then $\sigma(U) = \overline{\mathbb{D}}$.*

Proof. An isometry is always injective. Let us first consider the case when U is surjective (thus invertible). Using that $\|Uf\| = \|f\|$ for all f and also $\|U^{-1}g\| = \|U(U^{-1}g)\| = \|g\|$, we conclude that both U and U^{-1} have norm one. Let $z \in \mathbb{C}$ be with $|z| < 1$. Then $U - z\,\mathrm{Id} = (\mathrm{Id} - zU^{-1})U$ is invertible because $\|zU^{-1}\| < 1$. If $|z| > 1$, we have $U - z\,\mathrm{Id} = -(\mathrm{Id} - z^{-1}U)z$ which is also invertible. Thus, $\sigma(U)$ is included in the unit circle.

Now, let us consider the case when U is not surjective. Because $\|U\| = 1$, we know that $\sigma(U) \subset \overline{\mathbb{D}}$. Because U is not invertible, then $0 \in \sigma(U)$, hence $\sigma(U)$ has elements which are not on the unit circle. Thus, if the inclusion $\sigma(U) \subset \overline{\mathbb{D}}$ is strict, there must exist a point λ with $|\lambda| < 1$ which belongs to the boundary of $\sigma(U)$. We will now show that λ must be in the resolvent set of U, which would lead to a contradiction.

Since $\lambda \in \partial(\sigma(U))$, there must exist a sequence of points λ_n in the resolvent set of U such that $\lambda_n \to \lambda$ when $n \to \infty$. Since $|\lambda| < 1$, there exists $N > 1$ such that $|\lambda_n| \le (1 + |\lambda|)/2 < 1$ if $n > N$. Using the triangle inequality, we get

$$\|(U - \lambda_n\,\mathrm{Id})f\| \ge \|Uf\| - |\lambda_n|\|f\| \ge \frac{1 - |\lambda|}{2}\|f\|, \quad n > N.$$

Since $U - \lambda_n\,\mathrm{Id}$ is invertible, using this inequality with $f = (U - \lambda_n\,\mathrm{Id})^{-1}g$, we obtain

$$\|(U - \lambda_n\,\mathrm{Id})^{-1}\| \le \frac{2}{1 - |\lambda|}, \quad n > N.$$

This uniform bound and the identity

$$U - \lambda \,\mathrm{Id} = (\mathrm{Id} + (\lambda_n - \lambda)(U - \lambda_n \,\mathrm{Id})^{-1})(U - \lambda_n \,\mathrm{Id})$$

show that the right-hand side must be invertible if n is large enough, hence λ is in the resolvent set of U and cannot belong to the boundary of $\sigma(U)$. ■

3.3. Proof of (iv).

We know from (ii) that $u_1^{1/2}\mathcal{K}u_1^{-1/2}$ is an isometry on the Hilbert space $L^2([0,1])$. Then (1.7) implies that $\mathcal{P} = \mathcal{K}^*$ and

$$(u^{-1/2}\mathcal{P}u_1^{1/2})(u_1^{1/2}\mathcal{K}u_1^{-1/2}) = (u^{1/2}\mathcal{K}u_1^{-1/2})^*(u_1^{1/2}\mathcal{K}u_1^{-1/2}) = \mathrm{Id}\,. \tag{3.1}$$

The isometry $u_1^{1/2}\mathcal{K}u_1^{-1/2}$ has norm one. If $|z| < 1$, then ψ_z is different from zero and can be written with the help of a Neumann series. Finally,

$$\begin{aligned}\mathcal{P}\psi_z &= \mathcal{P}\psi_0 + \sum_{m\geq 1} z^m u_1^{1/2}(u_1^{-1/2}\mathcal{P}u_1^{1/2})(u_1^{1/2}\mathcal{K}u_1^{-1/2})^m u_1^{-1/2}\psi_0\\ &= \sum_{m\geq 1} z^m u_1^{1/2}(u_1^{1/2}\mathcal{K}u_1^{-1/2})^{m-1}u_1^{-1/2}\psi_0 = z\psi_z,\end{aligned}$$

where in the second equality we used $\mathcal{P}\psi_0 = 0$ and (3.1).

A. The greedy algorithm

Let $x \in [0,1)$. Applying the map T_β, we get that

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor \in [0,1),$$

where $\lfloor\,\cdot\,\rfloor$ is the floor function and $q < \beta = \beta_{n,q} < q+1$ in view of Lemma B.1 (i). By iterating the map T_β, we define the *j-th greedy coefficient* as

$$x_j := \lfloor \beta T_\beta^{(j-1)}(x) \rfloor \quad \text{for all } j \geq 1 \text{ with } T_\beta^0(x) := x. \tag{A.1}$$

The following lemma describes the greedy algorithm.

Lemma A.1. *With the definitions above, and with β as in (1.2), if $x \in [0,1)$ we have*

$$x = \sum_{j=1}^{\infty} x_j \beta^{-j}. \tag{A.2}$$

The scaled remainder $\beta^k\big(x-\sum_{j=1}^k x_j\beta^{-j}\big)$ *obeys*

$$\beta^k\Big(x-\sum_{j=1}^k x_j\beta^{-j}\Big)=T_\beta^k(x). \tag{A.3}$$

Moreover, the greedy coefficients satisfy three restrictions:

(1) $x_j\in\{0,1,\dots,q\}$ *for all* $j\ge 1$*;*

(2) $x_j=q$ *for* n *successive* j*'s cannot occur;*

(3) *it cannot happen that the sequence of* x_j*'s ends in the infinite sequence* $(c_1, c_2,\dots)$ *where* $c_{mn}=q-1$ *for all* $m\ge 1$*, and all the other* c_j*'s, with* j *not dividing* n*, are equal to* q*.*

Proof. (A.3) is true by definition for $k=1$. Assuming this equation for some $k\ge 1$, we have

$$\begin{aligned}T_\beta^{(k+1)}(x)&=T_\beta(T_\beta^k(x))=\beta T_\beta^k(x)-\lfloor\beta T_\beta^k\rfloor\\&=\beta^{k+1}\Big(x-\sum_{j=1}^k x_j\beta^{-j}\Big)-x_{k+1}\\&=\beta^{k+1}\Big(x-\sum_{j=1}^{k+1} x_j\beta^{-j}\Big).\end{aligned}$$

Since $T_\beta\colon[0,1)\to[0,1)$ and $\beta>1$, the series in (A.2) converges.

The first restriction on the x_j's follows from their definition:

$$0\le x_j=\lfloor\beta T_\beta^{(j-1)}(x)\rfloor\le\lfloor\beta\rfloor=q,$$

because of Lemma B.1 (i).

To prove the second restriction on the coefficients, suppose that there exists some $k\ge 0$ such that $x_{k+j}=q$, where $j\in\{1,\dots,n\}$. Using (1.2), we have

$$\sum_{j=1}^n q\beta^{-(k+j)}=\beta^{-k}.$$

If $k=0$, then $x\ge 1$, which is a contradiction. If $k\ge 1$, then using (A.3) and (A.2) we have

$$T_\beta^k(x)=\beta^k\Big(x-\sum_{j=1}^k x_j\beta^{-j}\Big)=\beta^k\Big(\sum_{j=k+1}^\infty x_j\beta^{-j}\Big)\ge\beta^k\Big(\sum_{j=k+1}^{n+k} q\beta^{-j}\Big)=1$$

contradicting $T_\beta\colon[0,1)\to[0,1)$.

In order to prove the third restriction, let us assume that there exists $x \in [0, 1)$ whose greedy expansion ends with $\beta^{-k} \sum_{j\geq 1} c_j \beta^{-j}$ for some $k \geq 0$, i.e. $x_{k+j} = c_j$ for $j \geq 1$. By repeatedly using (1.2) (see also Figure 2), we have

$$\begin{aligned} 1 &= \sum_{j=1}^{n-1} q\beta^{-j} + (q-1)\beta^{-n} + \beta^{-n} \\ &= \sum_{j=1}^{n-1} q\beta^{-j} + (q-1)\beta^{-n} + \beta^{-n}\Big(\sum_{j=1}^{n-1} q\beta^{-j} + (q-1)\beta^{-n}\Big) + \beta^{-2n} \\ &= \cdots = \sum_{j\geq 1} c_j \beta^{-j}, \end{aligned}$$

hence $x = \sum_{j=1}^{k} x_j \beta^{-j} + \beta^{-k}$ and thus by (A.3) $T_\beta^k(x) = 1$, contradiction. ∎

Lemma A.1 has shown that the greedy algorithm gives a unique output for the coefficients x_j defined in (A.1) for any number $x \in [0, 1)$, and these coefficients obey three necessary conditions. In the next lemma we will show, in particular, that any expansion for $x \in [0, 1)$ satisfying all these three conditions must be the greedy one.

Lemma A.2. *Suppose*

$$x = \sum_{j=1}^{\infty} \tilde{x}_j \beta^{-j} \tag{A.4}$$

where the coefficients $\tilde{x}_j \in \{0, 1, \ldots, q\}$ also satisfy the condition that no n consecutive coefficients equal q. Let $c_j = q - 1$ if n divides j, and $c_j = q$ otherwise. Let x_j be defined as in (A.1)*. Then one of the following possibilities occurs:*

(1) *$\tilde{x}_j = c_j$ for all j in which case $x = 1$;*

(2) *$x < 1$ with $\tilde{x}_j = x_j$ for all j, i.e., x is written in the greedy representation;*

(3) *$x < 1$ and there exists some $k \geq 1$ such that $\tilde{x}_j = x_j$ for $j < k$ (if $k \geq 2$), $\tilde{x}_k = x_k - 1$, and $\tilde{x}_{k+j} = c_j$ for $j \geq 1$. In this case, the finite sum $x = \sum_{j=1}^{k} x_j \beta^{-j}$ is the greedy representation of x which is different from* (A.4).

Proof. The largest possible value of $\sum_{j=1}^{\infty} \tilde{x}_j \beta^{-j}$, which can be achieved with the $\tilde{x}_j$ obeying the two restrictions of the current lemma, equals 1. This is the case if and only if $\tilde{x}_j = c_j$, for all j.

Assuming $x < 1$, suppose that the sequence $(\tilde{x}_1, \tilde{x}_2, \ldots)$ does not end in the infinite sequence $(c_1, c_2, \ldots)$ so that the scaled remainder, $\beta^k \sum_{j=k+1}^{\infty} \tilde{x}_j \beta^{-j} < 1$ for all $k \geq 1$ (we have already assumed this for $k = 0$). Then $\tilde{x}_j = x_j$ for all j: to see this, we have $x_1 = \lfloor \beta x \rfloor$ and $\beta x = \tilde{x}_1 + \beta \sum_{j=2}^{\infty} \tilde{x}_j \beta^{-j} = \tilde{x}_1 + t$ with $t \in [0, 1)$. Thus, $x_1 = \tilde{x}_1$. A simple induction gives $x_j = \tilde{x}_j$ for all j.

On the other hand, suppose k is the first integer such that $\beta^k \sum_{j=k+1}^{\infty} \tilde{x}_j \beta^{-j} = 1$. Then $\tilde{x}_{k+j} = c_j$, $j \geq 1$ and $\tilde{x}_j = x_j$, $j < k$, $\tilde{x}_k + 1 = x_k \leq q$. Thus, $\tilde{x}_k \leq q - 1$. If $\tilde{x}_k = q - 1$, the previous (if there are that many) $n - 1$ $\tilde{x}_j$'s cannot equal q because that would violate the definition of k. Thus, $x = \sum_{j=1}^{k} x_j \beta^{-j}$, the greedy representation, is a different representation of x. ∎

B. Properties of $\beta_{n,q}$

The following lemma is given for the sake of the reader and collects in one place a number of known results [5, 13].

Lemma B.1. *Let $n, q \in \mathbb{N}$ with $n \geq 2$ and $1 \leq q$. Let*

$$P_{n,q}(z) = z^n - q(z^{n-1} + z^{n-2} + \cdots + z + 1)$$

with $z \in \mathbb{C}$.

(i) *$P_{n,q}$ has only one positive root $\beta_{n,q}$, which also obeys $q < \beta_{n,q} < q + 1$.*

(ii) *All roots have algebraic multiplicity one.*

(iii) *The other roots of $P_{n,q}$ satisfy $(q/(q+2))^{1/n} < |z| < 1$. In particular, $\beta_{n,q}$ is a Pisot number.*

(iv) *Fix $\alpha \in (q, q+1)$. Then there exists $n_0 \geq 2$ such that $(q+1) - q\alpha^{-n} \leq \beta_{n,q} < q + 1$ for all $n \geq n_0$.*

Proof. (i) If $x > 0$, we define $f(x) := x^{-n} P_{n,q}(x) = 1 - q(x^{-1} + \cdots + x^{-n})$. We have that $f' > 0$, which means that it can have at most one positive root.

If $q = 1$, we have

$$f(1) = 1 - n < 0, \quad f(2) = 2^{-n} > 0$$

hence there exists a unique, simple root between 1 and 2.

For $q > 1$, we have

$$f(q) = 1 - \frac{1 - q^{-n}}{1 - q^{-1}} = \frac{q^{-n} - q^{-1}}{1 - q^{-1}} < 0,$$
$$f(q+1) = 1 - \frac{q}{q+1} \frac{1 - (q+1)^{-n}}{1 - (q+1)^{-1}} = (q+1)^{-n} > 0;$$

thus, there always exists a unique positive root $\beta_{n,q} \in (q, q+1)$.

(ii) Now, let us prove that all the other roots are also simple. If $z \neq 1$, we have

$$P_{n,q}(z) = z^n - q\frac{z^n - 1}{z - 1} = \frac{z^{n+1} - (q+1)z^n + q}{z - 1} =: \frac{Q_{n,q}(z)}{z - 1}.$$

Since $z = 1$ is not a root, $P_{n,q}(z)$ has the same roots (those different from 1) as $Q_{n,q}(z)$. If $z_1 \neq 1$ is a degenerate root of $P_{n,q}$, i.e., $P_{n,q}(z_1) = P'_{n,q}(z_1) = 0$, then we also have $Q_{n,q}(z_1) = Q'_{n,q}(z_1) = 0$. But

$$Q'_{n,q}(z) = (n+1)z^n - (q+1)nz^{n-1} = (n+1)z^{n-1}\Big(z - \frac{(q+1)n}{n+1}\Big)$$

and since 0 is not a root, we must have $z_1 = (q+1)n/(n+1)$, which is positive. But we know that $P_{n,q}$ only has a non-degenerate positive root, which is a contradiction.

(iii) We want to show that $Q_{n,q}$ has exactly n roots inside the closed unit complex disk. Let $F(z) = z^{n+1} + q$ and $G(z) = -(q+1)z^n$. If $|z| = 1+\varepsilon$ with $\varepsilon > 0$ small, we have

$$|F(z)| \le q + 1 + (n+1)\varepsilon + \mathcal{O}(\varepsilon^2), \quad |G(z)| = (q+1)(1+n\varepsilon) + \mathcal{O}(\varepsilon^2),$$

and since $nq > 1$, we have that $|G(z)| > |F(z)|$ on $|z| = 1+\varepsilon$ if ε is small enough. This implies that the function

$$H_t(z) := tF(z) + G(z), \quad H_0(z) = G(z), \quad H_1(z) = Q_{n,q}(z)$$

obeys $|H_t(z)| \ge |G(z)| - |F(z)| > 0$ on the circle $|z| = 1+\varepsilon$ for all $t \in [0,1]$. Thus, the number of zeros of H_t inside the disk $|z| \le 1+\varepsilon$ is constant in t and equals n. Taking the limit $\varepsilon \downarrow 0$, we conclude that $Q_{n,q}$ has exactly n zeros inside the complex closed unit disk. Now, if z is a zero with $|z| = 1$, we have

$$|z^{n+1} + q| = (q+1)|z^n| = q+1$$

which is possible only for $z^{n+1} = 1$. But then $(q+1)z^n = q+1$, hence $z^n = 1$. This implies that $z = 1$. Hence, $P_{n,q}$ has exactly $n-1$ complex roots inside the open unit disk.

Now, let z_1 be such a root with $|z_1| < 1$ and $Q_{n,q}(z_1) = 0$. Then

$$(q+1)|z_1|^n = |(q+1)z_1^n| \ge q - |z_1|^{n+1} > q - |z_1|^n$$

which leads to

$$|z_1|^n > \frac{q}{q+2}.$$

(iv) Fix any $n_0 \ge 2$ and let $n \ge n_0$. We have

$$\frac{1}{\beta_{n,q}} + \cdots + \frac{1}{\beta_{n,q}^{n_0}} \le \frac{1}{\beta_{n,q}} + \cdots + \frac{1}{\beta_{n,q}^{n}} = \frac{1}{q},$$

hence $\beta_{n,q} \ge \beta_{n_0,q}$. Also, $Q_{n,q}(\beta_{n,q}) = 0$, hence $\beta_{n,q}$ solves $\beta_{n,q} = q + 1 - q/\beta_{n,q}^n$. Thus,

$$q + 1 - \frac{q}{\beta_{n_0,q}^n} \le \beta_{n,q} < q+1, \quad 2 \le n_0 \le n.$$

Now, we can choose n_0 large enough such that $\beta_{n_0,q} > \alpha$ and we are done. ∎

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References

[1] A. Abdesselam, The weakly dependent strong law of large numbers revisited. *Grad. J. Math.* **3** (2018), no. 2, 94–97 Zbl 1497.60030 MR 3903054

[2] R. F. Allen and F. Colonna, Isometries and spectra of multiplication operators on the Bloch space. *Bull. Aust. Math. Soc.* **79** (2009), no. 1, 147–160 Zbl 1163.47027 MR 2486890

[3] F. Blanchard, β-expansions and symbolic dynamics. *Theoret. Comput. Sci.* **65** (1989), no. 2, 131–141 Zbl 0682.68081 MR 1020481

[4] A. Boyarsky and P. Góra, *Laws of chaos*. Invariant measures and dynamical systems in one dimension. Probab. Appl., Birkhäuser, Boston, MA, 1997 Zbl 0893.28013 MR 1461536

[5] A. Brauer, On algebraic equations with all but one root in the interior of the unit circle. *Math. Nachr.* **4** (1951), 250–257 Zbl 0042.01501 MR 0041975

[6] É. Charlier, C. Cisternino, and K. Dajani, Dynamical behavior of alternate base expansions. *Ergodic Theory Dynam. Systems* **43** (2023), no. 3, 827–860 Zbl 1525.11009 MR 4544145

[7] K. Dajani and C. Kalle, *A first course in ergodic theory*. Chapman & Hall/CRC, Boca Raton, FL, 2021 Zbl 1494.37001 MR 4701087

[8] P. Góra, Invariant densities for generalized β-maps. *Ergodic Theory Dynam. Systems* **27** (2007), no. 5, 1583–1598 Zbl 1123.37015 MR 2358979

[9] I. W. Herbst, J. Møller, and A. M. Svane, How many digits are needed? *Methodol. Comput. Appl. Probab.* **26** (2024), no. 1, article no. 5 Zbl 1539.60035 MR 4702653

[10] I. W. Herbst, J. Møller, and A. M. Svane, The asymptotic distribution of the scaled remainder for pseudo golden ratio expansions of a continuous random variable. *Methodol. Comput. Appl. Probab.* **27** (2025), no. 1, article no. 10 Zbl 08009211 MR 4858127

[11] V. Komornik, P. Loreti, and M. Pedicini, A quasi-ergodic approach to non-integer base expansions. *J. Number Theory* **254** (2024), 146–168 Zbl 1530.11010 MR 4636755

[12] A. Lasota and J. A. Yorke, On the existence of invariant measures for piecewise monotonic transformations. *Trans. Amer. Math. Soc.* **186** (1973), 481–488 (1974) Zbl 0298.28015 MR 0335758

[13] E. P. Miles, Jr., Generalized Fibonacci numbers and associated matrices. *Amer. Math. Monthly* **67** (1960), 745–752 Zbl 0103.27203 MR 0123521

[14] W. Parry, On the β-expansions of real numbers. *Acta Math. Acad. Sci. Hungar.* **11** (1960), 401–416 Zbl 0099.28103 MR 0142719

[15] M. Pedicini, Greedy expansions and sets with deleted digits. *Theoret. Comput. Sci.* **332** (2005), no. 1-3, 313–336 Zbl 1080.11009 MR 2122508

[16] A. Rényi, Representations for real numbers and their ergodic properties. *Acta Math. Acad. Sci. Hungar.* **8** (1957), 477–493 Zbl 0079.08901 MR 0097374
[17] S. Suzuki, Eigenfunctions of the Perron–Frobenius operators for generalized beta-maps. *Dyn. Syst.* **37** (2022), no. 1, 9–28 Zbl 1505.37038 MR 4408074
[18] P. Walters, Equilibrium states for β-transformations and related transformations. *Math. Z.* **159** (1978), no. 1, 65–88 Zbl 0357.28014 MR 0466492

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Horia D. Cornean
Department of Mathematical Sciences, Aalborg University, Thomas Manns Vej 23, 9220 Aalborg, Denmark; cornean@math.aau.dk

Ira W. Herbst
Department of Mathematics, University of Virginia, 141 Cabell Drive, Kerchof Hall, Charlottesville, VA 22904, USA; iwh@virginia.edu

Giovanna Marcelli
Dipartimento di Matematica e Fisica, Università di Roma Tre, L.go S. L. Murialdo 1, 00146 Roma, Italy; giovanna.marcelli@uniroma3.it