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Spectral instability of random Fredholm operators

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Abstract. If $A: D(A) \subset \mathcal{H} \to \mathcal{H}$ is an unbounded Fredholm operator of index 0 on a Hilbert space $\mathcal{H}$ with a dense domain $D(A)$, then its spectrum is either discrete or the entire complex plane. This spectral dichotomy plays a central role in the study of *magic angles* in twisted bilayer graphene. This paper proves that if such operators (with certain additional assumptions) are perturbed by certain random trace-class operators, their spectrum is discrete with high probability.

1. Introduction

This article describes the spectrum of random perturbations of Fredholm operators of index 0. If $A: D(A) \subset \mathcal{H} \to \mathcal{H}$ is an unbounded Fredholm operator of index 0 on a separable Hilbert space $\mathcal{H}$ with dense domain $D(A)$, then $\operatorname{Spec}(A)$ is either discrete or the entire complex plane (see Proposition 2.1). Here we denote $\operatorname{Spec}(A)$ as the spectrum of a linear operator on a Hilbert space, defined as the complement of the set of points $z \in \mathbb{C}$ such that $A - z$ is bijective.

A striking example of this dichotomy was presented in [23], where Seeley introduced a particularly simple family of operators with this property, defined as

$$A_{\text{Seeley}}: H^1(\mathbb{R}/2\pi\mathbb{Z}) \ni f \mapsto a(x)\partial_x f(x) + b(x) f(x) \in L^2(\mathbb{R}/2\pi\mathbb{Z}) \tag{1.1}$$

for $a, b \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ and $|a| > 0$. The spectrum of A_{Seeley} is either a discrete lattice, empty, or $\mathbb{C}$ depending on a and b.

While the case where $\operatorname{Spec}(A) = \mathbb{C}$ was considered pathological, the recent study of physical models in *twisted bilayer graphene* (TBG) have shown this case to be physically highly relevant. Indeed, this spectral dichotomy appears in the mathematical study of TBG in the so-called *chiral limit*, see [3,29] for details (which is the main motivation for this article). In this model, a Fredholm operator of index 0 is defined with a parameter $\beta \in \mathbb{C}$. The theory of TBG in the chiral limit defines *magic* twisting angles as the α such that the spectrum of the operator is $\mathbb{C}$ (this is further described in Section 1.1).

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For any Fredholm operator of index 0, it is easy to construct an arbitrarily small perturbation to make the spectrum of such an operator discrete (by mapping the kernel to the cokernel). The question that this paper aims to address is the following.

When is the spectrum of a random perturbation of Fredholm operators of index zero on a Hilbert space discrete?

The perturbations considered in this article are random trace-class perturbations. These types of random perturbations appear when numerically analyzing such operators. For example, if we restrict an operator A to an N-dimensional vector space[1] and project the image onto this same vector space, then we can easily compute the spectrum numerically. As we increase N, rounding errors in mathematical software can be modeled by finite-rank random perturbations (see also [6]).

In this paper (under certain conditions on the random perturbation), we provide a quantitative lower bound for the smallest singular value of the randomly perturbed operator (with certain probability). As a by-product, we see that these randomly perturbed operators have a discrete spectrum with high probability. We then apply our findings to the TBG model (see Theorem 1). A heuristic version of our main theorem is stated below; for a more precise version, see Theorem 2.

Theorem (Heuristic main result). *Suppose $A - z$ is a Fredholm operator of index* 0 *for all $z \in \Omega \subset \mathbb{C}$ (an open set) and Q_ω is a suitable random trace class perturbation (see* (2.1)*), then for sufficiently small $\delta > 0$, the set* $\operatorname{Spec}(A + \delta Q_\omega) \cap \Omega$ *is discrete with high probability.*

The general framework of our proof follows the works of Hager, Sjöstrand, and the authors [13, 15, 17, 30]. However, in all of those cases, the proofs rely on the semiclassical ellipticity of the symbol of the operator being studied. In this paper, we will work under more general assumptions of the operator to randomly perturb. In the case the operator is a pseudo-differential operator, we allow the principal symbol to be *not* semiclassicaly elliptic.

The spectral stability of our main result can be characterized more abstractly in the following way. Recall that $\lambda \in \mathbb{C}$ is a normal eigenvalue of A if $\lambda \in \operatorname{Spec}(A)$ is isolated and the kernel of $A - \lambda$ is finite dimensional. We then define the discrete spectrum of A as

$$\operatorname{Spec}_{\mathrm{disc}}(A) := \{\lambda \in \mathbb{C} : \lambda \text{ is a normal eigenvalue of } A\}$$

and its essential spectrum

$$\operatorname{Spec}_{\mathrm{ess}}(A) := \operatorname{Spec}(A) \setminus \operatorname{Spec}_{\mathrm{disc}}(A).$$

[1] For example, we could approximate $H^1(\mathbb{R}/2\pi\mathbb{Z})$ by the first N Fourier modes.

If A is a self-adjoint operator, and S is a relatively A-compact operator, then it is a classical theorem due to Weyl [31] that

$$\mathrm{Spec}_{\mathrm{ess}}(A) = \mathrm{Spec}_{\mathrm{ess}}(A + S).$$

It is well known that this does not hold in general for non-self-adjoint operators A. In fact, for non-self-adjoint operators A, there exists a maximal set [12] $\mathrm{Spec}_{\mathrm{ess,stab}}(A)$ contained in $\mathrm{Spec}_{\mathrm{ess}}(A)$ that is invariant under compact perturbations

$$\mathrm{Spec}_{\mathrm{ess,stab}}(A) = \bigcap_{S \text{ compact on } \mathcal{H}} \mathrm{Spec}(A + S).$$

One can verify that

$$\mathrm{Spec}_{\mathrm{ess,stab}}(A) = \{\lambda : A - \lambda \text{ is not a Fredholm operator of index } 0\}. \tag{1.2}$$

We recall that a Fredholm operator on a Hilbert space $\mathcal{H}$ is a closed linear operator $A\colon D(A) \subset \mathcal{H} \to \mathcal{H}$ such that $\mathrm{ran}(A)$ is closed and both its kernel and its cokernel are finite dimensional. Here, $D(A)$ is a dense linear subspace of $\mathcal{H}$ that is complete with respect to the norm $\|x\|_{D(A)} := \sqrt{\|x\|^2 + \|Ax\|^2}$.

In the case of self-adjoint operators A, Weyl's theorem gives

$$\mathrm{Spec}_{\mathrm{ess,stab}}(A) = \mathrm{Spec}_{\mathrm{ess}}(A).$$

Equation (1.2) raises the question about the stability of $\mathrm{Spec}_{\mathrm{ess}}(A) \setminus \mathrm{Spec}_{\mathrm{ess,stab}}(A)$ under generic perturbations. By definition, this part of the spectrum is not stable under all compact perturbations, but is it stable under suitable random perturbations of A?

This article considers random perturbations of Fredholm operators of index 0. As we shall see in the following, the spectrum of such operators satisfies a dichotomy on the set of z such that $A - z$ is Fredholm of index 0. We aim to provide quantitative estimates on the stability of the essential spectrum by estimating the probability the smallest singular value of $A - z$ is not too small.

Question 1.1. *How stable is the essential spectrum of non-self-adjoint Fredholm operators of index* 0 *under random trace-class perturbations?*

More specifically, for a fixed separable Hilbert space $\mathcal{H}$ with a dense subset $D(A)$, we consider unbounded Fredholm operators $A\colon D(A) \subset \mathcal{H} \to \mathcal{H}$ with index 0. For such operators, let

$$\rho_F(A) := \{z \in \mathbb{C} : A - z \text{ is a Fredholm operator}\}$$

denote the Fredholm domain of A. Because the Fredholm index is constant with respect to small perturbations (see for instance [9, Theorem C.5]), the Fredholm index is constant $\operatorname{ind}(A - z') = \operatorname{ind}(A - z)$ for z, z' in the same connected component of $\rho_F(A)$ and $\rho_F(A)$ is an open set. We then define the open set

$$\rho_F^{(0)}(A) := \{z \in \mathbb{C} : A - z \text{ is a Fredholm operator of index } 0\}. \tag{1.3}$$

Note that if A is Fredholm of index zero, then trivially $0 \in \rho_F^{(0)}(A)$.

Analytic Fredholm theory, see Proposition 2.1, shows that for each connected component $D \subset \rho_F^{(0)}(A)$, either $D \subset \operatorname{Spec}(A)$ or $\operatorname{Spec}(A) \cap D$ is discrete. We can always add an arbitrarily small (ad hoc) finite rank perturbation to A so that the second case holds.[2] The main result of this paper proves that the second case is generic (in a certain sense, but unlike the ad hoc perturbation), so that the stability in Question 1.1 fails dramatically.

As a random perturbation, we consider quasi-finite-rank random perturbations of the matrix elements of the operator. Although often not indicative of a physical noise profile, it is motivated by numerical algorithms, where one often considers finite-rank approximations of the full operator. Thus, a take on Question 1.1 from the perspective of a numerical analyst could be whether a noisy finite-rank implementation of a Fredholm operator of index 0 can still be expected to show traces of non-discrete spectra? As our main theorem and the illustration in Figure 2 show, the answer is negative.

1.1. The motivating example: Twisted bilayer graphene

A representative of the family of operators that motivated this paper appears in the mathematical study of *twisted bilayer graphene* (TBG) in the so-called *chiral limit,* see [3] for details. The operator is defined as

$$D_h(\beta) := \begin{pmatrix} 2hD_{\bar z} & \beta U(z) \\ \beta U(-z) & 2hD_{\bar z} \end{pmatrix} : H^1(\mathbb{C}/\Gamma; \mathbb{C}^2) \to L^2(\mathbb{C}/\Gamma; \mathbb{C}^2) \tag{1.4}$$

where $\beta \in \mathbb{C}$, $|\beta| = 1$ is the coupling parameter, $D_{\bar z} := (2i)^{-1}(\partial_{\operatorname{Re} z} + i\partial_{\operatorname{Im} z})$,

$$U(z) := \sum_{k=0}^{2} \omega^k e^{(z\bar\omega^k - \bar z\omega^k)/2},$$

$\omega := e^{2\pi i/3}$, $\Gamma := 4\pi(i\omega\mathbb{Z} \oplus i\omega^2\mathbb{Z})$, and $h \in \mathbb{R}_{>0}$ is proportional to the twisting angle between two stacked layers of graphene. It is proven in [3] that there exists a

[2] Indeed, without loss of generality, suppose that $0 \in D$. Then if the kernel of A has an orthonormal basis $\{u_i\}_{i=1}^N$, and the cokernel of A has an orthonormal basis $\{v_i\}_{i=1}^N$, then $A + \delta\sum_1^N v_i \otimes u_i$ is invertible for every $\delta > 0$, so that $0 \notin \operatorname{Spec}(A + \delta\sum_1^N v_i \otimes u_i)$ and thus spectrum of the perturbation of A is discrete in D.

discrete set $\mathcal{M} \subset \mathbb{C}$ such that

$$\operatorname{Spec}_{L^2(\mathbb{C}/\Gamma)} D_h(\beta) = \begin{cases} \mathbb{C}, & \beta/h \in \mathcal{M}, \\ \Gamma^*, & \beta/h \notin \mathcal{M}, \end{cases}$$

where Γ^*, the dual lattice of Γ, is a discrete set.[3] In the theory of twisted bilayer graphene, for fixed β, the twisting angle h is called *magic* if $\operatorname{Spec}(D_h(\beta)) = \mathbb{C}$. The set $\mathcal{M}$ has no explicit description, however it can be described as the set of eigenvalues of a certain auxiliary compact operator (see [3, Theorem 2] for details).

A consequence of our main result (Theorem 2) is that $D_h(\beta)$ does not exhibit any magic angles (with overwhelming probability) if perturbed by a suitable small random perturbation, as described below.

Theorem 1 (Application to TBG). *Suppose $\chi(z,\zeta) \in C_0^\infty(T^*(\mathbb{C}/\Gamma); [0,1])$ is identically 1 for $|\zeta| < C$, for some sufficiently large C, and*

$$Q_\omega := \begin{pmatrix} \operatorname{Op}_h(\chi) & 0 \\ 0 & \operatorname{Op}_h(\chi) \end{pmatrix} \circ \Big(\sum_{j,k} \alpha_{j,k} e_j \otimes e_k \Big) \circ \begin{pmatrix} \operatorname{Op}_h(\chi) & 0 \\ 0 & \operatorname{Op}_h(\chi) \end{pmatrix}$$

where $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{C}/\Gamma;\mathbb{C}^2)$, $\alpha_{j,k}$ are i.i.d. complex Gaussian random variables with mean 0 and variance 1, and $\operatorname{Op}_h(\chi)$ is the Weyl quantization of χ (see Definition A.4). Then for fixed β, if $0 < \delta < h^\kappa$, with $\kappa > 2$, $D_h(\beta) + \delta Q_\omega$ has discrete spectrum with probability at least

$$1 - C_1 e^{-C_2/h^{2\kappa}}$$

for positive constants C_1 and C_2 (i.e., with overwhelming probability).

In Figures 1 and 2 we provide numerical evidence supporting Theorem 1. In Figure 1, we compute the spectrum of a finite matrix truncation of $D_h(\beta)$ at a fixed magic angle (top left plot). The accumulation of eigenvalues near the origin should be interpreted as a finite-rank analogue of having the entire complex plane as spectrum. We then compute the spectrum at the same h and β, but with a small random perturbation. We observe that there is no accumulation of eigenvalues near the origin, suggesting that the magic angles have been washed out by randomness.

In Figure 2, we attempt to quantify this finite-rank analogue of having the entire complex plane as spectrum. We fix $\beta = 1$, vary h between 0 and 2.5, and measure the density of eigenvalues near the origin. Any spikes in the eigenvalues should correspond to magic angles. We then notice two spikes (corresponding to the first two

[3] The dual lattice of Γ is by definition all $k \in \mathbb{C}$ such that $(\gamma\bar{k} + \bar{\gamma}k) \in 4\pi\mathbb{Z}$ for all $\gamma \in \Gamma$, which can be explicitly computed as $\Gamma^* = 3^{-1/2}(\omega\mathbb{Z} \oplus \omega^2\mathbb{Z})$.

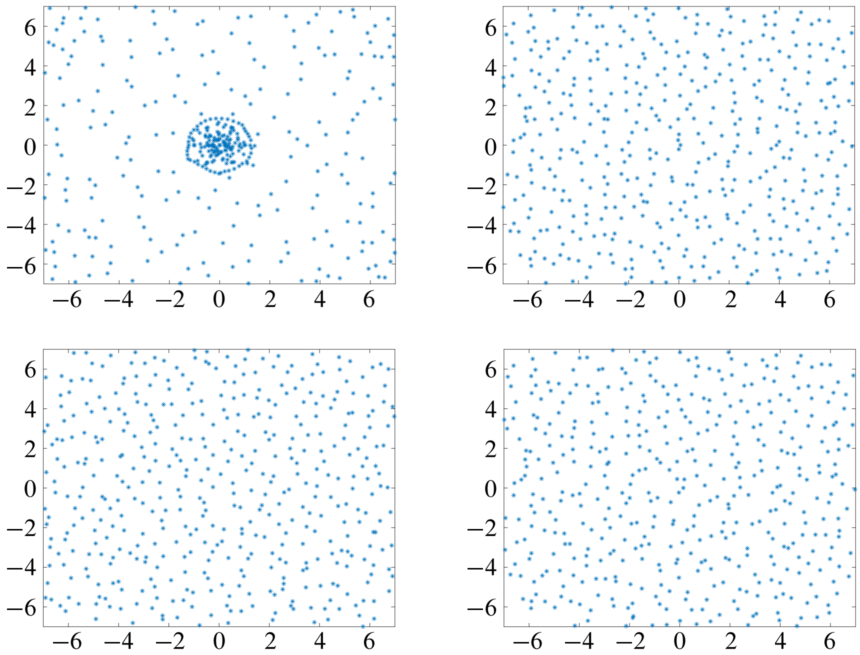

Figure 1. Spectrum of smallest 600 eigenvalues of finite matrix truncation of $D_h(\beta)$, with matrix size 13122, for largest magic h and $\beta = 1$ (top left). The accumulation of eigenvalues since $\mathrm{Spec}(D_h(\beta)) = \mathbb{C}$ in the center is clearly visible. Spectrum of finite matrix truncation of $D_h(\beta)$, with same h, β and random $\delta = 0, 0.01, 10^{-4}, 10^{-7}$ perturbation (clockwise). The accumulation of eigenvalues in the center gets resolved immediately.

magic angles) as shown in the first plot. We then do the same computation, but with a random perturbation added, and observe no spikes in eigenvalues.

Spectral analysis of random perturbations of non-self-adjoint operators is a rich field within the random matrix theory community. Indeed, Davies and Hager described the spectrum of randomly perturbed Jordan matrices [7]. Similarly, Guionnet, Wood, and Zeitouni described the empirical measure of eigenvalues of certain randomly perturbed non-self-adjoint operators [11]. Basak, Paquette, and Zeitouni considered random perturbations of banded and twisted Toeplitz matrices [1,2]. Sjöstrand and the author further described similar spectral properties for Toeplitz matrices and Toeplitz matrices [25, 26]. In various settings, the limiting spectral measure of randomly perturbed non-self-adjoint operators have been described. See [14] who considered perturbations of $hD_x + g(x)$, [30] who considered perturbations of quantizations of tori, and [17] who considered perturbations of quantizations of Kähler manifolds.

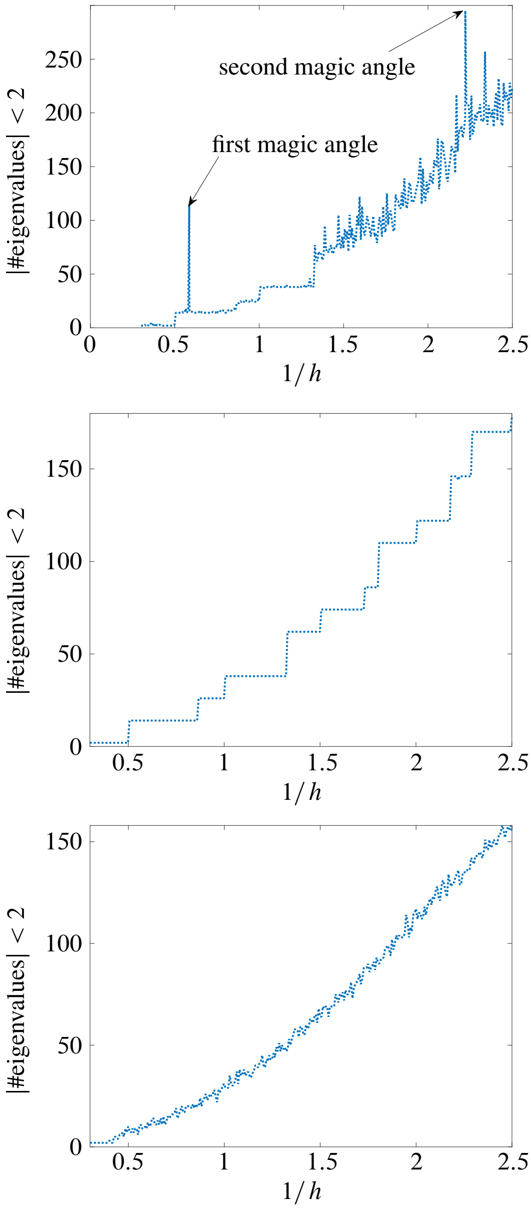

Figure 2. Counting the number of eigenvalues of a finite rank approximation of $D_h(1)$ (defined in (1.4)) for different h in a ball of radius 2 with random perturbations of size $\delta = 0, 10^{-5}, 10^{-1}$ from left to right. Magic angles at $1/h \approx 0.586, 2.221$ are clearly visible as spikes in the unperturbed figure (left) but washed out in the perturbed ones (center and right).

Outline of the paper. In Section 2 we provide the relevant background to state the general result of this paper (Theorem 2) as well as a more quantitative result (Theorem 3). In Section 3 we provide a proof of these results. Then in Section 4, we discuss the application of Theorem 2 to an operator appearing in the study of magic angles for twisted bilayer graphene, proving Theorem 1. This analysis requires the already known quantization procedure for matrix-valued functions on $\mathbb{C}/\Gamma$, which we felt worthwhile to outline separately in Appendix A.

Notation. For u, v elements of a Hilbert space, the operator $u \otimes v$ operates on $w \in \mathcal{H}$ by

$$(u \otimes v)w = u\langle w, v\rangle.$$

For a function f depending on a positive parameter $h > 0$ and $M \in \mathbb{N}$, we write $f = O(h^M)$ if there exist $h_0 > 0$ and $C_M > 0$ such that $|f| \leq C_M h^M$ for all $0 < h < h_0$. We write $f = O(h^\infty)$ if $f = O(h^M)$ for all $M \in \mathbb{N}$.

2. Setup of problem and main result

For the remainder of this paper, we let $\mathcal{H}$ denote a fixed separable Hilbert space. We start with a simple observation about the operators we consider.

Proposition 2.1. *Let A be a closed densely defined linear operator $A: D(A) \subset \mathcal{H} \to \mathcal{H}$ and $D \subset \rho_F^{(0)}(A)$ (defined in (1.3)) be a connected component. Then either $D \subset$* Spec(A) *or $D \cap$* Spec(A) *is discrete.*

Proof. Consider a connected component $D \subset \rho_F^{(0)}(A)$. By assumption, the Fredholm index of $A - z$ is zero for all $z \in D$. By analytic Fredholm theory, the proposition follows. Indeed, one can follow [9, Theorem C.8] which constructs a holomorphic function $f(z)$ with the property $f(z) = 0 \iff z \in \operatorname{Spec}(A)$. We note that in [9, Theorem C.8], A is assumed to be bounded, however the proof follows without modification for an unbounded A. ∎

We note that for $j \neq 0$, $\rho_F^{(j)}(A) \subset \operatorname{Spec}(A)$. This trivially follows by observing that if $z \in \rho_F^{(j)}(A)$ (for $j \neq 0$), then the kernel or cokernel of $A - z$ is non-trivial and thus $A - z$ is not bijective.

2.1. Random perturbation

Here we discuss the random perturbations considered in this work. Let $\{e_j^1 : j \in \mathbb{N}\}$ and $\{e_j^2 : j \in \mathbb{N}\}$ be two orthonormal bases of $\mathcal{H}$, and define

$$Q_\omega := S_1 \circ \Big(\sum_{j,k=1}^{\infty} \alpha_{j,k} e_j^1 \otimes e_k^2 \Big) \circ S_2 \tag{2.1}$$

where S_1 and S_2 are fixed Hilbert–Schmidt operators and $\alpha_{j,k}$ are independent identically distributed complex Gaussian random variables with mean 0 and variance 1. To make sense of the sum in (2.1), we decompose S_1 and S_2 using a singular value decomposition and use that the law of $(\alpha_{j,k})$ is invariant under unitary transformations, to rewrite Q_ω as

$$Q_\omega = \sum_{j,k} s_j^1 s_k^2 \alpha_{j,k} e_j^1 \otimes e_j^2$$

where $s_j^{1,2}$ are the singular values[4] of S_1 and S_2 respectively and $e_j^{1,2}$ are two potentially different (from the ones above) orthonormal bases of $\mathcal{H}$. Each term in this sum is an independent complex Gaussian random variable of mean zero and variance $s_j^1 s_k^2$. It can be shown that with probability 1, the Hilbert–Schmidt norm of Q_ω is finite (this follows from Lemma 3.1 and continuity of measures). And under the additional assumption that S_1 or S_2 is trace-class (we will assume in our main result that S_1 is trace-class), then Q_ω is almost surely trace-class (see (3.1)).

To analyze $A + \delta Q_\omega$, we require certain elliptic-type properties of S_1 and S_2. Because A is closed and densely defined, the adjoint of $A - z$ (for any fixed $z \in \mathbb{C}$), denoted by $(A - z)^*$ has dense domain

$$\begin{aligned} D((A-z)^*) := \{u \in \mathcal{H} : \text{there exists} v \in \mathcal{H} \text{ such that} \\ \langle (A-z)w, u\rangle = \langle w, v\rangle \text{ for all } w \in D((A-z))\}, \end{aligned}$$

see for instance [19, Theorem VIII.1]. We may then define the operator

$$(A-z)(A-z)^*$$

with domain

$$D((A-z)(A-z)^*) = \{u \in D((A-z)^*) : (A-z)u \in D((A-z))\}. \tag{2.2}$$

[4] If S is a Hilbert–Schmidt operator, we say $\{s_j\}_{j\in\mathbb{N}}$ are the *singular values* of S if s_j are the eigenvalues of $\sqrt{S^*S}$. Although not necessarily here, we assume s_j are decreasing to zero as $j \to \infty$.

A result of von Neumann (see for instance [16, Theorem 3.24]) states that the operator $(A-z)(A-z)^*$ with this domain is self-adjoint. For the following hypothesis to make sense, we assume that

$$\text{there exists } \tilde{z} \in \mathbb{C} \text{ such that } \operatorname{Spec}((A-\tilde{z})(A-\tilde{z})^*) \text{ is discrete.} \tag{2.3}$$

This assumption can be relaxed to $\operatorname{Spec}((A-\tilde{z})(A-\tilde{z})^*)$ being discrete near 0. We stress here that (2.3) is much weaker than $A-\tilde{z}$ having discrete spectrum (which we aim to prove for a perturbation of A). See Section 4 where we prove (2.3) for the operator $D_h(\beta)$ coming from TBG.

Our proof relies on constructing an auxiliary operator which has components projecting onto the singular vectors of $(A-z)$, which uses that the spectrum of $(A-z)(A-z)^*$ is discrete. If we only have that the spectrum of $(A-z)(A-z)^*$ is discrete near zero, we can still build an auxiliary operator by using spectral projectors, at the cost of requiring finer analysis. We will leave this added generality to future work.

Hypothesis 1. For $\alpha > 0$ and $\tilde{z} \in \mathbb{C}$ satisfying (2.3), we denote by $\{e_1, \dots, e_N\}$ and $\{f_1, \dots, f_N\}$ the eigenvectors of $(A-\tilde{z})(A-\tilde{z})^*$ and $(A-\tilde{z})^*(A-\tilde{z})$ (equipped with an analogously defined domain to (2.2)) with eigenvalues less than α, respectively. Let

$$\mathcal{H}_{1,\alpha} := \operatorname{span}(\{e_i : i = 1, \dots, N\}) \quad \text{and} \quad \mathcal{H}_{2,\alpha} := \operatorname{span}(\{f_i : i = 1, \dots, N\}).$$

Because our main motivation is to apply our result to semiclassical operators, we allow S_1 and S_2 to depend on a positive parameter h. We assume that for $S_1 = S_1(h,\alpha)$ and $S_2 = S_2(h,\alpha)$ there exists a $C_S = C_S(h,\alpha) > 0$ such that

$$\begin{aligned} \|S_1 v\| &\geq C_S \|v\| \quad \text{for all } v \in \mathcal{H}_{1,\alpha}, \\ \|S_2 w\| &\geq C_S \|w\| \quad \text{for all } w \in \mathcal{H}_{2,\alpha}. \end{aligned} \tag{2.4}$$

The set of pairs of operators (S_1, S_2) that satisfy these properties will be denoted by $\mathfrak{S}(\tilde{z}, \alpha)$.

We note that for a choice of α, the number of eigenvalues of $(A-\tilde{z})(A-\tilde{z})^*$ may be zero. In this case, $\mathcal{H}_{1,\alpha}$ and $\mathcal{H}_{2,\alpha}$ are empty, so we do not require additional assumptions for S_1 and S_2. Note that in this case, $\tilde{z} \notin \operatorname{Spec}(A)$, so before being randomly perturbed, $\operatorname{Spec}(A)$ is discrete within the connected component of $\rho_F^{(0)}(A)$ containing $\tilde{z}$. For such an A, Theorem 2 stated below still holds, i.e., that the perturbed operator has discrete spectrum with high probability.

We implicitly assumed in Hypothesis 1 that N is finite. This will always be true for A strictly unbounded (that is, when A is *not* bounded).

2.2. Main result

We now state the main result of our paper showing that random perturbations (of the form (2.1)) of closed linear operators $A\colon D(A) \subset \mathcal{H} \to \mathcal{H}$ immediately yield a discrete spectrum with high probability. Because we want to apply this to semiclassical operators that depend on a parameter $h > 0$, we state this result for such operators.

Theorem 2 (General result). *Suppose that the following statements are satisfied.*

(1) *Let $A = A(h)$ be a family of closed (possibly) unbounded operators $A\colon D(A) \subset \mathcal{H} \to \mathcal{H}$ indexed by a parameter $h \in \mathbb{R}_{>0}$.*

(2) *Setting D_j as disjoint, open, connected sets such that $\bigcup_{j\in\mathcal{J}} D_j \subset \rho_F^{(0)}(A)$ (with $\mathcal{J}$ an at most countable index set), then for all $j \in \mathcal{J}$, there exists $z_j \in D_j$ such that $(A - z_j)^*(A - z_j)$ has discrete spectrum.*

(3) *There are two families of Hilbert–Schmidt operators*

$$(S_1(h,\alpha), S_2(h,\alpha)) \in \bigcap_{j\in\mathcal{J}} \mathfrak{S}(z_j,\alpha)$$

(defined in (2.1) and satisfying Hypothesis 1) with constant $C_S = C_S(h,\alpha) > 0$ (as in (2.4)).

(4) *For each α and h, $S_1(h,\alpha)$ is a trace-class operator.*

Then there exists $C_0 > 0$ such that $\operatorname{Spec}(A + \delta Q_\omega) \cap \big(\bigcup_{j\in\mathcal{J}} D_j\big)$ *(for $\delta > 0$ and Q_ω defined in (2.1)) is discrete with probability at least*

$$\max\Big(1 - C_0 \exp\Big(\frac{C_0\|S_1\|\|S_1\|_{\mathrm{Tr}}\|S_2\|_{\mathrm{HS}}^2 - \alpha\delta^{-2}}{2\|S_1\|^{3/2}\|S_2\|}\Big), 0\Big) \tag{2.5}$$

where $\|\cdot\|_{\mathrm{Tr}}$ denotes the trace norm.

We have a few remarks about Theorem 2.

Assumption (4) can be replaced with $S_2(h,\alpha)$ being trace-class, in which case in (2.5) all norms of S_1 are replaced by the same norms of S_2 and vice versa.

The probabilistic bound (2.5) is close to 1 if

$$\frac{C_0\|S_1\|\|S_1\|_{\mathrm{Tr}}\|S_2\|_{\mathrm{HS}}^2 - \alpha\delta^{-2}}{2\|S_1\|^{3/2}\|S_2\|}$$

is a large negative number. This can be done by increasing α and/or decreasing δ. However, if α is large, the operators S_1 and S_2 may need to be modified to satisfy Hypothesis 1, which may increase $\|S_1\|_{\mathrm{Tr}}$ and $\|S_2\|_{\mathrm{HS}}$.

Alternatively, we can fix α and construct S_1 and S_2 to have norm 1, but at the cost of having large Hilbert–Schmidt and trace norms. This forces us to select a small δ.

One could artificially satisfy Hypothesis 1 by setting S_1 and S_2 to be the orthogonal projections onto $\mathrm{span}(\{e_1,\dots,e_N\})$ and $\mathrm{span}(\{f_1,\dots,f_N\})$ respectively. In this case, $\|S_1\| = \|S_2\| = 1$ and $\|S_1\|_{\mathrm{Tr}} = \|S_2\|_{\mathrm{HS}}^2 = N$. We would then find that $A+\delta Q_\omega$ has a discrete spectrum with probability close to 1 as long as $\delta \ll N^{-1}\sqrt{\alpha}$. Understanding how N depends on α on a case-by-case basis would further refine this estimate.

We furthermore note that Theorem 2 applies to unbounded closed operators such that $\rho_F^{(0)}(A)$ is non-empty (i.e., operators A such that there is a $z_0 \in \mathbb{C}$ such that $A - z_0$ is Fredholm of index 0) – not just Fredholm operators of index 0.

We also remark that for the previous discussed operators (Seeley's operator (1.1) and $D_h(\beta)$ (1.4)), $\rho_F^{(0)}(A) = \rho_F^{(0)}(D_h(\beta)) = \mathbb{C}$. Operators with several connected components of $\rho_F^{(0)}(A)$ do exist. For example, Toeplitz operators on the classical Hardy space $H^2(\mathbb{T})$ have Fredholm index related to the winding number of the image of the circle under the symbol [8, Theorem 7.26].

Assumption (2) in Theorem 2 will be satisfied for $D_h(\beta)$ by using the ellipticity at infinity of the operator and the compact embedding theorems of Sobolev spaces.

We next state a more quantitative version of Theorem 2. Instead of proving $A + \delta Q_\omega - z_0$ is invertible with certain probability (which would imply a discrete spectrum), we show that the smallest singular value cannot be too small (which implies invertibility).

Theorem 3. *Given the same assumptions as Theorem 2, for all $\delta > 0$ and $z_0 \in D_j$ (for some $j \in \mathcal{J}$), in the event that $\|Q_\omega\|_{\mathrm{HS}} < \infty$ (which occurs with probability 1) let $0 \le t_{1,\delta} \le t_{2,\delta} \le \cdots$ denote the singular values[5] of $A + \delta Q_\omega - z_0$, and fix $\alpha_{\max} > 0$. Then there exist constants $C_0, C_1, C_2 > 0$ such that for $a > 0$ satisfying*

$$a \le N^{NC_2}\delta^N \alpha^{-(N-2)/2} \tag{2.6}$$

we have that

$$\begin{aligned}\mathbb{P}(t_{1,\delta} \le a) \le C_0 \exp\Big(&\frac{C_0\|S_1\|\|S_1\|_{\mathrm{Tr}}\|S_2\|_{\mathrm{HS}}^2 - \alpha\delta^{-2}}{2\|S_1\|^{3/2}\|S_2\|}\Big)\\ &+ \frac{C_0\alpha^{2+(N-1)/2}}{N^{C_1 N}\delta^{N-2}C_S^{2N}\|S_1\|^{3/2}}a\end{aligned} \tag{2.7}$$

for $\alpha < \alpha_{\max}$ where C_S and $N(\alpha)$ are defined in Hypothesis 1.

Discreteness of the singular values of $A + \delta Q_\omega - z_0$ follows from the fact that the essential spectrum is invariant under compact perturbations. Indeed, let

$$\mathcal{A} := (A - z_0)^*(A - z_0) \quad \text{and} \quad \mathcal{B} := (A - z_0 + \delta Q_\omega)^*(A - z_0 + \delta Q_\omega).$$

[5]The fact that the singular values are discrete is not immediate and is discussed after the statement of the theorem.

If we can show that

$$(\mathcal{A}+i)^{-1}-(\mathcal{B}+i)^{-1} \\ = (\mathcal{B}+i)^{-1}(\delta(Q_\omega^*(A-z_0)+(A-z_0)^*Q_\omega+\delta Q_\omega^* Q_\omega))(\mathcal{A}+i)^{-1} \tag{2.8}$$

is compact, then because $\mathcal{A}$ has only discrete spectrum, then $\mathcal{B}$ has discrete spectrum (see [18, Section XIII.4, Corollary 1]). Compactness of (2.8) follows because the operators $(A-z_0)(\mathcal{A}+i)^{-1}$ and $(\mathcal{B}+i)^{-1}(A-z_0)^*$ are bounded (the first can be shown by taking the polar decomposition of A and using functional calculus, the second by the Kato–Rellich theorem) and Q_ω is compact.

We note that applications of Theorem 3 should be for a close to zero, and the required upper bound on a (2.6) is a technicality needed to apply a random matrix theory result. Heuristically, Theorem 3 states that as a goes to zero, the probability the smallest singular value of $A+\delta Q_\omega - z_0$ is less than a goes to zero. Indeed, the first term on the right-hand side of (2.7) is the same term appearing in (2.5), which is independent of a. In the case of TBG, this term is exponentially small in h (see Theorem 4).

The second term must also be treated on a case-by-case basis, in particular because it depends on N (which depends on α and h). In the case of TBG, if we fix α, let $\delta = h^4$, and construct S_1 and S_2 as in Theorem 4, we can apply [4, Proposition 4.2] which gives us that $N \asymp h^{-2}$. In this case, there exist positive constants C_3 and C_4 such that the second term on the right-hand side of (2.7) is bounded above by

$$\frac{ah^8}{(C_3h^2)^{h^{-2}C_4}}$$

which rapidly goes to zero if either a or h goes to zero.

Estimating the size of the smallest singular value of a randomly perturbed operator has been well studied in the context of random matrix theory. Sankar, Spielmann, and Teng in [22] estimate the probability that the smallest singular value of $X+\delta Q$ is small for X a deterministic matrix $N\times N$, Q a random matrix $N\times N$ with Gaussian i.i.d. entries, and $\delta>0$. There have been various extensions or related work, see for instance Rudelson and Vershynin [21] who estimate the size of the smallest singular value of a matrix whose entries are i.i.d. subgaussians, Tao and Vu [27, 28] where Q has i.i.d. entries of bounded second moment, and Cook [5] where Q has independent but not identically distributed entries under additional assumptions. See also Basak, Paquette, and Zeitouni [2, Remark 1.3] who describe random perturbations Q such that the smallest singular value of $X+\delta Q$ is small.

3. Proof of the main result

We begin by stating a lemma that provides a probabilistic norm bound of Q_ω as defined in (2.1), proven in [15, Remark 6.2] and [15, Equation 8.5].

Lemma 3.1. *There exists a* $C_0 > 0$ *such that for all* $a > 0$,

$$\mathbb{P}(\|Q_\omega\|_{\mathrm{HS}}^2 \geq a) \leq C_0 \exp\Big(\frac{C_0\|S_1\|_{\mathrm{HS}}^2\|S_2\|_{\mathrm{HS}}^2 - a}{2\|S_1\|\|S_2\|}\Big).$$

Under the assumption that S_1 is trace-class, we can write it as a product of Hilbert–Schmidt operators whose Hilbert–Schmidt norms are both the square root of the trace norm of S_1 (this is further expanded in the proof of Theorem 2). We can therefore deduce from Lemma 3.1 a bound on the trace norm of Q_ω:

$$\mathbb{P}(\|Q_\omega\|_{\mathrm{Tr}}^2 \geq a) \leq C_0 \exp\Big(\frac{C_0\|S_1\|_{\mathrm{Tr}}\|S_2\|_{\mathrm{HS}}^2 - a\|S_1\|_{\mathrm{Tr}}^{-1}}{2\|S_1\|^{1/2}\|S_2\|}\Big). \tag{3.1}$$

By continuity of measures, we get that almost surely Q_ω has finite trace norm.

Equation (3.1) follows by decomposing S_1 as $S_1 = S_{1,1}S_{1,2}$ (for appropriately chosen $S_{1,1}, S_{1,2}$ as discussed in the proof of Theorem 2), and using that

$$\|Q_\omega\|_{\mathrm{Tr}} \leq \|S_{1,1}\|_{\mathrm{HS}} \Big\| S_{1,2} \circ \Big(\sum_{j,k=1}^{\infty} \alpha_{j,k} e_j^1 \otimes e_k^2\Big) \circ S_2 \Big\|_{\mathrm{HS}}.$$

We now prove Theorem 2. This proof follows the method of proof of [15]. However, the main difference is that our operators do not necessarily have semiclassically elliptic symbols.

Proof of Theorem 2. We begin by selecting one $j \in \mathcal{J}$, setting $D := D_j$ and $z_0 := z_j$ given in the hypothesis. Next set $A^\delta := A + \delta Q_\omega$. Because Q_ω is compact, and compact perturbations of Fredholm operators are Fredholm operators with the same index, $A^\delta - z$ is Fredholm of index zero for all $z \in D$. Therefore, by Proposition 2.1, the spectrum of A^δ either contains a connected component $D \subset \rho_F^{(0)}(A)$ or $D \cap \mathrm{Spec}(A)$ is discrete.

Our overall goal is to show (with appropriate probability) that z_0 is not in the spectrum of A^δ, thus proving $D \cap \mathrm{Spec}(A^\delta)$ is discrete. To achieve this, we will build an auxiliary operator called a *Grushin problem.*

Step 1. *Build a Grushin problem for the unperturbed operator.* Equip

$$\Theta := (A - z_0)^*(A - z_0)$$

with the analogously defined domain to (2.2) and

$$\widetilde{\Theta} := (A - z_0)(A - z_0)^*$$

with domain (2.2) such that both operators are self-adjoint. Let $0 \le t_1^2 \le t_2^2 \le \cdots$ be the eigenvalues of Θ with an orthonormal basis of eigenvectors $\{e_i\}_{i\in\mathbb{N}}$ such that $t_i \in [0,\infty)$. Suppose $t_1 = t_2 = \cdots = t_n = 0$ and $t_{n+1} > 0$. Then because $(A - z_0)$ is Fredholm of index 0, and $\operatorname{Ker}(\Theta) = \operatorname{Ker}(A - z_0)$, the dimension of the kernel of $(A - z_0)^*$ has dimension n. Let $f_1, \dots, f_n$ be an orthonormal basis for this space, so that $\widetilde{\Theta} f_i = 0 = t_i^2 f_i$. If $t_i \neq 0$, then define

$$f_i := \frac{1}{t_i}(A - z_0)e_i.$$

Therefore,

$$\begin{aligned}\widetilde{\Theta} f_i &= (A - z_0)(A - z_0)^* f_i = \frac{1}{t_i}(A - z_0)(A - z_0)^*(A - z_0)e_i \\ &= t_i(A - z_0)e_i = t_i^2 f_i.\end{aligned}$$

We then get that $\{f_i\}_{i\in\mathbb{N}}$ is an orthonormal system of eigenvectors of $\widetilde{\Theta}$ corresponding to the same eigenvalues of Θ such that

$$(A - z_0)e_i = t_i f_i \quad \text{and} \quad (A - z_0)^* f_i = t_i e_i$$

for all $i \in \mathbb{N}$. We moreover claim that $\{f_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of eigenvectors of $\widetilde{\Theta}$ (which implies Θ and $\widetilde{\Theta}$ have the same eigenvalues). Indeed, suppose $f \in D((A - z_0)^*)$ is such that $\langle f, f_i\rangle = 0$ for all $i \in \mathbb{N}$; it suffices to show that $f = 0$. We have that $\langle (A - z_0)^* f, e_i\rangle = \langle f, (A - z_0)e_i\rangle$. If $i \le n$, then $(A - z_0)e_i = 0$ and if $i > n$, then $\langle f, (A - z_0)e_i\rangle = t_i\langle f, f_i\rangle = 0$. Therefore, $\langle (A - z_0)^* f, e_i\rangle = 0$, so that $f \in \operatorname{Ker}((A - z_0)^*) = \operatorname{Ker}\widetilde{\Theta}$. This implies $f = 0$ because $f_1, \dots, f_n$ span $\operatorname{Ker}\widetilde{\Theta}$. Because $D((A - z_0)^*)$ is dense in $\mathscr{H}$, we get by approximation that if $\langle f, f_i\rangle = 0$ for all $i \in \mathbb{N}$, then $f = 0$.

Let $\alpha > 0$, and define

$$N = \min\{j \in \mathbb{Z} : t_j^2 > \alpha\} - 1. \tag{3.2}$$

For now, we will assume that $N > 0$; the case when $N = 0$ will be treated at the end of this proof. Now, define $R_- = \sum_1^N f_i \otimes \delta_i$ and $R_+ = \sum_1^N \delta_i \otimes e_i$ (where δ_i is the standard basis of $\mathbb{C}^N$), so that

$$\mathcal{P} := \begin{pmatrix} A - z_0 & R_- \\ R_+ & 0 \end{pmatrix} : D(A) \times \mathbb{C}^N \to \mathscr{H} \times \mathbb{C}^N$$

is invertible with inverse

$$\mathcal{E} := \begin{pmatrix} \sum_{N+1}^\infty \frac{1}{t_i} e_i \otimes f_i & \sum_1^N e_i \otimes \delta_i \\ \sum_1^N \delta_i \otimes f_i & -\sum_1^N t_i \delta_i \otimes \delta_i \end{pmatrix} := \begin{pmatrix} E^0 & E_+^0 \\ E_-^0 & E_{-+}^0 \end{pmatrix} \tag{3.3}$$

as in [24, Section 4]. Using (3.2), we can bound (or explicitly compute) the norms of the block entries of $\mathcal{E}$ as

$$\begin{aligned} \|E^0\| &\le \frac{1}{\sqrt{\alpha}}, \qquad \|E_+^0\| = 1, \\ \|E_-^0\| &= 1, \qquad \|E_{-+}^0\| \le \sqrt{\alpha}. \end{aligned} \tag{3.4}$$

Step 2. *Build a Grushin problem for the perturbed operator.* We now define

$$\mathcal{P}^\delta := \begin{pmatrix} A^\delta - z_0 & R_- \\ R_+ & 0 \end{pmatrix} : D(A) \times \mathbb{C}^N \to \mathcal{H} \times \mathbb{C}^N$$

so that

$$\mathcal{P}^\delta \mathcal{E} = 1 + \begin{pmatrix} \delta Q_\omega E^0 & \delta Q_\omega E_+^0 \\ 0 & 0 \end{pmatrix} := 1 + K.$$

The term $1 + K$ can be inverted via a Neumann series provided that $\|K\| < 1$, which is satisfied if

$$\delta \alpha^{-1/2} \|Q_\omega\| < (1 - \varepsilon_0) < 1 \tag{3.5}$$

for some fixed $\varepsilon_0 \in (0, 1)$. When this is the case, $\mathcal{P}^\delta$ is bijective with inverse

$$\mathcal{E}^\delta := (\mathcal{P}^\delta)^{-1} = \begin{pmatrix} E^\delta & E_+^\delta \\ E_-^\delta & E_{-+}^\delta \end{pmatrix}.$$

By the Schur complement formula, $z_0 \notin \operatorname{Spec} A^\delta$ if and only if E_{-+}^δ is invertible. By construction, E_{-+}^δ is an $N \times N$ matrix. It now suffices to estimate the probability $\det E_{-+}^\delta$ is non-zero.

By the Neumann series construction, the terms in $\mathcal{E}^\delta$ can be computed by writing $\mathcal{E}^\delta = \mathcal{E}(1 + K)^{-1} = \mathcal{E} \sum_0^\infty (-K)^j$. In this case, we get that

$$E_{-+}^\delta = E_{-+}^0 + \delta E_-^0 Q_\omega E_+^0 + \sum_{j=2}^\infty E_-^0 (\delta Q_\omega E^0)^{j-1} \delta Q_\omega E_+^0. \tag{3.6}$$

Define

$$T := \delta^{-1} \Big(\sum_{j=2}^\infty E_-^0 (\delta Q_\omega E^0)^{j-1} \delta Q_\omega E_+^0 \Big), \quad \widehat{Q}_\omega := E_-^0 Q_\omega E_+^0 + T, \tag{3.7}$$

so that

$$E_{-+}^\delta = E_{-+}^0 + \delta \widehat{Q}_\omega. \tag{3.8}$$

Note that there exists a $C > 0$ such that

$$\|T\|_{\mathrm{HS}} \leq C \frac{\delta}{\sqrt{\alpha}} \|Q_\omega\|_{\mathrm{HS}}^2$$

by (3.4), (3.5), and (3.7), and the fact that the Hilbert–Schmidt norm of T is bounded by a constant times the norm of the first term in its summation expansion in (3.7).

Step 3. *Describe the law of* $\widehat{Q}_\omega$. We reiterate that $z_0 \notin \operatorname{Spec} A^\delta$ if and only if E_{-+}^δ is invertible. The remainder of the proof will show that E_{-+}^δ is invertible with high probability. By (3.8), E_{-+}^δ can be written as a sum of a deterministic $N \times N$ matrix and δ times a random $N \times N$ matrix $\widehat{Q}_\omega$. By (3.7), $\widehat{Q}_\omega$ has a leading order term (in δ) which we will describe in the following proposition.

Proposition 3.2. *There exist orthonormal bases* $\{f_j^1\}_{j=1}^N$ *and* $\{f_j^2\}_{j=1}^N$ *of* $\mathbb{C}^N$ *such that*

$$E_-^0 Q_\omega E_+^0 = \sum_{\substack{1 \leq j \\ k \leq N}} s_{1,j} s_{2,k} \alpha_{j,k} f_j^1 \otimes f_k^2$$

where $s_{1,j}, s_{2,k}$ *are such that* $0 < C_S \leq s_{1,j}, s_{2,k} \leq \max(\|S_1\|, \|S_2\|)$.

Proof. For a trace-class Fredholm operator B of index 0, its singular values are denoted by $s_1(B) \geq s_2(B) \geq \cdots$. These are the eigenvalues of $(B^*B)^{1/2}$ ordered in decreasing values and counting their multiplicities.

With this notation, let $s_{1,j}$ denote the singular values of $E_-^0 \circ S_1$ and let $s_{2,j}$ denote the singular values of $S_2 \circ E_+^0$. Note that $E_-^0 \circ S_1$ and $S_2 \circ E_+^0$ are both rank N operators, so that $s_{2,j} = 0$ for $j \geq N + 1$ and $E_-^0 \circ S_1$ has only N singular values. Both operators are bounded, so $s_{1,j}, s_{2,j} \leq \max(\|S_1\|, \|S_2\|)$.

By Hypothesis 1, we get a lower bound for the first N singular values. Indeed, if $(S_2 E_+^0)^*(S_2 E_+^0)u = (s_{2,N})^2 u$ with $\|u\| = 1$, then

$$(s_{2,N})^2 = \langle (s_{2,N})^2 u, u\rangle = \|S_2 E_+^0 u\|^2 \geq C_S^2 \|E_+^0 u\|^2 = C_S^2.$$

The last equality uses that $E_+^0 = \sum_1^N e_i \otimes \delta_i$ (recall (3.3)) so that E_+^0 is an isometry, and so $\|E_+^0 u\| = \|u\| = 1$. Because $s_{2,j}$ are decreasing in j, we get that $C_S \leq s_{2,j} \leq \max(\|S_1\|, \|S_2\|)$ for $j = 1, \dots, N$. A similar argument can be used to establish the same bound for each $s_{1,j}$.

Because the law of Gaussian ensembles is invariant under unitary conjugations (see for instance [15, Section 13]), we are free to choose any two orthonormal bases of $\mathscr{H}$ when defining Q_ω (in (2.1)). We will choose these bases in the following way. Let both f_j^1 and f_j^2 be orthonormal bases of $\mathbb{C}^N$ which are eigenfunctions of

$\sqrt{(E^0_- S_1)(E^0_- S_1)^*}$ and $\sqrt{(S_2 E^0_+)^*(S_2 E^0_+)}$, respectively, corresponding to eigenvalues $s_{1,j}$ and $s_{2,j}$. Then, let e^1_j and e^2_j (for $j \in \mathbb{N}$) be orthonormal bases of $\mathscr{H}$ such that

$$e^1_j = \frac{1}{s_{1,j}}(E^0_- S_1)^* f^1_j \quad \text{and} \quad e^2_j = \frac{1}{s_{2,j}}(S_2 E^0_+) f^2_j$$

for $j = 1, \dots, N$. By this construction, we see that

$$E^0_- Q_\omega E^0_+ = E^0_- S_1 \Big(\sum_{j,k=1}^{\infty} \alpha_{j,k} e^1_j \otimes e^2_k \Big) S_2 E^0_+ = \sum_{\substack{1 \le j \\ k \le N}} s_{1,j} s_{2,k} \alpha_{j,k} f^1_j \otimes f^2_k. \qquad \blacksquare$$

Let $T_1 = \operatorname{diag}(\{s_{1,j}\}_{1 \le j \le N})$ and $T_2 = \operatorname{diag}(\{s_{2,j}\}_{1 \le j \le N})$ where $s_{1,j}$ and $s_{2,j}$ are defined in Proposition 3.2. With this, we can rewrite the second term in (3.8) (with respect to the bases f^1_j and f^2_k) as

$$\delta \widehat{Q}_\omega = \delta T_1 \circ ((\alpha_{j,k})_{1 \le j,k \le N} + \widetilde{T}) \circ T_2 \tag{3.9}$$

where

$$\widetilde{T} = T_1^{-1} \circ T \circ T_2^{-1}. \tag{3.10}$$

Here, we recall T was defined in (3.7) as the lower order terms in the expansion of E^δ_{-+}.

Let $\hat{\mu}$ be the law of $(\alpha_{j,k})_{1 \le j,k \le N} + \widetilde{T}$. The following lemma, proven in [15, eq. 9.18], bounds the measure $\hat{\mu}$.

Recall from assumption (4) that S_1 is a trace-class operator, so it can be written as a product of Hilbert–Schmidt operators $S_{1,1}$ and $S_{1,2}$ (see for instance [20, Chapter 10]). Moreover, we can choose these Hilbert–Schmidt operators[6] such that

$$\|S_{1,1}\|_{\mathrm{HS}} = \|S_{1,2}\|_{\mathrm{HS}} = \sqrt{\|S_1\|_{\mathrm{Tr}}}$$

and $\|S_{1,1}\| = \|S_{1,2}\| = \sqrt{\|S_1\|}$.

Lemma 3.3. *For $M > 0$, define*

$$\mathcal{Q}_M := \Big\{ (\alpha_{j,k})_{j,k \in \mathbb{N}} : \Big\| S_{1,2} \circ \Big(\sum_{j,k=1}^{\infty} \alpha_{j,k} e^1_j \otimes e^2_k \Big) \circ S_2 \Big\|_{\mathrm{HS}} < M \Big\}. \tag{3.11}$$

[6] Indeed, by polar decomposition, $S_1 = \sum_j \lambda_j e_j \otimes f_j$ for e_j and f_j orthonormal bases of $\mathscr{H}$ and λ_j the singular values of S_1; then we can define $S_{1,1} := \sum_j \sqrt{\lambda_j} e_j \otimes f_j$ and $S_{1,2} := \sum_j \sqrt{\lambda_j} f_j \otimes f_j$.

Let μ_N *denote the probability measure on complex* $N \times N$ *matrices given by*

$$\mu_N = \prod_{j,k=1}^{N} \Big(e^{-|\alpha_{j,k}|^2} \frac{\mathrm{d}m(\alpha_{j,k})}{\pi} \Big)$$

where m *is the Lebesgue measure on* $\mathbb{C}$.

Then for $(\alpha)_{j,k\in\mathbb{N}}$ *in* $\mathcal{Q}_M$, *the law of* $(\alpha_{j,k})_{1\le j,k\le N} + \widetilde{T}$ *(as in* (3.10)*), denoted by* $\hat{\mu}$, *satisfies the following bound:*

$$\hat{\mu} \le \Big(1 + \mathcal{O}(1) \frac{\delta M^3}{\sqrt{\alpha}} \Big) \mu_N .$$

Concretely, if $A \subset \mathcal{Q}_M$ *is measurable, then* $\hat{\mu}(A) \le (1 + \mathcal{O}(1)\delta M^3 \alpha^{-1/2})\mu_N(A)$.

Step 4. *Estimate the probability* E_{-+}^{δ} *is invertible.* Observe by (3.8),

$$\det(E_{-+}^{\delta}) = \delta^N \det(\delta^{-1} E_{-+}^{\mathbf{0}} + \widehat{Q}_\omega). \tag{3.12}$$

Therefore, on $\mathcal{Q}_M$, the event that the right-hand side of (3.12) is zero will have probability zero. This is because the entries of the random matrix have probability densities that are absolutely continuous with respect to the Lebesgue measure (using Lemma 3.3), and the zero set of the characteristic polynomial has codimension 1.

Therefore, $\mathrm{Spec}(D \cap A^{\delta})$ is discrete as long as we can build a Grushin problem (which requires a norm bound on Q_ω, (3.5)) and $(\alpha_{j,k})_{j,k\in\mathbb{N}}$ in the random perturbation belongs to $\mathcal{Q}_M$. Concretely,

$$\mathbb{P}(\mathrm{Spec}(D \cap A^{\delta}) \text{ is discrete}) \ge \mathbb{P}(\{\|Q_\omega\| < \delta^{-1}\alpha^{1/2}\} \cap \mathcal{Q}_M). \tag{3.13}$$

If $M < \delta^{-1}\alpha^{1/2}\|S_{1,1}\|^{-1}$, then

$$\|Q_\omega\|_{\mathrm{HS}} \le \|S_{1,1}\| \Big\| S_{1,2} \circ \Big(\sum_{\alpha_{j,k}} e_j^1 \otimes e_k^2 \Big) \circ S_2 \Big\|_{\mathrm{HS}} < \delta^{-1}\alpha^{1/2}$$

so that $\mathcal{Q}_M \subset \{\|Q_\omega\| < \delta^{-1}\alpha^{1/2}\}$. Therefore, (3.13) becomes

$$\begin{aligned} \mathbb{P}(\mathrm{Spec}(D \cap A^{\delta}) \text{ is discrete}) &\ge \mathbb{P}(\mathcal{Q}_M) \\ &> 1 - C_0 \exp\Big(\frac{C_0 \|S_{1,2}\|_{\mathrm{HS}}^2 \|S_2\|_{\mathrm{HS}}^2 - M^2}{2\|S_{1,2}\|\|S_2\|} \Big). \end{aligned} \tag{3.14}$$

Here the second inequality follows from Lemma 3.1. We can select $\varepsilon \in (0,1)$ and set $M = \varepsilon\delta^{-1}\alpha^{1/2}\|S_{1,1}\|^{-1}$. In this case, we have

$$\frac{C_0 \|S_{1,2}\|_{\mathrm{HS}}^2 \|S_2\|_{\mathrm{HS}}^2 - M^2}{2\|S_{1,2}\|\|S_2\|} = \frac{C_0 \|S_{1,1}\|^2 \|S_{1,2}\|_{\mathrm{HS}}^2 \|S_2\|_{\mathrm{HS}}^2 - \varepsilon^2 \alpha \delta^{-2}}{2\|S_{1,1}\|^2 \|S_{1,2}\|\|S_2\|}$$

$$= \frac{C_0\|S_1\|\|S_{1,2}\|_{\mathrm{HS}}^2\|S_2\|_{\mathrm{HS}}^2 - \varepsilon^2\alpha\delta^{-2}}{2\|S_1\|^{3/2}\|S_2\|}$$
$$= \frac{C_0\|S_1\|\|S_1\|_{\mathrm{Tr}}\|S_2\|_{\mathrm{HS}}^2 - \varepsilon^2\alpha\delta^{-2}}{2\|S_1\|^{3/2}\|S_2\|}$$

where in the second equality, we use that $\|S_{1,1}\| = \|S_{1,2}\| = \|S_1\|^{1/2}$, and in the third equality, we use that $\|S_{1,2}\|_{\mathrm{HS}}^2 = \|S_1\|_{\mathrm{Tr}}$. We can then take the limit as $\varepsilon \to 1$, so that (3.14) becomes

$$\mathbb{P}(\mathrm{Spec}(D \cap A^\delta) \text{ is discrete}) \geq 1 - C_0 \exp\Big(\frac{C_0\|S_1\|\|S_1\|_{\mathrm{Tr}}\|S_2\|_{\mathrm{HS}}^2 - \alpha\delta^{-2}}{2\|S_1\|^{3/2}\|S_2\|}\Big).$$

Step 5. *Consider the case where* $N(\alpha) = 0$ *and extend the result to all* D_j. We recall, when building the Grushin problem, we assumed that $N > 0$ (where N was defined in (3.2) as the number of small singular values depending on α). If $N = 0$, then this implies that 0 is not a singular value, so $A - z_0$ is invertible. In this case, we can build the Grushin problem by setting $\mathcal{P} = A - z_0$ and $\mathcal{E} = \sum_1^\infty t_i^{-1} e_i \otimes f_i$. This allows us to build the inverse for $A + \delta Q_\omega - z_0$ as $\mathcal{E}(1 + \delta Q_\omega \mathcal{E})^{-1}$ with a Neumann series, provided that $\delta\alpha^{-1/2}\|Q\|_\omega < 1$. We then get $\mathbb{P}(\mathrm{Spec}(D \cap A^\delta) \text{ is discrete}) \geq \mathbb{P}(\{\|Q_\omega\| < \delta^{-1}\alpha^{1/2}\})$, which is bounded below by (3.14), and we get the same result.

For each $j \in \mathcal{J}$, let $\mathcal{E}_j$ be the event $\mathrm{Spec}(A^\delta) \cap D_j$ has discrete spectrum. But by the above discussion, if $M < \delta^{-1}\alpha^{1/2}\|S_{1,1}\|^{-1}$, then $\mathcal{E}_j \supset \mathcal{Q}_M$. Therefore,

$$\mathbb{P}\Big(\bigcap_{j\in\mathcal{J}} \mathcal{E}_j\Big) \geq \mathbb{P}(\mathcal{Q}_M)$$

which is estimated in (3.14). ∎

We now prove Theorem 3, the more quantitative version of Theorem 2, by estimating the size of the smallest singular value.

Proof of Theorem 3. We proceed in three steps.

Step 1. *Relate singular value to* E_{-+}^δ. We begin by relating the smallest singular value of $A^\delta - z_0$ to the absolute value of the determinant of E_{-+}^δ (as long as a Grushin problem can be constructed). We define $0 \leq t_{1,\delta} \leq t_{2,\delta} \leq \cdots$ as the increasing sequence of singular values of $A^\delta - z_0$. We can write

$$\mathbb{P}(t_{1,\delta} \leq a) \leq \mathbb{P}(\{t_{1,\delta} \leq a\} \cap \mathcal{Q}_M) + \mathbb{P}(\{t_{1,\delta} \leq a\} \cap \mathcal{Q}_M^c) \tag{3.15}$$

recalling the definition of $\mathcal{Q}_M$ given in (3.11). The second term of the right-hand side of (3.15) is bounded using Lemma 3.1:

$$\mathbb{P}(\{t_{1,\delta} \leq a\} \cap \mathcal{Q}_M^c) \leq \mathbb{P}(\mathcal{Q}_M^c) \leq C_0 \exp\Big(\frac{C_0\|S_{1,2}\|_{\mathrm{HS}}^2\|S_2\|_{\mathrm{HS}}^2 - M^2}{2\|S_{1,2}\|\|S_2\|}\Big). \tag{3.16}$$

As long as $M < \delta^{-1}\alpha^{1/2}\|S_{1,1}\|^{-1}$, we can set up the same Grushin problem as in the proof of Theorem 2 (recalling (3.5)). We now use a general fact about Grushin problems. Applying the Schur complement formula and properties of singular values (see for instance [30, Lemma 18]), we get that

$$\frac{t_1(E^\delta_{-+})}{t_1(E^\delta_{-+})\|E^\delta\| + \|E^\delta_-\|\|E^\delta_+\|} \leq t_{1,\delta} \tag{3.17}$$

where $0 \leq t_1(E^\delta_{-+}) \leq t_2(E^\delta_{-+}) \leq \cdots \leq t_N(E^\delta_{-+})$ are the singular values of E^δ_{-+}. By expanding the Neumann series expansion for the Grushin problem (similarly to (3.6)), we see that, in the event $\mathcal{Q}_M$, we have the following bounds:

$$t_1(E^\delta_{-+}) = \mathcal{O}(1), \quad \|E^\delta\| = \mathcal{O}(\alpha^{-1/2}),$$
$$\|E^\delta_-\| = \mathcal{O}(1), \quad \|E^\delta_+\| = \mathcal{O}(1),$$

which combined with (3.17), gives us

$$\alpha^{1/2} t_1(E^\delta_{-+}) \leq C t_{1,\delta} \tag{3.18}$$

for some constant $C > 0$ where we use that $\alpha < \alpha_{\max}$.

Next, observe that

$$\begin{aligned} |\det E^\delta_{-+}| &= \prod_{j=1}^N t_j(E^\delta_{-+}) \leq t_1(E^\delta_{-+})(t_N(E^\delta_{-+}))^{N-1} \\ &= t_1(E^\delta_{-+})\|E^\delta_{-+}\|^{N-1}. \end{aligned} \tag{3.19}$$

Within the event $\mathcal{Q}_M$, with $M < \delta^{-1}\alpha^{1/2}\|S_{1,1}\|^{-1}$, we can use (3.6) to get that $\|E^\delta_{-+}\| = \mathcal{O}(\alpha^{1/2})$. Using this, (3.19) can be rearranged as

$$t_1(E^\delta_{-+}) \geq \frac{|\det(E^\delta_{-+})|}{(C\sqrt{\alpha})^{N-1}} \tag{3.20}$$

for some (possibly new) constant $C > 0$. Therefore, combining (3.20) with (3.18), we get that

$$\{(t_{1,\delta} \leq a) \cap \mathcal{Q}_M\} \subset \{(|\det(E^\delta_{-+})| < aC^{N-1}\sqrt{\alpha}^{N-2}) \cap \mathcal{Q}_M\} \tag{3.21}$$

for some (possibly new) constant $C > 0$.

Step 2. *Estimate the probability that* $|\det(E^\delta_{-+})|$ *is small.* Recall, by (3.9),

$$E^\delta_{-+} = E_{-+} + \delta T_1 \circ ((\alpha_{j,k})_{1\leq j,k\leq N} + \widetilde{T}) \circ T_2,$$

where the definition of T is given in (3.7), $T_{1,2}$ are defined in the discussion preceding (3.9), and $\widetilde{T} := T_1^{-1} \circ T \circ T_2^{-1}$. This gives us

$$\begin{aligned}
&\{|\det E_{-+}^{\delta}| > aC^{N-1}\sqrt{\alpha}^{N-2}\} \\
&\quad = \{|\det((\alpha_{j,k})_{1\le j,k\le N} + T + \delta^{-1}T_1^{-1}E_{-+}^{0}T_2^{-1})| \\
&\qquad > aC^{N-1}\sqrt{\alpha}^{N-2}\delta^{-N}\det(T_1T_2)^{-1}\}.
\end{aligned}$$

We also recall that Lemma 3.3 estimates the law of $(\alpha_{j,k})_{1\le j,k\le N} + T$ by the law of an $N \times N$ Gaussian ensemble.

We next use the following Lemma from [15, Proposition 7.3] estimating the size of the determinant of a Gaussian ensemble added to a deterministic matrix.

Lemma 3.4. *If $V_\omega = (\alpha_{i,j})_{1\le i,j\le N}$ with $\alpha_{i,j} \sim \mathcal{N}_{\mathbb{C}}(0,1)$ i.i.d., and $D \in \mathbb{C}^{N\times N}$, then there exist $C_1, C_2 > 0$ (independent of D) such that for $c > 0$:*

$$\mathbb{P}(|\det(D + V_\omega)| \le c) \le C_1 c \exp\Big(-\frac{1}{2}\Big(C_2 + \Big(N - \frac{1}{2}\Big)\log N - 2N\Big)\Big) \tag{3.22}$$

as long as

$$c \le \exp\Big(\frac{C_2 + \big(N + \frac{1}{2}\big)\log(N) - 2N}{2}\Big) \tag{3.23}$$

for C_2 the same constant as in (3.22).

We now apply Lemmas 3.3 and 3.4 (setting $D = \delta^{-1}T_1^{-1}E_{-+}^{0}T_2^{-1}$ and assuming (3.23) holds for now) to get that

$$\begin{aligned}
&\mathbb{P}(\{|\det E_{-+}^{\delta}| \le aC^{N-1}\sqrt{\alpha}^{N-2}\} \cap \mathcal{Q}_M) \\
&\quad \le \Big(1 + \mathcal{O}(1)\frac{\delta M^3}{\sqrt{\alpha}}\Big)\mathbb{P}(|\det(V_\omega + D)| \le aC^{N-1}\sqrt{\alpha}^{N-2}\delta^{-N}\det(T_1T_2)^{-1}) \\
&\quad \le \Big(1 + \mathcal{O}(1)\frac{\delta M^3}{\sqrt{\alpha}}\Big)(aC^{N-1}\sqrt{\alpha}^{N-2}\delta^{-N}\det(T_1T_2)^{-1}) \\
&\qquad \cdot \exp\Big(-\frac{1}{2}\Big(C_2 + \Big(N - \frac{1}{2}\Big)\log(N) - 2N\Big)\Big).
\end{aligned}$$

Letting $M < \delta^{-1}\alpha^{1/2}\|S_{1,1}\|^{-1}$, we get that

$$\frac{\delta M^3}{\sqrt{\alpha}} < \frac{\alpha^2\delta^2}{\|S_{1,1}\|^3} = \frac{\alpha^2\delta^2}{\|S_1\|^{3/2}}$$

where we recall that $\|S_{1,1}\| = \|S_1\|^{1/2}$. By Proposition 3.2, the determinants of T_1 and T_2 are bounded below, so that

$$\det(T_1T_2)^{-1} \le C_S^{-2N}. \tag{3.24}$$

There exist positive constants C_3 and C_4 such that

$$\frac{-1}{2}\Big(\Big(N-\frac{1}{2}\Big)\log(N)-2N\Big) \le -C_3 N\log(N)+C_4. \tag{3.25}$$

Step 3. *Estimate the probability the smallest singular value is small.* We therefore get (by (3.21))

$$\mathbb{P}(\{t_{1,\delta}\le a\}\cap \mathcal{Q}_M)\le C\frac{\alpha^{2+(N-1)/2}}{\delta^{N-2}C_S^{2N}\|S_1\|^{3/2}}N^{-C_3N}a. \tag{3.26}$$

For (3.23) to hold, we require that

$$aC^{N-1}\sqrt{\alpha}^{N-2}\delta^{-N}\det(T_1T_2)^{-1}\le \exp\Big(\frac{C_2+\big(N+\frac{1}{2}\big)\log(N)-2N}{2}\Big). \tag{3.27}$$

Using (3.24) and a similar bound as in (3.25), (3.27) holds if

$$a\le \exp\Big(C_5N\log(N)+N\log(\delta)-\Big(\frac{N-2}{2}\Big)\log(\alpha)\Big)$$

for some $C_5>0$. Combining (3.26) with (3.16) (setting $M=\varepsilon\delta^{-1}\sqrt{\alpha}\|S_1\|^{-1/2}$) and taking the limit as $\varepsilon\to 1$ gives us the Theorem. ■

4. Applications to twisted bilayer graphene

In this section we build Hilbert–Schmidt operators to satisfy Hypothesis 1 for the operator

$$D_h(\beta)=\begin{pmatrix}2hD_{\bar z} & U(z)\\ U(-z) & 2hD_{\bar z}\end{pmatrix} \tag{4.1}$$

first defined in (1.4) recalling that $2D_{\bar z}:=-i(\partial_{\operatorname{Re} z}+i\partial_{\operatorname{Im} z})$. Here we absorb β into the definition of U, so that $U(z):=\beta\sum_{k=0}^{2}\omega^k\exp((z\bar\omega^k-\bar z\omega^k)/2)$.

By [3, Proposition 2.3], $D_h(\beta)$ is an unbounded Fredholm operator on the domain $H^1(\mathbb{C}/\Gamma;\mathbb{C}^2)$ with $\rho_F^{(0)}(D_h(\beta))=\mathbb{C}$ (recall $\rho_F^{(0)}$ is defined in equation (1.3)).

We will ultimately apply Theorem 2 to $D_h(\beta)$ which requires showing that the spectrum of $D_h(\beta)^*D_h(\beta)$ on its natural domain is discrete. This follows by the Sobolev embedding. In fact, $D_h(\beta)^*D_h(\beta)$ is a self-adjoint operator on $H^2(\mathbb{C}/\Gamma;\mathbb{C}^2)$, so that $(D_h(\beta)^*D_h(\beta)-i)^{-1}$ is a bounded operator from $L^2(\mathbb{C}/\Gamma)$ to $H^2(\mathbb{C}/\Gamma)$. According to the Rellich–Kondrachov Theorem, $H^2(\mathbb{C}/\Gamma)$ is compactly embedded in $L^2(\mathbb{C}/\Gamma)$ so that

$$(D_h(\beta)^*D_h(\beta)-i)^{-1}\colon L^2(\mathbb{C}/\Gamma)\to L^2(\mathbb{C}/\Gamma)$$

is a compact operator. Therefore, $(D_h(\beta)^* D_h(\beta) - i)^{-1}$ has discrete spectrum, which implies that $D_h(\beta)^* D_h(\beta)$ has discrete spectrum.

We want to construct operators S_1 and S_2 that satisfy Hypothesis 1 for the random perturbation Q_ω defined in (2.1).

Claim 4.1. *There exist constants* $C_1, C_2 > 0$ *such that if* $\chi(z,\zeta) \in C_0^\infty(T^*(\mathbb{C}/\Gamma); [0,1])$ *is identically* 1 *when* $|\zeta|^2 < C_1 C_2$, *then*

$$S_1 = S_2 = \operatorname{Op}_h \begin{pmatrix} \chi & 0 \\ 0 & \chi \end{pmatrix} \tag{4.2}$$

satisfy Hypothesis 1 *for* $\alpha < C_2$, *where* Op_h *is the quantization of functions on* $T^*(\mathbb{C}/\Gamma)$ *defined in Definition* A.4.

Proof. To show that S_1 and S_2 satisfy Hypothesis 1 for an α, we must show there exists a $C_S > 0$ such that $\|S_1 u\|, \|S_2 u\| \geq \|u\|$ for every $u \in H^2(\mathbb{C}/\Gamma; \mathbb{C}^2)$ satisfying $D_h(\beta)^* D_h(\beta) u = t_i^2 u$ for $t_i^2 < \alpha$. Let us now fix such a t_i and u.

To establish the lower bound, we will use the symbolic calculus of matrix-valued symbols (which is reviewed in Appendix A).

To apply this calculus, we use the following notation. We write elements of $T^*(\mathbb{C}/\Gamma)$ as $(x + iy, \xi + i\eta)$ where $\xi, \eta, x, y \in \mathbb{R}$ are such that $x + iy \in \mathbb{C}/\Gamma$. We will also write $\zeta = \xi + i\eta$ and $z = x + iy$.

The operator $D_h(\beta)$ (defined in (4.1)) has principal symbol

$$\begin{pmatrix} \zeta & U(z) \\ U(-z) & \zeta \end{pmatrix}.$$

Define $\Theta := (D_h(\beta))^*(D_h(\beta))$ with domain $H^2(\mathbb{C}/\Gamma; \mathbb{C}^2)$. From this, we see that $\Theta u = t_i^2 u$ and

$$q_0 := \sigma_0(\Theta) = \begin{pmatrix} |\zeta|^2 + |U(-z)|^2 & \bar{\zeta} U(z) + \zeta \overline{U(-z)} \\ \overline{U(z)} \zeta + \bar{\zeta} U(-z) & |\zeta|^2 + |U(z)|^2 \end{pmatrix}$$

where $\sigma_0(P)$ denotes the principal symbol of an operator P (see Definition A.5). We would like to build a $\chi \in C_0^\infty(T^*M; [0,1])$ such that a parametrix can be constructed for $\Theta - t_i^2 + i \operatorname{Op}_h(\chi I_2)$, where

$$I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Such a parametrix will then be used to show that $S_1 u = S_2 u = u + \mathcal{O}(h^\infty)$, which will provide the desired lower bound of $\|S_1 u\|$ and $\|S_2 u\|$. To construct such a parametrix, we must show that the determinant of the principal symbol of $\Theta - t_i^2 + i \operatorname{Op}_h(\chi I_2)$ is uniformly bounded from below.

Let $\lambda_\pm = \lambda_\pm(t_i)$ denote the eigenvalues of $q_0 - t_i^2 I_2$. Fix $\varepsilon \in (0, 1)$, $C_2 > 0$ sufficiently large (which we will specify later), and define

$$K := \{(z, \zeta) \in T^*(\mathbb{C}/\Gamma) : |\zeta|^2 \leq (1-\varepsilon)^{-1} C_2\}, \tag{4.3}$$

and let $\tilde{\chi} \in C_0^\infty(T^*(\mathbb{C}/\Gamma); [0, 1])$ be identically 1 on K (which is a compact set) and zero on the set

$$\{(z, \zeta) \in T^*(\mathbb{C}/\Gamma) : |\zeta|^2 > C_1 C_2\}$$

where $C_1 = c(1-\varepsilon)^{-1}$ and $c > 1$. We can then compute that

$$\begin{aligned} |\det(\sigma_0(\Theta - t_i^2 + i \operatorname{Op}_h(\chi I_2)))|^2 &= |\lambda_+\lambda_- + i\chi(\lambda_+ + \lambda_-) - \chi^2|^2 \\ &= (\lambda_+\lambda_-)^2 + \chi^4 + \chi^2(\lambda_+^2 + \lambda_-^2), \end{aligned} \tag{4.4}$$

which is bounded below by χ^4 so that for $(z, \zeta) \in K$, (4.4) is bounded below by 1.

We now aim to provide a lower bound of $\lambda_+\lambda_-$ for $(z, \zeta) \in K^c$. Note that

$$\lambda_+\lambda_- = \det(q_0 - t_i^2) = f_1(z, \zeta) f_2(z, \zeta) - f_3(z, \zeta) \tag{4.5}$$

where

$$\begin{aligned} f_1(z, \zeta) &:= |\zeta|^2 + |U(-z)|^2 - t_i^2, \\ f_2(z, \zeta) &:= |\zeta|^2 + |U(z)|^2 - t_i^2, \\ f_3(z, \zeta) &:= |\bar{\zeta} U(z) + \zeta \overline{U(-z)}|^2. \end{aligned}$$

Recalling that $|U(z)|$ is bounded, we see that $|f_3| \leq C|\zeta|^2$ where C depends on the maximum of $|U(z)|$. Furthermore, for $j = 1, 2$,

$$f_j(z, \zeta) \geq |\zeta|^2 - t_i^2 \geq |\zeta|^2 - \alpha \geq |\zeta|^2 - C_2 \geq \varepsilon|\zeta|^2 \tag{4.6}$$

using (4.3). Therefore, using (4.3), (4.5), and (4.6), we get that

$$\lambda_+\lambda_- \geq \varepsilon^2|\zeta|^4 - C|\zeta|^2 = |\zeta|^2(\varepsilon^2|\zeta|^2 - C) \geq (1-\varepsilon)^{-1} C_2(\varepsilon^2(1-\varepsilon)C_2 - C) > 0$$

as long as

$$C_2^2 > \frac{C}{\varepsilon^2(1-\varepsilon)}. \tag{4.7}$$

For a sharper lower bound on C_2, we may minimize the right-hand side of (4.7) by letting $\varepsilon = 2/3$, in which case we have $(\varepsilon^2(1-\varepsilon))^{-1} = 27/4$. In this case, we require $C_2 > \sqrt{27C}/2$, and C_1 in the statement of the Claim is 3.

Having shown a lower bound on the determinant of the principal symbol of $\Theta - t_i^2 + i \operatorname{Op}_h(\tilde{\chi} I_2)$, we can invert $\Theta - t_i^2 + i \operatorname{Op}_h(\tilde{\chi} I_2)$ using Proposition A.7. Let P be a parametrix of $\Theta - t_i^2 + i \operatorname{Op}_h(\tilde{\chi} I_2)$. Now, let $\chi \in C_0^\infty(T^*(\mathbb{C}/\Gamma); [0,1])$ be identically 1 on the support of $\tilde{\chi}$. Recalling that $\Theta u = t_i^2 u$, we get that

$$\begin{aligned}\operatorname{Op}_h(I_2 - \chi I_2)u &= \operatorname{Op}_h(I_2 - \chi I_2)P(\Theta - t_i^2 + i \operatorname{Op}_h(\tilde{\chi} I_2))u + \mathcal{O}_{L^2\to L^2}(h^\infty)\\ &= \operatorname{Op}_h(I_2 - \chi I_2)P(i \operatorname{Op}_h(\tilde{\chi} I_2))u + \mathcal{O}(h^\infty) = \mathcal{O}_{L^2\to L^2}(h^\infty).\end{aligned}$$

Here we use the fact that that the composition of symbols with disjoint support yields $\mathcal{O}_{L^2\to L^2}(h^\infty)$ errors. We then see that

$$\operatorname{Op}_h(\chi I_2)u = u + \mathcal{O}(h^\infty)$$

so we get that

$$\|\operatorname{Op}_h(\chi I_2)u\| \geq (1 + \mathcal{O}(h^\infty))\|u\|$$

so that $\operatorname{Op}_h(\chi I_2)$ satisfies Hypothesis 1. ∎

Theorem 4 (Perturbation of $D_h(\beta)$). *Suppose $D_h(\beta)$ is defined by (1.4), S_1 and S_2 are defined by (4.2), $0 < \delta < h^\kappa$ with $\kappa > 2$, and Q_ω is defined by (2.1). Then $D_h(\beta) + \delta Q_\omega$ has discrete spectrum with probability at least*

$$1 - C_1 \exp\Big(-\frac{C_2}{h^{2\kappa}}\Big)$$

for positive constants C_1 and C_2 (i.e., with overwhelming probability).

Proof. In Claim 4.1, we constructed S_1 and S_2 satisfying the hypotheses of Theorem 2. Moreover, because the symbols of S_1 and S_2 have compact support, they are both trace-class.

We therefore can apply Theorem 2 to get that there exists a $C_0 > 0$ such that $D(h) + \delta Q_\omega$ has discrete spectrum with probability at least

$$\max\Big(1 - C_0 \exp\Big(\frac{C_0\|S_1\|\|S_1\|_{\mathrm{Tr}}\|S_2\|_{\mathrm{HS}}^2 - \alpha\delta^{-2}}{2\|S_1\|^{3/2}\|S_2\|}\Big), 0\Big).$$

By construction, $\|S_1\|_{\mathrm{Tr}}, \|S_2\|_{\mathrm{HS}}^2 = \mathcal{O}(h^{-2})$, and $\|S_1\|, \|S_2\| \sim 1$. Therefore, if $0 < \delta < h^\kappa$ with $\kappa > 2$, we get that $D(h) + \delta Q_\omega$ has discrete spectrum with probability at least

$$1 - C_1 \exp\Big(-\frac{C_2}{h^{2\kappa}}\Big)$$

for positive constants C_1 and C_2. ∎

A. Quantization of matrix valued functions on $\mathbb{C}/\Gamma$

In this appendix, we provide the background and pseudo-differential calculus to construct operators S_1 and S_2 for our application to twisted bilayer graphene and, in particular, to $D_h(\beta)$ as defined in (1.4).

The ultimate goal is to prove a composition result for certain quantized operators (Proposition A.6) and a parametrix construction (Proposition A.7). We will prove this by using well-established results on the Weyl quantization of scalar-valued functions on $\mathbb{R}^d$.

The results of this section follow via mild modifications of standard results for pseudo-differential operators with scalar-valued symbols. The main difference in the matrix-valued symbol case is that when constructing a parametrix, the inverse of the symbol must be used. A careful verification of the usual parametrix construction (which relies on a well-defined symbol class with a calculus) is required.

A summary of this appendix is as follows.

(1) Define the class of functions ($S^k(T^*(\mathbb{C}/\Gamma); \mathbb{C}^2)$) we want to quantize (Definition A.2).

(2) Show that the usual h-Weyl quantization of these functions induces bounded linear maps between appropriate Sobolev spaces on $\mathbb{C}/\Gamma$ (Proposition A.3).

(3) Define the quantization of $A \in S^k(T^*(\mathbb{C}/\Gamma); \mathbb{C}^2)$ as the restriction of the h-Weyl quantization to a Sobolev space on $\mathbb{C}/\Gamma$ (Definition A.4).

(4) Use standard results about the h-Weyl quantization of scalar-valued functions on $\mathbb{R}^d$ to get a composition and parametrix result (Propositions A.6 and A.7).

We aim to quantize matrix-valued symbols on the cotangent space of $\mathbb{C}/\Gamma$. We recall that the lattice Γ is given by $4\pi(i\omega\mathbb{Z} \oplus i\omega^2\mathbb{Z})$ where $\omega = e^{2\pi i/3}$. The discussion in this section will work for any lattice of $\mathbb{R}^d$ spanned by d linearly independent vectors, however, we only consider $\Gamma \subset \mathbb{C}$ to keep the discussion as explicit as possible.

We first recall the h-Weyl quantization of symbols on $\mathbb{R}^{2d}$. For

$$a(x,\xi) \in S^0(\mathbb{R}^{2d}) := \{a(x,\xi) \in C^\infty(\mathbb{R}^{2d}) : \text{for all } \alpha \in \mathbb{N}^{2d} \text{ there exists } C_\alpha > 0 \text{ such that } |\partial^\alpha_{x,\xi} a(x,\xi)| \le C_\alpha\},$$

we define $\operatorname{Op}_h^w(a)$ acting on $u \in \mathcal{S}(\mathbb{R}^d)$ (Schwartz) by

$$\operatorname{Op}_h^w(a)u(x) := \frac{1}{2\pi h} \iint_{\mathbb{R}^{2d}} e^{\frac{i}{h}(x-y)\cdot\xi} a\Big(\frac{x+y}{2}, \xi\Big) u(y)\, \mathrm{d}y\, \mathrm{d}\xi \in \mathcal{S}(\mathbb{R}^d).$$

By duality, $\operatorname{Op}_h^w(a)$ extends to an operator from $\mathcal{S}'(\mathbb{R}^d)$ to itself. If a has certain decay properties, $\operatorname{Op}_h^w(a)$ can be shown to map boundedly between Sobolev spaces.

Identifying $\mathbb{C}$ with $\mathbb{R}^{2d}$, this allows us to quantize functions on $\mathbb{C}$. We can also consider matrix-valued symbols by quantizing each component individually. What requires some discussion is how to construct a quantization of symbols on $\mathbb{C}/\Gamma$ that map from some set of functions on $\mathbb{C}/\Gamma$ to some other set of functions on $\mathbb{C}/\Gamma$.

Let us begin by defining the space of symbols we wish to quantize. These symbols will be functions on the cotangent bundle $T^*(\mathbb{C}/\Gamma)$. The cotangent bundle $T^*(\mathbb{C}/\Gamma)$ can be identified with $(\mathbb{C}/\Gamma)\times\mathbb{R}^2$. Elements of $T^*(\mathbb{C}/\Gamma)$ can be written as $(z,(\xi,\eta))$ with this identification, where $z\in\mathbb{C}/\Gamma$ and $(\xi,\eta)\in\mathbb{R}^2$. We also identify $\mathbb{R}^2$ with $\mathbb{C}$ by writing $\zeta=\xi+i\eta$, so that elements of $T^*(\mathbb{C}/\Gamma)$ can also be written (z,ζ).

Definition A.1 ($S^k(T^*(\mathbb{C}/\Gamma))$). For $k\in\mathbb{R}$, we define $S^k(T^*(\mathbb{C}/\Gamma))$ as the set of functions f in $C^\infty(T^*(\mathbb{C}/\Gamma))$ such that for all $\alpha,\beta\in\mathbb{N}^2$, there exists $C_{\alpha,\beta}>0$ such that

$$|\partial^\alpha_{z,\bar z}\partial^\beta_{\zeta,\bar\zeta} f(z,\zeta)|\le C_{\alpha,\beta}(1+|\zeta|)^{k-|\beta|}.$$

The symbols we are interested in quantizing to build $D_h(\beta)$ are defined as follows.

Definition A.2 ($S^k(T^*(\mathbb{C}/\Gamma);\mathbb{C}^{2\times 2})$). For $k\in\mathbb{R}$, we say a matrix-valued function $A\in C^\infty(T^*(\mathbb{C}/\Gamma);\mathbb{C}^{2\times 2})$ is *in* $S^k(T^*(\mathbb{C}/\Gamma);\mathbb{C}^{2\times 2})$ if

$$A(z,\zeta)=\begin{pmatrix} A_{11}(z,\zeta) & A_{12}(z,\zeta)\\ A_{21}(z,\zeta) & A_{22}(z,\zeta)\end{pmatrix}$$

and $A_{ij}\in S^k(T^*(\mathbb{C}/\Gamma))$ for each i,j.

We define the Weyl quantization of matrix-valued symbols by quantizing element-wise

$$\mathrm{Op}^w_h(A):=\begin{pmatrix} \mathrm{Op}^w_h(A_{11}) & \mathrm{Op}^w_h(A_{12})\\ \mathrm{Op}^w_h(A_{21}) & \mathrm{Op}^w_h(A_{22})\end{pmatrix},$$

where $\mathrm{Op}^w_h(A_{ij})$ is defined as

$$\begin{aligned}&\mathrm{Op}^w_h(A_{ij})u(x_1+ix_2)\\ &:=\frac{1}{(2\pi h)^2}\int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} e^{\frac{i}{h}\langle(x-y),\xi\rangle}\widetilde{A}_{ij}\Big(\frac{x_1+y_1}{2}+i\frac{x_2+y_2}{2},\xi_1+i\xi_2\Big)u(y_1+iy_2)\,\mathrm{d}y\,\mathrm{d}\xi\end{aligned}$$

for $x\in\mathbb{R}^2$, $u\in\mathcal{S}(\mathbb{C})$, and $\widetilde{A}_{ij}$ is the Γ-periodization of A_{ij} in the z variable. This integral is well defined for $u\in\mathcal{S}(\mathbb{C})$. By duality, it extends to a linear operator from $\mathcal{S}'(\mathbb{C})$ to itself. Note that because each $\widetilde{A}_{ij}$ is periodic on Γ, $\mathrm{Op}^w_h(A_{ij})$ maps the Γ-periodic elements of $\mathcal{S}'(\mathbb{C})$ to the Γ-periodic elements of $\mathcal{S}'(\mathbb{C})$.

We next want to show that quantizations of elements of $S^k(T^*(\mathbb{C}/\Gamma);\mathbb{C}^{2\times 2})$ are bounded maps between certain Sobolev spaces on $\mathbb{C}/\Gamma$. This requires a brief digression into defining such Sobolev spaces.

Because $\Gamma \simeq \mathbb{Z}^2$, any element $u \in L^2(\mathbb{C}/\Gamma)$ has a Fourier series representation

$$u(z) = \sum_{k\in\Gamma^*} c_k e_k(z) \tag{A.1}$$

where for each $k \in \Gamma^*$

$$e_k(z) := (\mathrm{Vol}(\mathbb{C}/\Gamma))^{-1/2} e^{\frac{i}{2}(z\bar{k}+\bar{z}k)}.$$

These e_k form an orthonormal basis so that

$$\|u\|^2_{L^2(\mathbb{C}/\Gamma)} = \sum_{k\in\Gamma^*} |c_k|^2.$$

For $s \in \mathbb{R}$, define the h-dependent Sobolev space $H^s_h(\mathbb{C}/\Gamma)$ as the vector space of elements $u \in L^2(\mathbb{C}/\Gamma)$ such that

$$\|\pi_{\mathbb{C}/\Gamma} \circ \mathrm{Op}^w_h(\langle\xi\rangle^s) \circ \tau_{\mathbb{C}/\Gamma} u\|_{L^2(\mathbb{C}/\Gamma)} < \infty,$$

where

$$\langle\xi\rangle := (1+|\xi|^2)^{1/2},$$

$\pi_{\mathbb{C}/\Gamma}\colon L^2_{\mathrm{loc}}(\mathbb{C}) \to L^2(\mathbb{C}/\Gamma)$ is the restriction to the fundamental domain of Γ centered at the origin (which we abuse and call $\mathbb{C}/\Gamma$), $\tau_{\mathbb{C}/\Gamma}\colon L^2(\mathbb{C}/\Gamma) \to L^2_{\mathrm{loc}}(\mathbb{C})$ is the Γ-periodization of an element of $L^2(\mathbb{C}/\Gamma)$ to $L^2_{\mathrm{loc}}(\mathbb{C})$, and Op^w_h denotes the usual Weyl quantization on $\mathbb{R}^2$ which is identified with $\mathbb{C}$.

If u is written in the form (A.1) and $s \in \mathbb{R}_{\geq 0}$, then $u \in H^s_h(\mathbb{C}/\Gamma)$ if and only if

$$\sum_{k\in\Gamma^*} |hk|^{2s}|c_k|^2 < \infty. \tag{A.2}$$

We define the $H^s_h(\mathbb{C}/\Gamma)$ norm as the square root of the left-hand side of (A.2).

Proposition A.3. *Suppose $k, s \in \mathbb{R}$ and $A \in S^k(T^*(\mathbb{C}/\Gamma);\mathbb{C}^{2\times 2})$. Then $\pi_{\mathbb{C}/\Gamma} \circ \mathrm{Op}^w_h(A) \circ \tau_{\mathbb{C}/\Gamma}$ is a bounded map from $H^s_h(\mathbb{C}/\Gamma;\mathbb{C}^2)$ to $H^{s-k}_h(\mathbb{C}/\Gamma;\mathbb{C}^2)$.*

Proof. Here we adapt the argument from [32, Theorem 5.5].

Step 1. *Conjugate the symbol to reduce to $k = s = 0$.* It suffices to work componentwise. Let $a \in S^k(T^*(\mathbb{C}/\Gamma))$. It suffices to show that

$$\pi_{\mathbb{C}/\Gamma}\, \mathrm{Op}^w_h(\langle\xi\rangle^k)\, \mathrm{Op}^w_h(a)\, \mathrm{Op}^w_h(\langle\xi\rangle^{-s})\tau_{\mathbb{C}/\Gamma}\colon L^2(\mathbb{C}/\Gamma) \to L^2(\mathbb{C}/\Gamma) \tag{A.3}$$

is bounded. By the composition of Weyl quantizations (see for instance [32, Theorem 4.18]), there exists $\tilde{b} \in S^0(\mathbb{C})$ such that

$$\operatorname{Op}_h^w(\tilde{b}) = \operatorname{Op}_h^w(\langle\xi\rangle^k)\operatorname{Op}_h^w(\tau_{\mathbb{C}/\Gamma}a)\operatorname{Op}_h^w(\langle\xi\rangle^{-s}).$$

Because $\tau_{\mathbb{C}/\Gamma}a$ is Γ-periodic in $\mathbb{C}$ and is conjugated by symbols which are also Γ-periodic in $\mathbb{C}$, $\tilde{b}$ is Γ-periodic in $\mathbb{C}$. Let $b = \pi_{\mathbb{C}/\Gamma}\tilde{b} \in S^0(T^*(\mathbb{C}/\Gamma))$.

Step 2. *Decompose* $\operatorname{Op}_h^w(\tilde{b})$. Let $u \in L^2(\mathbb{C}/\Gamma)$ and define $\tilde{u} := \tau_{\mathbb{C}/\Gamma}u \in L^2_{\mathrm{loc}}(\mathbb{C})$. We now have, for $x \in \mathbb{C}/\Gamma$, that $\operatorname{Op}_h^w(\tilde{b})\tilde{u}(x)$ equals

$$\begin{aligned}&\frac{1}{(2\pi h)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{(i/h)\langle(x-y),\xi\rangle}\tilde{b}\Big(\frac{x_1+y_1}{2}+i\frac{x_2+y_2}{2},\xi_1+i\xi_2\Big)\tilde{u}(y_1+iy_2)\,\mathrm{d}y\,\mathrm{d}\xi\\ &\quad= \sum_{k\in\Gamma} B_k\tilde{u}(x)\end{aligned}$$

where $B_k\tilde{u}(x)$ is defined as

$$\begin{aligned}\frac{1}{(2\pi h)^2}\int_{\mathbb{R}^2}\int_{\mathbb{C}/\Gamma} e^{(i/h)\langle(x-y+k),\xi\rangle}b\Big(\frac{x_1+y_1-k_1}{2}+i\frac{x_2+y_2-k_2}{2},\xi_1+i\xi_2\Big)&\\ \cdot\,\tilde{u}(y_1+iy_2)\,\mathrm{d}y\,\mathrm{d}\xi&\end{aligned}$$

where $k = k_1 + ik_2$.

By periodicity of $\tilde{b}$, we see for each $k \in \Gamma$,

$$B_k = 1_{\mathbb{C}/\Gamma}T_{-k}\operatorname{Op}_h^w(\tilde{b})1_{\mathbb{C}/\Gamma} \tag{A.4}$$

where $T_{-k}v(x) := v(x+k)$ and $1_{\mathbb{C}/\Gamma}$ is the characteristic function on the fundamental domain of Γ.

Step 3. *Bound each component* B_k *for* k *away from zero.* For each $N \in \mathbb{N}$, x and y in the fundamental domain of Γ, and $|k| > 8\pi$ (so that $|x-y+k| \neq 0$), we can write

$$e^{(i/h)\langle x-y+k,\xi\rangle} = h^{2N}|x-y+k|^{-2N}|D_\xi|^{2N}e^{(i/h)\langle x-y+k,\xi\rangle}$$

so that by integration by parts (using that the Fourier transform of compactly supported functions will be Schwartz in ξ)

$$B_k = 1_{\mathbb{C}/\Gamma}T_{-k}\widetilde{B}_k 1_{\mathbb{C}/\Gamma}$$

with $\widetilde{B}_k u(x)$ defined as

$$\begin{aligned}\frac{1}{(2\pi h)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{\frac{i}{h}\langle x-y,\xi\rangle}\chi(x-k)\chi(y)h^{2N}|x-y|^{-2N}(|D_\xi|^{2N}\tilde{b})\Big(\frac{x+y}{2},\xi\Big)&\\ \cdot\,\tilde{u}(y)\,\mathrm{d}y\,\mathrm{d}\xi&\end{aligned}$$

with $\chi \in C_0^\infty(\mathbb{R}^2)$ identically 1 near the fundamental domain of Γ (identified with $\mathbb{R}^2$). We can then apply the Schur test to get that there exists a $C > 0$ independent of k such that

$$\|\widetilde{B}_k\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \le Ch^{2N}|k|^{-2N}$$

so that $\|B_k\|_{L^2(\mathbb{C}/\Gamma)\to L^2(\mathbb{C}/\Gamma)} = \mathcal{O}(h^\infty|k|^{-\infty})$.

From (A.4), B_k is bounded from $L^2(\mathbb{C}/\Gamma)$ to $L^2(\mathbb{C}/\Gamma)$ for $|k| \le 8\pi$. This, combined with the L^2 estimate for B_k away from zero gives (A.3). ∎

We therefore have the following quantization.

Definition A.4 (Quantization of $S^k(T^*(\mathbb{C}/\Gamma); \mathbb{C}^{2\times 2})$). For $k \in \mathbb{R}$ and

$$A \in S^k(T^*(\mathbb{C}/\Gamma); \mathbb{C}^{2\times 2}),$$

we define

$$\mathrm{Op}_h(A) := \pi_{\mathbb{C}/\Gamma} \circ \mathrm{Op}_h^w(A) \circ \tau_{\mathbb{C}/\Gamma}$$

We note that for $A \in S^k(T^*(\mathbb{C}/\Gamma); \mathbb{C}^{2\times 2})$, we get by Proposition A.3, that

$$\mathrm{Op}_h(A) \colon H^k(\mathbb{C}/\Gamma; \mathbb{C}^2) \to L^2(\mathbb{C}/\Gamma; \mathbb{C}^2).$$

For $k \in \mathbb{R}$, matrix-valued symbols $A, A_j \in S^k(T^*(\mathbb{C}/\Gamma); \mathbb{C}^{2\times 2})$ (with $j \in \mathbb{Z}_{\ge 0}$) depending on h, we write

$$A \sim \sum_0^\infty h^j A_j \tag{A.5}$$

in $S^k(T^*(\mathbb{C}/\Gamma); \mathbb{C}^{2\times 2})$ if for all $J \in \mathbb{N}$,

$$A - \sum_{j=0}^J h^j A_j \in h^{J+1} S^k(T^*(\mathbb{C}/\Gamma); \mathbb{C}^{2\times 2}).$$

Definition A.5 (Principal symbol). If $A \in S^k(T^*(\mathbb{C}/\Gamma); \mathbb{C}^{2\times 2})$ has an asymptotic expansion as in (A.5), then its *principal symbol* is the equivalence class of matrix-valued functions A_0 with the equivalence relation

$$A \sim B \iff A - B \in hS^k(T^*(\mathbb{C}/\Gamma); \mathbb{C}^{2\times 2}).$$

Proposition A.6 (Composition). *Suppose* $k, \ell \in \mathbb{R}$,

$$A \in S^k(T^*(\mathbb{C}/\Gamma); \mathbb{C}^{2\times 2}) \quad \textit{and} \quad B \in S^\ell(T^*(\mathbb{C}/\Gamma); \mathbb{C}^{2\times 2}).$$

Then there exists a $C \in S^{k+\ell}(T^*(\mathbb{C}/\Gamma);\mathbb{C}^{2\times 2})$ *such that*

$$\mathrm{Op}_h(A) \circ \mathrm{Op}_h(B) = \mathrm{Op}_h(C)$$

and

$$C \sim AB + \sum_1^\infty h^j \begin{pmatrix} (C_{11})_j & (C_{12})_j \\ (C_{21})_j & (C_{22})_j \end{pmatrix}.$$

Proof. By standard results about the composition of Weyl operators (see for instance [32, Section 4.3]), we have that

$$\mathrm{Op}_h^w(\widetilde{A}) \circ \mathrm{Op}_h^w(\widetilde{B}) = \mathrm{Op}_h^w(\widetilde{C})$$

where $\widetilde{C} = \widetilde{A}\#\widetilde{B} = \widetilde{A}\widetilde{B} + \mathcal{O}(h) \in S^{k+\ell}(T^*(\mathbb{C});\mathbb{C}^{2\times 2})$, and $\widetilde{A}$ and $\widetilde{B}$ denotes the periodization in the z variable of all matrix components. It is easy to verify from the formula for $a \# b$ (see, for instance, [32, Theorem 4.12]) that if a and b are Γ periodic in z, then $a \# b$ is Γ periodic in z. Therefore, we can restrict $\widetilde{C}$ to the fundamental domain of Γ in the variable z, which we denote by $C \in S^{k+\ell}(T^*(\mathbb{C}/\Gamma);\mathbb{C}^{2\times 2})$.

If $u \in H^{k+\ell}(\mathbb{C}/\Gamma;\mathbb{C}^2)$ and $\tilde{u} := \pi_{\mathbb{C}/\Gamma}^{-1} u$, then

$$\begin{aligned} \mathrm{Op}_h(A) \circ \mathrm{Op}_h(B)u &= (\mathrm{Op}_h^w(A)\,\mathrm{Op}_h^w(B)\tilde{u})|_{\mathbb{C}/\Gamma} \\ &= (\mathrm{Op}_h^w(C)\tilde{u})|_{\mathbb{C}/\Gamma} = \mathrm{Op}_h(C)u. \end{aligned}$$ ∎

Proposition A.7 (Parametrix construction). *If* $k \in \mathbb{R}$ *and* $F \in S^k(T^*(\mathbb{C}/\Gamma);\mathbb{C}^{2\times 2})$ *with principal symbol*

$$\begin{pmatrix} A(z,\zeta) & B(z,\zeta) \\ C(z,\zeta) & D(z,\zeta) \end{pmatrix}$$

such there exists a $c_0 > 0$ *with* $A(z,\zeta)D(z,\zeta) - B(z,\zeta)C(z,\zeta) > c_0$*, then there exists* $G \in S^{-k}(T^*(\mathbb{C}/\Gamma);\mathbb{C}^{2\times 2})$ *such that*

$$\mathrm{Op}_h(G) \circ \mathrm{Op}_h(F) = \mathrm{Op}_h(F) \circ \mathrm{Op}_h(G) = O(h^\infty)$$

and the principal symbol of G *is*

$$\begin{pmatrix} A(z,\zeta) & B(z,\zeta) \\ C(z,\zeta) & D(z,\zeta) \end{pmatrix}^{-1}.$$

Proof. Because we have a composition rule for this symbol class (Proposition A.6), the claim follows by an identical argument as the usual parametrix construction; see, for instance, [10, Theorem 4.1]. ∎

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