Improved semiclassical eigenvalue estimates for the Laplacian and the Landau Hamiltonian

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Abstract. The Berezin–Li–Yau and the Kröger inequalities show that Riesz means of order ≥ 1 of the eigenvalues of the Laplacian on a domain Ω of finite measure are bounded in terms of their semiclassical limit expressions. We show that these inequalities can be improved by a multiplicative factor that depends only on the dimension and the product $\sqrt{\Lambda}|\Omega|^{1/d}$, where Λ is the eigenvalue cut-off parameter in the definition of the Riesz mean. The same holds when $|\Omega|^{1/d}$ is replaced by a generalized inradius of Ω. Finally, we show similar inequalities in two dimensions in the presence of a constant magnetic field.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure. We denote by $-\Delta_\Omega^D$ and $-\Delta_\Omega^N$ the Dirichlet and Neumann realizations of $-\Delta$ in $L^2(\Omega)$, defined via quadratic forms; see [6, Section 3.1]. The famous Weyl asymptotics state that for every $\gamma \geq 0$ and $\sharp \in \{D, N\}$, one has

$$\mathrm{Tr}(-\Delta_\Omega^\sharp - \Lambda)_-^\gamma \sim L_{\gamma,d}^{\mathrm{sc}}|\Omega|\Lambda^{\gamma+d/2} \quad \text{as } \Lambda \to \infty. \tag{1.1}$$

Here, and in what follows, $L_{\gamma,d}^{\mathrm{sc}}$ denotes the so-called semiclassical constant

$$L_{\gamma,d}^{\mathrm{sc}} = \frac{\Gamma(1+\gamma)}{(4\pi)^{d/2}\Gamma(1+\gamma+d/2)},$$

and we use the notation $x_\pm = \frac{1}{2}(|x| \pm x)$. For $\gamma = 0$, $\mathrm{Tr}(-\Delta_\Omega^\sharp - \Lambda)_-^\gamma$ is interpreted as the number of eigenvalues $< \Lambda$. In the Neumann case, the validity of the asymptotics (1.1) requires some mild conditions on Ω. We refer to [6, Sections 3.2 and 3.3] for a proof and further background.

It is remarkable that for $\gamma \geq 1$ the asymptotics (1.1) are accompanied by uniform inequalities

$$\mathrm{Tr}(-\Delta_\Omega^{\mathrm{D}} - \Lambda)_-^\gamma \leq L_{\gamma,d}^{\mathrm{sc}}|\Omega|\Lambda^{\gamma+d/2} \leq \mathrm{Tr}(-\Delta_\Omega^{\mathrm{N}} - \Lambda)_-^\gamma \quad \text{for all } \Lambda \geq 0.$$

This was shown by Berezin [1], Li and Yau [26], and by Kröger [20] (see also [21]), respectively. The famous Pólya conjecture states that these uniform inequalities are valid for all $\gamma \geq 0$.

There is substantial literature on improvements of the Berezin–Li–Yau and Kröger inequalities, of which we cite [7, 10, 11, 13–15, 17, 18, 22–25, 31, 36, 38]. In most of these works, the respective inequalities are improved under the assumption that Ω belongs to a restricted class of open sets of finite measure. Typical assumptions are that Ω has a finite moment of inertia, that it is bounded, or that it admits a Hardy inequality. However, in many applications it is an important aspect of the Berezin–Li–Yau and Kröger inequalities that they are valid for *any* open subset $\Omega \subset \mathbb{R}^d$ of finite measure (which is necessary to state the inequalities). In this paper, our main result is that improved versions of the Berezin–Li–Yau and Kröger inequalities hold under the same minimal assumption, i.e., that Ω is an open set of finite measure.

Closest to this study, both from a technical and a conceptual point of view, are the work [25] by Li and Tang (where Ω is assumed to be bounded) and the work [7] by two of us (where Ω is assumed to have finite width). In particular, in the latter paper it is shown that there are constants $c, c' > 0$ such that for every open set $\Omega \subset \mathbb{R}^d$ of finite measure and finite width, one has

$$\mathrm{Tr}(-\Delta_\Omega^{\mathrm{D}} - \Lambda)_- \leq L_{1,d}^{\mathrm{sc}}|\Omega|\Lambda^{1+d/2}(1 - ce^{-c'\sqrt{\Lambda}w_\Omega}) \tag{1.2}$$

and

$$\mathrm{Tr}(-\Delta_\Omega^{\mathrm{N}} - \Lambda)_- \geq L_{1,d}^{\mathrm{sc}}|\Omega|\Lambda^{1+d/2}(1 + ce^{-c'\sqrt{\Lambda}w_\Omega}), \tag{1.3}$$

where $w_\Omega := \inf_{\omega \in \mathbb{S}^{d-1}}(\sup_{x \in \Omega} \omega \cdot x - \inf_{x \in \Omega} \omega \cdot x)$ is the width of Ω.

Our goal in this paper is three-fold. First, we extend the improved bounds from [7] to arbitrary open sets of finite measure, without any additional assumptions. Second, we will extend the improved bounds to Riesz exponents $\gamma > 1$. Third, we will prove corresponding improved bounds in the presence of a constant magnetic field in two dimensions.

Before stating our bounds, let us comment on the relative improvement in our bounds in comparison to the Berezin–Li–Yau and Kröger inequalities. As in (1.2) and (1.3), our relative remainder is exponentially small in the limit $\Lambda \to \infty$. This might appear rather disappointing as two-term semiclassical asymptotics suggest a relative improvement of order $1/\sqrt{\Lambda}$ in this regime (at least for sufficiently regular Ω) and indeed some of the above mentioned improvements capture at least an inverse

power-like dependence on Λ. However, both (1.2) and (1.3) provide a substantial improvement in the intermediate spectral regime $\sqrt{\Lambda} w_\Omega \lesssim 1$ and a corresponding statement holds for the results proved in this paper. In fact, the main application of (1.2) and (1.3) in [7] was to show that the validity of stronger improvements of the Berezin–Li–Yau and Kröger inequalities which had been proved in the regime of large Λ (by using semiclassical techniques as, for example, in [8]) could be extended to the entire range $\Lambda \geq 0$. We believe that the results obtained here can provide a useful tool in the regime when Ω has non-trivial geometry on the natural length-scale $1/\sqrt{\Lambda}$, in which case semiclassical techniques are not applicable.

We begin with the non-magnetic case, where we prove the following.

Theorem 1. *For any $d \geq 1$, there are constants $c, c' > 0$ such that for any open set $\Omega \subset \mathbb{R}^d$ of finite measure, any $\Lambda > 0$ and any $\gamma \geq 1$, one has*

$$\mathrm{Tr}(-\Delta_\Omega^{\mathrm{D}} - \Lambda)_-^\gamma \leq L_{\gamma,d}^{\mathrm{sc}} |\Omega| \Lambda^{\gamma+d/2} (1 - ce^{-c'\sqrt{\Lambda}|\Omega|^{1/d}})$$

and

$$\mathrm{Tr}(-\Delta_\Omega^{\mathrm{N}} - \Lambda)_-^\gamma \geq L_{\gamma,d}^{\mathrm{sc}} |\Omega| \Lambda^{\gamma+d/2} (1 + ce^{-c'\sqrt{\Lambda}|\Omega|^{1/d}}).$$

In fact, this theorem is a consequence of a more general result, Theorem 4, which we describe later in this introduction.

Let us turn to the case of a constant magnetic field in dimension $d = 2$. Let $B > 0$ and $A(x) := \frac{B}{2} \left(\begin{smallmatrix} x_2 \\ -x_1 \end{smallmatrix} \right)$ for $x \in \mathbb{R}^2$. Let $\Omega \subset \mathbb{R}^2$ be an open set and let H_Ω^{D} and H_Ω^{N} be the Dirichlet and Neumann realizations in $L^2(\Omega)$ of the Landau Hamiltonian

$$(-i\,\partial_{x_1} + A_1(x))^2 + (-i\,\partial_{x_2} + A_2(x))^2.$$

Specifically, these operators are defined via the quadratic form (see [6, Section 3.1])

$$u \mapsto \int_\Omega |\nabla u(x) + iA(x)u(x)|^2 \, dx.$$

with form domains $H_{A,0}^1(\Omega)$ and $H_A^1(\Omega)$, respectively. Here $H_A^1(\Omega)$ consists of all functions in $L^2(\Omega)$ such that the distribution $(\nabla + iA)u$ is square-integrable in Ω, and $H_{A,0}^1(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $H_A^1(\Omega)$.

We are interested in bounds on

$$\mathrm{Tr}(H_\Omega^\sharp - \Lambda)_-^\gamma$$

that correspond to the above-mentioned strengthened versions of the Berezin–Li–Yau and Kröger inequalities for the Laplace operator without magnetic field. The magnetic

Berezin–Li–Yau and Kröger inequalities were shown by Erdős–Loss–Vougalter [3] and the first author [5], respectively, and state that

$$\operatorname{Tr}(H_\Omega^{\mathrm{D}} - \Lambda)_-^\gamma \le |\Omega| \frac{B}{2\pi} \sum_{k=1}^\infty (B(2k-1) - \Lambda)_-^\gamma \quad \text{for all } \Lambda > 0$$

and

$$\operatorname{Tr}(H_\Omega^{\mathrm{N}} - \Lambda)_-^\gamma \ge |\Omega| \frac{B}{2\pi} \sum_{k=1}^\infty (B(2k-1) - \Lambda)_-^\gamma \quad \text{for all } \Lambda > 0.$$

Note that here, instead of the quantity $L_{\gamma,2}^{\mathrm{sc}} \Lambda^{\gamma+1}$, the quantity $\frac{B}{2\pi} \sum_{k=1}^\infty (B(2k-1) - \Lambda)_-^\gamma$ appears. This is crucial in applications where both Λ and B are considered as asymptotic parameters; see, for instance, [2, 4, 9, 28, 29, 34].

Our second main result is a strengthened version of the magnetic Berezin–Li–Yau and Kröger inequalities.

Theorem 2. *There are constants $c, c' > 0$ such that for any open set $\Omega \subset \mathbb{R}^2$ of finite measure, any $B > 0$, any $\Lambda > 0$ and any $\gamma \ge 1$ one has*

$$\operatorname{Tr}(H_\Omega^{\mathrm{D}} - \Lambda)_-^\gamma \le |\Omega| \frac{B}{2\pi} \sum_{k=1}^\infty (B(2k-1) - \Lambda)_-^\gamma (1 - ce^{-c'(\sqrt{\Lambda|\Omega|} + B|\Omega|)})$$

and

$$\operatorname{Tr}(H_\Omega^{\mathrm{N}} - \Lambda)_-^\gamma \ge |\Omega| \frac{B}{2\pi} \sum_{k=1}^\infty (B(2k-1) - \Lambda)_-^\gamma (1 + ce^{-c'(\sqrt{\Lambda|\Omega|} + B|\Omega|)}).$$

Again, this theorem is a consequence of a more general theorem that we describe momentarily.

These promised stronger results are stated in terms of a "regularized inradius." Define for $\theta \in (0, 1)$ the regularized inradius of a measurable set $\Omega \subset \mathbb{R}^d$ by

$$\rho_\theta(\Omega) := \inf\left\{\rho > 0 : \sup_{x \in \mathbb{R}^d} \frac{|\Omega \cap B_\rho(x)|}{|B_\rho(x)|} \le \theta\right\}. \tag{1.4}$$

Here, $B_\rho(x)$ denotes the open ball of radius ρ centered at $x \in \mathbb{R}^d$. The relevance of the regularized inradius $\rho_\theta(\Omega)$ was emphasized by Lieb [27] in his lower bound on the smallest Dirichlet eigenvalue of the Laplacian.

Remark 3. Let us summarize some facts about $\rho_\theta(\Omega)$.

(a) For any $\theta \in (0, 1)$ we have the bound

$$\rho_\theta(\Omega) \le \left(\frac{|\Omega|}{\theta|B_1|}\right)^{1/d}$$

Indeed, for any $x \in \mathbb{R}^d$ and $\rho \geq \rho_0 := \left(\frac{|\Omega|}{\theta|B_1|}\right)^{1/d}$ it holds that

$$|\Omega \cap B_\rho(x)| \leq |\Omega| = \theta|B_{\rho_0}(x)| \leq \theta|B_\rho(x)|.$$

(b) We claim that for any $d \in \mathbb{N}$ and any $\theta \in (0,1)$ there is a C such that for any set $\Omega \subset \mathbb{R}^d$ of finite width one has

$$\rho_\theta(\Omega) \leq C w_\Omega$$

Indeed, for any $\rho > w_\Omega/2, x \in \mathbb{R}^d$ it holds that

$$|\Omega \cap B_\rho(x)| \leq \sup_{z \in \mathbb{R}^d} |\{y = (y_1, \ldots, y_d) \in \mathbb{R}^d : |y_d| < w_\Omega/2\} \cap B_\rho(z)|$$

$$= |B_\rho(x)| \sup_{z \in \mathbb{R}^d} \frac{|\{y = (y_1, \ldots, y_d) \in \mathbb{R}^d : |y_d| < \frac{w_\Omega}{2\rho}\} \cap B_1(z)|}{|B_1|}.$$

Since the supremum on the right-hand side tends to 0 as $w_\Omega/(2\rho) \to 0$, for any given $\theta > 0$ there is a C such that it is $\leq \theta$ when $w_\Omega/(2\rho) \leq (2C)^{-1}$.

(c) If $\rho_{\text{in}}(\Omega) := \sup_{x \in \Omega} \text{dist}(x, \partial\Omega)$ denotes the inradius of Ω, then

$$\rho_\theta(\Omega) \geq \rho_{\text{in}}(\Omega) \quad \text{for all } \theta \in (0,1).$$

This follows from the fact that $\sup_{x \in \mathbb{R}^d} |\Omega \cap B_\rho(x)|/|B_\rho(x)| = 1$ for $\rho \in (0, \rho_{\text{in}}]$.

(d) From the Lebesgue differentiation theorem, $\lim_{\rho \to 0} \frac{|\Omega \cap B_\rho(x)|}{|B_\rho(x)|} = \mathbb{1}_\Omega(x)$ for almost every $x \in \mathbb{R}^d$. Therefore, $\rho_\theta(\Omega) = 0$ if and only if $|\Omega| = 0$.

(e) If $\rho_\theta(\Omega) < \infty$, then for all $x \in \mathbb{R}^d$ it holds that

$$|\Omega \cap B_{\rho_\theta(\Omega)}(x)| \leq \theta|B_{\rho_\theta(\Omega)}(x)|.$$

Indeed, there is a sequence $\{\rho_k\}_{k \geq 1} \subset [\rho_\theta(\Omega), \infty)$ such that $\lim_{k \to \infty} \rho_k = \rho_\theta(\Omega)$ and $|\Omega \cap B_\rho(x)| \leq \theta|B_\rho(x)|$ for each $k \geq 1$ and all $x \in \mathbb{R}^d$. Since $\Omega \cap B_{\rho_\theta(\Omega)}(x) \subseteq \Omega \cap B_{\rho_k}(x)$, we deduce that

$$|\Omega \cap B_{\rho_\theta(\Omega)}(x)| \leq |\Omega \cap B_{\rho_k}(x)| \leq \theta|B_{\rho_k}(x)| \quad \text{for all } k \geq 1, x \in \mathbb{R}^d. \quad (1.5)$$

The claimed inequality follows by taking the limit $k \to \infty$. Moreover, for every $\varepsilon \in (0,1]$ there is an $x_\varepsilon \in \mathbb{R}^d$ such that

$$|\Omega \cap B_{\rho_\theta(\Omega)}(x_\varepsilon)| \geq \theta(1 - \varepsilon)|B_{\rho_\theta(\Omega)}(x_\varepsilon)|.$$

Indeed, for each $\rho < \rho_\theta(\Omega)$ there is a $x'_\rho \in \mathbb{R}^d$ so that $|\Omega \cap B_\rho(x'_\rho)| \geq \theta|B_\rho(x'_\rho)|$, by inclusion and choosing $\rho \in [(1-\varepsilon)^{1/d}\rho_\theta(\Omega), \rho_\theta(\Omega))$ we find

$$|\Omega \cap B_{\rho_\theta(\Omega)}(x'_\rho)| \geq |\Omega \cap B_\rho(x'_\rho)| \geq \theta|B_\rho(x'_\rho)| \geq \theta(1-\varepsilon)|B_{\rho_\theta(\Omega)}(x'_\rho)|$$

which is the claimed inequality with $x_\varepsilon = x'_\rho$. If $|\Omega| < \infty$, one can extend the above inequality to $\varepsilon = 0$ by arguing that non-compactness of the set $\{x_\varepsilon\}_{\varepsilon \in (0,1)}$ would contradict the finiteness of $|\Omega|$.

We now state the stronger versions of Theorems 1 and 2.

Theorem 4. *For any $d \geq 1$ and $\theta \in (0,1)$ there are constants $c, c' > 0$ such that for any open set $\Omega \subset \mathbb{R}^d$ of finite measure, any $\Lambda > 0$ and any $\gamma \geq 1$ one has*

$$\mathrm{Tr}(-\Delta_\Omega^{\mathrm{D}} - \Lambda)_-^\gamma \leq L_{\gamma,d}^{\mathrm{sc}} |\Omega| \Lambda^{\gamma+d/2} (1 - ce^{-c'\sqrt{\Lambda}\rho_\theta(\Omega)})$$

and

$$\mathrm{Tr}(-\Delta_\Omega^{\mathrm{N}} - \Lambda)_-^\gamma \geq L_{\gamma,d}^{\mathrm{sc}} |\Omega| \Lambda^{\gamma+d/2} (1 + ce^{-c'\sqrt{\Lambda}\rho_\theta(\Omega)}).$$

Clearly, in view of Remark 3 (a), Theorem 1 is an immediate consequence of Theorem 4, taking any fixed value of θ. Similarly, the improved bound in [7, Proposition 2.1] is a consequence of Theorem 1 and Remark 3 (b), except for the fact that in [7] we also proved that the constants c, c' can be chosen independently of d.

In the magnetic case we have the corresponding result.

Theorem 5. *For any $\theta \in (0,1)$, there are constants $c, c' > 0$ such that for any open set $\Omega \subset \mathbb{R}^2$ with $|\Omega| < \infty$, any $B > 0$, any $\Lambda > 0$ and any $\gamma \geq 1$, one has*

$$\mathrm{Tr}(H_\Omega^{\mathrm{D}} - \Lambda)_-^\gamma \leq |\Omega| \frac{B}{2\pi} \sum_{k=1}^\infty (B(2k-1) - \Lambda)_-^\gamma (1 - ce^{-c'(\rho_\theta(\Omega)\sqrt{\Lambda} + \rho_\theta(\Omega)^2 B)})$$

and

$$\mathrm{Tr}(H_\Omega^{\mathrm{N}} - \Lambda)_-^\gamma \geq |\Omega| \frac{B}{2\pi} \sum_{k=1}^\infty (B(2k-1) - \Lambda)_-^\gamma (1 + ce^{-c'(\rho_\theta(\Omega)\sqrt{\Lambda} + \rho_\theta(\Omega)^2 B)}).$$

Again, in view of Remark 3 (a), Theorem 2 is an immediate consequence of Theorem 5, taking any fixed value of θ.

Let us explain the basic idea of the proof of Theorems 4 and 5. Following the classical proofs of the Berezin–Li–Yau and Kröger inequalities, as well as their magnetic analogues, we arrive at terms of the form

$$\|\mathbb{1}(L > \Lambda)J\psi\|^2,$$

where L is either the Laplacian $-\Delta$ in $L^2(\mathbb{R}^d)$ or the Landau Hamiltonian H in $L^2(\mathbb{R}^2)$, where ψ is an eigenfunction of either the Dirichlet or the Neumann restriction of these operators to a set Ω of finite measure and where J denotes the extension by zero of a function defined on Ω to $\mathbb{R}^d$. In the standard proofs, the above term is dropped since it is non-negative.

This is wasteful, since the term $\|\mathbb{1}(L > \Lambda)J\psi\|^2$ is never equal to zero. In fact, a version of the uncertainty principle says that a function ψ cannot be simultaneously localized in a set Ω of finite measure and supported in the spectral subspace ran $\mathbb{1}(L \le \Lambda)$. Thus, our task is to find a quantitative version of the uncertainty principle to obtain a positive lower bound on the term $\|\mathbb{1}(L > \Lambda)J\psi\|^2$. As we will show, such lower bounds can be deduced from inequalities known as 'spectral inequalities' in the control theory community and as the "Logvinenko–Sereda theorem" in harmonic analysis. For these inequalities the notion of a thick set is relevant and this is where our generalized inradius $\rho_\theta(\Omega)$ appears naturally.

The fact that our arguments in the non-magnetic and in the magnetic case are rather parallel underlines the general mechanism that leads from quantitative versions of the uncertainty principle to improved semiclassical eigenvalue bounds. Our strategy might be applicable in other settings as well, for instance in the setting of homogeneous spaces [35] or of the Heisenberg group [12].

This paper is organized as follows. Section 2 is devoted to quantitative versions of the uncertainty principle. The corresponding main results are stated in Section 2.1 and proved in Section 2.3, after having recalled some necessary spectral inequalities in Section 2.2. Section 3 and, in particular, Section 3.2 are devoted to the proof of our main results, Theorems 4 and 5, in the case $\gamma = 1$. In order to streamline the proofs in the magnetic and in the non-magnetic case, we present the identities underlying the Berezin–Li–Yau and Kröger arguments in an abstract setting in Section 3.1. Finally, in Section 4 we deduce the case $\gamma > 1$ in Theorems 4 and 5 from the case $\gamma = 1$.

2. An uncertainty principle

2.1. The key bound

The goal of this section is to prove bounds that quantify the fact that if a function $f \not\equiv 0$ is supported on a set of finite measure, then its "high-energy" part $\mathbb{1}(-\Delta > \Lambda)f$ cannot vanish identically. The same is true when $-\Delta$ is replaced by the Landau Hamiltonian. By quantifying this fact, we mean giving a lower bound on the norm of $\mathbb{1}(-\Delta > \Lambda)f$ in terms of the norm of f, and it is in these bounds that the regularized inradius $\rho_\theta(\Omega)$ enters.

The Fourier transform is denoted by

$$\widehat{f}(\xi) = (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x)\, dx, \quad \xi \in \mathbb{R}^d.$$

Theorem 6. *Let $d \ge 1$. For any $\theta \in (0, 1)$, there are constants $c, c' > 0$ such that for any measurable set $\Omega \subset \mathbb{R}^d$ with $\rho_\theta(\Omega) < \infty$, any $f \in L^2(\mathbb{R}^d)$ and any $\Lambda > 0$, one*

has

$$\int\limits_{|\xi|^2 > \Lambda} |\widehat{\mathbb{1}_\Omega f}(\xi)|^2 \, d\xi \geq c e^{-c' \rho_\theta(\Omega) \sqrt{\Lambda}} \|\mathbb{1}_\Omega f\|^2.$$

We now state a corresponding result in the magnetic case. As is well known, the Landau Hamiltonian $H = \left(-i \partial_{x_1} + \frac{B}{2} x_2\right)^2 + \left(-i \partial_{x_2} - \frac{B}{2} x_1\right)^2$ in $L^2(\mathbb{R}^2)$ can be diagonalized explicitly. Its spectrum consists of the eigenvalues $B(2k - 1)$, $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$, each of which has infinite multiplicity. We denote by Π_k the spectral projection of H corresponding to the eigenvalue $B(2k - 1)$.

We shall show the following theorem.

Theorem 7. *For any $\theta \in (0, 1)$, there are constants $c, c' > 0$ such that for any measurable set $\Omega \subset \mathbb{R}^2$ with $\rho_\theta(\Omega) < \infty$, any $f \in L^2(\mathbb{R}^2)$, any $B > 0$ and any $\Lambda > 0$, one has*

$$\sum_{B(2k-1)>\Lambda} \|\Pi_k \mathbb{1}_\Omega f\|^2 \geq c e^{-c'(\rho_\theta(\Omega)\sqrt{\Lambda} + \rho_\theta(\Omega)^2 B)} \|\mathbb{1}_\Omega f\|^2.$$

The remainder of this section is devoted to the proof of these two theorems. They will be deduced from results that are known in the literature as "spectral inequalities."

2.2. Known spectral inequalities

For $\rho > 0, \kappa \in (0, 1)$, we say that a set $S \subset \mathbb{R}^d$ is (ρ, κ)-*thick* if it is measurable and satisfies

$$|S \cap B_\rho(x)| \geq \kappa |B_\rho(x)| \quad \text{for all } x \in \mathbb{R}^d.$$

Theorem 8. *For any $d \in \mathbb{N}$ and $\kappa \in (0, 1)$, there are constants $c, c' > 0$ such that the following holds. Let $S \subset \mathbb{R}^d$ be a (ρ, κ)-thick set and let $\Lambda > 0$. Then for any $g \in \operatorname{ran} \mathbb{1}(-\Delta \leq \Lambda)$, we have*

$$\|\mathbb{1}_S g\|^2 \geq c e^{-c' \rho \sqrt{\Lambda}} \|g\|^2.$$

This result is due to Kacnel'son [16] and Logvinenko and Sereda [30], who provided a complete proof for $d = 1$. The necessary modifications for $d \geq 2$ can be found in [37, Lemma 2.1]. We also refer to the latter paper for a further discussion of its history. In particular, we mention the work of Kovrijkine [19], where the dependence of the constants c and c' on κ is quantified.

In the recent paper [32], M. Täufer and the third named author extended Theorem 8 to the case of a constant magnetic field in two dimensions.

Theorem 9. *For any $\kappa \in (0, 1)$, there are constants $c, c' > 0$ such that the following holds. Let $S \subset \mathbb{R}^2$ be a (ρ, κ)-thick set and let $B, \Lambda > 0$. Then for any $g \in \mathrm{ran}(H \leq \Lambda)$, we have*

$$\|\mathbb{1}_S g\|^2 \geq ce^{-c'(\rho\sqrt{\Lambda}+\rho^2 B)}\|g\|^2.$$

In [32] a more general version of thickness is used in terms of rectangles with side lengths ℓ_1, ℓ_2 playing the role of the disks $B_\rho(x)$ in the definition used here. However, any set that is (ρ, κ)-thick according to the definition used here can easily be verified to be thick in the sense of [32] with respect to the square Q_ρ with side length 2ρ, for which we have $B_\rho(x) \subset Q_\rho(x) = x + (-\rho, \rho)^2$ and $|B_\rho(x)| = (\pi/4)|Q_\rho|$.

2.3. Proof of Theorems 6 and 7

Theorems 8 and 9 deal with functions from the "low energy" subspaces $\mathrm{ran}\,\mathbb{1}(-\Delta \leq \Lambda)$ and $\mathrm{ran}\,\mathbb{1}(H \leq \Lambda)$, while Theorems 6 and 7 involve projections onto the "high energy" subspaces $\mathrm{ran}\,\mathbb{1}(-\Delta > \Lambda)$ and $\mathrm{ran}\,\mathbb{1}(H > \Lambda)$. The following lemma will be useful to pass from the first to the second setting. We will apply it with $P = \mathbb{1}(-\Delta > \Lambda)$ or $P = \mathbb{1}(H > \Lambda)$ and with Q defined through multiplication with $\mathbb{1}_\Omega$.

Lemma 10. *Let P, Q be selfadjoint projections in a Hilbert space $\mathcal{H}$. Then, for any $g \in \mathcal{H}$, it holds that*

$$\|PQg\|_{\mathcal{H}}^2\|(\mathbb{1} - P)Qg\|_{\mathcal{H}}^2 \geq \|Qg\|_{\mathcal{H}}^2\|(\mathbb{1} - Q)(\mathbb{1} - P)Qg\|_{\mathcal{H}}^2.$$

In particular, if $g \in \mathrm{ran}(Q) \setminus \mathrm{ran}(P)$, then

$$\|Pg\|_{\mathcal{H}}^2 \geq \|g\|_{\mathcal{H}}^2 \frac{\|(\mathbb{1} - Q)(\mathbb{1} - P)g\|_{\mathcal{H}}^2}{\|(\mathbb{1} - P)g\|_{\mathcal{H}}^2}.$$

Proof. Using the fact that P, Q are selfadjoint projections and applying the Cauchy–Schwarz inequality, one finds

$$
\begin{aligned}
\|(\mathbb{1} - P)Qg\|_{\mathcal{H}}^4 &= \langle Qg, (\mathbb{1} - P)Qg \rangle_{\mathcal{H}}^2 \\
&= \langle Qg, Q(\mathbb{1} - P)Qg \rangle_{\mathcal{H}}^2 \\
&\leq \|Qg\|_{\mathcal{H}}^2\|Q(\mathbb{1} - P)Qg\|_{\mathcal{H}}^2.
\end{aligned}
\tag{2.1}
$$

Recall that if P' is a selfadjoint projection in $\mathcal{H}$, then

$$\|h\|_{\mathcal{H}}^2 = \|P'h\|_{\mathcal{H}}^2 + \|(\mathbb{1} - P')h\|_{\mathcal{H}}^2 \quad \text{for all } h \in \mathcal{H}.$$

In particular, it holds that

$$\|(\mathbb{1} - P)Qg\|_{\mathcal{H}}^2 = \|Qg\|_{\mathcal{H}}^2 - \|PQg\|_{\mathcal{H}}^2,$$
$$\|Q(\mathbb{1} - P)Qg\|_{\mathcal{H}}^2 = \|(\mathbb{1} - P)Qg\|_{\mathcal{H}}^2 - \|(\mathbb{1} - Q)(\mathbb{1} - P)Qg\|_{\mathcal{H}}^2.$$

Thus, (2.1) is equivalent to

$$\|(\mathbb{1} - P)Qg\|_{\mathcal{H}}^2(\|Qg\|_{\mathcal{H}}^2 - \|PQg\|_{\mathcal{H}}^2)$$
$$\leq \|Qg\|_{\mathcal{H}}^2(\|(\mathbb{1} - P)Qg\|_{\mathcal{H}}^2 - \|(\mathbb{1} - Q)(\mathbb{1} - P)Qg\|_{\mathcal{H}}^2),$$

which, after rearranging the terms, yields the first estimate in the lemma.

The second estimate follows by noting that for $g \in \mathrm{ran}(Q) \setminus \mathrm{ran}(P)$ we have $Qg = g$ and $\|(\mathbb{1} - P)g\|_{\mathcal{H}} \neq 0$ since $\mathrm{ran}(P) = \ker(\mathbb{1} - P)$. ∎

We can now prove Theorems 6 and 7. As a preliminary remark, let us find a connection between the regularized inradius of a set and the thickness properties of its complement.

We claim that, if $\rho_\theta(\Omega) < \infty$, then $\Omega^\complement$ is $(\rho_\theta(\Omega), 1 - \theta)$-thick. Indeed, for any $x \in \mathbb{R}^d$ it follows from (1.5) that

$$|\Omega^\complement \cap B_{\rho_\theta(\Omega)}(x)| = |B_{\rho_\theta(\Omega)}(x)| - |\Omega \cap B_{\rho_\theta(\Omega)}| \geq (1 - \theta)|B_{\rho_\theta(\Omega)}(x)|.$$

Conversely, if $\Omega^\complement$ is (ρ, κ)-thick, then $\rho_{1-\kappa}(\Omega) \leq \rho$; indeed,

$$|\Omega \cap B_\rho(x)| = |B_\rho(x)| - |\Omega^\complement \cap B_\rho(x)| \leq (1 - \kappa)|B_\rho(x)| \quad \text{for all } x \in \mathbb{R}^d.$$

The proofs of Theorems 6 and 7 are completely analogous and we only give the details of the second one.

Proof of Theorem 7. Let P, Q be the selfadjoint projections on $L^2(\mathbb{R}^2)$ given by $P := \mathbb{1}(H > \Lambda)$ and Q defined by multiplication by $\mathbb{1}_\Omega$. Note that for these projections $\mathbb{1} - P = \mathbb{1}(H \leq \Lambda)$ and $\mathbb{1} - Q$ is given by multiplication by $\mathbb{1}_{\Omega^\complement}$.

Fix $f \in L^2(\mathbb{R}^2)$. If $\mathbb{1}_\Omega f \in \mathrm{ran}(P)$, then

$$\sum_{B(2k-1)>\Lambda} \|\Pi_k \mathbb{1}_\Omega f\|^2 = \|P\mathbb{1}_\Omega f\|^2 = \|\mathbb{1}_\Omega f\|^2$$

and the claimed inequality holds as soon as $c \leq 1$. If $\mathbb{1}_\Omega f \notin \mathrm{ran}(P)$, we can instead use the second bound in Lemma 10 applied to $g := \mathbb{1}_\Omega f \in \mathrm{ran}(Q) \setminus \mathrm{ran}(P)$ to deduce that

$$\sum_{B(2k-1)>\Lambda} \|\Pi_k \mathbb{1}_\Omega f\|^2 = \|P\mathbb{1}_\Omega f\|^2 \geq \|\mathbb{1}_\Omega f\|^2 \frac{\|\mathbb{1}_{\Omega^\complement}(\mathbb{1} - P)\mathbb{1}_\Omega f\|^2}{\|(\mathbb{1} - P)\mathbb{1}_\Omega f\|^2}.$$

We can now apply Theorem 9 to $g := (\mathbb{1} - P)\mathbb{1}_\Omega f$ and conclude that

$$\sum_{B(2k-1)>\Lambda} \|\Pi_k \mathbb{1}_\Omega f\|^2 \geq \|\mathbb{1}_\Omega f\|^2 c e^{-c'(\rho_\theta(\Omega)\sqrt{\Lambda} + \rho_\theta(\Omega)^2 B)},$$

with c, c' as in Theorem 9. Combining the two cases proves Theorem 7. ∎

3. Proof of the improved semiclassical bounds

3.1. Abstract formulas

We begin by proving two somewhat abstract identities which will be at the core of our proofs of our main theorems. Although not written down in the general form we prove them here, these identities are central in the proofs of the Berezin–Li–Yau [1,26] and Kröger [20, 21] inequalities, as well as their magnetic analogues proved in [3, 5]. We believe that the abstract formulation presented here captures that the underlying principle is very general, and that unifying the proofs will be beneficial for future applications in other contexts. One specific advantage of our formulation is that it completely avoids distinguishing into cases depending on the nature of the spectrum of the involved operators.

Lemma 11. *Let $\mathcal{H}$ and $\widehat{\mathcal{H}}$ be separable Hilbert spaces and $J\colon \mathcal{H} \to \widehat{\mathcal{H}}$ an isometry. Let L and $\widehat{L}$ be non-negative selfadjoint operators in $\mathcal{H}$ and $\widehat{\mathcal{H}}$, respectively, with spectral measures E and $\widehat{E}$ and quadratic form domains $\mathcal{Q}$ and $\widehat{\mathcal{Q}}$. Set $E^{\perp} = \mathbb{1} - E$ and $\widehat{E}^{\perp} = \mathbb{1} - \widehat{E}$ and let $\Lambda > 0$.*

(1) *If $J^*\widehat{\mathcal{Q}} \subset \mathcal{Q}$ and $\widehat{E}(\Lambda)J(L - \Lambda)J^*\widehat{E}(\Lambda)$ is trace class, then*

$$\mathrm{Tr}(L - \Lambda)_- = - \int (\lambda - \Lambda) \, d \, \mathrm{Tr} \, \widehat{E}(\Lambda) J E(\lambda) J^* \widehat{E}(\Lambda) + \mathcal{R}_< + \mathcal{R}_>,$$

where

$$\mathcal{R}_< = \int (\lambda - \Lambda)_- d \, \mathrm{Tr} \, \widehat{E}(\Lambda)^{\perp} J E(\lambda) J^* \widehat{E}(\Lambda)^{\perp},$$

$$\mathcal{R}_> = \int (\lambda - \Lambda)_+ \, d \, \mathrm{Tr} \, \widehat{E}(\Lambda) J E(\lambda) J^* \widehat{E}(\Lambda).$$

(2) *If $L = J^*\widehat{L}J$ in the sense of quadratic forms, then*

$$\mathrm{Tr}(L - \Lambda)_- = \int (\lambda - \Lambda)_- \, d \, \mathrm{Tr} \, J^* \widehat{E}(\lambda) J - \mathcal{R}'_< - \mathcal{R}'_>,$$

where

$$\mathcal{R}'_< := \int (\lambda - \Lambda)_+ d \, \mathrm{Tr} \, E(\Lambda) J^* \widehat{E}(\lambda) J E(\Lambda),$$

$$\mathcal{R}'_> := \int (\lambda - \Lambda)_- d \, \mathrm{Tr} \, E(\Lambda)^{\perp} J^* \widehat{E}(\lambda) J E(\Lambda)^{\perp}.$$

By the assumption $L = J^*\widehat{L}J$ in the sense of quadratic forms, we mean the following. Let q and $\widehat{q}$ be the quadratic forms of L and $\widehat{L}$, respectively. Then we assume $J\mathcal{Q} \subset \widehat{\mathcal{Q}}$ and $\widehat{q}[J\psi] = q[\psi]$ for all $\psi \in \mathcal{Q}$.

We have written the second identity in an elegant, but slightly ambiguous way because there could be cancellations of infinities, leaving the expressions undefined. The precise meaning is as follows. The identity in (2) is to be understood with $\mathcal{R}'_<$ and $\mathcal{R}'_>$ added to both sides. Then the left-hand side is a sum of three positive terms, while there is one positive term on the right-hand side. Identity (2) rewritten in this form holds in the sense of an equality of elements in $[0, \infty) \cup \{+\infty\}$, which is also the case for the identity in (1).

Proof. We begin with the proof of (1). In the sense of bounded operators in $\widehat{\mathcal{H}}$,

$$
\begin{aligned}
J(L - \Lambda)_- J^* &= (\widehat{E}(\Lambda) + \widehat{E}(\Lambda)^\perp) J(L - \Lambda)_- J^* (\widehat{E}(\Lambda) + \widehat{E}(\Lambda)^\perp) \\
&= \widehat{E}(\Lambda) J(L - \Lambda)_- J^* \widehat{E}(\Lambda) + \widehat{E}(\Lambda)^\perp J(L - \Lambda)_- J^* \widehat{E}(\Lambda)^\perp \\
&\quad + \widehat{E}(\Lambda) J(L - \Lambda)_- J^* \widehat{E}(\Lambda)^\perp + \widehat{E}(\Lambda)^\perp J(L - \Lambda)_- J^* \widehat{E}(\Lambda).
\end{aligned}
$$

In the sense of quadratic forms on $\widehat{\mathcal{H}}$, we can rewrite the first term on the right-hand side as

$$
\begin{aligned}
\widehat{E}(\Lambda) &J(L - \Lambda)_- J^* \widehat{E}(\Lambda) \\
&= -\widehat{E}(\Lambda) J(L - \Lambda) J^* \widehat{E}(\Lambda) + \widehat{E}(\Lambda) J(L - \Lambda)_+ J^* \widehat{E}(\Lambda).
\end{aligned}
$$

By assumption, $\widehat{E}(\Lambda) J(L - \Lambda) J^* \widehat{E}(\Lambda)$ is trace class and, therefore, bounded. Since the left-hand side is bounded, the same must be true for the second term on the right-hand side. Therefore, this identity holds in the sense of bounded operators on $\widehat{\mathcal{H}}$.

When combined, we have shown that

$$
\begin{aligned}
J(L - \Lambda)_- J^* &= - \widehat{E}(\Lambda) J(L - \Lambda) J^* \widehat{E}(\Lambda) + \widehat{E}(\Lambda) J(L - \Lambda)_+ J^* \widehat{E}(\Lambda) \\
&\quad + \widehat{E}(\Lambda)^\perp J(L - \Lambda)_- J^* \widehat{E}(\Lambda)^\perp + \widehat{E}(\Lambda) J(L - \Lambda)_- J^* \widehat{E}(\Lambda)^\perp \\
&\quad + \widehat{E}(\Lambda)^\perp J(L - \Lambda)_- J^* \widehat{E}(\Lambda).
\end{aligned}
\tag{3.1}
$$

As we can write $\widehat{\mathcal{H}}$ as the direct sum of $\mathrm{ran}(\widehat{E}(\Lambda))$ and $\mathrm{ran}(\widehat{E}(\Lambda)^\perp)$, there is a complete orthonormal set $\{\psi_k\}_{k \geq 1} \subset \widehat{\mathcal{H}}$ such that, for each k, $\psi_k \in \widehat{\mathcal{Q}}$ and one of $\widehat{E}(\Lambda)\psi_k$ and $\widehat{E}(\Lambda)^\perp \psi_k$ is zero. Thus, for each $K \geq 1$, the identity in (3.1) implies

$$
\begin{aligned}
\sum_{k=1}^K \langle \psi_k, J(L - \Lambda)_- J^* \psi_k \rangle_{\widehat{\mathcal{H}}} &= - \sum_{k=1}^K \langle \psi_k, \widehat{E}(\Lambda) J(L - \Lambda) J^* \widehat{E}(\Lambda) \psi_k \rangle_{\widehat{\mathcal{H}}} \\
&\quad + \sum_{k=1}^K \langle \psi_k, \widehat{E}(\Lambda) J(L - \Lambda)_+ J^* \widehat{E}(\Lambda) \psi_k \rangle_{\widehat{\mathcal{H}}} \\
&\quad + \sum_{k=1}^K \langle \psi_k, \widehat{E}(\Lambda)^\perp J(L - \Lambda)_- J^* \widehat{E}(\Lambda)^\perp \psi_k \rangle_{\widehat{\mathcal{H}}}.
\end{aligned}
$$

Note that all terms in each of the sums except for the first one on the right-hand side are non-negative. However, the limit as $K \to \infty$ of the first sum on the right-hand side exists by the assumption that $\widehat{E}(\Lambda) J (L - \Lambda) J^* \widehat{E}(\Lambda)$ is trace class. Therefore, by taking the limit $K \to \infty$, we conclude that

$$
\begin{aligned}
\mathrm{Tr}(J(L - \Lambda)_- J^*) = {}& -\mathrm{Tr}(\widehat{E}(\Lambda) J (L - \Lambda)_- J^* \widehat{E}(\Lambda)) \\
& + \mathrm{Tr}(\widehat{E}(\Lambda) J (L - \Lambda)_+ J^* \widehat{E}(\Lambda)) \\
& + \mathrm{Tr}(\widehat{E}(\Lambda)^\perp J (L - \Lambda)_- J^* \widehat{E}(\Lambda)^\perp).
\end{aligned}
$$

Here we used the definition of the trace of a non-negative, bounded operator; see [33, Section VI.6]. Using the cyclicity of the trace, that $J^* J = \mathbb{1}$, and rewriting the traces on the right-hand side in terms of spectral measures proves the claimed identity.

We turn now to the proof of (2). As bounded operators in $\mathcal{H}$, we have

$$
(L - \Lambda)_- + E(\Lambda) L E(\Lambda) = \Lambda E(\Lambda).
$$

Using $J^* J = \mathbb{1}$ and $L = J^* \widehat{L} J$, we can rewrite this as

$$
(L - \Lambda)_- + E(\Lambda) J^* \widehat{L} J E(\Lambda) = E(\Lambda) J^* \Lambda J E(\Lambda),
$$

Furthermore, using $x = (x - \Lambda)_+ + \min\{x, \Lambda\}$ for $x \geq 0$, we deduce that

$$
\begin{aligned}
E(\Lambda) J^* \widehat{L} J E(\Lambda) &= E(\Lambda) J^* ((\widehat{L} - \Lambda)_+ + \min\{\widehat{L}, \Lambda\}) J E(\Lambda) \\
&= E(\Lambda) J^* (\widehat{L} - \Lambda)_+ J E(\Lambda) + E(\Lambda) J^* \min\{\widehat{L}, \Lambda\} J E(\Lambda).
\end{aligned}
$$

Since the term on the left-hand side and the second term on the right-hand side are both bounded operators, the same must be true for the first term on the right-hand side. (The term on the left-hand side is bounded since it is equal to $E(\Lambda) L E(\Lambda)$.)

Similarly, writing $\Lambda = (x - \Lambda)_- + \min\{x, \Lambda\}$ for $x \geq 0$, we have that

$$
\begin{aligned}
E(\Lambda) J^* \Lambda J E(\Lambda) &= E(\Lambda) J^* ((\widehat{L} - \Lambda)_- + \min\{\widehat{L}, \Lambda\}) J E(\Lambda) \\
&= E(\Lambda) J^* (\widehat{L} - \Lambda)_- J E(\Lambda) + E(\Lambda) J^* \min\{\widehat{L}, \Lambda\} J E(\Lambda).
\end{aligned}
$$

Combining the above, we have shown that as bounded operators on $\mathcal{H}$,

$$
\begin{aligned}
&(L - \Lambda)_- + E(\Lambda) J^* (\widehat{L} - \Lambda)_+ J E(\Lambda) + E(\Lambda) J^* \min\{\widehat{L}, \Lambda\} J E(\Lambda) \\
&= E(\Lambda) J^* (\widehat{L} - \Lambda)_- J E(\Lambda) + E(\Lambda) J^* \min\{\widehat{L}, \Lambda\} J E(\Lambda).
\end{aligned}
$$

Since each of the terms are bounded operators, this is implies that

$$
(L - \Lambda)_- + E(\Lambda) J^* (\widehat{L} - \Lambda)_+ J E(\Lambda) = E(\Lambda) J^* (\widehat{L} - \Lambda)_- J E(\Lambda).
$$

Note that the operator

$$E(\Lambda)^\perp J^*(\hat{L} - \Lambda)_- JE(\Lambda)^\perp + E(\Lambda)J^*(\hat{L} - \Lambda)_- JE(\Lambda)^\perp$$
$$+ E(\Lambda)^\perp J^*(\hat{L} - \Lambda)_- JE(\Lambda)$$

is bounded, since $(\hat{L} - \Lambda)_-$ is bounded. Adding this operator to both sides, we obtain that as sums of bounded operators on $\mathcal{H}$,

$$J^*(\hat{L} - \Lambda)_- J = (L - \Lambda)_- + E(\Lambda)J^*(\hat{L} - \Lambda)_+ JE(\Lambda)$$
$$+ E(\Lambda)^\perp J^*(\hat{L} - \Lambda)_- JE(\Lambda)^\perp$$
$$+ E(\Lambda)J^*(\hat{L} - \Lambda)_- JE(\Lambda)^\perp$$
$$+ E(\Lambda)^\perp J^*(\hat{L} - \Lambda)_- JE(\Lambda).$$

The proof can now be completed in the same way as for the first identity in the lemma.
∎

Remark 12. It is quite possible that Lemma 11 remains valid without the assumption that $\mathcal{H}$ and $\hat{\mathcal{H}}$ are separable. This would probably involve a definition of the trace of a bounded non-negative operator via nets. Since the separability assumption is satisfied in all our examples, we content ourselves with this case.

3.2. Proof of main results

As mentioned above, the identities in Lemma 11 can be used to prove the inequalities of Berezin–Li–Yau, Kröger, Erdős–Loss–Vougalter and one of the authors. Indeed, as we shall see below, applying (2) of Lemma 11 with $\mathcal{H} = L^2(\Omega)$, $\hat{\mathcal{H}} = L^2(\mathbb{R}^d)$, L being either $-\Delta_\Omega^{\mathrm{D}}$ or H_Ω^{D}, $\hat{L}$ being $-\Delta$ or H, and $J : L^2(\Omega) \to L^2(\mathbb{R}^d)$ defined as extension by zero, and estimating $\mathcal{R}'_< \geq 0$, $\mathcal{R}'_> \geq 0$ yields the Dirichlet version of these inequalities. Similarly, applying (1) of Lemma 11 with $\mathcal{H} = L^2(\Omega)$, $\hat{\mathcal{H}} = L^2(\mathbb{R}^d)$, L being either $-\Delta_\Omega^{\mathrm{N}}$ or H_Ω^{N}, $\hat{L}$ being $-\Delta$ or H, and $J : L^2(\Omega) \to L^2(\mathbb{R}^d)$ defined as extension by zero, and estimating $\mathcal{R}_< \geq 0$, $\mathcal{R}_> \geq 0$ yields the Neumann version of these inequalities. To obtain our improved inequalities we shall utilize the uncertainty principles discussed in the first part of this paper to provide a positive lower bound for $\mathcal{R}_<$ and $\mathcal{R}'_<$, respectively.

In each case, we shall apply the uncertainty principles of Theorems 6 and 7 with f being an eigenfunction of the operator of interest (extended by zero to all of $\mathbb{R}^d$). As such, it is not unreasonable to believe that the conclusions of the uncertainty principles could be strengthened in this restricted class of functions. However, under the minimal assumption that $|\Omega| < \infty$, it is difficult to control global properties of the eigenfunctions in any uniform manner. Therefore, strengthening the conclusions of Theorems 6 and 7 for these eigenfunctions appears challenging without imposing further assumptions on Ω.

Proof of Theorem 4 *for* $-\Delta^{\mathrm{D}}_{\Omega}$, $\gamma = 1$. Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure.

Step 1. We recall that the finite measure assumption implies that $-\Delta^{\mathrm{D}}_{\Omega}$ has discrete spectrum, each of its eigenvalues has finite multiplicity and their only accumulation point is $+\infty$. Let $\{\psi_n\}_{n\geq 1}$ be a complete orthonormal set of eigenfunctions of $-\Delta^{\mathrm{D}}_{\Omega}$ and $\{\lambda_n\}_{n\geq 1}$ the corresponding eigenvalues.

We will apply (2) in Lemma 11 with $\mathcal{H} = L^2(\Omega)$, $\hat{\mathcal{H}} = L^2(\mathbb{R}^d)$, $L = -\Delta^{\mathrm{D}}_{\Omega}$, $\hat{L} = -\Delta$, and $J: L^2(\Omega) \to L^2(\mathbb{R}^d)$ defined by

$$J\varphi(x) = \begin{cases} \varphi(x) & \text{for } x \in \Omega, \\ 0 & \text{for } x \in \Omega^{\complement}. \end{cases}$$

Note that $L = J^*\hat{L}J$ in the sense of quadratic forms.

In the notation of Lemma 11, we have for any $\mu \geq 0$,

$$E(\mu) = \sum_{n:\lambda_n < \mu} \langle \psi_n, \cdot \rangle \psi_n \quad \text{and} \quad \hat{E}(\mu) = \mathcal{F}^{-1}(\mathbb{1}_{|\cdot|^2 < \mu} \widehat{(\cdot)}).$$

By Lemma 11 (2), it holds that

$$\mathrm{Tr}(-\Delta^{\mathrm{D}}_{\Omega} - \Lambda)_- = \int (\lambda - \Lambda)_- d\,\mathrm{Tr}\, J^*\hat{E}(\lambda)J - \mathcal{R}'_< - \mathcal{R}'_>.$$

Step 2. We compute the first term on the right-hand side. For any complete orthogonal set $\{\varphi_j\}_{j\geq 1} \subset L^2(\Omega)$, the fact that the Fourier transform is unitary combined with the Pythagorean theorem implies

$$\mathrm{Tr}\, J^*\hat{E}(\lambda)J = \sum_{j\geq 1} \langle J\varphi_j, \hat{E}(\lambda)J\varphi_j \rangle = \sum_{j\geq 1} \int_{|\xi|^2 < \lambda} |\hat{\varphi}_j(\xi)|^2 \, d\xi$$

$$= (2\pi)^{-d} \sum_{j\geq 1} \int_{|\xi|^2 < \lambda} |\langle \varphi_j, e^{i(\cdot)\cdot\xi} \rangle_{L^2(\Omega)}|^2 \, d\xi$$

$$= (2\pi)^{-d} \int_{|\xi|^2 < \lambda} \|e^{i(\cdot)\cdot\xi}\|^2_{L^2(\Omega)} \, d\xi = (2\pi)^{-d} |\Omega| \omega_d \lambda^{d/2}.$$

Therefore,

$$\int (\lambda - \Lambda)_- d\,\mathrm{Tr}\, J^*\hat{E}(\lambda)J = L^{\mathrm{sc}}_{1,d} |\Omega| \Lambda^{1+d/2}.$$

Step 3. Combining the formula from Step 1 with the computation in Step 2, we see that the Berezin–Li–Yau inequality follows from the trivial estimates $\mathcal{R}'_> \geq 0$, $\mathcal{R}'_< \geq 0$.

To do better, we provide a positive lower bound for $\mathcal{R}'_<$. In the case under consideration, the fact that $\{\psi_n\}_{n\geq 1}$ is a complete orthonormal set in $L^2(\Omega)$ entails that

$$
\begin{aligned}
\operatorname{Tr} E(\Lambda) J^* \hat{E}(\lambda) J E(\Lambda) &= \sum_n \langle \psi_n, E(\Lambda) J^* \hat{E}(\lambda) J E(\Lambda) \psi_n \rangle_{L^2(\Omega)} \\
&= \sum_{n:\lambda_n<\Lambda} \langle \psi_n, J^* \hat{E}(\lambda) J \psi_n \rangle_{L^2(\Omega)} \\
&= \sum_{n:\lambda_n<\Lambda} \int_{|\xi|^2<\lambda} |\hat{\psi}_n(\xi)|^2 \, d\xi \\
&= \operatorname{Tr}(-\Delta_\Omega^{\mathrm{D}} - \Lambda)_-^0 - \sum_{n:\lambda_n<\Lambda} \int_{|\xi|^2>\lambda} |\hat{\psi}_n(\xi)|^2 \, d\xi.
\end{aligned}
$$

We recall that $\operatorname{Tr}(-\Delta_\Omega^{\mathrm{D}} - \Lambda)_-^0$ denotes the number of eigenvalues $\lambda_n < \Lambda$. Therefore, by an integration by parts,

$$
\mathcal{R}'_< = \sum_{\lambda_n<\Lambda} \int_\Lambda^\infty \int_{|\xi|^2>\lambda} |\hat{\psi}_n(\xi)|^2 \, d\xi d\lambda.
$$

By Theorem 6 and the normalization $\|\psi_n\|_{L^2(\Omega)} = 1$, we have

$$
\int_{|\xi|^2>\lambda} |\hat{\psi}_n(\xi)|^2 \, d\xi \geq c e^{-c' \rho_\theta(\Omega)\sqrt{\lambda}}.
$$

Thus, bounding the resulting integral as in [7], we find

$$
\begin{aligned}
\mathcal{R}'_< &\geq \sum_{\lambda_n<\Lambda} \int_\Lambda^\infty c e^{-c' \rho_\theta(\Omega)\sqrt{\lambda}} d\lambda \\
&\geq c \operatorname{Tr}(-\Delta_\Omega^{\mathrm{D}} - \Lambda)_-^0 \int_\Lambda^{4\Lambda} e^{-c'\rho_\theta(\Omega)\sqrt{\lambda}} \, d\lambda \\
&\geq 3c \operatorname{Tr}(-\Delta_\Omega^{\mathrm{D}} - \Lambda)_-^0 \Lambda e^{-2c'\rho_\theta(\Omega)\sqrt{\Lambda}}.
\end{aligned}
$$

Since $\mathcal{R}'_> \geq 0$, we conclude that

$$
\operatorname{Tr}(-\Delta_\Omega^{\mathrm{D}} - \Lambda)_- \leq L_{1,d}^{\mathrm{sc}} |\Omega| \Lambda^{1+d/2} - 3c \operatorname{Tr}(-\Delta_\Omega^{\mathrm{D}} - \Lambda)_-^0 \Lambda e^{-2c'\rho_\theta(\Omega)\sqrt{\Lambda}}.
$$

To finish the proof, we distinguish two cases. The first case is where

$$
\operatorname{Tr}(-\Delta_\Omega^{\mathrm{D}} - \Lambda)_-^0 \geq \frac{1}{1+3c} L_{1,d}^{\mathrm{sc}} |\Omega| \Lambda^{d/2}.
$$

In this case, the previous estimate implies that

$$\mathrm{Tr}(-\Delta_\Omega^D - \Lambda)_- \leq L_{1,d}^{sc}|\Omega|\Lambda^{1+d/2}\Big(1 - \frac{3c}{1+3c}e^{-2c'\rho_\theta(\Omega)\sqrt{\Lambda}}\Big).$$

Meanwhile, when

$$\mathrm{Tr}(-\Delta_\Omega^D - \Lambda)_-^0 < \frac{1}{1+3c}L_{1,d}^{sc}|\Omega|\Lambda^{d/2},$$

we have

$$\mathrm{Tr}(-\Delta_\Omega^D - \Lambda)_- = \int_0^\Lambda \mathrm{Tr}(-\Delta_\Omega^D - \lambda)_-^0 \, d\lambda \leq \Lambda \, \mathrm{Tr}(-\Delta_\Omega^D - \Lambda)_-^0$$

$$< \frac{1}{1+3c}L_{1,d}^{sc}|\Omega|\Lambda^{1+d/2} = L_{1,d}^{sc}|\Omega|\Lambda^{1+d/2}\Big(1 - \frac{3c}{1+3c}\Big).$$

Thus, in both cases,

$$\mathrm{Tr}(-\Delta_\Omega^D - \Lambda)_- \leq L_{1,d}^{sc}|\Omega|\Lambda^{1+d/2}\Big(1 - \frac{3c}{1+3c}e^{-2c'\rho_\theta(\Omega)\sqrt{\Lambda}}\Big).$$

This proves Theorem 4 for $\gamma = 1$ in the Dirichlet case. ∎

Proof of Theorem 4 for $-\Delta_\Omega^N$, $\gamma = 1$. Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure.

Step 1. Let $\Sigma := \inf \sigma_{\mathrm{ess}}(-\Delta_\Omega^N)$. If $\Lambda > \Sigma$, the left-hand side of the claimed inequality is infinite and thus the inequality is trivially true. Therefore, in the rest of the proof we assume that $\Lambda \leq \Sigma$. Let $\{\lambda_n\}_{n\geq 1}$ be the set of eigenvalues of $-\Delta_\Omega^N$ that are less than $\Lambda \leq \Sigma$ and let $\{\psi_n\}_{n\geq 1}$ be an orthonormal set of eigenfunctions corresponding to these eigenvalues.

We aim to apply (1) of Lemma 11 with $\mathcal{H} = L^2(\Omega)$, $\hat{\mathcal{H}} = L^2(\mathbb{R}^d)$, $L = -\Delta_\Omega^N$, $\hat{L} = -\Delta$, and $J: L^2(\Omega) \to L^2(\mathbb{R}^d)$ defined as in the proof of the Dirichlet case.

In the notation of Lemma 11, we have for any $\mu \leq \Lambda$,

$$E(\mu) = \sum_{n:\lambda_n < \mu} \langle \psi_n, \cdot \rangle \psi_n$$

and, as before, for all $\mu \geq 0$,

$$\hat{E}(\mu) = \mathcal{F}^{-1}(\mathbb{1}_{|\cdot|^2 < \mu}\widehat{(\cdot)}).$$

Assuming for the moment that $\hat{E}(\Lambda)J(-\Delta_\Omega^N - \Lambda)J^*\hat{E}(\Lambda)$ is trace class, we deduce from Lemma 11 (1) that

$$\mathrm{Tr}(-\Delta_\Omega^N - \Lambda)_- = -\int (\lambda - \Lambda)d\,\mathrm{Tr}\,\hat{E}(\Lambda)JE(\lambda)J^*\hat{E}(\Lambda) + \mathcal{R}_< + \mathcal{R}_>.$$

Step 2. We compute the first term on the right-hand side and, showing that is finite, will justify that $\widehat{E}(\Lambda)J(-\Delta_\Omega^N - \Lambda)J^*\widehat{E}(\Lambda)$ is trace class. We note that

$$\operatorname{ran}(J^*\widehat{E}(\Lambda)) \subset H^1(\Omega) = \operatorname{dom}(\sqrt{-\Delta_\Omega^N + 1}).$$

Hence, for any complete orthogonal set $\{\varphi_j\}_{j\geq 1} \subset L^2(\mathbb{R}^d)$, we have

$$\int (\lambda - \Lambda)d \operatorname{Tr} \widehat{E}(\Lambda)JE(\lambda)J^*\widehat{E}(\Lambda)$$

$$= \sum_{j\geq 1} \int (\lambda - \Lambda)d \langle J^*\widehat{E}(\Lambda)\varphi_j, E(\lambda)J^*\widehat{E}(\Lambda)\varphi_j \rangle$$

$$= \sum_{j\geq 1} (\|\nabla(\widehat{E}(\Lambda)\varphi_j)\|_{L^2(\Omega)}^2 - \Lambda\|\widehat{E}(\Lambda)\varphi_j\|_{L^2(\Omega)}^2).$$

Since $\{\hat{\varphi}_j\}_{j\geq 1}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$, the Pythagorean theorem implies

$$\sum_{j\geq 1} \|\widehat{E}(\Lambda)\varphi_j\|_{L^2(\Omega)}^2 = (2\pi)^{-d} \int_\Omega \sum_{j\geq 1} |\langle \mathbb{1}_{|\cdot|^2 < \Lambda} e^{-ix\cdot(\cdot)}, \hat{\varphi}_j \rangle_{L^2(\mathbb{R}^d)}|^2 \, dx$$

$$= (2\pi)^{-d} \int_\Omega \|\mathbb{1}_{|\cdot|^2 < \Lambda} e^{-ix\cdot(\cdot)}\|_{L^2(\mathbb{R}^d)}^2 \, dx$$

$$= (2\pi)^{-d} |\Omega| \int_{|\xi|^2 < \Lambda} d\xi$$

and

$$\sum_{j\geq 1} \|\nabla(\widehat{E}(\Lambda)\varphi_j)\|_{L^2(\Omega)}^2 = (2\pi)^{-d} \int_\Omega \sum_{j\geq 1} \sum_{k=1}^d |\langle (\cdot)_k \mathbb{1}_{|\cdot|^2 < \Lambda} e^{-ix\cdot(\cdot)}, \hat{\varphi}_j \rangle_{L^2(\mathbb{R}^d)}|^2 \, dx$$

$$= (2\pi)^{-d} \int_\Omega \||\cdot| \mathbb{1}_{|\cdot|^2 < \Lambda} e^{-ix\cdot(\cdot)}\|_{L^2(\mathbb{R}^d)}^2 \, dx$$

$$= (2\pi)^{-d} |\Omega| \int_{|\xi|^2 < \Lambda} |\xi|^2 \, d\xi.$$

These two formulas show that $\widehat{E}(\Lambda)J(-\Delta_\Omega^N)J^*\widehat{E}(\Lambda)$ and $\widehat{E}(\Lambda)JJ^*\widehat{E}(\Lambda)$ are trace class under our assumptions. Moreover, they show

$$-\int (\lambda - \Lambda)d \operatorname{Tr} \widehat{E}(\Lambda)JE(\lambda)J^*\widehat{E}(\Lambda) = -(2\pi)^{-d}|\Omega| \int_{|\xi|^2 < \Lambda} (|\xi|^2 - \Lambda) \, d\xi$$

$$= L_{1,d}^{\mathrm{sc}}|\Omega|\Lambda^{1+d/2}.$$

Step 3. We observe that the Kröger inequality follows from the trivial estimates $\mathcal{R}_< \geq 0$, $\mathcal{R}_> \geq 0$. To do better, we provide a positive lower bound for $\mathcal{R}_<$.

For $\lambda \leq \Lambda \leq \Sigma$ and any complete ON set $\{\varphi_j\}_{j \geq 1}$, we find

$$
\begin{aligned}
\operatorname{Tr} \widehat{E}(\Lambda)^\perp J E(\lambda) J^* \widehat{E}(\Lambda)^\perp &= \sum_{j \geq 1} \langle \widehat{E}(\Lambda)^\perp \varphi_j, E(\lambda) \widehat{E}(\Lambda)^\perp \varphi_j \rangle_{L^2(\Omega)} \\
&= \sum_{j \geq 1} \sum_{n:\lambda_n < \lambda} |\langle \widehat{E}(\Lambda)^\perp \varphi_j, \psi_n \rangle_{L^2(\Omega)}|^2 \\
&= \sum_{j \geq 1} \sum_{n:\lambda_n < \lambda} |\langle \hat{\varphi}_j, \mathbb{1}_{|\cdot|^2 > \Lambda} \hat{\psi}_n \rangle_{L^2(\mathbb{R}^d)}|^2 \\
&= \sum_{n:\lambda_n < \lambda} \int_{|\xi|^2 > \Lambda} |\hat{\psi}_n(\xi)|^2 \, d\xi.
\end{aligned}
$$

Therefore, in the current case we find

$$
\begin{aligned}
\mathcal{R}_< &= \int (\lambda - \Lambda)_- \operatorname{Tr} \widehat{E}(\Lambda)^\perp J E(\lambda) J^* \widehat{E}(\Lambda)^\perp \\
&= \sum_{n:\lambda_n < \Lambda} (\Lambda - \lambda_n) \int_{|\xi|^2 > \Lambda} |\hat{\psi}_n(\xi)|^2 \, d\xi.
\end{aligned}
$$

By Theorem 6 and the normalization $\|\psi_n\|_{L^2(\Omega)} = 1$, we have

$$
\int_{|\xi|^2 > \Lambda} |\hat{\psi}_n(\xi)|^2 \, d\xi \geq c e^{-c' \rho_\theta(\Omega) \sqrt{\Lambda}}.
$$

Combining this with Kröger's inequality yields

$$
\mathcal{R}_< \geq c e^{-c' \rho_\theta(\Omega) \sqrt{\Lambda}} \operatorname{Tr}(-\Delta_\Omega^{\mathrm{N}} - \Lambda)_- \geq c L_{1,d}^{\mathrm{sc}} |\Omega| \Lambda^{1+d/2} e^{-c' \rho_\theta(\Omega) \sqrt{\Lambda}}.
$$

Since $\mathcal{R}_> \geq 0$, we have arrived at the inequality

$$
\operatorname{Tr}(-\Delta_\Omega^{\mathrm{N}} - \Lambda)_- \geq L_{1,d}^{\mathrm{sc}} |\Omega| \Lambda^{1+d/2} (1 + c e^{-c' \rho_\theta(\Omega) \sqrt{\Lambda}}),
$$

which completes the proof of Theorem 4 for $\gamma = 1$ in the Neumann case. ∎

Proof of Theorem 5 for H_Ω^{D}, $\gamma = 1$. Let $\Omega \subset \mathbb{R}^2$ be an open set of finite measure and let $B > 0$.

Step 1. Recall that the finite measure assumption implies that H_Ω^{D} has discrete spectrum, each of its eigenvalues has finite multiplicity and their only accumulation point is $+\infty$. Let $\{\psi_n\}_{n \geq 1}$ be the eigenfunctions of H_Ω^{D} and $\{\lambda_n\}_{n \geq 1}$ the corresponding eigenvalues.

As in the proof for the Dirichlet Laplace operator, we aim to apply Lemma 11 (2). In this case, we choose $\mathcal{H} = L^2(\Omega)$, $\hat{\mathcal{H}} = L^2(\mathbb{R}^2)$, $L = H_\Omega^D$, $\hat{L} = H$, and J defined as before through extension by zero. Note that in the sense of quadratic forms, $H_\Omega^D = J^* H J$.

We have for any $\mu \geq 0$,

$$E(\mu) = \sum_{n:\lambda_n < \mu} \langle \psi_n, \cdot \rangle \psi_n \quad \text{and} \quad \hat{E}(\mu) = \sum_{k:B(2k-1)<\mu} \Pi_k$$

By Lemma 11 (2),

$$\mathrm{Tr}(H_\Omega^D - \Lambda)_- = \int (\lambda - \Lambda)_- \, d \, \mathrm{Tr} \, J^* \hat{E}(\lambda) J - \mathcal{R}'_< - \mathcal{R}'_>.$$

Step 2. For the first term on the right-hand side, we have

$$\int (\lambda - \Lambda)_- \, d \, \mathrm{Tr} \, J^* \hat{E}(\lambda) J = \sum_{k:B(2k-1)<\Lambda} (\Lambda - B(2k-1)) \, \mathrm{Tr}(\mathbb{1}_\Omega \Pi_k \mathbb{1}_\Omega)$$

$$= \sum_{k:B(2k-1)<\Lambda} (\Lambda - B(2k-1)) \int_\Omega \Pi_k(x, x) \, dx$$

$$= \frac{B}{2\pi} |\Omega| \sum_{k:B(2k-1)<\Lambda} (\Lambda - B(2k-1)),$$

where we used the fact that the integral kernel of Π_k satisfies $\Pi_k(x, x) = B/(2\pi)$ for all $x \in \mathbb{R}^2$.

Step 3. Combining the formula from Step 1 with the computation in Step 2, we obtain

$$\mathrm{Tr}(H_\Omega^D - \Lambda)_- = |\Omega| \frac{B}{2\pi} \sum_{k=1}^\infty (B(2k-1) - \Lambda)_- - \mathcal{R}'_< - \mathcal{R}'_>.$$

The inequality of Erdős–Loss–Vougalter follows by bounding $\mathcal{R}'_< \geq 0$ and $\mathcal{R}'_> \geq 0$.

To get an improved inequality, we will still drop the term $\mathcal{R}'_>$, but we shall provide a positive lower bound for $\mathcal{R}'_<$. We write

$$\mathcal{R}'_< = \sum_{n:\lambda_n < \Lambda} \int (\lambda - \Lambda)_+ d \, \mathrm{Tr} \, E(\Lambda) J^* \hat{E}(\lambda) J E(\Lambda)$$

$$= \sum_{n:\lambda_n < \Lambda} \int (\lambda - \Lambda)_+ d \, \langle \psi_n, J^* \hat{E}(\lambda) J \psi_n \rangle$$

$$= \sum_{n:\lambda_n < \Lambda} \sum_{k \geq 1} (B(2k-1) - \Lambda)_+ \| \Pi_k \psi_n \|^2_{L^2(\mathbb{R}^2)}$$

$$= \int\limits_0^\infty \sum_{n:\lambda_n < \Lambda} \sum_{k:B(2k-1) \geq \Lambda + \eta} \|\Pi_k \psi_n\|^2_{L^2(\mathbb{R}^2)} \, d\eta.$$

For fixed $\eta > 0$, we bound

$$\sum_{n:\lambda_n < \Lambda} \sum_{k:B(2k-1) \geq \Lambda + \eta} \|\Pi_k \psi_n\|^2_{L^2(\mathbb{R}^2)}$$

$$\geq \mathrm{Tr}(H_\Omega^{\mathrm{D}} - \Lambda)_-^0 \inf_{n \geq 1} \sum_{k:B(2k-1) \geq \Lambda + \eta} \|\Pi_k \psi_n\|^2_{L^2(\mathbb{R}^2)}.$$

Now, by Theorem 7, there are constants $c, c' > 0$ depending only on θ such that

$$\sum_{k:B(2k-1) \geq \Lambda + \eta} \|\Pi_k \psi_n\|^2_{L^2(\mathbb{R}^2)} \geq c e^{-c'(\rho_\theta(\Omega)\sqrt{\Lambda + \eta} + \rho_\theta(\Omega)^2 B)}.$$

We obtain, as in [7] and the proof for $-\Delta_\Omega^{\mathrm{D}}$,

$$\mathcal{R}'_< \geq c \, \mathrm{Tr}(H_\Omega^{\mathrm{D}} - \Lambda)_-^0 \int\limits_0^\infty e^{-c'(\rho_\theta(\Omega)\sqrt{\Lambda + \eta} + \rho_\theta(\Omega)^2 B)} \, d\eta$$

$$\geq c \, \mathrm{Tr}(H_\Omega^{\mathrm{D}} - \Lambda)_-^0 \int\limits_0^{3\Lambda} e^{-c'(2\rho_\theta(\Omega)\sqrt{\Lambda} + \rho_\theta(\Omega)^2 B)} \, d\eta$$

$$\geq 3c\Lambda \, \mathrm{Tr}(H_\Omega^{\mathrm{D}} - \Lambda)_-^0 e^{-c'(2\rho_\theta(\Omega)\sqrt{\Lambda} + \rho_\theta(\Omega)^2 B)}.$$

Now, we distinguish two cases according to whether $\mathrm{Tr}(H_\Omega^{\mathrm{D}} - \Lambda)_-^0$ exceeds

$$\frac{1}{1 + 3c} \Lambda^{-1} |\Omega| \frac{B}{2\pi} \sum_{k=1}^\infty (B(2k-1) - \Lambda)_-$$

or not. Arguing similarly to proof of Theorem 4, we find that in either case

$$\mathrm{Tr}(H_\Omega^{\mathrm{D}} - \Lambda)_- \leq |\Omega| \frac{B}{2\pi} \sum_{k=1}^\infty (B(2k-1) - \Lambda)_- \left(1 - \frac{3c}{1 + 3c} e^{-c'(2\sqrt{|\Omega|\Lambda} + |\Omega|B)}\right).$$

This proves Theorem 5 for $\gamma = 1$ in the Dirichlet case. ∎

Proof of Theorem 5 for H_Ω^{N}, $\gamma = 1$. Let $\Omega \subset \mathbb{R}^2$ be an open set of finite measure and let $B > 0$.

Step 1. Let $\Sigma = \inf \sigma_{\mathrm{ess}}(H_\Omega^{\mathrm{N}})$. If $\Lambda > \Sigma$, the left-hand side of the inequality in the statement is infinite and the inequality is trivially true. Therefore, we in the rest of the proof assume that $\Lambda \leq \Sigma$. Let $\{\lambda_n\}_{n \geq 1}$ be the set of eigenvalues of H_Ω^{N} that are less than $\Lambda \leq \Sigma$ and let $\{\psi_n\}_{n \geq 1}$ be an orthonormal set of eigenfunctions corresponding to these eigenvalues.

We aim to apply (1) of Lemma 11 with $\mathcal{H} = L^2(\Omega)$, $\widehat{\mathcal{H}} = L^2(\mathbb{R}^2)$, $L = H_\Omega^{\mathrm{N}}$, $\widehat{L} = H$, and J defined as in the earlier proofs.

In the notation of Lemma 11, we have for any $\mu \leq \Lambda$,

$$E(\mu) = \sum_{n:\lambda_n < \mu} \langle \psi_n, \cdot \rangle \psi_n$$

and for all $\mu \geq 0$,

$$\widehat{E}(\mu) = \sum_{k:B(2k-1) < \mu} \Pi_k.$$

Assuming for the moment that $\widehat{E}(\Lambda) J (H_\Omega^{\mathrm{N}} - \Lambda) J^* \widehat{E}(\Lambda)$ is trace class, we deduce from Lemma 11 (1) that

$$\mathrm{Tr}(H_\Omega^{\mathrm{N}} - \Lambda)_- = -\int (\lambda - \Lambda) d \, \mathrm{Tr} \, \widehat{E}(\Lambda) J E(\lambda) J^* \widehat{E}(\Lambda) + \mathcal{R}_< + \mathcal{R}_>.$$

Step 2. We compute the first term on the right-hand side, in particular, showing that it is finite and thereby also justifying that $\widehat{E}(\Lambda) J (H_\Omega^{\mathrm{N}} - \Lambda) J^* \widehat{E}(\Lambda)$ is trace class. We note that

$$\widehat{E}(\Lambda) J E(\lambda) J^* \widehat{E}(\Lambda) = \sum_{\substack{k,k': B(2k-1) < \Lambda, \\ B(2k'-1) < \Lambda}} \Pi_k J E(\lambda) J^* \Pi_{k'},$$

Let $\{\varphi_j\}_{j \geq 1} \subset L^2(\mathbb{R}^d)$ be an orthonormal basis of $L^2(\mathbb{R}^2)$ satisfying $\Pi_k \varphi_j \in \{0, \varphi_j\}$ for each $k, j \geq 1$. Since $\mathrm{ran} \, J^* \Pi_k \subset H_A^1(\Omega) = \mathrm{dom}(\sqrt{H_\Omega^{\mathrm{N}} + 1})$, we have

$$\int (\lambda - \Lambda) d \, \mathrm{Tr} \, \widehat{E}(\Lambda) J E(\lambda) J^* \widehat{E}(\Lambda)$$

$$= \sum_{k:B(2k-1) < \Lambda} \sum_{j \geq 1} \int (\lambda - \Lambda) d \langle J^* \Pi_k \varphi_j, E(\lambda) J^* \Pi_k \varphi_j \rangle$$

$$= \sum_{k:B(2k-1) < \Lambda} \sum_{j \geq 1} (\|\nabla_A \Pi_k \varphi_j\|_{L^2(\Omega)}^2 - \Lambda \|\Pi_k \varphi_j\|_{L^2(\Omega)}^2).$$

Since $\{\varphi_j\}_{j \geq 1}$ is an orthonormal basis for $L^2(\mathbb{R}^2)$ and $\Pi_k(x, x) = B/(2\pi)$, we have

$$\sum_{j \geq 1} \|\Pi_k \varphi_j\|_{L^2(\Omega)}^2 = \mathrm{Tr}(\mathbb{1}_\Omega \Pi_k \mathbb{1}_\Omega) = \frac{B}{2\pi} |\Omega|.$$

The argument for the term involving ∇_A is a bit more complicated. Denote by $\Psi_{k,x}$ the restriction of $\Pi_k(\cdot, x)$ to Ω. We have for any $\psi \in L^2(\mathbb{R}^2)$ and $x \in \Omega$,

$$\Pi_k \psi(x) = \int_{\mathbb{R}^2} \Psi_{k,x'}(x) \psi(x') \, dx' = \langle \Psi_{k,\cdot}(x), \psi \rangle_{L^2(\mathbb{R}^2)}$$

and by smoothness of the kernel,

$$\nabla_A \Pi_k \psi(x) = \int_{\mathbb{R}^2} \nabla_{A,x} \Psi_{k,x'}(x) \psi(x') \, dx' = \langle \nabla_{A,x} \Psi_{k,\cdot}(x), \psi \rangle_{L^2(\mathbb{R}^2)}$$

Here $\nabla_{A,x}$ denotes the magnetic gradient acting with respect to the x variable. The completeness of $\{\varphi_j\}_{j \geq 1}$ implies that

$$\sum_{j \geq 1} \|\nabla_A \Pi_k \varphi_j\|_{L^2(\Omega)}^2 = \int_\Omega \int_{\mathbb{R}^2} |\nabla_{A,x} \Psi_{k,x'}(x)|^2 \, dx' dx.$$

Now, by an identity proved in [5], we have for every $x \in \Omega$,

$$\int_{\mathbb{R}^2} |(\nabla_{A,x} \Psi_{k,x'})(x)|^2 \, dx' = \frac{B}{2\pi} B(2k-1).$$

Consequently, we have shown that

$$-\int (\lambda - \Lambda) d \operatorname{Tr} \widehat{E}(\Lambda) J E(\lambda) J^* \widehat{E}(\Lambda) = |\Omega| \frac{B}{2\pi} \sum_{k=1}^{\infty} (B(2k-1) - \Lambda)_-.$$

We note that the argument that we have just given also shows that $\widehat{E}(\Lambda) J H_\Omega^N J^* \widehat{E}(\Lambda)$ and $\widehat{E}(\Lambda) J J^* \widehat{E}(\Lambda)$ are trace class.

Step 3. The semiclassical inequality obtained in [5] thus follows from the formula in Step 1 and the computation in Step 2 via the trivial estimates $\mathcal{R}_< \geq 0$, $\mathcal{R}_> \geq 0$. To do better, we yet again provide a positive lower bound for $\mathcal{R}_<$.

For $\lambda \leq \Lambda \leq \Sigma$ and any complete ON set $\{\varphi_j\}_{j \geq 1}$, we find

$$\operatorname{Tr} \widehat{E}(\Lambda)^\perp J E(\lambda) J^* \widehat{E}(\Lambda)^\perp$$
$$= \sum_{j \geq 1} \langle \widehat{E}(\Lambda)^\perp \varphi_j, E(\lambda) \widehat{E}(\Lambda)^\perp \varphi_j \rangle_{L^2(\Omega)}$$
$$= \sum_{j \geq 1} \sum_{n: \lambda_n < \lambda} |\langle \widehat{E}(\Lambda)^\perp \varphi_j, \psi_n \rangle_{L^2(\Omega)}|^2$$
$$= \sum_{j \geq 1} \sum_{n: \lambda_n < \lambda} \sum_{k: B(2k-1) \geq \Lambda} |\langle \Pi_k \varphi_j, \psi_n \rangle_{L^2(\Omega)}|^2$$

$$= \sum_{j \geq 1} \sum_{n:\lambda_n < \lambda} \sum_{k:B(2k-1) \geq \Lambda} |\langle \varphi_j, \Pi_k \psi_n \rangle_{L^2(\mathbb{R}^2)}|^2$$

$$= \sum_{n:\lambda_n < \lambda} \sum_{k:B(2k-1) \geq \Lambda} \|\Pi_k \psi_n\|_{L^2(\mathbb{R}^2)}^2.$$

Therefore, we have

$$\mathcal{R}_< = \int (\lambda - \Lambda)_- \operatorname{Tr} \hat{E}(\Lambda)^\perp J E(\lambda) J^* \hat{E}(\Lambda)^\perp$$

$$= \sum_{n:\lambda_n < \Lambda} (\Lambda - \lambda_n) \Big(\sum_{k:B(2k-1) \geq \Lambda} \|\Pi_k \psi_n\|_{L^2(\mathbb{R}^2)}^2 \Big).$$

By Theorem 7 and the semiclassical inequality for H_Ω^N proved in [5],

$$\mathcal{R}_< \geq c e^{-c'(\rho_\theta(\Omega)\sqrt{\Lambda} + \rho_\theta(\Omega)^2 B)} \sum_{n:\lambda_n < \Lambda} (\Lambda - \lambda_n)$$

$$= c e^{-c'(\rho_\theta(\Omega)\sqrt{\Lambda} + \rho_\theta(\Omega)^2 B)} \operatorname{Tr}(H_\Omega^N - \Lambda)_-$$

$$\geq c|\Omega| \frac{B}{2\pi} e^{-c'(\rho_\theta(\Omega)\sqrt{\Lambda} + \rho_\theta(\Omega)^2 B)} \sum_{k=1}^\infty (B(2k-1) - \Lambda)_-.$$

Since $\mathcal{R}_> \geq 0$, we have arrived at the claimed inequality

$$\operatorname{Tr}(H_\Omega^N - \Lambda)_- \geq c|\Omega| \frac{B}{2\pi} \sum_{k=1}^\infty (B(2k-1) - \Lambda)_-(1 + c e^{-c'(\rho_\theta(\Omega)\sqrt{\Lambda} + \rho_\theta(\Omega)^2 B)}).$$

This proves Theorem 5 for $\gamma = 1$ in the Neumann case. ∎

4. Consequences for Riesz means of higher order

So far, we have proved Theorems 4 and 5 for $\gamma = 1$. In this section we explain how to deduce the result for $\gamma > 1$ from this special case. We consider the non-magnetic and magnetic cases simultaneously.

Recall that by the Aizenman–Lieb identity, we have for any $\gamma > 1$ any $\Lambda \geq 0$, and any selfadjoint operator L,

$$\operatorname{Tr}(L - \Lambda)_-^\gamma = \gamma(\gamma - 1) \int_0^\Lambda (\Lambda - \lambda)^{\gamma - 2} \operatorname{Tr}(L - \lambda)_- \, d\lambda. \tag{4.1}$$

Define

$$G_\gamma^0(\Lambda) := L_{\gamma,d}^{\mathrm{sc}} \Lambda^{\gamma + d/2}$$

and

$$G_\gamma^B(\Lambda) := \frac{B}{2\pi} \sum_{k=1}^{\infty} (B(2k-1) - \Lambda)_-^\gamma \quad \text{for } B > 0.$$

(It is worth noting that $\lim_{B\to 0} G_\gamma^B(\Lambda) = G_\gamma^0(\Lambda)$ when $d = 2$.) For $\gamma > 1$ and $B \geq 0$, we have, similarly to (4.1),

$$G_\gamma^B(\Lambda) = \gamma(\gamma - 1) \int_0^\Lambda (\Lambda - \lambda)^{\gamma-2} G_1^B(\lambda) \, d\lambda.$$

Combining these formulas with Theorems 4 and 5, we infer that there are constants c, c' depending only on θ and d such that

$$\mathrm{Tr}(-\Delta_\Omega^{\mathrm{D}} - \Lambda)_-^\gamma \leq |\Omega| G_\gamma^0(\Lambda) - \mathcal{R}^0, \tag{4.2}$$

$$\mathrm{Tr}(-\Delta_\Omega^{\mathrm{N}} - \Lambda)_-^\gamma \geq |\Omega| G_\gamma^0(\Lambda) + \mathcal{R}^0, \tag{4.3}$$

$$\mathrm{Tr}(H_\Omega^{\mathrm{D}} - \Lambda)_-^\gamma \leq |\Omega| G_\gamma^B(\Lambda) - \mathcal{R}^B, \tag{4.4}$$

$$\mathrm{Tr}(H_\Omega^{\mathrm{N}} - \Lambda)_-^\gamma \geq |\Omega| G_\gamma^B(\Lambda) + \mathcal{R}^B, \tag{4.5}$$

where, for $B \geq 0$,

$$\mathcal{R}^B := c|\Omega|\gamma(\gamma - 1) \int_0^\Lambda (\Lambda - \lambda)^{\gamma-2} G_1^B(\lambda) e^{-c'(\rho_\theta(\Omega)\sqrt{\lambda} + \rho_\theta(\Omega)^2 B)} \, d\lambda.$$

Bounding $e^{-c'(\rho_\theta(\Omega)\sqrt{\lambda} + \rho_\theta(\Omega)^2 B)} \geq e^{-c'(\rho_\theta(\Omega)\sqrt{\Lambda} + \rho_\theta(\Omega)^2 B)}$ in the integrand, we find

$$\mathcal{R}^B \geq c e^{-c'(\rho_\theta(\Omega)\sqrt{\Lambda} + \rho_\theta(\Omega)^2 B)} |\Omega| G_\gamma^B(\Lambda).$$

Upon inserting this bound into (4.2)–(4.5), we obtain the bounds in Theorems 4 and 5 for $\gamma > 1$. This concludes the proof of these two theorems.

References

[1] F. A. Berezin, Covariant and contravariant symbols of operators (in Russian). *Izv. Akad. Nauk SSSR Ser. Mat.* **36** (1972), 1134–1167. English translation: *Math. USSR, Sb.* **6** (1972), no. 5, 1117–1151 Zbl 0247.47019 MR 0350504

[2] Y. Colin de Verdière, L'asymptotique de Weyl pour les bouteilles magnétiques. *Comm. Math. Phys.* **105** (1986), no. 2, 327–335 Zbl 0612.35102 MR 0849211

[3] L. Erdős, M. Loss, and V. Vougalter, Diamagnetic behavior of sums of Dirichlet eigenvalues. *Ann. Inst. Fourier (Grenoble)* **50** (2000), no. 3, 891–907 Zbl 0957.35104 MR 1779898

[4] S. Fournais, Atoms in strong magnetic fields. In *The physics and mathematics of Elliott Lieb.* The 90th anniversary. Vol. I, pp. 295–314, EMS Press, Berlin, 2022 Zbl 1500.81082 MR 4529872

[5] R. L. Frank, Remarks on eigenvalue estimates and semigroup domination. In *Spectral and scattering theory for quantum magnetic systems*, pp. 63–86, Contemp. Math. 500, Amer. Math. Soc., Providence, RI, 2009 Zbl 1183.81068 MR 2655143

[6] R. L. Frank, A. Laptev, and T. Weidl, *Schrödinger operators: eigenvalues and Lieb–Thirring inequalities*. Cambridge Stud. Adv. Math. 200, Cambridge University Press, Cambridge, 2023 MR 4496335

[7] R. L. Frank and S. Larson, Semiclassical inequalities for Dirichlet and Neumann Laplacians on convex domains. *Comm. Pure Appl. Math.* (2025) DOI https://doi.org/10.1002/cpa.70019

[8] R. L. Frank and S. Larson, Riesz means asymptotics for Dirichlet and Neumann Laplacians on Lipschitz domains. *Invent. Math.* **241** (2025), no. 3, 999–1079 MR 4946250 Zbl 08086119

[9] R. L. Frank, M. Loss, and T. Weidl, Pólya's conjecture in the presence of a constant magnetic field. *J. Eur. Math. Soc. (JEMS)* **11** (2009), no. 6, 1365–1383 Zbl 1179.35205 MR 2557138

[10] J. K. Freericks, E. H. Lieb, and D. Ueltschi, Segregation in the Falicov–Kimball model. *Comm. Math. Phys.* **227** (2002), no. 2, 243–279 Zbl 1004.82009 MR 1903646

[11] L. Geisinger, A. Laptev, and T. Weidl, Geometrical versions of improved Berezin–Li–Yau inequalities. *J. Spectr. Theory* **1** (2011), no. 1, 87–109 Zbl 1257.35138 MR 2820887

[12] A. M. Hansson and A. Laptev, Sharp spectral inequalities for the Heisenberg Laplacian. In *Groups and analysis*, pp. 100–115, London Math. Soc. Lecture Note Ser. 354, Cambridge Univ. Press, Cambridge, 2008 Zbl 1157.58008 MR 2528463

[13] E. M. Harrell, II, L. Provenzano, and J. Stubbe, Complementary asymptotically sharp estimates for eigenvalue means of Laplacians. *Int. Math. Res. Not. IMRN* (2021), no. 11, 8405–8450 Zbl 1473.35372 MR 4266145

[14] E. M. Harrell, II and J. Stubbe, Two-term, asymptotically sharp estimates for eigenvalue means of the Laplacian. *J. Spectr. Theory* **8** (2018), no. 4, 1529–1550 Zbl 1402.58020 MR 3870075

[15] E. M. Harrell, II and J. Stubbe, Partial retraction note to: "Two-term, asymptotically sharp estimates for eigenvalue means of the Laplacian". *J. Spectr. Theory* **11** (2021), no. 2, 903–904 Zbl 1475.58021 MR 4293495

[16] V. È. Kacnel'son, Equivalent norms in spaces of entire functions (in Russian). *Mat. Sb. (N.S.)* **92(134)** (1973), 34–54, 165. English translation: *Math. USSR, Sb.* **21** (1973), no. 1, 33–55 Zbl 0288.46024 MR 0338406

[17] H. Kovařík, S. Vugalter, and T. Weidl, Two-dimensional Berezin–Li–Yau inequalities with a correction term. *Comm. Math. Phys.* **287** (2009), no. 3, 959–981 Zbl 1197.35182 MR 2486669

[18] H. Kovařík and T. Weidl, Improved Berezin-Li-Yau inequalities with magnetic field. *Proc. Roy. Soc. Edinburgh Sect. A* **145** (2015), no. 1, 145–160 Zbl 1323.35116 MR 3304579

[19] O. Kovrijkine, Some results related to the Logvinenko–Sereda theorem. *Proc. Amer. Math. Soc.* **129** (2001), no. 10, 3037–3047 Zbl 0976.42004 MR 1840110

[20] P. Kröger, Upper bounds for the Neumann eigenvalues on a bounded domain in Euclidean space. *J. Funct. Anal.* **106** (1992), no. 2, 353–357 Zbl 0777.35044 MR 1165859

[21] A. Laptev, Dirichlet and Neumann eigenvalue problems on domains in Euclidean spaces. *J. Funct. Anal.* **151** (1997), no. 2, 531–545 Zbl 0892.35115 MR 1491551

[22] S. Larson, On the remainder term of the Berezin inequality on a convex domain. *Proc. Amer. Math. Soc.* **145** (2017), no. 5, 2167–2181 Zbl 1377.35206 MR 3611329

[23] S. Larson, Asymptotic shape optimization for Riesz means of the Dirichlet Laplacian over convex domains. *J. Spectr. Theory* **9** (2019), no. 3, 857–895 Zbl 1425.35126 MR 4003544

[24] S. Larson, Erratum to "Asymptotic shape optimization for Riesz means of the Dirichlet Laplacian over convex domains". *J. Spectr. Theory* **11** (2021), no. 4, 1987–1991 Zbl 1484.35300 MR 4349681

[25] L. Li and L. Tang, Some upper bounds for sums of eigenvalues of the Neumann Laplacian. *Proc. Amer. Math. Soc.* **134** (2006), no. 11, 3301–3307 Zbl 1111.35021 MR 2231915

[26] P. Li and S. T. Yau, On the Schrödinger equation and the eigenvalue problem. *Comm. Math. Phys.* **88** (1983), no. 3, 309–318 Zbl 0554.35029 MR 0701919

[27] E. H. Lieb, On the lowest eigenvalue of the Laplacian for the intersection of two domains. *Invent. Math.* **74** (1983), no. 3, 441–448 Zbl 0538.35058 MR 0724014

[28] E. H. Lieb, J. P. Solovej, and J. Yngvason, Asymptotics of heavy atoms in high magnetic fields. I. Lowest Landau band regions. *Comm. Pure Appl. Math.* **47** (1994), no. 4, 513–591 Zbl 0800.49041 MR 1272387

[29] E. H. Lieb, J. P. Solovej, and J. Yngvason, Asymptotics of heavy atoms in high magnetic fields. II. Semiclassical regions. *Comm. Math. Phys.* **161** (1994), no. 1, 77–124 Zbl 0807.47058 MR 1266071

[30] V. N. Logvinenko and J. F. Sereda, Equivalent norms in spaces of entire functions of exponential type (in Russian). *Teor. Funkciĭ Funkcional. Anal. i Priložen.* (1974), no. 20, 102–111, 175 Zbl 0312.46039 MR 0477719

[31] A. D. Melas, A lower bound for sums of eigenvalues of the Laplacian. *Proc. Amer. Math. Soc.* **131** (2003), no. 2, 631–636 Zbl 1015.58011 MR 1933356

[32] P. Pfeiffer and M. Täufer, Magnetic Bernstein inequalities and spectral inequality on thick sets for the Landau operator. *SIAM J. Control Optim.* **63** (2025), no. 4, 2659–2685 MR 4941911

[33] M. Reed and B. Simon, *Methods of modern mathematical physics.* I. *Functional analysis.* 2nd edn. Academic Press, New York, 1980 Zbl 0751959 MR 0459.46001

[34] A. V. Sobolev, Quasi-classical asymptotics for the Pauli operator. *Comm. Math. Phys.* **194** (1998), no. 1, 109–134 Zbl 0915.47052 MR 1628306

[35] R. S. Strichartz, Estimates for sums of eigenvalues for domains in homogeneous spaces. *J. Funct. Anal.* **137** (1996), no. 1, 152–190 Zbl 0848.58050 MR 1383015

[36] D. Ueltschi, Segregation in the asymmetric Hubbard model. *J. Statist. Phys.* **116** (2004), no. 1-4, 681–697 Zbl 1142.82320 MR 2082186

[37] G. Wang, M. Wang, C. Zhang, and Y. Zhang, Observable set, observability, interpolation inequality and spectral inequality for the heat equation in $\mathbb{R}^n$. *J. Math. Pures Appl. (9)* **126** (2019), 144–194 Zbl 1428.35146 MR 3950015

[38] T. Weidl, Improved Berezin–Li–Yau inequalities with a remainder term. In *Spectral theory of differential operators*, pp. 253–263, Amer. Math. Soc. Transl. Ser. 2 225, Amer. Math. Soc., Providence, RI, 2008 Zbl 1168.35404 MR 2509788

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