Port. Math. 83 (2026), 1–18
DOI 10.4171/PM/2155

Lefschetz properties and the Jacobian algebra of 3-dimensional hyperplane arrangements

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Abstract. In this article, we prove that the weak and strong Lefschetz properties hold, and, moreover, are equivalent, for any quotient ring at least 2-dimensional. Furthermore, we prove that the weak Lefschetz property holds for any dimension 1 almost complete intersection. We then apply the obtained results to the case of Jacobian ideals of hyperplane arrangements.

1. Introduction

In the past years, there has been an increasing interest in the weak and strong Lefschetz properties and, in particular, in understanding which homogeneous ideals fail to satisfy the Lefschetz properties. Given a graded $\mathbb{K}$- algebra R, we say that R has the *strong Lefschetz property (SLP)*, if the multiplication map $\times \ell^s \colon R_i \to R_{i+s}$ with a general linear form $\ell \in R_1$ has full rank for every $i \geq 0$ and $s \geq 1$. If it holds for any i and $s = 1$, we say that R has the *weak Lefschetz property (WLP)*.

This problem has been extensively studied for the case of Artinian algebras. In this context, it has been long known, thanks to the work of Stanley (see [28]), Watanabe [31] and Reid, Roberts and Roitman (see [26]), that every monomial complete intersection in characteristic zero has the SLP and that, if $I = (f_0, \dots, f_n)$ is a complete intersection, then R/I has the WLP for a general choice of homogeneous forms $f_0, \dots, f_n$ in a polynomial ring R. It remains an open problem to determine whether every complete intersection Artinian $\mathbb{K}$-algebra has the WLP or the SLP.

For an overview of the Lefschetz properties in the Artinian case, we refer the interested reader to [15, 18, 20].

In addition, this interest has been significantly stimulated by the strong connections between the WLP and SLP and various areas in algebraic geometry, commutative algebra, and combinatorics.

Mathematics Subject Classification 2020: 13D02 (primary); 14J60, 32S22, 52C35, 13N15 (secondary).
Keywords: Lefschetz property, hyperplane arrangements, freeness.

In particular, a deep connection has been identified between the Lefschetz properties and the studying of hyperplane arrangements. For instance, in the work of Di Gennaro, Ilardi and Vallés (see [9]), the stability of the derivation bundle associated with a line arrangement is shown to be related to the failure of the SLP. The key idea is that the failure of the SLP is equivalent to the existence of a line for which the splitting type of the associated bundle is unbalanced, thereby contradicting the stability condition. These ideas are further developed and generalized by Cook, Harbourne, Migliore and Nagel in [7], where the authors introduce the concept of an *unexpected curve*. This notion provides a subsequent characterization of the failure of the SLP and, as an application, allows to reinterpret the validity of Terao's conjecture, a longstanding open problem concerning the freeness of arrangements, in terms of the Lefschetz properties.

This connection with the concept of stability can be extended considering any homogeneous ideal and associating to it its *first syzygy bundle*. For instance, this approach has already been investigated by Brenner and Kaid in [6] and, more recently, by Beorchia and Miró-Roig for equigenerated complete intersections (see [3]).

It is therefore natural to ask what can be said in the non-Artinian case. In this direction, the goal of this paper is to continue and extend the study of the Lefschetz properties to non-Artinian algebras, a setting that has been already considered, for example, by the second and third named authors in [23] and [25] for hyperplane arrangements and by Dimca and Popescu in [10] for almost complete intersections, with a special emphasis on Jacobian ideals of projective hypersurfaces with isolated singularities.

This paper is organized as follows.

In Section 2, we recall the notions of generic initial ideals, weak and strong Lefschetz properties and their relations.

In Section 3, we focus on the case of non-Artinian algebras. First, we address the case of dimension greater than or equal to 2, providing equivalent conditions to ensure the SLP and, furthermore, proving that the strong and weak Lefschetz properties are equivalent in this setting (see Theorem 3.1). We then consider the 1-dimensional case, proving that WLP is always satisfied for almost complete intersections (see Theorem 3.3).

Finally, in Section 4, we apply our results to the case of hyperplane arrangements. In particular, we show that:

- WLP holds for any such algebra in three variables (see Theorem 4.10) and we determine in which degree it fails for n-spaces with $n > 3$ (see Theorem 4.12).
- SLP holds for any algebra given by a plus-one generated arrangement (see Theorem 4.16).

2. Preliminaries

In this section, we will introduce the required definitions, recall some known results and present some new ones that will be used throughout this work.

2.1. Generic initial ideals

The interested reader can find more details regarding this part in [8, Chapter 2], for example.

Let $\mathbb{K}$ be a field of characteristic 0 and consider the polynomial ring $S = \mathbb{K}[x_0, \dots, x_n]$. A monomial ideal I of S is said to be *strongly stable* if, for every monomial $t = x_0^{\alpha_0} \cdots x_n^{\alpha_n} \in I$ and every i, j such that $1 \le i < j \le n$ and $x_j \mid t$, the monomial $t' := x_i \cdot t / x_j \in I$, or, in other words, $t' = x_0^{\alpha_0} \cdots x_i^{\alpha_i+1} \cdots x_j^{\alpha_j-1} \cdots x_n^{\alpha_n} \in I$.

Consider a term ordering σ on S and f a non-zero polynomial in S and take

$$\mathrm{LT}_\sigma(f) = \max_\sigma\{\mathrm{Supp}(f)\}$$

being $\mathrm{Supp}(f)$ the set of all monomials appearing with non-zero coefficient in f. Given an ideal I in S, the *leading term ideal* of I (known also as the *initial ideal* of I), denoted by $\mathrm{LT}_\sigma(I)$, is defined as generated by

$$\{\mathrm{LT}_\sigma(f) \mid f \in I \setminus \{0\}\}.$$

Galligo proves in [14, Theorem 2] the following result, which relates the two notions we have recalled.

Theorem 2.1. *Let I be a homogeneous ideal of S, with σ a term ordering such that $x_0 >_\sigma x_1 >_\sigma \cdots >_\sigma x_n$. Then there exists a Zariski open set $U \subseteq \mathrm{GL}(n+1)$ and a strongly stable ideal J such that for each $g \in U$, $\mathrm{LT}_\sigma(g(I)) = J$.*

The strongly stable ideal J given in the previous theorem is called the *generic initial ideal* with respect to σ of I and it is denoted by $\mathrm{gin}_\sigma(I)$. In particular, when σ is the degree reverse lexicographic order, the ideal $\mathrm{gin}_\sigma(I)$ is simply denoted by $\mathrm{rgin}(I)$.

A homogeneous ideal I in S is *saturated* if the irrelevant maximal ideal $\mathfrak{m} = \langle x_0, \dots, x_n \rangle$ is not an associated prime ideal, i.e., $(I : \mathfrak{m}) = I$. For any homogeneous ideal I of S, the *saturation* of I, denoted I^{sat}, is defined by

$$I^{\mathrm{sat}} := \{f \in S \mid f\mathfrak{m}^s \subseteq I \text{ for some integer } s\}.$$

It is possible to deduce much information of an ideal from its generic initial ideal. In particular, given a homogeneous ideal I, the following results are useful for our work.

Remark 2.2. Let I be a homogeneous ideal of S, then:

(a) The Hilbert function of S/I coincides with the one of $S/\operatorname{rgin}(I)$ (see [4, Lemma 3.8]).

(b) Denote by $\operatorname{reg}(I)$ the Castelnuovo–Mumford regularity of I, that is the maximum of the numbers $d_i - i$, where $d_i = \max\{j \mid \beta_{i,j}(I) \neq 0\}$ and $\beta_{i,j}(I)$ are the graded Betti numbers of I. Then by [2, Theorem 2.4], $\operatorname{reg}(I) = \operatorname{reg}(\operatorname{rgin}(I))$. Moreover, if I is a strongly stable ideal, then $\operatorname{reg}(I)$ is the highest degree of a minimal generator of I (see [2]).

(c) We have that $\operatorname{rgin}(I^{\text{sat}}) = \operatorname{rgin}(I)_{x_n \to 0}$, i.e., $\operatorname{rgin}(I^{\text{sat}})$ is obtained from $\operatorname{rgin}(I)$ by evaluating each minimal generator at $(x_0, \dots, x_{n-1}, 1)$. Hence, if I is saturated, then $\operatorname{rgin}(I)$ has no minimal generators involving x_n (see [2]).

Finally, it can be proven that the saturation of an ideal "commutes" with the generic initial ideal construction.

Lemma 2.3. *Let I be a homogeneous ideal of S. Then* $\operatorname{rgin}(I^{\text{sat}}) = \operatorname{rgin}(I)^{\text{sat}}$.

Proof. Let t be a minimal generator of $\operatorname{rgin}(I^{\text{sat}})$. If $t \in \operatorname{rgin}(I)$, then clearly $t \in \operatorname{rgin}(I)^{\text{sat}}$.

If $t \notin \operatorname{rgin}(I)$, then there exists an $\alpha \geq 1$ such that $tx_n^{\alpha} \in \operatorname{rgin}(I)$. Since $\operatorname{rgin}(I)$ is a strongly stable ideal, then $tx_i^{\alpha} \in \operatorname{rgin}(I)$ for all $i = 0, \dots, n$, and hence $t \in \operatorname{rgin}(I)^{\text{sat}}$.

This shows that $\operatorname{rgin}(I^{\text{sat}}) \subseteq \operatorname{rgin}(I)^{\text{sat}}$.

On the other hand, let t be a minimal generator of $\operatorname{rgin}(I)^{\text{sat}}$. By definition, for each $i = 0, \dots, n$, there exists $\alpha_i \geq 0$ such that $tx_i^{\alpha_i} \in \operatorname{rgin}(I)$. In particular, $tx_n^{\alpha_n} \in \operatorname{rgin}(I)$ and hence $t \in \operatorname{rgin}(I^{\text{sat}})$. This shows that $\operatorname{rgin}(I^{\text{sat}}) \supseteq \operatorname{rgin}(I)^{\text{sat}}$. ∎

2.2. Lefschetz properties

The interested reader can find more details regarding this part in [20] and [15], for example.

Let R be a graded $\mathbb{K}$-algebra, and $R = \bigoplus_{i \geq 0} R_i$ its decomposition into homogeneous components with $\dim_{\mathbb{K}}(R_i) < \infty$.

(1) We say that R has the *weak Lefschetz property (WLP)*, if there exists an element $\ell \in R_1$ such that the multiplication map

$$\begin{aligned} \times\ell \colon R_i &\to R_{i+1}, \\ f &\mapsto \ell \cdot f \end{aligned}$$

has full-rank for every $i \geq 0$. In this case, ℓ is called a *weak Lefschetz element*.

(2) We say that R has the *strong Lefschetz property (SLP)*, if there exists an element $\ell \in R_1$ such that the multiplication map

$$\begin{aligned} \times \ell^s \colon R_i &\to R_{i+s}, \\ f &\mapsto \ell^s f \end{aligned}$$

has full-rank for every $i \geq 0$ and $s \geq 1$. In this case, ℓ is called a *strong Lefschetz element*.

Recall that the last two named authors showed that to check if a quotient algebra has either the SLP or the WLP, it is enough to check the quotient by strongly stable ideals.

Proposition 2.4 ([23, Proposition 2.9]). *Let I be a homogeneous ideal of S. Then S/I has the SLP (respectively the WLP) if and only if $S/\operatorname{rgin}(I)$ has the SLP (respectively the WLP) with Lefschetz element x_n.*

As an immediate consequence we have the following result.

Corollary 2.5. *Let I be a homogeneous saturated ideal of S. Then S/I has the SLP and it has an increasing Hilbert function.*

Proof. By Remark 2.2, item (c), $\operatorname{rgin}(I)$ has no minimal generators involving x_n. This implies that the map $\times x_n^s \colon (S/\operatorname{rgin}(I))_i \to (S/\operatorname{rgin}(I))_{i+s}$ is injective for every $i \geq 0$ and $s \geq 1$ that, in turn, implies that $S/\operatorname{rgin}(I)$ has the SLP with Lefschetz element x_n. By Proposition 2.4, S/I has also that SLP. The second part of the statement follows from Remark 2.2 (a), and the fact that the map $\times x_n^s \colon (S/\operatorname{rgin}(I))_i \to (S/\operatorname{rgin}(I))_{i+s}$ is injective for every $i \geq 0$ and $s \geq 1$. ■

In principle, in order to understand if a ring has the WLP, one has to check an infinite number of multiplication maps. However, the following result shows that if we are interested in the WLP or SLP, we can always reconduct to the Artinian case.

Theorem 2.6 ([25, Theorem 2.6]). *Let I be a homogeneous ideal of S. Then the following facts are equivalent:*

(1) *the graded ring S/I has the SLP (respectively the WLP);*

(2) *the graded Artinian ring S/J has the SLP (respectively the WLP), where $J = I + (x_0, \dots, x_n)^{\operatorname{reg}(I)+1}$.*

We will conclude this part introducing some technical results on the Lefschetz properties of a quotient ring, passing through the saturated ideal.

Let I be a homogeneous ideal of S. Since $I \subseteq I^{\text{sat}}$ and $S/I^{\text{sat}} \cong (S/I)/(I^{\text{sat}}/I)$, we can consider the following short exact sequence of standard graded S-modules

$$0 \to I^{\text{sat}}/I \to S/I \to S/I^{\text{sat}} \to 0. \tag{1}$$

Let $\ell \in S_1$ be a general linear form and $i \geq 0$. From the exact sequence (1) we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (I^{\text{sat}}/I)_i & \xrightarrow{\iota_i} & (S/I)_i & \xrightarrow{\rho_i} & (S/I^{\text{sat}})_i & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \times\ell} & & \downarrow{\scriptstyle \times\ell} & & \downarrow{\scriptstyle \times\ell} & & \\ 0 & \longrightarrow & (I^{\text{sat}}/I)_{i+1} & \xrightarrow{\iota_{i+1}} & (S/I)_{i+1} & \xrightarrow{\rho_{i+1}} & (S/I^{\text{sat}})_{i+1} & \longrightarrow & 0. \end{array} \tag{2}$$

In the subsequent results, we will describe how the injectivity and the surjectivity of the multiplication map are preserved when relating the quotient ring given by I and its saturation.

Lemma 2.7. *The multiplication map $\times\ell\colon (S/I)_i \to (S/I)_{i+1}$ is injective if and only if the multiplication map $\times\ell\colon (I^{\text{sat}}/I)_i \to (I^{\text{sat}}/I)_{i+1}$ is injective.*

The multiplication map $\times\ell\colon (S/I)_i \to (S/I)_{i+1}$ is surjective if and only if the maps $\times\ell\colon (I^{\text{sat}}/I)_i \to (I^{\text{sat}}/I)_{i+1}$ and $\times\ell\colon (S/I^{\text{sat}})_i \to (S/I^{\text{sat}})_{i+1}$ are surjective.

Proof. The statements follow from the Snake lemma applied to diagram (2), together with Corollary 2.5. Specifically, the injectivity of the map

$$\times\ell\colon (S/I^{\text{sat}})_i \to (S/I^{\text{sat}})_{i+1}$$

for all $i \geq 0$ implies that the kernels of the left and center multiplication maps are isomorphic. Moreover, it implies also that the cokernels of the three multiplication maps fit into a short exact sequence. ∎

3. Higher dimensional ideals

In this section, we study non-Artinian quotient rings, i.e., quotients of dimension greater than or equal to 1. We divide our analysis into two parts: first, we examine the case of quotient rings of dimension greater than or equal to 2, and then the case of 1-dimensional quotient rings that are almost complete intersections.

3.1. Dimension greater than or equal to 2

Theorem 3.1. *Let $n \geq 2$ and I be a homogeneous ideal of $S = \mathbb{K}[x_0, \dots, x_n]$. Assume that $\dim(S/I) \geq 2$. Then the following facts are equivalent:*

(1) *S/I has the SLP,*

(2) $\operatorname{projdim}(S/I) \leq n$*,*

(3) $\operatorname{depth}(S/I) \geq 1$*,*

(4) *the ideal* $\mathfrak{m} = \langle x_0, \ldots, x_n \rangle$ *is not in* $\operatorname{Ass}(S/I)$*, the associated prime ideals of* S/I*,*

(5) I *is saturated.*

Moreover, if S *has standard grading, then the previous properties are equivalent also to the following:*

(a) S/I *has the WLP.*

Proof. Since items (2), (3), (4) and (5) are known to be equivalent (see, for example, [12, Exercise 19.8] and [2, Section 1]), we just need to prove that (1) and (2) are equivalent.

Assume that S/I has the SLP. By Proposition 2.4, $S/\operatorname{rgin}(I)$ has the SLP with Lefschetz element x_n. By [23, Proposition 2.12], the Hilbert function of $S/\operatorname{rgin}(I)$ is unimodal. However, since $\dim(S/\operatorname{rgin}(I)) = \dim(S/I) \geq 2$, then the tail of the Hilbert function is strictly increasing, and hence the Hilbert function of $S/\operatorname{rgin}(I)$ is an increasing function. This implies that the multiplication map $\times x_n \colon (S/\operatorname{rgin}(I))_i \to (S/\operatorname{rgin}(I))_{i+1}$ is injective for any $i \geq 0$. As a consequence, $\operatorname{rgin}(I)$ has no minimal generators divisible by x_n, and hence $\operatorname{projdim}(S/I) = \operatorname{projdim}(S/\operatorname{rgin}(I)) \leq n$. This shows that (1) implies (2).

Assume now that $\operatorname{projdim}(S/I) \leq n$. By [13, Theorem A1.9],

$$H^0_{\mathfrak{m}}(S/I) \cong \operatorname{Ext}^{n+1}(S/I, S(-n-1)).$$

Since $\operatorname{projdim}(S/I) \leq n$, we have $\operatorname{Ext}^{n+1}(S/I, S(-n-1)) = 0$, and hence $I^{\text{sat}}/I \cong H^0_{\mathfrak{m}}(S/I) = 0$. This implies that I is a saturated ideal and we then conclude by Corollary 2.5. This shows that (2) implies (1).

Assume now that S has standard grading. By definition of SLP, clearly (1) implies (a). Assume that S/I has the WLP. By Proposition 2.4, $S/\operatorname{rgin}(I)$ has the WLP with Lefschetz element x_n. By [23, Proposition 2.10], the Hilbert function of $S/\operatorname{rgin}(I)$ is unimodal. Using the same argument as before, we can prove that (a) implies (2). ■

Remark 3.2. Notice that if we do not assume, in Theorem 3.1, that $\dim(S/I) \geq 2$, then we still obtain that (1) implies (a), and (2) implies (1) and (a) but the opposite implications are false, see [23, Example 5.9]. On the other hand, if we consider the ideal $I = \langle x^3, x^2y, xy^3, y^4, xy^2z \rangle$ in $S = \mathbb{K}[x, y, z]$, then S/I has the SLP with Lefschetz element z but $\operatorname{projdim}(S/I) = 3$.

3.2. Dimension 1 almost complete intersections

Let I be the ideal generated by homogeneous polynomials f_0, f_1, f_2, of degree respectively d_0, d_1, d_2, in $S = \mathbb{K}[x_0, x_1, x_2]$. Assume in addition that $\dim(S/I) = 1$

or, equivalently, that $V(I) = \{x \in \mathbb{P}^2 \mid f(x) = 0 \forall f \in I\} \subset \mathbb{P}^2$ is 0-dimensional (the case $\dim(S/I) = 0$ is already studied in [6]). Notice that under our assumptions, we are studying dimension 1 almost complete intersections, i.e., ideals of codimension 2 generated by three polynomials.

The goal of this part is to prove the following result.

Theorem 3.3. *The quotient ring S/I has the WLP.*

As observed in [10], we can describe the Hilbert–Poincaré series of S/I^{sat} as

$$HS_{S/I^{\text{sat}}}(t) = F/(1-t),$$

where $F \in \mathbb{Z}[t]$ and $F(1) \neq 0$. Moreover, $\deg(F)$ is the minimal degree p for which $\dim_{\mathbb{K}}((S/I^{\text{sat}})_k) = F(1)$ for all $k \geq p$.

Remark 3.4. Since S/I^{sat} is Cohen–Macaulay of dimension 1, we have that $\deg(F) = \operatorname{reg}(I^{\text{sat}}) - 1$.

We will now study the involved multiplication maps.

Lemma 3.5. *Let $\ell \in S_1$ be a general linear form. Then the multiplication map*

$$\times\ell \colon (S/I^{\text{sat}})_i \to (S/I^{\text{sat}})_{i+1}$$

is injective for any $0 \leq i \leq \deg(F) - 1$ *and it is bijective for any* $i \geq \deg(F)$.

Proof. By Corollary 2.5, the multiplication map is injective for all $i \geq 0$. However, since $\dim_{\mathbb{K}}((S/I^{\text{sat}})_i)$ is constant for any $i \geq \deg(F)$, this implies that the map is bijective for any $i \geq \deg(F)$. ∎

In order to study the multiplication map defined by

$$\times\ell \colon (I^{\text{sat}}/I)_i \to (I^{\text{sat}}/I)_{i+1}$$

we will relate it to the cohomology of the syzygy vector bundle of the ideal sheaf. Recalling the equivalence of categories between the one of vector bundles and the one of locally free sheaves, ([16, Example II.5.18]), we will use the terms "vector bundle" and "locally free sheaf" interchangeably.

Indeed, we can consider the short exact sequence

$$0 \to \mathcal{T} \to \bigoplus_{i=0}^{2} \mathcal{O}_{\mathbb{P}^2}(-d_i) \xrightarrow{M} \mathcal{I}_\Gamma \to 0,$$

where $M = [f_0 \ \ f_1 \ \ f_2]$, Γ denotes the zero locus of the ideal I and $\mathcal{T}$ is usually referred to as the first syzygy of the map defined by M. Notice that $\mathcal{T}$ is defined as the kernel of a surjective map between a locally free sheaf and a torsion free one. By [17, Proposition 1.1], we know that $\mathcal{T}$ is a reflexive sheaf on the projective plane, which implies that, furthermore, it is locally free (see [17, Corollary 1.4]).

Let $m(I)$ denote the minimal degree in which $\mathcal{T}$ admits a global section, that is, the integer such that $H^0(\mathcal{T}(m(I))) \neq 0$ and $H^0(\mathcal{T}(m(I)-1)) = 0$. Observe that $m(I)$ coincides with the minimal degree of a syzygy of the matrix M. Indeed, an element $s \in H^0(\mathcal{T}(m(I)))$ defines the following commutative diagram:

$$\begin{array}{ccccccc} \mathcal{O}_{\mathbb{P}^2}(-m(I)) & & & & & & \\ \downarrow s & \searrow^{[g_0,g_1,g_2]^t} & & & & & \\ \mathcal{T} & \longrightarrow & \bigoplus_{i=0}^{2} \mathcal{O}_{\mathbb{P}^2}(-d_i) & \xrightarrow{M} & \mathcal{I}_\Gamma & \longrightarrow & 0 \end{array}$$

with the relation $\sum_{i=0}^{2} f_i g_i = 0$ and such that $\deg(g_i) = m(I) - d_i$ for $i = 0, 1, 2$.

As observed in [27], this allows to have an identification

$$(I^{\text{sat}}/I)_i \cong (H^0_{\mathfrak{m}}(S/I))_i \cong H^1(\mathcal{T}(i)), \tag{3}$$

that leads to the following equality (see [10, Corollary 1.4]):

$$\deg(F) = d_0 + d_1 + d_2 - m(I) - 2.$$

Recall, that, for a torsion free sheaf $\mathcal{F}$ on $\mathbb{P}^n$, we can define its *slope* as $\mu(\mathcal{F}) := \frac{c_1(\mathcal{F})}{\mathrm{rk}(\mathcal{F})}$, i.e., the quotient of the degree given by its first Chern class by its rank. Furthermore, $\mathcal{F}$ is called *semistable* if, for every coherent subsheaf $0 \neq \mathcal{H} \subset \mathcal{F}$ with $0 < \mathrm{rk}(\mathcal{H}) < \mathrm{rk}(\mathcal{F})$, we have $\mu(\mathcal{H}) \leq \mu(\mathcal{F})$ and it is called *stable* when having a strict inequality. More details can be found, for example, in [21, Section II.1].

Proposition 3.6. *Let $\ell \in S_1$ be a general linear form. Consider the multiplication map*

$$\varphi_i \colon H^1(\mathcal{T}(i)) \xrightarrow{\times \ell} H^1(\mathcal{T}(i+1)). \tag{4}$$

Then:

- *If $\mathcal{T}$ is unstable, then φ_i is injective if $i < d_0 + d_1 + d_2 - m(I) - 1$ and it is surjective if $i > m(I) - 3$; moreover, these bounds are sharp.*
- *If $\mathcal{T}$ is either stable or semistable, then φ_i is injective if $i < \frac{d_0+d_1+d_2-2}{2}$, for $d_0 + d_1 + d_2$ even, and if $i < \frac{d_0+d_1+d_2-3}{2}$, for $d_0 + d_1 + d_2$ odd. Moreover, it is surjective if $i > \frac{d_0+d_1+d_2-6}{2}$, for $d_0 + d_1 + d_2$ even, and if $i > \frac{d_0+d_1+d_2-5}{2}$, for $d_0 + d_1 + d_2$ odd.*

Proof. If $\mathcal{T}$ is a direct sum of line bundles, then the statement is true because of the vanishing of $H^1(\mathcal{T}(i))$, for any $i \in \mathbb{Z}$. Therefore, from now on, we assume $\mathcal{T}$ to be indecomposable.

Let us study first the injectivity of the multiplication map (4).

Firstly, we consider the case that $\mathcal{T}$ is unstable. This implies that

$$-m(I) > \frac{-d_0 - d_1 - d_2}{2}.$$

Consider a generic line $L \subseteq \mathbb{P}^2$ defined by the linear form ℓ. Moreover, consider the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-m(I)) \to \mathcal{T} \to \mathcal{I}_Y(m(I) - d_0 - d_1 - d_2) \to 0,$$

being Y the vanishing locus of the minimal section, and its restriction to the line L

$$0 \to \mathcal{O}_L(-m(I)) \to \mathcal{T}_{|L} \to \mathcal{O}_L(m(I) - d_0 - d_1 - d_2) \to 0.$$

Since $d_0 + d_1 + d_2 - 2m(I) > 0$, this implies that

$$\begin{aligned}&\mathrm{Ext}^1(\mathcal{O}_L(m(I) - d_0 - d_1 - d_2), \mathcal{O}_L(-m(I)))\\ &\quad\cong H^1(\mathcal{O}_L(d_0 + d_1 + d_2 - 2m(I))) = 0,\end{aligned}$$

and, therefore, we obtain the splitting, for the generic line L,

$$\mathcal{T}_{|L} \cong \mathcal{O}_L(-m(I)) \oplus \mathcal{O}_L(m(I) - d_0 - d_1 - d_2).$$

Moreover, being $\mathcal{T}$ indecomposable, we have $Y \neq \emptyset$ and therefore $H^0(\mathcal{I}_Y(i)) = 0$ for all $i \leq 0$. This implies that $H^0(\mathcal{T}(i)) \cong H^0(\mathcal{O}_{\mathbb{P}^2}(i - m(I)))$ for all $i \leq d_0 + d_1 + d_2 - m(I)$.

Consider the short exact sequence

$$\begin{aligned}0 &\to \mathcal{T}(i) \to T(i+1)\\ &\to \mathcal{O}_L(i + 1 - m(I)) \oplus \mathcal{O}_L(i + 1 + m(I) - d_0 - d_1 - d_2) \to 0\end{aligned}$$

and its derived long exact sequence in cohomology

$$\begin{aligned}0 &\to H^0(\mathcal{T}(i)) \to H^0(T(i+1))\\ &\to H^0(\mathcal{O}_L(i + 1 - m(I))) \oplus H^0(\mathcal{O}_L(i + 1 + m(I) - d_0 - d_1 - d_2))\\ &\to H^1(\mathcal{T}(i)) \to H^1(T(i+1)) \to \cdots .\end{aligned}$$

If $i < d_0 + d_1 + d_2 - m(I) - 1$, then we have the following vanishing:

$$H^0(\mathcal{O}_L(i + 1 + m(I) - d_0 - d_1 - d_2)) = 0,$$

and dimension equivalence

$$\begin{aligned}h^0(\mathcal{T}(i+1)) - h^0(\mathcal{T}(i)) &= h^0(\mathcal{O}_{\mathbb{P}^2}(i + 1 - m(I))) - h^0(\mathcal{O}_{\mathbb{P}^2}(i - m(I)))\\ &= h^0(\mathcal{O}_L(i + 1 - m(I))).\end{aligned}$$

Therefore, we have that φ_i is injective for all $i < d_0 + d_1 + d_2 - m(I) - 1$.

Observe that, being $\mathcal{T}$ indecomposable, $\varphi_{(d_0+d_1+d_2-m(I)-1)}$ cannot be injective having a 1-dimensional kernel, hence the bound is sharp.

The map φ_i is surjective if and only if we have the short exact sequence in cohomology

$$\begin{aligned}0 &\to H^1(\mathcal{O}_L(i+1-m(I))) \oplus H^1(\mathcal{O}_L(i+1+m(I)-d_0-d_1-d_2)) \\ &\to H^2(\mathcal{T}(i)) \to H^2(T(i+1)) \to 0,\end{aligned}$$

which, by Serre duality, is equivalent to the following one:

$$\begin{aligned}0 &\to H^0(\mathcal{T}(d_0+d_1+d_2-i-4)) \xrightarrow{\varphi} H^0(\mathcal{T}(d_0+d_1+d_2-i-3)) \\ &\to H^1(\mathcal{O}_L(m(I)-i-3)) \oplus H^1(\mathcal{O}_L(-i-3-m(I)+d_0+d_1+d_2)) \to 0.\end{aligned}$$

Furthermore, this is equivalent to ask for the map $\varphi = \varphi_{d_0+d_1+d_2-i-4}$ to be injective. As seen before, this happens if and only if $d_0+d_1+d_2-i-4 \le \deg(F) = d_0+d_1+d_2-m(I)-2$. Therefore, we can state that φ_i is surjective if and only if $i > m(I)-3$.

Suppose now that $\mathcal{T}$ is either stable or semistable. This implies that

$$-m(I) \le \frac{-d_0-d_1-d_2}{2}.$$

If $d_0+d_1+d_2$ is even, due to Grauert–Mülich theorem (see [21, II, Corollary 2] for a reference), we have

$$\mathcal{T}_{|L} \cong \mathcal{O}_L((-d_0-d_1-d_2)/2)^2.$$

Considering the short exact sequence

$$0 \to \mathcal{T}(i) \to T(i+1) \to \mathcal{O}_L\Big(i+1+\frac{-d_0-d_1-d_2}{2}\Big)^2 \to 0,$$

we have that

$$H^0\Big(\mathcal{O}_L\Big(i+1+\frac{-d_0-d_1-d_2}{2}\Big)\Big) = 0$$

and therefore

$$\varphi_i \colon H^1(\mathcal{T}(i)) \to H^1(T(i+1))$$

is injective for all $i < \frac{d_0+d_1+d_2-2}{2}$.

Again, by Serre duality applied to the cohomology sequence, we have that φ_i is surjective if $i > \frac{d_0+d_1+d_2-6}{2}$.

If $d_0+d_1+d_2$ is odd, similarly to the even case, we have the following generic splitting type:

$$\mathcal{T}_{|L} \cong \mathcal{O}_L\Big(\frac{-d_0-d_1-d_2-1}{2}\Big) \oplus \mathcal{O}_L\Big(\frac{-d_0-d_1-d_2+1}{2}\Big)$$

which leads to the short exact sequence

$$0 \to \mathcal{T}(i) \to T(i+1)$$
$$\to \mathcal{O}_L\Big(i+1+\frac{-d_0-d_1-d_2-1}{2}\Big) \oplus \mathcal{O}_L\Big(i+1+\frac{-d_0-d_1-d_2+1}{2}\Big) \to 0.$$

This implies that the map φ_i is injective for all $i < \frac{d_0+d_1+d_2-3}{2}$.

Once again, by Serre duality applied to the cohomology sequence, we have that φ_i is surjective if $i \geq \frac{d_0+d_1+d_2-3}{2}$. ■

To finally prove the main result of this section, we need the following one, which is a direct consequence of the previous proposition.

Corollary 3.7. *Let $\ell \in S_1$ be a general linear form. Then the multiplication map*

$$\varphi_i \colon H^1(\mathcal{T}(i)) \xrightarrow{\times\ell} H^1(\mathcal{T}(i+1))$$

is injective for any $i \leq \deg(F)-1$ and it is surjective for any $i \geq \deg(F)$.

Proof. If $\mathcal{T}$ is not stable, we have that $\deg(F)+1 \leq \frac{d_0+d_1+d_2-2}{2}$ and $\deg(F) \geq m(I)-2$, which implies the injectivity and surjectivity of φ_i if, respectively, $i \leq \deg(F)$ and $i \geq \deg(F)$, due to Proposition 3.6.

If $\mathcal{T}$ is either semistable or stable, then $\deg(F) \leq \frac{d_0+d_1+d_2-4}{2}$, which implies, again by Proposition 3.6, that φ_i is injective for all $i \leq \deg(F)-1$. Finally, recall that the map $\times\ell\colon H^1(\mathcal{T}(i)) \to H^1(T(i+1))$ is surjective for all $i \geq \deg(F)$ by [10, Theorem 4.1]. ■

Proof of Theorem 3.3. By Corollary 3.7 and the identification described in diagram (3), the multiplication map

$$\times\ell\colon (I^{\text{sat}}/I)_i \to (I^{\text{sat}}/I)_{i+1}$$

is injective for any $i \leq \deg(F)-1$ and surjective for all $i \geq \deg(F)$.

Combined with Lemma 2.7, this implies that the multiplication map

$$\times\ell\colon (S/I)_i \to (S/I)_{i+1}$$

is also injective for any $i \leq \deg(F)-1$ and surjective for all $i \geq \deg(F)$. This proves the result. ■

As an immediate consequence, we obtain the following result.

Corollary 3.8. *Consider $p_0 = \min\{p \mid x_1^p \in \operatorname{rgin}(I)\}$. If $\operatorname{rgin}(I)$ has a minimal generator t that involves x_2, then $\deg(T) \geq p_0$.*

Proof. Suppose that there exists a minimal generator t of $\operatorname{rgin}(I)$, with $\deg(t) = g < p_0$ and $x_2 \mid t$. By Theorem 3.3, the quotient ring S/I has the WLP and hence $S/\operatorname{rgin}(I)$ has the WLP as well with Lefschetz element x_2. This implies that the multiplication map

$$\times x_2 \colon (S/\operatorname{rgin}(I))_{g-1} \to (S/\operatorname{rgin}(I))_g$$

has maximal rank. However, this map cannot be injective since $\frac{t}{x_2}$ belongs to its kernel and it cannot be surjective since x_1^g is a non-zero element of the cokernel. This leads to a contradiction with the existence of the element t. ■

4. Hyperplane arrangements and Lefschetz properties

The study of the Lefschetz properties has been lately of interest when considering the Jacobian ideal associated to a hyperplane arrangement. After recalling the necessary definitions and properties, we will describe in which cases the Lefschetz properties can be assured, due to the results in the previous section. For more details on hyperplane arrangements, see [22].

A finite set of affine hyperplanes $\mathcal{A} = \{H_1, \dots, H_d\}$ in $\mathbb{K}^{n+1}$ is called a *hyperplane arrangement*. For each hyperplane H_i we fix a defining linear polynomial $\alpha_i \in S$ such that $H_i = \alpha_i^{-1}(0)$, and let $Q(\mathcal{A}) = \prod_{i=1}^{d} \alpha_i$. An arrangement $\mathcal{A}$ is called *central* if each H_i contains the origin of $\mathbb{K}^{n+1}$. In this case, each $\alpha_i \in S$ is a linear homogeneous polynomial, and hence $Q(\mathcal{A})$ is homogeneous of degree d. Moreover, an arrangement $\mathcal{A}$ is called *essential* if there are $H_{i_1}, \dots, H_{i_{n+1}} \in \mathcal{A}$ such that $\operatorname{codim}(H_{i_1} \cap \dots \cap H_{i_{n+1}}) = n+1$.

We denote by $\operatorname{Der}_{\mathbb{K}^{n+1}} = \left\{\sum_{i=0}^{n} f_i \partial_{x_i} \mid f_i \in S\right\}$ the S-module of *polynomial vector fields* on $\mathbb{K}^{n+1}$ (or S-derivations). Let $\delta = \sum_{i=0}^{n} f_i \partial_{x_i} \in \operatorname{Der}_{\mathbb{K}^{n+1}}$. Then δ is said to be *homogeneous of polynomial degree* q if $f_0, \dots, f_n$ are homogeneous polynomials of degree q in S. In this case, we write $\operatorname{pdeg}(\delta) = q$.

Definition 4.1. Let $\mathcal{A}$ be a central arrangement in $\mathbb{K}^{n+1}$. Define the *module of vector fields logarithmic tangent* to $\mathcal{A}$ (or logarithmic vector fields) by

$$D(\mathcal{A}) = \{\delta \in \operatorname{Der}_{\mathbb{K}^{n+1}} \mid \delta(\alpha_i) \in \langle \alpha_i \rangle S,\ i = 1, \dots, d\}.$$

The module $D(\mathcal{A})$ is a graded S-module and $D(\mathcal{A}) = \{\delta \in \operatorname{Der}_{\mathbb{K}^{n+1}} \mid \delta(Q(\mathcal{A})) \in \langle Q(\mathcal{A}) \rangle S\}$.

Definition 4.2. A central arrangement $\mathcal{A}$ in $\mathbb{K}^{n+1}$ is said to be *free with exponents* $(e_1, \dots, e_{n+1})$ if and only if $D(\mathcal{A})$ is a free S-module and there exists a basis $\delta_1, \dots, \delta_{n+1} \in D(\mathcal{A})$ such that $\operatorname{pdeg}(\delta_i) = e_i$, or equivalently, $D(\mathcal{A}) \cong \bigoplus_{i=1}^{n+1} S(-e_i)$.

Given an arrangement $\mathcal{A}$ in $\mathbb{K}^l$, the *Jacobian ideal* $J(\mathcal{A})$ of $\mathcal{A}$ is the ideal of S generated by $Q(\mathcal{A})$ and all its partial derivatives. The Jacobian ideal has a central role in the study of free arrangements. In fact, we can characterize freeness by looking at $J(\mathcal{A})$ via the Terao's criterion. Terao shows this result for characteristic 0, but the statement holds true for any characteristic as shown in [30].

Theorem 4.3 ([29]). *A central arrangement $\mathcal{A}$ in $\mathbb{K}^{n+1}$ is free if and only if $S/J(\mathcal{A})$ is* 0 *or* $(n-1)$*-dimensional Cohen–Macaulay.*

In [5], the authors connect the study of generic initial ideals with that of hyperplane arrangements, providing a new characterization of freeness via the generic initial ideal of the Jacobian ideal.

Proposition 4.4 ([5]). *Let $\mathcal{A} = \{H_1, \dots, H_d\}$ be a central arrangement in $\mathbb{K}^{n+1}$. Then* $\operatorname{rgin}(J(\mathcal{A}))$ *coincides with S or its minimal generators include x_0^{d-1}, some positive power of x_1, and no monomials only in $x_2, \dots, x_n$.*

Theorem 4.5 ([5]). *Let $\mathcal{A} = \{H_1, \dots, H_d\}$ be a central arrangement in $\mathbb{K}^{n+1}$. Then $\mathcal{A}$ is free if and only if* $\operatorname{rgin}(J(\mathcal{A}))$ *coincides with S or it is minimally generated by*

$$x_0^{d-1}, x_0^{d-2}x_1^{\lambda_1}, \dots, x_1^{\lambda_{d-1}}$$

with $1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_{d-1}$ and $\lambda_{i+1} - \lambda_i = 1$ or 2*.*

The following conjecture appeared in [5].

Conjecture 4.6. *Let $\mathcal{A}$ be a central arrangement in $\mathbb{K}^{n+1}$, and consider $p_0 = \min\{p \mid x_1^p \in \operatorname{rgin}(J(\mathcal{A}))\}$. If $\operatorname{rgin}(J(\mathcal{A}))$ has a minimal generator t that involves x_2, then* $\deg(t) \geq p_0$.

As a consequence of our results, we can prove the following, evidence to the conjecture.

Theorem 4.7. *Conjecture 4.6 holds true for arrangements in $\mathbb{K}^3$.*

Proof. It follows directly from Corollary 3.8. ■

In [23] and then in [25], the authors studied the connection between the Jacobian algebra $S/J(\mathcal{A})$ of an arrangement $\mathcal{A}$ and the Lefschetz properties obtaining, in particular, the following results.

Proposition 4.8 ([23, Proposition 8.1]). *Let $\mathcal{A}$ be a central arrangement in $\mathbb{K}^2$. Then $S/J(\mathcal{A})$ has the SLP.*

Theorem 4.9 ([23, Theorem 8.3]). *Let $\mathcal{A}$ be a free arrangement in $\mathbb{K}^{n+1}$. Then $S/J(\mathcal{A})$ has the SLP.*

We can now extend the picture given by the mentioned results, showing that WLP is the best we can hope for when considering other families of hyperplane arrangements, specifying furthermore a bound for which it fails.

Theorem 4.10. *Let $\mathcal{A}$ be a central and essential arrangement in $\mathbb{K}^3$. Then $S/J(\mathcal{A})$ has the WLP.*

Proof. It follows directly from Theorem 3.3. ∎

Notice that there are arrangements in $\mathbb{K}^3$ such that their Jacobian algebra does not have the SLP.

Example 4.11. Let $\mathcal{A}$ be the arrangement in $\mathbb{R}^3$ with defining polynomial $Q(\mathcal{A}) = xz(x-z)(x-y)(y-z)(y-2z)(y-3z)(y-4z)$. In this case, we have that, looking at the Hilbert function, $\mathrm{HF}(S/\operatorname{rgin}(J(\mathcal{A})), 8) = \mathrm{HF}(S/\operatorname{rgin}(J(\mathcal{A})), 11) = 36$ and that $x^2y^6z^3$ is a minimal generator of $\operatorname{rgin}(J(\mathcal{A}))$. This shows that the multiplication map $\times z^3\colon (S/\operatorname{rgin}(J(\mathcal{A})))_8 \to (S/\operatorname{rgin}(J(\mathcal{A})))_{10}$ does not have maximal rank, and hence that z is not a Lefschetz element for $S/\operatorname{rgin}(J(\mathcal{A}))$. By [23, Lemma 2.8], $S/\operatorname{rgin}(J(\mathcal{A}))$ does not have the SLP. This implies that, by Proposition 2.4, $S/J(\mathcal{A})$ does not have the SLP as well.

Theorem 4.12. *Let $\mathcal{A}$ be a central and essential arrangement in $\mathbb{K}^{n+1}$ with $d = |\mathcal{A}|$. If $S/J(\mathcal{A})$ fails the WLP in degree k, then $k \geq d$.*

Proof. Since $J(\mathcal{A})$ is generated in degree $d-1$ then $S/J(\mathcal{A})$ has the WLP in any degree $k \leq d-2$. Generalizing [19, Theorem 3.2] to the non-Artinian case, if $S/J(\mathcal{A})$ fails the WLP in degree $d-1$, then the partial derivatives of $Q(\mathcal{A})$ are $\mathbb{K}$-dependent on every general central hyperplane H. This implies that there exists a logarithmic vector field δ of degree zero in $D(A^H)$, where $\mathcal{A}^H$ is the restriction of $\mathcal{A}$ to H. However, this is impossible since $\mathcal{A}^H$ is essential since $\mathcal{A}$ is also essential. ∎

In [1], Abe generalized the notions of free and nearly free (see [11]) to central and essential arrangements in any dimension. However, the definition can be given also for non-essential arrangements.

Definition 4.13. Let $\mathcal{A} = \{H_1, \dots, H_d\}$ be an arrangement in $\mathbb{K}^{n+1}$. We say that $\mathcal{A}$ is *plus-one generated* with *exponents* $\mathrm{POexp}(\mathcal{A}) = (a_1, \dots, a_{n+1})$ and *level* a if $D(\mathcal{A})$ has a minimal free resolution of the following form:

$$0 \to S(-a-1) \xrightarrow{(\alpha, g_1, \dots, g_{n+1})} S(-a) \oplus \left(\bigoplus_{i=1}^{n+1} S(-a_i) \right) \to D(\mathcal{A}) \to 0.$$

We will always consider the exponents in non-decreasing order.

Remark 4.14. Let $\mathcal{A}$ be a plus-one generated arrangement in $\mathbb{K}^{n+1}$ with exponents $\mathrm{POexp}(\mathcal{A}) = (a_1, \dots, a_{n+1})$ and level a. Since $\mathcal{A}$ is central, then there exists $k \geq 2$ such that $(a_1, \dots, a_{n+1}) = (0, \dots, 0, 1, a_k, \dots, a_{n+1})$. If $\mathcal{A}$ is essential, then $k = 2$. If $\mathcal{A}$ is non-essential, then $k \geq 3$.

Directly from the definition, we can show the following (see [24, Lemma 3.3]).

Lemma 4.15. *Let $\mathcal{A} = \{H_1, \dots, H_d\}$ be an arrangement in $\mathbb{K}^{n+1}$. Then $\mathcal{A}$ is a plus-one generated arrangement with exponents $\mathrm{POexp}(\mathcal{A}) = (a_1, \dots, a_{n+1})$ and level a if and only if $S/J(\mathcal{A})$ has a minimal free resolution of the form*

$$0 \to S(-d-a) \to S(-d-a+1) \oplus \left(\bigoplus_{i=k}^{n+1} S(-d-a_i+1) \right) \to S(-d+1)^{n-k+3} \to S.$$

Moreover, the map

$$\partial_3 \colon S(-d-a) \to S(-d-a+1) \oplus \left(\bigoplus_{i=k}^{n+1} S(-d-a_i+1) \right)$$

is defined by a matrix of the form $(\alpha, g_{i_1}, \dots, g_{i_{n-k+2}})$, where $1 \leq i_1 < \cdots < i_{n-k+2} \leq n+1$. Notice that $n-k+3$ coincides with the codimension of the center of $\mathcal{A}$ or, equivalently, the rank of $\mathcal{A}$.

To conclude, we show the following result.

Theorem 4.16. *Let $\mathcal{A}$ be a plus-one generated arrangement in $\mathbb{K}^{n+1}$ with $n \geq 3$. Then $S/J(\mathcal{A})$ has the SLP.*

Proof. It follows directly from Theorem 3.1. ∎

Funding. SM is partially supported by PID2024-157142NB-I00 and by the Spanish State Research Agency, through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M). MT is partially supported by the Perko Research Award.

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Received 2 November 2023; revised 9 September 2025.

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