

Maximal number of skew lines on Fermat surfaces

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Abstract. It is well known that the Fermat surface of degree $d \geq 3$ has $3d^2$ lines. However, it has not yet been established what is the maximal number of pairwise disjoint lines that it can have if $d \geq 4$. In this article, we show that the maximal number of skew lines on the Fermat surface of degree $d \geq 4$ is $3d$, either d even or d odd distinct of 5, otherwise ($d = 5$) it contains no more than 13 pairwise disjoint lines.

Introduction

It is well known that the Fermat surface of degree d in the complex projective space has $3d^2$ lines for $d \geq 3$, so it is a lower bound for ℓ_d , the maximal number of lines that a smooth surface of degree d in $\mathbb{P}^3$ can have (cf. Proposition 1.1). In fact, since 1882 it has been known that the so called Schur's quartic contains exactly 64 lines ([13]). And only in 1943, *B. Segre* proved that $\ell_4 = 64$ ([15], even though a gap was discovered in Segre's proof by *Rams–Schütt* in 2015 ([12]), the claim is still correct), but ℓ_d remains unknown for $d \geq 5$. In this regard, the articles of *Caporaso–Harris–Mazur* ([6]) and *Boissière–Sarti* ([4]) exhibited lower bounds for these numbers, which leads us to infer that $3d^2$ does not provide the maximal number of lines on a smooth surface of degree $d \geq 4$ in characteristic 0. On the other hand, according to *Bauer–Rams*, see [3], $11d^2 - 30d + 18$ is an upper bound for the maximal number of lines on a smooth surface of degree $d \geq 3$ in $\mathbb{P}^3(k)$ being k a field of characteristic 0 or of characteristic $p > d$. For example, the Fermat surface, defined by the vanishing of the polynomial $x^{q+1} + y^{q+1} + z^{q+1} + w^{q+1}$ on $\mathbb{P}^3(k)$ being k a field extension of $\mathbb{F}_{q^2}$ where $q = p^e$ for a prime p , contains $q^4 + q^3 + q + 1$ lines, which exceeds the Bauer–Rams's upper bound and leads the authors (cf. [5] and references there in) to conjecture that these Fermat surfaces may provide the maximal number of lines possible on a surface of a given degree in characteristic $p > 0$.

Another problem related to this is to determine the maximal number, $\varkappa_d$, of pairwise disjoint lines (or skew lines) that a smooth surface of degree d can have. In 1975,

Miyaoka gave the upper bound $\mathfrak{s}_d \leq 2d(d-2)$ if $d \geq 4$ ([9]). It is known that $\mathfrak{s}_3 = 6$, $\mathfrak{s}_4 = 16$ ([10]) and $\mathfrak{s}_6 = 48$ ([7]). Some lower bounds were given by *Rams* ([11]) and *Boissière–Sarti* ([4]). However, $\mathfrak{s}_d$ remains unknown for $d = 5$ and $d \geq 7$.

To the best of our knowledge, the maximal number of pairwise disjoint lines on Fermat surfaces is not explicitly stated in the modern literature. For instance, in *Rams*' article ([11]), it is mentioned: “Let us note that the Fermat surface F_d , i.e., the surface with $3d^2$ lines (the largest number known so far for $d \neq 4, 6, 8, 12, 20$), contains no family of $3d$ pairwise disjoint lines” but this statement is wrong and our article provides the correct answer. For Fermat surfaces over fields with characteristic $p \neq 0$, [5] provides certain bounds for $p = 2, 3$.

The aim of our work is to show, in an elementary and self-contained way, that the maximal number of pairwise disjoint lines on Fermat surfaces of degree $d \geq 3$ over the complex numbers is exactly $3d$ for any d even and d odd distinct from 3 and 5 (being such numbers 6 and 13, for $d = 3, 5$, respectively), according to Theorem 4.7.

In order to do that we first established a notation for the set of lines $\mathcal{L}^0$, $\mathcal{L}^1$ and $\mathcal{L}^2$ in F_d (see (1.1)), in such a way that we obtain the stratification (partition) $\mathcal{L}^0 \dot{\cup} \mathcal{L}^1 \dot{\cup} \mathcal{L}^2$ with $\#\mathcal{L}^i = d^2$, for $i = 0, 1, 2$ of these lines in F_d (cf. Proposition 1.1). Moreover, the relations established in Proposition 1.2, together with Proposition 2.2 give us enough conditions to study the intersection between the lines on families $\mathcal{L}^i$ and $\mathcal{L}^j$. Next, we check that the maximal number of pairwise disjoint lines on the family $\mathcal{L}^i$ is d for all i , which implies that $\mathfrak{s}(F_d) \leq 3d$ (being $\mathfrak{s}(F_d)$ the maximal number of pairwise disjoint lines that F_d can have). In fact, if d is even, then we easily get a family consisting of $3d$ pairwise disjoint lines on F_d (cf. Proposition 4.1), otherwise we are faced with a real/generalized “Sudoku game” to find such maximal set of pairwise disjoint lines on F_d (cf. Sections 3 and 4). To our surprise the case $d = 5$ was the only one (for $d \geq 4$) where there is no family with $3d$ skew lines.

Finally, we note that to study the maximal number of rational curves (in particular lines) which do not intersect one another on a surface is an important tool to classify surfaces in the projective space (cf. [1, 2, 10]), as well as to determine all the lines on a smooth surface from some set of its skew lines ([8]).

1. Lines on Fermat surfaces

Let F_d be the degree d Fermat surface in the projective complex space defined as the zeros locus of

$$x^d - y^d - z^d + w^d \in \mathbb{C}[x, y, z, w].$$

Set $\Phi(F_d) = \{\ell \subset F_d \mid \ell \text{ is a line}\}$. An easy verification allows us to conclude that $\mathcal{L}^j = \{L_{k,i}^j\} \subset \Phi(F_d)$ for $j = 0, 1, 2$ being

$$L_{k,i}^0 : \begin{cases} y = \eta^i x, \\ w = \eta^k z, \end{cases} \quad L_{k,i}^1 : \begin{cases} x = \eta^{k+i} z, \\ y = \eta^i w, \end{cases} \quad \text{and} \quad L_{k,i}^2 : \begin{cases} x = v\eta^i w, \\ y = v\eta^{k+i} z, \end{cases} \quad (1.1)$$

where η is a primitive d th root of the unity, v is a complex number such that $v^d = -1$ and $k, i \in \{0, 1, 2, \dots, d-1\}$.¹ Moreover, $\#(\mathcal{L}^j) = d^2$ for $j = 0, 1, 2$.

Proposition 1.1. *With the above notation $\Phi(F_d) = \mathcal{L}^0 \dot{\cup} \mathcal{L}^1 \dot{\cup} \mathcal{L}^2$. Thus $\#(\Phi(F_d)) = 3d^2$.*

Proof. Let us consider the line $L = Z(x, y)$ in $\mathbb{P}^3$. Note that we can stratify the lines in F_d studying their intersection with the line L , i.e.,

$$\Phi(F_d) = \{\ell \in \Phi(F_d) \mid \ell \cap L \neq \emptyset\} \dot{\cup} \{\ell \in \Phi(F_d) \mid \ell \cap L = \emptyset\}.$$

Let ℓ be a line in F_d . Have in mind that $F_d \cap L = \{[0 : 0 : 1 : \eta^j]\}_{j=0}^{d-1}$ where η is a primitive d th root of the unity. Therefore, according to $\ell \cap L \neq \emptyset$ or $\ell \cap L = \emptyset$ we have, respectively:

- (1) ℓ is determined by the points $p = [0 : 0 : 1 : \eta^k] \in L$ for exactly one value of $k \in \{0, \dots, d-1\}$ (since $L \notin \Phi(F_d)$) and $q = [\alpha : \beta : 0 : \gamma]$ with $\alpha, \beta, \gamma \in \mathbb{C}$ not all zero. Thus

$$\begin{aligned} \ell \subset F_d &\iff \alpha^d v^d - \beta^d v^d - u^d + (\eta^k u + \gamma v)^d = 0 \quad \forall [u : v] \in \mathbb{P}^1. \\ &\iff \begin{cases} \alpha^d - \beta^d + \gamma^d = 0, \\ \eta^{k(d-j)} \gamma^j = 0, \quad \text{for } j = 1, \dots, d-1. \end{cases} \\ &\iff \alpha^d - \beta^d = 0 \quad (\alpha\beta \neq 0) \quad \text{and} \quad \gamma = 0. \\ &\iff \ell = Z(w - \eta^k z, y - \eta^i x) = L_{k,i}^0 \in \mathcal{L}^0 \quad \text{with } \alpha^{-1}\beta = \eta^i. \end{aligned}$$

Therefore, $\mathcal{L}^0 = \{\ell \in \Phi(F_d) \mid \ell \cap L \neq \emptyset\}$.

- (2) If $\ell \cap L = \emptyset$, then we can assume that ℓ is defined by

$$x - \alpha z - \beta w \quad \text{and} \quad y - \gamma z - \delta w \quad \text{with } \alpha\delta - \beta\gamma \neq 0.$$

¹Here we use the indices i and $k+i$ to describe the lines on families $\mathcal{L}^1$ and $\mathcal{L}^2$ instead of simply i, k , because this simplifies the writing of incidence relations between the lines in F_d , as we will see later.

Thus,

$$\begin{aligned} \ell \subset \mathbb{F}_d &\iff (\alpha z + \beta w)^d - (\gamma z + \delta w)^d + z^d - w^d = 0. \\ &\iff \begin{cases} \alpha^d - \gamma^d + 1 = 0, \\ \beta^d - \delta^d - 1 = 0, \\ \alpha^{d-j} \beta^j - \gamma^{d-j} \delta^j = 0 \quad \text{for } j = 1, \dots, d-1. \end{cases} \end{aligned} \quad (1.2)$$

From (1.2) for $j = 1, 2$ (as $d \geq 3$), we get $\alpha^{d-1} \beta = \gamma^{d-1} \delta$ and $\alpha^{d-2} \beta^2 = \gamma^{d-2} \delta^2$, which implies that $\gamma^{d-2} \delta (\gamma \beta - \alpha \delta) = 0$. Therefore, $\gamma \delta = 0$. In fact, we have

$$\begin{cases} \gamma = 0 \implies \beta = 0 \implies \ell \in \mathcal{L}^2, \\ \delta = 0 \implies \alpha = 0 \implies \ell \in \mathcal{L}^1. \end{cases}$$

Finally, note that $[\eta^{k+i} : \eta^i : 1 : 1] \in L_{k,i}^1$ and $[\eta^{k+i} : \eta^i : 1 : 1] \notin L_{t,j}^2$ for any t, j . Thus, $\mathcal{L}^1 \cap \mathcal{L}^2 = \emptyset$. ■

Studying the intersections between the lines on $\mathbb{F}_d$

In what follows, we use the notation $a \equiv_d b$ instead of $a \equiv b \pmod{d}$ to indicate that a is congruent to b modulo d .

Proposition 1.2. *With the notation as in (1.1). For any $a, b, i, j, k, t \in \{0, 1, \dots, d-1\}$ it holds*

- (a) $L_{a,b}^0 \cap L_{k,i}^s \neq \emptyset \iff \begin{cases} a = k \text{ or } b = i & \text{if } s = 0, \\ b - a \equiv_d k & \text{if } s = 1, \\ b + a \equiv_d k & \text{if } s = 2. \end{cases}$
- (b) $L_{k,i}^s \cap L_{t,j}^{s_1} \neq \emptyset \iff \begin{cases} k + i \equiv_d t + j \text{ or } i = j & \text{if } s = s_1 \in \{1, 2\}, \\ v^2 \eta^{t+2j} = \eta^{k+2i} & \text{if } s = 1, s_1 = 2. \end{cases}$
- (c) *If d is odd, then we can choose $v = -1$ and it follows that*

$$L_{k,i}^1 \cap L_{t,j}^2 \neq \emptyset \iff k + 2i \equiv_d t + 2j.$$

- (d) *If d is even, then*

$$L_{k,i}^1 \cap L_{k,j}^2 = \emptyset \quad \text{for all } i, j.$$

Proof. The statements (a) and (b) are straightforward verification (from the definitions of the lines $L_{k,i}^s$ in (1.1)), and (c) follows from (b).

Now, let us consider $d \geq 4$ even and suppose that $L_{k,i}^1 \cap L_{k,j}^2 \neq \emptyset$ for some i, j . Thus, it follows from (b) that $v^2 \eta^{2j} = \eta^{2i}$, which implies that $v \eta^j = \pm \eta^i$. Here, if we compute the d th power of $v \eta^j = \pm \eta^i$, we lead to an absurd result. ■

The results of Proposition 1.2 are not novel. In fact, these intersection numbers were previously computed in ([14, equation (6)]). We became aware of this only after completing our own calculations.

2. Characterizing sets of skew lines on F_d

Let $\mathfrak{s}(X)$ be the *maximal number of skew lines* in $X \subseteq F_d$. The relations in the above proposition allow us to show the following result.

Corollary 2.1. $\mathfrak{s}(\mathcal{L}^s) = d$, for $s = 0, 1, 2$. In particular, $\mathfrak{s}(F_d) \leq 3d$.

Proof. From (a) in Proposition 1.2 we have that $L_{a,b}^0 \cap L_{k,i}^0 = \emptyset$ iff $a \neq k$ and $b \neq i$. Thus, any subset $C \subset \mathcal{L}^0$ of pairwise disjoint lines is constituted by lines $L_{a,b}^0$, of which the indices a are all distinct. Hence, $\mathfrak{s}(\mathcal{L}^0) \leq d$ (since $a \in \{0, \dots, d-1\}$). On the other hand $\{L_{a,a}^0\}_{a=0}^{d-1}$ is a family of d skew lines in $\mathcal{L}^0$. Therefore, $\mathfrak{s}(\mathcal{L}^0) = d$.

One more time, from (b) in Proposition 1.2 we have that $L_{k,i}^s \cap L_{t,j}^s = \emptyset$ iff $i \neq j$ and $k+i \not\equiv_d t+j$. Again, the condition $i \neq j$ (with $i, j \in \{0, 1, \dots, d-1\}$) implies that $\mathfrak{s}(\mathcal{L}^s) \leq d$ for $s = 1, 2$. However, $\{L_{k,i}^s\}_{i=0}^{d-1}$ is constituted by d skew lines in $\mathcal{L}^s$. Therefore, $\mathfrak{s}(\mathcal{L}^s) = d$ for $s = 1, 2$. Finally, note that $\mathfrak{s}(F_d) \leq \mathfrak{s}(\mathcal{L}^0) + \mathfrak{s}(\mathcal{L}^1) + \mathfrak{s}(\mathcal{L}^2) = 3d$. ■

From Corollary 2.1 we have the upper bound $3d$ for $\mathfrak{s}(F_d)$. So we are invited to look for maximal subsets of skew lines in F_d . In this regard, an important tool is the next proposition, which will establish some kind of *sudoku's* rule for our game². In fact, the lower bound $2d$ for $\mathfrak{s}(F_d)$ will be established in Corollary 2.3. From this point onward, we start playing (pay attention to the rules!).

Proposition 2.2. Let $R_d = \{0, 1, \dots, d-1\}$ and $r_d : \mathbb{Z} \rightarrow R_d$ the remainder³ function by d . Consider the functions

$$\begin{aligned} \psi_d : R_d \times R_d &\rightarrow R_d & \text{and} \quad \varphi_{d,\pm} : R_d \times R_d &\rightarrow R_d \\ (k, i) &\mapsto r_d(k + 2i) & (k, i) &\mapsto r_d(i \pm k). \end{aligned}$$

For $u \in R_d$, $s \in \{0, 1, 2\}$ define

$$\begin{aligned} D_u^s &:= \{L_{k,i}^s \in \mathcal{L}^s \mid (k, i) \in \psi_d^{-1}(u)\}, \quad \text{for } s = 1, 2; \\ D_{u,\pm}^0 &:= \{L_{k,i}^0 \in \mathcal{L}^0 \mid (k, i) \in \varphi_{d,\pm}^{-1}(u)\}. \end{aligned}$$

²The game is: given $d \geq 4$ find the maximal number of pairwise skew lines on the Fermat surface F_d .

³If $a \in \mathbb{Z}$, then $r_d(a) = r$ where $r \in R_d$ and $a \equiv_d r$.

It is verified that

- (a) the restriction of ψ_d and $\varphi_{d,\pm}$ to $R_d \times \{i\}$ is a bijection for all i ;
- (b) $\varphi_{d,\pm}|_{\{k\} \times R_d} : \{k\} \times R_d \rightarrow R_d$ is a bijection for all k ;
- (c) $\psi_d|_{\{k\} \times R_d} : \{k\} \times R_d \rightarrow R_d$ is a bijection for all k , if d is odd;
- (d) $\#\psi_d^{-1}(u) = d$ and $\#\varphi_{d,\pm}^{-1}(u) = d$ for all $u \in R_d$;
- (e) $D_{u,\pm}^0 \subset \mathcal{L}^0$ and $D_u^s \subset \mathcal{L}^s$ for $s = 1, 2$, are families of d skew lines.

Proof. It is left to the reader. ■

In the next corollary we find the lower bound $2d$ for $\mathfrak{s}(F_d)$.

Corollary 2.3. *Consider the above notation. For $0 \leq i < j \leq 2$, it follows that $\mathfrak{s}(\mathcal{L}^i \cup \mathcal{L}^j) = 2d$. Thus $2d \leq \mathfrak{s}(F_d) \leq 3d$.*

Proof. From Corollary 2.1 we have that $\mathfrak{s}(\mathcal{L}^0) = \mathfrak{s}(\mathcal{L}^1) = \mathfrak{s}(\mathcal{L}^2) = d$. Which implies that

$$\mathfrak{s}(\mathcal{L}^0 \cup \mathcal{L}^s) \leq 2d, \quad \text{for } s = 1, 2 \quad \text{and} \quad \mathfrak{s}(\mathcal{L}^1 \cup \mathcal{L}^2) \leq 2d.$$

Thus, it is enough to find a family of $2d$ skew lines on $\mathcal{L}^0 \cup \mathcal{L}^s$ for $s = 1, 2$ and on $\mathcal{L}^1 \cup \mathcal{L}^2$, respectively. For the first statement, from item (a) of Proposition 1.2, we conclude that the following two sets are constituted by $2d$ skew lines

$$\begin{aligned} &\{L_{0,0}^0, L_{1,1}^0, \dots, L_{d-1,d-1}^0, L_{1,0}^1, L_{1,1}^1, \dots, L_{1,d-1}^1\}; \\ &\{L_{0,0}^0, L_{1,d-1}^0, L_{2,d-2}^0, \dots, L_{d-1,1}^0, L_{1,0}^2, L_{1,1}^2, \dots, L_{1,d-1}^2\}. \end{aligned}$$

Now, for the second statement, we have that $\#D_0^1 = d$ and $\#D_1^2 = d$ in accordance with item (e) of Proposition 2.2. Moreover, by item (b) of Proposition 1.2 we have that $L_{k,i}^1 \cap L_{m,n}^2 = \emptyset$ for any $L_{k,i}^1 \in D_0^1$ and $L_{m,n}^2 \in D_1^2$. Therefore, $D_0^1 \cup D_1^2$ is a family of $2d$ skew lines in $\mathcal{L}^1 \cup \mathcal{L}^2$. ■

From now on, we will focus on capturing maximal subsets of skew lines in F_d , revisiting the conditions that must be satisfied by such subsets.

2.1. Rewriting conditions for subsets of skew lines in F_d

In order to find maximal sets of skew lines in F_d , we started by characterizing those subsets of skew lines in $\mathcal{L}^s$ for each $s = 0, 1, 2$ in terms of ψ_d and $\varphi_{d,\pm}$ (cf. Proposition 2.2), when it comes to the case.

Once again, from Proposition 2.2 we obtain the following two corollaries.

Corollary 2.4. Let $C \subset \Phi(F_d)$ and define $C^s := C \cap \mathcal{L}^s$ for $s \in \{0, 1, 2\}$.

- (a) C^0 is constituted by skew lines $\iff \begin{cases} C^0 = \{L_{a_1, b_1}^0, \dots, L_{a_m, b_m}^0\} \text{ with } \#C^0 = m, \\ 0 \leq a_1 < \dots < a_m \leq d-1 \text{ and there is} \\ \text{a permutation } \sigma \text{ of } R_d \text{ such that } \sigma(a_i) = b_i. \end{cases}$
- (b) C^1 is constituted by skew lines $\iff \begin{cases} C^1 = \{L_{a_1, b_1}^1, \dots, L_{a_m, b_m}^1\} \text{ with } \#C^1 = m, \\ 0 \leq b_1 < \dots < b_m \leq d-1 \text{ and } \varphi_{d,+} \\ \text{restricted to } \{(a_i, b_i)\}_{i=1}^m \text{ is injective.} \end{cases}$
- (c) C^2 is constituted by skew lines $\iff \begin{cases} C^2 = \{L_{a_1, b_1}^2, \dots, L_{a_m, b_m}^2\} \text{ with } \#C^2 = m, \\ 0 \leq b_1 < \dots < b_m \leq d-1 \text{ and } \varphi_{d,+} \\ \text{restricted to } \{(a_i, b_i)\}_{i=1}^m \text{ is injective.} \end{cases}$

Remark 2.5. Note that $\mathcal{L}_k^s = \{L_{k,i}^s \in \mathcal{L}^s \mid i \in R_d\}$ for $k \in R_d$ and $s \in \{0, 1, 2\}$ is constituted by d skew lines if $s \in \{1, 2\}$ (according to (b) in Proposition 1.2). Moreover,

$$\mathcal{L}^s = \mathcal{L}_0^s \dot{\cup} \dots \dot{\cup} \mathcal{L}_{d-1}^s \quad \text{for any } s \in \{0, 1, 2\}.$$

Now, we will concentrate our attention on the description of those subsets C^s of $\mathcal{L}^s$ consisting of skew lines such that $C^s \cup C^{s_1}$ is also formed by skew lines (for $0 \leq s < s_1 \leq 2$).

In what follows, for any subset $X \subseteq \Phi(F_d)$ we may identify the line $\mathcal{L}_{k,i}^s \in X$ with the pair (k, i) (which will be clear from the context). Having this in mind we will consider $\psi_d(X)$ and $\varphi_{d,\pm}(X)$.

Corollary 2.6. With the above notation. Assume that C^0, C^1 and C^2 consist of skew lines. Then we have:

- (a) $C^0 \cup C^1$ is constituted by skew lines $\iff C^1 \cap \mathcal{L}_k^1 = \emptyset$ for every $k \in \varphi_{d,-}(C^0)$.
- (b) $C^0 \cup C^2$ is constituted by skew lines $\iff C^2 \cap \mathcal{L}_k^2 = \emptyset$ for every $k \in \varphi_{d,+}(C^0)$.
- (c) For d odd, $C^1 \cup C^2$ is constituted by skew lines $\iff \psi_d(C^1) \cap \psi_d(C^2) = \emptyset$.

Remarks 2.7. Assume d odd. If $C^s \subset \mathcal{L}^s$ consists of skew lines for $s = 0, 1, 2$, then Corollary 2.6 allows to conclude that:

- (1) If $C^1 = \mathcal{L}_k^1$ (resp. $C^2 = \mathcal{L}_k^2$) for some $k \in R_d$, then $C^2 = \emptyset$ (resp. $C^1 = \emptyset$).
- (2) If $\varphi_{d,+}(C^0) = R_d$ (resp. $\varphi_{d,-}(C^0) = R_d$), then $C^2 = \emptyset$ (resp. $C^1 = \emptyset$)⁴.
- (3) $\#\psi_d(C^1) + \#\psi_d(C^2) \leq d$. In particular, if $\psi_d(C^1) = R_d$, then $C^2 = \emptyset$ and vice versa.

⁴It is also true for d even.

Let us see an example of a family of 13 skew lines in F_5 .

Example 2.8. Note that $C^0 = \{L_{0,4}^0, L_{2,0}^0, L_{3,1}^0, L_{4,3}^0\}$ consists of four skew lines (cf. (a) in Corollary 2.4). Furthermore, in the rows of the next table we register the values of $\varphi_{5,\pm}(C^0)$, respectively:

C^0	$L_{0,4}^0$	$L_{2,0}^0$	$L_{3,1}^0$	$L_{4,3}^0$
$\varphi_{5,-}$	4	3	3	4
$\varphi_{5,+}$	4	2	4	2

Now, having in mind Corollary 2.6 for the choice of $C^s \subset \mathcal{L}^s$ such that $C^0 \cup C^s$ is constituted by skew lines for $s = 1, 2$, it is necessary that

$$C^1 \cap \mathcal{L}_k^2 = \emptyset \quad \forall k \in \{3, 4\} \quad \text{and} \quad C^2 \cap \mathcal{L}_k^2 = \emptyset \quad \forall k \in \{2, 4\}.$$

So, $C^1 = \{L_{0,1}^1, L_{0,4}^1, L_{2,0}^1, L_{2,3}^1\}$ and $C^2 = \{L_{0,0}^2, L_{1,2}^2, L_{3,1}^2, L_{3,3}^2, L_{3,4}^2\}$ are admissible choices. As well as according to the information on the rows in the following two tables:

C^1	$L_{0,1}^1$	$L_{0,4}^1$	$L_{2,0}^1$	$L_{2,3}^1$	C^2	$L_{0,0}^2$	$L_{1,2}^2$	$L_{3,1}^2$	$L_{3,3}^2$	$L_{3,4}^2$
$\varphi_{5,+}$	1	4	2	0	$\varphi_{5,+}$	0	3	4	1	2
ψ_5	2	3	2	3	ψ_5	0	0	0	4	1

We have that C^s consists of skew lines for $s = 1, 2$ (cf. (b) and (c) in Corollary 2.4) and $\psi_5(C^1) \cap \psi_5(C^2) = \emptyset$, which implies that $C^1 \cup C^2$ also is formed by skew lines (cf. (c) in Corollary 2.6). Therefore, $C := C^0 \cup C^1 \cup C^2$ consists of 13 skew lines in F_5 . In fact, in Theorem 3.6, we will prove that $\varkappa(F_5) = 13$.

Remarks 2.9. Note that the correspondence between lines in $\mathcal{L}^s$ and pairs in $R_d \times R_d$ for each $s \in \{0, 1, 2\}$ allows us to associate to the d^2 lines in $\mathcal{L}^s$ (for each $s \in \{0, 1, 2\}$) the following $d \times d$ square matrix:

$$\begin{pmatrix} (0, 0) & (1, 0) & \cdots & (d-1, 0) \\ (0, 1) & (1, 1) & \cdots & (d-1, 1) \\ \vdots & \vdots & \cdots & \vdots \\ (0, d-1) & (1, d-1) & \cdots & (d-1, d-1) \end{pmatrix}. \quad (2.1)$$

Now, let us investigate the families of lines identified by the entries in the rows, columns, diagonals, and anti-diagonals of the matrix mentioned above. But first of all, it is important to make clear that:

For each $r \in R_d$

- (1) $\varphi_{d,-}^{-1}(r)$ will be named a *diagonal with remainder r* of the matrix in (2.1),
- (2) $\varphi_{d,+}^{-1}(r)$ will be named an *anti-diagonal with remainder r* of the matrix in (2.1).

So, for example we say that $L_{a,b}^s$ and L_{a_1,b_1}^s are in the same diagonal (resp. anti-diagonal) if $\varphi_{d,-}(a, b) = \varphi_{d,-}(a_1, b_1)$ (resp. $\varphi_{d,+}(a, b) = \varphi_{d,+}(a_1, b_1)$).

Note that

- (i) the family $\mathcal{L}_k^s$ is labeled by the pairs in the $(k + 1)$ -th column of the matrix in (2.1);
- (ii) any two lines labeled by pairs in the same row of the matrix in (2.1) meet;
- (iii) each of such diagonals and anti-diagonals determines d skew lines in $\mathcal{L}^0$;
- (iv) each diagonal (resp. anti-diagonal) with remainder r meets the column in the matrix (2.1) in exactly one pair (i.e., in exactly one line in $\mathcal{L}_k^s$ for $k \in R_d$);
- (v) Let $L_{a,b}^s, L_{a_1,b_1}^s \in \mathcal{L}^s$ be disjoint and d odd and $s \in \{0, 1, 2\}$. If $\varphi_{d,\pm}(a, b) = \varphi_{d,\pm}(a_1, b_1)$, then $\varphi_{d,\mp}(a, b) \neq \varphi_{d,\mp}(a_1, b_1)$. In other words, if $L_{a,b}^s, L_{a_1,b_1}^s$ are lines on the same diagonal, then they are in distinct anti-diagonal, and vice versa.

3. Computing $\mathfrak{s}(F_d)$ for $d \in \{3, 5\}$

Next we will exhibit the only two Fermat surfaces F_d satisfying $\mathfrak{s}(F_d) < 3d$.

3.1. Showing that $\mathfrak{s}(F_3) = 6$

Proposition 3.1. *Let C be a set of skew lines on Fermat cubic F_3 and consider $C^s = C \cap \mathcal{L}^s$ for each $s = 0, 1, 2$. If $\#C^s = 3$ for some s then there exists $k \in \{0, 1, 2\} \setminus \{s\}$ such that $C^k = \emptyset$.*

Proof. Next, we will subdivide in the following three cases:

- If $\#C^0 = 3$. By Proposition 1.2 we may assume that $C^0 = \{L_{0,b_0}^0, L_{1,b_1}^0, L_{2,b_2}^0\}$ with $b_j \in \{0, 1, 2\}$ and

$$b_0 \neq b_1, \quad b_0 \neq b_2, \quad b_1 \neq b_2. \quad (3.1)$$

We claim that $\#\varphi_{3,+}(C^0) = 1$ or $\#\varphi_{3,-}(C^0) = 1$. Note that $r_3(\{b_i, b_i + 1, b_i + 2\}) = R_3$, for any i . Thus $b_0 \equiv_3 b_1 + j$ for some $j = 0, 1, 2$. In fact, $b_0 \not\equiv_3 b_1$, so we have

$$\underbrace{b_0 \equiv_3 b_1 + 1}_{(i)} \quad \text{or} \quad \underbrace{b_0 \equiv_3 b_1 + 2}_{(ii)}$$

For (i), have in mind that $b_1 + 1 \equiv_3 b_2 + j$ for some $j = 0, 1, 2$. In fact, by equation (3.1) we have

$$b_1 + 1 \not\equiv_3 b_2 + 1 \quad \text{and} \quad b_1 + 1 \not\equiv_3 b_2.$$

Thus $b_0 \equiv_3 b_1 + 1 \equiv_3 b_2 + 2$ and consequently $\varphi_{3,+}(C^0) = \{b_0\}$, so $\#\varphi_{3,+}(C^0) = 1$.

For (ii) we used that $b_1 + 2 \equiv_3 b_2 + j$ for some $j = 0, 1, 2$. However, by equation (3.1) we have

$$b_1 + 2 \not\equiv_3 b_2 \quad \text{and} \quad b_1 + 2 \not\equiv_3 b_2 + 2.$$

Thus $b_0 \equiv_3 b_1 + 2 \equiv_3 b_2 + 1$, that is, $b_0 \equiv_3 b_1 - 1 \equiv_3 b_2 - 2$ and consequently $\varphi_{3,-}(C^0) = \{b_0\}$, so $\#\varphi_{3,-}(C^0) = 1$.

Finally, if $\#\varphi_{3,+}(C^0) = 1$ then $\#\varphi_{3,-}(C^0) = 3$ (cf. item (v) in Remarks 2.9). Which implies that $C^1 = \emptyset$ (cf. Remarks 2.7). Analogously, if $\#\varphi_{3,-}(C^0) = 1$ then $C^2 = \emptyset$.

- Assume $\#C^1 = 3$ and let $C^1 = \{L_{a_0,0}^1, L_{a_1,1}^1, L_{a_2,2}^1\}$ with $a_i \in \{0, 1, 2\}$ and

$$a_0 \not\equiv_3 a_1 + 1, \quad a_0 \not\equiv_3 a_2 + 2, \quad a_1 + 1 \not\equiv_3 a_2 + 2. \quad (3.2)$$

We will analyze the following two possibilities: $a_0 \equiv_3 a_1 + 2$ or $a_0 \not\equiv_3 a_1 + 2$.

$a_0 \equiv_3 a_1 + 2$ One more time have in mind that $a_1 + 2 \equiv_3 a_2 + j$ for some $j = 0, 1, 2$. In fact, it follows from equation (3.2)⁵ that

$$a_1 + 2 \not\equiv_3 a_2 \quad \text{and} \quad a_1 + 2 \not\equiv_3 a_2 + 2.$$

Thus $a_0 \equiv_3 a_1 + 2 \equiv_3 a_2 + 1$. This implies that $\#\psi_3(C^1) = 1$ and therefore $C^0 = \emptyset$.⁶

$a_0 \not\equiv_3 a_1 + 2$ In this case, we will show that $\#\psi_3(C^1) = 3$ (i.e., $r_3(\{a_0, a_1 + 2, a_2 + 1\}) = R_3$).

Since $a_0 \not\equiv_3 a_1 + 1$ (cf. (3.2)) then necessarily $a_0 \equiv_3 a_1$. On the other hand, note that

$$a_1 + 2 \equiv_3 a_2 + 1 \implies a_0 + 2 \equiv_3 a_2 + 1 \implies a_0 \equiv_3 a_2 + 2$$

and

$$a_0 \equiv_3 a_2 + 1 \implies a_1 + 1 \equiv_2 a_2 + 2$$

which are both absurd (cf. (3.2)). Therefore, $a_0 \not\equiv_3 a_1 + 2$, $a_0 \not\equiv_3 a_2 + 1$, $a_1 + 2 \not\equiv_3 a_2 + 1$ and this implies that $\psi_3(C^1) = R_3$. Furthermore, $C^2 = \emptyset$ (cf. Remarks 2.7). The case where $\#C^2 = 3$ is left as an exercise for the reader. ■

Corollary 3.2. *Let F_3 be the Fermat cubic surface. It follows that $\varkappa(F_3) = 6$.*

Proof. Let $C \subset F_3$ be a set of skew lines such that $\#C > 6$. Then $C \cap \mathcal{L}^i = C^i \neq \emptyset$, for each $i = 0, 1, 2$ (cf. Corollary 2.1). By Proposition 3.1 we may conclude that $\#C^i \leq 2$ for each $i = 0, 1, 2$ which contradicts the assumption on C . This implies that $\varkappa(F_3) \leq 6$. Now use Corollary 2.3. ■

⁵Note that $a_1 + 2 \equiv_3 a_2 \Rightarrow a_1 + 1 \equiv_3 a_2 + 2$. As well as, $a_1 + 2 \equiv_3 a_2 + 2 \Rightarrow a_0 \equiv_3 a_2 + 2$.

⁶ $\#\psi_3(C^1) = 1 \Rightarrow a_0 \equiv_3 a_1 + 2 \equiv_3 a_2 + 1 \Rightarrow \{a_0, a_1, a_2\} = R_3 \Rightarrow C^0 = \emptyset$.

3.2. Showing that $\mathfrak{s}(F_5) = 13$

Lemma 3.3. *Let $C^0 \in \mathcal{L}^0$ be a set of skew lines in F_5 . If $\#C^0 = 5$ then $\#\varphi_{5,+}(C^0) \geq 3$ or $\#\varphi_{5,-}(C^0) \geq 3$.*

Proof. We can assume that $C^0 = \{L_{a_0,b_0}^0, \dots, L_{a_4,b_4}^0\}$ with $\{a_0, \dots, a_4\} = \{b_0, \dots, b_4\} = \{0, \dots, 4\}$ (cf. Proposition 1.2). If $\#\varphi_{5,+}(C^0) \leq 2$ then, without loss of generality, we may assume that $a_0 + b_0 \equiv_5 a_1 + b_1 \equiv_5 a_2 + b_2$. This implies that $b_0 - a_0 \not\equiv_5 b_1 - a_1, b_0 - a_0 \not\equiv_5 b_2 - a_2$ and $b_1 - a_1 \not\equiv_5 b_2 - a_2$ (cf. (v) in Remarks 2.9). Therefore, $\#\varphi_{5,-}(C^0) \geq 3$ as we desired. ■

Lemma 3.4. *Let $C^s = \{L_{a_0,b_0}^s, \dots, L_{a_4,b_4}^s\} \subset \mathcal{L}^s$ be a set of skew lines in F_5 such that $\#C^s = 5$ for $s = 1, 2$. If $\#\{a_0, \dots, a_4\} \leq 3$ then $\#\psi_5(C^s) \geq 3$.*

Proof. We will divide the proof in three cases according to the $\#\{a_0, \dots, a_4\}$. The first case is $\#\{a_0, \dots, a_4\} = 1$. In this case, it follows that $\#\psi_5(C^s) = 5$ since $C^s = \mathcal{L}_k^s$ for some $k \in R_5$. The second one is when $\#\{a_0, \dots, a_4\} = 2$ and in this case at least three are equal, so we may assume that $a_0 = a_1 = a_2$. This implies that $\#\psi_5(C^s) \geq 3$.⁷ The last one occurs when $\#\{a_0, \dots, a_4\} = 3$. In this case, we have two possibilities (reordering indexes if necessary):

- (i) $a_0 = a_1 = a_2$ and $\#\{a_0, a_3, a_4\} = 3$, which implies $\#\psi_5(C^s) \geq 3$.
- (ii) $a_0 = a_1, a_2 = a_3$ and $\#\{a_0, a_2, a_4\} = 3$. In this case,

$$a_0 + 2b_0 \not\equiv_5 a_1 + 2b_1 \quad \text{and} \quad a_2 + 2b_2 \not\equiv_5 a_3 + 2b_3,$$

which implies that $\#\psi_5(C^s) \geq 2$. Let us suppose by absurd that $\#\psi_5(C^s) = 2$. So we may assume that

$$a_0 + 2b_0 \equiv_5 a_2 + 2b_2 \equiv_5 a_4 + 2b_4 \quad \text{and} \quad a_1 + 2b_1 \equiv_5 a_3 + 2b_3.$$

Now, note that

$$\begin{aligned} \sum_{i=0}^4 (a_i + b_i) &\equiv_5 \sum_{i=0}^4 a_i \equiv_5 \sum_{i=0}^4 (a_i + 2b_i) \equiv_5 3(a_0 + 2b_0) + 2(a_1 + 2b_1) \\ &\equiv_5 (a_0 + b_0) + 2a_0 + (a_1 + b_1) + a_1 + 3b_1 \end{aligned}$$

which implies that $(a_2 + b_2) + (a_3 + b_3) + (a_4 + b_4) \equiv_5 2a_0 + a_1 + 3b_1$. Having in mind that $r_5(\{a_i + b_i\}_{i=1}^5) = R_5$ (since $\#C^s = 5$), we have that

$$(a_2 + b_2) + (a_3 + b_3) + (a_4 + b_4) \equiv_5 4(a_0 + b_0) + 4(a_1 + b_1).$$

⁷Since, $a_0 + 2b_i \equiv_5 a_0 + 2b_j \Leftrightarrow b_i \equiv_5 b_j$ for $i \neq j$, and $i, j \in \{0, 1, 2\}$

Thus,

$$\begin{aligned}
 4(a_0 + b_0) + 4(a_1 + b_1) &\equiv_5 2a_0 + a_1 + 3b_1 \implies 3a_1 + b_1 + 2a_0 + 4b_0 \equiv_5 0 \\
 &\implies 2(a_0 + 2b_0) \equiv_5 2(a_1 + 2b_1) \\
 &\implies a_0 + 2b_0 \equiv_5 a_1 + 2b_1
 \end{aligned}$$

and this is an absurd. Therefore, $\#\psi_5(C^s) \geq 3$ for $s = 1, 2$. ■

Lemma 3.5. *Let C be a set of skew lines in F_5 , $C^i = C \cap \mathcal{L}^i$ for $i = 0, 1, 2$ such that $\#C^0 \geq 4$. If $C^s = \{L_{a_0, b_0}^s, \dots, L_{a_4, b_4}^s\}$ with $\#C^s = 5$ and $\#C^r = 4$ where $\{s, r\} = \{1, 2\}$ then $\#\{a_0, \dots, a_4\} \geq 3$.*

Proof. Note that, if $\#\{a_0, \dots, a_4\} = 1$, then $C^s = \mathcal{L}_k^s$ for some $k \in R_5$. And this implies that $C^r = \emptyset$ which is an absurd (cf. Remarks 2.7). Now, if $\#\{a_0, \dots, a_4\} = 2$, then we have two possibilities (reordering indexes if necessary):

(i) $a_0 = a_1 = a_2 = a_3$ and $a_0 \neq a_4$, (ii) $a_0 = a_1 = a_2$ and $a_0 \neq a_3 = a_4$.

In case (i), it follows that $\#\psi_5(C^s) \geq 4$ and $\#\psi_5(C^r) \in \{0, 1\}$ (since $\psi_5(C^s) \cap \psi_5(C^r) = \emptyset$). Hence, if $\#\psi_5(C^r) = 0$ then $C^r = \emptyset$, else $\#\psi_5(C^r) = 1$ which implies $\#\{a_0, \dots, a_4\} \leq 1$ and this is an absurd.⁸

For (ii) note that

$$\begin{aligned}
 0 &\equiv_5 \sum_{i=0}^4 (a_i + b_i) \equiv_5 3a_0 + 2a_3 \implies 3a_0 \equiv_5 3a_3 \implies a_0 \equiv_5 a_3 \\
 &\implies a_0 = a_3
 \end{aligned}$$

which is an absurd. ■

Theorem 3.6. *Let F_5 be the Fermat surface of degree 5. It follows that $\mathfrak{s}(F_5) = 13$.*

Proof. Let C be a set of skew lines in F_5 . Let us suppose that $\#C \geq 14$. In fact, it is enough to analyze the case $\#C = 14$. Since $\mathcal{L}^1, \mathcal{L}^2$ and $\mathcal{L}^3$ form a stratification of lines on F_5 and $C = \bigcup_{i=1}^3 C_i$ with $C^i = C \cap \mathcal{L}^i$, then $\#C = \sum_{i=1}^3 \#C_i$. On the other hand, it follows from Corollary 2.1 that $\#C_i \leq 5$ for $i = 0, 1, 2$. Thus just one of the next possibilities happens:

- (a) $\#C^0 = 4$ and $\#C^1 = \#C^2 = 5$;
- (b) $\#C^0 = 5$, $\#C^s = 5$, and $\#C^r = 4$ for $\{r, s\} = \{1, 2\}$.

⁸In fact, assume that $s = 1, r = 2$, $C^2 = \{L_{a'_0, b'_0}^2, \dots, L_{a'_3, b'_3}^2\}$. As $\#\psi_5(C^2) = 1$ then $\#\{a'_0, \dots, a'_3\} = 4$ (if $a'_0 = a'_1$, then $a'_0 + 2b'_0 \not\equiv_5 a'_1 + 2b'_1$ since $b'_0 \not\equiv_5 b'_1$, which is an absurd). Therefore, $\#\varphi_{5,+}(C^0) = 1$ (cf. Corollary 2.6) which implies that $\#\varphi_{5,-}(C^0) \geq 4$. Hence, $\#\{a_0, \dots, a_4\} \leq 1$.

For (a) let us consider $C^1 = \{L_{a_0, b_0}^1, \dots, L_{a_4, b_4}^1\}$ and $C^2 = \{L_{a'_0, b'_0}^2, \dots, L_{a'_4, b'_4}^2\}$. Note that

$$\#\{a_0, \dots, a_4\} \leq 3 \quad \text{or} \quad \#\{a_0, \dots, a_4\} > 3.$$

The last inequality cannot occur because other way

- (i) if $\#\{a_0, \dots, a_4\} = 5$ then $\#\varphi_{5,-}(C^0) = 0$ and this implies that $C^0 = \emptyset$;
- (ii) if $\#\{a_0, \dots, a_4\} = 4$, then $\#\varphi_{5,-}(C^0) = 1$. Hence, $\#\varphi_{5,+}(C^0) = 4$. Therefore, $\#\{a'_0, \dots, a'_4\} = 1$, which implies that $\#\psi_5(C^2) = 5$. Furthermore, $\#\psi_5(C^1) = 0$ which is an absurd.

Therefore, $\#\{a_0, \dots, a_4\} \leq 3$. Analogously, we may conclude that $\#\{a'_0, \dots, a'_4\} \leq 3$. It follows from Lemma 3.4 that $\#\psi_5(C^1) \geq 3$ and $\#\psi_5(C^2) \geq 3$, which is an absurd by Remarks 2.7.

For (b), let us assume that $\#C^0 = \#C^1 = 5$ and $\#C^2 = 4$ (the other case is analogous). Let us consider $C^0 = \{L_{a_0, b_0}^0, \dots, L_{a_4, b_4}^0\}$, $C^1 = \{L_{a'_0, b'_0}^1, \dots, L_{a'_4, b'_4}^1\}$ and $C^2 = \{L_{a''_0, b''_0}^2, \dots, L_{a''_3, b''_3}^2\}$. Using arguments analogous to cases (i) and (ii) we may conclude⁹ that $\#\{a'_0, \dots, a'_4\} \leq 3$. Now, it follows from Lemma 3.5 that $\#\{a'_0, \dots, a'_4\} = 3$. So $\#(\varphi_{5,-}(C^0)) \leq 2$. Now, we will analyze all three possibilities:

- (iii) if $\#\varphi_{5,-}(C^0) = 0$, then $C^0 = \emptyset$;
- (iv) if $\#\varphi_{5,-}(C^0) = 1$, then $\#\varphi_{5,+}(C^0) = 5$ and this implies that $C^2 = \emptyset$;
- (v) if $\#\varphi_{5,-}(C^0) = 2$, then we may assume that

$$b_0 - a_0 \equiv_5 b_1 - a_1 \equiv_5 b_2 - a_2 \equiv_5 b_3 - a_3 \quad \text{and} \quad b_0 - a_0 \not\equiv_5 b_4 - a_4 \quad (3.3)$$

or

$$b_0 - a_0 \equiv_5 b_1 - a_1 \equiv_5 b_2 - a_2 \quad \text{and} \quad b_0 - a_0 \not\equiv_5 b_4 - a_4 \equiv_5 b_3 - a_3 \quad (3.4)$$

By equation (3.3), we may conclude that $\#(\varphi_{5,+}(C^0)) \geq 4$. So, $\#\{a''_0, \dots, a''_3\} = 1$. This implies that $\#\psi_5(C^2) = 4$ and consequently $\#\psi_5(C^1) = 1$ by Remarks 2.7. And this is an absurd by Lemma 3.4. Finally, by equation (3.4) it follows that

$$\begin{aligned} \sum_{i=0}^4 (b_i - a_i) &\equiv_5 0 \implies 3(b_0 - a_0) + 2(b_4 - a_4) \equiv_5 0 \\ &\implies 3(b_0 - a_0) \equiv_5 3(b_4 - a_4) \\ &\implies b_0 - a_0 \equiv_5 b_4 - a_4 \end{aligned}$$

which is an absurd. Therefore, $\#C \leq 13$ for any set C of skew lines in F_5 . On the other hand, Example 2.8 shows an example with 13 skew lines, thus $s(F_5) = 13$. ■

⁹If $\#\{a'_0, \dots, a'_4\} = 5$ then $C^0 = \emptyset$. If $\#\{a'_0, \dots, a'_4\} = 4$ then $\#\varphi_{5,-}(C^0) = 1$. Hence $\#\varphi_{5,+}(C^0) = 5$ and this implies that $C^2 = \emptyset$, which is an absurd.

4. Addressing the case $d \geq 4$ and $d \neq 5$

From Corollary 2.3, we have that $2d \leq \mathfrak{s}(F_d) \leq 3d$ for any $d \geq 3$. For $d \geq 4$ even, we have that $\mathfrak{s}(F_d) = 3d$, as we prove in the next proposition. However, for $d \geq 7$ odd, we will divide our study into two cases: $d \equiv_4 1$ and $d \equiv_4 3$ being $d \geq 7$.

Proposition 4.1. *Let $d \geq 4$ even. If $C^0 = \{L_{a,a}^0\}_{a=0}^{d-1}$ and $C^s = \{L_{1,i}^s\}_{i=0}^{d-1}$ for $s = 1, 2$, then $C^0 \cup C^1 \cup C^2$ consists of $3d$ skew lines in F_d .*

Proof. It follows from Corollary 2.4 that C^s consists of d skew lines for each $s \in \{0, 1, 2\}$.¹⁰ On the other hand, $\varphi_{d,-}(C^0) = \{0\}$ and $\varphi_{d,+}(C^0) = \{0, 2, \dots, 2d-2\}$, which implies that $C^0 \cup C^s$ consists of skew lines for $s = 1, 2$, respectively (cf. Corollary 2.6). Finally, note that statement (d) in Proposition 1.2 assures us that $C^1 \cup C^2$ also is formed by skew lines. Therefore, $C^0 \cup C^1 \cup C^2$ consists of $3d$ skew lines in F_d . ■

Below we will discuss some more examples that led us to believe that $\mathfrak{s}(F_d) = 3d$ for $d \geq 7$ odd.

Example 4.2. For $d=7$, let us consider $C^0 = \{L_{0,0}^0, L_{1,3}^0, L_{2,2}^0, L_{3,5}^0, L_{4,4}^0, L_{5,6}^0, L_{6,1}^0\}$ which consists of seven skew lines (cf. (a) in Corollary 2.4). Furthermore, in the rows of the next table we register the values of $\varphi_{7,\pm}(C^0)$, respectively:

C^0	$L_{0,0}^0$	$L_{1,3}^0$	$L_{2,2}^0$	$L_{3,5}^0$	$L_{4,4}^0$	$L_{5,6}^0$	$L_{6,1}^0$
$\varphi_{7,-}$	0	2	0	2	0	1	2
$\varphi_{7,+}$	0	4	4	1	1	4	0

Now, having in mind Corollary 2.6 for the choice of $C^s \subset \mathcal{L}^s$ such that $C^0 \cup C^s$ is constituted by skew lines for $s = 1, 2$, it is necessary that

$$C^1 \cap \mathcal{L}_k^2 = \emptyset \quad \forall k \in \{0, 1, 2\} \quad \text{and} \quad C^2 \cap \mathcal{L}_k^2 = \emptyset \quad \forall k \in \{0, 1, 4\}.$$

So, $C^1 = \{L_{4,0}^1, L_{4,2}^1, L_{5,3}^1, L_{5,4}^1, L_{5,5}^1, L_{6,1}^1, L_{6,6}^1\}$ and $C^2 = \{L_{2,0}^2, L_{2,5}^2, L_{2,6}^2, L_{3,1}^2, L_{3,2}^2, L_{3,3}^2, L_{6,4}^2\}$ are admissible choices. As well, according to the information on the rows in the following two tables:

C^1	$L_{4,0}^1$	$L_{4,2}^1$	$L_{5,3}^1$	$L_{5,4}^1$	$L_{5,5}^1$	$L_{6,1}^1$	$L_{6,6}^1$
$\varphi_{7,+}$	4	6	1	2	3	0	5
ψ_7	4	1	4	6	1	1	4

C^2	$L_{2,0}^2$	$L_{2,5}^2$	$L_{2,6}^2$	$L_{3,1}^2$	$L_{3,2}^2$	$L_{3,3}^2$	$L_{6,4}^2$
$\varphi_{7,+}$	2	0	1	4	5	6	3
ψ_7	2	5	0	5	0	2	0

¹⁰Note that $\varphi_{d,+}(C^1) = R_d = \varphi_{d,+}(C^2)$ and $\#C^s = d$ for all s .

we have that C^s consists of skew lines for $s = 1, 2$ (cf. (b) and (c) in Corollary 2.4) and $\psi_7(C^1) \cap \psi_7(C^2) = \emptyset$, which implies that $C^1 \cup C^2$ also is formed by skew lines (cf. (c) in Corollary 2.6). Therefore, $C := C^0 \cup C^1 \cup C^2$ consists of 21 skew lines in F_7 . Thus, $\varkappa(F_7) = 21$ (since $\varkappa(F_7) \leq 21$).

Let us go now to case $d = 9$ and $d = 11$.

Example 4.3. The next tables contain the necessary information to conclude that those 27 lines (in F_9) bellow are pairwise disjoint.

C^0	$L_{4,1}^0$	$L_{5,2}^0$	$L_{6,3}^0$	$L_{7,7}^0$	$L_{8,8}^0$	$L_{0,0}^0$	$L_{1,4}^0$	$L_{2,5}^0$	$L_{3,6}^0$
$\varphi_{9,-}$	6	6	6	0	0	0	3	3	3
$\varphi_{9,+}$	5	7	0	5	7	0	5	7	0

C^1	$L_{5,0}^1$	$L_{5,1}^1$	$L_{1,2}^1$	$L_{1,3}^1$	$L_{7,4}^1$	$L_{5,5}^1$	$L_{2,6}^1$	$L_{2,7}^1$	$L_{8,8}^1$
$\varphi_{9,+}$	5	6	3	4	2	1	8	0	7
ψ_9	5	7	5	7	6	6	5	7	6

C^2	$L_{1,0}^2$	$L_{1,1}^2$	$L_{6,2}^2$	$L_{6,3}^2$	$L_{2,4}^2$	$L_{2,5}^2$	$L_{6,6}^2$	$L_{6,7}^2$	$L_{6,8}^2$
$\varphi_{9,+}$	1	2	8	0	6	7	3	4	5
ψ_9	1	3	1	3	1	3	0	2	4

Now, we show the tables for the lines in F_{11} :

C^0	$L_{5,0}^0$	$L_{6,1}^0$	$L_{7,2}^0$	$L_{8,3}^0$	$L_{9,7}^0$	$L_{10,8}^0$	$L_{0,9}^0$	$L_{1,4}^0$	$L_{2,5}^0$	$L_{3,6}^0$	$L_{4,10}^0$
$\varphi_{11,-}$	6	6	6	6	9	9	9	3	3	3	6
$\varphi_{11,+}$	5	7	9	0	5	7	9	5	7	9	3

C^1	$L_{8,0}^1$	$L_{8,1}^1$	$L_{4,2}^1$	$L_{4,3}^1$	$L_{0,4}^1$	$L_{0,5}^1$	$L_{7,6}^1$	$L_{7,7}^1$	$L_{4,8}^1$	$L_{1,9}^1$	$L_{1,10}^1$
$\varphi_{11,+}$	8	9	6	7	4	5	2	3	1	10	0
ψ_{11}	8	10	8	10	8	10	8	10	9	8	10

C^2	$L_{1,0}^2$	$L_{1,1}^2$	$L_{1,2}^2$	$L_{6,3}^2$	$L_{6,4}^2$	$L_{6,5}^2$	$L_{2,6}^2$	$L_{8,7}^2$	$L_{8,8}^2$	$L_{8,9}^2$	$L_{8,10}^2$
$\varphi_{11,+}$	1	2	3	9	10	0	8	4	5	6	7
ψ_{11}	1	3	5	1	3	5	3	0	2	4	6

In Propositions 4.4 and 4.5, the indices a, b in the notation $L_{a,b}^s$ are always to be considered modulo d .

Proposition 4.4. Let $d = 2n + 1$ with $n = 2k$ and $k \geq 3$. Consider the families

$$\begin{aligned}
 C^0 &= \{L_{1+i,2k+i}^0\}_{i=0}^k \cup \{L_{k+i,k+i}^0\}_{i=2}^{k-1} \cup \{L_{2k+i,1+i}^0\}_{i=0}^k \cup \{L_{3k+i,3k+i}^0\}_{i=1}^{k+1}, \\
 C^1 &= \{L_{2k+1,2k+i}^1\}_{i=1}^{k-1} \cup \{L_{3k+1,2k}^1\} \cup \{L_{1,k+i}^1\}_{i=0}^{k-1} \cup \{L_{2k+1,i}^1\}_{i=0}^{k-1} \\
 &\quad \cup \{L_{3k+2,4k}^1\} \cup \{L_{2,3k+i}^1\}_{i=0}^{k-1},
 \end{aligned}$$

$$C^2 = \{L_{1,i}^2\}_{i=0}^{k-1} \cup \{L_{2k+2,3k+i}^2\}_{i=0}^k \cup \{L_{2,2k+i}^2\}_{i=0}^{k-1} \cup \{L_{2k+2,k+i}^2\}_{i=0}^{k-1}.$$

It is verified that $C^0 \cup C^1 \cup C^2$ consists of $3d$ skew lines in F_d .

Proof. Let us divide the proof into four steps.

Step 1: C^0 is constituted by d skew lines. First of all, note that C^0 is defined by four strata below

$$C^0 = \underbrace{\{L_{1+i,2k+i}^0\}_{i=0}^k}_{(i)} \cup \underbrace{\{L_{k+i,k+i}^0\}_{i=2}^{k-1}}_{(ii)} \cup \underbrace{\{L_{2k+i,1+i}^0\}_{i=0}^k}_{(iii)} \cup \underbrace{\{L_{3k+i,3k+i}^0\}_{i=1}^{k+1}}_{(iv)},$$

where the stratum (ii) is non-empty if and only if $k \geq 3$ (so, $d \neq 5$ and $d \neq 9$). Furthermore, we have that the label t in each $L_{t,j}^0 \in C^0$ is varying throughout the set

$$\underbrace{\{1, \dots, k+1\}}_{(i)} \cup \underbrace{\{k+2, \dots, 2k-1\}}_{(ii)} \cup \underbrace{\{2k, \dots, 3k\}}_{(iii)} \cup \underbrace{\{3k+1, \dots, 4k, 4k+1 \equiv_d 0\}}_{(iv)}. \quad (4.1)$$

And the label j throughout the set

$$\underbrace{\{1, \dots, k+1\}}_{(iii)} \cup \underbrace{\{k+2, \dots, 2k-1\}}_{(ii)} \cup \underbrace{\{2k, \dots, 3k\}}_{(i)} \cup \underbrace{\{3k+1, \dots, 4k, 4k+1 \equiv_d 0\}}_{(iv)}. \quad (4.2)$$

Since the sets in (4.1) and (4.2) are equal to R_d , it follows that C^0 is constituted by d skew lines.

Step 2: $C^0 \cup C^s$ is constituted by skew lines for $s = 1, 2$. Next, we display the values of $\varphi_{d,\pm}$ over C^0 (using the stratification (i), ..., (iv) for C^0).

$C^0/(i)$	$L_{1,2k}^0$	$L_{2,2k+1}^0$	$L_{3,2k+2}^0$	$\cdots$	$L_{k-1,3k-2}^0$	$L_{k,3k-1}^0$	$L_{1+k,3k}^0$
$\varphi_{d,-}$	$2k-1$	$2k-1$	$2k-1$	$\cdots$	$2k-1$	$2k-1$	$2k-1$
$\varphi_{d,+}$	$2k+1$	$2k+3$	$2k+5$	$\cdots$	$4k-3$	$4k-1$	$4k+1 \equiv_d 0$

$C^0/(ii)$	$L_{k+2,k+2}^0$	$L_{k+3,k+3}^0$	$\cdots$	$L_{2k-3,2k-3}^0$	$L_{2k-2,2k-2}^0$	$L_{2k-1,2k-1}^0$
$\varphi_{d,-}$	0	0	$\cdots$	0	0	0
$\varphi_{d,+}$	$2k+4$	$2k+6$	$\cdots$	$4k-6$	$4k-4$	$4k-2$

$C^0/(iii)$	$L_{2k,1}^0$	$L_{2k+1,2}^0$	$L_{2k+2,3}^0$	$\cdots$	$L_{3k-2,k-1}^0$	$L_{3k-1,k}^0$	$L_{3k,1+k}^0$
$\varphi_{d,-}$	$2k+2$	$2k+2$	$2k+2$	$\cdots$	$2k+2$	$2k+2$	$2k+2$
$\varphi_{d,+}$	$2k+1$	$2k+3$	$2k+5$	$\cdots$	$4k-3$	$4k-1$	0

$C^0/(iv)$	$L_{3k+1,3k+1}^0$	$L_{3k+2,3k+2}^0$	$\cdots$	$L_{4k-1,4k-1}^0$	$L_{4k,4k}^0$	$L_{0,0}^0$
$\varphi_{d,-}$	0	0	$\cdots$	0	0	0
$\varphi_{d,+}$	$2k+1$	$2k+3$	$\cdots$	$4k-3$	$4k-1$	0

Now, having in mind Corollary 2.6 for the choice of $C^s \subset \mathcal{L}^s$ such that $C^0 \cup C^s$ is constituted by skew lines for $s = 1, 2$ and the tables (involving C^0) above, it is necessary that

$$\begin{cases} C^1 \cap \mathcal{L}_t^1 = \emptyset & \forall t \in \{0, 2k-1, 2k+2\}, \\ C^2 \cap \mathcal{L}_t^2 = \emptyset & \forall t \in \{2k+j\}_{j=1}^{2k+1} - \{2k+2, 4k\}. \end{cases} \quad (4.3)$$

Now, it is a straightforward verification to see that the label t in each $L_{t,j}^s \in C^s$ belongs to the set

$$\{1, 2, 2k+1, 3k+1, 3k+2\} \quad \text{for } s = 1 \quad \text{and} \quad \{1, 2, 2k+2\} \quad \text{for } s = 2.$$

Thus, using (4.3) we concluded that $C^0 \cup C^s$ is constituted by skew lines for $s = 1, 2$.

Step 3: C^s is constituted by d skew lines for $s = 1, 2$. Let us stratify C^1 as follows: $C^1 = A_1 \dot{\cup} A_2 \dot{\cup} A_3 \dot{\cup} A_4 \dot{\cup} A_5 \dot{\cup} A_6$ where

$$\begin{aligned} A_1 &:= \{L_{2k+1,2k+i}^1\}_{i=1}^{k-1}, & A_2 &:= \{L_{3k+1,2k}^1\}, & A_3 &:= \{L_{1,k+i}^1\}_{i=0}^{k-1}, \\ A_4 &:= \{L_{2k+1,i}^1\}_{i=0}^{k-1}, & A_5 &:= \{L_{3k+2,4k}^1\}, & A_6 &:= \{L_{2,3k+i}^1\}_{i=0}^{k-1}. \end{aligned} \quad (4.4)$$

Note that the label j in each $L_{t,j}^1 \in C^1$ is varying throughout the set

$$\underbrace{\{0, \dots, k-1\}}_{A_4}, \underbrace{\{k, \dots, 2k-1\}}_{A_3}, \underbrace{\{2k\}}_{A_2}, \underbrace{\{2k+1, \dots, 3k-1\}}_{A_1}, \underbrace{\{3k, \dots, 4k-1\}}_{A_6}, \underbrace{\{4k\}}_{A_5},$$

which is equal to R_d . Furthermore, $\varphi_{d,+}(C^1)$ is given by

A_1	$L_{2k+1,2k+1}^1$	$L_{2k+1,2k+2}^1$	$\cdots$	$L_{2k+1,3k-3}^1$	$L_{2k+1,3k-2}^1$	$L_{2k+1,3k-1}^1$
$\varphi_{d,+}$	1	2	$\cdots$	$k-3$	$k-2$	$k-1$

A_2	$L_{3k+1,2k}^1$
$\varphi_{d,+}$	k

A_3	$L_{1,k}^1$	$L_{1,k+1}^1$	$L_{1,k+2}^1$	$\cdots$	$L_{1,2k-3}^1$	$L_{1,2k-2}^1$	$L_{1,2k-1}^1$
$\varphi_{d,+}$	$k+1$	$k+2$	$k+3$	$\cdots$	$2k-2$	$2k-1$	$2k$

A_4	$L_{2k+1,0}^1$	$L_{2k+1,1}^1$	$L_{2k+1,2}^1$	$\cdots$	$L_{2k+1,k-3}^1$	$L_{2k+1,k-2}^1$	$L_{2k+1,k-1}^1$
$\varphi_{d,+}$	$2k+1$	$2k+2$	$2k+3$	$\cdots$	$3k-2$	$3k-1$	$3k$

A_5	$L_{3k+2,4k}^1$
$\varphi_{d,+}$	$3k+1$

A_6	$L_{2,3k}^1$	$L_{2,3k+1}^1$	$L_{2,3k+2}^1$	$\cdots$	$L_{2,4k-3}^1$	$L_{2,4k-2}^1$	$L_{2,4k-1}^1$
$\varphi_{d,+}$	$3k+2$	$3k+3$	$3k+4$	$\cdots$	$4k-1$	$4k$	0

Thus, $\varphi_{d,+}(C^1) = R_d$. Taking into account the established facts, we may use Corollary 2.4 to conclude that C^1 is constituted by d skew lines.

In a similar way, let us consider the following stratification for C^2 : $C^2 = B_1 \dot{\cup} B_2 \dot{\cup} B_3 \dot{\cup} B_4$ where

$$\begin{aligned} B_1 &:= \{L_{1,i}^2\}_{i=0}^{k-1}, & B_2 &:= \{L_{2k+2,3k+i}^2\}_{i=0}^k, \\ B_3 &:= \{L_{2,2k+i}^2\}_{i=0}^{k-1}, & B_4 &:= \{L_{2k+2,k+i}^2\}_{i=0}^{k-1}. \end{aligned} \quad (4.5)$$

Note that the label j in each $L_{i,j}^2 \in C^2$ is varying throughout the set

$$\underbrace{\{0, \dots, k-1\}}_{B_1}, \underbrace{\{k, \dots, 2k-1\}}_{B_4}, \underbrace{\{2k, \dots, 3k-1\}}_{B_3}, \underbrace{\{3k, \dots, 4k\}}_{B_2},$$

which is equal to R_d . Furthermore, $\varphi_{d,+}(C^2)$ is given by:

B_1	$L_{1,0}^2$	$L_{1,1}^2$	$L_{1,2}^2$	$\dots$	$L_{1,k-3}^2$	$L_{1,k-2}^2$	$L_{1,k-1}^2$
$\varphi_{d,+}$	1	2	3	$\dots$	$k-2$	$k-1$	k
B_2	$L_{2k+2,3k}^2$	$L_{2k+2,3k+1}^2$	$\dots$	$L_{2k+2,4k-2}^2$	$L_{2k+2,4k-1}^2$	$L_{2k+2,4k}^2$	
$\varphi_{d,+}$	$k+1$	$k+2$	$\dots$	$2k-1$	$2k$	$2k+1$	
B_3	$L_{2,2k}^2$	$L_{2,2k+1}^2$	$L_{2,2k+2}^2$	$\dots$	$L_{2,3k-3}^2$	$L_{2,3k-2}^2$	$L_{2,3k-1}^2$
$\varphi_{d,+}$	$2k+2$	$2k+3$	$2k+4$	$\dots$	$3k-1$	$3k$	$3k+1$
B_4	$L_{2k+2,k}^2$	$L_{2k+2,k+1}^2$	$\dots$	$L_{2k+2,2k-3}^2$	$L_{2k+2,2k-2}^2$	$L_{2k+2,2k-1}^2$	
$\varphi_{d,+}$	$3k+2$	$3k+3$	$\dots$	$4k-1$	$4k$	0	

Thus, $\varphi_{d,+}(C^2) = R_d$. Again, using Corollary 2.4, we concluded that C^2 is constituted by d skew lines.

Step 4: $C^1 \cup C^2$ is constituted by $2d$ skew lines. Having in mind (c) in Corollary 2.6, it is enough to prove that $\psi_d(C^1) \cap \psi_d(C^2) = \emptyset$. So, we will use again the stratification for C^1 in (4.4) and C^2 in (4.5) to display the computation of $\psi_d(C^1)$ and $\psi_d(C^2)$ below:

A_1	$L_{2k+1,2k+1}^1$	$L_{2k+1,2k+2}^1$	$\cdots$	$L_{2k+1,3k-3}^1$	$L_{2k+1,3k-2}^1$	$L_{2k+1,3k-1}^1$	
ψ_d	$2k+2$	$2k+4$	$\cdots$	$4k-6$	$4k-4$	$4k-2$	
A_2	$L_{3k+1,2k}^1$						
ψ_d	$3k$						
A_3	$L_{1,k}^1$	$L_{1,k+1}^1$	$L_{1,k+2}^1$	$\cdots$	$L_{1,2k-3}^1$	$L_{1,2k-2}^1$	$L_{1,2k-1}^1$
ψ_d	$2k+1$	$2k+3$	$2k+5$	$\cdots$	$4k-5$	$4k-3$	$4k-1$

A ₄	$L_{2k+1,0}^1$	$L_{2k+1,1}^1$	$L_{2k+1,2}^1$	$\cdots$	$L_{2k+1,k-3}^1$	$L_{2k+1,k-2}^1$	$L_{2k+1,k-1}^1$
ψ_d	$2k+1$	$2k+3$	$2k+5$	$\cdots$	$4k-5$	$4k-3$	$4k-1$

A ₅	$L_{3k+2,4k}^1$
ψ_d	$3k$

A ₆	$L_{2,3k}^1$	$L_{2,3k+1}^1$	$L_{2,3k+2}^1$	$\cdots$	$L_{2,4k-3}^1$	$L_{2,4k-2}^1$	$L_{2,4k-1}^1$
ψ_d	$2k+1$	$2k+3$	$2k+5$	$\cdots$	$4k-5$	$4k-3$	$4k-1$

So,

$$\psi_d(C^1) = \{2k+1, 2k+2, \dots, 4k-2, 4k-1\}. \quad (4.6)$$

B ₁	$L_{1,0}^2$	$L_{1,1}^2$	$L_{1,2}^2$	$\cdots$	$L_{1,k-3}^2$	$L_{1,k-2}^2$	$L_{1,k-1}^2$
ψ_d	1	3	5	$\cdots$	$2k-5$	$2k-3$	$2k-1$

B ₂	$L_{2k+2,3k}^2$	$L_{2k+2,3k+1}^2$	$\cdots$	$L_{2k+2,4k-2}^2$	$L_{2k+2,4k-1}^2$	$L_{2k+2,4k}^2$
ψ_d	0	2	$\cdots$	$2k-4$	$2k-2$	$2k$

B ₃	$L_{2,2k}^2$	$L_{2,2k+1}^2$	$L_{2,2k+2}^2$	$\cdots$	$L_{2,3k-3}^2$	$L_{2,3k-2}^2$	$L_{2,3k-1}^2$
ψ_d	1	3	5	$\cdots$	$2k-5$	$2k-3$	$2k-1$

B ₄	$L_{2k+2,k}^2$	$L_{2k+2,k+1}^2$	$\cdots$	$L_{2k+2,2k-3}^2$	$L_{2k+2,2k-2}^2$	$L_{2k+2,2k-1}^2$
ψ_d	1	3	$\cdots$	$2k-5$	$2k-3$	$2k-1$

Therefore,

$$\psi_d(C^2) = \{0, 1, 2, \dots, 2k-1, 2k\}. \quad (4.7)$$

Thus, from (4.6) and (4.7) we have that $\psi_d(C^1) \cap \psi_d(C^2) = \emptyset$. ■

Proposition 4.5. *Let $d = 2n + 1$ with $n = 2k + 1$ and $k \geq 3$. Consider the families*

$$\begin{aligned} C^0 &= \{L_{2k+1+i,0}^0\}_{i=0}^{k+1} \cup \{L_{k+1+i,3k+i}^0\}_{i=1}^{k+2} \cup \{L_{2+i,2k-1+i}^0\}_{i=0}^{k+1} \\ &\quad \cup \{L_{k+4+i,k+2+i}^0\}_{i=0}^{k-4}, \\ C^1 &= \{L_{2k+1,2k+i}^1\}_{i=1}^{k-1} \cup \{L_{3k+1,2k}^1\} \cup \{L_{1,k+i}^1\}_{i=0}^{k-1} \cup \{L_{2k+1,i}^1\}_{i=0}^{k-1} \\ &\quad \cup \{L_{3k+2,4k}^1\} \cup \{L_{2,3k+i}^1\}_{i=0}^{k-1}, \\ C^2 &= \{L_{3,i}^2\}_{i=0}^k \cup \{L_{2k+4,k+1+i}^2\}_{i=0}^k \cup \{L_{2,2k+2+i}^2\}_{i=0}^k \cup \{L_{2k+4,3k+3+i}^2\}_{i=0}^{k-1}. \end{aligned}$$

It is verified that $C^0 \cup C^1 \cup C^2$ consists of $3d$ skew lines in F_d .

Proof. It is analogous to the proof presented in the Proposition 4.4. ■

The next Corollary follows from the previous Propositions 4.4, 4.5 and the Examples 4.2, 4.3.

Corollary 4.6. Assume $d \geq 7$ odd. Then $\mathfrak{s}(F_d) = 3d$.

Theorem 4.7. Let F_d be the Fermat surface of degree $d \geq 3$. If $\mathfrak{s}(F_d)$ is the maximal number of skew lines in F_d , then $\mathfrak{s}(F_d) = 3d$ for all $d \neq 3, 5$. Being $\mathfrak{s}(F_3) = 6$ and $\mathfrak{s}(F_5) = 13$.

Proof. For $d \in \{3, 5\}$ see Section 3. For the other cases, see the previous results in this section. ■

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