

# Zero mean curvature surfaces in isotropic space with planar curvature lines

Joseph Cho and Masaya Hara

**Abstract.** We give a comprehensive account of zero mean curvature surfaces with planar curvature lines in isotropic 3-space. After providing a complete classification of all such surfaces, we show that they belong to a 1-parameter family of surfaces. We then investigate their relationship to Thomsen-type surfaces in isotropic 3-space, those zero mean curvature surfaces in isotropic 3-space that are also affine minimal.

## 1. Introduction

In surface theory, surfaces with specific geometric constraints—such as constant curvature, embeddedness, or compactness—are highly sought after, with particular interest in those satisfying multiple constraints. Among these, the condition of having *planar curvature lines* has drawn significant attention from both classical and modern geometers. The study of surfaces with planar curvature lines began in the middle of 19th century with the works of Monge [28], Joachimsthal [21], and Bonnet [8, 10], who treated surfaces as graphs of functions of two variables. However, it was Enneper [18, 19] and his students [7, 24, 50] in the latter half of the 19th century who introduced analytical methods for studying surfaces of constant curvature with planar curvature lines, and these methods had a profound impact on modern surface theory. Notably, following the discovery of Wente tori [52] in 1986, Abresch [1] and Walter [51] independently observed that Wente tori possess one family of planar curvature lines, providing a simplifying ansatz that enabled explicit descriptions of these tori following the approach taken by Enneper.

Classical geometers were also interested in minimal surfaces with planar curvature lines, with their classification completed by Bonnet [9] and Enneper [17]. For this reason, such minimal surfaces are often referred to as *Bonnet minimal surfaces*. (For

---

*Mathematics Subject Classification 2020:* 53A10 (primary); 53A15, 53A35, 53B30 (secondary).

*Keywords:* planar curvature line, zero mean curvature surface, affine minimal surface, isotropic geometry.

a modern treatment, see [29, Section 2.6].) Interestingly, Bonnet minimal surfaces have an alternative characterization from the perspective of affine differential geometry found in 1923: a minimal surface is a Bonnet minimal surface if and only if its conjugate minimal surface is a Thomsen surface, a minimal surface that is also affine minimal [49]. Through analytical methods, it was later demonstrated that Thomsen surfaces form a 1-parameter family of surfaces [3, 37], which in turn implies that Bonnet minimal surfaces also constitute a 1-parameter family (see also [12]).

Similar considerations were given to zero mean curvature surfaces in Minkowski space. The classification of Bonnet-type maximal surfaces is provided in [23] in 2015, while their connection to Thomsen-type maximal surfaces is explored in [26], and they are also shown to form a 1-parameter family in [13]. Thomsen-type timelike minimal surfaces were first studied in [25], and Bonnet-type timelike minimal surfaces received a comprehensive treatment, including their classification, in [2] in 2020.

In this paper, we shift our focus to *simply isotropic space* to explore analogous problems. Isotropic plane geometry first emerged in early 20th century as one of the Cayley–Klein geometries through a particular choice of the absolute quadric in projective geometry [4, 5]. The study of surface theory within *simply isotropic geometry*, which we refer to as *isotropic geometry* hereafter, followed soon after in the 1940s, with foundational works by Strubecker [41–44]. Notably, a Weierstrass-type representation for zero mean curvature surfaces in isotropic space was introduced in [42]. (For a more contemporary account of isotropic geometry, see [35].) In recent years, interest in surface theory within isotropic geometry has seen renewed attention, particularly due to its applications in architecture [20, 22, 27, 33, 34, 48].

This paper aims to provide an exhaustive account of zero mean curvature surfaces with planar curvature lines in isotropic space. This problem was considered by Strubecker [47] as part of his examination of zero mean curvature surfaces in isotropic space that are also affine minimal. In the work, Strubecker finds notable examples of such surfaces, and shows that the conjugate zero mean curvature surfaces are zero mean curvature surfaces with planar curvature lines. (We also note here that reflection principles across geodesic planar curvature lines in isotropic planes for zero mean curvature surfaces in isotropic space have been discussed in [45].)

We will consider this topic with an emphasis on the constraint of having planar curvature lines, allowing us to use modern analytical methods modeled after the works of Abresch and Walter [1, 51] (also used in [2, 12, 13]). Thus, after briefly describing surface theory of isotropic 3-spaces in Section 2, we will focus on zero mean curvature surfaces with planar curvature lines in Section 3. In particular, we will use the aforementioned analytical methods to give an analytic classification of all zero mean curvature surfaces with planar curvature lines in Theorem 3.5. Then, using the Weierstrass-type representation for zero mean curvature surfaces in isotropic space [42] (see also [15, 32, 35, 36, 38]), we will recover the Weierstrass data for these

surfaces, thereby obtaining a complete classification in Theorem 3.13. After noting few geometric facts about zero mean curvature surfaces with planar curvature lines in Section 4, we show that these surfaces constitute a 1-parameter family of surfaces in Section 5 on the level of Weierstrass-data (see Theorem 5.2).

Finally, in Section 6, we explore zero mean curvature surfaces in isotropic space that are also affine minimal, and show that they are conjugate zero mean curvature surfaces of those zero mean curvature surfaces with planar curvature lines in Theorem 6.1, recovering the result by Strubecker in [47]. This result quickly leads to a complete classification of these surfaces in Theorem 6.2, and also allows us to conclude immediately that these surfaces also constitute a 1-parameter family of surfaces in Theorem 6.3.

## 2. Preliminaries

We first briefly describe the isotropic 3-space and the surface theory within by following the viewpoint taken in [11].

### 2.1. Isotropic 3-space

Denote by  $\mathbb{R}^{3,1}$  the Minkowski 4-space equipped with the bilinear form  $\langle \cdot, \cdot \rangle$  of signature  $(- + + +)$ , and let  $\mathcal{L}$  denote the light cone, i.e.,

$$\mathcal{L} = \{X \in \mathbb{R}^{3,1} : \langle X, X \rangle = 0\}.$$

Choosing some  $\mathfrak{p}, \tilde{\mathfrak{p}} \in \mathcal{L}$  with  $\langle \mathfrak{p}, \tilde{\mathfrak{p}} \rangle = 1$ , the isotropic 3-space  $\mathbb{I}^3$  is given by

$$\mathbb{I}^3 = \{X \in \mathbb{R}^{3,1} : \langle X, \mathfrak{p} \rangle = 0\}.$$

To consider coordinates, we will normalize  $\mathfrak{p}$  and  $\tilde{\mathfrak{p}}$  as

$$\mathfrak{p} = (1, 0, 0, 1)^t \quad \text{and} \quad \tilde{\mathfrak{p}} = \frac{1}{2}(-1, 0, 0, 1)^t,$$

and the simply isotropic space, referred to as isotropic space hereafter, is given by

$$\mathbb{I}^3 = \{(\ell, x, y, \ell)^t \in \mathbb{R}^{3,1}\},$$

so that the induced metric is

$$ds_{\mathbb{I}^3}^2 = dx^2 + dy^2.$$

The isometries of  $\mathbb{I}^3$  are then induced by the isometries of  $\mathbb{R}^{3,1}$ : denoting the set of orientation-preserving isometries of the Minkowski 4-space that fix the origin as

$SO(3, 1)$ , the set of orientation-preserving isometries of  $\mathbb{I}^3$ , denoted by  $SO(2, 0, 1)$ , are given by

$$SO(2, 0, 1) := \{A \in SO(3, 1) : A\mathfrak{p} = \mathfrak{p}\}.$$

This in particular preserves the *co-metric* of  $\mathbb{I}^3$  (see, for example, [14, 36]), as any isotropic vector in  $\mathbb{I}^3$  is parallel to  $\mathfrak{p}$ , and hence is preserved under the action.

We will often refer to the coordinate space  $\{(x, y, \ell)^t\}$  as isotropic 3-space, and refer to the  $\ell$ -direction as *vertical*.

**2.2. Surface theory**

Now for some simply-connected domain  $\Sigma$ , let  $X : \Sigma \rightarrow \mathbb{I}^3 \subset \mathbb{R}^{3,1}$  be a space-like immersion so that they are admissible immersions in isotropic space, that is, the induced metric on the tangent planes of  $X$  is Riemannian. Thus, we may locally find conformal coordinates  $(u, v) \in \Sigma$  with induced conformal structure  $z = u + iv$  so that there is some  $\omega : \Sigma \rightarrow \mathbb{R}$  with

$$ds^2 = e^{2\omega}(du^2 + dv^2) = e^{2\omega} dz d\bar{z}. \tag{2.1}$$

This complexification also gives us the Wirtinger derivatives

$$\partial_z = \frac{1}{2}(\partial_u - i\partial_v), \quad \text{and} \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_u + i\partial_v).$$

Viewing  $X$  as a surface of codimension two in  $\mathbb{R}^{3,1}$ , every fiber of the normal bundle of  $X$  has signature  $(1, 1)$  with  $\mathfrak{p}$  being a constant null section of the normal bundle. Therefore, we may find a unique  $n : \Sigma \rightarrow \mathcal{L}$  so that

$$\langle dX, n \rangle = 0 \quad \text{and} \quad \langle n, \mathfrak{p} \rangle = 1.$$

Such  $n$  is called the *lightlike Gauss map* of  $X$ , and can be used to calculate the second fundamental form via

$$\mathbb{II} = L du^2 + 2M du dv + N dv^2$$

with

$$L := \langle X_{uu}, n \rangle, \quad M := \langle X_{uv}, n \rangle, \quad N := \langle X_{vv}, n \rangle.$$

From the lightlike Gauss map, the parabolic Gauss map  $\nu$  [42, §76] can be obtained via

$$\nu := g - \tilde{\mathfrak{p}}.$$

In fact, the second fundamental form can also be calculated from the parabolic Gauss map [31] via

$$L = -\langle X_u, \nu_u \rangle, \quad M = -\langle X_u, \nu_v \rangle = -\langle X_v, \nu_u \rangle, \quad N = -\langle X_v, \nu_v \rangle.$$

The mean curvature  $H$  is then given by

$$H = \frac{1}{2}e^{-2\omega}(L + N) = \frac{1}{2}e^{-2\omega}\langle X_{uu} + X_{vv}, n \rangle = 2e^{-2\omega}\langle X_{z\bar{z}}, n \rangle,$$

while the Hopf differential is

$$Q dz^2 = \frac{1}{4}(L - N - 2iM) dz^2 = \langle X_{zz}, n \rangle dz^2.$$

The Gauss–Weingarten equation now reads

$$\begin{cases} X_{zz} = 2\omega_z X_z + Qp, \\ X_{z\bar{z}} = \frac{1}{2}e^{2\omega}Hp, \\ n_z = -HX_z - 2e^{-2\omega}QX_{\bar{z}}, \end{cases}$$

so that the Gauss–Codazzi equations are

$$\begin{cases} \omega_{z\bar{z}} = 0, & \text{(Gauss equation)} \\ H_z = 2e^{-2\omega}Q_{\bar{z}}. & \text{(Codazzi equation)} \end{cases}$$

Now assume that  $X$  has zero mean curvature so that  $H \equiv 0$ . Then the Codazzi equation implies that the Hopf differential factor is holomorphic. As we are interested in zero mean curvature surfaces with planar curvature lines, we reparametrize as in [6, Lemma 2.3.2] to assume without loss of generality that  $Q = -\frac{1}{2}$  so that  $(u, v)$  are conformal curvature line coordinates, or *isothermic coordinates*. Here, we are assuming that there are no umbilics in  $\Sigma$ .

The Gauss–Weingarten equations then simplify to

$$\begin{cases} X_{uu} = \omega_u X_u - \omega_v X_v - p, \\ X_{uv} = \omega_v X_u + \omega_u X_v, \\ X_{vv} = -\omega_u X_u + \omega_v X_v + p, \\ n_u = e^{-2\omega}X_u, \\ n_v = -e^{-2\omega}X_v, \end{cases}$$

while the compatibility equation becomes

$$\omega_{uu} + \omega_{vv} = 0.$$

### 3. Classification of zero mean curvature surfaces with planar curvature lines

In this section, we will obtain an analytic classification for zero mean curvature surfaces with planar curvature lines using the techniques from [1,2,12,13,51]. Then using the Weierstrass-type representation for zero mean curvature surfaces in isotropic space [42, §88], we will recover canonical Weierstrass data for these surfaces.

#### 3.1. Analytic classification

We first calculate the condition on  $\omega$  for a zero mean curvature surface  $X$  to have planar curvature lines.

**Lemma 3.1.** *For an umbilic-free zero mean curvature surface in isotropic space, the following are equivalent:*

- (1)  *$u$ -curvature lines are planar.*
- (2)  *$v$ -curvature lines are planar.*
- (3)  $\omega_{uv} + \omega_u\omega_v = 0$ .

*Proof.* We have that  $u$ -curvature lines are planar if and only if they have zero torsion; thus, they are planar if and only if  $X_u, X_{uu}, X_{uuu}$  are linearly dependent [40, equation (2.13)]. Calculating that

$$X_{uuu} = AX_u + (-2\omega_u\omega_v - \omega_{uv})X_v - \omega_u p$$

for some function  $A$ , we have that  $X_u, X_{uu}, X_{uuu}$  are linearly dependent if and only if

$$0 = - \begin{vmatrix} 1 & 0 & 0 \\ \omega_u & -\omega_v & -1 \\ A & -2\omega_u\omega_v - \omega_{uv} & -\omega_u \end{vmatrix} = \omega_{uv} + \omega_u\omega_v.$$

One can show similarly that  $v$ -curvature lines are planar if and only if  $\omega_{uv} + \omega_u\omega_v = 0$ . ■

Thus, every zero mean curvature surface with planar curvature lines corresponds to a solution of the following system of partial differential equations for  $\omega$ :

$$\begin{cases} \omega_{uu} + \omega_{vv} = 0, & \text{(compatibility condition)} & (3.1a) \\ \omega_{uv} + \omega_u\omega_v = 0. & \text{(zero mean curvature and planarity condition)} & (3.1b) \end{cases}$$

Note that  $\omega \equiv c$  for some real constant  $c$  is a trivial solution to (3.1); to obtain non-trivial solutions to the system of partial differential equations (3.1), we reduce the problem to a system of ordinary differential equations.

**Proposition 3.2.** *The non-trivial solutions to (3.1) are given by*

$$e^{\omega(u,v)} = \frac{f(u)^2 + g(v)^2}{f_u(u) + g_v(v)}, \tag{3.2}$$

where  $f(u)$  and  $g(v)$  are single-variable functions such that

$$\begin{cases} f(u) := \omega_u e^\omega, \\ g(v) := \omega_v e^\omega, \end{cases} \tag{3.3a}$$

$$\tag{3.3b}$$

satisfying the system of ordinary differential equations

$$\begin{cases} f_{uu} = af, \\ f_u^2 = af^2 + b, \end{cases} \tag{3.4a}$$

$$\tag{3.4b}$$

$$\begin{cases} g_{vv} = -ag, \\ g_v^2 = -ag^2 + b, \end{cases} \tag{3.4c}$$

$$\tag{3.4d}$$

for some real constants  $a, b$  such that  $a^2 + b^2 \neq 0$ .

*Proof.* Since  $\omega \equiv c$  is a trivial solution to (3.1), we assume that  $\omega \neq c$ . From (3.1b), we see that

$$\frac{\omega_{uv}}{\omega_v} = -\omega_u$$

so that integrating both sides with respect to  $u$  gives

$$\log \omega_v = -\omega + k_1(v)$$

for some function  $k_1(v)$ . Therefore, we have

$$g(v) := e^{k_1(v)} = \omega_v e^\omega.$$

Similarly, (3.1b) also implies that

$$f(u) := e^{k_2(u)} = \omega_u e^\omega$$

for some function  $k_2(u)$ . Thus, we have by (3.1a) that

$$\begin{aligned} 0 &= \omega_{uu} + \omega_{vv} = (fe^{-\omega})_u + (ge^{-\omega})_v \\ &= f_u e^{-\omega} - f\omega_u e^{-\omega} + g_v e^{-\omega} - g\omega_v e^{-\omega} = e^{-\omega}(f_u - f\omega_u + g_v - g\omega_v), \end{aligned}$$

which in turn tells us

$$0 = f_u - f\omega_u + g_v - g\omega_v = f_u - f^2 e^{-\omega} + g_v - g^2 e^{-\omega}$$

giving us (3.2).

Now we will show that  $f$  and  $g$  satisfies (3.4). Taking the  $u$ -derivative of (3.2), we see by (3.3a) that

$$f = \omega_u e^\omega = \frac{2ff_u(f_u + g_v) - (f^2 + g^2)f_{uu}}{(f_u + g_v)^2}$$

so that

$$f(f_u^2 - g_v^2) - (f^2 + g^2)f_{uu} = 0.$$

Thus, we have

$$f_{uu} = af \tag{3.5}$$

where

$$a := \frac{f_u^2 - g_v^2}{f^2 + g^2}. \tag{3.6}$$

Note that since  $f_{uu}$  and  $f$  are functions of  $u$  alone, (3.5) implies  $a_v = 0$ .

On the other hand, taking the  $v$ -derivative of (3.2) and using (3.3b), we have

$$g = \omega_v e^\omega = \frac{2gg_v(f_u + g_v) - (f^2 + g^2)g_{vv}}{(f_u + g_v)^2},$$

which implies

$$-g(f_u^2 - g_v^2) - (f^2 + g^2)g_{vv} = 0.$$

Therefore, we have by (3.6) that

$$g_{vv} = -\frac{f_u^2 - g_v^2}{f^2 + g^2}g = -ag,$$

but since  $g_{vv}$  and  $g$  are functions of  $v$  alone, we have  $a_u = 0$ . Thus we conclude that  $a$  is some real constant, giving us (3.4a) and (3.4c).

Finally, multiplying (3.4a) by  $2f_u$  and integrating by  $u$  gives us

$$f_u^2 = af^2 + b,$$

while multiplying (3.4c) by  $2g_v$  and integrating by  $v$  implies

$$g_v^2 = -ag^2 + \tilde{b}$$

for some constants of integration  $b$  and  $\tilde{b}$ . Thus by (3.6), we have

$$b - \tilde{b} = f_u^2 - g_v^2 - a(f^2 + g^2) = 0,$$

giving us (3.4b) and (3.4d). ■

Before we solve (3.4) explicitly, we derive an additional condition for  $b$ .

**Lemma 3.3.** *The constant  $b$  in (3.4) is non-negative, that is,  $b \geq 0$ .*

*Proof.* Suppose for contradiction that  $b < 0$ . If  $a \geq 0$ , then (3.4d) gives us

$$g_v^2 = -ag^2 + b < 0,$$

which is a contradiction. If  $a < 0$ , then (3.4b) implies

$$f_u^2 = af^2 + b < 0,$$

which is also a contradiction. ■

We also note here that by switching the parameters  $(u, v)$ , the system of ordinary differential equations (3.4) is symmetric in  $f$  and  $g$ : thus, we may assume without loss of generality that  $a \geq 0$ . Under these conditions of  $a \geq 0$  and  $b \geq 0$ , let us take the constants  $\alpha := \sqrt{a}$  and  $\beta := \sqrt{b}$ , and in the case  $\alpha \neq 0$ , we can solve (3.4a) and (3.4c) to obtain the general solutions

$$\begin{cases} f = C_1 e^{\alpha u} + C_2 e^{-\alpha u}, & (3.7a) \\ g = C_3 e^{i\alpha v} + C_4 e^{-i\alpha v}, & (3.7b) \end{cases}$$

where  $C_1, C_2, C_3$ , and  $C_4$  are constants such that  $C_1, C_2 \in \mathbb{R}$  and  $C_3 = \bar{C}_4 \in \mathbb{C}$  since  $f$  and  $g$  are real-valued functions. Then using (3.4b) and (3.4d), we obtain additional relations on the constants,

$$C_1 C_2 = -\frac{\beta^2}{4\alpha^2} = -C_3 C_4 = -|C_3|^2. \quad (3.8)$$

Using these relations, we can identify the initial conditions for  $f$  and  $g$  using the following lemma.

**Lemma 3.4.** *The solution  $f$  (resp.  $g$ ) to (3.4) has a zero if and only if  $b > 0$  or  $f \equiv 0$  (resp.  $g \equiv 0$ ).*

*Proof.* To show one direction, suppose that there exists some  $u_0$  such that  $f(u_0) = 0$ . Using Lemma 3.3 we know that  $b \geq 0$ ; thus, we have only to see that if  $b = 0$  then  $f \equiv 0$ . When  $b = 0$  so that  $\beta = 0$  (and  $\alpha > 0$ ), (3.8) implies that

$$C_1 C_2 = 0.$$

Assuming without loss of generality that  $C_2 = 0$ , we have  $f = C_1 e^{\alpha u}$  which has a zero only if  $C_1 = 0$ , that is,  $f \equiv 0$ . The statement for  $g$  is proven similarly.

To show the other direction, we only have to see that if  $b > 0$  then  $f$  and  $g$  have a zero. So let  $f$  be of the form (3.7a) satisfying (3.8) with  $\beta > 0$ . Then, if we let

$$u_0 := \frac{1}{2\alpha} \log\left(\frac{\beta^2}{4C_1^2\alpha^2}\right) \in \mathbb{R},$$

we can check directly that  $f(u_0) = 0$ .

On the other hand, if  $g$  takes the form (3.7b) satisfying (3.8) with  $\beta > 0$ , then we may write

$$\bar{C}_4 = C_3 = \frac{\beta}{2\alpha} e^{i\theta}$$

for some  $\theta \in \mathbb{R}$ . Then, if we have

$$v_0 = \frac{1}{2\alpha}(\pi - 2\theta),$$

we can verify that  $g(v_0) = 0$ . ■

Now we are in a position to solve for  $f$  and  $g$  explicitly using (3.7) with (3.8).

**Case (a):** Let  $\alpha > 0$  and  $\beta = 0$ . Then we have by (3.8) that

$$C_1 C_2 = 0 = |C_3|^2$$

so that we obtain

$$\begin{cases} f = C e^{\pm\alpha u}, \\ g = 0, \end{cases}$$

for some nonzero real constant  $C$ . By considering a coordinate change in  $u \mapsto -u$ , we may assume without loss of generality that  $f = C e^{\alpha u}$  which in turn implies that  $C > 0$  since

$$0 < e^\omega = \frac{f^2 + g^2}{f_u + g_v} = \frac{C}{\alpha} e^{\alpha u}.$$

Then a parameter change of the form of  $u \mapsto -u - \frac{1}{\alpha} \log C$  allows us to simplify so that

$$\begin{cases} f = e^{-\alpha u}, \\ g = 0. \end{cases}$$

**Case (b):** If  $\alpha = 0$  and  $\beta > 0$ , then (3.4) becomes

$$\begin{cases} f_{uu} = 0, \\ f_u^2 = \beta^2, \end{cases} \quad \text{and} \quad \begin{cases} g_{vv} = 0, \\ g_v^2 = \beta^2, \end{cases}$$

and using Lemma 3.4, we have that

$$\begin{cases} f = \pm\beta u, \\ g = \pm\beta v. \end{cases}$$

Considering coordinate changes and the fact that  $f_u + g_v \neq 0$  from (3.2), we may assume without loss of generality that

$$\begin{cases} f = \beta u, \\ g = \beta v. \end{cases}$$

**Case (c):** Finally, if  $\alpha > 0$  and  $\beta > 0$ , then, using Lemma 3.4 on general solutions (3.7), we may assume  $f(0) = g(0) = 0$ , so that

$$0 = C_1 + C_2 = C_3 + C_4.$$

Thus, (3.8) tells us

$$C_1 = \pm \frac{\beta}{2\alpha} \quad \text{and} \quad C_3 = \pm \frac{i\beta}{2\alpha},$$

and the solutions are given by

$$\begin{cases} f = \pm \frac{\beta}{\alpha} \sinh \alpha u, \\ g = \pm \frac{\beta}{\alpha} \sin \alpha v. \end{cases}$$

Again, by considering coordinate changes and shift in parameters  $v \mapsto v + \frac{\pi}{\alpha}$ , we may assume

$$\begin{cases} f = \frac{\beta}{\alpha} \sinh \alpha u, \\ g = \frac{\beta}{\alpha} \sin \alpha v. \end{cases}$$

We summarize the discussions in the section in the following classification of all zero mean curvature surfaces with planar curvature lines (see also Figure 1).

**Theorem 3.5.** *Let  $X : \Sigma \rightarrow \mathbb{I}^3$  be a non-planar zero mean curvature immersion conformally parametrized by  $(u, v) \in \Sigma$  so that*

$$ds^2 = e^{2\omega}(du^2 + dv^2)$$

*with normalized Hopf differential factor  $Q dz^2 = -\frac{1}{2} dz^2$ . Then  $X$  has planar curvature lines if and only if  $\omega$  satisfies one of the following cases:*

Case (1) *If  $\omega$  is not constant, then*

$$e^{\omega(u,v)} = \frac{f(u)^2 + g(v)^2}{f_u(u) + g_v(v)}$$

where

$$\begin{cases} f_{uu} = \alpha^2 f, \\ f_u^2 = \alpha^2 f^2 + \beta^2, \\ g_{vv} = -\alpha^2 g, \\ g_v^2 = -\alpha^2 g^2 + \beta^2, \end{cases}$$

for  $\alpha \geq 0$  and  $\beta \geq 0$  such that  $\alpha^2 + \beta^2 \neq 0$ .

Case (1a) For  $\alpha > 0$  and  $\beta = 0$ , we have

$$\begin{cases} f = e^{-\alpha u}, \\ g = 0 \end{cases}$$

so that

$$e^\omega = -\frac{1}{\alpha}e^{-\alpha u},$$

and thus the surface is a rotational surface.

Case (1b) For  $\alpha = 0$  and  $\beta > 0$ , we have

$$\begin{cases} f = \beta u, \\ g = \beta v, \end{cases}$$

implying

$$e^\omega = \frac{\beta}{2}(u^2 + v^2),$$

and thus the surface is non-periodic in both the  $u$ -direction and the  $v$ -direction.

Case (1c) For  $\alpha > 0$  and  $\beta > 0$ , we have

$$\begin{cases} f = \frac{\beta}{\alpha} \sinh \alpha u, \\ g = \frac{\beta}{\alpha} \sin \alpha v, \end{cases}$$

giving rise to

$$e^\omega = \frac{\beta}{\alpha^2}(\cosh \alpha u - \cos \alpha v),$$

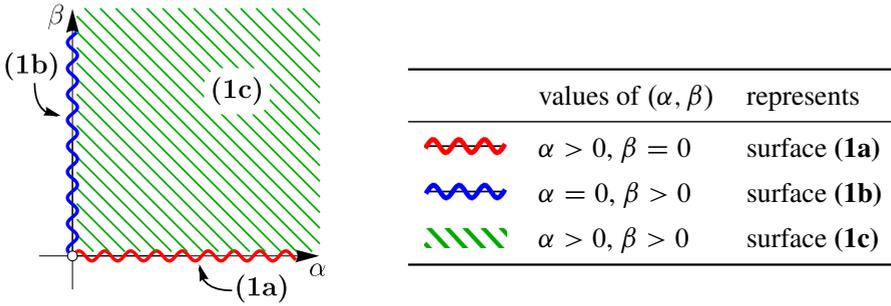
and thus the surface is non-periodic in the  $u$ -direction, but periodic in the  $v$ -direction.

Case (2) We have that  $\omega \equiv c$  for some  $c \in \mathbb{R}$ .

**Remark 3.6.** Note that we have dropped the umbilic-free condition as the solutions to (3.1) as given in Theorem 3.5 extend globally.

### 3.2. Recovering the Weierstrass data

As in the cases of zero mean curvatures surfaces in Euclidean space and Minkowski 3-space, zero mean curvature surfaces in isotropic 3-space also enjoy a *Weierstrass-type representation* (cf. [42, §88], see also [11, 15, 32, 36, 38]), given as follows.



**Figure 1.** Bifurcation diagram of zero mean curvature surfaces with planar curvature lines according to Theorem 3.5.

**Fact 3.7.** Any zero mean curvature surface  $X : \Sigma \rightarrow \mathbb{I}^3$  over a simply-connected domain  $\Sigma$  can be locally be represented as

$$X = \operatorname{Re} \int (1, -i, h)\eta$$

for some meromorphic function  $h$  and holomorphic 1-form  $\eta$  with holomorphic  $h^2\eta$ . Then the zero mean curvature surface has the induced metric

$$ds^2 = |\eta|^2$$

with the Hopf differential

$$Q dz^2 = \frac{1}{2}\eta dh$$

while the lightlike Gauss map is given by

$$n = -\frac{1}{2}(1 + |h|^2, 2 \operatorname{Re} h, -2 \operatorname{Im} h, -1 + |h|^2).$$

We call  $(h, \eta)$  the Weierstrass data.

Furthermore, the conjugate zero mean curvature surface of a given zero mean curvature surface can be defined as follows using Weierstrass data.

**Definition 3.8** ([42, §89]). Let  $X : \Sigma \rightarrow \mathbb{I}^3$  be a zero mean curvature surface represented by the Weierstrass data  $(h, \eta)$ . A family of zero mean curvature surfaces  $X^\theta : \Sigma \rightarrow \mathbb{I}^3$  given by the Weierstrass data  $(h, e^{i\theta}\eta)$  for  $\theta \in \mathbb{R}$  is called the *associate family* of the zero mean curvature surface  $X$ . A special member of the associate family  $\tilde{X} = X^{\frac{\pi}{2}}$  given by the Weierstrass data  $(h, i\eta)$  is referred to as the *conjugate zero mean curvature surface* of  $X$ . The pair of surfaces  $X$  and  $\tilde{X}$  are referred to as *conjugate zero mean curvature pair*.

We make two important remarks regarding Weierstrass representations.

**Remark 3.9.** The Weierstrass data of a zero mean curvature surface change as isometries are applied to the surface. Therefore, the Weierstrass data are not uniquely determined from the intrinsic data of a given zero mean curvature surface. However, if different Weierstrass data give the same intrinsic data of the zero mean curvature surfaces then the two surfaces are still congruent up to isometries of isotropic 3-space.

**Remark 3.10.** We note that the conformal factor  $\omega$  and the Hopf differential  $Q dz^2$  are given by the Weierstrass data. Thus, if a pair of zero mean curvature surfaces share the same conformal factor, while the Hopf differential differs by a factor of  $i$ , then we can deduce that the two zero mean curvature surfaces are a conjugate zero mean curvature pair, up to isometries of isotropic space.

Under the current setting of normalizing  $Q = -\frac{1}{2}$ , we have that the holomorphic 1-form in the Weierstrass representation must take the form

$$\eta = -\frac{1}{h_z} dz. \tag{3.9}$$

To recover the Weierstrass data from the conformal factor, we will use the following result [39, §4].

**Fact 3.11.** *Let  $\eta : \Sigma \rightarrow \mathbb{C}$  be a holomorphic function with some given modulus  $|\eta|^2 = R^2$ . Then the holomorphic function can be recovered via*

$$\eta(z) = \frac{1}{\eta(z_0)} R^2 \left( \frac{z + \bar{z}_0}{2}, \frac{z - \bar{z}_0}{2i} \right)$$

for some  $z_0 \in \Sigma$ .

**Remark 3.12.** The holomorphic function recovered from its modulus via Fact 3.11 is not unique, as one can multiply the holomorphic function by a unit length complex constant. This freedom can be observed in Fact 3.11 in the choice of  $\eta(z_0)$ , as

$$|\eta(z_0)|^2 = R^2(\text{Re}(z_0), \text{Im}(z_0)),$$

and thus

$$\eta(z_0) = e^{i\theta} R(\text{Re}(z_0), \text{Im}(z_0))$$

for some  $\theta \in \mathbb{R}$ . However, by Remark 3.9, we are free to choose any  $\theta$  in recovering the Weierstrass data.

Using Fact 3.11, Remark 3.12, and (3.9), we recover the Weierstrass data as follows.

**Case (1a):** Let  $\alpha > 0$  and  $\beta = 0$ . In this case, the metric is

$$|\eta_{(\alpha,0)}|^2 = \frac{1}{\alpha^2} e^{-2\alpha u} |dz|^2.$$

Therefore, we may take

$$\eta_{(\alpha,0)} = -\frac{1}{\alpha} e^{-\alpha z} dz,$$

so that by (3.9), we have

$$h_{(\alpha,0)} = e^{\alpha z}.$$

**Case (1b):** If we have  $\alpha = 0$  and  $\beta > 0$ , then the metric is given by

$$|\eta_{(0,\beta)}|^2 = \frac{\beta^2}{4} (u^2 + v^2)^2 |dz|^2,$$

so that we may take

$$\eta_{(0,\beta)} = \frac{\beta z^2}{2} dz,$$

and thus

$$h_{(0,\beta)} = \frac{2}{\beta z}.$$

**Case (1c):** Now, let  $\alpha > 0$  and  $\beta > 0$ , so that the metric becomes

$$|\eta_{(\alpha,\beta)}|^2 = \frac{\beta^2}{\alpha^4} (\cosh \alpha u - \cos \alpha v)^2 |dz|^2.$$

Thus, we take

$$\eta_{(\alpha,\beta)} = \frac{2\beta}{\alpha^2} \sinh^2 \frac{\alpha z}{2} dz,$$

so that

$$h_{(\alpha,\beta)} = \frac{\alpha}{\beta} \coth \frac{\alpha z}{2}. \tag{3.10}$$

**Case (2):** Finally, let  $\omega \equiv -c$  for some constant  $c \in \mathbb{R}$ . Then the metric is

$$|\eta_c|^2 = e^{-2c}$$

so that we may take

$$\eta_c = -e^{-c} dz,$$

implying that

$$h_c = e^c z.$$

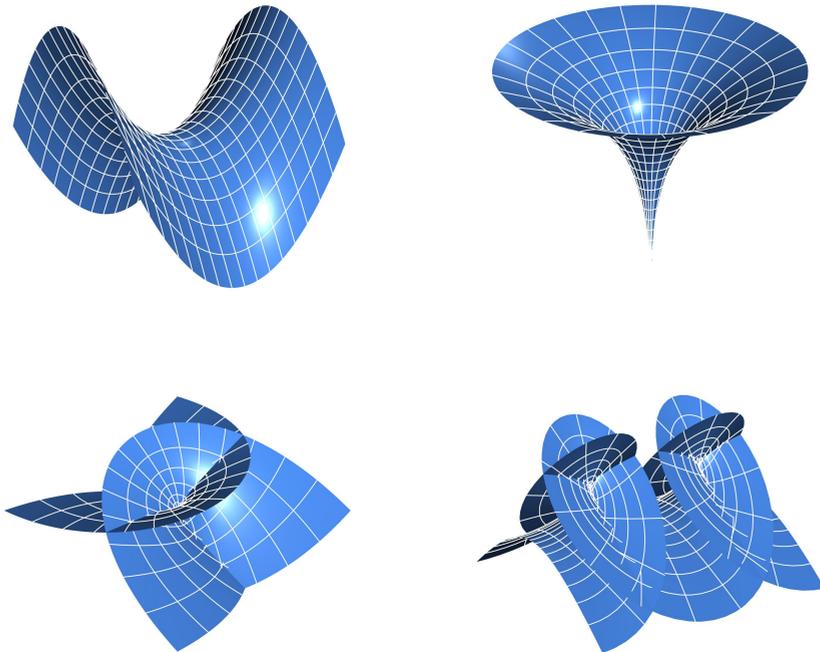
Noting that the change of coordinates  $\tilde{h}(z) := h(Cz)$  with  $\tilde{\eta}$  given by (3.9) corresponds to the surface changing by

$$\tilde{X}(u, v) = \frac{1}{C^2} X(Cu, Cv),$$

so that it is a reparametrization of the surface up to some homothety factor, we summarize our classification of zero mean curvature surfaces given with their Weierstrass data (see also Figure 2) in the following theorem.

**Theorem 3.13.** *Let  $X : \Sigma \rightarrow \mathbb{I}^3$  be a zero mean curvature immersion with planar curvature lines. Then  $X$  must be a piece of one, and only one, of*

- *plane  $(0, 1 dz)$ ,*
- *trivial Enneper-type surface  $(z, 1 dz)$ ,*
- *catenoid  $(e^z, e^{-z} dz)$ ,*
- *Enneper-type surface  $(\frac{1}{z}, z^2 dz)$ , or*
- *one of Bonnet-type surfaces  $(\gamma \coth z, \frac{1}{\gamma} \sinh^2 z dz)$  for any constant  $\gamma > 0$ ,*  
*given with the respective Weierstrass data, up to isometries and homotheties of the isotropic space.*



**Figure 2.** Examples of zero mean curvature surfaces with planar curvature lines in isotropic space classified in Theorem 3.13: trivial Enneper surface and catenoid on the top row; Enneper surface and Bonnet-type surface on the bottom row.

### 4. Geometry of zero mean curvature surfaces with planar curvature lines

Here, we note a few geometric facts about zero mean curvature surfaces with planar curvature lines.

#### 4.1. Trivial Enneper-type surface

The zero mean curvature surface pertaining to the trivial solution to the integrability condition, that is,  $\omega \equiv c$ , has been studied in the context of higher order Enneper-type surfaces in isotropic 3-space in [46, §9], and appears as the trivial example of such surfaces. Assuming without loss of generality that  $c = 0$ , the parametrization of the surface reads

$$X_0(u, v) = -\left(u, v, \frac{1}{2}(u^2 - v^2)\right)^t$$

so that every curvature line of the surface is in the shape of a parabola in Euclidean geometry. These are the *parabolic circles* [40, §11] of the isotropic 3-space, those curves contained in isotropic planes with constant curvature. Thus, every curvature line of the trivial Enneper surface is a circle in isotropic 3-space, so that the surface is a doubly-channel surface.

In particular, the parametrization also shows that all the (parabolic) circles in a family of curvature lines belong to planes that are parallel to each other. This alludes to the fact that the surface may be invariant under certain actions of isometries of the isotropic 3-space.

Isometries of the isotropic 3-space that fix the origin are inherited from the isometries of  $\mathbb{R}^{3,1}$  via

$$SO(2, 0, 1) := \{A \in SO(3, 1) : A\mathbf{p} = \mathbf{p}\}$$

acting on the isotropic 3-space. To view parabolic circles as orbits under action of isometries, let us consider the subgroup  $P_{\mathbf{v}} < SO(2, 0, 1)$  given by

$$P_{\mathbf{v}} := \{A \in SO(2, 0, 1) : A\mathbf{v} = \mathbf{v}, \langle \mathbf{v}, \mathbf{v} \rangle = 1\}.$$

Normalizing so that  $\mathbf{v} = (0, 1, 0, 0)^t =: e_1$ , the subgroup  $P_{\mathbf{v}}$  can be parametrized via

$$v \mapsto \begin{pmatrix} 1 + \frac{v^2}{2} & 0 & v & -\frac{v^2}{2} \\ 0 & 1 & 0 & 0 \\ -v & 0 & -1 & v \\ \frac{v^2}{2} & 0 & v & 1 - \frac{v^2}{2} \end{pmatrix} =: P_{e_1}(v).$$

Now consider the following (well-defined) action on  $\mathbb{I}^3$ :

$$X \mapsto P_{e_1,r}(v)X := P_{e_1}(v)(X + r\tilde{\mathbf{p}}) - r\tilde{\mathbf{p}}.$$

This action acts as an isometry of  $\mathbb{I}^3$ ; furthermore, applying this action to the origin  $O = (0, 0, 0, 0)^t$ , we see that

$$P_{e_1,r}(v)(O) = \left(-\frac{r}{2}v^2, 0, rv, -\frac{r}{2}v^2\right)^t,$$

that is, parabolic circles are obtained as orbits under the action of  $P_{v,r}$ . We refer to such actions as *parabolic rotations*.

Defining  $e_2 := (0, 0, 1, 0)^t$ , we can now similarly check that  $P_{e_2}$  can be parametrized via

$$u \mapsto \begin{pmatrix} 1 + \frac{u^2}{2} & -u & 0 & -\frac{u^2}{2} \\ u & -1 & 0 & -u \\ 0 & 0 & 1 & 0 \\ \frac{u^2}{2} & -u & 0 & 1 - \frac{u^2}{2} \end{pmatrix} =: P_{e_2}(u),$$

and calculate that

$$P_{e_2,1}(u) \circ P_{e_1,-1}(v)(O) = -\left(\frac{1}{2}(u^2 - v^2), u, v, \frac{1}{2}(u^2 - v^2)\right)^t.$$

Therefore, we not only observe that the trivial Enneper-type surface is invariant under parabolic rotations, but also conclude that the surface is generated by applying two of such rotations to a single point. As every rotational surface must have planar curvature lines, we therefore also recover the result of [16, Proposition 3] for the case of zero mean curvature surfaces.

### 4.2. Axial directions

As in the case of zero mean curvature surfaces in Euclidean space and Minkowski space [2, 12, 13], we show that zero mean curvature surfaces with planar curvature lines in isotropic 3-space also have *axial directions* following the techniques from [51]. Axial direction is the unique direction contained in the planes of a family of planar curvature lines, that is, there are some constants  $w_1, w_2 \in \mathbb{I}^3$  such that

$$w_1 \in \text{span}\{X_u, X_{uu}\} \quad \text{and} \quad w_2 \in \text{span}\{X_v, X_{vv}\}$$

for all  $(u, v) \in \Sigma$ . However, unlike the cases of zero mean curvature surfaces in Euclidean space or Minkowski space, the difficulty of finding this direction lies in the fact that there is no cross product structure in isotropic 3-space.

To obtain a characterization of a direction  $w \in \mathbb{I}^3$  being contained in a plane  $P \subset \mathbb{I}^3$ , recall from [11, §2.1.1] that planes in  $\mathbb{I}^3$  are given by  $m \in \mathbb{R}^{3,1}$  with  $\langle m, m \rangle = 0$  and  $\langle m, p \rangle = 1$  via

$$P_{m,q} = \{x \in \mathbb{I}^3 : \langle x, m \rangle = q\}.$$

When viewing the plane as a surface,  $m$  is the constant lightlike Gauss map of  $P_{m,q}$ . Thus, if  $w \in P_{m,q}$ , then we must have that

$$\langle m, w \rangle = 0.$$

Denoting by  $P_1(v)$  the plane containing the  $u$ -curvature line at  $v$ , that is,

$$P_1(v) = \text{span}\{X_u(u, v), X_{uu}(u, v)\},$$

we first find the (unnormalized) lightlike Gauss map  $m_1(v)$  of  $P_1(v)$  by requiring that

$$\langle m_1, m_1 \rangle = 0, \quad \langle m_1, X_u \rangle = 0, \quad \langle m_1, X_{uu} \rangle = 0, \quad \langle m_1, \mathfrak{p} \rangle = \omega_v,$$

where we have assumed  $\omega_v \neq 0$ . Therefore, we can find

$$m_1 = -e^{-2\omega} X_v + \omega_v n - \frac{e^{-2\omega}}{2\omega_v} \mathfrak{p}.$$

Similarly, denoting by  $P_2(u)$  the plane containing the  $v$ -curvature line at  $u$ , we can calculate the (unnormalized) lightlike Gauss map  $m_2(u)$  of  $P_2(u)$  as

$$m_2 = e^{-2\omega} X_u + \omega_u n - \frac{e^{-2\omega}}{2\omega_u} \mathfrak{p}.$$

Then we have

$$\begin{aligned} m_{1,v} &= \omega_u e^{-2\omega} X_u + \omega_{vv} n + \frac{\omega_{vv} e^{-2\omega}}{2\omega_v^2} \mathfrak{p}, \\ m_{2,u} &= -\omega_v e^{-2\omega} X_v + \omega_{uu} n + \frac{\omega_{uu} e^{-2\omega}}{2\omega_u^2} \mathfrak{p}. \end{aligned}$$

Now, if we define  $w_1, w_2 \in \mathbb{I}^3$  via

$$\begin{aligned} w_1 &:= \omega_{uu} X_u - \omega_{uv} X_v + \omega_u \mathfrak{p}, \\ w_2 &:= \omega_{uv} X_u - \omega_{vv} X_v + \omega_v \mathfrak{p}, \end{aligned}$$

then we can show

$$\langle m_1, w_1 \rangle = \omega_{uv} + \omega_v \omega_u = 0 \quad \text{and} \quad \langle m_2, w_2 \rangle = \omega_{uv} + \omega_v \omega_u = 0,$$

while

$$\langle m_{1,v}, w_1 \rangle = \omega_{uu} \omega_u + \omega_{vv} \omega_u = 0 \quad \text{and} \quad \langle m_{2,u}, w_2 \rangle = \omega_v \omega_{vv} + \omega_v \omega_{uu} = 0.$$

Finally, using (3.1) and their partial derivatives, we can show that

$$\begin{aligned} w_{1,u} &= (\omega_{uuu} + \omega_{uu} \omega_u - \omega_{uv} \omega_v) X_u - (\omega_{uvv} + \omega_{uu} \omega_v + \omega_{uv} \omega_u) X_v = 0, \\ w_{1,v} &= (\omega_{uvv} + \omega_{uu} \omega_v + \omega_{uv} \omega_u) X_u - (\omega_{uvv} - \omega_{uu} \omega_u + \omega_{uv} \omega_v) X_v = 0, \end{aligned}$$

$$\begin{aligned} w_{2,u} &= (\omega_{uuv} + \omega_{uv}\omega_u - \omega_{vv}\omega_v)X_u - (\omega_{uvv} + \omega_{uv}\omega_v + \omega_{vv}\omega_u)X_v = 0, \\ w_{2,v} &= (\omega_{uvv} + \omega_{uv}\omega_v + \omega_{vv}\omega_u)X_u - (\omega_{vvv} - \omega_{uv}\omega_u + \omega_{vv}\omega_v)X_v = 0. \end{aligned}$$

Therefore, we have shown that  $w_1$  and  $w_2$  are constant vectors contained in the planes defined by  $m_1$  and  $m_2$ , respectively, and we summarize as follows.

**Proposition 4.1.** *If  $f(u)$  (resp.  $g(v)$ ) is not identically equal to zero, then there is a unique constant direction  $w_1$  (resp.  $w_2$ ) contained in all the planes of  $u$ -curvature lines (resp.  $v$ -curvature lines), given by*

$$\begin{cases} w_1 = \omega_{uu}X_u - \omega_{uv}X_v + \omega_u\mathfrak{p}, \\ w_2 = \omega_{uv}X_u - \omega_{vv}X_v + \omega_v\mathfrak{p}. \end{cases}$$

Furthermore, if  $w_1$  and  $w_2$  both exist, then  $w_1$  is orthogonal to  $w_2$ . We call  $w_1$  and  $w_2$  the axial directions of the surface.

*Proof.* We only need to verify that

$$0 = \langle w_1, w_2 \rangle = \omega_{uv}e^{2\omega}(\omega_{uu} + \omega_{vv})$$

which vanishes by (3.1a). ■

To see the causality of the axial directions, using (3.3) tells us

$$\begin{aligned} \langle w_1, w_1 \rangle &= e^{2\omega}\omega_{uv}^2 + e^{2\omega}\omega_{uu}^2 = f^2g^2e^{-2\omega} + (f_u - f^2e^{-\omega})^2 \\ &= f^2((f^2 + g^2)e^{-2\omega} - 2f_ue^{-\omega}) + f_u^2 = -f^2\frac{f_u^2 - g_v^2}{f^2 + g^2} + f_u^2 \\ &= -f^2e^{-\omega}(f_u - g_v) + \alpha^2f^2 + \beta^2 = \beta^2, \end{aligned}$$

where we have used the fact that

$$e^\omega = \frac{f^2 + g^2}{f_u + g_v} = \frac{(f^2 + g^2)(f_u - g_v)}{f_u^2 - g_v^2} = \frac{f_u - g_v}{\alpha^2}.$$

coming from (3.4). Similarly, one can calculate that

$$\langle w_2, w_2 \rangle = \beta^2.$$

## 5. Deformation of zero mean curvature surfaces with planar curvature lines

We will now show that the class of zero mean curvature surfaces with planar curvature lines can be described via a continuous deformation, mirroring the result of zero

mean curvature surfaces in Euclidean and Minkowski space [2, 12, 13]. In this paper, we will consider a deformation to be continuous with respect to a parameter, if the surface converges uniformly component-wise with respect to the parameter on compact subdomains. To consider this, we will show that the deformation exists on the level of Weierstrass data for these surfaces.

Note that with the Weierstrass data given in (3.10), we have a deformation connecting Bonnet-type surfaces to the Enneper-type surface while keeping the planar curvature line condition, as

$$\begin{aligned} \lim_{\alpha \searrow 0} h_{(\alpha, \beta)} &= \lim_{\alpha \searrow 0} \frac{\alpha}{\beta} \coth \frac{\alpha z}{2} = \frac{2}{\beta z} = h_{(0, \beta)}, \\ \lim_{\alpha \searrow 0} \eta_{(\alpha, \beta)} &= \lim_{\alpha \searrow 0} \frac{2\beta}{\alpha^2} \sinh^2 \frac{\alpha z}{2} dz = \frac{\beta z^2}{2} dz = \eta_{(0, \beta)}. \end{aligned}$$

Thus, we only need to show that there exists a continuous deformation keeping the planarity of curvature lines to the trivial Enneper surface and the catenoid.

### 5.1. Deformation to the trivial Enneper surface

To obtain a deformation to the trivial Enneper surface, let us reconsider the case  $\alpha > 0$  and  $\beta > 0$ , and take solutions to (3.4) as

$$\begin{cases} f = \frac{\beta}{\alpha} \sinh \alpha u, \\ g = -\frac{\beta}{\alpha} \sin \alpha v. \end{cases}$$

Taking the path  $\beta = \frac{\alpha^2}{2}$  (see Figure 3), we then have that

$$e^{2\omega} = \frac{1}{4}(\cosh \alpha u + \cos \alpha v)^2.$$

Thus we may recover the holomorphic 1-form of the Weierstrass data to be

$$\eta_\alpha = -\cosh^2 \frac{\alpha z}{2} dz$$

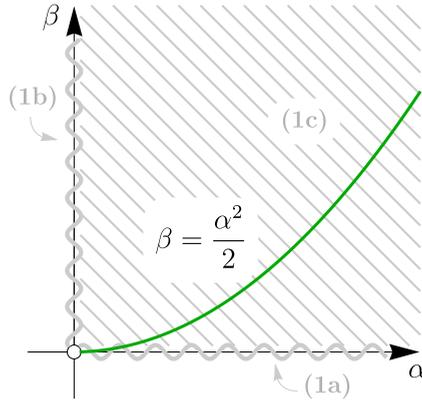
so that by (3.9),

$$h_\alpha = \frac{2}{\alpha} \tanh \frac{\alpha z}{2}.$$

With these Weierstrass data, we observe that

$$\lim_{\alpha \searrow 0} h_\alpha = z = h_0 \quad \text{and} \quad \lim_{\alpha \searrow 0} \eta_\alpha = -dz = \eta_0,$$

corresponding to the Weierstrass data of the trivial Enneper surface.



**Figure 3.** Path on classification diagram taken to find a deformation to the trivial Enneper surface. For the meaning of the diagram, please refer to Figure 1.

In fact, one can obtain an explicit parametrization  $X_\alpha$  using the Weierstrass data  $(h_\alpha, \eta_\alpha)$  and choosing the correct initial condition as

$$X_\alpha = -\frac{1}{2\alpha^2} \begin{pmatrix} \alpha(\sinh \alpha u \cos \alpha v + \alpha u) \\ \alpha(\cosh \alpha u \sin \alpha v + \alpha v) \\ 2(\cosh \alpha u \cos \alpha v - 1) \end{pmatrix},$$

and check that

$$\lim_{\alpha \searrow 0} X_\alpha = - \begin{pmatrix} u \\ v \\ \frac{1}{2}(u^2 - v^2) \end{pmatrix}.$$

### 5.2. Deformation to the catenoid

The Weierstrass data given Theorem 3.13 alone are not sufficient to obtain a deformation to the catenoid. This is due to the fact that we have normalized the initial conditions of  $f$  and  $g$  to have zeroes, which excludes the case of catenoids as the solutions to (3.4) corresponding to the case of catenoids, i.e.,  $f = e^{-\alpha u}$ , have no zeroes.

To find a suitable alternative initial condition, we observe that in every case of Theorem 3.5,  $g$  has a zero, while there is always some  $u_0$  such that  $f(u_0) = 1$ . Therefore, in considering the problem of continuous deformation, we will revisit the problem and solve for  $f$  using (3.4a) and (3.4c) assuming that  $f(0) = 1$ .

However, note that in Theorem 3.5,  $f$  corresponding to case (1a) already satisfies the new initial condition, while  $f$  in case (1b) can be easily modified to be  $f = -\beta u + 1$  with  $g = -\beta v$ . So let us consider case (1c), where  $\alpha > 0$  and  $\beta > 0$ . Then, using the general solution (3.7a), we have

$$1 = f(0) = C_1 + C_2$$

so that (3.8) implies

$$C_1 = \frac{\alpha - \sqrt{\alpha^2 + \beta^2}}{2\alpha}.$$

Therefore, we have

$$f = \cosh \alpha u - \frac{\sqrt{\alpha^2 + \beta^2}}{\alpha} \sinh \alpha u.$$

Summarizing will leads us to the following result.

**Lemma 5.1.** *Explicit solutions to (3.4) with  $f(0) = 1$  and  $g(0) = 0$  are given by:*

Case (1a) *For  $\alpha > 0$  and  $\beta = 0$ , the solutions are given by*

$$\begin{cases} f(\alpha, 0) = e^{-\alpha u}, \\ g(\alpha, 0) = 0. \end{cases}$$

Case (1b) *For  $\alpha = 0$  and  $\beta > 0$ , we have*

$$\begin{cases} f(0, \beta) = -\beta u + 1, \\ g(0, \beta) = -\beta v. \end{cases}$$

Case (1c) *For  $\alpha > 0$  and  $\beta > 0$ , we calculate that*

$$\begin{cases} f(\alpha, \beta) = \cosh \alpha u - \frac{\sqrt{\alpha^2 + \beta^2}}{\alpha} \sinh \alpha u, \\ g(\alpha, \beta) = -\frac{\beta}{\alpha} \sin \alpha v. \end{cases}$$

*With these solutions, we have*

$$\begin{aligned} \lim_{\alpha \rightarrow 0} f(\alpha, \beta) &= f(0, \beta), & \lim_{\alpha \rightarrow 0} g(\alpha, \beta) &= g(0, \beta), \\ \lim_{\beta \rightarrow 0} f(\alpha, \beta) &= f(\alpha, 0), & \lim_{\beta \rightarrow 0} g(\alpha, \beta) &= g(\alpha, 0). \end{aligned}$$

To obtain Weierstrass data for the case  $\alpha > 0$  and  $\beta > 0$ , note that

$$|\tilde{\eta}_{(\alpha, \beta)}|^2 = e^{2\omega} = \left( \frac{1}{\alpha^2} (\alpha \sinh \alpha u - \sqrt{\alpha^2 + \beta^2} \cosh \alpha u + \beta \cos \alpha v) \right)^2.$$

Using Fact 3.11 and Remark 3.12, we take

$$\tilde{\eta}_{(\alpha,\beta)} = \frac{\alpha - i\beta}{\alpha^2(\alpha + i\beta)}(\alpha \sinh \alpha z - \sqrt{\alpha^2 + \beta^2} \cosh \alpha z + \beta) dz.$$

Thus, taking the correct constant of integration, we have

$$\tilde{h}_{(\alpha,\beta)} = \frac{\alpha(\alpha + i\beta)(\beta e^{\alpha z} + \sqrt{\alpha^2 + \beta^2} - \alpha)}{\beta(\alpha - i\beta)(\alpha - \beta \sinh \alpha z)}.$$

Let us define constants  $r \in (0, \infty)$  and  $\theta \in (0, \frac{\pi}{2})$  so that  $\alpha = r \cos \theta$  and  $\beta = r \sin \theta$ .

Then we have

$$\tilde{h}_{(r,\theta)} = \frac{2e^{2i\theta} \cos \theta}{e^{-r \cos \theta z}(\cos \theta + 1) - \sin \theta}$$

and

$$\tilde{\eta}_{(r,\theta)} = \frac{e^{-2i\theta}(\cos \theta \sinh(r \cos \theta z) - \cosh(r \cos \theta z) + \sin \theta)}{r \cos^2 \theta} dz.$$

For these Weierstrass data, we can verify that

$$\lim_{\theta \searrow 0} \tilde{h}_{(r,\theta)} = e^{rz} \quad \text{and} \quad \lim_{\theta \searrow 0} \tilde{\eta}_{(r,\theta)} = -\frac{1}{r} e^{-rz}$$

while

$$\lim_{\theta \nearrow \frac{\pi}{2}} \tilde{h}_{(r,\theta)} = \frac{2}{rz - 1} \quad \text{and} \quad \lim_{\theta \nearrow \frac{\pi}{2}} \tilde{\eta}_{(r,\theta)} = \frac{(rz - 1)^2}{2r}.$$

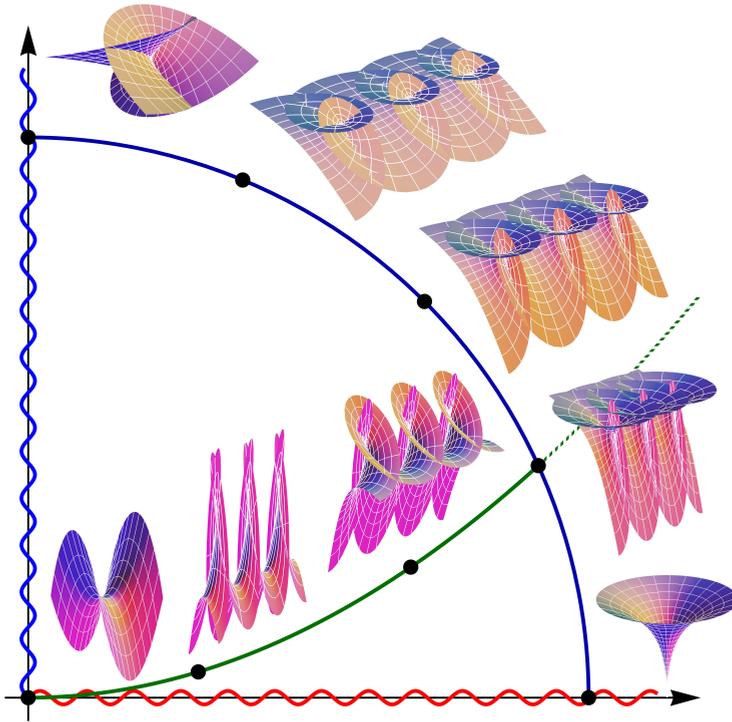
Therefore, the Weierstrass data  $(\tilde{h}_{(r,\theta)}, \tilde{\eta}_{(r,\theta)} dz)$  provide a continuous deformation connecting catenoid, Bonnet-type surfaces, and Enneper-type surfaces, allowing us to conclude as in the next theorem (see also Figure 4).

**Theorem 5.2.** *There exists a continuous deformation consisting exactly of the zero mean curvature surfaces with planar curvature lines in isotropic 3-space.*

**Remark 5.3.** We note here that since the holomorphic 1-form and its norm are changing in the deformation of Theorem 5.2, this deformation is not an isometric deformation. However it is a deformation that keeps the planarity of all curvature lines.

## 6. Zero mean curvature surfaces that are also affine minimal

Zero mean curvature surfaces in Euclidean and Minkowski space with planar curvature lines enjoy a certain relationship to those zero mean curvature surfaces that are also affine minimal [2, 26, 49]. In this section, we investigate the analogous relationship in the case of isotropic 3-space and recover the result by Strubecker [47, Satz 9].



**Figure 4.** Continuous deformation of zero mean curvature surfaces keeping planar curvature lines. For the meaning of the diagram, please refer to Figure 1.

Using this result, we will obtain a classification of all zero mean curvature surfaces in isotropic 3-space that are also affine minimal, and conclude that they also constitute a 1-parameter family of surfaces. (For a detailed introduction to affine differential geometry, we refer the readers to [30].)

Let  $X : \Sigma \rightarrow \mathbb{I}^3$  be a non-planar zero mean curvature immersion in isotropic 3-space viewed as an affine surface. If we define

$$\begin{aligned} \tilde{L} &:= \det(X_u, X_v, X_{uu}), \\ \tilde{M} &:= \det(X_u, X_v, X_{uv}), \\ \tilde{N} &:= \det(X_u, X_v, X_{vv}), \end{aligned}$$

then for non-degenerate surfaces such that  $\tilde{L}\tilde{N} - \tilde{M}^2$  does not vanish on  $\Sigma$ , the Berwald–Blaschke metric of the affine surface is given by

$$d\tilde{s}^2 = \frac{1}{|\tilde{L}\tilde{N} - \tilde{M}^2|^{1/4}} (\tilde{L} du^2 + 2\tilde{M} du dv + \tilde{N} dv^2).$$

In particular, since zero mean curvature implies that the Gaussian curvature is negative, we may assume without loss of generality that  $(u, v) \in \Sigma$  are asymptotic coordinates, i.e., we normalize the Hopf differential so that  $Q = -\frac{i}{2}$ . Then the Gauss-Weingarten equations read

$$\begin{cases} X_{uu} = \omega_u X_u - \omega_v X_v, \\ X_{uv} = \omega_v X_u + \omega_u X_v + \mathfrak{p}, \\ X_{vv} = -\omega_u X_u + \omega_v X_v, \\ n_u = -e^{-2\omega} X_v, \\ n_v = -e^{-2\omega} X_u, \end{cases} \tag{6.1}$$

with the compatibility condition

$$\omega_{uu} + \omega_{vv} = 0.$$

Under these assumptions, we have  $\tilde{L} = \tilde{N} = 0$ . To find  $\tilde{M}$ , we first split  $\mathbb{I}^3 = \mathbb{R}^2 \oplus \langle \mathfrak{p} \rangle$ , and write  $X = x + \chi \mathfrak{p}$  for some  $x : \Sigma \rightarrow \mathbb{R}^2$  and  $\chi : \Sigma \rightarrow \mathbb{R}$ . Then, we calculate

$$\tilde{M} = \det(X_u, X_v, \omega_v X_u + \omega_u X_v + \mathfrak{p}) = \det(X_u, X_v, \mathfrak{p}) = \det(x_u, x_v, \mathfrak{p}),$$

and thus  $\tilde{M}$  represents the area of the parallelogram spanned by  $x_u$  and  $x_v$  in  $\mathbb{R}^2$ . By conformality (2.1),  $x_u$  and  $x_v$  span a square with side length  $e^\omega$ , and thus the metric becomes

$$d\tilde{s}^2 = 2\sqrt{\tilde{M}} du dv = 2e^\omega du dv.$$

The affine normal vector field  $\tilde{n}$  is then given by

$$\tilde{n} := e^{-\omega} X_{uv},$$

from which the affine shape operator  $\tilde{S}$  of  $X$  follows via

$$d\tilde{n} = -dX \circ \tilde{S}.$$

Then the definitions of affine mean curvature  $\tilde{H}$  and affine Gaussian curvature  $\tilde{K}$  can be given as

$$\tilde{H} = \frac{1}{2} \operatorname{tr} \tilde{S}, \quad \text{and} \quad \tilde{K} = \det \tilde{S}.$$

In particular, we can verify directly that

$$\begin{aligned} \tilde{n}_u &= e^{-\omega} ((\omega_{uv} + \omega_u \omega_v) X_u + (\omega_{uu} - \omega_v^2) X_v), \\ \tilde{n}_v &= e^{-\omega} ((\omega_{vv} - \omega_u^2) X_u + (\omega_{uv} + \omega_u \omega_v) X_v), \end{aligned}$$

and thus

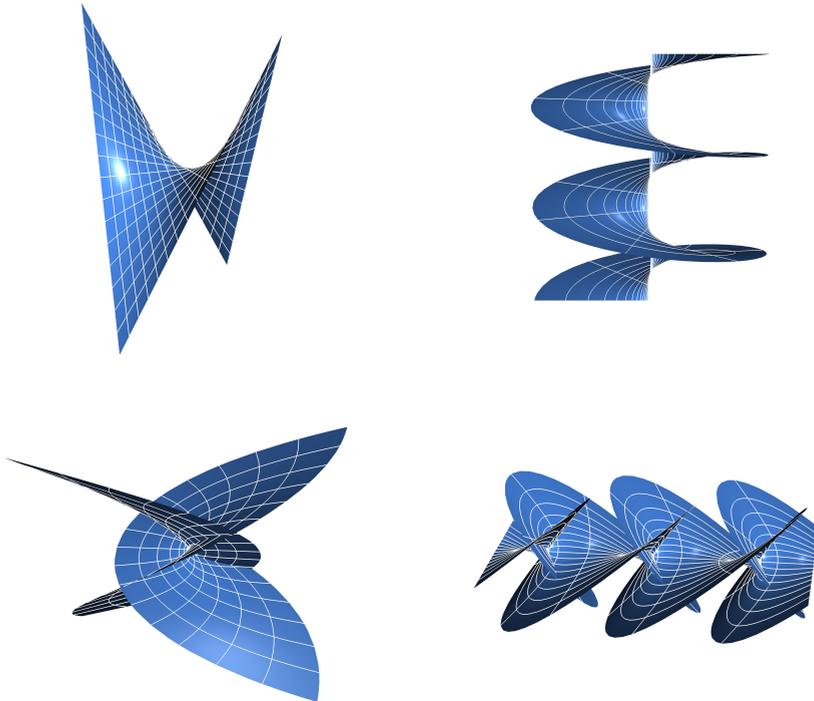
$$\tilde{H} = -e^{-\omega}(\omega_{uv} + \omega_u\omega_v).$$

Therefore, every zero mean curvature surface in isotropic space that is also affine minimal corresponds to the solution to the following system of partial differential equations:

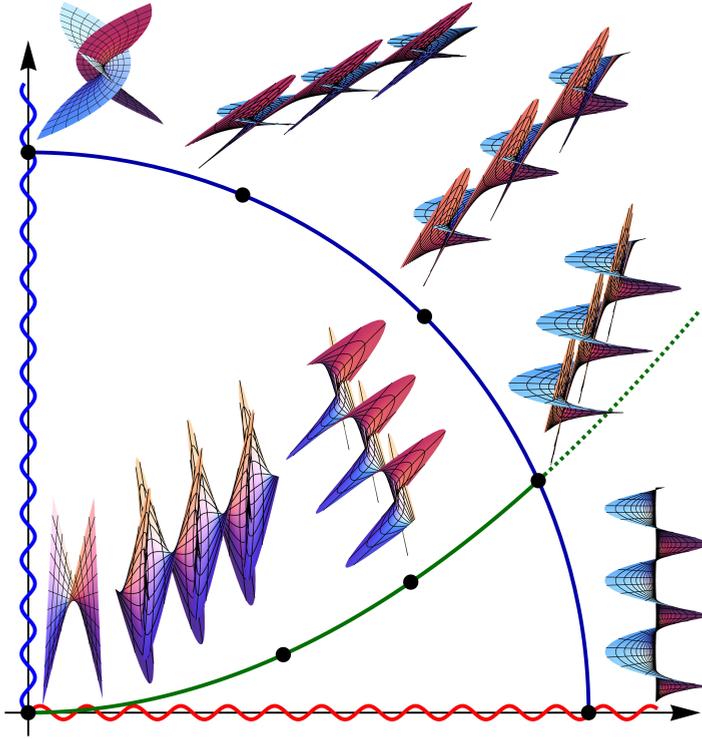
$$\begin{cases} \omega_{uu} + \omega_{vv} = 0, & \text{(compatibility condition)} & (6.2a) \\ \omega_{uv} + \omega_u\omega_v = 0, & \text{(zero mean curvature and affine minimality condition)} & (6.2b) \end{cases}$$

which is identical to the system of partial differential equations that corresponds to zero mean curvature surfaces with planar curvature lines (3.1).

Thus, every zero mean curvature surface with planar curvature lines (with normalized Hopf differential factor  $Q = -\frac{1}{2}$ ) corresponds to a zero mean curvature surface that is also affine minimal (with normalized Hopf differential factor  $Q = -\frac{i}{2}$ ), allowing us to conclude as follows from Remark 3.10.



**Figure 5.** Zero mean curvature surfaces that are also affine minimal in isotropic space: trivial Enneper surface and helicoid on the top row; Enneper surface and Thomsen-type surface on the bottom row.



**Figure 6.** Continuous deformation of zero mean curvature surfaces keeping affine minimality. For the meaning of the diagram, please refer to Figure 1.

**Theorem 6.1** ([47, Satz 9]). *Every zero mean curvature surface in isotropic 3-space that is also an affine minimal surface is a conjugate zero mean curvature surface of a zero mean curvature surface with planar curvature lines.*

Using the classification of zero mean curvature surfaces with planar curvature lines in Theorem 3.13, we also obtain a classification of all zero mean curvature surfaces in isotropic 3-space that are also affine minimal (see also Figure 5).

**Theorem 6.2.** *Let  $X : \Sigma \rightarrow \mathbb{I}^3$  be a zero mean curvature immersion that is also affine minimal. Then  $X$  must be a piece of one, and only one, of*

- plane  $(0, 1 \, dz)$ ,
- trivial Enneper-type surface  $(z, i \, dz)$ ,
- helicoid  $(e^z, i e^{-z} \, dz)$ ,
- Enneper-type surface  $(\frac{1}{z}, i z^2 \, dz)$ , or

- one of Thomsen-type surfaces ( $\gamma \coth z, \frac{i}{\gamma} \sinh^2 z$ ) for some constant  $\gamma > 0$  given with the respective Weierstrass data, up to isometries and homotheties of the isotropic space.

Furthermore, we can also conclude from Theorem 5.2 as follows, obtaining the isotropic space analogue of results in [2, 3, 13, 37] (see also Figure 6).

**Theorem 6.3.** *There exists a continuous deformation consisting exactly of the zero mean curvature surfaces that are also affine minimal in isotropic 3-space.*

**Acknowledgments.** The authors would like to express their gratitude to Shintaro Akamine, Yuta Ogata, and the referee for their helpful comments. The first author gratefully acknowledges the support from Handong Global University, New Faculty Grant. The second author gratefully acknowledges the support from JST SPRING, Grant Number JPMJSP2148, including the opportunity to visit TU Wien, Austria.

## References

- [1] U. Abresch, [Constant mean curvature tori in terms of elliptic functions](#). *J. Reine Angew. Math.* **374** (1987), 169–192 Zbl [0597.53003](#) MR [0876223](#)
- [2] S. Akamine, J. Cho, and Y. Ogata, [Analysis of timelike Thomsen surfaces](#). *J. Geom. Anal.* **30** (2020), no. 1, 731–761 Zbl [1439.53006](#) MR [4058536](#)
- [3] W. Barthel, R. Volkmer, and I. Haubitz, [Thomsensche Minimalflächen – analytisch und anschaulich](#). *Results Math.* **3** (1980), no. 2, 129–154 Zbl [0451.53007](#) MR [0600998](#)
- [4] H. Beck, Zur Geometrie in der Minimalebene. *Sitzungsber. Berl. Math. Ges.* **12** (1913), 14–30 Zbl [44.0632.01](#)
- [5] L. Berwald, [Über Bewegungsinvarianten und elementare Geometrie in einer Minimalebene](#). *Monatsh. Math. Phys.* **26** (1915), no. 1, 211–228 Zbl [45.0220.02](#) MR [1548649](#)
- [6] A. I. Bobenko and U. Eitner, [Painlevé equations in the differential geometry of surfaces](#). Lecture Notes in Math. 1753, Springer, Berlin, 2000 Zbl [0970.53002](#) MR [1806600](#)
- [7] G. Bockwoldt, [Ueber die Enneper’schen Flächen mit constantem positivem Krümmungsmaas, bei denen die eine Schaar der Krümmungslinien von ebenen Curven gebildet wird](#). Ph.D. thesis, Göttingen, 1878 Zbl [10.0495.02](#)
- [8] O. Bonnet, Mémoire sur la théorie générale des surfaces. *J. Éc. Polytech.* **32** (1848), 1–146
- [9] O. Bonnet, Observations sur les surfaces minima. *C. R. Acad. Sci. Paris* **41** (1855), 1057–1058
- [10] O. Bonnet, Mémoire sur la théorie des surfaces applicables sur une surface donnée. *J. Éc. Polytech.* **42** (1867), 1–151
- [11] J. Cho, D. Lee, W. Lee, and S.-D. Yang, [Spinor representation in isotropic 3-space via Laguerre geometry](#). *Results Math.* **79** (2024), no. 1, article no. 8 Zbl [1540.53020](#) MR [4660756](#)

- [12] J. Cho and Y. Ogata, [Deformation of minimal surfaces with planar curvature lines](#). *J. Geom.* **108** (2017), no. 2, 463–479 Zbl [1381.53023](#) MR [3667234](#)
- [13] J. Cho and Y. Ogata, [Deformation and singularities of maximal surfaces with planar curvature lines](#). *Beitr. Algebra Geom.* **59** (2018), no. 3, 465–489 Zbl [1396.53081](#) MR [3844639](#)
- [14] L. C. B. da Silva, Differential geometry of invariant surfaces in simply isotropic and pseudo-isotropic spaces. *Math. J. Okayama Univ.* **63** (2021), 15–52 Zbl [1468.53005](#) MR [4174857](#)
- [15] L. C. B. da Silva, [Holomorphic representation of minimal surfaces in simply isotropic space](#). *J. Geom.* **112** (2021), no. 3, article no. 35 Zbl [1481.53013](#) MR [4318435](#)
- [16] L. C. B. da Silva and R. López, [Catenaries and singular minimal surfaces in the simply isotropic space](#). *Results Math.* **78** (2023), no. 5, article no. 204 Zbl [1546.53008](#) MR [4627386](#)
- [17] A. Enneper, Analytisch-geometrische Untersuchungen. *Z. Math. Phys.* **9** (1864), 96–125
- [18] A. Enneper, Analytisch-geometrische Untersuchungen. *Nachr. Königl. Ges. Wiss. Georg-Augusts-Univ. Göttingen* **1868** (1868), no. 11, 258–276, 421–433
- [19] A. Enneper, Untersuchungen über die Flächen mit planen und sphärischen Krümmungslinien. *Abh. Königl. Ges. Wissensch. Göttingen* **23** (1878), 1–96 Zbl [12.0579.01](#)
- [20] C. Jiang, C. Wang, X. Tellier, J. Wallner, and H. Pottmann, [Planar panels and planar supporting beams in architectural structures](#). *ACM Trans. Graph.* **42** (2022), no. 2, 19:1–17
- [21] F. Joachimsthal, [Demonstrationes theorematum ad superficies curvas spectantium](#). *J. Reine Angew. Math.* **30** (1846), 347–350 Zbl [030.0873cj](#) MR [1578476](#)
- [22] M. Kilian, D. Pellis, J. Wallner, and H. Pottmann, [Material-minimizing forms and structures](#). *ACM Trans. Graph.* **36** (2017), no. 6, 173:1–12
- [23] M. L. Leite, [Surfaces with planar lines of curvature and orthogonal systems of cycles](#). *J. Math. Anal. Appl.* **421** (2015), no. 2, 1254–1273 Zbl [1298.53051](#) MR [3258318](#)
- [24] E. Lenz, [Ueber die Enneper'schen Flächen constanten negativen Krümmungsmaasses mit einem Systeme ebener Krümmungslinien](#). Ph.D. thesis, Göttingen, 1879
- [25] M. A. Magid, [Timelike Thomsen surfaces](#). *Results Math.* **20** (1991), no. 3-4, 691–697 Zbl [0760.53006](#) MR [1145303](#)
- [26] F. Manhart, [Bonnet–Thomsen surfaces in Minkowski geometry](#). *J. Geom.* **106** (2015), no. 1, 47–61 Zbl [1327.53011](#) MR [3320877](#)
- [27] C. Millar, T. Mitchell, A. Mazurek, A. Chhabra, A. Beghini, J. N. Clelland, A. McRobie, and W. F. Baker, [On designing plane-faced funicular gridshells](#). *Int. J. Space Struct.* **38** (2023), no. 1, 40–63
- [28] G. Monge, *Application de l'analyse à la géométrie*. Cinquième édition edn., Bachelier, Paris, 1850
- [29] J. C. C. Nitsche, *Vorlesungen über Minimalflächen*. Grundlehren Math. Wiss. 199, Springer, Berlin-New York, 1975 Zbl [0319.53003](#) MR [0448224](#)
- [30] K. Nomizu and T. Sasaki, *Affine differential geometry*. Cambridge Tracts in Math. 111, Cambridge University Press, Cambridge, 1994 Zbl [0834.53002](#) MR [1311248](#)
- [31] B. Pavković, An interpretation of the relative curvatures for surfaces in the isotropic space. *Glasnik Mat. Ser. III* **15(35)** (1980), no. 1, 149–152 Zbl [0436.53014](#) MR [0586298](#)

- [32] M. Pember, [Weierstrass-type representations](#). *Geom. Dedicata* **204** (2020), 299–309  
Zbl [1433.53015](#) MR [4056704](#)
- [33] H. Pottmann, P. Grohs, and N. J. Mitra, [Laguerre minimal surfaces, isotropic geometry and linear elasticity](#). *Adv. Comput. Math.* **31** (2009), no. 4, 391–419 Zbl [1425.65034](#)  
MR [2558260](#)
- [34] H. Pottmann and Y. Liu, [Discrete surfaces in isotropic geometry](#). In *Mathematics of surfaces XII. 12th IMA international conference, Sheffield, UK, September 4–6, 2007. Proceedings*, edited by R. Martin et al., pp. 341–363, Lecture Notes in Computer Science 4647, Springer, Berlin, 2007 Zbl [1163.68359](#)
- [35] H. Sachs, *Isotrope Geometrie des Raumes*. Friedr. Vieweg & Sohn, Braunschweig, 1990  
Zbl [0703.51001](#) MR [1059891](#)
- [36] Y. Sato, [d-minimal surfaces in three-dimensional singular semi-Euclidean space  \$\mathbb{R}^{0,2,1}\$](#) . *Tamkang J. Math.* **52** (2021), no. 1, 37–67 Zbl [1483.53011](#) MR [4209565](#)
- [37] H. Schaal, [Die Ennepersche Minimalfläche als Grenzfall der Minimalfläche von G. Thomsen](#). *Arch. Math. (Basel)* **24** (1973), 320–322 Zbl [0261.53005](#) MR [0319068](#)
- [38] J. J. Seo and S.-D. Yang, [Zero mean curvature surfaces in isotropic three-space](#). *Bull. Korean Math. Soc.* **58** (2021), no. 1, 1–20 Zbl [1476.53036](#) MR [4206079](#)
- [39] W. T. Shaw, [Recovering holomorphic functions from their real or imaginary parts without the Cauchy–Riemann equations](#). *SIAM Rev.* **46** (2004), no. 4, 717–728 Zbl [1068.30001](#)  
MR [2124683](#)
- [40] K. Strubecker, [Differentialgeometrie des isotropen Raumes. I. Theorie der Raumkurven](#). *Akad. Wiss. Wien, S.-B. IIA* **150** (1941), 1–53 Zbl [67.0666.02](#) MR [0018957](#)
- [41] K. Strubecker, [Differentialgeometrie des isotropen Raumes. II. Die Flächen konstanter Relativkrümmung  \$K = rt - s^2\$](#) . *Math. Z.* **47** (1942), 743–777 Zbl [68.0439.01](#)  
MR [0016238](#)
- [42] K. Strubecker, [Differentialgeometrie des isotropen Raumes. III. Flächentheorie](#). *Math. Z.* **48** (1942), 369–427 Zbl [0027.25301](#) MR [0009145](#)
- [43] K. Strubecker, [Differentialgeometrie des isotropen Raumes. IV. Theorie der flächentreuen Abbildungen der Ebene](#). *Math. Z.* **50** (1944), 1–92 Zbl [0063.07216](#) MR [0016990](#)
- [44] K. Strubecker, [Differentialgeometrie des isotropen Raumes. V. Zur Theorie der Eiliniien](#). *Math. Z.* **51** (1949), 525–573 Zbl [0032.18303](#) MR [0029201](#)
- [45] K. Strubecker, [Über Potentialflächen](#). *Arch. Math.* **5** (1954), 32–38 Zbl [0055.15302](#)  
MR [0063081](#)
- [46] K. Strubecker, [Über das isotrope Gegenstück  \$z = \frac{3}{2} \cdot J\(x + iy\)^{2/3}\$  der Minimalfläche von Enneper](#). *Abh. Math. Sem. Univ. Hamburg* **44** (1975), 152–174 (1976)  
Zbl [0321.53013](#) MR [0397580](#)
- [47] K. Strubecker, [Über die Minimalflächen des isotropen Raumes, welche zugleich Affinminimalflächen sind](#). *Monatsh. Math.* **84** (1977), no. 4, 303–339 Zbl [0376.53003](#)  
MR [0467561](#)
- [48] X. Tellier, C. Douthe, O. Baverel, and L. Hauswirth, [Designing funicular grids with planar quads using isotropic Linear-Weingarten surfaces](#). *Int. J. Solids Struct.* **264** (2023), article no. 112028

- [49] G. Thomsen, [Über affine Geometrie XXXIX. Abh. Math. Sem. Univ. Hamburg](#) **2** (1923), no. 1, 71–73 Zbl [49.0527.04](#) MR [3069411](#)
- [50] M. Voretzsch, *Untersuchung einer speciellen Fläche constanter mittlerer Krümmung, bei welcher die eine der beiden Schaaren der Krümmungslinien von ebenen Curven gebildet wird.* Ph.D. thesis, Göttingen, 1883 Zbl [15.0731.01](#)
- [51] R. Walter, [Explicit examples to the  \$H\$ -problem of Heinz Hopf.](#) *Geom. Dedicata* **23** (1987), no. 2, 187–213 Zbl [0615.53050](#) MR [0892400](#)
- [52] H. C. Wente, [Counterexample to a conjecture of H. Hopf.](#) *Pacific J. Math.* **121** (1986), no. 1, 193–243 Zbl [0586.53003](#) MR [0815044](#)

Received 12 November 2024; revised 4 May 2025.

**Joseph Cho**

Global Leadership School, Handong Global University, 558 Handong-ro Buk-gu,  
37554 Pohang, Republic of Korea; [jcho@handong.edu](mailto:jcho@handong.edu)

**Masaya Hara**

Department of Mathematics, Graduate School of Science, Kobe University, 1-1 Rokkodai-cho,  
Nada-ku, Kobe 657-8501, Japan; [mhara@math.kobe-u.ac.jp](mailto:mhara@math.kobe-u.ac.jp)