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Height functions on a cuspidal edge with a distinguished singular curve

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Abstract. We classify submersions on a cuspidal edge M in $\mathbb{R}^3$ with a distinguished singular curve S in it. We use this classification to study the contact of the pair (M, S) with planes, which is measured by the singularities of the height functions on (M, S). The discriminants of the versal deformations of the submersions obtained are also described.

1. Introduction

The beginning of the XXI century has witnessed a great impulse in the study of differential geometry of singular surfaces. Differential geometers and singularity theorists have got together to define new types of curvature for singular points or understand the behaviour of classical curvatures when approaching a singular point, amongst other approaches. Good examples of this are [11, 13, 15–17, 19, 21].

One way of understanding the geometry of an object through singularity theory techniques is studying its contact with flat objects such as planes or lines. For the former, one must study the singularities of the height functions on certain directions, which codify the type of contact of the object with the plane orthogonal to that direction. More precisely, consider planes orthogonal to a direction $\mathbf{v}$ in the unit sphere in $\mathbb{R}^3$. These planes are the fibres of the function $h_{\mathbf{v}}(p) = \langle p, \mathbf{v} \rangle$. The contact of a surface M with these planes at a certain point is measured by the singularities of $h_{\mathbf{v}} \circ \phi$ at the origin, where ϕ is a parametrisation of M. By varying $\mathbf{v}$, we get the family of height functions $H : U \times S^2 \to \mathbb{R}$ on M given by

$$H((x, y), \mathbf{v}) = \langle \phi(x, y), \mathbf{v} \rangle.$$

In this setting, the model surfaces in $\mathbb{R}^3$ are planes and the parametrisation ϕ is taken in general form. However, in this paper, we follow the approach in [7] and invert the situation: we fix a model surface and consider its contact with fibres of submersions.

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For this we must classify submersions up to changes of coordinates in the source that preserve the model surface, i.e., we must classify submersions under the geometric subgroup $\mathcal{R}(X)$. Many authors have studied the contact with planes of singular surfaces such as the cross-cap (or Whitney umbrella) [7, 10], the cuspidal edge [20], the swallowtail [9], the cuspidal cross-cap [18] or even singular surfaces in $\mathbb{R}^4$ [6].

In this paper, we consider the cuspidal edge with a distinguished singular curve in it. Such a setting appears in different situations such as in bifurcation diagrams of certain caustics [1]. In Section 2, we describe some generic geometry of the cuspidal edge, introduce the classification tools and compare our setting to Zakalyukin's contact flag singularities [22]. We remark that our setting can be considered as an example of corner flag contact singularities (corner in the sense of [3]). In Section 3, we obtain the module of vector fields tangent to both the cuspidal edge and its distinguished curve and classify submersions which preserve not only the cuspidal edge, but the curve too. For this classification we use an adopted version of Arnold's spectral sequence method for reduction to normal forms [5]. In particular, the method is a version of [3, Lemma 8.1] and [2, Lemma 2.10]. We study the types of contact in Section 4 and relate them to certain curvatures of cuspidal edges. In Section 5, we analyze different discriminants of the versal deformations of the submersions obtained in our classification. These discriminants model the duals of the pair given by the cuspidal edge and the distinguished curve.

2. Preliminaries

We go over some elements of the geometry of the cuspidal edge and establish some notation for the classification of germs of functions on the cuspidal edge equipped with a singular curve.

2.1. Geometric cuspidal edge

Let $U \subseteq \mathbb{R}^2$ be an open set with local coordinates (x, y). Let $f : U \to \mathbb{R}^3$ be given by $f(x, y) = (x, y^2, y^3)$. A cuspidal edge is any surface which admits a parametrisation $\mathcal{A}$-equivalent (changes of variables in source and target) to f. In [14], a normal form for a general cuspidal edge M is given under diffeomorphisms in the source and isometries in the target

$$\phi(x, y) = \Big(x, a(x) + \frac{1}{2}y^2, b_1(x) + y^2 b_2(x) + y^3 b_3(x, y)\Big), \tag{2.1}$$

such that $a(0) = a'(0) = 0$, $b_1(0) = b_1'(0) = 0$, $b_2(0) = 0$, $b_3(0) \neq 0$, where

$$a(x) = \frac{1}{2}a_{20}x^2 + \frac{1}{6}a_{30}x^3 + \frac{1}{24}a_{40}x^4 + O(5);$$

$$
\begin{aligned}
b_1(x) &= \frac{1}{2}b_{20}x^2 + \frac{1}{6}b_{30}x^3 + \frac{1}{24}b_{40}x^4 + O(5);\\
b_2(x) &= \frac{1}{2}b_{12}x + \frac{1}{6}b_{22}x^2 + \frac{1}{24}b_{32}x^3 + O(4);\\
b_3(x,y) &= \frac{1}{6}b_{03} + \frac{1}{6}b_{13}x + \frac{1}{24}b_{04}y + \frac{1}{24}b_{23}x^2 + \frac{1}{24}b_{14}xy\\
&\quad + \frac{1}{120}b_{05}y^2 + O(3).
\end{aligned}
$$

In the terminology of [7], such an M is called a *geometric cuspidal edge* and the image of f is called the *standard cuspidal edge*. The set of singular points of M is the line $y = 0$ and its image is parametrised as $\gamma_1(x) = \phi(x,0) = (x, a(x), b_1(x))$, and will be denoted by $\Sigma \subseteq M$. The tangential direction of M at the origin along Σ, which will be denoted by $\mathcal{T}_M$, is parallel to the vector $\mathbf{v}_1 = (1,0,0)$.

In [14], amongst many other invariants, Martins and Saji study the curvature and torsion of Σ at the origin as a space curve and the cuspidal curvature (see [16]). Using the normal form (2.1), we have

$$
\begin{aligned}
&\text{the curvature of } \Sigma: && \kappa_\Sigma(0) = \sqrt{a_{20}^2 + b_{20}^2};\\
&\text{the torsion of } \Sigma: && \tau_\Sigma(0) = \frac{a_{20}b_{30} - b_{20}a_{30}}{a_{20}^2 + b_{20}^2};\\
&\text{the cuspidal curvature:} && \kappa_c(0) = b_{03}.
\end{aligned}
$$

Denote the coordinates of the target space $\mathbb{R}^3$ by $z = (u, v, w)$. Then, the tangent cone to M is the plane $w = 0$. The osculating plane of Σ at the origin is orthogonal to the vector $(0, -b_{20}, a_{20})$ and it coincides with the tangent cone if $b_{20} = 0$.

Let $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^3, 0);\ t \mapsto \gamma(t)$ be a singular curve in $\mathbb{R}^3$, that is, $\gamma'(0) = (0,0,0)$. The point 0 is called of A-type if $\gamma''(0) \neq 0$. In this case, the singular curvature of γ at 0 (see [12, 18]) is defined by

$$
\kappa_{\text{sing}}(\gamma) = \frac{|\gamma''(0) \times \gamma'''(0)|}{|\gamma''(0)|^{\frac{5}{2}}}.
$$

The intersection of the plane $u = 0$ with the geometric cuspidal edge M is a singular space curve having a cusp at 0, which will be denoted by S and parametrised as

$$
\gamma_2(y) = \phi(0, y) = \left(0, \frac{1}{2}y^2, y^3 b_2(0, y)\right). \tag{2.2}
$$

The limiting tangential direction $\mathcal{T}_S$ of γ_2 at the origin is parallel to the vector $\mathbf{v}_2 = (0,1,0)$. Clearly, 0 is an A-type point of γ_2 and hence its singular curvature is

$$
\kappa_{\text{sing}}(\gamma_2) = |b_{03}|.
$$

Notice that in this case $\kappa_{\text{sing}}(\gamma_2)$ coincides with the cuspidal curvature κ_c defined in [16].

By $\widetilde{M}$, we shall mean the pair (M, S) consisting of the geometric cuspidal edge M and the singular curve S in it, as described above.

2.2. Classification tools

Consider germs of smooth functions $g : (\mathbb{R}^n, 0) \to \mathbb{R}$ in local coordinates $x = (x_1, x_2, \ldots, x_n)$. Let $\mathcal{E}_x$ be the ring of all such germs and denote by $\mathcal{M}_x$ the maximal ideal in $\mathcal{E}_x$.

Let $(V, 0)$ be the germ of an algebraic variety in $(\mathbb{R}^n, 0)$. Recall that two function germs g_1 and g_2 are $\mathcal{R}(V)$-equivalent if there is a diffeomorphism-germ $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $g_2 = g_1 \circ \varphi$ and φ preserves V (i.e., $\varphi(V) \subseteq V$).

Let θ_n be the set of germs of the vector fields on $\mathbb{R}^n$. Let $I(V) = \langle h_1, h_2, \ldots, h_s \rangle$, $h_i \in \mathcal{E}_x$, be the ideal of germs of functions that vanish on V. The module of all vector fields in θ_n tangent to V is denoted by $\Theta(V)$ and defined as

$$\Theta(V) = \left\{ \xi \in \theta_n : \exists r_{ij} \in \mathcal{E}_x \text{ such that } \xi(h_j) = \sum_{i=1}^{s} r_{ij} h_i, \forall j = 1, 2, \ldots, s \right\}.$$

It is evident that if $\xi \in \Theta(V)$ then its flow φ_t preserves V.

Let $\Theta^0(V) = \{\xi \in \Theta(V) : \xi(0) = 0\}$. Then, the tangent space and the extended tangent space to the $\mathcal{R}(V)$-orbit of g at the germ g are given by

$$L\mathcal{R}(V).g = \left\{ \xi(g) : \xi \in \Theta^0(V) \right\}$$

and

$$L_e\mathcal{R}(V).g = \left\{ \xi(g) : \xi \in \Theta(V) \right\},$$

respectively.

The $\mathcal{R}_e^+$-codimension of g is defined as $d(g, \mathcal{R}_e^+(V)) = \dim_{\mathbb{R}}(\mathcal{M}_n / L_e\mathcal{R}(X).g)$.

Our goal in Section 3 is to classify germs of functions g with respect to $\mathcal{R}(V)$-equivalence where V is the germ of the pair consisting of the model cuspidal edge and a distinguished singular curve in it such that $d(g, \mathcal{R}_e^+(V)) \leq 2$. The basic method of the classification and prenormal forms is based on Arnold's spectral sequence [5].

2.3. Comparison to Zakalyukin's flag contact singularities

A flag in an affine space A^n is a set F_s of nested affine subspaces $L_1^{k_1} \subset \cdots \subset L_s^{k_s} \subset A^n$. Zakalyukin studied in [22] the contact of the flag F_s with the image of a smooth mapping $p : M^m \to A^n$, where M is a smooth manifold. The tangent space to the F_s-orbit he used was $T_p = dp(\theta_m) + p^*(\Theta_{F_s})$, where p^* is the regular pullback. When

$n = 3$, $s = 1$, and $k_1 = 2$, if p are generic maps from surfaces to $\mathbb{R}^3$, the flag contact singularities give the contact of the cross-cap with planes, which was done in [7]. If p are generic 2-dimensional wavefronts, the flag contact singularities give the contact of the cuspidal edge or the swallowtail with planes, which were done in [20] and [9], respectively.

The approach can be reversed: On certain stratified singular surfaces such as wavefronts, the stratification induces a flag in the well-defined tangent space. For instance, for a cuspidal edge, besides the tangent cone (which is a plane), there is a distinguished line in the tangent cone (the tangent direction to the singular set), or for the swallowtail, in the tangent cone one may consider the tangent direction in the limit to the cuspidal edges and the point in it given by the origin.

When classifying submersions on such wavefronts in order to study contact with planes, since the vector fields tangent to the wavefront are tangent to the strata, in a way, one is analyzing contact with a flag. The contacts of these wavefronts with planes are related to the contact of an embedded surface with the flag induced by the wavefront.

This is better understood using Damon's theorem on transverse fibre squares (see [8] for details). Suppose the image of the wavefront V is the discriminant of a map $F : (R^3, 0) \to (R^3, 0)$. Let $i : (R^2, 0) \to (R^3, 0)$ be an embedding transversal to F and let $g : (R^3, 0) \to \mathbb{R}$ be a submersion such that $g \circ i = 0$. Then

$$\dim_{\mathbb{R}} \frac{\theta(g)}{dg(\Theta_V) + \langle g \rangle} = \dim_{\mathbb{R}} \frac{\theta(i)}{di(\theta_2) + i^*(\Theta_V)},$$

where $\theta(g)$ are germs of vector fields along g (similarly for i). On one hand $dg(\Theta_V) = L_e\mathcal{R}(V).g$ and on the other hand $di(\theta_2) + i^*(\Theta_V)$ is Zakalyukin's tangent space for the embedding i. Considering the relation of Θ_V with Θ_{F_s}, this shows that the classification of submersions preserving V is closely related to the classification of embeddings with respect to the flag F_s.

In our setting, where we consider the pair $\widetilde{M} = (M, S)$, however, the tangent direction to the distinguished curve S is transversal to the tangent direction to the singular curve Σ, and therefore the induced geometry is not related to contact with a certain flag. As we will see in the next Section, we want tangent vector fields which preserve the cuspidal edge surface and that preserve the distinguished curve, which can be interpreted as an intersection of different flags. In this sense, our work can be seen as an example of corner flag contact singularities (see [3] for details on corner singularities).

3. Functions on a cuspidal edge equipped with singular curve

Let $(f, \overline{\gamma}_2)$, $f(x, y) = (x, y^2, y^3)$, $\overline{\gamma}_2(y) = f(0, y) = (0, y^2, y^3)$, be the pair of the $\mathcal{A}$-normal form of a cuspidal edge and a distinguished singular curve in it determined by the equation $x = 0$. Denote by $\widetilde{X}$ the pair (X, B), where $X = f(\mathbb{R}^2, 0)$ and $B = \overline{\gamma}_2(\mathbb{R}, 0)$. In this case, the defining equation of X is $h_1(u, v, w) = v^3 - w^2$ and the defining equations of B are h_1 and $h_2(u, v, w) = u$.

A diffeomorphism $\varphi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ will be said to preserve $\widetilde{X}$ if and only if $\varphi(X) \subseteq X$ and $\varphi(B) \subseteq B$. Two function germs $g_1, g_2 : (\mathbb{R}^3) \to (\mathbb{R}, 0)$ are $\mathcal{R}(\widetilde{X})$-equivalent if there is a diffeomorphism φ of the ambient space such that $g_2 = g_1 \circ \varphi$ and φ preserves $\widetilde{X}$.

A vector field $\xi \in \theta_3$ is said to be tangent to $\widetilde{X}$ if and only if $\xi(I(X)) \subseteq I(X)$ and $\xi(I(B)) \subseteq I(B)$, where $I(X) = \langle h_1 \rangle$ and $I(B) = \langle h_1, h_2 \rangle$. Such vector fields can be integrated to obtain diffeomorphisms that preserve $\widetilde{X}$.

Let $\Theta(\widetilde{X}) \subset \theta_3$ be the $\mathcal{E}_z$-module of all vector fields tangent to $\widetilde{X}$.

Proposition 3.1. *The module $\Theta(\widetilde{X})$ is generated by the vector fields*

$$\begin{aligned}\xi_1 &= u\frac{\partial}{\partial u},\\ \xi_2 &= (v^3 - w^2)\frac{\partial}{\partial u},\\ \xi_3 &= 2v\frac{\partial}{\partial v} + 3w\frac{\partial}{\partial w},\\ \xi_4 &= 2w\frac{\partial}{\partial v} + 3v^2\frac{\partial}{\partial w}.\end{aligned}$$

Proof. For notation purposes, we write $(u, v, w) = (X_1, X_2, X_3)$. Vector fields $\xi = \sum_{i=1}^{3} \xi_i \frac{\partial}{\partial X_i}$ on $\mathbb{R}^3$ such that $\xi(I(X)) \subseteq I(X)$ are computed in [20, Proposition 3.1], call that module $\Theta(X)$. Now we are looking for vector fields $\xi = \sum_{i=1}^{3} \xi_i \frac{\partial}{\partial X_i}$ such that for each $j = 1, 2$ there exist functions $r_i(X_1, X_2, X_3)$ such that

$$\sum_{i=1}^{3} \xi_i \frac{\partial h_j}{\partial X_i} = \sum_{i=1}^{2} r_i h_i.$$

Consider, for $j = 1, 2$, the map $\Phi_j : \mathcal{E}_3^5 \to \mathbb{R}$ given by

$$\Phi_j(\xi, r) = \sum_{i=1}^{3} \xi_i \frac{\partial h_j}{\partial X_i} - \sum_{i=1}^{2} r_i h_i,$$

where $\xi = (\xi_1, \xi_2, \xi_3) \in \mathcal{E}_3^3$ and $r = (r_1, r_2) \in \mathcal{E}_3^2$. Let $K_j = \ker \Phi_j$. Let $\pi : \mathcal{E}_3^5 \to \mathcal{E}_3^3$ be the canonical projection given by $\pi(\xi, r) = \xi$. Let $B_j = \pi(K_j)$. Then

$$\Theta(B) = B_1 \cap B_2.$$

Finally,

$$\Theta(\widetilde{X}) = \Theta(X) \cap \Theta(B).$$

In order to obtain the K_j, we use syzygies in the computer package Singular. It can be checked that all the vector fields obtained by this method are, in fact, liftable, i.e., there exists a vector field η on $\mathbb{R}^2$ such that $dh_i(\eta) = \xi \circ h_i$, and are therefore tangent to $\widetilde{X}$. ∎

Corollary 3.2. *For $g \in \mathcal{E}_z$, the tangent space to the $\mathcal{R}(\widetilde{X})$-orbit of g at the germ g is defined as*

$$L\mathcal{R}(\widetilde{X}).g = \left\{(uA_1 + (v^3 - w^2)A_2)\frac{\partial g}{\partial u} + (2vP + 2wQ)\frac{\partial g}{\partial v} + (3wP + 3v^2Q)\frac{\partial g}{\partial w} : A_1, A_2, P, Q \in \mathcal{E}_z\right\}.$$

Remark 3.3. The extended tangent space $L_e\mathcal{R}(\widetilde{X}).g$ coincides with $L\mathcal{R}(\widetilde{X}).g$.

We will classify germs of submersions $g : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ with respect to the $\mathcal{R}(\widetilde{X})$-equivalence relation. The classification method is described in the following lemma. We review first some concepts from [4].

Consider a fixed and convenient Newton diagram $\Gamma \subset \mathbb{Z}^n_{\geq 0}$. Each face Γ_i of Γ specifies a quasihomogeneity type $\boldsymbol{\alpha}_i = (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in})$ in which the degree of the monomials $\boldsymbol{x}^{\boldsymbol{k}} = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ with exponent lying on Γ_i is equal to one, that is $\langle \boldsymbol{k}, \boldsymbol{\alpha}_i \rangle = \alpha_{i1}k_1 + \cdots + \alpha_{in}k_n = 1$. Here $\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in} \in \mathbb{N}$ and they are called weights of the variables $x_1, x_2, \ldots, x_n$, respectively.

A monomial $\boldsymbol{x}^{\boldsymbol{k}}$ is said to have Newton degree d if $d = \min_j \langle \boldsymbol{k}, \boldsymbol{\alpha}_j \rangle$. The monomials of Newton degree d are exactly those whose exponents lie in the diagram $d\Gamma$ obtained from Γ through a homothetic transformation of ratio d.

The Newton order d of a power series is the smallest of the Newton degrees of the monomials that appear in it. The series of order at least d form an ideal $\mathcal{S}_j$ in the ring $\mathcal{E}_x$. The ideals $\mathcal{S}_j$ yield the Newton filtration in $\mathcal{E}_x$, that is, $\mathcal{S}_0 = \mathcal{E}_x$, $\mathcal{S}_k \subseteq \mathcal{S}_l$ whenever $k > l$.

The sum of the terms of Newton degree d of a power series g of order d will be referred to as *the principal part* of g.

For $g \in \mathcal{E}_z$, let $g = g_0 + \widetilde{g}$ be a decomposition of a function germ into its principal part g_0 of the Newton degree N and higher order terms $\widetilde{g}$. We assume that the $\mathcal{R}_e^+$-codimension of g_0 is finite, that is, $d(g_0, \mathcal{R}_e^+(X)) < \infty$. Denote by $\dot{u}$, $\dot{v}$ and $\dot{w}$ the components of the vector field ξ that belong to $\Theta(\widetilde{X})$. The following is a version of [3, Lemma 8.1] and [2, Lemma 2.10].

Lemma 3.4. *Consider a monomial basis of the linear space $\mathcal{E}_z / L\mathcal{R}(\widetilde{X}).g_0$. Let $e_1(z), e_2(z), \ldots, e_s(z)$ be all its elements of Newton degrees higher than N.*

Suppose that for any $\omega \in \mathcal{S}_\beta \setminus \mathcal{S}_{>\beta}$, $\beta > N$*:*

(1) *There is a vector field* $\xi = \dot{u}\frac{\partial}{\partial u} + \dot{v}\frac{\partial}{\partial v} + \dot{w}\frac{\partial}{\partial w} \in \Theta(\widetilde{X})$ *, such that*

$$\omega = \frac{\partial g_0}{\partial u}\dot{u} + \frac{\partial g_0}{\partial v}\dot{v} + \frac{\partial g_0}{\partial w}\dot{w} + \widehat{\omega} + \sum_{i=1}^{s} c_i e_i(z),$$

where $\widehat{\omega} \in \mathcal{S}_{>\beta}$ *and* $c_i \in \mathbb{R}$.

(2) *Moreover, for any* δ, $N < \delta < \beta$, *and any* $\psi \in \mathcal{S}_\delta$ *the expression*

$$E(\psi, \omega) = \frac{\partial \psi}{\partial u}\dot{u} + \frac{\partial \psi}{\partial v}\dot{v} + \frac{\partial \psi}{\partial w}\dot{w}$$

belongs to $\mathcal{S}_\beta$.

Then any germ $g = g_0 + \widetilde{g}$ *is* $\mathcal{R}(\widetilde{X})$*-equivalent to a germ* $g_0 + \sum_{i=1}^{s} a_i e_i$*, where* $a_i \in \mathbb{R}$.

Remark 3.5. By removing the requirement that g_0 has finite codimension, the proof of Lemma 3.4 shows that any function $g = g_0 + \widetilde{g}$ is $\mathcal{R}(\widetilde{X})$-equivalent to a similar form

$$g_0 + \sum a_i e_i + \widetilde{\psi},$$

where $\widetilde{\psi}$ belongs to sufficiently large power of the maximal ideal.

Two families of germs of functions $H_1, H_2 : (\mathbb{R}^3 \times \mathbb{R}^l, (0,0)) \to (\mathbb{R}, 0)$ are called P-$\mathcal{R}^+(\widetilde{X})$-equivalent if there exists a germ of a diffeomorphism $\Phi : (\mathbb{R}^3 \times \mathbb{R}^l, (0,0)) \to (\mathbb{R}^3 \times \mathbb{R}^l, (0,0))$, having the form $\Phi(z,u) = (\varphi(z,u), \chi(u))$, and a function germ $C : (\mathbb{R}^l, 0) \to \mathbb{R}$ such that

$$H_2(z,u) = H_1 \circ \Phi(z,u) + C(u).$$

Let $G(z,u)$ be a deformation of $g \in \mathcal{E}_z$. Then, G is called $\mathcal{R}^+(\widetilde{X})$-*versal* if for any other deformation H of g there is a map germ $\Phi(z,u)$ (which need not necessarily be a diffeomorphism) and a germ of a function C as above with

$$H(z,u) = G \circ \Phi(z,u) + C(u).$$

Given a deformation $G(z,u)$, where $u \in \mathbb{R}^l$, the initial speeds $\dot{G}_i$ of G are defined by

$$\dot{G}_i = \frac{\partial G}{\partial u_i}(z,0), \quad \forall i \in \{1, 2, \ldots, l\}.$$

The following result is a version of [7, Theorem 3].

Proposition 3.6. *A deformation* G *of a germ of a function* g *on* $\widetilde{X}$ *is* $\mathcal{R}^+(\widetilde{X})$*-versal if and only if*

$$L\mathcal{R}(\widetilde{X}).g + \mathbb{R}\{1, \dot{G}_1, \ldots, \dot{G}_l\} = \mathcal{E}_z.$$

Normal form	$d(f, \mathcal{R}^+(\widetilde{X}))$	$\mathcal{R}(\widetilde{X})$-versal deformation
$u \pm v$	0	$u \pm v$
$u + w$	1	$u + w + \lambda_1 v$
$\pm v \pm u^2$	1	$\pm v \pm u^2 + \lambda_1 u$
$u \pm v^2$	2	$u \pm v^2 + \lambda_2 w + \lambda_1 v$
$\pm v + u^3$	2	$\pm v + u^3 + \lambda_2 u^2 + \lambda_1 u$
$w + uv + au^2, a \neq 0$	$2^{(*)}$	$w + uv + au^2 + \lambda_2 u + \lambda_1 v$

(*): a is a modulus and the codimension is that of the stratum.

Table 1. Germs of submersions in $\mathcal{M}_z$ of $\mathcal{R}^+(\widetilde{X})$-codimension ≤ 2.

The $\mathcal{R}(\widetilde{X})$-classification of germs of submersions is stated as follows.

Theorem 3.7. *Any function germ g at the origin with $\mathcal{R}^+(\widetilde{X})$-codimension ≤ 2 (of the stratum in the presence of moduli) is $\mathcal{R}(\widetilde{X})$-equivalent to one of the germs in Table 1.*

Proof. After integrating the 1-jets of the vector fields from $\Theta(\widetilde{X})$, the linear changes of coordinates that were obtained are

$$\phi_1(u, v, w) = (u, e^{2c_1}v, e^{3c_1}w);$$
$$\phi_2(u, v, w) = (e^{c_2}u, v, w);$$
$$\phi_3(u, v, w) = (u, v + c_3 w, w);$$

where $c_i \in \mathbb{R}$. Note that the diffeomorphisms $\phi_4(u, v, w) = (-u, v, w)$ and $\phi_5(u, v, w) = (u, v, -w)$ preserve $\widetilde{X}$.

Now, let $g = g_0 + \widetilde{g}$ be the decomposition of g into its 1-jet $g_0 = au + bv + cw$, with $a, b, c \in \mathbb{R}$, and $\widetilde{g} \in \mathcal{M}_z^2$. We shall classify all germs g with $\mathcal{R}^+(\widetilde{X})$-codimension ≤ 2 (of the stratum in the presence of moduli).

We start by normalising the 1-jet using the above linear changes of coordinates. For simplicity, we shall say two function germs are equivalent if they are $\mathcal{R}(\widetilde{X})$-equivalent.

Suppose $a \neq 0$. If $b \neq 0$, then g_0 is equivalent to $u \pm v$, otherwise, g_0 is equivalent to $u + w$ or u. Next, assume $a = 0$ but $b \neq 0$, then g_0 is equivalent to $\pm v$. Finally, if $a = b = 0$, then, clearly, g_0 is equivalent to w. Thus, the orbits of the space of 1-jets are $u \pm v$, $u + w$, u, $\pm v$ and w.

We shall demonstrate how to apply Lemma 3.4 and Remark 3.5 in the first case and all other cases may be handled in a similar manner.

Let $g = g_0 + \widetilde{g}$, where the principal part is $g_0 = u \pm v$ and $\widetilde{g} \in \mathcal{M}_{u,v,w}^2$. Note that g_0 is quasi-homogeneous with respect to weights $\alpha_u = \alpha_v = \alpha_w = 1$, and its Newton degree $N = d(g_0) = 1$. So, $g_0 \in \mathcal{S}_1$. Also note that, by Hadamard's lemma,

$$\widetilde{g}(u, v, w) = ug_1(u, v, w) + vg_2(0, v, w) + wg_3(0, 0, w), \tag{3.1}$$

where $g_1 \in \mathcal{M}_{u,v,w}$, $g_2 \in \mathcal{M}_{v,w}$ and $g_3 \in \mathcal{M}_w$. Now the $\mathcal{R}(\widetilde{X})$-tangent space to the orbit of g_0 at g_0 is

$$L\mathcal{R}(\widetilde{X}).g_0 = \{(uA_1 + (v^3 - w^2)A_2) \pm (2vP + 2wQ) : A_1, A_2, P, Q \in \mathcal{E}_z\}.$$

Thus, we get $\operatorname{mod} L\mathcal{R}(\widetilde{X}).g_0$: $u \equiv 0$ and $v \equiv 0$ and $w \equiv 0$. It follows that

$$\mathcal{E}_z / L\mathcal{R}(\widetilde{X}).g_0 \simeq \mathbb{R}.$$

Let $h = uh_1(u, v, w) + vh_2(0, v, w) + wh_3(0, 0, w)$ be any homogeneous polynomial of degree m, from the right-hand side of (3.1) and hence $h \in \mathcal{S}_m$. Note that h_i is a polynomial of degree $m - 1$ in the respective variables. Clearly, h is contained in $L\mathcal{R}(\widetilde{X}).g_0$, by setting $A_1 = h_1$, $A_2 = 0$, $P = \frac{\pm 1}{2}h_2$ and $Q = \frac{\pm 1}{2}h_3$. This implies the existence of a vector field $\xi = \dot{u}\frac{\partial}{\partial u} + \dot{v}\frac{\partial}{\partial v} + \dot{w}\frac{\partial}{\partial w} \in \Theta(\widetilde{X})$, such that $h = \frac{\partial g_0}{\partial u}\dot{u} + \frac{\partial g_0}{\partial v}\dot{v} + \frac{\partial g_0}{\partial w}\dot{w}$, and constraint (1) of Lemma 3.4 is fulfilled. For the second constraint of the lemma, let ψ be any polynomial of degree $s < m$, so that $\psi \in \mathcal{S}_s$. Hence, the expression

$$E(\psi, \omega) = \frac{\partial \psi}{\partial u}\dot{u} + \frac{\partial \psi}{\partial v}\dot{v} + \frac{\partial \psi}{\partial w}\dot{w}$$

belongs to $\mathcal{S}_m$. Therefore, the conclusion of the lemma implies that g is equivalent to $u \pm v$. By similar considerations one can show that $u + w + \widetilde{g}$, where $\widetilde{g} \in \mathcal{M}_{u,v,w}$, is equivalent to $u + w$. The codimensions of $u \pm v$ and $u + w$ are 0 and 1, respectively. An $\mathcal{R}^+(\widetilde{X})$-versal deformation of the latter germ is $u + w + \lambda_1 v$, with $\lambda_1 \in \mathbb{R}$.

Next, consider the principal part $g_0 = u$. The $\mathcal{R}(\widetilde{X})$-tangent space to the orbit of g_0 at g_0 is

$$L\mathcal{R}(\widetilde{X}).g_0 = \{uA_1 + (v^3 - w^2)A_2 : A_1, A_2 \in \mathcal{E}_z\}.$$

Thus, we get $\operatorname{mod} L\mathcal{R}(\widetilde{X}).g_0$: $u \equiv 0$ and $w^2 \equiv v^3$. It follows that

$$\mathcal{E}_z / L\mathcal{R}(\widetilde{X}).g_0 \simeq \{h_1(v) + wh_2(v) : h_1, h_2 \in \mathcal{E}_v\}.$$

Using Remark 3.5 and taking into account the constraints on $\widetilde{g}$, it is evident that g is equivalent to a germ of the form $h = u + \widetilde{h}_1(v) + w\widetilde{h}_2(v)$, where $\widetilde{h}_1 \in \mathcal{M}_v^2$ and $\widetilde{h}_2 \in \mathcal{M}_v$. If $\widetilde{h}_1$ contains a term dv^2, $d \neq 0$, then h is equivalent to the germ $u \pm v^2$ which has codimension 2 and an $\mathcal{R}^+(\widetilde{X})$-versal deformation is $u \pm v^2 + \lambda_2 w + \lambda_1 v$, $\lambda_1, \lambda_2 \in \mathbb{R}$. Otherwise, in the most general case, h has codimension greater than 2.

By similar arguments to the previous case, we can show that any germ g with the 1-jet being $\pm v$ is equivalent to a germ of the form $h^* = \pm v + h_2^*(u)$, where $h_2^* \in \mathcal{M}_u^2$. Let $h_2^*(u) = d_2u^2 + d_3u^3 + h_4^*(u)$, where $d_2, d_3 \in \mathbb{R}$ and $h_4^* \in \mathcal{M}_u^4$. If

$d_2 \neq 0$, then h^* is equivalent to $\pm v \pm u^2$ and its codimension is 1. Next, if $d_2 = 0$ but $d_3 \neq 0$, then h^* is equivalent to $\pm v + u^3$ and its codimension is 2. An $\mathcal{R}^+(\widetilde{X})$-versal deformation of the germs $\pm v \pm u^2$ and $\pm v + u^3$ may be given as $\pm v \pm u^2 + \mu u$ and $\pm v + u^3 + \lambda_2 u^2 + \lambda_1 u$, respectively, where $\mu, \lambda_1, \lambda_2 \in \mathbb{R}$.

Finally, consider the 1-jet $g_0 = w$. Then, g is equivalent to a germ of the form $\widehat{h} = w + v\widehat{h}_1(u) + \widehat{h}_2(u)$, where $\widehat{h}_1 \in \mathcal{M}_u$ and $\widehat{h}_2 \in \mathcal{M}_u^2$. Let $\widehat{h}_1 = d_1 u + \widehat{h}_3(u)$, $\widehat{h}_3 \in \mathcal{M}_u^2$, $0 \neq d_1 \in \mathbb{R}$ and $\widehat{h}_2 = c_2 u^2 + \widehat{h}_4(u)$, $\widehat{h}_4 \in \mathcal{M}_u^3$, $0 \neq c_2 \in \mathbb{R}$. Consider the principal part $h_0 = w + d_1 uv + c_2 u^2$. Using ϕ_1 and ϕ_4, one can reduce g_0 to $\widetilde{g}_0 = w + uv + au^2$, where $0 \neq a \in \mathbb{R}$. The $\mathcal{R}(\widetilde{X})$-tangent space to the orbit of $\widetilde{g}_0$ at $\widetilde{g}_0$ is the set

$$\big\{(2uv + 3w)P + (2uw + 3v^2)Q + (uv + 2au^2)A_1 + (v^3 - w^2)(v + 2au)A_2 \\ : A_1, A_2, P, Q \in \mathcal{E}_z\big\}.$$

Hence, we have

$$\mathcal{E}_z / L\mathcal{R}(\widetilde{X}).\widetilde{g}_0 \simeq \big\{a_0 + a_1 u + a_2 v + a_3 u^2 : a_i \in \mathbb{R}\big\}.$$

By Lemma 3.4, the germ $\widehat{h}$ is equivalent to $w + uv + au^2$, where $0 \neq a \in \mathbb{R}$ is a modulus. In this case, the codimension of $\widehat{h}$ is 2 and an $\mathcal{R}^+(\widetilde{X})$-versal deformation may be taken as $w + uv + au^2 + \lambda_2 u + \lambda_1 v$. If $d_1 = 0$ or $c_2 = 0$, then $\widehat{h}$ has codimension greater than 2. This finishes the proof of the theorem. ■

4. Height functions and contact with planes

Recall that we have two distinguished tangential directions on $\widetilde{M} = (M, S)$ at the origin. The first one is the standard tangential direction $\mathcal{T}_M$ along the singular points Σ of M. The other one is the limiting tangential direction $\mathcal{T}_S$ at a cusp point of S.

Let S^2 be the two sphere. Then the family of height functions $H : \widetilde{M} \times S^2 \to \mathbb{R}$ on $\widetilde{M}$ is defined by $H((x, y), \mathbf{v}) = H_{\mathbf{v}}(x, y) = \phi(x, y) \cdot \mathbf{v}$, where $\cdot$ is the scalar product. Following the approaches discussed in [20], the contact of $\widetilde{M}$ along $\mathcal{T}_M$ or $\mathcal{T}_S$ at p with planes $\pi_{\mathbf{v}}$ through p and orthogonal to the direction $\mathbf{v}$ is measured by the singularities of the height functions $H_{\mathbf{v}}$ on $\widetilde{M}$. The contact of $\widetilde{M}$ with $\pi_{\mathbf{v}}$ is described by that of the fibre $g = 0$ with model $\widetilde{X}$, where g as in Theorem 3.7. Moreover, $H_{\mathbf{v}}$ can have singularities of $\mathcal{R}_e^+$-codimension ≤ 2 (of the stratum) at any point $p \in \widetilde{M}$ and for any $\mathbf{v} \in S^2$, due to transversality constraints (see the Appendix in [7]).

We shall take M and S as in (2.1) and (2.2), respectively. If we set $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ then

$$H_{\mathbf{v}}(x, y) = x v_1 + \Big(a(x) + \frac{1}{2} y^2\Big) v_2 + (b_1(x) + y^2 b_2(x) + y^3 b_3(x, y)) v_3.$$

In particular, the restrictions of $H_{\mathbf{v}}$ to the singular set Σ and S are given by

$$H_{\mathbf{v}}(x,0) = \frac{1}{2}(a_{20}v_2 + b_{20}v_3)x^2 + \frac{1}{6}(a_{30}v_2 + b_{30}v_3)x^3 + \frac{1}{24}(a_{40}v_2 + b_{40}v_3)x^4 + O(5),$$

and

$$H_{\mathbf{v}}(0,y) = \frac{1}{2}y^2 v_2 + \Big(\frac{1}{6}b_{03}y^3 + \frac{1}{24}b_{04}y^4 + \frac{1}{120}b_{05}y^5\Big)v_3 + O(6),$$

respectively.

The precise description of the nature of the contact between $\widetilde{M}$ and $\pi_{\mathbf{v}}$ is provided below.

If $\pi_{\mathbf{v}}$ is transverse to both $\mathcal{T}_M$ and $\mathcal{T}_S$ then the contact of $\widetilde{M}$ with $\pi_{\mathbf{v}}$ at the origin is the same as that of the zero fibre of $g = u \pm v$ with the model $\widetilde{X}$. On the other hand, if $\pi_{\mathbf{v}}$ contains both $\mathcal{T}_M$ and $\mathcal{T}_S$ then the contact is described by that of the fibre $g = w + uv + au^2$, and the height function $H_{\mathbf{v}}(x, y)$ has singularities of type A_2.

Proposition 4.1. *The height function $H_{\mathbf{v}}(0, y)$ along the singular curve S can have singularities of types $A_i^{\pm}$, $i = 1, 2, 3$, which are modeled by the contact of the zero fibre of the submersions $g = u \pm v$, $g = u + w$ and $g = u \pm v^2$, respectively, with the model $\widetilde{X}$. The geometrical meaning can be described as follows:*

$A_1^{\pm}$: $\pi_{\mathbf{v}}$ *is not a tangent plane of S;*
$A_2^{\pm}$: $\pi_{\mathbf{v}}$ *is not the osculating plane of S and* $\kappa_{\text{sing}}(S)(= \kappa_c) \neq 0$;
$A_3^{\pm}$: $\pi_{\mathbf{v}}$ *is the osculating plane of S and* $\kappa_{\text{sing}}(S) = 0$, $\kappa'_{\text{sing}}(S) \neq 0$.

Proof. In all cases, the height function along S is singular. It exhibits singularities of type $A_1^{\pm}$ when $v_2 \neq 0$. This condition occurs when $\pi_{\mathbf{v}}$ does not include $\mathcal{T}_S$ and has been indicated by $g = u \pm v$ earlier.

Assume $\pi_{\mathbf{v}}$ contains $\mathcal{T}_S$ at the origin. The plane $\pi_{\mathbf{v}}$ is transversal to $\mathcal{T}_M$ when $v_1 \neq 0$, and hence it is not the tangent cone. Then the contact of $\pi_{\mathbf{v}}$ with $\widetilde{M}$ is described by the contact of the zero fibre of $g = u + w$ and $g = u \pm v^2$, with the model $\widetilde{X}$.

For the height function $H_{\mathbf{v}}(0, y)$ along S to have $A_2^{\pm}$-singularities, we must have

$$v_2 = 0 \quad \text{and} \quad b_{03}v_3 \neq 0.$$

The second constraint implies that the cuspidal curvature of S at the origin $\kappa_{\text{sing}}(S)$ $(= \kappa_c) = |b_{03}| \neq 0$. The contact of a germ $\widetilde{M}$ with $\pi_{\mathbf{v}}$ is the same as that of the zero fibre $g = u + w$.

The function $H_{\mathbf{v}}(0, y)$ has an $A_3^{\pm}$-singularity if

$$v_2 = 0, \quad b_{03} = 0 \quad \text{and} \quad b_{04}v_3 \neq 0.$$

This means that $\kappa_{\text{sing}}(S) = 0$ but $\kappa'_{\text{sing}}(S) = |b_{04}| \neq 0$ at the origin. The contact of a germ $\widetilde{M}$ with $\pi_{\mathbf{v}}$ is the same as that of the zero fibre $g = u \pm v^2$. ■

Proposition 4.2. *The height function $H_{\mathbf{v}}(x, 0)$ along the cuspidal edge Σ can have singularities of types $A_k^{\pm}$-singularity, $k = 1, 2$ which are modeled by the contact of the zero fibre of the submersions $g = \pm v \pm u^{k+1}$, $k = 1, 2$, respectively, with the model $\widetilde{X}$. The geometrical meaning can be described as follows:*

$$A_1^{\pm} : \pi_{\mathbf{v}} \text{ is not the osculating plane of } \Sigma;$$
$$A_2^{\pm} : \pi_{\mathbf{v}} \text{ is the osculating plane of } \Sigma \text{ and } \tau_{\Sigma}(0) \neq 0$$

Proof. Note that the function $H_{\mathbf{v}}(x, 0)$ is singular at the origin if and only if $v_1 = 0$. Let $\pi_{\mathbf{v}}$ be a member of a pencil of planes that contains $\mathcal{T}_M$ at the origin. Then the contact of a generic $\widetilde{M}$ along the singular set Σ with $\pi_{\mathbf{v}}$ is the same as that of the zero fibre of $g = \pm v \pm u^{k+1}$, $k = 1, 2$, with the model $\widetilde{X}$. In particular, $H_{\mathbf{v}}(x, 0)$ has an $A_1^{\pm}$-singularity when

$$v_2 a_{20} + v_3 b_{20} \neq 0.$$

This means that the plane $\pi_{\mathbf{v}}$ is not the osculating plane of Σ (i.e., the limiting normal curvature from [14] $\kappa_n(0) = b_{20} \neq 0$). The contact is described by the fibre of $g = \pm v \pm u^2$. On the other hand, $H_{\mathbf{v}}(x, 0)$ has an A_2-singularity when

$$v_2 a_{20} + v_3 b_{20} = 0 \quad \text{and} \quad a_{20} b_{30} - a_{30} b_{20} \neq 0.$$

This implies that $\pi_{\mathbf{v}}$ is the osculating plane of Σ and $\tau_{\Sigma}(0) \neq 0$. It is described by the fibre $g = \pm v \pm u^3$. ■

5. The geometry of functions on a cuspidal edge equipped with a singular curve and their dual

Consider the model pair $\widetilde{X}$. For a deformation $F : (\mathbb{R}^3 \times \mathbb{R}^2, 0) \to \mathbb{R}; (z, \lambda) \mapsto F(z, \lambda)$ of a germ $g \in \mathcal{E}_z$ on $\widetilde{X}$, we proceed to study the family $G(x, y, \lambda) = F(f(x, y), \lambda)$, and define the discriminant of the family G as the set

$$\mathcal{D}_1(F) = \left\{(\lambda, G(x, y, \lambda)) \in \mathbb{R}^2 \times \mathbb{R} : \frac{\partial G}{\partial x} = \frac{\partial G}{\partial y} = 0 \text{ at } (x, y, \lambda)\right\};$$

as well as the discriminant of the family G restricted to the singular set Σ and the singular curve B as the sets

$$\mathcal{D}_2(F) = \left\{(\lambda, G(x, 0, \lambda)) \in \mathbb{R}^2 \times \mathbb{R} : \frac{\partial G}{\partial x} = 0 \text{ at } (x, 0, \lambda)\right\}$$

and

$$\mathcal{D}_3(F) = \left\{(\lambda, G(0, y, \lambda)) \in \mathbb{R}^2 \times \mathbb{R} : \frac{\partial G}{\partial y} = 0 \text{ at } (0, y, \lambda)\right\},$$

respectively.

It can be shown that if F_1 and F_2 are P-$\mathcal{R}^+(\widetilde{X})$-equivalent deformations then the sets $\mathcal{D}_i(F_1)$ and $\mathcal{D}_i(F_2)$ are diffeomorphic for all $i \in \{1, 2, 3\}$. It therefore suffices to calculate the sets $\mathcal{D}_i(F)$ for the versal deformations $F(z, \lambda)$ of the germs $F(z, 0) = g$ listed in Table 1.

- For the versal deformation $F(z, \lambda) = u \pm v$, the fibre $g = 0$ is a plane transverse to the tangent cone of X, $\mathcal{T}_X$ and $\mathcal{T}_B$ at the origin. The sets $\mathcal{D}_1(F)$ and $\mathcal{D}_2(F)$ are empty. However, the set $\mathcal{D}_3(F)$ is a plane, parametrised by $(\lambda_1, \lambda_2) \mapsto (\lambda_1, \lambda_2, 0)$.
- For the versal deformation $F(z, \lambda) = u + w + \lambda_1 v$, the fibre $g = 0$ is a plane which is tangent to $\mathcal{T}_B$ and transverses both to the tangent cone of X and to $\mathcal{T}_X$ at the origin. The singularity of $g(f(0, y)) = y^3$ determines the contact between $\mathcal{T}_B$ and the fibre $g = 0$, which is classified as type A_2. We have $G(x, y, \lambda) = x + y^3 + \lambda_1 y^2$. Hence, the sets $\mathcal{D}_1(F)$ and $\mathcal{D}_2(F)$ are both empty. The set $\mathcal{D}_3(F)$ is a union of the plane $(\lambda_1, \lambda_2) \mapsto (\lambda_1, \lambda_2, 0)$, and a smooth surface parametrised by $(y, \lambda_2) \mapsto (\frac{-3}{2} y, \lambda_2, \frac{-1}{2} y^3)$ (Figure 1).
- For the versal deformation $F(z, \lambda) = u \pm v^2 + \lambda_2 w + \lambda_1 v$, the fibre $g = 0$ is a parabolic cylinder which is tangent to $\mathcal{T}_B$ and transverse to both the tangent cone of X and $\mathcal{T}_X$. The contact between $\mathcal{T}_B$ and the fibre $g = 0$ is established by the singularity of $g(f(0, y)) = \pm y^4$ and hence of type $A_3^{\pm}$. We have $G(x, y, \lambda) = x \pm y^4 + \lambda_2 y^3 + \lambda_1 y^2$. Therefore, the sets $\mathcal{D}_1(F)$ and $\mathcal{D}_2(F)$ are both empty. On the singular curve B we have $G(0, y, \lambda) = \pm y^4 + \lambda_2 y^3 + \lambda_1 y^2$, and so $\frac{\partial G}{\partial y} = 0$ when $y = 0$ or $\lambda_1 = \mp 2y^2 - \frac{3}{2}\lambda_2 y$. Thus, the set $\mathcal{D}_3(F)$ is a union of the plane $(\lambda_1, \lambda_2) \mapsto (\lambda_1, \lambda_2, 0)$, and a surface parametrised by $(y, \lambda_2) \mapsto (\mp 2y^2 - \frac{3}{2}\lambda_2 y, \lambda_2, \mp y^4 - \frac{1}{2}\lambda_2 y^3)$ which is cuspidal cross-cap (Figure 2).

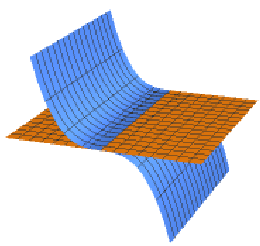

Figure 1. The set $\mathcal{D}_3(F)$ of the versal deformation $F(z, \lambda) = u + w + \lambda_1 v$.

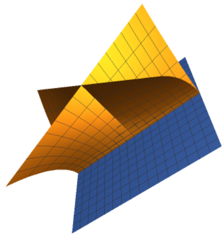

Figure 2. The set $\mathcal{D}_3(F)$ of the versal deformation $F(z,\lambda) = u \pm v^2 + \lambda_2 w + \lambda_1 v$.

- For the versal deformations $F(z,\lambda) = \pm v \pm u^2 + \lambda_1 u$ or $F(z,\lambda) = \pm v \pm u^3 + \lambda_2 u^2 + \lambda_1 u$, both fibres $g = \pm v \pm u^k$, $k = 2, 3$ are tangent to $\mathcal{T}_X$ and transverse $\mathcal{T}_B$ and the tangent cone. The singularities of $g(f(x,0)) = \pm x^k$ measure the contact of $\mathcal{T}_X$ with the fibre $g = 0$, and therefore are classified as being of type $A^{\pm}_{k-1}$, $k = 2, 3$.
  (1) When $k = 2$, the set $\mathcal{D}_1(F)$ coincides with $\mathcal{D}_2(F)$ and is parametrised by $(x, \lambda_2) \mapsto (\mp 2x, \lambda_2, \mp x^2)$, which is a regular surface. The set $\mathcal{D}_3(F)$ is the plane $(\lambda_1, \lambda_2) \mapsto (\lambda_1, \lambda_2, 0)$ (Figure 3).
  (2) When $k = 3$, the set $\mathcal{D}_1(F)$ coincides with $\mathcal{D}_2(F)$ and is parametrised by $(x, \lambda_2) \mapsto (\mp 3x^2 - 2\lambda_2 x, \lambda_2, \mp 2x^3 - \lambda_2 x^2)$, which is a cuspidal edge. The set $\mathcal{D}_3(F)$ is the plane $(\lambda_1, \lambda_2) \mapsto (\lambda_1, \lambda_2, 0)$ (Figure 4).
- For the versal deformation $F(z,\lambda) = w + uv + au^2 + \lambda_2 u + \lambda_1 v$, the tangent plane to the fibre $g = 0$, which is a hyperbolic paraboloid, coincides with the tangent cone and contains both $\mathcal{T}_X$ and $\mathcal{T}_B$. Since $g(f(x,y)) = y^3 + xy^2 + ax^2$, $a \neq 0$, the contact of the fibre $g = 0$ with $\mathcal{T}_X$ and $\mathcal{T}_B$ is of type A_2.
  We have $G(x,y,\lambda) = y^3 + xy^2 + ax^2 + \lambda_2 x + \lambda_1 y^2$. Therefore, $\frac{\partial G}{\partial x} = 0$ when $\lambda_2 = -y^2 - 2ax$. On the other hand, $\frac{\partial G}{\partial y} = 0$ when either $y = 0$ or $\lambda_1 = -\frac{3}{2}y - x$.

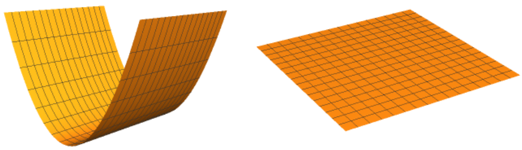

Figure 3. The set $\mathcal{D}_i(F)$ of the versal deformation $F(z,\lambda) = \pm v \pm u^2 + \lambda_1 u$. In the left figure are the sets $\mathcal{D}_1(F)$ and $\mathcal{D}_2(F)$, and the right figure is $\mathcal{D}_3(F)$.

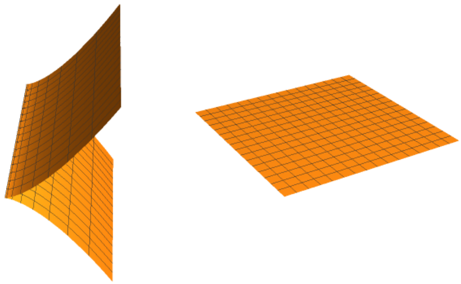

Figure 4. The set $\mathcal{D}_i(F)$ of the versal deformation $F(z,\lambda) = \pm v \pm u^3 + \lambda_2 u^2 + \lambda_1 u$. In the left figure are the sets $\mathcal{D}_1(F)$ and $\mathcal{D}_2(F)$, and the right figure is $\mathcal{D}_3(F)$.

Hence, the set $\mathcal{D}_1(F)$ is a union of two intersecting regular surfaces parametrised by

$$(x, \lambda_1) \mapsto (\lambda_1, -2ax, -ax^2)$$

and

$$(x, y) \mapsto \left(-\frac{3}{2}y - x, -y^2 - 2ax, -\frac{1}{2}y^3 - xy^2 - ax^2\right).$$

The set $\mathcal{D}_2(F)$ is a regular surface parametrised by $(x, \lambda_1) \mapsto (\lambda_1, -2ax, -ax^2)$, which is exactly one of the components of $\mathcal{D}_1(F)$. The set $\mathcal{D}_3(F)$ is a union of a regular surface parametrised by $(\lambda_2, y) \mapsto (-\frac{3}{2}y, \lambda_2, -\frac{1}{2}y^3)$ and a plane tangent to it at the origin (Figure 5).

The discriminants may be used for the examination of the dual of the cuspidal edge equipped with a singular curve as explained below.

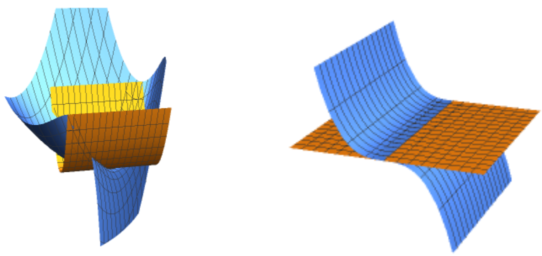

Figure 5. The set $\mathcal{D}_i(F)$ of the versal deformation $F(z,\lambda) = \pm v \pm u^3 + \lambda_2 u^2 + \lambda_1 u$. In the left figure are the sets $\mathcal{D}_1(F)$ and $\mathcal{D}_2(F)$, and the right figure is $\mathcal{D}_3(F)$.

We have the sets

$$\mathcal{D}_1(H) = \left\{(\mathbf{v}, H_{\mathbf{v}}(x, y)) \in S^2 \times \mathbb{R} : \frac{\partial H}{\partial x} = \frac{\partial H}{\partial y} = 0 \text{ at } (x, y, \mathbf{v})\right\},$$

$$\mathcal{D}_2(H) = \left\{(\mathbf{v}, H_{\mathbf{v}}(x, 0)) \in S^2 \times \mathbb{R} : \frac{\partial H}{\partial x} = 0 \text{ at } (x, 0, \mathbf{v})\right\}$$

and

$$\mathcal{D}_3(H) = \left\{(\mathbf{v}, H_{\mathbf{v}}(0, y)) \in S^2 \times \mathbb{R} : \frac{\partial H}{\partial y} = 0 \text{ at } (0, y, \mathbf{v})\right\}.$$

- If $\pi_{\mathbf{v}}$ is transverse to $\mathcal{T}_S$ and contains $\mathcal{T}_M$ but is not the tangent cone, then the set $\mathcal{D}_1(H)$ coincides with the set $\mathcal{D}_2(H)$ and describes locally the dual of the curve Σ.
- When $\pi_{\mathbf{v}}$ is the tangent cone containing both $\mathcal{T}_S$ and $\mathcal{T}_M$, then the set $\mathcal{D}_1(H)$ consists of two components. One of them is $\mathcal{D}_2(H)$ (the dual of Σ) and the other is the proper dual of M which is the surface consisting of the tangent planes to M away from points on Σ together with their limits at points on Σ, i.e., the tangent cones at points in Σ. The set $\mathcal{D}_3(H)$ also consists of two components. One of them is the dual of the singular point of S and the other is the proper dual of S.
- If $\pi_{\mathbf{v}}$ is a member of the pencil containing $\mathcal{T}_S$ but it is transverse to $\mathcal{T}_M$ and the tangent cone of M, then $\mathcal{D}_1(H)$ and $\mathcal{D}_2(H)$ are empty. However, the set $\mathcal{D}_3(H)$ consists of two components. One of them is the dual of the singular point of S and the other is the proper dual of S.

If the contact of $\widetilde{M}$ with $\pi_{\mathbf{v}}$ is described by that of the fibre $g = 0$ with the model $\widetilde{X}$, with g as in Theorem 3.7, then $\mathcal{D}_i(H)$ is diffeomorphic to $\mathcal{D}_i(F)$, for $i = 1, 2, 3$, where F is an $\mathcal{R}^+(\widetilde{X})$-versal deformation of g.

Proposition 5.1. *Let $\widetilde{M}$ be a pair of a generic cuspidal edge in $\mathbb{R}^3$ equipped with a singular curve given by $u = 0$. Then, the calculations and figures in Section 5 give the models, up to diffeomorphisms, of $\mathcal{D}_i(H)$.*

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