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Cyclic cohomology of entwining structures

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Abstract. In this paper, we introduce and study a cyclic cohomology theory $H_\lambda^\bullet(A, C, \psi)$ for an entwining structure (A, C, ψ) over a field k. We then provide a complete description of the cocycles and the coboundaries in this theory using entwined traces applied to dg-entwining structures over (A, C, ψ). We then apply these descriptions to construct a pairing $H_\lambda^m(A, C, \psi) \otimes H_\lambda^n(A', C', \psi') \to H_\lambda^{m+n}(A \otimes A', C \otimes C', \psi \otimes \psi')$, where (A, C, ψ) and (A', C', ψ') are entwining structures.

1. Introduction

An entwining structure, as introduced by Brzeziński and Majid [8], consists of an algebra A, a coalgebra C and a map $\psi : C \otimes A \to A \otimes C$ satisfying certain conditions. Together, an entwining structure (A, C, ψ) behaves like a bialgebra or more generally, a comodule algebra over a bialgebra, as pointed out by Brzeziński [5]. There is also a well-developed theory of modules over entwining structures, with applications to diverse objects such as Doi–Hopf modules, Yetter–Drinfeld modules and coalgebra Galois extensions (see, for instance, [1, 3, 4, 6, 7, 9–11, 14, 16]).

In [5], Brzeziński introduced the Hochschild complex $\mathcal{C}^\bullet(A, C, \psi)$ of an entwining structure and proceeded to construct Gerstenhaber-like structures on the cohomology groups. The starting point of this paper was to find a corresponding cyclic cohomology theory $H_\lambda^\bullet(A, C, \psi)$ for an entwining structure. We then study the cocycles and coboundaries in this theory using differential graded algebras in a manner similar to Connes [12, 13].

Similar to the classical approach of Connes [13], we take our "cyclic complex" $\mathcal{C}_\lambda^\bullet(A, C, \psi)$ to be a certain subcomplex of the Hochschild complex $\mathcal{C}^\bullet(A, C, \psi)$ of Brzeziński [5]. In [13], Connes showed that the cocycles and coboundaries in the cyclic cohomology of an algebra A can be described using traces on differential graded algebras over A. Accordingly, we show that the cocycles $Z_\lambda^\bullet(A, C, \psi)$ in our theory can be expressed as characters of "entwined traces" applied to dg-entwining structures over (A, C, ψ). We also obtain a description of the coboundaries $B_\lambda^\bullet(A, C, \psi)$ in terms of

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characters of "vanishing cycles" over (A, C, ψ). These descriptions are then applied to construct a pairing

$$H_\lambda^m(A, C, \psi) \otimes H_\lambda^n(A', C', \psi') \longrightarrow H_\lambda^{m+n}(A \otimes A', C \otimes C', \psi \otimes \psi')$$

on cyclic cohomology groups. We mention here that we have previously studied in [2] a modified version of the Hochschild theory of Brzeziński [5] for entwining structures. In the future, we hope to study further the cohomology groups of entwining structures, on the lines of the usual cohomology theories for rings.

We now describe the paper in more detail. For an element $c \otimes a \in C \otimes A$, we will always suppress the summation and write $\psi(c \otimes a) = a_\psi \otimes c^\psi \in A \otimes C$. We begin in Section 2 by introducing the cyclic complex $\mathcal{C}_\lambda^\bullet(A, C, \psi)$ of an entwining structure (A, C, ψ). As a vector space, $\mathcal{C}_\lambda^n(A, C, \psi)$ consists of all k-linear maps $g : C \otimes A^{\otimes n+1} \to k$ such that

$$g(c \otimes a_1 \otimes \cdots \otimes a_{n+1}) = (-1)^n g(c^\psi \otimes a_2 \otimes a_3 \otimes \cdots \otimes a_{n+1} \otimes a_{1\psi}) \tag{1.1}$$

for $c \in C$ and $a_1, \ldots, a_{n+1} \in A$. We show (see Theorem 2.2) that $\mathcal{C}_\lambda^\bullet(A, C, \psi)$ is a subcomplex of the Hochschild complex of Brzeziński [5]. We denote by $H_\lambda^\bullet(A, C, \psi)$ the cohomology groups of $\mathcal{C}_\lambda^\bullet(A, C, \psi)$.

In Section 3, we consider dg-entwining structures over (A, C, ψ). A dg-entwining structure $((R^\bullet, D^\bullet), C, \Psi^\bullet)$ consists of a (not necessarily unital) dg-algebra $(R^\bullet, D^\bullet)$ and an entwining $\Psi^\bullet : C \otimes R^\bullet \to R^\bullet \otimes C$ that is a morphism of complexes. Along with an algebra morphism $\rho : A \to R^0$ that is compatible with the respective entwinings $\psi : C \otimes A \to A \otimes C$ and $\Psi^0 : C \otimes R^0 \to R^0 \otimes C$, we say that $((R^\bullet, D^\bullet), C, \Psi^\bullet)$ is a dg-entwining over (A, C, ψ). In particular, we show that $\psi : C \otimes A \to A \otimes C$ may be extended to produce a dg-entwining structure $((\Omega^\bullet A, d^\bullet), C, \widehat{\psi})$ over (A, C, ψ), where $(\Omega^\bullet A, d^\bullet)$ is the universal differential graded algebra associated to A. Further, we show (see Theorem 3.4) that $((\Omega^\bullet A, d^\bullet), C, \widehat{\psi})$ is universal among dg-entwining structures over (A, C, ψ).

Suppose $((R^\bullet, D^\bullet), C, \Psi^\bullet)$ is a dg-entwining structure over (A, C, ψ). By an n-dimensional closed graded entwined trace on $((R^\bullet, D^\bullet), C, \Psi^\bullet)$, we will mean a linear map $T : C \otimes R^n \to k$ which satisfies

$$T(c \otimes D(r)) = 0 \quad T(c \otimes r'r'') = (-1)^{ij} T(c^\Psi \otimes r'' r'_\Psi) \tag{1.2}$$

for all $c \in C$, $r \in R^{n-1}$ and $r' \in R^i$, $r'' \in R^j$ such that $i + j = n$. Together, the datum $((R^\bullet, D^\bullet), C, \Psi^\bullet, \rho, T)$ will be referred to as an n-dimensional entwined cycle over (A, C, ψ). In Theorem 4.5, we show that each cyclic cocycle $g \in Z_\lambda^n(A, C, \psi)$ may be expressed as the character of an n-dimensional entwined cycle over (A, C, ψ).

Let $M_r(A)$ be the ring of $(r \times r)$-matrices with entries in A. Then, $\psi : C \otimes A \to A \otimes C$ extends in an obvious manner to an entwining

$$C \otimes M_r(A) = C \otimes (A \otimes M_r(k)) \longrightarrow (A \otimes M_r(k)) \otimes C = M_r(A) \otimes C \tag{1.3}$$

that we continue to denote by ψ. In Section 5, we show Morita invariance for Hochschild cohomology groups $HH^\bullet(A,C,\psi)$ of matrix rings. For this, we show that the morphisms on the Hochschild complex induced by the inclusion $\mathrm{inc}_1 : A \to M_r(A)$ and the generalized trace $\mathrm{tr} : M_r(A)^{\otimes n+1} \to A^{\otimes n+1}$, $n \geq 0$ are homotopy inverses of each other. It follows (see Proposition 5.5) that we have mutually inverse isomorphisms

$$\begin{aligned} \mathrm{inc}_1^\bullet : HH^\bullet(M_r(A),C,\psi) &\longrightarrow HH^\bullet(A,C,\psi) \\ \mathrm{tr}^\bullet : HH^\bullet(A,C,\psi) &\longrightarrow HH^\bullet(M_r(A),C,\psi) \end{aligned} \tag{1.4}$$

of Hochschild cohomology groups.

The Morita invariance for cyclic cohomology groups $H_\lambda^\bullet(A,C,\psi)$ of matrix rings is shown in Section 6. For this, we consider the subspace $\mathcal{J}^n(A,C,\psi) \subseteq \mathcal{C}^n(A,C,\psi)$ consisting of maps $g : C \otimes A^{\otimes n+1} \to k$ satisfying

$$g(c \otimes a_1 \otimes \cdots \otimes a_{n+1}) = g(c^{\psi^{n+1}} \otimes a_{1\psi} \otimes a_{2\psi} \otimes \cdots \otimes a_{n+1\psi}) \tag{1.5}$$

for $c \in C$ and $a_1, \ldots, a_{n+1} \in A$. We check that $\mathcal{J}^\bullet(A,C,\psi)$ is a subcomplex of $\mathcal{C}^\bullet(A,C,\psi)$ and that there are induced maps $\mathrm{inc}_1^\bullet : \mathcal{J}^\bullet(M_r(A),C,\psi) \to \mathcal{J}^\bullet(A,C,\psi)$ and $\mathrm{tr}^\bullet : \mathcal{J}^\bullet(A,C,\psi) \to \mathcal{J}^\bullet(M_r(A),C,\psi)$ which are homotopy inverses of each other. We also show that $\mathcal{J}^\bullet(A,C,\psi)$ is a cocyclic module such that $\mathcal{C}_\lambda^\bullet(A,C,\psi)$ is the subspace invariant under the action of the cyclic operator on $\mathcal{J}^\bullet(A,C,\psi)$. It follows (see Theorem 6.4) that we have mutually inverse isomorphisms

$$\begin{aligned} \mathrm{inc}_1^\bullet : H_\lambda^\bullet(M_r(A),C,\psi) &\longrightarrow H_\lambda^\bullet(A,C,\psi) \\ \mathrm{tr}^\bullet : H_\lambda^\bullet(A,C,\psi) &\longrightarrow H_\lambda^\bullet(M_r(A),C,\psi) \end{aligned} \tag{1.6}$$

of cyclic cohomology groups.

The main purpose of Section 7 is to obtain a description for the space $B_\lambda^\bullet(A,C,\psi)$ of coboundaries in $\mathcal{C}_\lambda^\bullet(A,C,\psi)$. We consider the group $\mathbb{U}(A)$ of units of A and take the subcollection

$$\mathbb{U}_\psi(A) := \{x \in \mathbb{U}(A) \mid \psi(c \otimes x) = x \otimes c \text{ for every } c \in C\}. \tag{1.7}$$

We verify that $\mathbb{U}_\psi(A)$ is a subgroup of $\mathbb{U}(A)$. We also show that conjugation by an element $x \in \mathbb{U}_\psi(A)$ induces the identity map on cyclic cohomology groups $H_\lambda^\bullet(A,C,\psi)$. Using the Morita invariance established in Section 6, we now obtain a set of sufficient conditions for the cyclic cohomology of an entwining structure to be zero. Accordingly (see Definition 7.9), an n-dimensional entwined cycle $((R^\bullet, D^\bullet), C, \Psi^\bullet, \rho, T)$ is said to be vanishing if (R^0, C, Ψ^0) satisfies these conditions.

We now take $k = \mathbb{C}$. In Theorem 7.10, we show that a cocycle $g \in Z_\lambda^n(A,C,\psi)$ is a coboundary if and only if it is the character of an n-dimensional entwined vanishing cycle over (A,C,ψ). In particular, the entwined vanishing cycle corresponding to a coboundary $g \in B_\lambda^\bullet(A,C,\psi)$ is constructed with the help of a certain algebra $\mathbf{C}$ of infinite matrices with complex entries used in [13]. Taken together, Theorems 4.5 and 7.10 provide a complete description of the cocycles and the coboundaries in the cyclic theory of entwined structures, developed in a manner similar to Connes [13]. Our final result is Theorem 7.11, where we apply these descriptions to construct a pairing

$$H^m_\lambda(A, C, \psi) \otimes H^n_\lambda(A', C', \psi') \longrightarrow H^{m+n}_\lambda(A \otimes A', C \otimes C', \psi \otimes \psi') \qquad m, n \geq 0, \tag{1.8}$$

where (A, C, ψ) and (A', C', ψ') are entwining structures.

2. Cyclic cohomology of an entwining structure

Let k be a field. Throughout this section and the rest of this paper, we let A be a unital algebra over k and let C be a counital coalgebra over k. The product on A will be denoted by $\theta : A \otimes A \to A$. The coproduct $\Delta : C \to C \otimes C$ will always be expressed using Sweedler notation $\Delta(c) = c_1 \otimes c_2$ for any $c \in C$. The counit on C will be denoted by $\varepsilon : C \to k$. For the sake of convenience, we will denote the tensor powers $A^{\otimes n}$ of the algebra A simply by A^n. Similarly, an element of $C \otimes A^{\otimes n}$ will be denoted simply by $(c, a_1, \dots, a_n)$. We now recall the notion of an entwining structure, introduced by Brzeziński and Majid in [8].

Definition 2.1. Let k be a field. An entwining structure (A, C, ψ) over k consists of a unital k-algebra A, a counital k-coalgebra C and a k-linear map $\psi : C \otimes A \to A \otimes C$ satisfying the following conditions:

$$\begin{aligned} \psi(c \otimes \theta(a \otimes b)) &= \psi(c \otimes ab) = (ab)_\psi \otimes c^\psi = a_\psi b_\psi \otimes c^{\psi\psi} \\ &= ((\theta \otimes \mathrm{id}_C) \circ (\mathrm{id}_A \otimes \psi) \circ (\psi \otimes \mathrm{id}_A))(c \otimes a \otimes b) \\ (\mathrm{id}_A \otimes \Delta)(\psi(c \otimes a)) &= a_\psi \otimes \Delta(c^\psi) = a_{\psi_\psi} \otimes c_1^\psi \otimes c_2^\psi \\ &= ((\psi \otimes \mathrm{id}_C) \circ (\mathrm{id}_C \otimes \psi))(\Delta(c) \otimes a) \\ a_\psi \varepsilon(c^\psi) &= \varepsilon(c)a \quad 1_\psi \otimes c^\psi = 1 \otimes c. \end{aligned} \tag{2.1}$$

Here, the summation has been suppressed by writing $\psi(c \otimes a) = a_\psi \otimes c^\psi$ for any $c \in C$ and $a \in A$.

In this paper, if A' is a non-unital algebra, we will still say that $(A', C, \psi : C \otimes A' \to A' \otimes C)$ is an entwining structure if it satisfies all the conditions in (2.1) except for the last condition $1_\psi \otimes c^\psi = 1 \otimes c$.

Given an entwining structure (A, C, ψ), Brzeziński [5] introduced the Hochschild complex $\mathcal{C}^\bullet((A, C, \psi); M)$ of (A, C, ψ) with coefficients in an A-bimodule M:

$$\begin{aligned} \mathcal{C}^n((A, C, \psi); M) &= \mathrm{Hom}(C \otimes A^n, M) \\ \delta^n : \mathrm{Hom}(C \otimes A^n, M) &\longrightarrow \mathrm{Hom}(C \otimes A^{n+1}, M) \\ \delta^n(f)(c, a_1, \ldots, a_{n+1}) &= a_{1\psi} \cdot f(c^\psi, a_2, \ldots, a_{n+1}) \\ &\quad + \sum_{i=1}^{n} (-1)^i f(c, a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) \\ &\quad + (-1)^{n+1} f(c, a_1, \ldots, a_n) \cdot a_{n+1}. \end{aligned} \tag{2.2}$$

The cohomology of this complex will be denoted by $HH^\bullet((A, C, \psi); M)$. In particular, when $M = A^* = \mathrm{Hom}(A, k)$ is made into an A-bimodule as follows:

$$(a \cdot f \cdot a')(a'') := f(a'a''a) \quad f \in A^* = \mathrm{Hom}(A, k) \quad a, a', a'' \in A, \tag{2.3}$$

this complex will be denoted by $\mathcal{C}^\bullet(A, C, \psi)$, and its cohomology groups will be denoted by $HH^\bullet(A, C, \psi)$. It is immediate that an element $f \in \mathcal{C}^n(A, C, \psi) = \mathrm{Hom}(C \otimes A^n, A^*)$ may also be expressed as a linear map $g : C \otimes A^{n+1} \to k$ by setting

$$g(c, a_1, \ldots, a_{n+1}) = f(c, a_1, \ldots, a_n)(a_{n+1}). \tag{2.4}$$

We now define a subspace $\mathcal{C}^n_\lambda(A, C, \psi) \subseteq \mathcal{C}^n(A, C, \psi) = \mathrm{Hom}(C \otimes A^n, A^*)$ by taking the collection of all $f \in \mathrm{Hom}(C \otimes A^n, A^*)$ that satisfy

$$f(c, a_1, \ldots, a_n)(a_{n+1}) = (-1)^n f(c^\psi, a_2, \ldots, a_{n+1})(a_{1\psi}) \tag{2.5}$$

for every $(c, a_1, \ldots, a_{n+1}) \in C \otimes A^{n+1}$. Equivalently, using (2.4), the space $\mathcal{C}^n_\lambda(A, C, \psi)$ may also be described as the collection of all $g \in \mathrm{Hom}(C \otimes A^{n+1}, k)$ such that

$$g(c, a_1, \ldots, a_{n+1}) = (-1)^n g(c^\psi, a_2, a_3, \ldots, a_{n+1}, a_{1\psi}). \tag{2.6}$$

Theorem 2.2. *Let (A, C, ψ) be an entwining structure. Then, $(\mathcal{C}^\bullet_\lambda(A, C, \psi), \delta^\bullet)$ is a subcomplex of the Hochschild complex of (A, C, ψ).*

Proof. We consider $f \in \mathcal{C}^n_\lambda(A, C, \psi)$. We need to verify that $\delta^n(f) \in \mathcal{C}^{n+1}_\lambda(A, C, \psi)$, that is,

$$\delta^n(f)(c, a_1, \ldots, a_{n+1})(a_{n+2}) = (-1)^{n+1} \delta^n(f)(c^\psi, a_2, \ldots, a_{n+2})(a_{1\psi}). \tag{2.7}$$

Using the description of the differential in (2.2) and the A-bimodule structure of A^* described in (2.3), we see that

$$\begin{aligned}
&\delta^n(f)(c, a_1, \dots, a_{n+1})(a_{n+2}) \\
&\quad = f(c^\psi, a_2, \dots, a_{n+1})(a_{n+2}a_{1\psi}) \\
&\qquad + \sum_{i=1}^{n}(-1)^i f(c, a_1, \dots, a_i a_{i+1}, \dots, a_{n+1})(a_{n+2}) \\
&\qquad + (-1)^{n+1} f(c, a_1, \dots, a_n)(a_{n+1}a_{n+2}). \\
&\delta^n(f)(c^\psi, a_2, \dots, a_{n+2})(a_{1\psi}) \\
&\quad = f(c^{\psi\psi}, a_3, \dots, a_{n+2})(a_{1\psi}a_{2\psi}) \\
&\qquad + \sum_{i=2}^{n+1}(-1)^{i-1} f(c^\psi, a_2, \dots, a_i a_{i+1}, \dots, a_{n+2})(a_{1\psi}) \\
&\qquad + (-1)^{n+1} f(c^\psi, a_2, \dots, a_{n+1})(a_{n+2}a_{1\psi}).
\end{aligned} \tag{2.8}$$

Applying condition (2.5), we obtain

$$\begin{aligned}
&\sum_{i=2}^{n}(-1)^i f(c, a_1, \dots, a_i a_{i+1}, \dots, a_{n+1})(a_{n+2}) \\
&\qquad = (-1)^{n+1}\sum_{i=2}^{n}(-1)^{i-1} f(c^\psi, a_2, \dots, a_i a_{i+1}, \dots, a_{n+2})(a_{1\psi})
\end{aligned} \tag{2.9}$$

as well as

$$(-1)^{n+1} f(c, a_1, \dots, a_n)(a_{n+1}a_{n+2}) = (-1)^{n+1}(-1)^n f(c^\psi, a_2, \dots, a_{n+1}a_{n+2})(a_{1\psi}). \tag{2.10}$$

Finally, using (2.5) as well as the properties of an entwining structure in (2.1), we obtain

$$\begin{aligned}
-f(c, a_1a_2, \dots, a_{n+1})(a_{n+2}) &= (-1)^{n+1} f(c^\psi, a_3, \dots, a_{n+2})((a_1a_2)_\psi) \\
&= (-1)^{n+1} f(c^{\psi\psi}, a_3, \dots, a_{n+2})(a_{1\psi}a_{2\psi}).
\end{aligned} \tag{2.11}$$

The result is now clear from (2.8)–(2.11). ∎

Definition 2.3. Suppose (A, C, ψ) is an entwining structure. Then, we will say that $(\mathcal{C}^\bullet_\lambda(A, C, \psi), \delta^\bullet)$ is the cyclic complex of (A, C, ψ) and the cyclic cohomology groups will be denoted by $H^\bullet_\lambda(A, C, \psi)$. The cocycles and coboundaries in $(\mathcal{C}^\bullet_\lambda(A, C, \psi), \delta^\bullet)$ will be denoted, respectively, by $Z^\bullet_\lambda(A, C, \psi)$ and $B^\bullet_\lambda(A, C, \psi)$.

It is clear that the Hochschild complex $\mathcal{C}^\bullet(A, C, \psi)$ may be rewritten with terms $\mathrm{Hom}(C \otimes A^{\otimes n+1}, k)$. In that case, the Hochschild differential may be expressed as follows:

$$\begin{aligned}
\mathcal{C}^n(A, C, \psi) &= \mathrm{Hom}(C \otimes A^{n+1}, k) \\
\delta^n : \mathrm{Hom}(C \otimes A^{n+1}, k) &\longrightarrow \mathrm{Hom}(C \otimes A^{n+2}, k) \\
\delta^n(g)(c, a_1, \dots, a_{n+2}) &= g(c^\psi, a_2, \dots, a_{n+1}, a_{n+2}a_{1\psi}) \\
&\quad + \sum_{i=1}^{n+1} (-1)^i g(c, a_1, \dots, a_i a_{i+1}, \dots, a_{n+2}).
\end{aligned} \tag{2.12}$$

In the rest of this section, we will explain the intuition behind the complex $\mathcal{C}^\bullet_\lambda(A, C, \psi)$ in terms of traces and derivations. When $C = k$ and $\psi = \mathrm{id}$, we recover the complex $\mathcal{C}^\bullet_\lambda(A)$ of Connes [13] which computes the cyclic cohomology groups $H^\bullet_\lambda(A)$ of A. Let $Z^\bullet_\lambda(A)$ denote the cocycles in $\mathcal{C}^\bullet_\lambda(A)$. A trace on A consists of a k-linear map $s : A \to k$ satisfying $s(a_1 a_2) = s(a_2 a_1)$ for all $a_1, a_2 \in A$. It is easy to see that $Z^0_\lambda(A) = H^0_\lambda(A)$ is the space of all such traces on A.

Let M be an A-bimodule. Then, the counterpart of the notion of a trace is a k-linear map $t : M \to k$ satisfying $t(am) = t(ma)$ for all $a \in A$, $m \in M$. If (A, C, ψ) is an entwining structure, then $A \otimes C$ becomes an A-bimodule by setting

$$a_1 \cdot (a_2 \otimes c) := a_1 a_2 \otimes c \qquad (a_2 \otimes c) \cdot a_1 := a_2 a_{1\psi} \otimes c^\psi \tag{2.13}$$

for $a_1, a_2 \in A$, $c \in C$. Accordingly, a trace on the A-bimodule $A \otimes C$ consists of a linear map $t : A \otimes C \to k$ which satisfies

$$t(a_1 a_2 \otimes c) - t(a_2 a_{1\psi} \otimes c^\psi) = t(a_1 \cdot (a_2 \otimes c)) - t((a_2 \otimes c) \cdot a_1) = 0 \tag{2.14}$$

for $a_1, a_2 \in A$, $c \in C$. If we set $a_2 = 1$ in (2.14), we get $t(a_1 \otimes c) = t(a_{1\psi} \otimes c^\psi)$. It is now clear that if we define $f : C \otimes A \to k$ by setting $f(c \otimes a) := t(a \otimes c)$, we get $f \in \mathcal{C}^0_\lambda(A, C, \psi) \subseteq \mathcal{C}^0(A, C, \psi)$ and $\delta^0(f) = 0$, that is, $f \in Z^0_\lambda(A, C, \psi)$. In other words, we have

$$Z^0_\lambda(A, C, \psi) = H^0_\lambda(A, C, \psi) = \{\text{space of } A\text{-bimodule traces on } (A \otimes C)\}. \tag{2.15}$$

For example, if H is a Hopf algebra and A is a right H-comodule algebra, it may be verified that the map $\psi_{H,A} : H \otimes A \to A \otimes H$ given by $h \otimes a \mapsto a_{(0)} \otimes h a_{(1)}$ is an entwining structure. Here $a \mapsto a_{(0)} \otimes a_{(1)}$ denotes the right H-coaction on A in Sweedler notation. Then, a trace $t_{H,A} \in Z^0_\lambda(A, H, \psi_{H,A})$ is determined by a map $t_{H,A} : A \otimes H \to k$ which satisfies $t_{H,A}(ab \otimes h) - t_{H,A}(b a_{(0)} \otimes h a_{(1)}) = 0$ for $a, b \in A$, $h \in H$.

We also know how to obtain a cyclic 1-cocycle using a derivation. For an algebra A, let $s : A \to k$ be a trace on A and $\partial : A \to A$ be a derivation such that $s \circ \partial = 0$. Then, it is easy to check that the map $A \otimes A \to k$ that takes $a_1 \otimes a_2$ to $s(a_1 \partial(a_2))$ determines a cyclic 1-cocycle on A, that is, an element of $Z^1_\lambda(A)$.

To construct a cyclic 1-cocycle on (A, C, ψ), we can proceed as follows. We fix a trace $t : A \otimes C \to k$ on the A-bimodule $A \otimes C$ as in (2.14). For example, if $s : A \to k$ is a trace on A, we can verify that $s \otimes \varepsilon : A \otimes C \to k$ satisfies the condition in (2.14). We consider a k-linear derivation $\partial : A \to A$ on A such that $0 = t \circ (\partial \otimes C) : A \otimes C \to k$. We now set

$$g : C \otimes A \otimes A \longrightarrow k \quad g(c \otimes a_1 \otimes a_2) := t(a_1\partial(a_2) \otimes c) \tag{2.16}$$

for $a_1, a_2 \in A$, $c \in C$. For $c \in C$, $a_1, a_2 \in A$, we note that

$$\begin{aligned} g(c \otimes a_1 \otimes a_2) + g(c^\psi \otimes a_2 \otimes a_{1\psi}) &= t(a_1\partial(a_2) \otimes c) + t(a_2\partial(a_{1\psi}) \otimes c^\psi) \\ &= t(a_1\partial(a_2) \otimes c) + t(\partial(a_2 a_{1\psi}) \otimes c^\psi) \\ &\quad - t(\partial(a_2)a_{1\psi} \otimes c^\psi) \\ &= t(a_1\partial(a_2) \otimes c) - t(\partial(a_2)a_{1\psi} \otimes c^\psi) = 0, \end{aligned}$$

where the last equality follows from (2.14). From this, it is clear that $g \in \mathcal{C}^1_\lambda(A, C, \psi)$. For $c \in C$, $a_1, a_2, a_3 \in A$, we can also check that

$$\begin{aligned} \delta^1(g)(c \otimes a_1 \otimes a_2 \otimes a_3) &= g(c^\psi \otimes a_2 \otimes a_3 a_{1\psi}) \\ &\quad - g(c \otimes a_1 a_2 \otimes a_3) + g(c \otimes a_1 \otimes a_2 a_3) \\ &= t(a_2\partial(a_3 a_{1\psi}) \otimes c^\psi) - t(a_1 a_2 \partial(a_3) \otimes c) \\ &\quad + t(a_1\partial(a_2 a_3) \otimes c) \\ &= t(a_2\partial(a_3)a_{1\psi} \otimes c^\psi) + t(a_2 a_3 \partial(a_{1\psi}) \otimes c^\psi) \\ &\quad - t(a_1 a_2 \partial(a_3) \otimes c) + t(a_1\partial(a_2)a_3 \otimes c) \\ &\quad + t(a_1 a_2 \partial(a_3) \otimes c) \\ &= t(a_1 a_2 \partial(a_3) \otimes c) - t(\partial(a_2 a_3)a_{1\psi} \otimes c^\psi) \\ &\quad + t(\partial(a_2 a_3 a_{1\psi}) \otimes c^\psi) - t(a_1 a_2 \partial(a_3) \otimes c) \\ &\quad + t(a_1\partial(a_2)a_3 \otimes c) + t(a_1 a_2 \partial(a_3) \otimes c) \\ &= -t(\partial(a_2 a_3)a_{1\psi} \otimes c^\psi) + t(a_1\partial(a_2)a_3 \otimes c) \\ &\quad + t(a_1 a_2 \partial(a_3) \otimes c) \\ &= -t(a_1\partial(a_2 a_3) \otimes c) + t(a_1\partial(a_2)a_3 \otimes c) \\ &\quad + t(a_1 a_2 \partial(a_3) \otimes c) = 0. \end{aligned}$$

Hence, $g \in Z^1_\lambda(A, C, \psi)$. We also notice that there is a morphism

$$\mathrm{Der}^0_t(A) \longrightarrow Z^1_\lambda(A, C, \psi), \tag{2.17}$$

where $\mathrm{Der}^0_t(A) \subseteq \mathrm{Der}(A)$ is the subspace of k-linear derivations $\partial : A \to A$ such that $t \circ (\partial \otimes C) = 0$. We can now use a similar idea to obtain cyclic n-cocycles on (A, C, ψ). Let $t : A \otimes C \to k$ be a trace for the A-bimodule structure on $(A \otimes C)$ induced by the entwining (A, C, ψ) as in (2.14). If $\partial_1, \dots, \partial_n \in \mathrm{Der}^0_t(A)$ is a mutually commuting family of derivations, we set

$$\begin{aligned} h : C \otimes A^{\otimes n+1} &\longrightarrow k \\ h(c \otimes a_1 \otimes \cdots \otimes a_{n+1}) &:= \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \\ &\qquad \cdot t(a_1 \partial_{\pi(1)}(a_2)\partial_{\pi(2)}(a_3)\cdots\partial_{\pi(n)}(a_{n+1}) \otimes c) \end{aligned} \tag{2.18}$$

for $a_1, \ldots, a_{n+1} \in A$, $c \in C$. Indeed, since the derivations $\{\partial_i\}_{1\le i\le n}$ commute with each other, we see that

$$\begin{aligned} &h(c \otimes a_1 \otimes \cdots \otimes a_{n+1}) \\ &\quad= \sum_{\pi \in S_n} \operatorname{sgn}(\pi) t(a_1 \partial_{\pi(1)}(a_2)\partial_{\pi(2)}(a_3)\cdots\partial_{\pi(n)}(a_{n+1}) \otimes c) \\ &\quad= \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \Big(t(a_1 \partial_{\pi(1)}(a_2 \partial_{\pi(2)}(a_3)\cdots\partial_{\pi(n)}(a_{n+1})) \otimes c) \\ &\qquad\qquad - t\Big(\Big(a_1 a_2 \sum_{2\le j\le n} \partial_{\pi(2)}(a_3)\cdots\partial_{\pi(1)}\partial_{\pi(j)}(a_{j+1})\cdots \\ &\qquad\qquad\qquad \partial_{\pi(n)}(a_{n+1})\Big) \otimes c\Big)\Big) \\ &\quad= \sum_{\pi \in S_n} \operatorname{sgn}(\pi) t(a_1 \partial_{\pi(1)}(a_2 \partial_{\pi(2)}(a_3)\cdots\partial_{\pi(n)}(a_{n+1})) \otimes c) \\ &\quad= \sum_{\pi \in S_n} \operatorname{sgn}(\pi) t(\partial_{\pi(1)}(a_2 \partial_{\pi(2)}(a_3)\cdots\partial_{\pi(n)}(a_{n+1}))a_{1\psi} \otimes c^{\psi}) \\ &\quad= -\sum_{\pi \in S_n} \operatorname{sgn}(\pi) t(a_2 \partial_{\pi(2)}(a_3)\cdots\partial_{\pi(n)}(a_{n+1})\partial_{\pi(1)}(a_{1\psi}) \otimes c^{\psi}) \\ &\quad= (-1)^n h(c^{\psi} \otimes a_2 \otimes \cdots \otimes a_{n+1} \otimes a_{1\psi}). \end{aligned}$$

This means that $h \in \mathcal{C}^n_\lambda(A, C, \psi)$. We can also check that $\delta^n(h) = 0$. Thus, we can get a cyclic n-cocycle $h \in Z^n_\lambda(A, C, \psi)$ from a family of mutually commuting derivations and a trace on the A-bimodule $A \otimes C$ determined by the entwining (A, C, ψ).

We also see that if (A, C, ψ) is an entwining structure and A' is any k-algebra, we have an induced entwining structure $(A \otimes A', C, \psi_{A'})$. Here, $\psi_{A'}$ is obtained by extending $\psi : C \otimes A \to A \otimes C$ to $\psi_{A'} : C \otimes A \otimes A' \to A \otimes A' \otimes C$ by setting $\psi_{A'}(c \otimes a \otimes a') := a_\psi \otimes a' \otimes c^\psi$ for $a \in A$, $a' \in A'$, $c \in C$. Now if $t' \in Z^0_\lambda(A')$ is a trace on A' and $t : A \otimes C \to k$ is a trace that satisfies (2.14), it is clear that

$$t_{A'} : A \otimes A' \otimes C \longrightarrow k \quad a \otimes a' \otimes c \mapsto t'(a')t(a \otimes c) \tag{2.19}$$

satisfies condition (2.14) for the entwining structure $(A \otimes A', C, \psi_{A'})$. In other words, we have a pairing

$$Z^0_\lambda(A, C, \psi) \otimes Z^0_\lambda(A') \longrightarrow Z^0_\lambda(A \otimes A', C, \psi_{A'}). \tag{2.20}$$

In the remaining part of this paper, our main objective will be to obtain a general pairing of cyclic cohomologies

$$H^m_\lambda(A, C, \psi) \otimes H^n_\lambda(A', C', \psi') \longrightarrow H^{m+n}_\lambda(A \otimes A', C \otimes C', \psi \otimes \psi') \qquad m, n \geq 0 \tag{2.21}$$

for entwining structures (A, C, ψ) and (A', C', ψ').

3. Entwining of the universal differential graded algebra

We continue with (A, C, ψ) being an entwining structure over k. We begin this section by considering an entwining structure where the algebra is differential graded.

Definition 3.1. Let $(R^\bullet, D^\bullet)$ be a differential (non-negatively) graded, not necessarily unital k-algebra, and let C be a counital k-coalgebra. A dg-entwining structure over k consists of a k-linear map

$$\Psi^\bullet : C \otimes R^\bullet \longrightarrow R^\bullet \otimes C$$

of degree zero such that

(1) $\Psi^\bullet : (C \otimes R^\bullet, \mathrm{id}_C \otimes D^\bullet) \to (R^\bullet \otimes C, D^\bullet \otimes \mathrm{id}_C)$ is a morphism of complexes, that is,

$$D^n(r_\Psi) \otimes c^\Psi = (D^n \otimes C)(\Psi^n(c \otimes r)) = \Psi^{n+1}(c \otimes D^n(r)) = D^n(r)_\Psi \otimes c^\Psi$$

for $c \in C$, $r \in R^n$.

(2) the tuple (R, C, Ψ) is an entwining structure.

Definition 3.2. Let (A, C, ψ) be an entwining structure. A dg-entwining structure over (A, C, ψ) consists of a dg-entwining $((R^\bullet, D^\bullet), C, \Psi^\bullet)$ and a k-algebra morphism $\rho : A \to R^0$ such that we have a commutative diagram:

$$\begin{array}{ccc} C \otimes A & \xrightarrow{\psi} & A \otimes C \\ {\scriptstyle \mathrm{id}_C \otimes \rho}\downarrow & & \downarrow{\scriptstyle \rho \otimes \mathrm{id}_C} \\ C \otimes R^0 & \xrightarrow{\Psi^0} & R^0 \otimes C \end{array} \tag{3.1}$$

Given the k-algebra A, we now consider the algebra $\widetilde{A} := A \oplus k$ with multiplication given by

$$(a + \mu) \cdot (a' + \nu) = (aa' + \mu a' + \nu a) + \mu\nu$$

for a, $a' \in A$ and scalars μ, $\nu \in k$. It is clear that $\widetilde{A}$ is also a unital algebra, with $1 \in k$ being the unit. However, we note that the canonical inclusion $A \hookrightarrow \widetilde{A}$ of algebras is not necessarily unital.

We now consider the universal differential graded algebra $(\Omega^\bullet A, d^\bullet)$ associated to A (see [13, §II.1]). As a graded vector space, it is given by setting $\Omega^n A = \widetilde{A} \otimes A^{\otimes n}$ for $n > 0$ and $\Omega^0 A = A$. For $n > 0$, an element in $\widetilde{A} \otimes A^{\otimes n}$ is a linear combination of terms of the form

$$(a_0 + \mu) da_1 \cdots da_n \quad a_i \in A,\ \mu \in k. \tag{3.2}$$

By abuse of notation, we will use $(a_0 + \mu)da_1 \cdots da_n$ to denote an element of $\Omega^n A$ even for $n = 0$. In this case, it will be understood that $\mu = 0$. The multiplication in ΩA is determined by

$$\begin{aligned} a_0 da_1 \cdots da_n &= (a_0) \cdot (da_1) \cdot \ldots \cdot (da_n) \quad (da) \cdot a' = d(aa') - a(da') \\ da_1 \cdots da_n &= (da_1) \cdot \ldots \cdot (da_n) \end{aligned} \tag{3.3}$$

for $a, a', a_0, \ldots, a_n \in A$. More generally, for elements $p_0, \ldots, p_i, q_0, \ldots, q_j \in A$ and μ, $\nu \in k$, we have

$$\begin{aligned} &((p_0 + \mu) dp_1 \cdots dp_i) \cdot ((q_0 + \nu) dq_1 \cdots dq_j) \\ &\quad = (p_0 + \mu) \Big(dp_1 \cdots dp_{i-1} d(p_i q_0) dq_1 \cdots dq_j \\ &\qquad\qquad + \sum_{l=1}^{i-1} (-1)^{i-l} dp_1 \cdots d(p_l p_{l+1}) \cdots dp_i dq_0 dq_1 \cdots dq_j \Big) \\ &\qquad\qquad + (-1)^i (p_0 + \mu) p_1 dp_2 \cdots dp_i dq_0 dq_1 \cdots dq_j \\ &\qquad\qquad + \nu (p_0 + \mu) dp_1 \cdots dp_i dq_1 \cdots dq_j. \end{aligned} \tag{3.4}$$

The differential on ΩA is determined by setting

$$d((a_0 + \mu) da_1 \cdots da_n) = da_0 da_1 \cdots da_n. \tag{3.5}$$

We also define a morphism

$$\begin{aligned} \widehat{\psi} : C \otimes \Omega A &\longrightarrow \Omega A \otimes C \\ \widehat{\psi}(c \otimes ((a_0 + \mu) da_1 \cdots da_n)) &= (a_{0\psi} da_{1\psi} \cdots da_{n\psi}) \otimes c^{\psi^{n+1}} \\ &\quad + \mu (da_{1\psi} \cdots da_{n\psi}) \otimes c^{\psi^n}. \end{aligned} \tag{3.6}$$

In particular, we have $(da)_\psi \otimes c^\psi = \widehat{\psi}(c \otimes da) = da_\psi \otimes c^\psi$.

Proposition 3.3. *Let (A, C, ψ) be an entwining structure. Then, $((\Omega^\bullet A, d^\bullet), C, \widehat{\psi})$ is a dg-entwining structure over (A, C, ψ).*

Proof. From (3.5) and (3.6), it is evident that $\widehat{\psi}$ is a morphism of complexes. It is also clear that

$$\begin{aligned}
&\widehat{\psi}(c \otimes (a_0)\cdot(da_1)\cdot\ldots\cdot(da_n)) \\
&\quad = \widehat{\psi}(c \otimes a_0 da_1 \cdots da_n) = (a_{0\psi} da_{1\psi} \cdots da_{n\psi}) \otimes c^{\psi^{n+1}} \\
&\quad = ((a_{0\psi})\cdot(da_{1\psi})\ldots\cdot(da_{n\psi})) \otimes c^{\psi^{n+1}} \widehat{\psi}(c \otimes (da_1)\cdot\ldots\cdot(da_n)) \\
&\quad = \widehat{\psi}(c \otimes da_1 \cdots da_n) = (da_{1\psi} \cdots da_{n\psi}) \otimes c^{\psi^n} \\
&\quad = ((da_{1\psi})\ldots\cdot(da_{n\psi})) \otimes c^{\psi^{n+1}}
\end{aligned} \tag{3.7}$$

for $a_0, \ldots, a_n \in A$. Further, for $a, a' \in A$, we have

$$\begin{aligned}
\widehat{\psi}(c \otimes ((da)\cdot a')) &= \widehat{\psi}(c \otimes d(aa')) - \widehat{\psi}(c \otimes (a(da'))) \\
&= d(aa')_\psi \otimes c^\psi - (a_\psi da'_\psi) \otimes c^{\psi\psi} \\
&= d(a_\psi a'_\psi) \otimes c^{\psi\psi} - (a_\psi da'_\psi) \otimes c^{\psi\psi} \\
&= ((da_\psi)\cdot a'_\psi) \otimes c^{\psi\psi} = ((da)_\psi \cdot a'_\psi) \otimes c^{\psi\psi}.
\end{aligned} \tag{3.8}$$

Together, (3.7) and (3.8) show that $\widehat{\psi}$ is well behaved with respect to the multiplication on ΩA. The other conditions in (2.1) for $(\Omega A, C, \widehat{\psi})$ to be an entwining structure may also be verified by direct computation. Finally, it is clear that the maps $\widehat{\psi}^0 : C \otimes A = C \otimes \Omega^0 A \to \Omega^0 A \otimes C = A \otimes C$ and $\psi : C \otimes A \to A \otimes C$ are identical. This proves the result. ∎

Theorem 3.4. *Let $((R^\bullet, D^\bullet), C, \Psi^\bullet)$ be a dg-entwining structure over (A, C, ψ) consisting of a k-algebra homomorphism $\rho : A \to R^0$. Then, there is an induced morphism $\widehat{\rho} : (\Omega^\bullet A, d^\bullet) \to (R^\bullet, D^\bullet)$ of dg-algebras such that $\widehat{\rho}|_A = \rho : A \to R^0$, and we have a commutative diagram:*

$$\begin{array}{ccc}
C \otimes \Omega A & \xrightarrow{\widehat{\psi}} & \Omega A \otimes C \\
{\scriptstyle \mathrm{id}_C \otimes \widehat{\rho}} \downarrow & & \downarrow {\scriptstyle \widehat{\rho} \otimes \mathrm{id}_C} \\
C \otimes R & \xrightarrow{\Psi} & R \otimes C
\end{array} \tag{3.9}$$

Proof. From the universal property of $(\Omega^\bullet A, d^\bullet)$ (see [13, §II.1]), we know that there is a unique morphism $\widehat{\rho} : (\Omega^\bullet A, d^\bullet) \to (R^\bullet, D^\bullet)$ of dg-algebras such that $\widehat{\rho}|_A = \rho : \Omega^0 A = A \to R^0$. In particular, $\widehat{\rho}$ is described as follows:

$$\begin{aligned}
&\widehat{\rho}((a_0 + \mu) da_1 \cdots da_n) \\
&\quad = (\rho(a_0))\cdot(D\rho(a_1))\cdot\ldots\cdot(D\rho(a_n)) + \mu(D\rho(a_1))\cdot\ldots\cdot(D\rho(a_n))
\end{aligned} \tag{3.10}$$

for $a_0, \ldots, a_n \in A$ and $n > 0$, where the products on the right-hand side are taken in R. For the sake of convenience, we will suppress the morphism ρ and often write $\rho(a) \in R^0$

simply as a for any $a \in A$. For $c \in C$, we now compute

$$\begin{aligned}
&((\widehat{\rho} \otimes \mathrm{id}_C) \circ \widehat{\psi})(c \otimes (a_0 + \mu) da_1 \cdots da_n) \\
&\quad = (\widehat{\rho} \otimes \mathrm{id}_C)((a_{0\psi} da_{1\psi} \cdots da_{n\psi}) \otimes c^{\psi^{n+1}} + \mu(da_{1\psi} \cdots da_{n\psi}) \otimes c^{\psi^n}) \\
&\quad = (a_{0\psi}) \cdot (Da_{1\psi}) \cdot \ldots \cdot (Da_{n\psi}) \otimes c^{\psi^{n+1}} + \mu(Da_{1\psi}) \cdot \ldots \cdot (Da_{n\psi}) \otimes c^{\psi^n} \\
&(\Psi \circ (\mathrm{id}_C \otimes \widehat{\rho}))(c \otimes (a_0 + \mu) da_1 \cdots da_n) \\
&\quad = \Psi(c \otimes ((a_0) \cdot (Da_1) \cdot \ldots \cdot (Da_n)) + c \otimes (\mu(Da_1) \cdot \ldots \cdot (Da_n))) \\
&\quad = ((a_0)_\Psi \cdot (Da_1)_\Psi \cdot \ldots \cdot (Da_n)_\Psi) \otimes c^{\Psi^{n+1}} + (\mu(Da_1)_\Psi \cdot \ldots \cdot (Da_n)_\Psi) \otimes c^{\Psi^n} \\
&\quad = ((a_{0\Psi}) \cdot (Da_{1\Psi}) \cdot \ldots \cdot (Da_{n\Psi})) \otimes c^{\Psi^{n+1}} + (\mu(Da_{1\Psi}) \cdot \ldots \cdot (Da_{n\Psi})) \otimes c^{\Psi^n} \\
&\quad = ((a_{0\psi}) \cdot (Da_{1\psi}) \cdot \ldots \cdot (Da_{n\psi})) \otimes c^{\psi^{n+1}} + (\mu(Da_{1\psi}) \cdot \ldots \cdot (Da_{n\psi})) \otimes c^{\psi^n},
\end{aligned}$$

where the replacement of Ψ by ψ in the last equality follows from the commutativity of (3.1). We have now shown that the diagram (3.9) is commutative. ■

4. Entwined traces and classes in cyclic cohomology

In this section, we will show that cocycles in $(\mathcal{C}^\bullet_\lambda(A, C, \psi), \delta^\bullet)$ correspond to certain kinds of traces on dg-entwining structures over (A, C, ψ). We continue to suppress the morphism $\rho : A \to R^0$ when working with a dg-entwining structure $((R^\bullet, D^\bullet), C, \Psi^\bullet)$ over (A, C, ψ). We begin by introducing the notion of an entwined trace.

Definition 4.1. Let $((R^\bullet, D^\bullet), C, \Psi^\bullet)$ be a dg-entwining structure. An n-dimensional closed graded entwined trace for $((R^\bullet, D^\bullet), C, \Psi^\bullet)$ consists of a linear morphism

$$T : C \otimes R^n \longrightarrow k \tag{4.1}$$

satisfying the following conditions:

(1) For any $c \in C$ and $r \in R^{n-1}$, we have

$$T(c \otimes D(r)) = 0. \tag{4.2}$$

(2) For $r \in R^i$, $r' \in R^j$ with $i + j = n$ and any $c \in C$, we have

$$T(c \otimes rr') = (-1)^{ij} T(c^\Psi \otimes r' r_\Psi). \tag{4.3}$$

Definition 4.2. Let (A, C, ψ) be an entwining structure. An n-dimensional entwined cycle over (A, C, ψ) is a tuple

$$((R^\bullet, D^\bullet), C, \Psi^\bullet, T, \rho), \tag{4.4}$$

where

(1) $\rho : A \to R^0$ is a morphism of k-algebras making $((R^\bullet, D^\bullet), C, \Psi^\bullet)$ a dg-entwining structure over (A, C, ψ).

(2) the linear map $T : C \otimes R^n \to k$ is an n-dimensional closed graded entwined trace on $((R^\bullet, D^\bullet), C, \Psi^\bullet)$.

Definition 4.3. Let (A, C, ψ) be an entwining structure, and let $((R^\bullet, D^\bullet), C, \Psi^\bullet, T, \rho)$ be an n-dimensional entwined cycle over (A, C, ψ). Then, we define the character of the cycle $((R^\bullet, D^\bullet), C, \Psi^\bullet, T, \rho)$ to be the element $g \in \mathcal{C}^n(A, C, \psi)$ determined by

$$g(c \otimes a_1 \otimes \cdots \otimes a_{n+1}) := T(c \otimes (\rho(a_1) \cdot D(\rho(a_2)) \cdots D(\rho(a_{n+1}))))$$

for any $c \otimes a_1 \otimes \cdots \otimes a_{n+1} \in C \otimes A^{n+1}$.

Proposition 4.4. *Let (A, C, ψ) be an entwining structure, and let $g : C \otimes A^{\otimes n+1} \to k$ be a linear morphism. Then, the following are equivalent:*

(1) *There is an n-dimensional entwined cycle $((R^\bullet, D^\bullet), C, \Psi^\bullet, T, \rho)$ over (A, C, ψ) such that*

$$g(c, a_0, \ldots, a_n) = T(c \otimes (\rho(a_0) \cdot D\rho(a_1) \cdot \ldots \cdot D\rho(a_n))) \quad a_i \in A,\ c \in C. \tag{4.5}$$

(2) *There exists a closed graded entwined trace $t : C \otimes \Omega^n A \to k$ of dimension n on $((\Omega^\bullet A, d^\bullet), C, \widehat{\psi})$ such that*

$$g(c, a_0, \ldots, a_n) = t(c \otimes a_0 da_1 \cdots da_n) \quad a_i \in A,\ c \in C. \tag{4.6}$$

Proof. (1) $\Rightarrow$ (2): By Theorem 3.4, we obtain a morphism $\widehat{\rho} : (\Omega^\bullet A, d^\bullet) \to (R^\bullet, D^\bullet)$ of dg-algebras extending $\rho : A \to R^0$. We define $t : C \otimes \Omega^n A \to k$ by setting

$$t(c \otimes ((a_0 + \mu)da_1 \cdots da_n)) = T(c \otimes \widehat{\rho}((a_0 + \mu)da_1 \cdots da_n)) \quad a_i \in A,\ c \in C. \tag{4.7}$$

In particular, when $\mu = 0$, we get

$$t(c \otimes a_0 da_1 \cdots da_n) = T(c \otimes (\rho(a_0) \cdot D\rho(a_1) \cdot \ldots \cdot D\rho(a_n))) = g(c, a_0, \ldots, a_n). \tag{4.8}$$

We have to verify that t satisfies conditions (4.2) and (4.3) in Definition 4.1. First, we note that

$$\begin{aligned} & t(c \otimes d((a_0 + \mu)da_1 \cdots da_{n-1})) \\ &= t(c \otimes da_0 \cdots da_{n-1}) \\ &= T(c \otimes (D\rho(a_0) \cdot \ldots \cdot D\rho(a_{n-1}))) \\ &= T(c \otimes D(\rho(a_0) \cdot D\rho(a_1) \cdot \ldots \cdot D\rho(a_{n-1}))) = 0. \end{aligned} \tag{4.9}$$

This proves condition (4.2). Now, for $\alpha \in \Omega^i A$ and $\alpha' \in \Omega^j A$ with $i + j = n$ and for any $c \in C$, we have

$$\begin{aligned} & t(c \otimes \alpha \cdot \alpha') \\ &= T(c \otimes \widehat{\rho}(\alpha \cdot \alpha')) = T(c \otimes \widehat{\rho}(\alpha) \cdot \widehat{\rho}(\alpha')) = (-1)^{ij} T(c^\Psi \otimes \widehat{\rho}(\alpha') \cdot \widehat{\rho}(\alpha)_\Psi) \\ &= (-1)^{ij} T(c^{\widehat{\psi}} \otimes \widehat{\rho}(\alpha') \cdot \widehat{\rho}(\alpha_{\widehat{\psi}})) = (-1)^{ij} T(c^{\widehat{\psi}} \otimes \widehat{\rho}(\alpha' \cdot \alpha_{\widehat{\psi}})) \\ &= (-1)^{ij} t(c^{\widehat{\psi}} \otimes (\alpha' \cdot \alpha_{\widehat{\psi}})), \end{aligned}$$

where the equality $c^{\Psi} \otimes \widehat{\rho}(\alpha') \cdot \widehat{\rho}(\alpha)_{\Psi} = c^{\widehat{\psi}} \otimes \widehat{\rho}(\alpha') \cdot \widehat{\rho}(\alpha_{\widehat{\psi}})$ follows from the commutativity of the diagram in (3.9). This shows that t also satisfies condition (4.3).

(2) $\Rightarrow$ (1): From Proposition 3.3, we already know that $((\Omega^\bullet A, d^\bullet), C, \widehat{\psi})$ is a dg-entwining structure over (A, C, ψ). Since $t : C \otimes \Omega^n A \to k$ is a closed graded entwined trace of dimension n, it follows that $((\Omega^\bullet A, d^\bullet), C, \widehat{\psi}, t, \mathrm{id}_A)$ is an n-dimensional entwined cycle over (A, C, ψ). We also have

$$t(c \otimes a_0 \cdot da_1 \cdot \ldots \cdot da_n) = t(c \otimes a_0 da_1 \cdots da_n) = g(c, a_0, \ldots, a_n)$$

for $c \in C$ and $a_i \in A$. ∎

Theorem 4.5. *Let (A, C, ψ) be an entwining structure, and let $g : C \otimes A^{\otimes n+1} \to k$ be a linear morphism. Then, the following are equivalent:*

(1) *There is an n-dimensional entwined cycle $((R^\bullet, D^\bullet), C, \Psi^\bullet, T, \rho)$ over (A, C, ψ) such that*

$$g(c, a_0, \ldots, a_n) = T(c \otimes (\rho(a_0) \cdot D\rho(a_1) \cdot \ldots \cdot D\rho(a_n))) \quad a_i \in A,\ c \in C. \tag{4.10}$$

(2) *There exists a closed graded entwined trace $t : C \otimes \Omega^n A \to k$ of dimension n on $((\Omega^\bullet A, d^\bullet), C, \widehat{\psi})$ such that*

$$g(c, a_0, \ldots, a_n) = t(c \otimes a_0 da_1 \cdots da_n) \quad a_i \in A,\ c \in C. \tag{4.11}$$

(3) $g \in Z^n_\lambda(A, C, \psi)$.

Proof. From Proposition 4.4, we already know that (1) and (2) are equivalent.

(3) $\Rightarrow$ (2): We know that $g \in \mathcal{C}^n_\lambda(A, C, \psi)$. We define $t : C \otimes \Omega^n A \to k$ by setting

$$t(c \otimes (a_0 + \mu) da_1 \cdots da_n) := g(c, a_0, a_1, \ldots, a_n) \tag{4.12}$$

for $c \in C$, $a_i \in A$ and $\mu \in k$. We note that

$$\begin{aligned} &t(c \otimes d((a_0 + \mu) da_1 \cdots da_{n-1})) \\ &\quad = t(c \otimes da_0 \cdots da_{n-1}) = g(c, 0, a_0, \ldots, a_{n-1}) = 0. \end{aligned} \tag{4.13}$$

We now consider elements $(p_0 + \mu) dp_1 \cdots dp_i \in \Omega^i A$ and $(q_0 + \nu) dq_1 \cdots dq_j \in \Omega^j A$ with $i + j = n$. Using the expression for the product on ΩA given in (3.4), we obtain

$$\begin{aligned} &t(c \otimes ((p_0 + \mu) dp_1 \cdots dp_i) \cdot ((q_0 + \nu) dq_1 \cdots dq_j)) \\ &\quad = g(c, p_0, p_1, \ldots, p_{i-1}, p_i q_0, q_1, \ldots, q_j) \\ &\qquad + \sum_{l=1}^{i-1} (-1)^{i-l} g(c, p_0, p_1, \ldots, p_l p_{l+1}, \ldots, p_i, q_0, q_1, \ldots, q_j) \\ &\qquad + (-1)^i g(c, p_0 p_1, p_2, \ldots, p_i, q_0, \ldots, q_j) \\ &\qquad + (-1)^i \mu g(c, p_1, p_2, \ldots, p_i, q_0, \ldots, q_j) \\ &\qquad + \nu g(c, p_0, p_1, \ldots, p_i, q_1, \ldots, q_j). \end{aligned} \tag{4.14}$$

On the other hand, we have

$$\begin{aligned} &t(c^{\widehat{\psi}} \otimes ((q_0+\nu)dq_1\cdots dq_j)\cdot((p_0+\mu)dp_1\cdots dp_i)_{\widehat{\psi}})\\ &\quad = t(c^{\psi^{i+1}} \otimes (q_0dq_1\cdots dq_j)\cdot(p_{0\psi}dp_{1\psi}\cdots dp_{i\psi}))\\ &\qquad + \nu t(c^{\psi^{i+1}} \otimes (dq_1\cdots dq_j)\cdot(p_{0\psi}dp_{1\psi}\cdots dp_{i\psi}))\\ &\qquad + \mu t(c^{\psi^{i}} \otimes ((q_0+\nu)dq_1\cdots dq_j)\cdot(dp_{1\psi}\cdots dp_{i\psi})). \end{aligned} \tag{4.15}$$

We need to verify that

$$\begin{aligned} &t(c\otimes((p_0+\mu)dp_1\cdots dp_i)\cdot((q_0+\nu)dq_1\cdots dq_j))\\ &\quad = (-1)^{ij} t(c^{\widehat{\psi}} \otimes ((q_0+\nu)dq_1\cdots dq_j)\cdot((p_0+\mu)dp_1\cdots dp_i)_{\widehat{\psi}}). \end{aligned} \tag{4.16}$$

For this, we compare one by one the terms in (4.14) and (4.15) using the relations (4.12), (4.13), the product on ΩA described in (3.4) and the property of $g \in \mathcal{C}^n_\lambda(A, C, \psi)$ from (2.6). First, we note that

$$\begin{aligned} &\mu t(c^{\psi^{i}} \otimes ((q_0+\nu)dq_1\cdots dq_j)\cdot(dp_{1\psi}\cdots dp_{i\psi}))\\ &\quad = \mu g(c^{\psi^{i}}, q_0, q_1, \ldots, q_j, p_{1\psi}, \ldots, p_{i\psi})\\ &\quad = (-1)^{ij}(-1)^{i}\mu g(c, p_1, p_2, \ldots, p_i, q_0, \ldots, q_j). \end{aligned} \tag{4.17}$$

Next, we have

$$\begin{aligned} &\nu t(c^{\psi^{i+1}} \otimes (dq_1\cdots dq_j)\cdot(p_{0\psi}dp_{1\psi}\cdots dp_{i\psi}))\\ &\quad = (-1)^{j}\nu g(c^{\psi^{i+1}}, q_1, \ldots, q_j, p_{0\psi}, \ldots, p_{i\psi})\\ &\quad = (-1)^{ij}\nu g(c, p_0, p_1, \ldots, p_i, q_1, \ldots, q_j). \end{aligned} \tag{4.18}$$

We also have

$$\begin{aligned} &t(c^{\psi^{i+1}} \otimes (q_0dq_1\cdots dq_j)\cdot(p_{0\psi}dp_{1\psi}\cdots dp_{i\psi}))\\ &\quad = g(c^{\psi^{i+1}}, q_0, \ldots, q_{j-1}, q_j p_{0\psi}, p_{1\psi}, \ldots, p_{i\psi})\\ &\qquad + \sum_{l=1}^{j-1}(-1)^{j-l} g(c^{\psi^{i+1}}, q_0, q_1, \ldots, q_l q_{l+1}, \ldots, q_j, p_{0\psi}, \ldots, p_{i\psi})\\ &\qquad + (-1)^{j} g(c^{\psi^{i+1}}, q_0q_1, q_2, \ldots, q_j, p_{0\psi}, \ldots, p_{i\psi})\\ &\quad = (-1)^{ni} g(c^{\psi}, p_1, \ldots, p_i, q_0, \ldots, q_{j-1}, q_j p_{0\psi})\\ &\qquad + \sum_{l=1}^{j-1}(-1)^{j-l}(-1)^{n(i+1)} g(c, p_0, p_1, \ldots, p_i, q_0, q_1, \ldots, q_l q_{l+1}, \ldots, q_j)\\ &\qquad + (-1)^{j+n(i+1)} g(c, p_0, \ldots, p_i, q_0q_1, q_2, \ldots, q_j). \end{aligned} \tag{4.19}$$

The result of (4.16) now follows from the fact that $g \in Z^n_\lambda(A, C, \psi)$ satisfies $\delta(g) = 0$, where δ is the Hochschild differential as described in (2.12).

(2) $\Rightarrow$ (3): We have an n-dimensional closed graded entwined trace $t : C \otimes \Omega^n A \to k$ such that

$$g(c, a_0, \ldots, a_n) = t(c \otimes a_0 da_1 \cdots da_n) \quad a_i \in A,\ c \in C. \tag{4.20}$$

Since t is a closed graded entwined trace, we note that $t(c \otimes da_1 \cdots da_n) = t(c \otimes d(a_1 da_2 \cdots da_n)) = 0$. Hence,

$$g(c, a_0, \ldots, a_n) = t(c \otimes (a_0 + \mu) da_1 \cdots da_n) \quad a_i \in A,\ c \in C,\ \mu \in k. \tag{4.21}$$

To show that $g \in Z^n_\lambda(A, C, \psi)$, we have to verify that

$$\begin{aligned} &\delta(g)(c, p_1, \ldots, p_{n+2}) = 0 \\ &\quad g(c, p_1, \ldots, p_{n+1}) = (-1)^n g(c^\psi, p_2, \ldots, p_{n+1}, p_{1\psi}) \end{aligned} \tag{4.22}$$

for any $p_i \in A$ and $c \in C$. Here, δ denotes the Hochschild differential as described in (2.12). We now have

$$\begin{aligned} &g(c, p_1, \ldots, p_{n+1}) - (-1)^n g(c^\psi, p_2, \ldots, p_{n+1}, p_{1\psi}) \\ &\quad = t(c \otimes p_1 dp_2 \cdots dp_{n+1}) - (-1)^n t(c^\psi \otimes p_2 dp_3 \cdots dp_{n+1} dp_{1\psi}) \\ &\quad = t(c \otimes p_1 dp_2 \cdots dp_{n+1}) + t(c \otimes (dp_1)(p_2 dp_3 \cdots dp_{n+1})) \\ &\quad = t(c \otimes d(p_1 p_2) dp_3 \cdots dp_{n+1}) = 0. \end{aligned} \tag{4.23}$$

Applying (4.23), we now see that

$$\begin{aligned} &t(c^{\psi^{n+1}} \otimes (p_{n+2}) \cdot (p_{1\psi} dp_{2\psi} \cdots dp_{(n+1)\psi})) \\ &\quad = (-1)^n g(c^\psi, p_2, \ldots, p_{n+1}, p_{n+2} p_{1\psi}). \end{aligned} \tag{4.24}$$

Applying (4.23), we can reverse the arguments in (4.14) to see that

$$\begin{aligned} &t(c \otimes (p_1 dp_2 \cdots dp_{n+1})(p_{n+2})) \\ &\quad = g(c, p_1, \ldots, p_{n+1} p_{n+2}) + \sum_{l=1}^{n-1} (-1)^{n-l} g(c, p_1, p_2, \ldots, p_{l+1} p_{l+2}, \ldots, p_{n+2}) \\ &\qquad + (-1)^n g(c, p_1 p_2, \ldots, p_{n+1}, p_{n+2}). \end{aligned} \tag{4.25}$$

From (4.24) and (4.25) and the fact that $t(c \otimes (p_1 dp_2 \cdots dp_{n+1})(p_{n+2})) - t(c^{\psi^{n+1}} \otimes (p_{n+2}) \cdot (p_{1\psi} dp_{2\psi} \cdots dp_{(n+1)\psi})) = 0$, it now follows that $\delta(g) = 0$. ∎

5. Morita invariance of Hochschild cohomology

Let (A, C, ψ) be an entwining structure. We construct a presimplicial module $\mathcal{C}_\bullet = \mathcal{C}_\bullet(A, C, \psi)$ as follows:

$$\{\mathcal{C}_n(A, C, \psi) = C \otimes A^{\otimes n+1}\}_{n\geq 0} \qquad \{d_i : \mathcal{C}_n \longrightarrow \mathcal{C}_{n-1}\}_{0\leq i\leq n}$$
$$d_i(c, a_1, \ldots, a_{n+1}) := \begin{cases} (c^\psi, a_2, \ldots, a_{n+1}a_{1\psi}) & \text{if } i = 0, \\ (c, a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) & \text{if } 0 < i \leq n. \end{cases} \tag{5.1}$$

Lemma 5.1. *The collection* $\mathcal{C}_\bullet = \mathcal{C}_\bullet(A, C, \psi)$ *along with the maps in* (5.1) *forms a presimplicial module.*

Proof. We need to verify (see, for instance, [15, §1.0.6]) that $d_i d_j = d_{j-1} d_i$ for $0 \leq i < j \leq n$. This is obvious for $j > 1$. Since $j > i \geq 0$, the only remaining case is that of $i = 0$ and $j = 1$. In that case, we have

$$\begin{aligned} d_0 d_1(c, a_1, \ldots, a_{n+1}) &= d_0(c, a_1 a_2, \ldots, a_{n+1}) \\ &= (c^\psi, a_3, \ldots, a_{n+1}(a_1 a_2)_\psi) \\ &= (c^{\psi\psi}, a_3, \ldots, a_{n+1} a_{1\psi} a_{2\psi}) \\ &= d_0 d_0(c, a_1, \ldots, a_{n+1}). \end{aligned} \tag{5.2}$$

This proves the result. ∎

We continue to denote by $\mathcal{C}_\bullet(A, C, \psi)$ the complex corresponding to this presimplicial module, equipped with standard differential $\sum_{i=0}^n (-1)^i d_i$. The homology groups of this complex will be denoted by $HH_\bullet(A, C, \psi)$. From (2.12), it is evident that

$$\mathcal{C}^\bullet(A, C, \psi) = \mathrm{Hom}(\mathcal{C}_\bullet(A, C, \psi), k). \tag{5.3}$$

We will now show that the complexes $\mathcal{C}^\bullet(A, C, \psi)$ and $\mathcal{C}^\bullet(M_r(A), C, \psi)$ are quasi-isomorphic for any $r \geq 1$, where $M_r(A)$ is the ring of $(r \times r)$-matrices with entries in A.

First, we extend the entwining $\psi : C \otimes A \to A \otimes C$ to a map (still denoted by ψ)

$$\begin{gathered} \psi : C \otimes M_2(A) \longrightarrow M_2(A) \otimes C \\ \psi(c \otimes (a \otimes E_{ij}(1))) = (a_\psi \otimes E_{ij}(1)) \otimes c^\psi \quad a \in A,\ c \in C, \end{gathered} \tag{5.4}$$

where $\{E_{ij}(1)\}_{1\leq i,j\leq r}$ is the $(r \times r)$-matrix whose (i, j)-th entry is 1 and all others are 0. It is immediate that $(M_r(A), C, \psi)$ is an entwining structure.

For any $1 \leq p \leq r$, we have a (not necessarily unital) inclusion of rings

$$\mathrm{inc}_p : A \longrightarrow M_r(A) \quad a \mapsto E_{pp}(a) \tag{5.5}$$

inducing a morphism of complexes $\mathrm{inc}_{p\bullet} : \mathcal{C}_\bullet(A, C, \psi) \to \mathcal{C}_\bullet(M_r(A), C, \psi)$. On the other hand, consider the generalized trace map (see, for instance, [15, §1.2.1])

$$\begin{aligned} \mathrm{tr} : M_r(A)^{\otimes n+1} &\longrightarrow A^{\otimes n+1} \\ \mathrm{tr}(X^1, \ldots, X^{n+1}) &= \sum (X^1_{i_1 i_2}) \otimes (X^2_{i_2 i_3}) \otimes \cdots \otimes (X^{n+1}_{i_{n+1} i_1}), \end{aligned} \tag{5.6}$$

where the sum is taken over all possible tuples $(i_1, \ldots, i_{n+1})$. Writing $M_r(A) = A \otimes M_r(k)$, the generalized trace can be expressed as (see, for instance, [15, §1.2.2])

$$\mathrm{tr}(a_1u_1 \otimes \cdots \otimes a_{n+1}u_{n+1}) = \mathrm{tr}(u_1 \cdots u_{n+1})(a_1 \otimes \cdots \otimes a_{n+1}), \tag{5.7}$$

where $a_i \in A$ and $u_i \in M_r(k)$. The generalized trace can be extended to a map (still denoted by tr) as follows:

$$\begin{aligned} \mathrm{tr} : C \otimes M_r(A)^{\otimes n+1} &\longrightarrow C \otimes A^{\otimes n+1} \\ \mathrm{tr}(c \otimes a_1u_1 \otimes \cdots \otimes a_{n+1}u_{n+1}) &= \mathrm{tr}(u_1 \cdots u_{n+1})(c \otimes a_1 \otimes \cdots \otimes a_{n+1}). \end{aligned} \tag{5.8}$$

Lemma 5.2. *The generalized trace induces a natural morphism of complexes* $\mathrm{tr}_\bullet : \mathcal{C}_\bullet(M_r(A), C, \psi) \to \mathcal{C}_\bullet(A, C, \psi)$.

Proof. It suffices to show that the generalized trace commutes with the face maps d_i of the presimplicial modules. From (5.8), this is obvious for $i > 0$. For $i = 0$, we have

$$\begin{aligned} d_0 \circ \mathrm{tr}(c, a_1u_1, \ldots, a_{n+1}u_{n+1}) &= \mathrm{tr}(u_1 \cdots u_{n+1})(c^\psi, a_2, \ldots, a_{n+1}a_{1\psi}) \\ &= \mathrm{tr}(u_2 \cdots u_{n+1}u_1)(c^\psi, a_2, \ldots, a_{n+1}a_{1\psi}) \\ &= \mathrm{tr} \circ d_0(c, a_1u_1, \ldots, a_{n+1}u_{n+1}), \end{aligned} \tag{5.9}$$

where $a_i \in A$ and $u_i \in M_r(k)$. ∎

Proposition 5.3. *The maps* $\mathrm{inc}_{1\bullet} : \mathcal{C}_\bullet(A, C, \psi) \to \mathcal{C}_\bullet(M_r(A), C, \psi)$ *and the maps* $\mathrm{tr}_\bullet : \mathcal{C}_\bullet(M_r(A), C, \psi) \to \mathcal{C}_\bullet(A, C, \psi)$ *are homotopy inverses to each other.*

Proof. We have

$$\begin{aligned} (\mathrm{tr}_\bullet \circ \mathrm{inc}_{1\bullet})(c \otimes a_1 \otimes \cdots \otimes a_{n+1}) &= \mathrm{tr}_\bullet(c \otimes a_1E_{11}(1) \otimes \cdots \otimes a_{n+1}E_{11}(1)) \\ &= c \otimes a_1 \otimes \cdots \otimes a_{n+1} \end{aligned}$$

which shows that $\mathrm{tr}_\bullet \circ \mathrm{inc}_{1\bullet} = \mathrm{id}$. Therefore, it remains to show that $\mathrm{inc}_{1\bullet} \circ \mathrm{tr}_\bullet \sim \mathrm{id}$.

For each $n \geq 0$, we define k-linear maps

$$\{h_i \mid \mathcal{C}_n(M_r(A), C, \psi) \longrightarrow \mathcal{C}_{n+1}(M_r(A), C, \psi)\}_{0 \leq i \leq n}$$

given by

$$\begin{aligned} &h_i(c \otimes a_1u_1 \otimes \cdots \otimes a_{n+1}u_{n+1}) \\ &\quad := c \otimes \sum_{1 \leq k,l,\ldots,p,q,s \leq r} a_1E_{11}(u_{1kl}) \otimes a_2E_{11}(u_{2lm}) \otimes \cdots \otimes a_iE_{11}(u_{ipq}) \\ &\qquad \otimes 1_AE_{1q}(1) \otimes a_{i+1}u_{i+1} \otimes \cdots \otimes a_{n+1}E_{s1}(u_{n+1sk}) \end{aligned}$$

for $1 \leq i \leq n$ and

$$h_0(c \otimes a_1u_1 \otimes \cdots \otimes a_{n+1}u_{n+1}) := c \otimes \sum_{1\leq k,s\leq r} 1_A E_{1k}(1) \otimes a_1u_1 \otimes \cdots \otimes a_nu_n \otimes a_{n+1}E_{s1}(u_{n+1sk}).$$

We will now prove that $h = \sum_{i=0}^{n}(-1)^i h_i$ is a homotopy between $\text{inc}_{1\bullet} \circ \text{tr}_\bullet$ and the identity. To do so, we check the following relations (see [15, §1.0.8]):

$$\begin{aligned} d_ih_j &= h_{j-1}d_i \quad \text{for } i < j \\ d_ih_i &= d_ih_{i-1} \quad \text{for } 0 < i \leq n \\ d_ih_j &= h_jd_{i-1} \quad \text{for } i > j+1 \\ d_0h_0 &= \text{id} \quad \text{and} \quad d_{n+1}h_n = \text{inc}_{1\bullet} \circ \text{tr}_\bullet. \end{aligned} \tag{5.10}$$

We have

$$\begin{aligned} &d_0h_0(c \otimes a_1u_1 \otimes \cdots \otimes a_{n+1}u_{n+1}) \\ &= d_0\Big(c \otimes \sum_{1\leq k,s\leq r} 1_A E_{1k}(1) \otimes a_1u_1 \otimes \cdots \otimes a_nu_n \otimes a_{n+1}E_{s1}(u_{n+1sk})\Big) \\ &= c^\psi \otimes a_1u_1 \otimes \cdots \otimes a_nu_n \otimes \sum_{1\leq k,s\leq r} a_{n+1}E_{s1}(u_{n+1sk})(1_AE_{1k}(1))_\psi \\ &= c \otimes a_1u_1 \otimes \cdots \otimes a_nu_n \otimes \sum_{1\leq k,s\leq r} a_{n+1}E_{s1}(u_{n+1sk})E_{1k}(1) \\ &= c \otimes a_1u_1 \otimes \cdots \otimes a_{n+1}u_{n+1}. \end{aligned}$$

The third equality follows from the fact that $\psi(c \otimes 1_AE_{pq}(1)) = 1_AE_{pq}(1) \otimes c$ for all $1 \leq p, q \leq r$. The fourth equality follows from the fact that $\sum_{1\leq k,s\leq r} E_{s1}(u_{n+1sk})E_{1k}(1) = u_{n+1}$.

Further, using the fact that $E_{1q}(1)E_{s1}(u_{n+1sk}) = 0$ unless $q = s$, we have

$$\begin{aligned} &d_{n+1}h_n(c \otimes a_1u_1 \otimes \cdots \otimes a_{n+1}u_{n+1}) \\ &= d_{n+1}\Big(c \otimes \sum_{1\leq k,l,\dots,p,q,s\leq r} a_1E_{11}(u_{1kl}) \otimes a_2E_{11}(u_{2lm}) \otimes \cdots \otimes a_nE_{11}(u_{npq}) \\ &\qquad \otimes 1_AE_{1q}(1) \otimes a_{n+1}E_{s1}(u_{n+1sk})\Big) \\ &= c \otimes \sum_{1\leq k,l,\dots,p,q,s\leq r} a_1E_{11}(u_{1kl}) \otimes a_2E_{11}(u_{2lm}) \otimes \cdots \otimes a_nE_{11}(u_{npq}) \\ &\qquad \otimes a_{n+1}E_{1q}(1)E_{s1}(u_{n+1sk}) \\ &= c \otimes \sum_{1\leq k,l,\dots,p,q\leq r} a_1E_{11}(u_{1kl}) \otimes a_2E_{11}(u_{2lm}) \otimes \cdots \otimes a_nE_{11}(u_{npq}) \\ &\qquad \otimes a_{n+1}E_{1q}(1)E_{q1}(u_{n+1qk}) \end{aligned}$$

$$
\begin{aligned}
&= c \otimes \sum_{1\le k,l,\dots,p,q\le r} a_1 E_{11}(u_{1kl}) \otimes a_2 E_{11}(u_{2lm}) \otimes \cdots \otimes a_n E_{11}(u_{npq}) \\
&\quad \otimes a_{n+1} E_{11}(u_{n+1qk}) \\
&= (c \otimes a_1 E_{11}(1) \otimes \cdots \otimes a_{n+1} E_{11}(1)) \sum_{1\le k,l,\dots,p,q\le r} (u_{1kl} u_{2lm} \cdots u_{n+1qk}) \\
&= (c \otimes a_1 E_{11}(1) \otimes \cdots \otimes a_{n+1} E_{11}(1)) \sum_{1\le k\le r} (u_1 u_2 \cdots u_{n+1})_{kk} \\
&= (c \otimes a_1 E_{11}(1) \otimes \cdots \otimes a_{n+1} E_{11}(1)) \operatorname{tr}(u_1 \cdots u_{n+1}) \\
&= (\mathrm{inc}_{1\bullet} \circ \mathrm{tr}_\bullet)(c \otimes a_1 u_1 \otimes \cdots \otimes a_{n+1} u_{n+1}).
\end{aligned}
$$

Now, for $0 < i < j$, we have

$$
\begin{aligned}
&d_i h_j (c \otimes a_1 u_1 \otimes \cdots \otimes a_{n+1} u_{n+1}) \\
&\quad = d_i \Big(c \otimes \sum_{1\le k,l,\dots,p,q,s\le r} a_1 E_{11}(u_{1kl}) \otimes a_2 E_{11}(u_{2lm}) \otimes \cdots \otimes a_j E_{11}(u_{j\,pq}) \\
&\qquad\qquad \otimes 1_A E_{1q}(1) \otimes a_{j+1} u_{j+1} \otimes \cdots \otimes a_{n+1} E_{s1}(u_{n+1sk}) \Big) \\
&\quad = c \otimes \sum_{1\le k,l,\dots,p,q,s\le r} a_1 E_{11}(u_{1kl}) \otimes a_2 E_{11}(u_{2lm}) \\
&\qquad \otimes \cdots \otimes a_i E_{11}(u_{i\,tt'}) a_{i+1} E_{11}(u_{i+1\,t'n}) \otimes \cdots \otimes a_j E_{11}(u_{j\,pq}) \\
&\qquad \otimes 1_A E_{1q}(1) \otimes a_{j+1} u_{j+1} \otimes \cdots \otimes a_{n+1} E_{s1}(u_{n+1sk}) \\
&\quad = c \otimes \sum_{1\le k,l,\dots,p,q,s\le r} a_1 E_{11}(u_{1kl}) \otimes a_2 E_{11}(u_{2lm}) \\
&\qquad \otimes \cdots \otimes a_i a_{i+1} E_{11}((u_i u_{i+1})_{tn}) \otimes \cdots \otimes a_j E_{11}(u_{j\,pq}) \\
&\qquad \otimes 1_A E_{1q}(1) \otimes a_{j+1} u_{j+1} \otimes \cdots \otimes a_{n+1} E_{s1}(u_{n+1sk}) \\
&\quad = h_{j-1}\big(c \otimes a_1 u_1 \otimes a_2 u_2 \otimes \cdots \otimes a_i a_{i+1} u_i u_{i+1} \otimes \cdots \otimes a_{n+1} u_{n+1}\big) \\
&\quad = h_{j-1} d_i (c \otimes a_1 u_1 \otimes \cdots \otimes a_{n+1} u_{n+1}).
\end{aligned}
$$

Moreover, for $j > 0$,

$$
\begin{aligned}
&d_0 h_j (c \otimes a_1 u_1 \otimes \cdots \otimes a_{n+1} u_{n+1}) \\
&\quad = d_0 \Big(c \otimes \sum_{1\le k,l,\dots,p,q,s\le r} a_1 E_{11}(u_{1kl}) \otimes a_2 E_{11}(u_{2lm}) \otimes \cdots \otimes a_j E_{11}(u_{j\,pq}) \\
&\qquad\qquad \otimes 1_A E_{1q}(1) \otimes a_{j+1} u_{j+1} \otimes \cdots \otimes a_{n+1} E_{s1}(u_{n+1sk}) \Big) \\
&\quad = c^{\psi} \otimes \sum_{1\le k,l,\dots,p,q,s\le r} a_2 E_{11}(u_{2lm}) \otimes \cdots \otimes a_j E_{11}(u_{j\,pq}) \otimes 1_A E_{1q}(1) \\
&\qquad \otimes a_{j+1} u_{j+1} \otimes \cdots \otimes a_{n+1} E_{s1}(u_{n+1sk}) (a_1 E_{11}(u_{1kl}))_{\psi}
\end{aligned}
$$

$$
\begin{aligned}
&= c^{\psi} \otimes \sum_{1\leq k,l,\ldots,p,q,s\leq r} a_2E_{11}(u_{2lm}) \otimes \cdots \otimes a_jE_{11}(u_{j_{pq}}) \otimes 1_AE_{1q}(1) \\
&\qquad \otimes a_{j+1}u_{j+1} \otimes \cdots \otimes a_{n+1}a_{1\psi}E_{s1}(u_{n+1sk})E_{11}(u_{1kl}) \\
&= c^{\psi} \otimes \sum_{1\leq k,l,\ldots,p,q,s\leq r} a_2E_{11}(u_{2lm}) \otimes \cdots \otimes a_jE_{11}(u_{j_{pq}}) \otimes 1_AE_{1q}(1) \\
&\qquad \otimes a_{j+1}u_{j+1} \otimes \cdots \otimes a_{n+1}a_{1\psi}E_{s1}((u_{n+1}u_1)_{sl}) \\
&= h_{j-1}(c^{\psi} \otimes a_2u_2 \otimes \cdots \otimes a_{n+1}a_{1\psi}u_{n+1}u_1) \\
&= h_{j-1}d_0(c \otimes a_1u_1 \otimes \cdots \otimes a_{n+1}u_{n+1}).
\end{aligned}
$$

Using the equality $\sum_{q=1}^{r} E_{11}(u_{pq})E_{1q}(1) = E_{1p}(1)u$, we have for $0 < i \leq n$

$$
\begin{aligned}
&d_ih_i(c \otimes a_1u_1 \otimes \cdots \otimes a_{n+1}u_{n+1}) \\
&\quad = d_i\Bigg(c \otimes \sum_{1\leq k,l,\ldots,p,q,s\leq r} a_1E_{11}(u_{1kl}) \otimes a_2E_{11}(u_{2lm}) \otimes \cdots \otimes a_iE_{11}(u_{i_{pq}}) \\
&\qquad\qquad \otimes 1_AE_{1q}(1) \otimes a_{i+1}u_{i+1} \otimes \cdots \otimes a_{n+1}E_{s1}(u_{n+1sk})\Bigg) \\
&\quad = c \otimes \sum_{1\leq k,l,\ldots,p,q,s\leq r} a_1E_{11}(u_{1kl}) \otimes a_2E_{11}(u_{2lm}) \otimes \cdots \otimes a_iE_{11}(u_{i_{pq}})E_{1q}(1) \\
&\qquad \otimes a_{i+1}u_{i+1} \otimes \cdots \otimes a_{n+1}E_{s1}(u_{n+1sk}) \\
&\quad = c \otimes \sum_{1\leq k,l,\ldots,p,s\leq r} a_1E_{11}(u_{1kl}) \otimes a_2E_{11}(u_{2lm}) \otimes \cdots \otimes a_iE_{1p}(1)u_i \\
&\qquad \otimes a_{i+1}u_{i+1} \otimes \cdots \otimes a_{n+1}E_{s1}(u_{n+1sk}) \\
&\quad = d_i\Bigg(c \otimes \sum_{1\leq k,l,\ldots,p,s\leq r} a_1E_{11}(u_{1kl}) \otimes a_2E_{11}(u_{2lm}) \otimes \cdots \otimes 1_AE_{1p}(1) \\
&\qquad\qquad \otimes a_iu_i \otimes a_{i+1}u_{i+1} \otimes \cdots \otimes a_{n+1}E_{s1}(u_{n+1sk})\Bigg) \\
&\quad = d_ih_{i-1}(c \otimes a_1u_1 \otimes \cdots \otimes a_{n+1}u_{n+1}).
\end{aligned}
$$

For $i > j + 1$, it may be similarly verified that $d_ih_j = h_jd_{i-1}$. This proves the result. ∎

Theorem 5.4. *The morphisms*

$$
\begin{aligned}
\mathrm{inc}_{1\bullet} &: HH_\bullet(A, C, \psi) \longrightarrow HH_\bullet(M_r(A), C, \psi) \\
\mathrm{tr}_\bullet &: HH_\bullet(M_r(A), C, \psi) \longrightarrow HH_\bullet(A, C, \psi)
\end{aligned}
$$

are mutually inverse isomorphisms of Hochschild homologies.

For each $n \geq 0$, we now obtain k-linear maps

$$
\{h^i \mid \mathcal{C}^{n+1}(M_r(A), C, \psi) \longrightarrow \mathcal{C}^n(M_r(A), C, \psi)\}_{0\leq i\leq n} \tag{5.11}
$$

given by $h^i(f) = f \circ h_i$. Explicitly, for $1 \leq i \leq n$, we have

$$\begin{aligned}(h^i(f))&(c \otimes a_1u_1 \otimes a_{n+1}u_{n+1})\\ &= f\Big(c \otimes \sum_{1\leq k,l,\dots,p,q,s\leq r} a_1E_{11}(u_{1kl}) \otimes a_2E_{11}(u_{2lm})\\ &\qquad \otimes \cdots \otimes a_iE_{11}(u_{ipq}) \otimes 1_AE_{1q}(1) \otimes a_{i+1}u_{i+1}\\ &\qquad \otimes \cdots \otimes a_nu_n \otimes a_{n+1}E_{s1}(u_{n+1sk})\Big)\end{aligned} \tag{5.12}$$

and

$$\begin{aligned}(h^0(f))&(c \otimes a_1u_1 \otimes a_{n+1}u_{n+1})\\ &= f\Big(c \otimes \sum_{1\leq k,s\leq r} 1_AE_{1k}(1) \otimes a_1u_1 \otimes \cdots \otimes a_nu_n \otimes a_{n+1}E_{s1}(u_{n+1sk})\Big).\end{aligned} \tag{5.13}$$

Proposition 5.5. *The maps* $\mathrm{tr}^\bullet : \mathcal{C}^\bullet(A, C, \psi) \to \mathcal{C}^\bullet(M_r(A), C, \psi)$ *and the maps* $\mathrm{inc}_1^\bullet$: $\mathcal{C}^\bullet(M_r(A), C, \psi) \to \mathcal{C}^\bullet(A, C, \psi)$ *are homotopy inverses to each other. In particular, the morphisms*

$$\begin{aligned}\mathrm{tr}^\bullet &: HH^\bullet(A, C, \psi) \longrightarrow HH^\bullet(M_r(A), C, \psi)\\ \mathrm{inc}_1^\bullet &: HH^\bullet(M_r(A), C, \psi) \longrightarrow HH^\bullet(A, C, \psi)\end{aligned}$$

are mutually inverse isomorphisms of Hochschild cohomologies.

6. Invariant subcomplex and Morita invariance of cyclic cohomology

Let (A, C, ψ) be an entwining structure. The dual $\mathcal{C}^\bullet(A, C, \psi)$ of $\mathcal{C}_\bullet(A, C, \psi)$ is a pre-cosimplicial module, equipped with maps $\{\delta_i \mid \mathcal{C}^n(A, C, \psi) \to \mathcal{C}^{n+1}(A, C, \psi)\}_{0\leq i\leq n+1}$. For each $n \geq 0$, we set $\mathcal{I}^n(A, C, \psi) \subseteq \mathcal{C}^n(A, C, \psi)$ to be the collection of morphisms $g \in \mathrm{Hom}(C \otimes A^{n+1}, k)$ satisfying

$$g(c \otimes a_1 \otimes \cdots \otimes a_{n+1}) = g(c^{\psi^{n+1}} \otimes a_{1\psi} \otimes \cdots \otimes a_{n+1\psi})$$

for every $c \in C$ and $a_1, \dots, a_{n+1} \in A$.

Lemma 6.1. *Let* (A, C, ψ) *be an entwining structure. Then,* $\mathcal{I}^\bullet(A, C, \psi)$ *is a subcomplex of the Hochschild complex* $\mathcal{C}^\bullet(A, C, \psi)$.

Proof. We will show that δ_i for $0 \le i \le n+1$ restricts to $\mathcal{J}^\bullet(A, C, \psi)$. For any $n \ge 0$ and $g \in \mathcal{J}^n(A, C, \psi)$, we have

$$\begin{aligned}(\delta_0 g)(c \otimes a_1 \otimes \cdots \otimes a_{n+2}) &= g(c^\psi \otimes a_2 \otimes \cdots \otimes a_{n+2}a_{1\psi}) \\ &= g(c^{\psi^{n+2}} \otimes a_{2\psi} \otimes \cdots \otimes (a_{n+2}a_{1\psi})_\psi) \\ &= g(c^{\psi^{n+3}} \otimes a_{2\psi} \otimes \cdots \otimes a_{n+2\psi}a_{1\psi\psi}) \\ &= (\delta_0 g)(c^{\psi^{n+2}} \otimes a_{1\psi} \otimes a_{2\psi} \otimes \cdots \otimes a_{n+2\psi}).\end{aligned}$$

Hence, $\delta_0 g \in \mathcal{J}^{n+1}(A, C, \psi)$. Moreover, for $1 \le i \le n+1$, we have

$$\begin{aligned}(\delta_i g)(c \otimes a_1 \otimes \cdots \otimes a_{n+2}) &= g(c \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+2}) \\ &= g(c^{\psi^{n+1}} \otimes a_{1\psi} \otimes \cdots \otimes (a_i a_{i+1})_\psi \otimes \cdots \otimes a_{n+2\psi}) \\ &= g(c^{\psi^{n+2}} \otimes a_{1\psi} \otimes \cdots \otimes a_{i\psi}a_{i+1\psi} \otimes \cdots \otimes a_{n+2\psi}) \\ &= (\delta_i g)(c^{\psi^{n+2}} \otimes a_{1\psi} \otimes a_{2\psi} \otimes \cdots \otimes a_{n+2\psi}).\end{aligned}$$

This shows that $\delta_i g \in \mathcal{J}^{n+1}(A, C, \psi)$ for each $1 \le i \le n+1$. This proves the result. ∎

We will refer to $\mathcal{J}^\bullet(A, C, \psi)$ as the invariant subcomplex of $\mathcal{C}^\bullet(A, C, \psi)$. For each $n \ge 0$, we define the k-linear maps $\{\sigma_j \mid \mathcal{J}^{n+1}(A, C, \psi) \to \mathcal{J}^n(A, C, \psi)\}_{0 \le j \le n}$ given by

$$(\sigma_j f)(c \otimes a_1 \otimes \cdots \otimes a_{n+1}) = f(c \otimes a_1 \otimes \cdots a_j \otimes 1_A \otimes a_{j+1} \otimes \cdots a_{n+1}). \tag{6.1}$$

We also define the cyclic operator $\tau_n : \mathcal{J}^n(A, C, \psi) \to \mathcal{J}^n(A, C, \psi)$ as follows:

$$(\tau_n g)(c \otimes a_1 \otimes \cdots \otimes a_{n+1}) = (-1)^n g(c^\psi \otimes a_2 \otimes \cdots \otimes a_{n+1} \otimes a_{1\psi}). \tag{6.2}$$

Proposition 6.2. *The object $\mathcal{J}^\bullet(A, C, \psi)$ is a cocyclic module.*

Proof. From the proof of Lemma 6.1, we know that $\mathcal{J}^\bullet(A, C, \psi)$ is precosimplicial module. Together with the maps in (6.1), it may be easily verified that $\left(\mathcal{J}^\bullet(A, C, \psi), \delta_i^\bullet, \sigma_j^\bullet\right)$ is a cosimplicial module.

From (6.2), we have for $g \in \mathcal{J}^n(A, C, \psi)$,

$$\begin{aligned}(\tau_n^{n+1} g)(c \otimes a_1 \otimes \cdots \otimes a_{n+1}) &= (-1)^{n(n+1)} g(c^{\psi^{n+1}} \otimes a_{1\psi} \otimes \cdots \otimes a_{n+1\psi}) \\ &= g(c \otimes a_1 \otimes \cdots \otimes a_{n+1}).\end{aligned}$$

It remains therefore to verify the following identities:

$$\begin{aligned}\delta_i \tau_{n-1} &= -\tau_n \delta_{i-1} \qquad 1 \le i \le n \\ \delta_0 &= (-1)^n \tau_n \delta_n \\ \sigma_i \tau_{n+1} &= -\tau_n \sigma_{i-1} \qquad 1 \le i \le n \\ \sigma_0 \tau_{n+1}^2 &= (-1)^n \tau_n \sigma_n.\end{aligned}$$

For $1 < i \leq n$ and $g \in \mathcal{J}^{n-1}(A, C, \psi)$, we have

$$\begin{aligned}
&(\delta_i \tau_{n-1} g)(c \otimes a_1 \otimes \cdots \otimes a_{n+1}) \\
&\quad = (\tau_{n-1} g)(c \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots a_{n+1}) \\
&\quad = (-1)^{n-1} g(c^{\psi} \otimes a_2 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots a_{n+1} \otimes a_{1\psi}) \\
&\quad = (-1)^{n-1} (\delta_{i-1} g)(c^{\psi} \otimes a_2 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots a_{n+1} \otimes a_{1\psi}) \\
&\quad = -(\tau_n \delta_{i-1} g)(c \otimes a_1 \otimes \cdots \otimes a_{n+1}).
\end{aligned}$$

For $i = 1$, we have

$$\begin{aligned}
&(\delta_1 \tau_{n-1} g)(c \otimes a_1 \otimes \cdots \otimes a_{n+1}) \\
&\quad = (\tau_{n-1} g)(c \otimes a_1 a_2 \otimes a_3 \otimes \cdots a_{n+1}) \\
&\quad = (-1)^{n-1} g(c^{\psi} \otimes a_3 \otimes \cdots \otimes a_{n+1} \otimes (a_1 a_2)_{\psi}) \\
&\quad = (-1)^{n-1} (\delta_0 g)(c^{\psi} \otimes a_2 \otimes \cdots \otimes a_{n+1} \otimes a_{1\psi}) \\
&\quad = -(\tau_n \delta_0 g)(c \otimes a_1 \otimes \cdots \otimes a_{n+1}).
\end{aligned}$$

Clearly, $\delta_0 = (-1)^n \tau_n \delta_n$. It may also be easily verified that $\sigma_i \tau_{n+1} = -\tau_n \sigma_{i-1}$ for $1 \leq i \leq n$. Further, for any $g' \in \mathcal{J}^{n+1}(A, C, \psi)$,

$$\begin{aligned}
&(\sigma_0 \tau_{n+1}^2 g')(c \otimes a_1 \otimes \cdots \otimes a_{n+1}) \\
&\quad = (\tau_{n+1}^2 g')(c \otimes 1_A \otimes a_1 \otimes \cdots \otimes a_{n+1}) \\
&\quad = (-1)^{n+1} (\tau_{n+1} g')(c \otimes a_1 \otimes \cdots \otimes a_{n+1} \otimes 1_A) \\
&\quad = g'(c^{\psi} \otimes a_2 \otimes \cdots \otimes a_{n+1} \otimes 1_A \otimes a_{1\psi}) \\
&\quad = (\sigma_n g')(c^{\psi} \otimes a_2 \otimes \cdots \otimes a_{n+1} \otimes a_{1\psi}) \\
&\quad = (-1)^n (\tau_n \sigma_n g')(c \otimes a_1 \otimes \cdots \otimes a_{n+1}).
\end{aligned}$$

This proves the result. ∎

Proposition 6.3. *We now show that the invariant subcomplex is Morita invariant.*

(1) *The maps* $\mathrm{inc}_1^\bullet : \mathcal{C}^\bullet(M_r(A), C, \psi) \to \mathcal{C}^\bullet(A, C, \psi)$ *and* $\mathrm{tr}^\bullet : \mathcal{C}^\bullet(A, C, \psi) \to \mathcal{C}^\bullet(M_r(A), C, \psi)$ *restrict to the corresponding invariant subcomplexes. In other words, we have morphisms*

$$\mathrm{inc}_1^\bullet : \mathcal{J}^\bullet(M_r(A), C, \psi) \longrightarrow \mathcal{J}^\bullet(A, C, \psi) \qquad \mathrm{tr}^\bullet : \mathcal{J}^\bullet(A, C, \psi) \longrightarrow \mathcal{J}^\bullet(M_r(A), C, \psi).$$

(2) *The morphisms* $\mathrm{inc}_1^\bullet : \mathcal{J}^\bullet(M_r(A), C, \psi) \to \mathcal{J}^\bullet(A, C, \psi)$ *and* $\mathrm{tr}^\bullet : \mathcal{J}^\bullet(A, C, \psi) \to \mathcal{J}^\bullet(M_r(A), C, \psi)$ *are homotopy inverses of each other. Hence, the invariant subcomplexes* $\mathcal{J}^\bullet(M_r(A), C, \psi)$ *and* $\mathcal{J}^\bullet(A, C, \psi)$ *are quasi-isomorphic.*

Proof. (1) For any $n \geq 0$ and $f \in \mathcal{J}^n(M_r(A), C, \psi)$, we have, for $a_i \in A$, $c \in C$,

$$\begin{aligned}
(\mathrm{inc}_1^n(f))(c \otimes a_1 \otimes \cdots \otimes a_{n+1}) &= f(c \otimes a_1 E_{11}(1) \otimes \cdots \otimes a_{n+1} E_{11}(1)) \\
&= f(c^{\psi^{n+1}} \otimes a_{1\psi} E_{11}(1) \otimes \cdots \otimes a_{n+1\psi} E_{11}(1)) \\
&= (\mathrm{inc}_1^n(f))(c^{\psi^{n+1}} \otimes a_{1\psi} \otimes \cdots \otimes a_{n+1\psi}).
\end{aligned}$$

Hence, $\mathrm{inc}_1^n(f) \in \mathcal{J}^n(A, C, \psi)$. Further, for any $g \in \mathcal{J}^n(A, C, \psi)$, we have, for $a_i \in A$, $u_i \in M_r(k)$, $c \in C$,

$$\begin{aligned}
&(\mathrm{tr}^n(g))(c \otimes a_1u_1 \otimes \cdots \otimes a_{n+1}u_{n+1}) \\
&\quad = g(c \otimes a_1 \otimes \cdots \otimes a_{n+1})\,\mathrm{tr}(u_1u_2\cdots u_{n+1}) \\
&\quad = g(c^{\psi^{n+1}} \otimes a_{1\psi} \otimes \cdots \otimes a_{n+1\psi})\,\mathrm{tr}(u_1u_2\cdots u_{n+1}) \\
&\quad = (\mathrm{tr}^n(g))(c^{\psi^{n+1}} \otimes a_{1\psi}u_1 \otimes \cdots \otimes a_{n+1\psi}u_{n+1}).
\end{aligned}$$

Therefore, $\mathrm{tr}^n(g) \in \mathcal{J}^n(M_r(A), C, \psi)$.

(2) We will now show that the homotopy $h = \sum_{i=0}^n (-1)^i h^i$ as constructed in (5.12) and (5.13) restricts to be a homotopy between the maps $\mathrm{tr}^\bullet \circ \mathrm{inc}_1^\bullet$ and $\mathrm{id}_{\mathcal{J}^n(M_r(A),C,\psi)}$. For any $f \in \mathcal{J}^{n+1}(M_r(A), C, \psi)$ and $1 \le i \le n$, we have

$$\begin{aligned}
&(h^i(f))(c \otimes a_1u_1 \otimes \cdots \otimes a_{n+1}u_{n+1}) \\
&\quad = f\Bigg(c \otimes \sum_{1\le k,l,\ldots,p,q,s\le r} a_1E_{11}(u_{1kl}) \otimes a_2E_{11}(u_{2lm}) \otimes \cdots \otimes a_iE_{11}(u_{ipq}) \\
&\qquad\qquad \otimes 1_AE_{1q}(1) \otimes a_{i+1}u_{i+1} \otimes \cdots \otimes a_nu_n \otimes a_{n+1}E_{s1}(u_{n+1sk})\Bigg) \\
&\quad = f\Bigg(c^{\psi^{n+2}} \otimes \sum_{1\le k,l,\ldots,p,q,s\le r} a_{1\psi}E_{11}(u_{1kl}) \otimes a_{2\psi}E_{11}(u_{2lm}) \\
&\qquad\qquad \otimes \cdots \otimes a_{i\psi}E_{11}(u_{ipq}) \otimes (1_A)_\psi E_{1q}(1) \otimes a_{i+1\psi}u_{i+1} \\
&\qquad\qquad \otimes \cdots \otimes a_{n\psi}u_n \otimes a_{n+1\psi}E_{s1}(u_{n+1sk})\Bigg) \\
&\quad = f\Bigg(c^{\psi^{n+1}} \otimes \sum_{1\le k,l,\ldots,p,q,s\le r} a_{1\psi}E_{11}(u_{1kl}) \otimes a_{2\psi}E_{11}(u_{2lm}) \\
&\qquad\qquad \otimes \cdots \otimes a_{i\psi}E_{11}(u_{ipq}) \otimes 1_AE_{1q}(1) \otimes a_{i+1\psi}u_{i+1} \\
&\qquad\qquad \otimes \cdots \otimes a_{n\psi}u_n \otimes a_{n+1\psi}E_{s1}(u_{n+1sk})\Bigg) \\
&\quad = (h^i(f))(c^{\psi^{n+1}} \otimes a_{1\psi}u_1 \otimes \cdots \otimes a_{n\psi}u_n \otimes a_{n+1\psi}u_{n+1})
\end{aligned}$$

and

$$\begin{aligned}
&(h^0(f))(c \otimes a_1u_1 \otimes \cdots \otimes a_{n+1}u_{n+1}) \\
&\quad = f\Bigg(c \otimes \sum_{1\le k,s\le r} 1_AE_{1k}(1) \otimes a_1u_1 \otimes \cdots \otimes a_nu_n \otimes a_{n+1}E_{s1}(u_{n+1sk})\Bigg) \\
&\quad = f\Bigg(c^{\psi^{n+1}} \otimes \sum_{1\le k,s\le r} 1_AE_{1k}(1) \otimes a_{1\psi}u_1 \otimes \cdots \otimes a_{n\psi}u_n \otimes a_{n+1\psi}E_{s1}(u_{n+1sk})\Bigg) \\
&\quad = (h^0(f))(c^{\psi^{n+1}} \otimes a_{1\psi}u_1 \otimes \cdots \otimes a_{n\psi}u_n \otimes a_{n+1\psi}u_{n+1}).
\end{aligned}$$

This proves the result. ∎

We now observe that $\mathcal{C}^\bullet_\lambda(A,C,\psi) \subseteq \mathcal{I}^\bullet(A,C,\psi)$. For any $g \in \mathcal{C}^n_\lambda(A,C,\psi)$, we have

$$\begin{aligned} g(c\otimes a_1\otimes\cdots\otimes a_{n+1}) &= (-1)^n g(c^\psi\otimes a_2\otimes\cdots\otimes a_{n+1}\otimes a_{1\psi}) \\ &= (-1)^{n(n+1)} g(c^{\psi^{n+1}}\otimes a_{1\psi}\otimes\cdots\otimes a_{n+1\psi}) \end{aligned} \tag{6.3}$$

for every $c\in C$ and $a_1,\ldots,a_{n+1}\in A$. In fact, we observe that

$$\mathcal{C}^\bullet_\lambda(A,C,\psi) = \{g\in\mathcal{I}^\bullet(A,C,\psi) \mid \tau_\bullet g = g\}. \tag{6.4}$$

Theorem 6.4. *We have mutually inverse isomorphisms*

$$\mathrm{inc}_1^\bullet : H^\bullet_\lambda(M_r(A),C,\psi) \longrightarrow H^\bullet_\lambda(A,C,\psi) \qquad \mathrm{tr}^\bullet : H^\bullet_\lambda(A,C,\psi) \longrightarrow H^\bullet_\lambda(M_r(A),C,\psi)$$

of cyclic cohomology groups.

Proof. By Proposition 6.3, $\mathrm{inc}_1^\bullet : \mathcal{I}^\bullet(M_r(A),C,\psi)\to\mathcal{I}^\bullet(A,C,\psi)$ and $\mathrm{tr}^\bullet : \mathcal{I}^\bullet(A,C,\psi)\to\mathcal{I}^\bullet(M_r(A),C,\psi)$ are obtained by restricting the dual of maps $\mathrm{inc}_{1\bullet} : \mathcal{C}_\bullet(A,C,\psi)\to\mathcal{C}_\bullet(M_r(A),C,\psi)$ and $\mathrm{tr}_\bullet : \mathcal{C}_\bullet(M_r(A),C,\psi)\to\mathcal{C}_\bullet(A,C,\psi)$ to their respective invariant subcomplexes. It is easily verified that $\mathrm{inc}_1^\bullet$ and $\mathrm{tr}^\bullet$ commute with the cyclic operators on $\mathcal{I}^\bullet(A,C,\psi)$ and $\mathcal{I}^\bullet(M_r(A),C,\psi)$. This shows that $\mathrm{inc}_1^\bullet : \mathcal{I}^\bullet(M_r(A),C,\psi)\to\mathcal{I}^\bullet(A,C,\psi)$ and $\mathrm{tr}^\bullet : \mathcal{I}^\bullet(A,C,\psi)\to\mathcal{I}^\bullet(M_r(A),C,\psi)$ induce morphisms of the double complexes computing the cyclic cohomology of $\mathcal{I}^\bullet(A,C,\psi)$ and $\mathcal{I}^\bullet(M_r(A),C,\psi)$.

From Proposition 6.3, we know that $\mathrm{inc}_1^\bullet$ and $\mathrm{tr}^\bullet$ induce mutually inverse isomorphisms of the Hochschild cohomologies of $\mathcal{I}^\bullet(A,C,\psi)$ and $\mathcal{I}^\bullet(M_r(A),C,\psi)$. Since $\mathcal{C}^\bullet_\lambda(A,C,\psi) = \{g\in\mathcal{I}^\bullet(A,C,\psi) \mid \tau_\bullet g = g\}$, the result now follows from the Hochschild to cyclic spectral sequence. ∎

7. Vanishing cycles and coboundaries

Let (A,C,ψ) and (A',C',ψ') be entwining structures. A morphism of entwining structures from (A,C,ψ) to (A',C',ψ') is a pair (α,γ), where $\alpha : A\to A'$ is a k-algebra morphism and $\gamma : C\to C'$ is a k-coalgebra morphism such that

$$(\alpha\otimes\gamma)\circ\psi = \psi'\circ(\gamma\otimes\alpha). \tag{7.1}$$

Lemma 7.1. *Let $(\alpha,\gamma) : (A,C,\psi)\to(A',C',\psi')$ be a morphism of entwining structures. Then, (α,γ) induces a morphism of complexes $F^\bullet(\alpha,\gamma) : \mathcal{C}^\bullet(A',C',\psi')\to\mathcal{C}^\bullet(A,C,\psi)$ and a morphism*

$$F^\bullet(\alpha,\gamma) : H^\bullet_\lambda(A',C',\psi') \longrightarrow H^\bullet_\lambda(A,C,\psi)$$

of entwined cyclic cohomologies.

Proof. Given $(\alpha,\gamma) : (A,C,\psi)\to(A',C',\psi')$, we have a morphism

$$F_n(\alpha,\gamma) := \gamma\otimes\alpha^{n+1} : C\otimes A^{n+1} \longrightarrow C'\otimes A'^{n+1}.$$

We denote its dual by $F^n(\alpha,\gamma) : \mathrm{Hom}(C' \otimes A'^{n+1}, k) \to \mathrm{Hom}(C \otimes A^{n+1}, k)$. We first need to show that the following diagram commutes for all $n \geq 0$ and $0 \leq i \leq n+1$:

$$\begin{array}{ccc} \mathrm{Hom}(C' \otimes A'^{n+1}, k) & \xrightarrow{\delta_i} & \mathrm{Hom}(C' \otimes A'^{n+2}, k) \\ {\scriptstyle F^n(\alpha,\gamma)}\downarrow & & \downarrow{\scriptstyle F^{n+1}(\alpha,\gamma)} \\ \mathrm{Hom}(C \otimes A^{n+1}, k) & \xrightarrow{\delta_i} & \mathrm{Hom}(C \otimes A^{n+2}, k) \end{array} \tag{7.2}$$

For any $f' \in \mathrm{Hom}(C' \otimes A'^{n+1}, k)$, we have, for $c \in C, a_1, \ldots, a_{n+2} \in A$,

$$\begin{aligned} &(F^{n+1}(\alpha,\gamma)\delta_0 f)(c \otimes a_1 \otimes \cdots \otimes a_{n+2}) \\ &\quad = (\delta_0 f)(\gamma(c) \otimes \alpha(a_1) \otimes \cdots \otimes \alpha(a_{n+2})) \\ &\quad = f((\gamma(c))^{\psi'} \otimes \alpha(a_2) \otimes \cdots \otimes \alpha(a_{n+2})(\alpha(a_1))_{\psi'}) \\ &\quad = f(\gamma(c^{\psi}) \otimes \alpha(a_2) \otimes \cdots \otimes \alpha(a_{n+2}a_{1\psi})) \\ &\quad = (\delta_0 F^n(\alpha,\gamma) f)(c \otimes a_1 \otimes \cdots \otimes a_{n+2}). \end{aligned} \tag{7.3}$$

The third equality in (7.3) follows by using (7.1) and the fact that α is an algebra map. Similarly, it may be easily verified that $F^{n+1}(\alpha,\gamma)\delta_i = \delta_i F^n(\alpha,\gamma)$ for each $1 \leq i \leq n+1$. Finally, if $f' \in \mathcal{C}^n_\lambda(A', C', \psi')$, we have

$$\begin{aligned} &(F^n(\alpha,\gamma) f')(c \otimes a_1 \otimes \cdots \otimes a_{n+1}) \\ &\quad = f'(\gamma(c) \otimes \alpha(a_1) \otimes \cdots \otimes \alpha(a_{n+1})) \\ &\quad = (-1)^n f'(\gamma(c)^{\psi'} \otimes \alpha(a_2) \otimes \cdots \otimes \alpha(a_{n+1}) \otimes \alpha(a_1)_{\psi'}) \\ &\quad = (-1)^n f'(\gamma(c^{\psi}) \otimes \alpha(a_2) \otimes \cdots \otimes \alpha(a_{n+1}) \otimes \alpha(a_{1\psi})) \\ &\quad = (-1)^n (F^n(\alpha,\gamma) f')(c^{\psi} \otimes a_2 \otimes \cdots \otimes a_{n+1} \otimes a_{1\psi}). \end{aligned} \tag{7.4}$$

Hence, $(F^n(\alpha,\gamma) f') \in \mathcal{C}^n_\lambda(A, C, \psi)$. This proves the result. ■

Remark 7.2. The definition in (7.1) and the proof of Lemma 7.1 make sense even if the k-algebra morphism $\alpha : A \to A'$ is not unital.

Suppose that we have morphisms $(\alpha_1, \gamma_1) : (A, C, \psi) \to (A', C', \psi')$ and $(\alpha_2, \gamma_2) : (A', C', \psi') \to (A'', C'', \psi'')$ of entwining structures. Then, we note that $F^\bullet(\alpha_2 \circ \alpha_1, \gamma_2 \circ \gamma_1) = F^\bullet(\alpha_1, \gamma_1) \circ F^\bullet(\alpha_2, \gamma_2)$.

For any algebra A, let $\mathbb{U}(A) := \{x \in A \mid \exists y \in A \text{ such that } xy = yx = 1_A\}$ be the group of units of A. Given an entwining structure (A, C, ψ), we set

$$\mathbb{U}_\psi(A) := \{x \in \mathbb{U}(A) \mid \psi(c \otimes x) = x \otimes c \text{ for every } c \in C\}.$$

Lemma 7.3. *Let (A, C, ψ) be an entwining structure. Then, $\mathbb{U}_\psi(A)$ is a subgroup of $\mathbb{U}(A)$.*

Proof. Clearly, $1_A \in \mathbb{U}_\psi(A)$. Let $x, x' \in \mathbb{U}_\psi(A)$. Then, for any $c \in C$, we have

$$\psi(c \otimes xx') = (\theta \otimes \mathrm{id}_C)(\mathrm{id}_A \otimes \psi)(\psi \otimes \mathrm{id}_A)(c \otimes x \otimes x') = xx' \otimes c,$$

where $\theta : A \otimes A \to A$ is the product on A. Hence, $xx' \in \mathbb{U}_\psi(A)$.

Now let $x \in \mathbb{U}_\psi(A)$, and let $y \in \mathbb{U}(A)$ be its inverse. We will show that $y \in \mathbb{U}_\psi(A)$. For this, we set $\psi(c \otimes y) = \sum_i y_i \otimes c_i \in A \otimes C$. Then, we have

$$1_A \otimes c = \psi(c \otimes xy) = (\theta \otimes \mathrm{id}_C)(\mathrm{id}_A \otimes \psi)(\psi \otimes \mathrm{id}_A)(c \otimes x \otimes y) = \sum_i xy_i \otimes c_i.$$

Therefore, $y \otimes c = \sum_i yxy_i \otimes c_i = \sum_i y_i \otimes c_i = \psi(c \otimes y)$. Hence, $y \in \mathbb{U}_\psi(A)$. ∎

Lemma 7.4. *Let (A, C, ψ) be an entwining structure and let $x \in \mathbb{U}_\psi(A)$. Then,*

(1) *the pair $(\phi_x, \mathrm{id}_C) : (A, C, \psi) \to (A, C, \psi)$ is a morphism of entwining structures, where $\phi_x : A \to A$ is the inner automorphism given by $\phi_x(a) := xax^{-1}$ for all $a \in A$.*

(2) *the pair $(\Phi_x, \mathrm{id}_C) : (M_2(A), C, \psi) \to (M_2(A), C, \psi)$ is a morphism of entwining structures, where $\Phi_x : M_2(A) \to M_2(A)$ is the inner automorphism given by*

$$\Phi_x \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & x^{-1} \end{pmatrix}.$$

Proof. (1) Since $x, x^{-1} \in \mathbb{U}_\psi(A)$, we have

$$(\psi \circ (\mathrm{id}_C \otimes \phi_x))(c \otimes a) = \psi(c \otimes xax^{-1}) = xa_\psi x^{-1} \otimes c^\psi = ((\phi_x \otimes \mathrm{id}_C) \circ \psi)(c \otimes a)$$

for any $c \in C$ and $a \in A$. This shows that (ϕ_x, id_C) is a morphism of entwining structures.

(2) For any $c \in C$ and $\left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right) \in M_2(A)$, we have

$$\begin{aligned}(\psi \circ (\mathrm{id}_C \otimes \Phi_x))\left(c \otimes \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) &= \psi\left(c \otimes \begin{pmatrix} a_{11} & a_{12}x^{-1} \\ xa_{21} & xa_{22}x^{-1} \end{pmatrix}\right) \\ &= \begin{pmatrix} a_{11\psi} & a_{12\psi}x^{-1} \\ xa_{21\psi} & xa_{22\psi}x^{-1} \end{pmatrix} \otimes c^\psi \\ &= ((\Phi_x \otimes \mathrm{id}_C) \circ \psi)\left(c \otimes \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right).\end{aligned}$$

Hence, (Φ_x, id_C) is a morphism of entwining structures. ∎

For $x \in \mathbb{U}_\psi(A)$, we will always denote by $\phi_x : A \to A$ and $\Phi_x : M_2(A) \to M_2(A)$ the inner automorphisms described in Lemma 7.4.

Let (A, C, ψ) and (A', C', ψ') be entwining structures. Then, it may be easily seen that the tuple $(A \otimes A', C \otimes C', \psi \otimes \psi')$ is also an entwining structure with the entwining $\psi \otimes \psi' : (C \otimes C') \otimes (A \otimes A') \to (A \otimes A') \otimes (C \otimes C')$ given by

$$(\psi \otimes \psi')(c \otimes c' \otimes a \otimes a') = a_\psi \otimes a'_{\psi'} \otimes c^\psi \otimes c'^{\psi'}$$

for any $c \otimes c' \in C \otimes C'$ and $a \otimes a' \in A \otimes A'$.

Lemma 7.5. *Let (A, C, ψ) and (A', C', ψ') be entwining structures. Let $x \in \mathbb{U}_\psi(A)$ and $x' \in \mathbb{U}_{\psi'}(A')$. Then, $x \otimes x' \in \mathbb{U}_{\psi\otimes\psi'}(A \otimes A')$.*

Proof. This follows immediately from the definitions. ■

We now show that conjugation by a unit $x \in \mathbb{U}_\psi(A)$ induces the identity on cyclic cohomology of (A, C, ψ).

Proposition 7.6. *Let (A, C, ψ) be an entwining structure. For any $x \in \mathbb{U}_\psi(A)$, the pair $(\phi_x, \mathrm{id}_C) : (A, C, \psi) \to (A, C, \psi)$ induces the identity map on $H^\bullet_\lambda(A, C, \psi)$.*

Proof. According to Lemma 7.1, we have morphisms $\mathrm{inc}_1^\bullet = F^\bullet(\mathrm{inc}_1, \mathrm{id}_C)$, $\mathrm{inc}_2^\bullet = F^\bullet(\mathrm{inc}_2, \mathrm{id}_C) : H^\bullet_\lambda(M_r(A), C, \psi) \to H^\bullet_\lambda(A, C, \psi)$. By Theorem 6.4, the maps

$$\mathrm{inc}_1^\bullet : H^\bullet_\lambda(M_r(A), C, \psi) \longrightarrow H^\bullet_\lambda(A, C, \psi) \qquad \mathrm{tr}^\bullet : H^\bullet_\lambda(A, C, \psi) \longrightarrow H^\bullet_\lambda(M_r(A), C, \psi)$$

are mutually inverse isomorphisms. Therefore,

$$\begin{aligned} F^\bullet(\mathrm{inc}_2, \mathrm{id}_C) \circ (F^\bullet(\mathrm{inc}_1, \mathrm{id}_C))^{-1} &= F^\bullet(\mathrm{inc}_2, \mathrm{id}_C) \circ \mathrm{tr}^\bullet \\ &= F^\bullet(\mathrm{tr} \circ \mathrm{inc}_2, \mathrm{id}_C) = \mathrm{id}_{H^\bullet_\lambda(A,C,\psi)}. \end{aligned} \tag{7.5}$$

We notice that we have the following commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{\mathrm{inc}_1} & M_2(A) & \xleftarrow{\mathrm{inc}_2} & A \\ {\scriptstyle \mathrm{id}_A}\downarrow & & \downarrow{\scriptstyle \Phi_x} & & \downarrow{\scriptstyle \phi_x} \\ A & \xrightarrow{\mathrm{inc}_1} & M_2(A) & \xleftarrow{\mathrm{inc}_2} & A \end{array}$$

Using Lemma 7.4, we know that (ϕ_x, id_C) and (Φ_x, id_C) are morphisms of entwining structures. Therefore, we obtain the following commutative diagram:

$$\begin{array}{ccccc} H^\bullet_\lambda(A, C, \psi) & \xleftarrow{F^\bullet(\mathrm{inc}_1, \mathrm{id}_C)} & H^\bullet_\lambda(M_2(A), C, \psi) & \xrightarrow{F^\bullet(\mathrm{inc}_2, \mathrm{id}_C)} & H^\bullet_\lambda(A, C, \psi) \\ {\scriptstyle F^\bullet(\mathrm{id}_A, \mathrm{id}_C)}\uparrow & & \uparrow{\scriptstyle F^\bullet(\Phi_x, \mathrm{id}_C)} & & \uparrow{\scriptstyle F^\bullet(\phi_x, \mathrm{id}_C)} \\ H^\bullet_\lambda(A, C, \psi) & \xleftarrow{F^\bullet(\mathrm{inc}_1, \mathrm{id}_C)} & H^\bullet_\lambda(M_2(A), C, \psi) & \xrightarrow{F^\bullet(\mathrm{inc}_2, \mathrm{id}_C)} & H^\bullet_\lambda(A, C, \psi) \end{array}$$

Therefore, using (7.5), we get

$$\begin{aligned} F^\bullet(\phi_x, \mathrm{id}_C) = (F^\bullet(\mathrm{inc}_2, \mathrm{id}_C)) \circ F^\bullet(\mathrm{inc}_1, \mathrm{id}_C)^{-1} \circ F^\bullet(\mathrm{id}_A, \mathrm{id}_C) \\ \circ (F^\bullet(\mathrm{inc}_1, \mathrm{id}_C)) \circ F^\bullet(\mathrm{inc}_2, \mathrm{id}_C)^{-1} = \mathrm{id}_{H^\bullet_\lambda(A,C,\psi)}. \end{aligned}$$ ■

Proposition 7.7. *Let (A, C, ψ) be an entwining structure. Suppose that there is an algebra morphism $\nu : A \to A$ and an element $X \in \mathbb{U}_\psi(M_2(A))$ such that*

(1) *the pair* $(\nu, \mathrm{id}_C) : (A, C, \psi) \to (A, C, \psi)$ *is a morphism of entwining structures, that is,* $(\nu \otimes \mathrm{id}_C) \circ \psi = \psi \circ (\mathrm{id}_C \otimes \nu)$.

(2) *the inner automorphism* $\phi_X = X(\_\_)X^{-1} : M_2(A) \to M_2(A)$ *satisfies*

$$\phi_X \begin{pmatrix} a & 0 \\ 0 & \nu(a) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \nu(a) \end{pmatrix}$$

for all $a \in A$. *Then,* $H^\bullet_\lambda(A, C, \psi) = 0$.

Proof. Let $\alpha : A \to M_2(A)$ and $\beta : A \to M_2(A)$ be the algebra morphisms defined by

$$\alpha(a) = \begin{pmatrix} a & 0 \\ 0 & \nu(a) \end{pmatrix} = \mathrm{inc}_1(a) + (\mathrm{inc}_2 \circ \nu)(a)$$
$$\beta(a) = \begin{pmatrix} 0 & 0 \\ 0 & \nu(a) \end{pmatrix} = (\mathrm{inc}_2 \circ \nu)(a)$$

for $a \in A$. Since $(\mathrm{inc}_2 \circ \nu) \otimes \mathrm{id}_C = (\mathrm{inc}_2 \otimes \mathrm{id}_C) \circ (\nu \otimes \mathrm{id}_C)$ and using the fact that the pairs (ν, id_C), $(\mathrm{inc}_1, \mathrm{id}_C)$ and $(\mathrm{inc}_2, \mathrm{id}_C)$ are morphisms of entwining structures, we see that the pairs (α, id_C) and (β, id_C) are also morphisms of entwining structures from (A, C, ψ) to $(M_2(A), C, \psi)$.

Applying Lemma 7.1, we now have morphisms on cohomology groups

$$F^\bullet(\alpha, \mathrm{id}_C) : H^\bullet_\lambda(M_2(A), C, \psi) \longrightarrow H^\bullet_\lambda(A, C, \psi)$$
$$F^\bullet(\beta, \mathrm{id}_C) : H^\bullet_\lambda(M_2(A), C, \psi) \longrightarrow H^\bullet_\lambda(A, C, \psi).$$

Further, using assumption (2), we have $\phi_X \circ \alpha = \beta$. Applying Proposition 7.6 with $X \in \mathbb{U}_\psi(M_2(A))$, we obtain

$$F^\bullet(\beta, \mathrm{id}_C) = F^\bullet(\phi_X \circ \alpha, \mathrm{id}_C) = F^\bullet(\alpha, \mathrm{id}_C) \circ F^\bullet(\phi_X, \mathrm{id}_C) = F^\bullet(\alpha, \mathrm{id}_C). \tag{7.6}$$

Now let $g \in Z^n_\lambda(A, C, \psi)$. Using the isomorphism of cohomology groups in Theorem 6.4, we have $\widetilde{g} := \mathrm{tr}^n(g) = g \circ \mathrm{tr} \in Z^n_\lambda(M_2(A), C, \psi)$. Let $[\widetilde{g}]$ denote the cohomology class of $\widetilde{g}$. Then, using (7.6), we have $F^n(\alpha, \mathrm{id}_C)[\widetilde{g}] = F^n(\beta, \mathrm{id}_C)[\widetilde{g}]$ in $H^\bullet_\lambda(A, C, \psi)$. In other words,

$$\widetilde{g} \circ F_n(\alpha, \mathrm{id}_C) - \widetilde{g} \circ F_n(\beta, \mathrm{id}_C) \in B^n_\lambda(A, C, \psi)$$

so that

$$\widetilde{g} \circ F_n(\alpha, \mathrm{id}_C) - \widetilde{g} \circ F_n(\beta, \mathrm{id}_C) = \widetilde{g} \circ F_n(\mathrm{inc}_1, \mathrm{id}_C) = g \in B^n_\lambda(A, C, \psi).$$

This proves the result. ∎

For the remainder of this paper, we assume $k = \mathbb{C}$. Let $\mathbf{C}$ be the algebra of infinite matrices with complex entries $(a_{ij})_{i,j \geq 1}$ (see [13, p. 103]) satisfying the following two conditions:

(i) The set $\{a_{ij} \mid i, j \geq 1\}$ is finite.

(ii) The number of non-zero entries in each row or each column is bounded.

Let (A, C, ψ) be an entwining structure. Then, ψ extends to an entwining $C \otimes \mathbf{C} \otimes A \to \mathbf{C} \otimes A \otimes C$, which we continue to denote by ψ and determined by $c \otimes U \otimes a \mapsto U \otimes a_\psi \otimes c^\psi$ for any $c \in C$, $U \in \mathbf{C}$ and $a \in A$. Thus, $(\mathbf{C} \otimes A, C, \psi)$ is also an entwining structure. Similarly, ψ can also be extended to obtain an entwining structure $(M_2(\mathbf{C} \otimes A), C, \psi)$.

Lemma 7.8. *Let (A, C, ψ) be an entwining structure. Then, $H^\bullet_\lambda(\mathbf{C} \otimes A, C, \psi) = 0$.*

Proof. We will show that the entwining structure $(\mathbf{C} \otimes A, C, \psi)$ satisfies the assumptions in Proposition 7.7. By the result in [13, p. 104], we know that there exist an algebra morphism $\nu : \mathbf{C} \to \mathbf{C}$ and a unit $X \in M_2(\mathbf{C})$ such that the corresponding inner automorphism $\phi_X : M_2(\mathbf{C}) \to M_2(\mathbf{C})$ satisfies

$$\phi_X \begin{pmatrix} U & 0 \\ 0 & \nu(U) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \nu(U) \end{pmatrix} \tag{7.7}$$

for all $U \in \mathbf{C}$.

The map ν extends to an algebra morphism $\nu \otimes \mathrm{id}_A : \mathbf{C} \otimes A \to \mathbf{C} \otimes A$ such that

$$(\nu \otimes \mathrm{id}_A \otimes \mathrm{id}_C) \circ \psi = \psi \circ (\mathrm{id}_C \otimes \nu \otimes \mathrm{id}_A).$$

Hence, the pair $(\nu \otimes \mathrm{id}_A, \mathrm{id}_C)$ is a morphism of entwining structures $(\mathbf{C} \otimes A, C, \psi) \to (\mathbf{C} \otimes A, C, \psi)$. Moreover, under the identification $M_2(\mathbf{C} \otimes A) \cong M_2(\mathbf{C}) \otimes A$, we have the unit $X \otimes 1_A \in M_2(\mathbf{C} \otimes A)$. By definition, $\psi(c \otimes X \otimes 1_A) = X \otimes 1_A \otimes c$. Hence, $X \otimes 1_A \in \mathbb{U}_\psi(M_2(\mathbf{C}) \otimes A) = \mathbb{U}_\psi(M_2(\mathbf{C} \otimes A))$.

Clearly, $\phi_{X \otimes 1_A} = \phi_X \otimes \mathrm{id}_A : M_2(\mathbf{C}) \otimes A \to M_2(\mathbf{C}) \otimes A$. It then follows using (7.7) that

$$(\phi_X \otimes \mathrm{id}_A) \begin{pmatrix} U \otimes a & 0 \\ 0 & (\nu \otimes \mathrm{id}_A)(U \otimes a) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & (\nu \otimes \mathrm{id}_A)(U \otimes a) \end{pmatrix}$$

for any $U \otimes a \in \mathbf{C} \otimes A$. Hence, the entwining structure $(\mathbf{C} \otimes A, C, \psi)$ satisfies the assumptions in Proposition 7.7 and therefore, $H^\bullet_\lambda(\mathbf{C} \otimes A, C, \psi) = 0$. ∎

Definition 7.9. Let (A, C, ψ) be an entwining structure and $((R^\bullet, D^\bullet), C, \Psi^\bullet, T, \rho)$ be an n-dimensional entwined cycle over (A, C, ψ). Suppose that R^0 is a unital k-algebra and that $\Psi^0(c \otimes 1_{R^0}) = 1_{R^0} \otimes c$ for each $c \in C$. Then, we say that the cycle $((R^\bullet, D^\bullet), C, \Psi^\bullet, T, \rho)$ is vanishing if the entwining structure (R^0, C, Ψ^0) satisfies the assumptions in Proposition 7.7.

As a consequence of Theorem 4.5, we observe that the character of an n-dimensional entwined cycle over (A, C, ψ) always lies in $Z^n_\lambda(A, C, \psi)$. We will now describe the coboundaries $B^n_\lambda(A, C, \psi)$.

Theorem 7.10. *Let (A, C, ψ) be an entwining structure, and let $g \in Z^n_\lambda(A, C, \psi)$. Then, the following are equivalent:*

(1) $g \in B^n_\lambda(A, C, \psi)$.

(2) *g is the character of an n-dimensional entwined vanishing cycle over (A, C, ψ).*

Proof. (1) $\Rightarrow$ (2). Let $g \in B^n_\lambda(A, C, \psi)$. Then, $g = \delta(g')$ for some $g' \in \mathcal{C}^{n-1}_\lambda(A, C, \psi)$. Extending g', we obtain an element $\widehat{g}' \in \mathcal{C}^{n-1}(\mathbf{C} \otimes A, C, \psi)$ as follows:

$$\widehat{g}'(c \otimes (U_1 \otimes a_1) \otimes \cdots \otimes (U_n \otimes a_n)) := g'(c \otimes (U_1)_{11}a_1 \otimes \cdots \otimes (U_n)_{11}a_n)$$

for any $c \in C$ and $(U_1 \otimes a_1) \otimes \cdots \otimes (U_n \otimes a_n) \in (\mathbf{C} \otimes A)^n$. We have

$$\begin{aligned}
&\widehat{g}'(c^\psi \otimes (U_2 \otimes a_2) \otimes \cdots \otimes (U_n \otimes a_n) \otimes (U_1 \otimes a_1)_\psi) \\
&\quad = \widehat{g}'(c^\psi \otimes (U_2 \otimes a_2) \otimes \cdots \otimes (U_n \otimes a_n) \otimes (U_1 \otimes a_{1\psi})) \\
&\quad = g'(c^\psi \otimes (U_2)_{11}a_2 \otimes \cdots \otimes (U_n)_{11}a_n \otimes (U_1)_{11}a_{1\psi}) \\
&\quad = (-1)^{n-1} g'(c \otimes (U_1)_{11}a_1 \otimes (U_2)_{11}a_2 \otimes \cdots \otimes (U_n)_{11}a_n) \\
&\quad = (-1)^{n-1} \widehat{g}'(c \otimes (U_1 \otimes a_1) \otimes \cdots \otimes (U_n \otimes a_n)).
\end{aligned}$$

Hence, $\widehat{g}' \in \mathcal{C}^{n-1}_\lambda(\mathbf{C} \otimes A, C, \psi)$.

We now set $\widehat{g}'' := \delta(\widehat{g}') \in Z^n_\lambda(\mathbf{C} \otimes A, C, \psi)$. We also consider the algebra morphism $\rho : A \to \mathbf{C} \otimes A$ given by $a \mapsto I \otimes a$, where I is the identity matrix in $\mathbf{C}$. By the implication (3) $\Rightarrow$ (2) in Theorem 4.5, there exists an n-dimensional closed graded entwined trace t on the dg-entwining structure $((\Omega^\bullet(\mathbf{C} \otimes A), d^\bullet), C, \widehat{\psi})$ satisfying in particular that

$$\begin{aligned}
&t(c \otimes \rho(a_1) d(\rho(a_2)) \cdots d(\rho(a_{n+1}))) \\
&\quad = \widehat{g}''(c \otimes \rho(a_1) \otimes \cdots \otimes \rho(a_{n+1})) \\
&\quad = \widehat{g}''(c \otimes I \otimes a_1 \otimes \cdots \otimes I \otimes a_{n+1}) \\
&\quad = \delta(\widehat{g}')(c \otimes I \otimes a_1 \otimes \cdots \otimes I \otimes a_{n+1}) \\
&\quad = \delta(g')(c \otimes a_1 \otimes \cdots \otimes a_{n+1}) \\
&\quad = g(c \otimes a_1 \otimes \cdots \otimes a_{n+1}).
\end{aligned} \tag{7.8}$$

Since $(\rho, \mathrm{id}_C) : (A, C, \psi) \to (\mathbf{C} \otimes A, C, \psi)$ is a morphism of entwining structures, the tuple $((\Omega^\bullet(\mathbf{C} \otimes A), d^\bullet), C, \widehat{\psi}, t, \rho)$ is also an n-dimensional entwined cycle over (A, C, ψ). We notice that $\Omega^0(\mathbf{C} \otimes A) = \mathbf{C} \otimes A$ is unital and $\widehat{\psi}(c \otimes (I \otimes 1_A)) = (I \otimes 1_A) \otimes c$ for each $c \in C$. From the proof of Lemma 7.8, we now know that $((\Omega^\bullet(\mathbf{C} \otimes A), d^\bullet), C, \widehat{\psi}, t, \rho)$ is a vanishing cycle. From (7.8) it is clear that $g \in B^n_\lambda(A, C, \psi)$ is the character of this vanishing cycle.

(2) $\Rightarrow$ (1). Let $g \in Z^n_\lambda(A, C, \psi) \subseteq \mathcal{C}^n_\lambda(A, C, \psi)$ be the character of an n-dimensional entwined vanishing cycle $((R^\bullet, D^\bullet), C, \Psi^\bullet, T, \rho)$ over the entwining structure (A, C, ψ). Then, by definition, the tuple (R^0, C, Ψ^0) is an entwining structure. We define $f \in \mathcal{C}^n(R^0, C, \Psi^0)$ by setting

$$f(c \otimes r_1 \otimes \cdots \otimes r_{n+1}) := T(c \otimes r_1 D(r_2) \cdots D(r_{n+1}))$$

for any $c \otimes r_1 \otimes \cdots \otimes r_{n+1} \in C \otimes (R^0)^{\otimes n+1}$. Since $\Psi^\bullet$ is a morphism of complexes and T is a closed graded entwined trace of dimension n, we have

$$\begin{aligned} & f(c^{\Psi^0} \otimes r_2 \otimes \cdots \otimes r_{n+1} \otimes r_{1\Psi^0}) \\ &= T(c^{\Psi^0} \otimes r_2 D(r_3) \cdots D(r_{n+1}) D(r_{1\Psi^0})) \\ &= T(c^{\Psi^1} \otimes r_2 D(r_3) \cdots D(r_{n+1}) D(r_1)_{\Psi^1}) \\ &= (-1)^{n-1} T(c \otimes D(r_1) r_2 D(r_3) \cdots D(r_{n+1})) \\ &= (-1)^{n-1} (T(c \otimes D(r_1 r_2) D(r_3) \cdots D(r_{n+1})) \\ &\qquad - T(c \otimes r_1 D(r_2) D(r_3) \cdots D(r_{n+1}))) \\ &= (-1)^n T(c \otimes r_1 D(r_2) D(r_3) \cdots D(r_{n+1})) \\ &= (-1)^n f(c \otimes r_1 \otimes \cdots \otimes r_{n+1}). \end{aligned}$$

This shows that $f \in \mathcal{C}^n_\lambda(R^0, C, \Psi^0)$. Moreover, using the implication (1) $\Rightarrow$ (3) in Theorem 4.5, we see that $f \in Z^n_\lambda(R^0, C, \Psi^0)$. Since $((R^\bullet, D^\bullet), C, \Psi^\bullet, T, \rho)$ is a vanishing cycle, we know that $H^\bullet_\lambda(R^0, C, \Psi^0) = 0$. Hence, $f = \delta f'$ for some $f' \in \mathcal{C}^{n-1}_\lambda(R^0, C, \Psi^0)$.

Since $((R^\bullet, D^\bullet), C, \Psi^\bullet)$ is a dg-entwining structure over (A, C, ψ), we have

$$(\rho \otimes \mathrm{id}_C) \circ \psi = \Psi^0 \circ (\mathrm{id}_C \otimes \rho), \tag{7.9}$$

that is, the pair $(\rho, \mathrm{id}_C) : (A, C, \psi) \to (R^0, C, \Psi^0)$ is a morphism of entwining structures. As in the proof of Lemma 7.1, it is clear that we have an induced morphism of complexes

$$F^\bullet(\rho, \mathrm{id}_C) : \mathcal{C}^\bullet_\lambda(R^0, C, \Psi^0) \longrightarrow \mathcal{C}^\bullet_\lambda(A, C, \psi).$$

We set $f'' := F^{n-1}(\rho, \mathrm{id}_C)(f') \in \mathcal{C}^{n-1}_\lambda(A, C, \psi)$. Then,

$$f''(c \otimes a_1 \otimes \cdots \otimes a_n) = f'(c \otimes \rho(a_1) \otimes \cdots \otimes \rho(a_n))$$

for any $c \otimes a_1 \otimes \cdots \otimes a_n \in C \otimes A^n$. Since $F^\bullet(\rho, \mathrm{id}_C)$ is a morphism of complexes, we must have

$$(\delta f'')(c \otimes a_1 \otimes \cdots \otimes a_{n+1}) = (\delta f')(c \otimes \rho(a_1) \otimes \cdots \otimes \rho(a_{n+1}))$$

for any $c \otimes a_1 \otimes \cdots \otimes a_{n+1} \in C \otimes A^{n+1}$. Thus, we obtain

$$\begin{aligned} g(c \otimes a_1 \otimes \cdots \otimes a_{n+1}) &= T(c \otimes \rho(a_1) D(\rho(a_2)) \cdots D(\rho(a_{n+1}))) \\ &= f(c \otimes \rho(a_1) \otimes \cdots \otimes \rho(a_{n+1})) \\ &= (\delta f')(c \otimes \rho(a_1) \otimes \cdots \otimes \rho(a_{n+1})) \\ &= (\delta f'')(c \otimes a_1 \otimes \cdots \otimes a_{n+1}). \end{aligned}$$

Hence, $g = \delta f''$, where $f'' \in \mathcal{C}^{n-1}_\lambda(A, C, \psi)$. This proves the result. ∎

Taken together, Theorems 4.5 and 7.10 give a complete description of the cocycles and coboundaries of the complex $\mathcal{C}^\bullet_\lambda(A, C, \psi)$ in terms of entwined cycles over (A, C, ψ). We conclude by applying these descriptions to obtain a pairing on cyclic cohomologies.

Theorem 7.11. *Let (A, C, ψ) and (A', C', ψ') be entwining structures. Then, we have a pairing*

$$H_\lambda^m(A, C, \psi) \otimes H_\lambda^n(A', C', \psi') \longrightarrow H_\lambda^{m+n}(A \otimes A', C \otimes C', \psi \otimes \psi')$$

for any $m, n \geq 0$.

Proof. Let $g \in Z_\lambda^m(A, C, \psi)$ and $g' \in Z_\lambda^n(A', C', \psi')$. Then, by Theorem 4.5, we know that g (resp. g') must be the character of an m (resp. n)-dimensional entwined cycle $((R^\bullet, D^\bullet), C, \Psi^\bullet, T, \rho)$ (resp. $((R'^\bullet, D'^\bullet), C', \Psi'^\bullet, T', \rho')$) over (A, C, ψ) (resp. (A', C', ψ')).

We know that $((R \otimes R')^\bullet, (D \otimes D')^\bullet)$ is a differential graded algebra, where $(R \otimes R')^p = \bigoplus_{i+j=p} R^i \otimes R'^j$ for $p \geq 0$. The multiplication and differential on $R \otimes R'$ are given, respectively, by

$$\begin{aligned}(r_1 \otimes r_1')(r_2 \otimes r_2') &= (-1)^{\deg(r_1')\deg(r_2)}(r_1 r_2 \otimes r_1' r_2')\\ (D \otimes D')(r \otimes r') &= D(r) \otimes r' + (-1)^{\deg(r)} r \otimes D'(r')\end{aligned}$$

for homogeneous elements $r_1, r_2, r \in R$ and $r_1', r_2', r' \in R'$. Using the fact that Ψ is a map of degree zero and both Ψ, Ψ' are morphisms of complexes, we have

$$\begin{aligned}&((D \otimes D' \otimes \mathrm{id}_{C \otimes C'}) \circ (\Psi \otimes \Psi'))(c \otimes c' \otimes r \otimes r')\\ &\quad = (D \otimes D' \otimes \mathrm{id}_{C \otimes C'})(r_\Psi \otimes r'_{\Psi'} \otimes c^\Psi \otimes c'^{\Psi'})\\ &\quad = D(r_\Psi) \otimes r'_{\Psi'} \otimes c^\Psi \otimes c'^{\Psi'} + (-1)^{\deg(r_\Psi)} r_\Psi \otimes D'(r'_{\Psi'}) \otimes c^\Psi \otimes c'^{\Psi'}\\ &\quad = D(r)_\Psi \otimes r'_{\Psi'} \otimes c^\Psi \otimes c'^{\Psi'} + (-1)^{\deg(r)} r_\Psi \otimes D'(r')_{\Psi'} \otimes c^\Psi \otimes c'^{\Psi'}\\ &\quad = (\Psi \otimes \Psi')(c \otimes c' \otimes (D \otimes D')(r \otimes r')),\end{aligned}$$

where $c \otimes c' \in C \otimes C'$ and r, r' are homogeneous elements of R and R', respectively.

This shows that $\Psi \otimes \Psi'$ is a morphism of complexes. Thus, we obtain a dg-entwining structure $(((R \otimes R')^\bullet, (D \otimes D')^\bullet), C \otimes C', \Psi \otimes \Psi')$. It is also clear that the tuple $\big(((R \otimes R')^\bullet, (D \otimes D')^\bullet), C \otimes C', \Psi \otimes \Psi', \rho \otimes \rho'\big)$ is a dg-entwining structure over $(A \otimes A', C \otimes C', \psi \otimes \psi')$. We will now construct a closed graded entwined trace of dimension $(m + n)$ on this dg-entwining structure.

For this, we consider the k-linear map $T \otimes T' : (C \otimes C') \otimes (R \otimes R')^{m+n} \to k$ given by

$$(T \otimes T')\bigg(\bigoplus_{i+j=m+n} (c \otimes c' \otimes r_i \otimes r_j')\bigg) := T(c \otimes r_m)T'(c \otimes r_n')$$

for any $c \otimes c' \in C \otimes C'$ and $\bigoplus_{i+j=m+n} r_i \otimes r_j' \in (R \otimes R')^{m+n}$.

It may be easily verified that $T \otimes T'$ satisfies the condition in (4.2). Further, using the fact that T and T' are graded entwined traces, we have

$$\begin{aligned}
&(T \otimes T')(c \otimes c' \otimes (r \otimes r')(s \otimes s'))\\
&\quad = (-1)^{\deg(r')\deg(s)}(T \otimes T')(c \otimes c' \otimes rs \otimes r's')\\
&\quad = (-1)^{\deg(r')\deg(s)}T(c \otimes (rs)_m)T'(c \otimes (r's')_n)\\
&\quad = (-1)^{\deg(r')\deg(s)}(-1)^{\deg(r)\deg(s)}(-1)^{\deg(r')\deg(s')}\\
&\qquad \cdot T(c^{\Psi} \otimes (sr_{\Psi})_m)T'(c'^{\Psi'} \otimes (s'r'_{\Psi'})_n)\\
&\quad = (-1)^{\deg(r')\deg(s)}(-1)^{\deg(r)\deg(s)}(-1)^{\deg(r')\deg(s')}\\
&\qquad (-1)^{\deg(s')\deg(r_{\Psi})}(T \otimes T')(c^{\Psi} \otimes c'^{\Psi'} \otimes (s \otimes s')(r_{\Psi} \otimes r'_{\Psi'}))\\
&\quad = (-1)^{\deg(r\otimes r')\deg(s\otimes s')}(T \otimes T')((c \otimes c')^{\Psi\otimes\Psi'} \otimes (s \otimes s')(r \otimes r')_{\Psi\otimes\Psi'})
\end{aligned}$$

for any $c \otimes c' \in C \otimes C'$ and homogeneous elements $r, s \in R$ and $r', s' \in R'$. This shows that $T \otimes T'$ satisfies the condition in (4.3) Hence, $T \otimes T'$ is an $(m+n)$-dimensional closed graded entwined trace.

Therefore, $(((R \otimes R')^\bullet, (D \otimes D')^\bullet), C \otimes C', \Psi \otimes \Psi', T \otimes T', \rho \otimes \rho')$ is an $(m+n)$-dimensional entwined cycle over the entwining structure $(A \otimes A', C \otimes C', \psi \otimes \psi')$. Using Theorem 4.5, we know that the character of this cycle, denoted by $g \otimes g'$, lies in $Z_\lambda^{m+n}(A \otimes A', C \otimes C', \psi \otimes \psi')$. The association $(g, g') \mapsto g \otimes g'$ gives a pairing

$$\xi : Z_\lambda^m(A, C, \psi) \otimes Z_\lambda^n(A', C', \psi') \longrightarrow Z_\lambda^{m+n}(A \otimes A', C \otimes C', \psi \otimes \psi'). \tag{7.10}$$

From the equivalence of (1) and (2) in Theorem 4.5, it is clear that this pairing does not depend on the choice of the cycles $((R^\bullet, D^\bullet), C, \Psi^\bullet, T, \rho)$ and $((R'^\bullet, D'^\bullet), C', \Psi'^\bullet, T', \rho')$ determining g and g', respectively.

To induce the pairing on cohomologies, it suffices to show that ξ restricts to a pairing

$$B_\lambda^m(A, C, \psi) \otimes Z_\lambda^n(A', C', \psi') \longrightarrow B_\lambda^{m+n}(A \otimes A', C \otimes C', \psi \otimes \psi').$$

Let $g \in B_\lambda^m(A, C, \psi)$. Then, by Theorem 7.10, we know that g is the character of an m-dimensional entwined vanishing cycle $((R^\bullet, D^\bullet), C, \Psi^\bullet, T, \rho)$ over (A, C, ψ). In particular, by Definition 7.9, we know that R^0 is unital and $\Psi^0(c \otimes 1_{R^0}) = 1_{R^0} \otimes c$ for each $c \in C$. Using the implication (1) $\Rightarrow$ (2) in Theorem 4.5, we might as well assume that R'^0 is unital and that $\Psi'^0(c' \otimes 1_{R'^0}) = 1_{R'^0} \otimes c'$ for each $c' \in C'$. In fact, we might even assume that $(R'^\bullet, D'^\bullet) = (\Omega^\bullet A', d'^\bullet)$.

It now suffices to show that the cycle $\big(((R \otimes R')^\bullet, (D \otimes D')^\bullet), C \otimes C', \Psi \otimes \Psi', T \otimes T', \rho \otimes \rho'\big)$ used in (7.10) is a vanishing cycle. In other words, we need to verify that the entwining structure $(R^0 \otimes R'^0, C \otimes C', \Psi^0 \otimes \Psi'^0)$ satisfies the assumptions in Proposition 7.7.

Since $((R^\bullet, D^\bullet), C, \Psi^\bullet, T, \rho)$ is a vanishing cycle, there exist an algebra morphism $\nu : R^0 \to R^0$ and a unit $X \in \mathbb{U}_{\Psi^0}(M_2(R^0))$ satisfying the assumptions in Proposition 7.7.

Extending ν, we have the algebra morphism $\nu \otimes \mathrm{id}_{R'^0} : R^0 \otimes R'^0 \to R^0 \otimes R'^0$. Identifying $M_2(R^0 \otimes R'^0) \cong M_2(R^0) \otimes R'^0$ and using Lemma 7.5, we have $X \otimes 1_{R'^0} \in \mathbb{U}_{\Psi^0 \otimes \Psi'^0}(M_2(R_0) \otimes R'^0) \cong \mathbb{U}_{\Psi^0 \otimes \Psi'^0}(M_2(R_0 \otimes R'^0))$. Clearly, the pair $(\nu \otimes \mathrm{id}_{R'^0}, \mathrm{id}_C \otimes \mathrm{id}_{C'})$ is a morphism of entwining structures. Identifying $\phi_{X \otimes 1_{R'^0}} = \phi_X \otimes \mathrm{id}_{R'^0} : M_2(R^0 \otimes R'^0) \to M_2(R^0 \otimes R'^0)$, we also see that

$$(\phi_X \otimes \mathrm{id}_{R'^0}) \begin{pmatrix} r \otimes r' & 0 \\ 0 & (\nu \otimes \mathrm{id}_{R'^0})(r \otimes r') \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & (\nu \otimes \mathrm{id}_{R'^0})(r \otimes r') \end{pmatrix}$$

for any $r \otimes r' \in R^0 \otimes R'^0$.

Thus, all the assumptions in Proposition 7.7 are satisfied by the entwining structure $(R^0 \otimes R'^0, C \otimes C', \Psi^0 \otimes \Psi'^0)$. Hence, $(((R \otimes R')^\bullet, (D \otimes D')^\bullet), C \otimes C', \Psi \otimes \Psi', T \otimes T', \rho \otimes \rho')$ is a vanishing cycle. This proves the result. ∎

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References

[1] J. Y. Abuhlail, Dual entwining structures and dual entwined modules. *Algebr. Represent. Theory* **8** (2005), no. 2, 275–295 Zbl 1103.16022 MR 2162287

[2] M. Balodi, A. Banerjee, and A. Naolekar, Weak comp algebras and cup products in secondary Hochschild cohomology of entwining structures. [v1] 2019, [v4] 2021, arXiv:1909.05476v4

[3] M. Balodi, A. Banerjee, and S. Ray, Entwined modules over linear categories and Galois extensions. *Israel J. Math.* **241** (2021), no. 2, 623–692 Zbl 1466.18006 MR 4242540

[4] T. Brzeziński, On modules associated to coalgebra Galois extensions. *J. Algebra* **215** (1999), no. 1, 290–317 Zbl 0936.16030 MR 1684158

[5] T. Brzeziński, The cohomology structure of an algebra entwined with a coalgebra. *J. Algebra* **235** (2001), no. 1, 176–202 Zbl 0984.16035 MR 1807661

[6] T. Brzeziński, The structure of corings: induction functors, Maschke-type theorem, and Frobenius and Galois-type properties. *Algebr. Represent. Theory* **5** (2002), no. 4, 389–410 Zbl 1025.16017 MR 1930970

[7] T. Brzeziński, The structure of corings with a grouplike element. In *Noncommutative geometry and quantum groups (Warsaw, 2001)*, pp. 21–35, Banach Center Publ. 61, Polish Acad. Sci. Inst. Math., Warsaw, 2003 Zbl 1066.16034 MR 2024419

[8] T. Brzeziński and S. Majid, Coalgebra bundles. *Comm. Math. Phys.* **191** (1998), no. 2, 467–492 Zbl 0899.55016 MR 1604340

[9] D. Bulacu, S. Caenepeel, and B. Torrecillas, Frobenius and separable functors for the category of entwined modules over cowreaths, II: applications. *J. Algebra* **515** (2018), 236–277 Zbl 1472.16028 MR 3859966

[10] D. Bulacu, S. Caenepeel, and B. Torrecillas, Frobenius and separable functors for the category of entwined modules over cowreaths, I: general theory. *Algebr. Represent. Theory* **23** (2020), no. 3, 1119–1157 Zbl 1447.16031 MR 4109150

[11] S. Caenepeel and E. De Groot, Modules over weak entwining structures. In *New trends in Hopf algebra theory (La Falda, 1999)*, pp. 31–54, Contemp. Math. 267, American Mathematical Society, Providence, RI, 2000 Zbl 0978.16033 MR 1800705

[12] A. Connes, Cohomologie cyclique et foncteurs Ext^n. *C. R. Acad. Sci. Paris Sér. I Math.* **296** (1983), no. 23, 953–958 Zbl 0534.18009 MR 0777584
[13] A. Connes, Non-commutative differential geometry. *Publ. Math. Inst. Hautes Études Sci.* **62** (1985), no. 1, 41–144 Zbl 0592.46056 MR 0823176
[14] L. Jia, The sovereign structure on categories of entwined modules. *J. Pure Appl. Algebra* **221** (2017), no. 4, 867–874 Zbl 1376.16033 MR 3574212
[15] J.-L. Loday, *Cyclic homology*. 2nd edn., Grundlehren Math. Wiss. 301, Springer, Berlin, Heidelberg, 1998 Zbl 0885.18007 MR 1600246
[16] P. Schauenburg, Doi–Koppinen Hopf modules versus entwined modules. *New York J. Math.* **6** (2000), 325–329 Zbl 0969.16015 MR 1800353

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