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Quantum SU(3) as the C^*-algebra of a 2-graph

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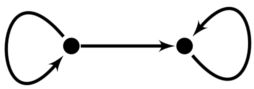
Abstract. We show that, for $q \in (0,1)$, the C^*-algebra $\mathrm{SU}_q(3)$ is isomorphic to a rank 2 graph C^*-algebra (in the sense of Kumjian and Pask), and describe the graph in terms of its skeleton and commutation relations. Moreover, this isomorphism is $\mathbb{T}^2$-equivariant with respect to the right action on $\mathrm{SU}_q(3)$ and the gauge action coming from the 2-graph.

1. introduction

We observe that certain quantised algebras of continuous functions on homogeneous spaces can be given the structure of graph C^*-algebras. A well-known example is the case of the Soibelman–Vaksman quantum odd spheres [11]. In particular, $\mathrm{SU}_q(2)$ – the algebra of quantised continuous functions on SU(2) introduced by S. L. Woronowicz in [12] – is isomorphic to the graph C^*-algebra of the directed graph



([4, Proposition 2.1]). A natural question to ask would be if a similar result holds for $\mathrm{SU}_q(n)$ when $n \geq 3$. However, the primitive spectrum of these C^*-algebras will topologically contain the space $\mathbb{T}^{n-1}$ and it is known that such C^*-algebras cannot come from directed graphs (see the [4, end of the introduction]).

In [6] Alex Kumjian and David Pask generalised the notion of a C^*-algebra constructed from a directed graph, by introducing higher-rank graphs together with their associated Cuntz–Krieger algebras. In a graph of degree k (or k-graph for short), the usual dots-and-arrow presentation is replaced by a countable category $\mathcal{G}$ together with a functor $\mathrm{r}\colon \mathcal{G} \to \mathbb{N}^k$ called the *rank functor* (the semi-group $\mathbb{N}^k$ is considered as a category with one element). In this context, a 1-graph is equivalent to a category generated by a directed graph.

In this paper we show that the compact quantum group $\mathrm{SU}_q(3)$ (as a C^*-algebra) is isomorphic to a C^*-algebra coming from a 2-graph $\mathrm{r}\colon \mathcal{S}u(3) \to \mathbb{N}^2$, whose 2-skeleton

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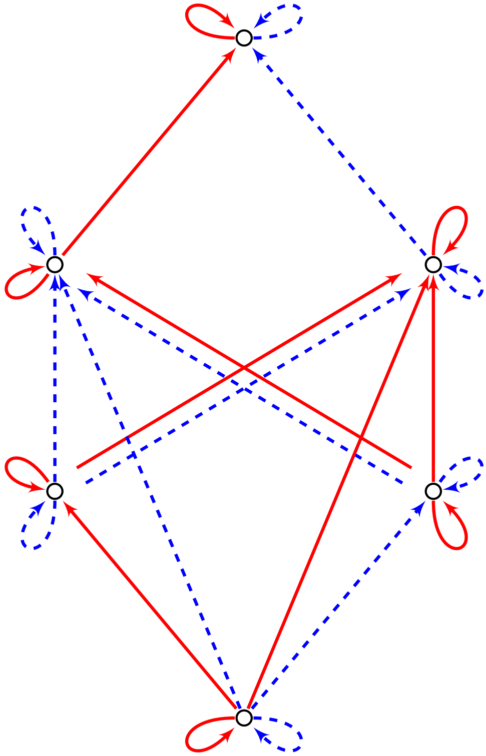
(i.e., the set of objects in $\mathcal{S}u(3)$ together with the pre-images of morphisms $\mathrm{r}^{-1}(1,0)$ and $\mathrm{r}^{-1}(0,1)$) can be visualised by the coloured directed graph



We get this graph by passing to the limit $q \to 0$ for an set of $\mathrm{SU}_q(3)$-generators under the usual faithful Soibelman representation. This is similar to how the graph of $\mathrm{SU}_q(2)$ is derived, where one passes to the limit $q \to 0$ on appropriate generators, and then show that the resulting limits still generates the same C^*-algebra. However, due to the additional dimension, some further steps are needed. Under the Soibelman representation, we realise $\mathrm{SU}_q(3)$ as a C^*-subalgebra of $C^*(S)^{\otimes 3} \otimes C(\mathbb{T}) \otimes C(\mathbb{T})$ (here $C^*(S)$ denotes the C^*-algebra generated by the forward shift $S \in \mathcal{B}(\ell^2(\mathbb{N}))$, i.e., the Toeplitz algebra), and the issue is that these C^*-subalgebras varies as a function of q, even though they are all isomorphic (see [3, 8] for the latter claim). In order to then get an isomorphism with the limit at 0, one must first straighten out the field of C^*-algebras. Unfortunately, this method is non-constructive and based on a lifting lemma, so that, in comparison with the $\mathrm{SU}_q(2)$-case, it is much less clear how the 2-graph "sits" inside of $\mathrm{SU}_q(3)$.

The topic of this paper connects with some similar recent investigations. In [7], M. Matassa and R. Yuncken showed that, for any compact semisimple Lie group K, if one looks at an appropriate $*$-subalgebra of its quantised coordinate ring $\mathcal{O}_q[K]$ (of elements regular at $q = 0$), then, under the faithful Soibelman representation, the limit $q \to 0$ exists for every element in this subalgebra, and the resulting $*$-algebra can be described as a

Kumjian–Pask algebra coming from a higher rank graph (with the rank of the graph equal the rank of K). In particular, in the case of SU(3), Matassa and Yuncken derive the same 2-graph as we do here. Whereas our 2-graph was the result of empirical investigations, they derive theirs from the theory of crystal bases. It thus seems reasonable to expect that the results here can be generalised to hold for every compact semisimple Lie group K.

Another recent approach to understand the limit $q = 0$ in the case of $\mathrm{SU}_q(n)$ was given by M. Giri and A. K. Pal in [2]. They gave a description of $\mathrm{SU}_0(n)$ in terms of generators and relations. They focus in particular on the case $\mathrm{SU}_0(3)$ and classifies its irreducible representations.

1.1. Higher rank graph C^*-algebras

Graph C^*-algebras of higher rank are a generalization of graph C^*-algebras that were introduced by Kumjian and Pask in [6]. We give the definition. For a small category $\mathcal{C}$, we let $\mathcal{C}_{\mathrm{obj}}$ denote the set of objects of $\mathcal{C}$ and $\mathcal{C}_{\mathrm{arr}}$ the set of arrow of $\mathcal{C}$.

Definition 1 (Graph of rank k). Consider the abelian semigroup $\mathbb{N}^k$ (we will assume that $0 \in \mathbb{N}$). Viewing $\mathbb{N}^k$ as a category with one object, with arrows the elements in $\mathbb{N}^k$, and with composition of arrows $\mathbf{a}, \mathbf{b} \in \mathbb{N}^k$ is the sum $\mathbf{a} + \mathbf{b} \in \mathbb{N}^k$. A *graph of rank k* (or k-graph for short) is a countable category $\mathcal{F}$ together with a functor $\mathrm{r}\colon \mathcal{F} \to \mathbb{N}^k$, called the *rank functor*, such that if $\mathrm{r}(f) = \mathbf{a} + \mathbf{b} \in \mathbb{N}^k$, for $\mathbf{a}, \mathbf{b} \in \mathbb{N}^k$ then there are *unique* arrows g and h such that $\mathrm{r}(g) = \mathbf{a}$, $\mathrm{r}(h) = \mathbf{b}$ and $f = g \circ h$.

It is easy to see from the definitions that the pre-image $\mathrm{r}^{-1}(0)$ is the set of identity arrows of $\mathcal{F}$. Moreover, the category $\mathcal{F}$ has the property that if $f = g \circ h = g' \circ h$, then $g = g'$, and similarly if $f = g \circ h = g \circ h'$, then $h = h'$, i.e., every arrow in $\mathcal{F}$ is both a monomorphism and an epimorphism.

Following [6], we introduce some notations and terminology about higher rank graph. Let $\mathcal{F}$ be a graph of rank k.

- For $\mathbf{n} \in \mathbb{N}^k$, let $\mathcal{F}^{\mathbf{n}} := \mathrm{r}^{-1}(\mathbf{n})$. It is easy to see that we can identify $\mathcal{F}^{\mathbf{0}}$ with $\mathcal{F}_{\mathrm{obj}}$.
- For $f \in \mathcal{F}_{\mathrm{arr}}$, let $r(f) := \mathrm{cod}(f) \in \mathcal{F}_{\mathrm{obj}}$ and $s(f) := \mathrm{dom}(f) \in \mathcal{F}_{\mathrm{obj}}$ (range and source maps).
- For $E \subseteq \mathcal{F}_{\mathrm{arr}}$ and $f \in \mathcal{F}_{\mathrm{arr}}$, let $fE := \{f \circ g \mid g \in E,\ r(g) = s(f)\}$ and similarly $Ef := \{g \circ f \mid s(g) = r(f)\, g \in E\}$. Moreover, for $v \in \mathcal{F}_{\mathrm{obj}}$ we write vE for the set $\mathrm{Id}_v\, \mathcal{F} = \{g \in E \mid r(g) = v\}$, i.e., the set of elements in E with range v. Similarly we write Ev for the set $\{g \in E \mid s(g) = v\}$.
- If $f \in \mathcal{F}_{\mathrm{arr}}$ and $\mathbf{n} \leq \mathbf{m} \leq \mathbf{j} = \mathrm{r}(f)$ in the partial ordering of $\mathbb{N}^k$, then we write $f(0, \mathbf{n})$, $f(\mathbf{n}, \mathbf{m})$, and $f(\mathbf{m}, \mathbf{j})$ for the unique paths of degree $\mathbf{n}$, $\mathbf{m} - \mathbf{n}$, and $\mathbf{j} - \mathbf{m}$ respectively, such that $f = f(0, \mathbf{n}) \circ f(\mathbf{n}, \mathbf{m}) \circ f(\mathbf{m}, \mathbf{j})$.
- We call a k-graph $\mathcal{F}$ *row-finite* if $|v\, \mathcal{F}^{\mathbf{n}}| < \infty$ for all $\mathbf{n} \in \mathbb{N}^k$ and $v \in \mathcal{F}_{\mathrm{obj}}$.

- For $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$, we write $\mathcal{F}^{\leq \mathbf{n}}$ for the set
$$\mathcal{F}^{\leq \mathbf{n}} = \{f \in \mathcal{F} \mid \mathrm{r}(f) \leq \mathbf{n} \text{ and } \mathrm{r}(f)_i < n_i \implies s(f)\,\mathcal{F}^{e_i} = \emptyset\},$$
where the subscript i denote the i-th coordinate in $\mathbb{N}^k$ and $e_i \in \mathbb{N}^k$ is the element with a 1 in the i-th coordinate and 0 elsewhere.
- We say that $\mathcal{F}$ is *locally convex* if, whenever $j \neq i$, $f \in \mathcal{F}^{e_j}$, $g \in \mathcal{F}^{e_i}$, and $r(f) = r(g)$, we have $s(f)\,\mathcal{F}^{e_i} \neq \emptyset$ and $s(g)\,\mathcal{F}^{e_j} \neq \emptyset$.

Remark. The higher rank graphs considered in this text will have the property that $v\,\mathcal{F}^{e_i} \neq \emptyset$ and $\mathcal{F}^{e_i}\,v \neq \emptyset$, for every $v \in \mathcal{F}_{\mathrm{obj}}$ and $i = 1, \ldots, k$. Thus, they will be locally convex, and we will have $\mathcal{F}^{\leq \mathbf{n}} = \mathcal{F}^{\mathbf{n}}$.

We give the definition of a higher-rank graph C^*-algebra.

Definition 2. Let $\mathcal{F}$ be a locally convex, row-finite graph of rank k. The C^*-algebra $C^*(\mathcal{F})$ is the universal enveloping C^*-algebra generated by orthogonal projections $\{P_\sigma \mid \sigma \in \mathcal{F}_{\mathrm{obj}}\}$ and partial isometries $\{S_f \mid f \in \mathcal{F}_{\mathrm{arr}}\}$ subject to the Cuntz–Krieger relations:

(CK1) the projections $\{P_v \mid v \in \mathcal{F}_{\mathrm{obj}}\}$ are all mutually orthogonal,

(CK2) $S_f S_g = S_{f \circ g}$ whenever $s(f) = r(g)$,

(CK3) $S_f^* S_f = P_{s(f)}$ for all $f \in \mathcal{F}_{\mathrm{arr}}$, and

(CK4) $P_v = \sum_{f \in v\,\mathcal{F}^{\leq \mathbf{n}}} S_f S_f^*$ for all $v \in \mathcal{F}_{\mathrm{obj}}$ and $\mathbf{n} \in \mathbb{N}^k$.

For a k-graph $\mathcal{F}$, we have a natural homomorphism $\alpha\colon \mathbb{T}^k \to \mathrm{Aut}(C^*(\mathcal{F}))$ determined on generators as
$$\alpha_z(S_f) = t^{\mathrm{r}(f)} S_f,$$
where $t^{\mathrm{r}(f)} = (t_1^{\mathrm{r}(f)_1}, \ldots, t_k^{\mathrm{r}(f)_k})$ (with $t^0 = 1$). Clearly, by (CK2) and the functorial property of r for every $t \in \mathbb{T}^k$, this map extends to an automorphism of $C^*(\mathcal{F})$.

Definition 3. The set of homomorphisms $\{\alpha_t \mid t \in \mathbb{T}^k\}$ is called the *gauge action* on $C^*(\mathcal{F})$.

A useful result in the representation theory of higher rank graph C^*-algebras is the "Gauge Invariance Uniqueness Theorem" due to A. Kumjian and D. Pask.

Theorem 4 ([6, Theorem 3.4]). *Let $\mathcal{F}$ be a k-graph, let B be a C^*-algebra, and let $\pi\colon C^*(\mathcal{F}) \to B$ be a homomorphism. If we have an action $\beta\colon \mathbb{T}^k \to \mathrm{Aut}(B)$ such that $\pi \circ \alpha_t = \beta_t \circ \pi$ for all $t \in \mathbb{T}^k$. Then π is faithful if and only if $\pi(P_\sigma) \neq 0$ for all $\sigma \in \mathcal{F}^0$.*

2. $\mathrm{SU}_q(3)$ as a rank-2 graph C^*-algebra

The proof of the main result is divided into two steps.

(i) Give a proper definition of the C^*-algebra

$$\mathrm{SU}_0(3) := \lim_{q \to 0} \mathrm{SU}_q(3),$$

and finding an appropriate 2-graph $\mathcal{S}u(3)$ such that $\mathrm{SU}_0(3) \cong C^*(\mathcal{S}u(3))$.

(ii) Show that $\mathrm{SU}_0(3) \cong \mathrm{SU}_q(3)$ for all $0 < q < 1$.

Remark. Step (i) was done more generally in [7]. The obstacle to generalizing the main result here is to prove (ii).

Here, we prove (i) by explicitly finding a set of partial isometries that generates $\mathrm{SU}_0(3)$, and that can be seen as the edges of a 2-graph $\mathcal{S}u(3)$. We then show that $\mathrm{SU}_0(3)$ is the universal C^*-representation for this graph. We then prove (ii) by extending the proof from [3] to the $q = 0$ case.

2.1. The Hopf $*$-algebra $\mathbb{C}[\mathrm{SU}(n)]_q$

Recall the definitions of the $*$-Hopf algebras $\mathbb{C}[\mathrm{SU}(n)]_q$, for $q \in (0, 1)$. It is defined by generators $\{t_{ij} \mid i, j = 1, \ldots, n\}$, along with a unit I, subject to the relations

$$\begin{aligned} t_{ij}t_{kl} - q t_{kl}t_{ij} &= 0 && \text{for } i = k \text{ and } j < l, \text{ or } i < k \text{ and } j = l, \\ t_{ij}t_{kl} - t_{kl}t_{ij} &= 0 && \text{if } i < k \text{ and } j > l, \\ t_{ij}t_{kl} - t_{kl}t_{ij} - (q - q^{-1})t_{il}t_{kj} &= 0, && \text{for } i < k \text{ and } j < l, \end{aligned}$$

$$\det_q \mathbf{t} = I.$$

Here

$$\det_q \mathbf{t} = \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} t_{1,\sigma(1)} \cdots t_{n,\sigma(n)}$$

(where S_n is the permutations of n elements and ℓ the length of the permutation) is the q-determinant of the matrix $\mathbf{t} = (t_{i,j})_{i,j=1}^n$. The comultiplication Δ, the counit ε, the antipode S and the involution $*$ are defined as follows:

$$\Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj}, \quad \varepsilon(t_{ij}) = \delta_{ij}, \quad S(t_{ij}) = (-q)^{i-j} \det_q \mathbf{t}_{ji},$$

and

$$t_{ij}^* = (-q)^{j-i} \det_q \mathbf{t}_{ij},$$

where $\mathbf{t}_{ij}$ is the matrix derived from $\mathbf{t}$ by discarding its i-th row and j-th column.

2.2. Representation theory of $\mathbb{C}[\mathrm{SU}(3)]_q$

We give a quick rundown of the representation theory of $\mathbb{C}[\mathrm{SU}(3)]_q$. These results are standards and the proofs of the statements here can be found in [5].

For two $\rho_i\colon \mathbb{C}[\mathrm{SU}(3)]_q \to \mathcal{B}(\mathrm{H}_i)$, $i = 1, 2$, we use the box-times notation to denote their tensor product

$$\rho_1 \boxtimes \rho_2 := (\rho_1 \otimes \rho_2) \circ \Delta\colon \mathbb{C}[\mathrm{SU}(3)]_q \to \mathcal{B}(\mathrm{H}_1 \otimes \mathrm{H}_2).$$

We give the usual construction of $\mathbb{C}[\mathrm{SU}(3)]_q$-representations. Let

$$C_q, S, D_q\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$$

be the operators defined on the natural orthonormal basis $\{x_j\}_{j\in\mathbb{N}}$ as follows:

$$Sx_n = x_{n+1}, \quad C_q x_n = \sqrt{1-q^{2n}}\,x_n, \quad D_q x_n = q^n x_n. \tag{2.1}$$

By [12], the map

$$\pi_q(t_{11}) = S^*C_q, \quad \pi_q(t_{12}) = -qD_q, \quad \pi_q(t_{21}) = D_q, \quad \pi_q(t_{22}) = C_q S$$

extends to a $*$-representation of $\mathbb{C}[\mathrm{SU}(2)]_q$. Let $C^*(S) \subseteq \mathcal{B}(\ell^2(\mathbb{N}))$ be the C^*-algebra generated by S (so that $C^*(S)$ is the Toeplitz algebra). From the expressions (2.1) for C_q and D_q, it is easy to see that $C_q, D_q \in C^*(S)$, and, hence, that

$$\pi_q(\mathbb{C}[\mathrm{SU}(2)]_q) \subseteq C^*(S).$$

For $i = 1, 2$, we have the two homomorphims

$$\mathbb{C}[\mathrm{SU}(3)]_q \xrightarrow{\vartheta_i} \mathbb{C}[\mathrm{SU}(2)]_q$$

corresponding, respectively, to the two embeddings

$$\begin{bmatrix} \mathrm{SU}(2) & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{SU}(2) \end{bmatrix} \subseteq \mathrm{SU}(3),$$

and that are determined on the generators as

$$\vartheta_1(t_{jk}) = \begin{cases} t_{jk} & \text{if } 1 \le j,k \le 2, \\ \delta_{jk} I & \text{otherwise,} \end{cases} \qquad \vartheta_2(t_{jk}) = \begin{cases} t_{(j-1)(k-1)} & \text{if } 2 \le j,k \le 3, \\ \delta_{jk} I & \text{otherwise.} \end{cases}$$

For $i = 1, 2$, we define the two representations

$$\pi_i^{(q)} = \pi_q \circ \vartheta_i\colon \mathbb{C}[\mathrm{SU}(3)]_q \to C^*(S). \tag{2.2}$$

Moreover, we have two natural homomorphisms $\tau_i\colon \mathbb{C}[\mathrm{SU}(3)]_q \to C(\mathbb{T})$ for $i = 1, 2$ determined by compositions

$$\tau_i\colon \mathbb{C}[\mathrm{SU}(3)]_q \xrightarrow{\pi_i^{(q)}} C^*(S) \xrightarrow{\mathcal{K}\sim} C^*(S)/\mathcal{K} \cong C(\mathbb{T}),$$

using the standard isomorphism $C^*(S)/\mathcal{K} \cong C(\mathbb{T})$, where $\mathcal{K} \subseteq \mathcal{B}(\ell^2(\mathbb{N}))$ denotes the compact operators. To be specific, we consider the image of S under the homomorphism $C^*(S) \to C(\mathbb{T})$ to be the coordinate function z. Thus, as D_q and $C_q - I$ are compact, we have (using the letter I again for the identity in $C(\mathbb{T})$)

$$\tau_1(t_{11}) = \bar{z}, \quad \tau_1(t_{22}) = z, \quad \tau_1(t_{33}) = I,$$
$$\tau_2(t_{11}) = I, \quad \tau_2(t_{22}) = \bar{z}, \quad \tau_2(t_{33}) = z,$$

and $\tau_i(t_{kj}) = 0$ for all other indices.

The homomorphism $\tau_2 \boxtimes \tau_1 \colon \mathbb{C}[\mathrm{SU}(3)]_q \to C(\mathbb{T}) \otimes C(\mathbb{T})$ corresponds to the maximal torus restriction $\mathbb{T}^2 \subseteq \mathrm{SU}(3)$. We denote the universal enveloping C^*-algebra of $\mathbb{C}[\mathrm{SU}(3)]_q$ by $\mathrm{SU}_q(3)$. From the representation theory, we get that $\mathrm{SU}_q(3)$ is isomorphic to the closure of $\mathbb{C}[\mathrm{SU}(3)]_q$ under the faithful representation

$$\Xi_q := (\pi_1^{(q)} \boxtimes \pi_2^{(q)} \boxtimes \pi_1^{(q)}) \boxtimes (\tau_2 \boxtimes \tau_1) \colon \mathbb{C}[\mathrm{SU}(3)]_q \to C^*(S)^{\otimes 3} \otimes C(\mathbb{T}) \otimes C(\mathbb{T}). \tag{2.3}$$

In this paper, we will usually identify $\mathbb{C}[\mathrm{SU}(3)]_q$ with $\mathrm{Im}\ \Xi_q$ and the C^*-algebra $\mathrm{SU}_q(3)$ with its closure. This is to be able to simultaneously consider all these algebras as subalgebras of $C^*(S)^{\otimes 3} \otimes C(\mathbb{T}) \otimes C(\mathbb{T})$.

2.3. The algebras $\lim_{q\to 0} \mathbb{C}[\mathrm{SU}(3)]_q$ and $\lim_{q\to 0} \mathrm{SU}_q(3)$

To make sense of the C^*-algebras $\mathbb{C}[\mathrm{SU}(3)]_0$ and $\mathrm{SU}_0(3)$ in the limit $q \to 0$, we want to take an appropriate set of generators of $\mathbb{C}[\mathrm{SU}(3)]_q$, where we have that the limits $q \to 0$ under the representation (2.3) actually exists. We then define $\mathbb{C}[\mathrm{SU}(3)]_0$ as $*$-algebra that these limits generates, and $\mathrm{SU}_0(3)$ its C^*-closure.

Let $U_q(\mathfrak{su}(3))$ be the Jimbo–Drinfeld q-deformation of the universal algebra of $\mathfrak{su}(3)$, and let V_+ denote the set of dominant weights. For $\lambda \in V_+$, we let M_λ^q denote the $U_q(\mathfrak{su}(3))$-module with highest weight corresponding to λ. We let $\omega_i \in V_+$, for $i = 1, 2$, denote the fundamental weights, ordered such that

$$\omega_1 = \Big(1, -\frac{1}{2}, -\frac{1}{2}\Big), \quad \omega_2 = \Big(\frac{1}{2}, \frac{1}{2}, -1\Big).$$

As $\dim M_{\omega_i}^q = 3$, for $i = 1, 2$, we let $\{\xi_{\omega_i}^1, \xi_{\omega_i}^2, \xi_{\omega_i}^3\} \subseteq M_{\omega_i}^q$ be an orthonormal basis, such that $\xi_{\omega_i}^1$ generates the highest weight space, $\xi_{\omega_i}^3$ generates the lowest.

Using the identification of $\mathbb{C}[\mathrm{SU}(3)]_q$ with the reduced dual of $U_q(\mathfrak{su}(3))$, we consider the elements in $\mathbb{C}[\mathrm{SU}(3)]_q$ defined as

$$C_j^{\omega_i}(\cdot) = \langle \cdot\ \xi_{\omega_i}^j, \xi_{\omega_i}^1 \rangle, \quad i = 1, 2,\ j = 1, 2, 3. \tag{2.4}$$

By [5, Theorem 2.2.1], the elements (2.4) generates $\mathbb{C}[\mathrm{SU}(3)]_q$ as a $*$-algebra. Moreover, by perhaps multiplying these element with an appropriate value in $\mathbb{T}$, it is not hard to see that one can express these elements in the usual set of generators t_{ij} as

$$C_j^{\omega_1} = t_{j1}, \quad C_j^{\omega_2} = (-q)^{(1-j)} t_{(4-j)3}^*. \tag{2.5}$$

We omit the proof of this statement, as we will do explicit calculations involving the right-hand sides of (2.5). It is a simple calculation to show that these generates $\mathbb{C}[\mathrm{SU}(3)]_q$ as a $*$-algebra. Formula (2.4) is there to give a more structural understanding of what is going on and to align this text with the framework from [7]. We will thus assume that (2.5) holds, and consider it as the definition of $C_j^{\omega_i}$.

Let $P \in \mathcal{B}(\ell^2(\mathbb{N}))$ denote the orthogonal projection onto the subspace generated by e_0. We let $Q = I - P$. Moreover, let $z \in C(\mathbb{T})$ be the coordinate function. Taking $q \to 0$, we have limits in norm (we spare the reader the standard – but tedious – calculations)

$$\Xi_q(C_1^{\omega_2})^* \rightsquigarrow I \otimes S \otimes I \otimes z \otimes I = \hat{A}, \tag{2.6a}$$
$$\Xi_q(C_2^{\omega_2})^* \rightsquigarrow S \otimes P \otimes I \otimes z \otimes I = \hat{B}, \tag{2.6b}$$
$$\Xi_q(C_3^{\omega_2})^* \rightsquigarrow P \otimes P \otimes I \otimes z \otimes I = \hat{C}, \tag{2.6c}$$
$$\Xi_q(C_1^{\omega_1})^* \rightsquigarrow S \otimes I \otimes S \otimes I \otimes z = \hat{X}, \tag{2.6d}$$
$$\Xi_q(C_2^{\omega_1})^* \rightsquigarrow P \otimes I \otimes S \otimes I \otimes z + S^* \otimes S \otimes P \otimes I \otimes z = \hat{Y}, \tag{2.6e}$$
$$\Xi_q(C_3^{\omega_1})^* \rightsquigarrow I \otimes P \otimes P \otimes I \otimes z = \hat{Z}. \tag{2.6f}$$

These are all partial isometries, as

$$\hat{A}^*\hat{A} = I \otimes I \otimes I \otimes I \otimes I, \quad \hat{A}\hat{A}^* = I \otimes Q \otimes I \otimes I \otimes I,$$
$$\hat{B}^*\hat{B} = I \otimes P \otimes I \otimes I \otimes I, \quad \hat{B}\hat{B}^* = Q \otimes P \otimes I \otimes I \otimes I,$$
$$\hat{C}^*\hat{C} = \hat{C}\hat{C}^* = P \otimes P \otimes I \otimes I \otimes I;$$

$$\hat{X}^*\hat{X} = I \otimes I \otimes I \otimes I \otimes I, \quad \hat{X}\hat{X}^* = Q \otimes I \otimes Q \otimes I \otimes I,$$
$$\hat{Y}^*\hat{Y} = P \otimes I \otimes I \otimes I \otimes I + Q \otimes I \otimes P \otimes I \otimes I,$$
$$\hat{Y}\hat{Y}^* = P \otimes I \otimes Q \otimes I \otimes I + I \otimes Q \otimes P \otimes I \otimes I,$$
$$\hat{Z}^*\hat{Z} = \hat{Z}\hat{Z}^* = I \otimes P \otimes P \otimes I \otimes I.$$

Recalling that a partial isometry v is called *quasinormal* if $vv^* \leq v^*v$, we see from the above that $\hat{A}, \hat{B}, \hat{C}, \hat{X}, \hat{Y}, \hat{Z}$ are in fact quasinormal partial isometries.

Moreover, we see that

$$\hat{A}\hat{A}^* + \hat{B}\hat{B}^* + \hat{C}\hat{C}^* = I = \hat{X}\hat{X}^* + \hat{Y}\hat{Y}^* + \hat{Z}\hat{Z}^*.$$

Definition 5. We denote the $*$-algebra generated by the limits (2.6) by $\mathbb{C}[\mathrm{SU}(3)]_0$, and denote its norm-closure by $\mathrm{SU}_0(3)$.

The right action β_t, for $t = (t_1, t_2) \in \mathbb{T}^2$ of the maximal torus on $C_j^{\omega_i}$, is induced by $\beta_t(C_j^{\omega_2}) = t_1 C_j^{\omega_2}$ and $\beta_t(C_j^{\omega_1}) = t_2 C_j^{\omega_1}$. Taking any non-commutative polynomials F in the variables $C_j^{\omega_i}, (C_j^{\omega_i})^*$, we get from the limit

$$\lim_{q\to 0} \|\Xi_q(F)\| = \lim_{q \to 0} \|\Xi_q(\beta_t(F))\|$$

(since F was arbitrary) that β_t extends to an automorphism of $\mathrm{SU}_0(3)$ induced by the actions

$$\{\widehat{A}, \widehat{B}, \widehat{C}\} \mapsto \{t_1\widehat{A}, t_1\widehat{B}, t_1\widehat{C}\}, \quad \{\widehat{X}, \widehat{Y}, \widehat{Z}\} \mapsto \{t_2\widehat{X}, t_2\widehat{Y}, t_2\widehat{Z}\}, \quad \text{for } t = (t_1, t_2). \tag{2.7}$$

From (2.7), it follows that $\beta_{ts} = \beta_t \circ \beta_s$. Moreover, as the map $t \in \mathbb{T}^2 \mapsto \beta_t(\widehat{R})$ is norm-continuous when $\widehat{R} \in \{\widehat{A}, \widehat{B}, \widehat{C}, \widehat{X}, \widehat{Y}, \widehat{Z}\}$, we get from a standard approximation argument that also $t \mapsto \beta_t(a)$ is norm-continuous for every $a \in \mathrm{SU}_0(3)$. Thus, we have an injective homomorphism $\beta_t \colon \mathbb{T}^2 \to \mathrm{Aut}(\mathrm{SU}_0(3))$ that is point-norm continuous. By using the natural identification

$$\mathcal{T}^{\otimes 3} \otimes C(\mathbb{T}) \otimes C(\mathbb{T}) \cong C(\mathbb{T}^2 : \mathcal{T}^{\otimes 3})$$

(i.e., continuous functions $\mathbb{T}^2 \to \mathcal{T}^{\otimes 3}$) we see that, for $f(z_1, z_2) \in \mathrm{SU}_q(3) \subseteq C(\mathbb{T}^2 : \mathcal{T}^{\otimes 3})$ and all $q \in [0, 1)$, we have

$$\beta_t(f(z_1, z_2)) = f(t_1 z_1, t_2 z_2). \tag{2.8}$$

Thus, β is actually the restriction of the $\mathbb{T}^2$-action $(f(z_1, z_2)) \mapsto f(t_1 z_1, t_2 z_2)$ on the ambient C^*-algebra (identified with) $C(\mathbb{T}^2 : \mathcal{T}^{\otimes 3})$. We can thus, with a slight abuse of notation, use the same symbol β to denote all these group actions.

2.4. The coloured graph of $\mathrm{SU}_0(3)$

We make a set of projections that will be corresponding to the nodes of our graph and that will be labelled by the set $\{AX, AY, BX, BZ, CY, CZ\}$:

$$\widehat{P}_{CZ} = \widehat{C}\widehat{C}^*\widehat{Z}\widehat{Z}^* = P \otimes P \otimes P \otimes I \otimes I, \tag{2.9a}$$
$$\widehat{P}_{CY} = \widehat{C}\widehat{C}^*\widehat{Y}\widehat{Y}^* = P \otimes P \otimes Q \otimes I \otimes I, \tag{2.9b}$$
$$\widehat{P}_{BZ} = \widehat{B}\widehat{B}^*\widehat{Z}\widehat{Z}^* = Q \otimes P \otimes P \otimes I \otimes I, \tag{2.9c}$$
$$\widehat{P}_{BX} = \widehat{B}\widehat{B}^*\widehat{X}\widehat{X}^* = Q \otimes P \otimes Q \otimes I \otimes I, \tag{2.9d}$$
$$\widehat{P}_{AY} = \widehat{A}\widehat{A}^*\widehat{Y}\widehat{Y}^* = P \otimes Q \otimes Q \otimes I \otimes I + I \otimes Q \otimes P \otimes I \otimes I, \tag{2.9e}$$
$$\widehat{P}_{AX} = \widehat{A}\widehat{A}^*\widehat{X}\widehat{X}^* = Q \otimes Q \otimes Q \otimes I \otimes I. \tag{2.9f}$$

A quick calculation shows that these projections actually adds up to the identity.

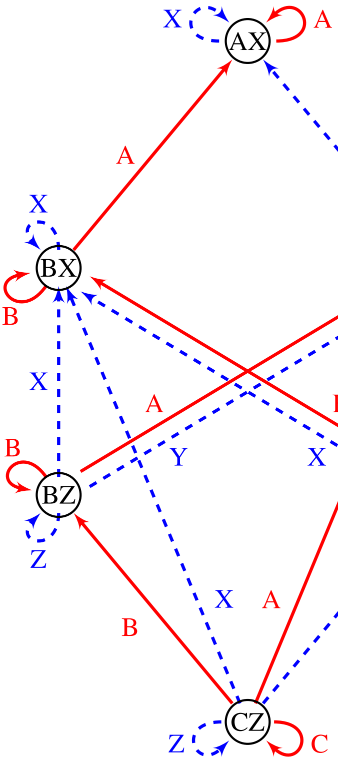
We will construct a directed 2-coloured graph $\mathcal{G} = (V, E, r, s)$ for which these projections will correspond to the vertices of. The vertices will be labelled by the corresponding index in the projection. Thus, the $V = \{AX, AY, BX, BZ, CY, CZ\}$. The set of edges E is defined and labelled by $\{A, B, C, X, Y, Z\}$ in the following way. There is directed edge

$$v_1 \xrightarrow{e} v_2,$$

labelled by $e \in \{A, B, C, X, Y, Z\}$ if we have

$$\widehat{P}_{v_2}\hat{e}\widehat{P}_{v_1} \neq 0.$$

Moreover, we will colour the edges either red or blue (dashed) depending on if the label takes values in the sets $\{A, B, C\}$ or $\{X, Y, Z\}$ respectively. It is easily verifiable that resulting graph $\mathcal{G}$ looks as follows:



(2.10)

This will be the blueprint for the 2-graph $\mathcal{S}u(3)$ that we construct below. There are further relations in the compositions of the operators $\widehat{A}, \ldots, \widehat{Z}$, that is not captured by the graph (2.10). As an example, it is not hard to see that $\widehat{Y}\widehat{B} = \widehat{A}\widehat{Z}$. The same relations will hold for all admissible compositions of morphisms labelled as such in the 2-graph category.

Remark. This text is quite heavy on the calculations, and, all throughout it, we will use (2.10) as a calculation tool.

2.5. Construction of the 2-graph $\mathcal{S}u(3)$

We will use the methods from [6, Section 6] to construct the 2-graph. The basis for their construction is a directed coloured graph, such that the transition matrices commutes for different colours, together with further commutation rules whenever there is ambiguity in the arrow composition.

Let us consider the complex vector space $\mathbb{C}^6$. We identify the canonical basis vectors $v_1, \dots, v_6$ with the vertices of (2.10) in the following way:

$$1 = CZ, \quad 2 = BZ, \quad 3 = CY, \quad 4 = BX, \quad 5 = AY, \quad 6 = AX.$$

We define the transition matrices M_R and M_B (R for red and B for blue) such that M_R has a 1 at the (i, j) index if in (2.10) we have a red arrow $i \to j$ between the corresponding vertices. For M_B, we proceed in a similar way. With this definition, we have

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad M_B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

A calculation gives that

$$M_R M_B = M_B M_R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{2.11}$$

so that these matrices fulfill the requirements in [6, Section 6]. We see that in (2.11), we have 4 indices with 2's in them, and we need to resolve these ambiguities.

Let us define

$$A^{ij}_{RB} = \{(a, x) \mid a \text{ a red arrow, } x \text{ blue, such that } r(a) = i, s(a) = r(x) \text{ and } s(x) = j\}.$$

and we define A^{ij}_{BR} similarly, but with the colours switched. From (2.11), we deduce that the sets A^{ij}_{RB} and A^{ij}_{BR} contains the same number of elements for all indices i, j. For each i, j, we will construct a bijection

$$\phi_{ij} \colon A^{ij}_{BR} \to A^{ij}_{RB}.$$

It is clear from (2.11) that we only need to specify ϕ_{ij} for $i, j = (1, 4), (1, 5), (2, 4), (3, 5)$, i.e., for

$$(CZ) \to (AY), \quad (BZ) \to (AY), \tag{2.12}$$

$$(CZ) \to (BX), \quad (CY) \to (BX). \tag{2.13}$$

We get these from the relations satisfied by the operators (2.6) as (for the arrows with the labels)

$$Z \circ A = B \circ Y, \quad Y \circ A = A \circ Y, \quad \text{for } \phi_{15}, \phi_{35} \text{ (the arrows from (2.12))}, \tag{2.14a}$$

$$Y \circ B = C \circ X, \quad X \circ B = B \circ X, \quad \text{for } \phi_{14}, \phi_{24} \text{ (the arrows from (2.13))}. \tag{2.14b}$$

From [6, Section 6], we now get the following result.

Proposition 6. *This input-data uniquely determines a* 2*-graph* $\mathcal{S}u(3)$.

Thus, $\mathcal{S}u(3)$ is the 2-graph with vertices the nodes in (2.10), the blue and red arrows, the sets $\mathrm{r}^{-1}(1,0)$ and $\mathrm{r}^{-1}(0,1)$, respectively, and where we have the additional relations between the red and blue arrows determined by (2.14).

2.6. Proof that $C^*(\mathcal{S}u(3)) \cong \mathrm{SU}_0(3)$

We define operators $A, B, C, X, Y, Z \in C^*(\mathcal{S}u(3))$ by letting these be the sum of all the partial isometries in $C^*(\mathcal{S}u(3))$ corresponding to the edges in (2.10) labelled respectively. We denote the orthogonal projections in $C^*(\mathcal{S}u(3))$ coming from the vertices by P_v, with $v \in \{AX, AY, BX, BZ, CY, CZ\}$.

Lemma 7. *The elements* A, B, C, X, Y, Z *are generating* $C^*(\mathcal{S}u(3))$.

Proof. Note that all the red or blue arrows with range a particular vertex has the same labels. By inspection of the graph (2.10), and (CK4), we have

$$AA^* = P_{AX} + P_{AY} \quad BB^* = P_{BX} + P_{BZ} \quad CC^* = P_{CY} + P_{CZ}; \tag{2.15a}$$

$$XX^* = P_{AX} + P_{BX} \quad YY^* = P_{AY} + P_{CY} \quad ZZ^* = P_{BZ} + P_{CZ}. \tag{2.15b}$$

From the list (2.15), we see that with $m \in \{A, B, C\}$ and $n \in \{X, Y, Z\}$, we have

$$mm^*nn^* = P_{mn}.$$

It follows that we can recover all the edges in (2.10). ■

Unsurprisingly, this is the same formula for the projection as (2.9). We also see from (2.15) that

$$AA^* + BB^* + CC^* = I = XX^* + YY^* + ZZ^*.$$

Lemma 8. A, B, C, X, Y, Z *are all quasinormal partial isometries. In particular,* A *and* X *are isometries. Moreover, the following relations hold:*

$$BA = B^*A = 0, \qquad YA = AY, \qquad CA = C^*A = 0, \tag{2.16a}$$

$$ZB = BZ, \qquad YB = AZ, \qquad ZA = Z^*A = 0, \tag{2.16b}$$

$$CB = C^*B = 0, \qquad AX = XA, \qquad AX^* = X^*A; \tag{2.16c}$$

$$YX = Y^*X = 0, \qquad BX = XB, \qquad ZX = Z^*X = 0, \tag{2.16d}$$

$$CY = YC, \qquad BY = XC, \qquad CX = C^*X = 0, \tag{2.16e}$$

$$ZY = Z^*Y = 0. \tag{2.16f}$$

Proof. Notice that, by the symmetry of the graph (2.10) and the relations (2.14), we can switch A, B, C with X, Y, Z. By this argument, the relations below the gap in (2.16) follows from the ones above.

Let us first show the claims of quasinormality and isometry. We know from Lemma 7 that $A, \dots, Z$ are partial isometries. Moreover, as every vertex is a source vertex for an A or X arrow, it follows that A and X are actually isometries. Quasinormality (i.e., $vv^* \leq v^*v$) is easy to see from (CK3) and (CK4), as, for example in the case of Y, the set of range vertices for Y-labelled arrows is a subset of the source vertices. Similarly for the others.

Assume for this paragraph that $A, \dots, Z$ again denote the arrow-labels in (2.10). The definition of $\mathcal{S}u(3)$, in particular (2.14), shows that $YA = AY$, $ZB = BZ$, $YB = AZ$, and $AX = XA$ for all composable arrows with these labels. Some diagram inspection shows that the same is true for the corresponding operators.

We deduce that $BA = B^*A = CA = C^*A = ZA = Z^*A = 0$ from the fact that in (2.10) no arrow with label A has its range on any vertex that is the source or range of an arrow labelled either B, C, or Z. We have $CB = C^*B = 0$ for the same reason.

Lets prove the last relation $AX^* = X^*A$. Some diagram chasing of (2.10) shows that the projection onto ker A^* is given by $P = P_{BX} + P_{BZ} + P_{CY} + P_{CZ}$. By (2.10), this subspace is invariant under X. It follows that

$$PXP = XP \implies (I - P)X(I - P) = (I - P)X.$$

As $AA^* = I - P$, we get from $AX = XA$ that

$$XA^* = A^*(AX)A^* = A^*(XA)A^* = A^*X(I - P).$$

Then

$$A^*X(I - P) = A^*AA^*X(I - P) = A^*(I - P)X(I - P) = A^*(I - P)X = A^*X. \qquad \blacksquare$$

Assume that we have a Hilbert space H and a faithful representation

$$\rho\colon C^*(\mathcal{S}u(3)) \to \mathcal{B}(\mathrm{H}).$$

Let us denote $\mathcal{E} = \rho(P_{CZ})\mathrm{H}$. To make things appear more concise, we will suppress ρ and write x instead of $\rho(x)$ for $x \in C^*(\mathcal{S}u(3))$. This is safe as ρ was assumed to be faithful.

By conditions (CK3)–(CK4), we have that $\mathcal{E}$ is invariant under the operators C and Z, and, moreover, the restrictions to this subspace are unitary. Thus, let us denote $\widetilde{C} := C|_{\mathcal{E}}$ and $\widetilde{Z} := Z|_{\mathcal{E}}$. Clearly, $\widetilde{C}, \widetilde{Z} \in \mathcal{B}(\mathcal{E})$ are commuting unitary operators. Moreover, as the representation is assumed to be faithful, it follows from the gauge action that we have an isomorphism $C^*(\widetilde{C}, \widetilde{Z}) \cong C(\mathbb{T}) \otimes C(\mathbb{T})$, and, moreover, that this isomorphism can be taken so that $\widetilde{C} \mapsto z \otimes I$ and $\widetilde{Z} \mapsto I \otimes z$.

Consider the closed subspaces

$$\mathcal{E}(\mathbf{k}) = A^k B^j Y^m \widetilde{C}^{-(k+j)} \widetilde{Z}^{-m} \mathcal{E}, \quad \text{for } \mathbf{k} = (j, k, m) \in \mathbb{N}^3. \tag{2.17}$$

Note that the negative exponents are not a problem as $\widetilde{C}$ and $\widetilde{Z}$ are unitary.

Lemma 9. *We have* $\mathcal{E}(\mathbf{k})\perp\mathcal{E}(\mathbf{n})$ *for* $\mathbf{k}\neq\mathbf{n}$, *and a unitary isometry*

$$W\colon\bigoplus_{\mathbf{k}\in\mathbb{N}^3}\mathcal{E}(\mathbf{k})\to\ell^2(\mathbb{N})^{\otimes 3}\otimes\mathcal{E}, \tag{2.18}$$

given by the formula

$$A^kB^jY^m\tilde{C}^{-(k+j)}\tilde{Z}^{-m}x\mapsto e_j\otimes e_k\otimes e_m\otimes x,\quad \textit{for } x\in\mathcal{E}. \tag{2.19}$$

Proof. As $\tilde{C}$ and $\tilde{Z}$ are unitary, we can safely ignore these factors in (2.17) when we prove the orthogonality and non-zero-ness of the subspaces $\mathcal{E}(\mathbf{k})$. An inspection of the graph gives that

$$\begin{aligned} A^*A &= I, & AA^* &= P_{AX}+P_{AY},\\ B^*B &= P_{BX}+P_{BZ}+P_{CY}+P_{CZ}, & BB^* &= P_{BX}+P_{BZ},\\ Y^*Y &= P_{AY}+P_{BZ}+P_{CY}+P_{CZ}, & YY^* &= P_{AY}+P_{CY}.\end{aligned}$$

It follows from this that A,B,Y are isometries when restricted to the ranges of A^*A, B^*B, Y^*Y, respectively (note that these subspaces also reduces their respective operator). Moreover, the Wold–von Neumann decomposition gives, with $\mathcal{E}_A=(I-P_{AX}-P_{AY})\mathrm{H}$, $\mathcal{E}_B=(P_{CY}+P_{CZ})\mathrm{H}$, and $\mathcal{E}_Y=(P_{AY}+P_{CY})\mathrm{H}$, for $R\in\{A,B,Y\}$ and all $k\in\mathbb{N}$ a unitary operator

$$R\colon R^k\mathcal{E}_R\to R^{k+1}\mathcal{E}_R,$$

and, moreover,

$$R^m\mathcal{E}_R\perp R^j\mathcal{E}_R,\quad \text{for } m\neq j.$$

An inspection of (2.10) gives

$$\mathcal{E}\subseteq\mathcal{E}_Y,\quad Y^k\mathcal{E}\subseteq\mathcal{E}_B,\quad B^k\mathcal{E}_B\subseteq\mathcal{E}_A,\quad \text{for all } k\in\mathbb{N}.$$

From this, we can then deduce that for, all $\mathbf{k}=(j,k,m)\in\mathbb{N}^3$, the map

$$A^kB^jY^m\colon\mathcal{E}\to\mathcal{E}(\mathbf{k})$$

is unitary, and, moreover, that $\mathcal{E}(\mathbf{k})\perp\mathcal{E}(\mathbf{n})$ for $\mathbf{k}\neq\mathbf{n}$. In particular, it follows that W determined by (2.19) is unitary. ∎

Proposition 10. *The subspace* $\bigoplus_{\mathbf{k}\in\mathbb{N}^3}\mathcal{E}(\mathbf{k})$ *is reducing the images of* $A,\dots,Z$. *Moreover, if* W *is the isomorphism* (2.18), *then*

$$WAW^*=I\otimes S\otimes I\otimes\tilde{C}, \tag{2.20a}$$

$$WBW^*=S\otimes P\otimes I\otimes\tilde{C}, \tag{2.20b}$$

$$WCW^*=P\otimes P\otimes I\otimes\tilde{C}; \tag{2.20c}$$

$$WXW^*=S\otimes I\otimes S\otimes\tilde{Z}, \tag{2.20d}$$

$$WYW^*=P\otimes I\otimes S\otimes\tilde{Z}+S^*\otimes S\otimes P\otimes\tilde{Z}, \tag{2.20e}$$

$$WZW^*=I\otimes P\otimes P\otimes\tilde{Z}. \tag{2.20f}$$

Proof. We Will use the notation

$$[x](j,k,m) := A^k B^j Y^m \widetilde{C}^{-(k+j)} \widetilde{Z}^{-m} x, \quad \text{for } x \in \mathcal{E}.$$

By using the list of relations (2.16) from Lemma 8, it is simple to calculate the actions of $A, \dots, Z$, and $A^* - Z^*$ on this element as

$$A[x](j,k,m) = [\widetilde{C}x](j,k+1,m)$$

$$A^*[x](j,k,m) = \begin{cases} 0 & \text{if } k=0, \\ [\widetilde{C}^*x](j,k-1,m) & \text{otherwise,} \end{cases}$$

$$B[x](j,k,m) = \begin{cases} 0 & \text{if } k>0, \\ [\widetilde{C}x](j+1,0,m) & \text{otherwise,} \end{cases}$$

$$B^*[x](j,k,m) = \begin{cases} 0 & \text{if } k>0 \text{ or } m=0, \\ [\widetilde{C}^*x](j-1,0,m) & \text{otherwise,} \end{cases}$$

$$C[x](j,k,m) = \begin{cases} 0 & \text{if } j,k>0, \\ [\widetilde{C}x](0,0,m) & \text{otherwise,} \end{cases}$$

$$C^*[x](j,k,m) = \begin{cases} 0 & \text{if } j,k>0, \\ [\widetilde{C}^*x](0,0,m) & \text{otherwise,} \end{cases}$$

$$\begin{aligned} X[x](j,k,m) &= XA^k B^j Y^m \widetilde{C}^{-(k+j)} \widetilde{Z}^{-m} x \\ &= A^k B^j XY^m \widetilde{C}^{-(k+j)} \widetilde{Z}^{-m} \\ &= A^k B^j (XC) Y^m \widetilde{C}^{-(k+j+1)} \widetilde{Z}^{-m} x \\ &= A^k B^j (BY) Y^m \widetilde{C}^{-(k+j+1)} \widetilde{Z}^{-(m+1)} \widetilde{Z} x \\ &= [\widetilde{Z}x](j+1,k,m+1), \\ X^*[x](j,k,m) &= X^* A^k B^j Y^m \widetilde{C}^{-(k+j)} \widetilde{Z}^{-m} x \\ &= A^k X^* B^j Y^m \widetilde{C}^{-(k+j)} \widetilde{Z}^{-m} x. \end{aligned} \tag{2.21}$$

We break down the calculation of the right-hand side of (2.21) into three cases:

(i) if $j=0$, then $A^k X^* Y^m \widetilde{C}^{-k} \widetilde{Z}^{-m} x = 0$,

(ii) if $m=0$, then $A^k X^* B^j \widetilde{C}^{-(k+j)} x = A^k (X^* Z) B^j \widetilde{C}^{-(k+j)} \widetilde{Z}^{-1} x = 0$,

(iii) if $j,m>0$, then

$$\begin{aligned} A^k X^* B^j Y^m \widetilde{C}^{-(k+j)} \widetilde{Z}^{-m} x &= A^k X^* B^{j-1} (BY) Y^{m-1} \widetilde{C}^{-(k+j)} \widetilde{Z}^{-m} x \\ &= A^k X^* B^{j-1} (XC) Y^{m-1} \widetilde{C}^{-(k+j)} \widetilde{Z}^{-m} x \\ &= A^k (X^* X) B^{j-1} Y^{m-1} \widetilde{C}^{-(k+j-1)} \widetilde{Z}^{-(m-1)} \widetilde{Z}^* x \\ &= A^k B^{j-1} Y^{m-1} \widetilde{C}^{-(k+j-1)} \widetilde{Z}^{-(m-1)} \widetilde{Z}^* x \\ &= [\widetilde{Z}^* x](j-1,k,m-1). \end{aligned}$$

It is easy to see (again using (2.16)) that the action by Z, Z^* is given as

$$Z[x](j,k,m) = \begin{cases} 0 & \text{if } k, m > 0, \\ [\widetilde{Z}x](j,0,0) & \text{otherwise,} \end{cases}$$

$$Z^*[x](j,k,m) = \begin{cases} 0 & \text{if } k, m > 0, \\ [\widetilde{Z}^*x](j,0,0), & \text{otherwise.} \end{cases}$$

To calculate the action of Y, we break this into the cases $j = 0$ and $j > 0$. We then have:

(1) if $j = 0$, then

$$\begin{aligned} Y[x](0,k,m) &= YA^k Y^m \widetilde{C}^{-k} \widetilde{Z}^{-m} x \\ &= A^k Y^{m+1} \widetilde{C}^{-k} \widetilde{Z}^{-(m+1)} \widetilde{Z} x \\ &= [\widetilde{Z}x](0,k,m+1); \end{aligned}$$

(2) if $j > 0$, then

$$\begin{aligned} &YA^k B^j Y^m \widetilde{C}^{-(k+j)} \widetilde{Z}^{-m} x \\ &= A^k (YB) B^{j-1} Y^m \widetilde{C}^{-(k+j)} \widetilde{Z}^{-m} x \\ &= A^k (AZ) B^{j-1} Y^m \widetilde{C}^{-(k+j)} \widetilde{Z}^{-m} x \\ &= A^{k+1} B^{j-1} Z Y^m \widetilde{C}^{-(k+j)} \widetilde{Z}^{-m} x \\ &= \begin{cases} A^{k+1} B^{j-1} (ZY) Y^{m-1} \widetilde{C}^{-(k+j)} \widetilde{Z}^{-m} x = 0 & \text{if } m > 0, \\ A^{k+1} B^{j-1} \widetilde{C}^{-(k+j)} \widetilde{Z} x = [\widetilde{Z}x](j-1,k+1,0) & \text{if } m = 0. \end{cases} \end{aligned}$$

We have shown that $\bigoplus_{\mathbf{k}\in\mathbb{N}^3} \mathcal{E}(\mathbf{k})$ is an invariant subspace for XX^* and ZZ^*. As $XX^* + YY^* + ZZ^* = I$ it follows that its invariant under YY^* as well. Moreover, a calculation using (2.10) gives that

$$Y^*Y = P_{CZ} + P_{CY} + P_{BZ} + P_{AY} = YY^* + CC^*ZZ^* + BB^*ZZ^*,$$

and thus by what we have proven, the subspace $\bigoplus_{\mathbf{k}\in\mathbb{N}^3} \mathcal{E}(\mathbf{k})$ reduces Y^*Y as well. Since we have that Y is a quasinormal partial isometry, it follows that $\bigoplus_{\mathbf{k}\in\mathbb{N}^3} \mathcal{E}(\mathbf{k})$ is also invariant under Y^*. Formulas (2.20) now follows from the above calculations. ∎

Proposition 11. *We have an isomorphim* $\phi\colon C^*(\mathcal{S}u(3)) \to \mathrm{SU}_0(3)$ *that intertwines the gauge action* $C^*(\mathcal{S}u(3))$ *with the right action on* $\mathrm{SU}_0(3)$.

Proof. From formulas (2.20), as well as the comments about $C^*(\widetilde{C}, \widetilde{Z})$ made before the statement of Proposition 10, we have a surjective homomorphism $\phi\colon C^*(\mathcal{S}u(3)) \to \mathrm{SU}_0(3)$. The gauge action α_t, for $t \in \mathbb{T}^2$ on $C^*(\mathcal{S}u(3))$ acts on $A, \ldots, Z$ for $t = (t_1, t_2)$ as

$$\begin{aligned} \{A, B, C\} &\mapsto \{t_1 A, t_1 B, t_1 C\}, \\ \{X, Y, Z\} &\mapsto \{t_2 X, t_2 Y, t_2 Z\}. \end{aligned}$$

From (2.7), recall that we had the homomorphism $\beta\colon \mathbb{T}^2 \to \mathrm{Aut}(\mathrm{SU}_0(3))$ with action induced by (2.7). It follows that we have $\phi \circ \alpha_t = \beta_t \circ \phi$. Clearly, $\phi(P_\sigma) \neq 0$ for all $\sigma \in \mathcal{S}u(3)_{\mathrm{obj}}$. We then get from Theorem 4 that ϕ is faithful, and thus an isomorphism. ■

2.7. Proof that $\mathrm{SU}_0(3) \cong \mathrm{SU}_q(3)$

Recall that we consider the C^*-algebra $\mathrm{SU}_0(3)$ as a closed subalgebra of $C^*(S)^{\otimes 3} \otimes C(\mathbb{T}) \otimes C(\mathbb{T})$. Proposition 11 gave us the result that $\mathrm{SU}_0(3) \cong C^*(\mathcal{S}u(3))$, and thus in order to show that $\mathrm{SU}_q(3)$ is the graph C^*-algebra of $\mathcal{S}u(3)$, we need to prove that we have an isomorphism $\mathrm{SU}_0(3) \cong \mathrm{SU}_q(3)$. In order to do this, we essentially retrace the proof of the q-independence of $\mathrm{SU}_q(3)$ from [3] (or [8]), and extend it to the limit $q \to 0$.

In this section, we aim to prove the following result.

Theorem 12. *For all $q \in (0,1)$, we have an isomorphism $\mathrm{SU}_0(3) \cong \mathrm{SU}_q(3)$ that is equivariant with respect to the right actions.*

We first need some lemmas.

Lemma 13. *The image of $\mathrm{SU}_q(3)$ in*

$$((C^*(S) \otimes C^*(S) \otimes C(\mathbb{T})) \oplus (C(\mathbb{T}) \otimes C^*(S) \otimes C^*(S))) \otimes C(\mathbb{T}) \otimes C(\mathbb{T})$$

under the representation

$$\Lambda_q = ((\pi_1^{(q)} \boxtimes \pi_2^{(q)} \boxtimes \tau_1) \oplus (\tau_1 \boxtimes \pi_2^{(q)} \boxtimes \pi_1^{(q)})) \boxtimes (\tau_2 \boxtimes \tau_1)$$

does not depend on q. Moreover, this C^*-algebra is generated by the limits $q \to 0$ of the image of the generators $C_j^{\omega_i}$, for $i = 1, 2$, and $j = 1, 2, 3$.

Proof. This is verified by a straightforward calculation. Recall that the generators $C_j^{\omega_i}$ are given by the formulas

$$C_j^{\omega_1} = t_{j1}, \quad C_j^{\omega_2} = (-q)^{(1-j)} t^*_{(4-j)3}.$$

We calculate the images of these under Λ_q as

$$\Lambda_q(C_1^{\omega_1})^* = ((C_q S \otimes I \otimes z) \oplus (z \otimes I \otimes C_q S)) \otimes I \otimes z, \tag{2.22a}$$
$$\Lambda_q(C_2^{\omega_1})^* = ((D_q \otimes I \otimes z) \oplus (\bar{z} \otimes C_q S \otimes D_q)) \otimes I \otimes z, \tag{2.22b}$$
$$\Lambda_q(C_3^{\omega_1})^* = ((0 \otimes 0 \otimes 0) \oplus (I \otimes D_q \otimes D_q)) \otimes I \otimes z; \tag{2.22c}$$

$$\Lambda_q(C_1^{\omega_2})^* = ((I \otimes C_q S \otimes I) \oplus (I \otimes C_q S \otimes I)) \otimes z \otimes I, \tag{2.22d}$$
$$\Lambda_q(C_2^{\omega_2})^* = ((C_q S \otimes D_q \otimes I) \oplus (z \otimes D_q \otimes I)) \otimes z \otimes I, \tag{2.22e}$$
$$\Lambda_q(C_3^{\omega_2})^* = ((D_q \otimes D_q \otimes I) \oplus (0 \otimes 0 \otimes 0)) \otimes z \otimes I. \tag{2.22f}$$

Notice that all images have norm-continuous limits at $q = 0$. We denote these limits by $\Lambda_0(C_j^{\omega_i})^*$. It is not hard to see that the two C^*-algebras

$$\mathcal{A}_1 = C^*(\Lambda_q(C_1^{\omega_1})^*, \Lambda_q(C_2^{\omega_1})^*, \Lambda_q(C_3^{\omega_1})^*),$$
$$\mathcal{A}_2 = C^*(\Lambda_q(C_1^{\omega_2})^*, \Lambda_q(C_2^{\omega_2})^*, \Lambda_q(C_3^{\omega_2})^*)$$

are independent on q and generated by the limits $q \to 0$.

We prove this for $\mathcal{A}_1$. The case of $\mathcal{A}_2$ is treated similarly. Take the limits $q \to 0$ to get

$$\begin{aligned}\Lambda_0(C_1^{\omega_1})^* &= ((S \otimes I \otimes z) \oplus (z \otimes I \otimes S)) \otimes I \otimes z,\\ \Lambda_0(C_2^{\omega_1})^* &= ((P \otimes I \otimes z) \oplus (\bar{z} \otimes S \otimes P)) \otimes I \otimes z,\\ \Lambda_0(C_3^{\omega_1})^* &= ((0 \otimes 0 \otimes 0) \oplus (I \otimes P \otimes P)) \otimes I \otimes z.\end{aligned}$$

First, we show that $\Lambda_0(C_i^{\omega_1})^* \in \mathcal{A}_1$ for $i = 1, 2, 3$. Clearly, $\Lambda_q(C_1^{\omega_1})\Lambda_q(C_1^{\omega_1})^*$ is invertible. It follows that

$$\begin{aligned}\Lambda_q(C_1^{\omega_1})^*(\Lambda_q(C_1^{\omega_1})\Lambda_q(C_1^{\omega_1})^*)^{-\frac{1}{2}} &= ((S \otimes I \otimes z) \oplus (z \otimes I \otimes S)) \otimes I \otimes z\\ &= \Lambda_0(C_1^{\omega_1})^* \in \mathcal{A}_1.\end{aligned}$$

From this, we get that also

$$\begin{aligned}&\Lambda_0(C_1^{\omega_1})\Lambda_0(C_1^{\omega_1})^* - \Lambda_0(C_1^{\omega_1})^*\Lambda_0(C_1^{\omega_1})\\ &\quad= ((P \otimes I \otimes I) \oplus (I \otimes I \otimes P)) \otimes I \otimes I \in \mathcal{A}_1.\end{aligned}$$

Thus,

$$\begin{aligned}&(((P \otimes I \otimes I) \oplus (I \otimes I \otimes P)) \otimes I \otimes I)\Lambda_q(C_2^{\omega_1})^*\\ &\quad= ((P \otimes I \otimes z) \oplus (\bar{z} \otimes C_q S \otimes P)) \otimes I \otimes z \in \mathcal{A}_1. \end{aligned} \tag{2.23}$$

It is not hard to see that 0 is an isolated point in the spectrum of the absolute value of (2.23). Thus, the partial isometry of its polar decomposition lies in $\mathcal{A}_1$, so that

$$((P \otimes I \otimes z) \oplus (\bar{z} \otimes S \otimes P)) \otimes I \otimes z = \Lambda_0(C_2^{\omega_1})^* \in \mathcal{A}_1$$

From (2.22), it is also clear that 1 is an isolated point in the spectrum of the operator $\Lambda_q(C_3^{\omega_1})\Lambda_q(C_3^{\omega_1})^*$, and so it follows from the spectral theorem that $((0 \otimes 0 \otimes 0) \oplus (I \otimes P \otimes P)) \otimes I \otimes I \in \mathcal{A}_1$. Thus, we have

$$\begin{aligned}&((0 \otimes 0 \otimes 0) \oplus (I \otimes P \otimes P))\Lambda_q(C_3^{\omega_1})^*\\ &\quad= ((0 \otimes 0 \otimes 0) \oplus (I \otimes P \otimes P)) \otimes I \otimes z = \Lambda_0(C_3^{\omega_1})^* \in \mathcal{A}_1.\end{aligned}$$

This finishes the proof that $\Lambda_0(C_i^{\omega_1})^* \in \mathcal{A}_1$ for $i = 1, 2, 3$.

That these elements also generates $\mathcal{A}_1$ follows from the easily-verified norm-convergent formulas

$$\begin{aligned}\Lambda_q(C_1^{\omega_1})^* &= \Lambda_0(C_1^{\omega_1})^*(\Lambda_q(C_1^{\omega_1})\Lambda_q(C_1^{\omega_1})^*)^{\frac{1}{2}}\\ &= \Lambda_0(C_1^{\omega_1})^*((1-q^2)\sum_{k=0}^{\infty} q^{2k}(\Lambda_0(C_1^{\omega_1})^*)^k(\Lambda_0(C_1^{\omega_1}))^k)^{\frac{1}{2}},\end{aligned}$$

$$\begin{aligned}&\Lambda_q(C_2^{\omega_1})^*\\ &=\sum_{k=0}^{\infty} q^k \Lambda_0(C_1^{\omega_1})^{*k}((1-q^2)\sum_{j=1}^{\infty} q^{2(j-1)}\Lambda_0(C_2^{\omega_1})^{*j}\Lambda_0(C_2^{\omega_1})^j)^{\frac{1}{2}}\Lambda_0(C_2^{\omega_1})^*\Lambda_0(C_1^{\omega_1})^k,\end{aligned}$$

$$\Lambda_q(C_3^{\omega_1})^* = \sum_{k,j=0}^{\infty} q^{k+j}((\Lambda_0(C_1^{\omega_1})^*)^k(\Lambda_0(C_2^{\omega_1})^*)^j)\Lambda_0(C_3^{\omega_1})^*((\Lambda_0(C_2^{\omega_1}))^j(\Lambda_0(C_1^{\omega_1}))^k). \qquad \blacksquare$$

By [5], we have that $\mathrm{SU}_q(3)$ is a C^*-algebra of type I. Thus, the C^*-algebra from Lemma 13 is type I as well, by virtue of being the image of $\mathrm{SU}_q(3)$ under a homomorphism.

One of the central idea in G. Nagy paper [8] (e.g., to show that the C^*-algebras $\mathrm{SU}_q(3)$ are all isomorphic) was to use the following lifting result, based on the theory developed in [10], to extend the trivial isomorphisms from Lemma 13, to isomorphisms between the full C^*-algebras.

Lemma 14 ([8, Lemma 2]). *Let* H *be a separable Hilbert space, let* $\mathcal{K}$ *be the space of compact operators on* H*, let* $Q(\mathrm{H}) = \mathcal{B}(\mathrm{H})/\mathcal{K}$ *be the Calkin algebra, and let* $p\colon \mathcal{B}(\mathrm{H}) \to Q(\mathrm{H})$ *be the quotient map. Suppose* A *is a fixed separable* C^**-algebra of type* I *and* $\psi_q\colon A \to Q(\mathrm{H})$, $q \in [0,1]$, *is a point-norm continuous family of injective* $*$*-homomorphisms. Denote*

$$\mathfrak{A}_q := \psi_q(A), \quad M_q := p^{-1}(\mathfrak{A}_q).$$

Then there exists a family of injective $*$*-homomorphisms* $\Psi_q\colon M_0 \to \mathcal{B}(\mathrm{H})$, $q \in [0,1]$, *with the following properties:*

(a) $\Psi_q(M_0) = M_q$ *for* $q \in [0,1]$ *and* $\Psi_0 = \mathrm{Id}_{M_0}$*;*

(b) *the family* $\Psi_q\colon M_0 \to \mathcal{B}(\mathrm{H})$, $q \in [0,1]$ *is point-norm continuous;*

(c) *for every* $q \in [0,1]$, *the diagram*

$$\begin{array}{ccc} M_0 & \xrightarrow{\ \Psi_q\ } & M_q \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ \mathfrak{A}_0 & \xrightarrow[\psi_q\circ\psi_0^{-1}]{} & \mathfrak{A}_q \end{array} \tag{2.24}$$

is commutative.

By using similar arguments (or rather half the arguments) as in [3, Lemma 2], one can extend this lemma to the half-closed interval $[0,1)$. We remark that this extension is not really needed for the purpose here, as we are mostly interested in establish an isomorphism between $\mathrm{SU}_0(3)$ and $\mathrm{SU}_q(3)$, and thus need to only consider the interval $[0,q]$.

Lemma 15. *Let* $\mathcal{K} \subseteq C^*(S)^{\otimes 3}$ *denotes the* C^**-algebra of compact operators. We have* $\mathcal{K} \otimes C(\mathbb{T}) \otimes C(\mathbb{T}) \subseteq \mathrm{SU}_0(3)$.

Proof. This is a calculation done using the operators (2.6). We have $(\hat{C}^*\hat{C})(\hat{Z}^*\hat{Z}) = P \otimes P \otimes P \otimes I \otimes I \in \mathrm{SU}_0(3)$. Thus, also

$$\hat{C}(P \otimes P \otimes P \otimes I \otimes I) = P \otimes P \otimes P \otimes z \otimes I \in \mathrm{SU}_0(3),$$
$$\hat{Z}(P \otimes P \otimes P \otimes I \otimes I) = P \otimes P \otimes P \otimes I \otimes z \in \mathrm{SU}_0(3).$$

We then get from the Stone–Weierstrass theorem that $P \otimes P \otimes P \otimes C(\mathbb{T}) \otimes C(\mathbb{T}) \subseteq \mathrm{SU}_0(3)$. Assume that we have the usual inclusion $C(\mathbb{T}) \subseteq \mathcal{B}(L^2(\mathbb{T}))$. A simple calculation shows that, for all $(j,k,m) \in \mathbb{N}^3$, we have

$$(\hat{A}^k \hat{B}^j \hat{Y}^m \hat{C}^{-(m+k)} \hat{Z}^{-j})(x_0 \otimes x_0 \otimes x_0 \otimes 1 \otimes 1) = x_j \otimes x_k \otimes x_m \otimes 1 \otimes 1.$$

From this, it follows easily that $\mathcal{K} \otimes C(\mathbb{T}) \otimes C(\mathbb{T}) \subseteq \mathrm{SU}_0(3)$. ∎

After this lemma, we can now proceed with the proof of Theorem 12.

Proof of Theorem 12. We have the $\mathrm{SU}_q(3)$-representation

$$\Pi_q = \pi_1^{(q)} \boxtimes \pi_2^{(q)} \boxtimes \pi_1^{(q)} \colon \mathrm{SU}_q(3) \to C^*(S)^{\otimes 3} \subseteq \mathcal{B}(\ell^2(\mathbb{N})^{\otimes 3}),$$

where $\pi_i^{(q)} \colon \mathrm{SU}_q(3) \to C^*(S)$ is as in (2.2). By [5], this representation is irreducible. As $\mathrm{SU}_q(3)$ is type I, we have

$$\mathcal{K} \subseteq \operatorname{Im} \Pi_q \subseteq C^*(S)^{\otimes 3}$$

(where again $\mathcal{K}$ denotes the compact operators). Consider now the homomorphism defined as the composition

$$\mathrm{SU}_q(3) \xrightarrow{\Pi_q} \mathcal{B}(\ell^2(\mathbb{N})^{\otimes 3}) \xrightarrow{\sim\mathcal{K}} \mathcal{B}(\ell^2(\mathbb{N})^{\otimes 3})/\mathcal{K} = \mathcal{Q}(\ell^2(\mathbb{N})^{\otimes 3}). \tag{2.25}$$

We can deduce from the $\mathrm{SU}_q(3)$-representation theory ([9, Theorem 4.1 (ii)]) that the kernel of the representation (2.25) is the same as for the representation

$$\Phi_q = (\pi_1^{(q)} \boxtimes \pi_2^{(q)} \boxtimes \tau_1) \oplus (\tau_1 \boxtimes \pi_2^{(q)} \boxtimes \pi_1^{(q)}).$$

By evaluating the C^*-algebra from Lemma 13 at the point $(1,1) \in \mathbb{T}^2$ at the factor $C(\mathbb{T}) \otimes C(\mathbb{T})$, one sees that the image of Φ_q is actually independent of q, and that it is generated as a C^*-algebra by the limits $\lim_{q\to 0} \Phi_q(C_j^{\omega_i})$.

We denote $A := \operatorname{Im} \Phi_q$. By comparison of kernels, it follows from the representation theory of $\mathrm{SU}_q(3)$ that we have an injective homomorphism $\varphi_q \colon A \to \mathcal{Q}(\ell^2(\mathbb{N})^{\otimes 3})$ such that the following diagram commutes:

$$\begin{array}{ccc} C(\mathrm{SU}_3)_q & \xrightarrow{\Phi_q} & A \\ {\scriptstyle \Pi_q}\downarrow & & \downarrow{\scriptstyle \varphi_q} \\ \mathcal{B}(\ell^2(\mathbb{N})^{\otimes 3}) & \xrightarrow[\sim\mathcal{K}]{} & \mathcal{Q}(\ell^2(\mathbb{N})^{\otimes 3}) \end{array} \tag{2.26}$$

Let $F(\mathbf{t}(q))$ be a fixed non-commutative polynomial in the set $\{C_j^{\omega_i}, (C_j^{\omega_i})^* \mid j = 1, 2, i = 1, 2, 3\}$. Such elements are dense in $\mathrm{SU}_q(3)$ for $q \in [0, 1)$. As the images of the generators $C_j^{\omega_i}$ under Φ_q and Π_q depends norm-continuously on $q \in [0, 1)$, it follows that so does $\Phi_q(F(\mathbf{t}(q)))$ and $\Pi_q(F(\mathbf{t}(q)))$. In particular, we have norm-limits

$$\lim_{q\to 0} \Phi_q(F(\mathbf{t}(q))) = \Phi_0(F(\mathbf{t}(0))),$$
$$\lim_{q\to 0} \Pi_q(F(\mathbf{t}(q))) = \Pi_0(F(\mathbf{t}(0))).$$

We claim that φ_q depends point-norm continuously on $q \in [0, 1)$. To prove continuity at $q \in (0, 1)$, we consider elements in A of the form $\Phi_q(F(\mathbf{t}(q)))$. We have

$$\begin{aligned}
&\|\varphi_q(\Phi_q(F(\mathbf{t}(q)))) - \varphi_{q+\epsilon}(\Phi_q(F(\mathbf{t}(q))))\| \\
&\quad \le \|\varphi_q(\Phi_q(F(\mathbf{t}(q)))) - \varphi_{q+\epsilon}(\Phi_{q+\epsilon}(F(\mathbf{t}(q+\epsilon))))\| \\
&\qquad + \|\varphi_{q+\epsilon}(\Phi_{q+\epsilon}(F(\mathbf{t}(q+\epsilon)))) - \varphi_{q+\epsilon}(\Phi_q(F(\mathbf{t}(q))))\| \\
&\quad \le \|\Pi_q(F(\mathbf{t}(q))) - \Pi_{q+\epsilon}(F(\mathbf{t}(q+\epsilon)))\| + \|\Phi_{q+\epsilon}(F(\mathbf{t}(q+\epsilon))) - \Phi_q(F(\mathbf{t}(q)))\|,
\end{aligned}$$

and by the above comments, both terms in the last expression $\to 0$ as $|\epsilon| \to 0$. The claim follows as these elements are dense in A. We show the existence of the point-norm limit $\lim_{q\to 0} \varphi_q$, and that φ_0 maps $\Phi_0(F(\mathbf{t}(0)))$ to $\Pi_0(F(\mathbf{t}(0))) + \mathcal{K}$. For $\epsilon > 0$, we have

$$\begin{aligned}
&\|\Pi_0(F(\mathbf{t}(0))) + \mathcal{K} - \varphi_\epsilon(\Phi_0(F(\mathbf{t}(0))))\| \\
&\quad \le \|\Pi_0(F(\mathbf{t}(0))) + \mathcal{K} - \varphi_\epsilon(\Phi_\epsilon(F(\mathbf{t}(\epsilon))))\| \\
&\qquad + \|\varphi_\epsilon(\Phi_\epsilon(F(\mathbf{t}(\epsilon)))) - \varphi_\epsilon(\Phi_0(F(\mathbf{t}(0))))\| \\
&\quad \le \|\Pi_0(F(\mathbf{t}(0))) - \Pi_\epsilon(F(\mathbf{t}(\epsilon)))\| + \|\Phi_\epsilon(F(\mathbf{t}(\epsilon))) - \Phi_0(F(\mathbf{t}(0)))\|,
\end{aligned}$$

and both these last terms converges $\to 0$ as $\epsilon \to 0$. Again, by the density of $\Phi_0(F(\mathbf{t}(0)))$, the point-norm limit $\varphi_0 \colon A \to \mathcal{Q}(\ell^2(\mathbb{N})^{\otimes 3})$ exists and is an injective homomorphism (as it is a point-norm limit of injective homomorphisms). We can thus apply Lemma 14 to get isomorphisms

$$\Gamma_q \colon p^{-1}(\varphi_0(A)) \to p^{-1}(\varphi_q(A)) = \Phi_q(C(\mathrm{SU}_3)_q).$$

Let us denote $M := p^{-1}(\varphi_0(A))$. By Lemma 15, evaluating the $C(\mathbb{T}) \otimes C(\mathbb{T})$-factors in $\mathrm{SU}_0(3)$ at a point shows that the C^*-algebra generated by elements $\Pi_0(F(\mathbf{t}(0)))$ contains $\mathcal{K}$. Thus, it coincides with M. In particular, we have that

$$\mathrm{SU}_0(3) \subseteq M \otimes C(\mathbb{T}) \otimes C(\mathbb{T}) \subseteq C^*(S)^{\otimes 3} \otimes C(\mathbb{T}) \otimes C(\mathbb{T}).$$

As

$$(\pi_1^{(q)} \boxtimes \pi_2^{(q)} \boxtimes \pi_1^{(q)}) \boxtimes (\tau_2 \boxtimes \tau_1) = \Xi_q = \Pi_q \boxtimes (\tau_2 \boxtimes \tau_1)$$

gives a faithful representation of $\mathrm{SU}_q(3)$, we can consider this C^*-algebra as a sub-C^*-algebra

$$\mathrm{SU}_q(3) \subseteq \Pi_q(\mathrm{SU}_q(3)) \otimes C(\mathbb{T}) \otimes C(\mathbb{T}) \subseteq C^*(S)^{\otimes 3} \otimes C(\mathbb{T}) \otimes C(\mathbb{T}).$$

We now show that $\Gamma_q \otimes \iota \otimes \iota$ restricts to an isomorphism

$$\Gamma_q \otimes \iota \otimes \iota\colon \mathrm{SU}_0(3) \to \mathrm{SU}_q(3), \quad \text{for } q \in (0,1). \tag{2.27}$$

To do this, note that we have the ideal $\mathcal{K} \otimes C(\mathbb{T}) \otimes C(\mathbb{T}) \subseteq \mathrm{SU}_q(3)$ for $q \in [0,1)$. For $q \in (0,1)$, this is standard theory ([3, Lemma 16]); and, for $q = 0$, this is Lemma 15. By the diagram (2.24), the isomorphisms Γ_q fixes $\mathcal{K}$, so that (2.27) fixes $\mathcal{K} \otimes C(\mathbb{T}) \otimes C(\mathbb{T})$. Thus, to show that (2.27) is an isomorphism, it is enough to check this modulo this ideal. By (2.24), this is the same as showing that $\varphi_q \circ \varphi_0^{-1} \otimes \iota \otimes \iota$ restricts to a isomorphism

$$(\varphi_q \circ \varphi_0^{-1} \otimes \iota \otimes \iota)\colon \mathrm{SU}_0(3)/\mathcal{K} \otimes C(\mathbb{T}) \otimes C(\mathbb{T}) \to \mathrm{SU}_q(3)/\mathcal{K} \otimes C(\mathbb{T}) \otimes C(\mathbb{T}). \tag{2.28}$$

Notice that, for $q \in [0,1)$, we have

$$\mathrm{SU}_q(3)/\mathcal{K} \otimes C(\mathbb{T}) \otimes C(\mathbb{T}) \subseteq \varphi_q(A) \otimes C(\mathbb{T}) \otimes C(\mathbb{T}),$$

so that (2.28) can be reduced to the question if the following equality holds:

$$\begin{aligned}(\varphi_0^{-1} \otimes \iota \otimes \iota)&(\mathrm{SU}_0(3)/\mathcal{K} \otimes C(\mathbb{T}) \otimes C(\mathbb{T})) \\ &= (\varphi_q^{-1} \otimes \iota \otimes \iota)(\mathrm{SU}_q(3)/\mathcal{K} \otimes C(\mathbb{T}) \otimes C(\mathbb{T}))\end{aligned} \tag{2.29}$$

as sub-C^*-algebras of $A \otimes C(\mathbb{T}) \otimes C(\mathbb{T})$. Due to (2.26), the question of equality in (2.29) is the same as whether we have independence of q of the images of

$$\Phi_q \boxtimes \tau_2 \boxtimes \tau_1 = ((\pi_1^{(q)} \boxtimes \pi_2^{(q)} \boxtimes \tau_1) \oplus (\tau \boxtimes \pi_2^{(q)} \boxtimes \pi_1^{(q)})) \boxtimes \tau_2 \boxtimes \tau_1.$$

However, this was proven in Lemma 13. Thus, (2.29) holds, and hence $\Gamma_q \otimes \iota \otimes \iota$ induces an isomorphism $\mathrm{SU}_0(3) \to \mathrm{SU}_q(3)$. By (2.8), we have for all $t \in \mathbb{T}$ that

$$(\Gamma_q \otimes \iota \otimes \iota) \circ \beta_t = \beta_t \circ (\Gamma_q \otimes \iota \otimes \iota). \qquad \blacksquare$$

Remark. Since Γ_q fixes the compacts, it can be written as $\Gamma_q(a) = U_q^* a U_q$ for some unitary $U_q \in \mathcal{B}(\ell^2(\mathbb{N})^{\otimes 3})$ ([1, Corollary 1.10]).

By combining Proposition 11 with Theorem 12, we have proven the main result here.

Theorem 16. *For every $q \in (0,1)$, we have an isomorphism $\phi_q\colon C^*(\mathcal{S}u(3)) \to \mathrm{SU}_q(3)$, intertwining the gauge action on $C^*(\mathcal{S}u(3))$ with the right action on $\mathrm{SU}_q(3)$.*

Proof. Take $\phi_q = (\Gamma_q \otimes \iota \otimes \iota) \circ \phi$, where ϕ is the isomorphism from Proposition 11 and $\Gamma_q \otimes \iota \otimes \iota$ is the isomorphism from the proof of Theorem 12. $\blacksquare$

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