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SL(2, Z) modular forms and Witten genus in odd dimensions

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Abstract. Using $\mathrm{SL}(2,\mathbf{Z})$ modular forms introduced in Liu (1996) and Chen–Han–Zhang (2011), we construct some modular forms over $\mathrm{SL}(2,\mathbf{Z})$ and some modular forms over $\Gamma^0(2)$ and $\Gamma_0(2)$ in odd dimensions. In parallel, we obtain some new cancellation formulas for odd-dimensional spin manifolds and odd-dimensional spin^c manifolds, respectively. As corollaries, we get some divisibility results of index of the Toeplitz operators on spin manifolds and spin^c manifolds.

Contents

1. Introduction

In [1], Alvarez-Gaumé and Witten discovered a formula that represents the beautiful relationship between the top components of the Hirzebruch $\widehat{L}$-form and $\widehat{A}$-form of a 12-dimensional smooth Riemannian manifold. This formula is called the “miraculous cancellation” formula for gravitational anomalies. In [13], Liu established higher-dimensional “miraculous cancellation” formulas for $(8k+4)$-dimensional Riemannian manifolds by developing modular invariance properties of characteristic forms. These formulas can be used to deduce some divisibility results. In [11,12], Han and Zhang established some more general cancellation formulas that involve a complex line bundle over each $(8k+4)$-dimensional smooth Riemannian manifold. This formula was applied to spin^c manifolds, then an analytic Ochanine congruence formula was derived. In [17], Wang obtained some new anomaly cancellation formulas by studying the modular invariance of some characteristic forms. This formula was applied to spin manifolds and spin^c manifolds, then

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some results on divisibilities on spin manifolds and spin^c manifolds were derived. Moreover, Han, Liu and Zhang, using the Eisenstein series, derived a more general cancellation formula [8]. In [9], the authors showed that both of the Green–Schwarz anomaly factorization formula for the gauge group $E_8 \times E_8$ and the Horava–Witten anomaly factorization formula for the gauge group E_8 could be derived through modular forms of weight 14. This answered a question of J. H. Schwarz. They also established generalizations of these decomposition formulas and obtained a new Horava–Witten type decomposition formula on 12-dimensional manifolds. In [7], Han, Huang, Liu and Zhang introduced a modular form of weight 14 over $\mathrm{SL}(2,\mathbf{Z})$ and a modular form of weight 10 over $\mathrm{SL}(2,\mathbf{Z})$, and they got some interesting anomaly cancellation formulas on 12-dimensional manifolds. In [18, 19], Wang obtained some new anomaly cancellation formulas by studying some $\mathrm{SL}(2,\mathbf{Z})$ modular forms. Some divisibility results of the index of the twisted Dirac operator were obtained.

In [14], Liu introduced a modular form of a $4k$-dimensional spin manifold with a weight of $2k$. In [5], Chen, Han and Zhang defined an integral modular form of weight $2k$ for a $4k$-dimensional spin^c manifold and an integral modular form of weight $2k$ for a $(4k+2)$-dimensional spin^c manifold. A natural question is whether we can get some interesting cancellation formulas and more results on divisibilities in odd dimensions. In [15,16], Liu and Wang go through studying modular invariance properties of some characteristic forms to get some new anomaly cancellation formulas on $(4k-1)$-dimensional manifolds. And they derive some results on divisibilities on $(4k-1)$-dimensional spin manifolds and congruences on $(4k-1)$-dimensional spin^c manifolds. Inspired by this, we introduce some modular forms of weight $2k$ over $\mathrm{SL}(2,\mathbf{Z})$ and some modular forms of weight $2k$ over $\Gamma^0(2)$ and $\Gamma_0(2)$ in odd dimensions, respectively, through the $\mathrm{SL}(2,\mathbf{Z})$ modular forms introduced in [14] and [5]. In parallel, we derive some new anomaly cancellation formulas and some divisibility results over spin manifolds and spin^c manifolds in odd dimensions.

The structure of this paper is briefly described below: In Section 2, we introduce some definitions and basic concepts that we will use in the paper. In Section 3, we prove some generalized cancellation formulas over $\mathrm{SL}(2,\mathbf{Z})$ in odd dimensions. Finally, in Section 4, we obtain some generalized cancellation formulas over $\Gamma^0(2)$ and $\Gamma_0(2)$ in odd dimensions.

2. Characteristic forms and modular forms

The purpose of this section is to review the necessary knowledge on characteristic forms and modular forms that we are going to use.

2.1. Characteristic forms

Let M be a Riemannian manifold. Let ∇^{TM} be the associated Levi-Civita connection on TM and $R^{TM} = (\nabla^{TM})^2$ be the curvature of ∇^{TM}. Let $\widehat{A}(TM, \nabla^{TM})$ and $\widehat{L}(TM, \nabla^{TM})$ be the Hirzebruch characteristic forms defined, respectively, by (cf. [20])

$$\widehat{A}(TM, \nabla^{TM}) = \det{}^{\frac{1}{2}}\left(\frac{\frac{\sqrt{-1}}{4\pi}R^{TM}}{\sinh\left(\frac{\sqrt{-1}}{4\pi}R^{TM}\right)}\right), \tag{2.1}$$

$$\widehat{L}(TM, \nabla^{TM}) = \det{}^{\frac{1}{2}}\left(\frac{\frac{\sqrt{-1}}{2\pi}R^{TM}}{\tanh\left(\frac{\sqrt{-1}}{4\pi}R^{TM}\right)}\right). \tag{2.2}$$

Let E, F be two Hermitian vector bundles over M carrying Hermitian connection ∇^E, ∇^F, respectively. Let $R^E = (\nabla^E)^2$ (resp. $R^F = (\nabla^F)^2$) be the curvature of ∇^E (resp. ∇^F). If we set the formal difference $G = E - F$, then G carries an induced Hermitian connection ∇^G in an obvious sense. We define the associated Chern character form as

$$\mathrm{ch}(G, \nabla^G) = \mathrm{Tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}R^E\right)\right] - \mathrm{Tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}R^F\right)\right]. \tag{2.3}$$

For any complex number t, let

$$\wedge_t(E) = \mathbf{C}|_M + tE + t^2 \wedge^2(E) + \cdots, \quad S_t(E) = \mathbf{C}|_M + tE + t^2 S^2(E) + \cdots$$

denote, respectively, the total exterior and symmetric powers of E, which live in $K(M)[\![t]\!]$. The following relations between these operations hold:

$$S_t(E) = \frac{1}{\wedge_{-t}(E)}, \quad \wedge_t(E - F) = \frac{\wedge_t(E)}{\wedge_t(F)}. \tag{2.4}$$

Moreover, if $\{\omega_i\}$, $\{\omega_j'\}$ are formal Chern roots for Hermitian vector bundles E, F, respectively, then

$$\mathrm{ch}(\wedge_t(E)) = \prod_i (1 + e^{\omega_i} t). \tag{2.5}$$

Then we have the following formulas for Chern character forms:

$$\mathrm{ch}(S_t(E)) = \frac{1}{\prod_i (1 - e^{\omega_i} t)}, \quad \mathrm{ch}(\wedge_t(E - F)) = \frac{\prod_i (1 + e^{\omega_i} t)}{\prod_j (1 + e^{\omega_j'} t)}. \tag{2.6}$$

If W is a real Euclidean vector bundle over M carrying a Euclidean connection ∇^W, then its complexification $W_{\mathbf{C}} = W \otimes \mathbf{C}$ is a complex vector bundle over M carrying a canonical induced Hermitian metric from that of W, as well as a Hermitian connection $\nabla^{W_{\mathbf{C}}}$ induced from ∇^W. If E is a vector bundle (complex or real) over M, set $\widetilde{E} = E - \dim E$ in $K(M)$ or $KO(M)$.

2.2. Some properties of the Jacobi theta functions and modular forms

We first recall that the four Jacobi theta functions are defined as follows (cf. [3]):

$$\theta(v,\tau)=2q^{\frac{1}{8}}\sin(\pi v)\prod_{j=1}^{\infty}\left[(1-q^{j})\left(1-e^{2\pi\sqrt{-1}v}q^{j}\right)\left(1-e^{-2\pi\sqrt{-1}v}q^{j}\right)\right], \tag{2.7}$$

$$\theta_1(v,\tau)=2q^{\frac{1}{8}}\cos(\pi v)\prod_{j=1}^{\infty}\left[(1-q^{j})\left(1+e^{2\pi\sqrt{-1}v}q^{j}\right)\left(1+e^{-2\pi\sqrt{-1}v}q^{j}\right)\right], \tag{2.8}$$

$$\theta_2(v,\tau)=\prod_{j=1}^{\infty}\left[(1-q^{j})\left(1-e^{2\pi\sqrt{-1}v}q^{j-\frac{1}{2}}\right)\left(1-e^{-2\pi\sqrt{-1}v}q^{j-\frac{1}{2}}\right)\right], \tag{2.9}$$

$$\theta_3(v,\tau)=\prod_{j=1}^{\infty}\left[(1-q^{j})\left(1+e^{2\pi\sqrt{-1}v}q^{j-\frac{1}{2}}\right)\left(1+e^{-2\pi\sqrt{-1}v}q^{j-\frac{1}{2}}\right)\right], \tag{2.10}$$

where $q=e^{2\pi\sqrt{-1}\tau}$ with $\tau\in\mathbf{H}$, the upper half complex plane. Let

$$\theta'(0,\tau)=\left.\frac{\partial\theta(v,\tau)}{\partial v}\right|_{v=0}. \tag{2.11}$$

Then the following Jacobi identity (cf. [3]) holds:

$$\theta'(0,\tau)=\pi\theta_1(0,\tau)\theta_2(0,\tau)\theta_3(0,\tau). \tag{2.12}$$

Denote

$$\mathrm{SL}(2,\mathbf{Z})=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\mid a,b,c,d\in\mathbf{Z},\ ad-bc=1\right\}$$

the modular group. Let $S=\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$, $T=\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ be the two generators of $\mathrm{SL}(2,\mathbf{Z})$. They act on $\mathbf{H}$ by $S\tau=-\frac{1}{\tau}$, $T\tau=\tau+1$. One has the following transformation laws of theta functions under the actions of S and T (cf. [3]):

$$\theta(v,\tau+1)=e^{\frac{\pi\sqrt{-1}}{4}}\theta(v,\tau),\qquad \theta\left(v,-\frac{1}{\tau}\right)=\frac{1}{\sqrt{-1}}\left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}}\cdot e^{\pi\sqrt{-1}\tau v^2}\theta(\tau v,\tau), \tag{2.13}$$

$$\theta_1(v,\tau+1)=e^{\frac{\pi\sqrt{-1}}{4}}\theta_1(v,\tau),\qquad \theta_1\left(v,-\frac{1}{\tau}\right)=\left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}}e^{\pi\sqrt{-1}\tau v^2}\theta_2(\tau v,\tau), \tag{2.14}$$

$$\theta_2(v,\tau+1)=\theta_3(v,\tau),\qquad \theta_2\left(v,-\frac{1}{\tau}\right)=\left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}}e^{\pi\sqrt{-1}\tau v^2}\theta_1(\tau v,\tau), \tag{2.15}$$

$$\theta_3(v,\tau+1)=\theta_2(v,\tau),\qquad \theta_3\left(v,-\frac{1}{\tau}\right)=\left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}}e^{\pi\sqrt{-1}\tau v^2}\theta_3(\tau v,\tau), \tag{2.16}$$

$$\theta'(v,\tau+1)=e^{\frac{\pi\sqrt{-1}}{4}}\theta'(v,\tau),\qquad \theta'\left(0,-\frac{1}{\tau}\right)=\frac{1}{\sqrt{-1}}\left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}}\tau\theta'(0,\tau). \tag{2.17}$$

Definition 2.1. A modular form over Γ, a subgroup of $SL(2, \mathbf{Z})$, is a holomorphic function $f(\tau)$ on $\mathbf{H}$ such that

$$f(g\tau) := f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi(g)(c\tau+d)^k f(\tau) \quad \forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \tag{2.18}$$

where $\chi : \Gamma \to \mathbf{C}^\star$ is a character of Γ, and k is called the weight of f.

Let

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}) \mid c \equiv 0 \ (\mathrm{mod}\ 2) \right\},$$
$$\Gamma^0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}) \mid b \equiv 0 \ (\mathrm{mod}\ 2) \right\}$$

be the two modular subgroups of $SL(2, \mathbf{Z})$. It is known that the generators of $\Gamma_0(2)$ are T, ST^2ST, and the generators of $\Gamma^0(2)$ are STS, T^2STS (cf. [3]).

If Γ is a modular subgroup, let $\mathcal{M}_{\mathbf{R}}(\Gamma)$ denote the ring of modular forms over Γ with real Fourier coefficients. Writing $\theta_j = \theta_j(0, \tau)$, $1 \le j \le 3$, we introduce six explicit modular forms (cf. [13]):

$$\delta_1(\tau) = \frac{1}{8}(\theta_2^4 + \theta_3^4), \qquad \varepsilon_1(\tau) = \frac{1}{16}\theta_2^4\theta_3^4,$$
$$\delta_2(\tau) = -\frac{1}{8}(\theta_1^4 + \theta_3^4), \quad \varepsilon_2(\tau) = \frac{1}{16}\theta_1^4\theta_3^4,$$
$$\delta_3(\tau) = \frac{1}{8}(\theta_1^4 - \theta_2^4), \qquad \varepsilon_3(\tau) = -\frac{1}{16}\theta_1^4\theta_2^4.$$

They have the following Fourier expansions in $q^{\frac{1}{2}}$:

$$\delta_1(\tau) = \frac{1}{4} + 6q + \cdots, \qquad \varepsilon_1(\tau) = \frac{1}{16} - q + \cdots,$$
$$\delta_2(\tau) = -\frac{1}{8} - 3q^{\frac{1}{2}} + \cdots, \quad \varepsilon_2(\tau) = q^{\frac{1}{2}} + \cdots,$$
$$\delta_3(\tau) = -\frac{1}{8} + 3q^{\frac{1}{2}} + \cdots, \quad \varepsilon_3(\tau) = -q^{\frac{1}{2}} + \cdots,$$

where the "$\cdots$" terms are the higher degree terms, all of which have integral coefficients. They also satisfy the transformation laws:

$$\delta_2\left(-\frac{1}{\tau}\right) = \tau^2\delta_1(\tau), \quad \varepsilon_2\left(-\frac{1}{\tau}\right) = \tau^4\varepsilon_1(\tau), \tag{2.19}$$
$$\delta_2(\tau+1) = \delta_3(\tau), \qquad \varepsilon_2(\tau+1) = \varepsilon_3(\tau). \tag{2.20}$$

Lemma 2.2 ([13]). *Let $\delta_1(\tau)$ (resp. $\varepsilon_1(\tau)$) be a modular form of weight 2 (resp. 4) over $\Gamma_0(2)$, $\delta_2(\tau)$ (resp. $\varepsilon_2(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma^0(2)$, while $\delta_3(\tau)$ (resp. $\varepsilon_3(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma_\theta(2)$, and moreover, $\mathcal{M}_{\boldsymbol{R}}(\Gamma^0(2)) = \boldsymbol{R}[\delta_2(\tau), \varepsilon_2(\tau)]$.*

3. Some modular forms and Witten genus over SL(2, Z) in odd dimensions

In this section, we will construct some modular forms over $\mathrm{SL}(2, \mathbf{Z})$ in odd dimensions. Furthermore, we calculate and derive some new cancellation formulas for odd-dimensional spin manifolds and odd-dimensional spin^c manifolds, respectively. Meanwhile, we will derive some divisibility results of index of the Toeplitz operators on spin manifolds and spin^c manifolds.

3.1. Some modular forms and Witten genus in spin manifolds

Let M be a $(4k-1)$-dimensional spin manifold and $\triangle(M)$ be the spinor bundle. Let $\widetilde{T_{\mathbf{C}}M} = T_{\mathbf{C}}M - \dim M$. Set

$$\Theta_1(T_{\mathbf{C}}M) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbf{C}}M}) \otimes \bigotimes_{m=1}^{\infty} \wedge_{q^m}(\widetilde{T_{\mathbf{C}}M}), \tag{3.1}$$

$$\Theta_2(T_{\mathbf{C}}M) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbf{C}}M}) \otimes \bigotimes_{m=1}^{\infty} \wedge_{-q^{m-\frac{1}{2}}}(\widetilde{T_{\mathbf{C}}M}), \tag{3.2}$$

$$\Theta_3(T_{\mathbf{C}}M) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbf{C}}M}) \otimes \bigotimes_{m=1}^{\infty} \wedge_{q^{m-\frac{1}{2}}}(\widetilde{T_{\mathbf{C}}M}). \tag{3.3}$$

We recall the odd Chern character of a smooth map g from M to the general linear group $\mathrm{GL}(N, \mathbf{C})$ with N a positive integer (see [20]). Let d denote a trivial connection on $\mathbf{C}^N|_M$. We will denote by $c_g(M, [g])$ the cohomology class associated to the closed n-form

$$c_n(\mathbf{C}^N|_M, g, d) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{(n+1)}{2}} \mathrm{Tr}[(g^{-1}dg)^n]. \tag{3.4}$$

The odd Chern character form $\mathrm{ch}(\mathbf{C}^N|_M, g, d)$ associated to g and d by definition is

$$\mathrm{ch}(\mathbf{C}^N|_M, g, d) = \sum_{n=1}^{\infty} \frac{n!}{(2n+1)!} c_{2n+1}((\mathbf{C}^N|_M, g, d)). \tag{3.5}$$

Let the connection ∇_u on the trivial bundle $\mathbf{C}^N|_M$ be defined by

$$\nabla_u = (1-u)d + ug^{-1} \cdot d \cdot g, \quad u \in [0, 1]. \tag{3.6}$$

Then we have

$$d\,\mathrm{ch}(\mathbf{C}^N|_M, g, d) = \mathrm{ch}(\mathbf{C}^N|_M, d) - \mathrm{ch}(\mathbf{C}^N|_M, g^{-1} \cdot d \cdot g). \tag{3.7}$$

Now let $g : M \to \mathrm{SO}(N)$, and we assume that N is even and large enough. Let E denote the trivial real vector bundle of rank N over M. We equip E with the canonical trivial metric and trivial connection d. Set

$$\nabla_u = d + ug^{-1}dg, \quad u \in [0, 1].$$

Let R_u be the curvature of ∇_u, then

$$R_u = (u^2 - u)(g^{-1}dg)^2. \tag{3.8}$$

We also consider the complexification of E, and g extends to a unitary automorphism of $E_{\mathbf{C}}$. The connection ∇_u extends to a Hermitian connection on $E_{\mathbf{C}}$ with curvature still given by (3.6). Let $\Delta(E)$ be the spinor bundle of E, which is a trivial Hermitian bundle of rank $2^{\frac{N}{2}}$. We assume that g has a lift to the spin group $\mathrm{Spin}(N) : g^{\Delta} : M \to \mathrm{Spin}(N)$. So g^{Δ} can be viewed as an automorphism of $\Delta(E)$ preserving the Hermitian metric. We lift d on E to be a trivial Hermitian connection d^{Δ} on $\Delta(E)$, then

$$\nabla_u^{\Delta} = (1-u)d^{\Delta} + u(g^{\Delta})^{-1} \cdot d^{\Delta} \cdot g^{\Delta}, \quad u \in [0, 1] \tag{3.9}$$

lifts ∇_u on E to $\Delta(E)$. Let $Q_j(E)$, $j = 1, 2, 3$ be the virtual bundles defined as follows:

$$Q_1(E) = \triangle(E) \otimes \bigotimes_{n=1}^{\infty} \wedge_{q^n}(\widetilde{E_{\mathbf{C}}}), \tag{3.10}$$

$$Q_2(E) = \bigotimes_{n=1}^{\infty} \wedge_{-q^{n-\frac{1}{2}}}(\widetilde{E_{\mathbf{C}}}), \tag{3.11}$$

$$Q_3(E) = \bigotimes_{n=1}^{\infty} \wedge_{q^{n-\frac{1}{2}}}(\widetilde{E_{\mathbf{C}}}). \tag{3.12}$$

Let g on E have a lift $g^{Q(E)}$ on $Q(E)$ and ∇_u have a lift $\nabla_u^{Q(E)}$ on $Q(E)$. Following [10], we defined $\mathrm{ch}(Q(E), g^{Q(E)}, d, \tau)$ as follows:

$$\mathrm{ch}(Q(E), \nabla_0^{Q(E)}, \tau) - \mathrm{ch}(Q(E), \nabla_1^{Q(E)}, \tau) = -d\, \mathrm{ch}(Q(E), g^{Q(E)}, d, \tau), \tag{3.13}$$

where

$$Q(E) = Q_1(E) \otimes Q_2(E) \otimes Q_3(E),$$

and

$$\mathrm{ch}(Q(E), g^{Q(E)}, d, \tau) = -\frac{2^{\frac{N}{2}}}{8\pi^2} \int_0^1 \mathrm{Tr}[g^{-1}dg(A)]du, \tag{3.14}$$

with

$$A = \frac{\theta_1'(R_u/(4\pi^2), \tau)}{\theta_1(R_u/(4\pi^2), \tau)} + \frac{\theta_2'(R_u/(4\pi^2), \tau)}{\theta_2(R_u/(4\pi^2), \tau)} + \frac{\theta_3'(R_u/(4\pi^2), \tau)}{\theta_3(R_u/(4\pi^2), \tau)}.$$

By [10, Proposition 2.2], it follows that if $c_3(E_{\mathbf{C}}, g, d) = 0$, then for any integer $r \geq 1$, we obtain $\mathrm{ch}(Q(E), g^{Q(E)}, d, \tau + 1)^{(4r-1)} = \mathrm{ch}(Q(E), g^{Q(E)}, d, \tau)^{(4r-1)}$ and $\mathrm{ch}\big(Q(E), g^{Q(E)}, d, -\frac{1}{\tau}\big)^{(4r-1)} = \tau^{2r}\, \mathrm{ch}(Q(E), g^{Q(E)}, d, \tau)^{(4r-1)}$. Therefore, we can deduce $\mathrm{ch}(Q(E), g^{Q(E)}, d, \tau)^{(4r-1)}$ are modular forms of weight $2r$ over $\mathrm{SL}(2, \mathbf{Z})$. Let

$$\begin{aligned} Q(\nabla^{TM}, g, d, \tau) = \{&\widehat{A}(TM, \nabla^{TM})\, \mathrm{ch}([\triangle(M) \otimes \Theta_1(T_{\mathbf{C}}M) + 2^{2k}\Theta_2(T_{\mathbf{C}}M) \\ &+ 2^{2k}\Theta_3(T_{\mathbf{C}}M)])\, \mathrm{ch}(Q(E), g^{Q(E)}, d, \tau)\}^{(4k-1)}. \end{aligned} \tag{3.15}$$

Let $g : M \to \mathrm{SO}(N)$ and N is even and large enough. Let E denote the trivial real vector bundle of rank N over M. We have the following theorem.

Theorem 3.1. *Let* $\dim M = 4k-1$. *Suppose* g *has a lift to the spin group* $\mathrm{Spin}(N)$: $g^{\Delta} : M \to \mathrm{Spin}(N)$. *If* $c_3(E, g, d) = 0$, *then for any integer* $p, r \geq 1$, $Q(\nabla^{TM}, g, d, \tau)$ *is a modular form over* $\mathrm{SL}(2, \mathbf{Z})$ *with the weight* $2p + 2r = 2k$.

Proof. Let

$$Q(M,\tau) = \{\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}([\triangle(M)\otimes\Theta_1(T_{\mathbf{C}}M) + 2^{2k}\Theta_2(T_{\mathbf{C}}M) + 2^{2k}\Theta_3(T_{\mathbf{C}}M)])\}^{(4p)}, \tag{3.16}$$

then we have

$$Q(\nabla^{TM}, g, d, \tau) = Q(M,\tau)\cdot \mathrm{ch}(Q(E), g^{Q(E)}, d, \tau)^{(4r-1)}. \tag{3.17}$$

Let $\{\pm 2\pi\sqrt{-1}x_j\}$, $(1 \leq j \leq 2k-1)$ be the formal Chern roots for $T_{\mathbf{C}}M$, then we have

$$\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M))\,\mathrm{ch}(\Theta_1(T_{\mathbf{C}}M)) = \prod_{j=1}^{2k-1}\frac{2x_j\theta'(0,\tau)}{\theta(x_j,\tau)}\frac{\theta_1(x_j,\tau)}{\theta_1(0,\tau)}, \tag{3.18}$$

$$\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(2^{2k}\Theta_2(T_{\mathbf{C}}M)) = \prod_{j=1}^{2k-1}\frac{2x_j\theta'(0,\tau)}{\theta(x_j,\tau)}\frac{\theta_2(x_j,\tau)}{\theta_2(0,\tau)}, \tag{3.19}$$

$$\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(2^{2k}\Theta_3(T_{\mathbf{C}}M)) = \prod_{j=1}^{2k-1}\frac{2x_j\theta'(0,\tau)}{\theta(x_j,\tau)}\frac{\theta_3(x_j,\tau)}{\theta_3(0,\tau)}. \tag{3.20}$$

So we have

$$Q(M,\tau) = \left(\prod_{j=1}^{2k-1}\frac{2x_j\theta'(0,\tau)}{\theta(x_j,\tau)} \cdot\left(\prod_{j=1}^{2k-1}\frac{\theta_1(x_j,\tau)}{\theta_1(0,\tau)} + \prod_{j=1}^{2k-1}\frac{\theta_2(x_j,\tau)}{\theta_2(0,\tau)} + \prod_{j=1}^{2k-1}\frac{\theta_3(x_j,\tau)}{\theta_3(0,\tau)}\right)\right)^{(4p)}. \tag{3.21}$$

By [10, Proposition 2.2], and (3.16)–(3.21), we have

$$Q(\nabla^{TM}, g, d, \tau+1) = Q(M,\tau)\cdot\mathrm{ch}(Q(E), g^{Q(E)}, d, \tau)^{(4r-1)} = Q(\nabla^{TM}, g, d, \tau)$$

and

$$\begin{aligned} Q\left(\nabla^{TM}, g, d, -\frac{1}{\tau}\right) &= \tau^{2p}Q(M,\tau)\cdot\tau^{2r}\,\mathrm{ch}(Q(E), g^{Q(E)}, d, \tau)^{(4r-1)} \\ &= \tau^{2k}Q(\nabla^{TM}, g, d, \tau), \end{aligned}$$

so $Q(\nabla^{TM}, g, d, \tau)$ is a modular form over $\mathrm{SL}(2, \mathbf{Z})$ with the weight $2p + 2r = 2k$. ∎

Remark. In Theorem 3.1, g has a lift to the spin group $\mathrm{Spin}(N) : g^{\Delta} : M \to \mathrm{Spin}(N)$, which is satisfied when M is simply connected and $c_3(E, g, d) = 0$ if $\mathbf{H}^3(M, \mathbb{R}) = 0$. For more specific details, see [10].

Remark. All the conclusions given later hold true under this condition that g has a lift to the spin group $\mathrm{Spin}(N)$ and will not be elaborated here.

Set

$$\begin{aligned}\left\langle Q(\nabla^{TM}, g, d, \tau), [M]\right\rangle = &-\mathrm{Ind}(T \otimes (\triangle(M) \otimes \Theta_1(T_{\mathbf{C}}M) + 2^{2d}\Theta_2(T_{\mathbf{C}}M)\\ &+ 2^{2d}\Theta_3(T_{\mathbf{C}}M)) \otimes (Q(E), g^{Q(E),d})),\end{aligned} \tag{3.22}$$

where $\mathrm{Ind}(T \otimes \cdots)$ denotes the index of the Toeplitz operator. Clearly, $\Theta_1(T_{\mathbf{C}}M) \otimes Q(E), \Theta_2(T_{\mathbf{C}}M) \otimes Q(E)$ and $\Theta_2(T_{\mathbf{C}}M) \otimes Q(E)$ admit formal Fourier expansion in $q^{\frac{1}{2}}$ as

$$\begin{aligned}\Theta_1(T_{\mathbf{C}}M) \otimes Q(E) &= B_0^1(T_{\mathbf{C}}M, E_{\mathbf{C}}) + B_1^1(T_{\mathbf{C}}M, E_{\mathbf{C}})q + B_2^1(T_{\mathbf{C}}M, E_{\mathbf{C}})q^2 + \cdots,\\ \Theta_2(T_{\mathbf{C}}M) \otimes Q(E) &= B_0^2(T_{\mathbf{C}}M, E_{\mathbf{C}}) + B_1^2(T_{\mathbf{C}}M, E_{\mathbf{C}})q^{\frac{1}{2}} + B_2^2(T_{\mathbf{C}}M, E_{\mathbf{C}})q + \cdots,\\ \Theta_3(T_{\mathbf{C}}M) \otimes Q(E) &= B_0^3(T_{\mathbf{C}}M, E_{\mathbf{C}}) + B_1^3(T_{\mathbf{C}}M, E_{\mathbf{C}})q^{\frac{1}{2}} + B_2^3(T_{\mathbf{C}}M, E_{\mathbf{C}})q + \cdots,\end{aligned}$$

where the B_j^i are elements in the semi-group formally generated by Hermitian vector bundles over M. Moreover, they carry canonically induced Hermitian connections. If

$$B_j^i(T_{\mathbf{C}}M, E) = B_{j,1}^i(T_{\mathbf{C}}M) \otimes B_{j,2}^i(E),$$

we let

$$\widetilde{\mathrm{ch}}(B_j^i(T_{\mathbf{C}}M, E)) = \mathrm{ch}(B_{j,1}^i(T_{\mathbf{C}}M))\, \mathrm{ch}(B_{j,2}^i(E), g^{B_{j,2}^i(E)}, d).$$

If ω is a differential form over M, we denote by $\omega^{(4k-1)}$ its top degree component. Our main results include the following theorem.

Theorem 3.2. *When* $\dim M = 7$ *and* $c_3(E, g, d) = 0$, *we have*

$$\begin{aligned}&\{\widehat{A}(TM, \nabla^{TM})[\mathrm{ch}(\triangle(M) \otimes 2\widetilde{T_{\mathbf{C}}M})\, \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d)\\ &\qquad + \mathrm{ch}(\triangle(M))\, \mathrm{ch}(\Delta(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)]\\ &\qquad + 32\widehat{A}(TM, \nabla^{TM})[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M} + \wedge^2\widetilde{T_{\mathbf{C}}M})\, \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d)\\ &\qquad + \mathrm{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)]\}^{(7)}\\ &= 240[\widehat{A}(TM, \nabla^{TM})\, \mathrm{ch}(\triangle(M))\, \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d)\\ &\qquad + 32\widehat{A}(TM, \nabla^{TM})\, \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d)],\end{aligned} \tag{3.23}$$

$$\begin{aligned}&\{\widehat{A}(TM, \nabla^{TM})[\mathrm{ch}(\triangle(M) \otimes (2\widetilde{T_{\mathbf{C}}M} + \wedge^2\widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M} + S^2\widetilde{T_{\mathbf{C}}M}))\\ &\qquad \cdot \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) + \mathrm{ch}(\triangle(M) \otimes 2\widetilde{T_{\mathbf{C}}M})\, \mathrm{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}}\end{aligned}$$

$$
\begin{aligned}
&- \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d) + \mathrm{ch}(\triangle(M))\,\mathrm{ch}(\triangle(E) \otimes (\wedge^2 \widetilde{E_{\mathbf{C}}} \\
&\quad \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} + 2 \wedge^4 \widetilde{E_{\mathbf{C}}} - 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^3 \widetilde{E_{\mathbf{C}}} + 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \\
&\quad \otimes \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}} + \wedge^2 \widetilde{E_{\mathbf{C}}}), g, d)] \\
&+ 32\widehat{A}(TM, \nabla^{TM})[\mathrm{ch}(\wedge^4 \widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M} \\
&\quad + S^2 \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) \\
&\quad + \mathrm{ch}(\widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M}) \cdot \mathrm{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \\
&\qquad \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d) \\
&\quad + \mathrm{ch}(\triangle(E) \otimes (\wedge^2 \widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} + 2 \wedge^4 \widetilde{E_{\mathbf{C}}} - 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^3 \widetilde{E_{\mathbf{C}}} \\
&\qquad + 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \\
&\qquad + \widetilde{E_{\mathbf{C}}} + \wedge^2 \widetilde{E_{\mathbf{C}}}), g, d)]\}^{(7)} \\
&= 2160[\widehat{A}(TM, \nabla^{TM})\,\mathrm{ch}(\triangle(M))\,\mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) \\
&\quad + 32\widehat{A}(TM, \nabla^{TM})\,\mathrm{ch}(\triangle(E), g^{\triangle(E)}, d)].
\end{aligned}
\tag{3.24}
$$

Proof. It is well known that modular forms over $\mathrm{SL}(2, \mathbf{Z})$ can be expressed as polynomials of the Eisenstein series $E_4(\tau)$ and $E_6(\tau)$, where

$$E_4(\tau) = 1 + 240q + 2160q^2 + 6720q^3 + \cdots, \tag{3.25}$$

$$E_6(\tau) = 1 - 504q - 16632q^2 - 122976q^3 + \cdots. \tag{3.26}$$

Their weights are 4 and 6, respectively. When $\dim M = 7$, then $Q(\nabla^{TM}, g, d, \tau)$ is a modular form over $\mathrm{SL}(2, \mathbf{Z})$ with the weight 4. By (3.1)–(3.3) and (3.10)–(3.12), we have

$$
\begin{aligned}
\Theta_1(T_{\mathbf{C}}M) = 1 + 2q\widetilde{T_{\mathbf{C}}M} + q^2(2\widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} \\
\otimes \widetilde{T_{\mathbf{C}}M} + S^2 \widetilde{T_{\mathbf{C}}M}) + O(q^3),
\end{aligned}
\tag{3.27}
$$

$$
\begin{aligned}
\Theta_2(T_{\mathbf{C}}M) = 1 - q^{\frac{1}{2}} \widetilde{T_{\mathbf{C}}M} + q(\widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M}) \\
- q^{\frac{3}{2}}(\wedge^3 \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M}) \\
+ q^2(\wedge^4 \widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M} \\
+ S^2 \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M}) + O\big(q^{\frac{5}{2}}\big),
\end{aligned}
\tag{3.28}
$$

$$
\begin{aligned}
\Theta_3(T_{\mathbf{C}}M) = 1 + q^{\frac{1}{2}} \widetilde{T_{\mathbf{C}}M} + q(\widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M}) \\
+ q^{\frac{3}{2}}(\wedge^3 \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M}) \\
+ q^2(\wedge^4 \widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M} \\
+ S^2 \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M}) + O\big(q^{\frac{5}{2}}\big),
\end{aligned}
\tag{3.29}
$$

$$
\begin{aligned}
Q(E) = \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) + q\,\mathrm{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \\
\otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d) + q^2\,\mathrm{ch}(\triangle(E) \otimes (\wedge^2 \widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}}
\end{aligned}
$$

$$+ 2\wedge^4 \widetilde{E_{\mathbf{C}}} - 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^3 \widetilde{E_{\mathbf{C}}} + 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}} + \wedge^2 \widetilde{E_{\mathbf{C}}}), g, d) + \cdots, \tag{3.30}$$

so

$$\begin{aligned}
Q(\nabla^{TM}, g, d, \tau) = {} & [\widehat{A}(TM, \nabla^{TM}) \operatorname{ch}(\triangle(M)) \operatorname{ch}(\triangle(E), g^{\triangle(E)}, d) \\
& + 2^{2k+1} \widehat{A}(TM, \nabla^{TM}) \cdot \operatorname{ch}(\triangle(E), g^{\triangle(E)}, d)]^{(4k-1)} \\
& + q\{\widehat{A}(TM, \nabla^{TM})[\operatorname{ch}(\triangle(M) \otimes 2\widetilde{T_{\mathbf{C}}M}) \cdot \operatorname{ch}(\triangle(E), g^{\triangle(E)}, d) \\
& \quad + \operatorname{ch}(\triangle(M)) \operatorname{ch}(\triangle(E) \otimes (2\wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \\
& \quad \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)] + 2^{2k+1} \widehat{A}(TM, \nabla^{TM})[\operatorname{ch}(\widetilde{T_{\mathbf{C}}M} \\
& \quad + \wedge^2 \widetilde{T_{\mathbf{C}}M}) \operatorname{ch}(\triangle(E), g^{\triangle(E)}, d) \\
& \quad + \operatorname{ch}(\triangle(E) \otimes (2\wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)]\}^{(4k-1)} \\
& + q^2\{\widehat{A}(TM, \nabla^{TM}) \cdot [\operatorname{ch}(\triangle(M) \otimes (2\widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M} \\
& \quad + \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M} + S^2 \widetilde{T_{\mathbf{C}}M})) \operatorname{ch}(\triangle(E), g^{\triangle(E)}, d) \\
& \quad + \operatorname{ch}(\triangle(M) \otimes 2\widetilde{T_{\mathbf{C}}M}) \operatorname{ch}(\triangle(E) \otimes (2\wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \\
& \quad \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d) + \operatorname{ch}(\triangle(M)) \\
& \quad \cdot \operatorname{ch}(\triangle(E) \otimes (\wedge^2 \widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} + 2\wedge^4 \widetilde{E_{\mathbf{C}}} - 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^3 \widetilde{E_{\mathbf{C}}} \\
& \quad + 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}} \\
& \quad + \wedge^2 \widetilde{E_{\mathbf{C}}}), g, d)] \\
& + 2^{2k+1} \widehat{A}(TM, \nabla^{TM})[\operatorname{ch}(\wedge^4 \widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M} \\
& \quad + \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M} + S^2 \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M}) \operatorname{ch}(\triangle(E), g^{\triangle(E)}, d) \\
& \quad + \operatorname{ch}(\widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M}) \operatorname{ch}(\triangle(E) \otimes (2\wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \\
& \quad \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d) + \operatorname{ch}(\triangle(E) \otimes (\wedge^2 \widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} \\
& \quad + 2\wedge^4 \widetilde{E_{\mathbf{C}}} - 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^3 \widetilde{E_{\mathbf{C}}} + 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} \\
& \quad - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}} + \wedge^2 \widetilde{E_{\mathbf{C}}}), g, d)]\}^{(4k-1)} + \cdots.
\end{aligned} \tag{3.31}$$

When $\dim M = 7$, then $Q(\nabla^{TM}, g, d, \tau)$ must be a multiple of

$$E_4(\tau) = 1 + 240q + 2160q^2 + 6720q^3 + \cdots.$$

By (3.25) and (3.31), we compare the coefficients of $1, q, q^2$. We get Theorem 3.2. ∎

Corollary 3.3. *Let M be a 7-dimensional spin manifold without boundary. Suppose M is simply connected and $\mathbf{H}^3(M,\mathbb{R})=0$, then*

$$\begin{aligned}\operatorname{Ind}(T\otimes(\triangle(M)\otimes 2\widetilde{T_{\mathbf{C}}M}\otimes(\triangle(E),g^{\triangle(E)},d)+\triangle(M)\operatorname{ch}(\triangle(E)\\ \otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)))\equiv 0 \mod 16\mathbf{Z},\end{aligned}\tag{3.32}$$

$$\begin{aligned}&\operatorname{Ind}(T\otimes(\triangle(M)\otimes((2\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{T_{\mathbf{C}}M}+S^2\widetilde{T_{\mathbf{C}}M})\\ &\otimes(\triangle(E),g^{\triangle(E)},d)+2\widetilde{T_{\mathbf{C}}M}\otimes((2\wedge^2\widetilde{E_{\mathbf{C}}}\\ &-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)+(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}\\ &+2\wedge^4\widetilde{E_{\mathbf{C}}}-2\widetilde{E_{\mathbf{C}}}\otimes\wedge^3\widetilde{E_{\mathbf{C}}}+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}\\ &-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}\\ &+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d))))\equiv 0 \mod 16\mathbf{Z}.\end{aligned}\tag{3.33}$$

Remark. In the even-dimensional case, Rocklin's theorem says that on the 4-dimensional spin closed manifold, $\operatorname{Ind}(d+\delta)_+=\sigma(M)\equiv 0(\operatorname{mod} 16)$, where $d+\delta$ is the signature operator and $\sigma(M)$ is the signature. Ochanine's theorem says that on the $(8k+4)$-dimensional spin manifold, $\operatorname{Ind}(d+\delta)_+=\sigma(M)\equiv 0(\operatorname{mod} 16)$. If M is an $(8k+4)$-dimensional closed spin manifold, then the twisted signature $\operatorname{Sig}(M,T)$ is divisible by 256. If M is an $8k$-dimensional closed spin manifold, then $\operatorname{Sig}(M,T)$ is divisible by 2048. If M is an $(8k+4)$-dimensional closed spin manifold, when $\dim M=4$, the twisted signature $\operatorname{Sig}(M,T\otimes T)$ is divisible by $256\cdot 7$; when $\dim M=8k+4, k\geq 1$, the twisted signature $\operatorname{Sig}(M,T\otimes T)$ is divisible by 256 (resp. [4]), where T is the tangent bundle. So Index mod 16 is very meaningful. So in the case of odd-dimensional spin manifolds, our Corollaries 3.3 and 3.5 may be considered as the analogy of the above theorems.

Theorem 3.4. *When* $\dim M=11$ *and* $c_3(E,g,d)=0$*, we have*

$$\begin{aligned}&\{\widehat{A}(TM,\nabla^{TM})[\operatorname{ch}(\triangle(M)\otimes 2\widetilde{T_{\mathbf{C}}M})\operatorname{ch}(\triangle(E),g^{\triangle(E)},d)\\ &\quad+\operatorname{ch}(\triangle(M))\operatorname{ch}(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)]\\ &\quad+128\widehat{A}(TM,\nabla^{TM})[\operatorname{ch}(\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M})\operatorname{ch}(\triangle(E),g^{\triangle(E)},d)\\ &\quad+\operatorname{ch}(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)]\}^{(11)}\\ &=-504[\widehat{A}(TM,\nabla^{TM})\operatorname{ch}(\triangle(M))\operatorname{ch}(\triangle(E),g^{\triangle(E)},d)\\ &\quad+128\widehat{A}(TM,\nabla^{TM})\operatorname{ch}(\triangle(E),g^{\triangle(E)},d)],\end{aligned}\tag{3.34}$$

$$\begin{aligned}&\{\widehat{A}(TM,\nabla^{TM})[\operatorname{ch}(\triangle(M)\otimes(2\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{T_{\mathbf{C}}M}+S^2\widetilde{T_{\mathbf{C}}M}))\\ &\quad\cdot\operatorname{ch}(\triangle(E),g^{\triangle(E)},d)+\operatorname{ch}(\triangle(M)\otimes 2\widetilde{T_{\mathbf{C}}M})\operatorname{ch}(\triangle(E)\\ &\quad\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)\\ &\quad+\operatorname{ch}(\triangle(M))\operatorname{ch}(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}+2\wedge^4\widetilde{E_{\mathbf{C}}}-2\widetilde{E_{\mathbf{C}}}\otimes\wedge^3\widetilde{E_{\mathbf{C}}}\\ &\quad+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d)]\end{aligned}$$

$$
\begin{aligned}
&+128\cdot\widehat{A}(TM,\nabla^{TM})[\mathrm{ch}(\wedge^4\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}\\
&\qquad\otimes\widetilde{T_{\mathbf{C}}M}+S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\
&\quad+\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)\\
&\quad+\mathrm{ch}(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}+2\wedge^4\widetilde{E_{\mathbf{C}}}-2\widetilde{E_{\mathbf{C}}}\otimes\wedge^3\widetilde{E_{\mathbf{C}}}\\
&\qquad+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d)]\}^{(11)}\\
&=-16632[\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M))\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\
&\qquad+128\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)].
\end{aligned}\tag{3.35}
$$

Proof. When $\dim M=11$, then $Q(\nabla^{TM},g,d,\tau)$ is a modular form over SL(2, **Z**) with the weight 6, so

$$Q(\nabla^{TM},g,d,\tau)=\lambda E_6(\tau),\tag{3.36}$$

where λ is degree 11 forms. When $\dim M=11$, direct computations show that

$$
\begin{aligned}
Q(\nabla^{TM},g,d,\tau)&=[\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M))\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\
&\quad+128\widehat{A}(TM,\nabla^{TM})\cdot\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)]^{(11)}\\
&\quad+q\{\widehat{A}(TM,\nabla^{TM})[\mathrm{ch}(\triangle(M)\otimes 2\widetilde{T_{\mathbf{C}}M})\cdot\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\
&\qquad+\mathrm{ch}(\triangle(M))\,\mathrm{ch}(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)]\\
&\quad+128\widehat{A}(TM,\nabla^{TM})[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M})\cdot\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\
&\qquad+\mathrm{ch}(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)]\}^{(11)}\\
&\quad+q^2\{\widehat{A}(TM,\nabla^{TM})\cdot[\mathrm{ch}(\triangle(M)\otimes(2\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}\\
&\qquad\otimes\widetilde{T_{\mathbf{C}}M}+S^2\widetilde{T_{\mathbf{C}}M}))\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)+\mathrm{ch}(\triangle(M)\\
&\qquad\otimes 2\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)\\
&\qquad+\mathrm{ch}(\triangle(M))\,\mathrm{ch}(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}+2\wedge^4\widetilde{E_{\mathbf{C}}}\\
&\qquad-2\widetilde{E_{\mathbf{C}}}\otimes\wedge^3\widetilde{E_{\mathbf{C}}}+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\\
&\qquad+\widetilde{E_{\mathbf{C}}}+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d)]+128\widehat{A}(TM,\nabla^{TM})\\
&\qquad\cdot[\mathrm{ch}(\wedge^4\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}\\
&\qquad\quad\otimes\widetilde{T_{\mathbf{C}}M}+S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M})\cdot\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\
&\qquad\quad+\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M})\cdot\mathrm{ch}(\triangle(E)\\
&\qquad\qquad\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)\\
&\qquad\quad+\mathrm{ch}(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}+2\wedge^4\widetilde{E_{\mathbf{C}}}-2\widetilde{E_{\mathbf{C}}}\\
&\qquad\qquad\otimes\wedge^3\widetilde{E_{\mathbf{C}}}+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\\
&\qquad\qquad+\widetilde{E_{\mathbf{C}}}+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d)]\}^{(11)}+\cdots.
\end{aligned}\tag{3.37}
$$

In (3.36), we compare the coefficients of (3.36), and we get three equations about λ. By (3.26), (3.36) and (3.37), we get Theorem 3.4. ∎

Corollary 3.5. *Let M be an 11-dimensional spin manifold without boundary. Suppose M is simply connected and $\mathbf{H}^3(M,\mathbb{R})=0$, then*

$$\begin{aligned}&\mathrm{Ind}(T\otimes(\triangle(M)\otimes 2\widetilde{T_{\mathbf{C}}M}\otimes(\triangle(E),g^{\triangle(E)},d)\\&\quad+\triangle(M)\,\mathrm{ch}(\Delta(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)))\equiv 0 \mod 8\mathbf{Z},\end{aligned}\tag{3.38}$$

$$\begin{aligned}&\mathrm{Ind}(T\otimes(\triangle(M)\otimes((2\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{T_{\mathbf{C}}M}+S^2\widetilde{T_{\mathbf{C}}M})\\&\quad\otimes(\triangle(E),g^{\triangle(E)},d)+2\widetilde{T_{\mathbf{C}}M}\otimes((2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\\&\quad+\widetilde{E_{\mathbf{C}}}),g,d)+(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}+2\wedge^4\widetilde{E_{\mathbf{C}}}\\&\quad-2\widetilde{E_{\mathbf{C}}}\otimes\wedge^3\widetilde{E_{\mathbf{C}}}+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\\&\quad\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d))))\equiv 0 \mod 8\mathbf{Z}.\end{aligned}\tag{3.39}$$

Theorem 3.6. *When $\dim M=15$ and $c_3(E,g,d)=0$, we have*

$$\begin{aligned}&\{\widehat{A}(TM,\nabla^{TM})[\mathrm{ch}(\triangle(M)\otimes 2\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\&\quad+\mathrm{ch}(\triangle(M))\,\mathrm{ch}(\Delta(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)]\\&\quad+512\widehat{A}(TM,\nabla^{TM})[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\Delta(E),g^{\Delta(E)},d)\\&\quad+\mathrm{ch}(\Delta(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)]\}^{(15)}\\&=480[\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M))\,\mathrm{ch}(\triangle(E),g^{\Delta(E)},d)\\&\quad+512\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(E),g^{\Delta(E)},d)],\end{aligned}\tag{3.40}$$

$$\begin{aligned}&\{\widehat{A}(TM,\nabla^{TM})[\mathrm{ch}(\triangle(M)\otimes(2\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{T_{\mathbf{C}}M}+S^2\widetilde{T_{\mathbf{C}}M}))\\&\quad\cdot\mathrm{ch}(\triangle(E),g^{\Delta(E)},d)+\mathrm{ch}(\triangle(M)\otimes 2\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}\\&\quad-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)+\mathrm{ch}(\triangle(M))\,\mathrm{ch}(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}\\&\quad+2\wedge^4\widetilde{E_{\mathbf{C}}}-2\widetilde{E_{\mathbf{C}}}\otimes\wedge^3\widetilde{E_{\mathbf{C}}}+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\\&\quad+\widetilde{E_{\mathbf{C}}}+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d)]\\&\quad+512\widehat{A}(TM,\nabla^{TM})[\mathrm{ch}(\wedge^4\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{T_{\mathbf{C}}M}\\&\quad+S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M})\cdot\mathrm{ch}(\triangle(E),g^{\Delta(E)},d)\\&\quad+\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\Delta(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\\&\quad+\widetilde{E_{\mathbf{C}}}),g,d)+\mathrm{ch}(\Delta(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}\\&\quad+2\wedge^4\widetilde{E_{\mathbf{C}}}-2\widetilde{E_{\mathbf{C}}}\otimes\wedge^3\widetilde{E_{\mathbf{C}}}+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}\\&\quad-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d)]\}^{(15)}\\&=61920[\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M))\,\mathrm{ch}(\triangle(E),g^{\Delta(E)},d)\\&\quad+512\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(E),g^{\Delta(E)},d)].\end{aligned}\tag{3.41}$$

Proof. When $\dim M = 15$, then $Q(\nabla^{TM}, g, d, \tau)$ is a modular form over $\mathrm{SL}(2, \mathbf{Z})$ with the weight 8, so

$$Q(\nabla^{TM}, g, d, \tau) = \lambda E_4(\tau)^2, \tag{3.42}$$

where λ is degree 15 forms. By (3.25), we have

$$E_4(\tau)^2 = 1 + 480q + 61920q^2 + \cdots. \tag{3.43}$$

When $\dim M = 15$, direct computations show that

$$\begin{aligned}
Q(\nabla^{TM}, g, d, \tau) = {} & [\widehat{A}(TM, \nabla^{TM})\,\mathrm{ch}(\triangle(M))\,\mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) \\
& + 512\widehat{A}(TM, \nabla^{TM}) \cdot \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d)]^{(15)} \\
& + q\{\widehat{A}(TM, \nabla^{TM})[\mathrm{ch}(\triangle(M) \otimes 2\widetilde{T_{\mathbf{C}}M}) \cdot \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) \\
& + \mathrm{ch}(\triangle(M))\,\mathrm{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \\
& + \widetilde{E_{\mathbf{C}}}), g, d)] + 512\widehat{A}(TM, \nabla^{TM}) \\
& \cdot [\mathrm{ch}(\widetilde{T_{\mathbf{C}}M} + \wedge^2\widetilde{T_{\mathbf{C}}M}) \cdot \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) \\
& + \mathrm{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)]\}^{(15)} \\
& + q^2\{\widehat{A}(TM, \nabla^{TM})[\mathrm{ch}(\triangle(M) \otimes (2\widetilde{T_{\mathbf{C}}M} + \wedge^2\widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} \\
& \otimes \widetilde{T_{\mathbf{C}}M} + S^2\widetilde{T_{\mathbf{C}}M})) \\
& \cdot \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) + \mathrm{ch}(\triangle(M) \otimes 2\widetilde{T_{\mathbf{C}}M}) \\
& \cdot \mathrm{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \\
& + \widetilde{E_{\mathbf{C}}}), g, d) + \mathrm{ch}(\triangle(M)) \\
& \cdot \mathrm{ch}(\triangle(E) \otimes (\wedge^2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2\widetilde{E_{\mathbf{C}}} + 2 \wedge^4 \widetilde{E_{\mathbf{C}}} \\
& - 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^3\widetilde{E_{\mathbf{C}}} + 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2\widetilde{E_{\mathbf{C}}} \\
& - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}} \\
& + \wedge^2\widetilde{E_{\mathbf{C}}}), g, d)] \\
& + 512\widehat{A}(TM, \nabla^{TM})[\mathrm{ch}(\wedge^4\widetilde{T_{\mathbf{C}}M} + \wedge^2\widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} \\
& \otimes \widetilde{T_{\mathbf{C}}M} + S^2\widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M}) \\
& \cdot \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) + \mathrm{ch}(\widetilde{T_{\mathbf{C}}M} \\
& + \wedge^2\widetilde{T_{\mathbf{C}}M}) \cdot \mathrm{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} \\
& - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d) \\
& + \mathrm{ch}(\triangle(E) \otimes (\wedge^2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2\widetilde{E_{\mathbf{C}}} \\
& + 2 \wedge^4 \widetilde{E_{\mathbf{C}}} - 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^3\widetilde{E_{\mathbf{C}}} + 2\widetilde{E_{\mathbf{C}}} \\
& \otimes \wedge^2\widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}} \\
& + \wedge^2\widetilde{E_{\mathbf{C}}}), g, d)]\}^{(15)} + \cdots.
\end{aligned} \tag{3.44}$$

By (3.42)–(3.44), we compare the coefficients of $1, q, q^2$. We get Theorem 3.6. ∎

Corollary 3.7. *Suppose M is a 15-dimensional spin manifold without boundary. If $c_3(E,g,d)=0$, then*

$$\begin{aligned}\operatorname{Ind}(T\otimes(\triangle(M)\otimes 2\widetilde{T_{\mathbf{C}}M}\otimes(\triangle(E),g^{\triangle(E)},d)\\ +\triangle(M)\operatorname{ch}(\Delta(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)))\equiv 0 \mod 32\mathbf{Z},\end{aligned} \tag{3.45}$$

$$\begin{aligned}\operatorname{Ind}(T\otimes(\triangle(M)\otimes((2\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{T_{\mathbf{C}}M}+S^2\widetilde{T_{\mathbf{C}}M})\\ \otimes(\triangle(E),g^{\triangle(E)},d)+2\widetilde{T_{\mathbf{C}}M}\otimes((2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\\ \otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)+(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}\\ +2\wedge^4\widetilde{E_{\mathbf{C}}}-2\widetilde{E_{\mathbf{C}}}\otimes\wedge^3\widetilde{E_{\mathbf{C}}}+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\\ \otimes\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d))))\equiv 0 \mod 32\mathbf{Z}.\end{aligned} \tag{3.46}$$

Theorem 3.8. *When* $\dim M=19$ *and* $c_3(E,g,d)=0$, *we have*

$$\begin{aligned}\{\widehat{A}(TM,\nabla^{TM})[\operatorname{ch}(\triangle(M)\otimes 2\widetilde{T_{\mathbf{C}}M})\operatorname{ch}(\triangle(E),g^{\triangle(E)},d)\\ +\operatorname{ch}(\triangle(M))\operatorname{ch}(\Delta(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)]\\ +2048\widehat{A}(TM,\nabla^{TM})[\operatorname{ch}(\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M})\operatorname{ch}(\triangle(E),g^{\triangle(E)},d)\\ +\operatorname{ch}(\Delta(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)]\}^{(19)}\\ =-264[\widehat{A}(TM,\nabla^{TM})\operatorname{ch}(\triangle(M))\operatorname{ch}(\triangle(E),g^{\triangle(E)},d)\\ +2048\widehat{A}(TM,\nabla^{TM})\operatorname{ch}(\triangle(E),g^{\triangle(E)},d)],\end{aligned} \tag{3.47}$$

$$\begin{aligned}\{\widehat{A}(TM,\nabla^{TM})[\operatorname{ch}(\triangle(M)\otimes(2\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{T_{\mathbf{C}}M}+S^2\widetilde{T_{\mathbf{C}}M}))\\ \cdot\operatorname{ch}(\triangle(E),g^{\triangle(E)},d)+\operatorname{ch}(\triangle(M)\otimes 2\widetilde{T_{\mathbf{C}}M})\operatorname{ch}(\Delta(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}\\ -\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)+\operatorname{ch}(\triangle(M))\operatorname{ch}(\Delta(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}\\ +2\wedge^4\widetilde{E_{\mathbf{C}}}-2\widetilde{E_{\mathbf{C}}}\otimes\wedge^3\widetilde{E_{\mathbf{C}}}+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\\ +\widetilde{E_{\mathbf{C}}}+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d)]\\ +2048\widehat{A}(TM,\nabla^{TM})[\operatorname{ch}(\wedge^4\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{T_{\mathbf{C}}M}\\ +S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M})\operatorname{ch}(\triangle(E),g^{\triangle(E)},d)\\ +\operatorname{ch}(\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M})\cdot\operatorname{ch}(\Delta(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\\ \otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)+\operatorname{ch}(\Delta(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}\\ +2\wedge^4\widetilde{E_{\mathbf{C}}}-2\widetilde{E_{\mathbf{C}}}\otimes\wedge^3\widetilde{E_{\mathbf{C}}}+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}\\ -\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d)]\}^{(19)}\\ =-117288[\widehat{A}(TM,\nabla^{TM})\operatorname{ch}(\triangle(M))\operatorname{ch}(\triangle(E),g^{\triangle(E)},d)\\ +2048\widehat{A}(TM,\nabla^{TM})\operatorname{ch}(\triangle(E),g^{\triangle(E)},d)].\end{aligned} \tag{3.48}$$

Proof. When $\dim M = 19$, then $Q(\nabla^{TM}, g, d, \tau)$ is a modular form over SL(2, **Z**) with the weight 10, so

$$Q(\nabla^{TM}, g, d, \tau) = \lambda E_4(\tau)E_6(\tau), \tag{3.49}$$

where λ is degree 19 forms. By (3.25) and (3.26), we have

$$E_4(\tau)E_6(\tau) = 1 - 264q - 117288q^2 + \cdots . \tag{3.50}$$

When $\dim M = 19$, direct computations show that

$$\begin{aligned}
Q(\nabla^{TM}, g, d, \tau) = {} & [\widehat{A}(TM, \nabla^{TM})\,\mathrm{ch}(\triangle(M))\,\mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) \\
& + 2048\widehat{A}(TM, \nabla^{TM}) \cdot \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d)]^{(19)} \\
& + q\{\widehat{A}(TM, \nabla^{TM})[\mathrm{ch}(\triangle(M) \otimes 2\widetilde{T_{\mathbf{C}}M}) \cdot \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) \\
& \qquad + \mathrm{ch}(\triangle(M))\,\mathrm{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \\
& \qquad + \widetilde{E_{\mathbf{C}}}), g, d)] + 2048\widehat{A}(TM, \nabla^{TM}) \\
& \cdot [\mathrm{ch}(\widetilde{T_{\mathbf{C}}M} + \wedge^2\widetilde{T_{\mathbf{C}}M}) \cdot \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) \\
& + \mathrm{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)]\}^{(19)} \\
& + q^2\{\widehat{A}(TM, \nabla^{TM}) \cdot [\mathrm{ch}(\triangle(M) \otimes (2\widetilde{T_{\mathbf{C}}M} + \wedge^2\widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} \\
& \qquad \otimes \widetilde{T_{\mathbf{C}}M} + S^2\widetilde{T_{\mathbf{C}}M})) \\
& \qquad \cdot \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) + \mathrm{ch}(\triangle(M) \otimes 2\widetilde{T_{\mathbf{C}}M}) \\
& \qquad \cdot \mathrm{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \\
& \qquad + \widetilde{E_{\mathbf{C}}}), g, d) + \mathrm{ch}(\triangle(M)) \\
& \qquad \cdot \mathrm{ch}(\triangle(E) \otimes (\wedge^2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2\widetilde{E_{\mathbf{C}}} + 2 \wedge^4 \widetilde{E_{\mathbf{C}}} \\
& \qquad - 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^3\widetilde{E_{\mathbf{C}}} + 2\widetilde{E_{\mathbf{C}}} \\
& \qquad \otimes \wedge^2\widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \\
& \qquad + \widetilde{E_{\mathbf{C}}} + \wedge^2\widetilde{E_{\mathbf{C}}}), g, d)] \\
& + 2048\widehat{A}(TM, \nabla^{TM})[\mathrm{ch}(\wedge^4\widetilde{T_{\mathbf{C}}M} + \wedge^2\widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} \\
& \qquad \otimes \widetilde{T_{\mathbf{C}}M} + S^2\widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M}) \\
& \qquad \cdot \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) + \mathrm{ch}(\widetilde{T_{\mathbf{C}}M} \\
& \qquad + \wedge^2\widetilde{T_{\mathbf{C}}M}) \cdot \mathrm{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} \\
& \qquad - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d) \\
& \qquad + \mathrm{ch}(\triangle(E) \otimes (\wedge^2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2\widetilde{E_{\mathbf{C}}} \\
& \qquad + 2 \wedge^4 \widetilde{E_{\mathbf{C}}} - 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^3\widetilde{E_{\mathbf{C}}} \\
& \qquad + 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2\widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \\
& \qquad \otimes \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}} \\
& \qquad + \wedge^2\widetilde{E_{\mathbf{C}}}), g, d)]\}^{(19)} + \cdots .
\end{aligned} \tag{3.51}$$

By (3.49)–(3.51), we compare the coefficients of $1, q, q^2$. We get Theorem 3.8. ∎

Corollary 3.9. *Suppose M is a 19-dimensional spin manifold without boundary. If $c_3(E,g,d)=0$, then*

$$\begin{aligned}&\mathrm{Ind}(T\otimes(\triangle(M)\otimes 2\widetilde{T_{\mathbf{C}}M}\otimes(\triangle(E),g^{\triangle(E)},d)\\&\qquad+\triangle(M)\,\mathrm{ch}(\Delta(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)))\equiv 0 \mod 8\mathbf{Z},\end{aligned}\tag{3.52}$$

$$\begin{aligned}&\mathrm{Ind}(T\otimes(\triangle(M)\otimes((2\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{T_{\mathbf{C}}M}+S^2\widetilde{T_{\mathbf{C}}M})\\&\qquad\otimes(\triangle(E),g^{\triangle(E)},d)+2\widetilde{T_{\mathbf{C}}M}\otimes((2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\\&\qquad+\widetilde{E_{\mathbf{C}}}),g,d)+(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}+2\wedge^4\widetilde{E_{\mathbf{C}}}\\&\qquad-2\widetilde{E_{\mathbf{C}}}\otimes\wedge^3\widetilde{E_{\mathbf{C}}}+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\\&\qquad+\widetilde{E_{\mathbf{C}}}+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d)))_+)\equiv 0 \mod 8\mathbf{Z}.\end{aligned}\tag{3.53}$$

Theorem 3.10. *When* $\dim M=23$ *and* $c_3(E,g,d)=0$, *we have*

$$\begin{aligned}&\{\widehat{A}(TM,\nabla^{TM})[\mathrm{ch}(\triangle(M)\otimes(2\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{T_{\mathbf{C}}M}+S^2\widetilde{T_{\mathbf{C}}M}))\\&\qquad\cdot\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)+\mathrm{ch}(\triangle(M)\otimes 2\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}\\&\qquad-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)+\mathrm{ch}(\triangle(M))\,\mathrm{ch}(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}\\&\qquad+2\wedge^4\widetilde{E_{\mathbf{C}}}-2\widetilde{E_{\mathbf{C}}}\otimes\wedge^3\widetilde{E_{\mathbf{C}}}+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\\&\qquad+\widetilde{E_{\mathbf{C}}}+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d)]\\&\qquad+8192\widehat{A}(TM,\nabla^{TM})[\mathrm{ch}(\wedge^4\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{T_{\mathbf{C}}M}\\&\qquad+S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\&\qquad+\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\\&\qquad\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)+\mathrm{ch}(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}\\&\qquad+2\wedge^4\widetilde{E_{\mathbf{C}}}-2\widetilde{E_{\mathbf{C}}}\otimes\wedge^3\widetilde{E_{\mathbf{C}}}+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}\\&\qquad-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d)]\}^{(23)}\\&=\{196560[\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M))\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\&\qquad+8192\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)]\\&\qquad-24\{\widehat{A}(TM,\nabla^{TM})\cdot[\mathrm{ch}(\triangle(M)\otimes 2\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\&\qquad+\mathrm{ch}(\triangle(M))\,\mathrm{ch}(\Delta(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\\&\qquad+\widetilde{E_{\mathbf{C}}}),g,d)]\\&\qquad+8192\widehat{A}(TM,\nabla^{TM})\cdot[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\&\qquad+\mathrm{ch}(\Delta(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\\&\qquad+\widetilde{E_{\mathbf{C}}}),g,d)]\}\}^{(23)}.\end{aligned}\tag{3.54}$$

Proof. When $\dim M = 23$, then $Q(\nabla^{TM}, g, d, \tau)$ is a modular form over SL(2, **Z**) with the weight 12, so

$$Q(\nabla^{TM}, g, d, \tau) = \lambda_1 E_4(\tau)^3 + \lambda_2 E_6(\tau)^2, \tag{3.55}$$

where λ_1, λ_2 are degree 23 forms. By (3.25) and (3.26), we have

$$E_4(\tau)^3 = 1 + 720q + 179280q^2 + \cdots, \tag{3.56}$$

$$E_6(\tau)^2 = 1 - 1008q + 220752q^2 + \cdots. \tag{3.57}$$

When $\dim M = 23$, direct computations show that

$$\begin{aligned}
Q(\nabla^{TM}, g, d, \tau) = {} & [\widehat{A}(TM, \nabla^{TM}) \operatorname{ch}(\triangle(M)) \operatorname{ch}(\triangle(E), g^{\triangle(E)}, d) \\
& + 8192\widehat{A}(TM, \nabla^{TM}) \cdot \operatorname{ch}(\triangle(E), g^{\triangle(E)}, d)]^{(23)} \\
& + q\{\widehat{A}(TM, \nabla^{TM})[\operatorname{ch}(\triangle(M) \otimes 2\widetilde{T_{\mathbf{C}}M}) \cdot \operatorname{ch}(\triangle(E), g^{\triangle(E)}, d) \\
& \quad + \operatorname{ch}(\triangle(M)) \operatorname{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \\
& \quad + \widetilde{E_{\mathbf{C}}}), g, d)] + 8192\widehat{A}(TM, \nabla^{TM}) \\
& \quad \cdot [\operatorname{ch}(\widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M}) \operatorname{ch}(\triangle(E), g^{\triangle(E)}, d) \\
& \quad + \operatorname{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)]\}^{(23)} \\
& + q^2\{\widehat{A}(TM, \nabla^{TM}) \cdot [\operatorname{ch}(\triangle(M) \otimes (2\widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} \\
& \quad \otimes \widetilde{T_{\mathbf{C}}M} + S^2 \widetilde{T_{\mathbf{C}}M})) \\
& \quad \cdot \operatorname{ch}(\triangle(E), g^{\triangle(E)}, d) + \operatorname{ch}(\triangle(M) \otimes 2\widetilde{T_{\mathbf{C}}M}) \\
& \quad \cdot \operatorname{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \\
& \quad + \widetilde{E_{\mathbf{C}}}), g, d) + \operatorname{ch}(\triangle(M)) \\
& \quad \cdot \operatorname{ch}(\triangle(E) \otimes (\wedge^2 \widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} + 2 \wedge^4 \widetilde{E_{\mathbf{C}}} \\
& \quad - 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^3 \widetilde{E_{\mathbf{C}}} + 2\widetilde{E_{\mathbf{C}}} \\
& \quad \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \\
& \quad + \widetilde{E_{\mathbf{C}}} + \wedge^2 \widetilde{E_{\mathbf{C}}}), g, d)] \\
& \quad + 8192\widehat{A}(TM, \nabla^{TM})[\operatorname{ch}(\wedge^4 \widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} \\
& \quad \otimes \widetilde{T_{\mathbf{C}}M} + S^2 \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M}) \\
& \quad \cdot \operatorname{ch}(\triangle(E), g^{\triangle(E)}, d) + \operatorname{ch}(\widetilde{T_{\mathbf{C}}M} \\
& \quad + \wedge^2 \widetilde{T_{\mathbf{C}}M}) \cdot \operatorname{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} \\
& \quad - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d) \\
& \quad + \operatorname{ch}(\triangle(E) \otimes (\wedge^2 \widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} \\
& \quad + 2 \wedge^4 \widetilde{E_{\mathbf{C}}} - 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^3 \widetilde{E_{\mathbf{C}}} \\
& \quad + 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}}
\end{aligned}$$

$$\otimes \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}} + \wedge^2 \widetilde{E_{\mathbf{C}}}), g, d)]\}^{(23)} + \cdots . \tag{3.58}$$

In (3.58), we compare the coefficients of 1, q, q^2, and we get three equations about λ_1, λ_2. By (3.55)–(3.58), we get Theorem 3.10. ∎

Corollary 3.11. *Suppose M is a 23-dimensional spin manifold without boundary. If $c_3(E, g, d) = 0$, then*

$$\begin{aligned}\operatorname{Ind}(T \otimes (\triangle(M) \otimes ((2\widetilde{T_{\mathbf{C}}M} + \wedge^2\widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M} + S^2\widetilde{T_{\mathbf{C}}M})& \\ \otimes (\triangle(E), g^{\triangle(E)}, d) + 2\widetilde{T_{\mathbf{C}}M} \otimes ((2\wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}}& \\ \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d) + (\triangle(E) \otimes (\wedge^2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2\widetilde{E_{\mathbf{C}}} + 2\wedge^4 \widetilde{E_{\mathbf{C}}}& \\ - 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^3\widetilde{E_{\mathbf{C}}} + 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2\widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}}& \\ \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}} + \wedge^2\widetilde{E_{\mathbf{C}}}), g, d)))) \equiv 0 \mod 16\mathbf{Z}.&\end{aligned} \tag{3.59}$$

3.2. Some modular forms and Witten genus in spin^c manifolds

Let M be closed oriented spin^c manifold and L be the complex line bundle associated to the given spin^c structure on M. Denote by $c = c_1(L)$ the first Chern class of L. Also, we use $L_{\mathbf{R}}$ for the notation of L, when it is viewed as an oriented real plane bundle. Let $\Theta(T_{\mathbf{C}}M, L_{\mathbf{R}} \otimes \mathbf{C})$ be the virtual complex vector bundle over M defined by

$$\begin{aligned}\Theta(T_{\mathbf{C}}M, L_{\mathbf{R}} \otimes \mathbf{C}) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbf{C}}M}) \otimes \bigotimes_{m=1}^{\infty} \wedge_{q^m}(\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}})& \\ \otimes \bigotimes_{r=1}^{\infty} \wedge_{-q^{r-\frac{1}{2}}}(\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \otimes \bigotimes_{s=1}^{\infty} \wedge_{q^{s-\frac{1}{2}}}(\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}).&\end{aligned}$$

Let $\dim M = 4k - 1$ and $y = -\frac{\sqrt{-1}}{2\pi}c$. Set

$$\begin{aligned}\widetilde{Q}(\nabla^{TM}, \nabla^L, g, d, \tau) = \Big\{\widehat{A}(TM, \nabla^{TM}) \exp\Big(\frac{c}{2}\Big) \operatorname{ch}(\Theta(T_{\mathbf{C}}M, L_{\mathbf{R}} \otimes \mathbf{C}))& \\ \cdot \operatorname{ch}(Q(E), g^{Q(E)}, d, \tau)\Big\}^{(4k-1)}.&\end{aligned} \tag{3.60}$$

When $c = 0$, $\Theta(T_{\mathbf{C}}M) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbf{C}}M})$ be the Witten bundle over M. And

$$\widetilde{Q}(\nabla^{TM}, g, d, \tau) = \{\widehat{A}(TM, \nabla^{TM}) \operatorname{ch}(\Theta(T_{\mathbf{C}}M)) \operatorname{ch}(Q(E), g^{Q(E)}, d, \tau)\}^{(4k-1)}$$

is called the Witten form in odd dimensions.

Then

$$\widetilde{Q}(M, L, \tau) = \left(\prod_{j=1}^{2k-1} \frac{x_j \theta'(0, \tau)}{\theta(x_j, \tau)} \left(\frac{\theta_1(y, \tau)}{\theta_1(0, \tau)} \frac{\theta_2(y, \tau)}{\theta_2(0, \tau)} \frac{\theta_3(y, \tau)}{\theta_3(0, \tau)}\right)\right)^{(4p)}, \tag{3.61}$$

and

$$\widetilde{Q}(\nabla^{TM}, \nabla^{L}, g, d, \tau) = (\widetilde{Q}(M, L, \tau) \cdot \mathrm{ch}(Q(E), g^{Q(E)}, d, \tau))^{(4k-1)}. \tag{3.62}$$

By equations (2.13)–(2.17), we have $\widetilde{Q}(\nabla^{TM}, \nabla^{L}, g, d, \tau + 1) = \widetilde{Q}(\nabla^{TM}, \nabla^{L}, g, d, \tau)$ and $\widetilde{Q}(\nabla^{TM}, \nabla^{L}, g, d, -\frac{1}{\tau}) = \tau^{2k}\widetilde{Q}(\nabla^{TM}, \nabla^{L}, g, d, \tau)$ if $3p_1(L) - p_1(M) = 0$ and $c_3(E, g, d) = 0$. So we get the following theorem.

Theorem 3.12. *Let* $\dim M = 4k - 1$. *If* $3p_1(L) - p_1(M) = 0$ *and* $c_3(E, g, d) = 0$, *then* $\widetilde{Q}(\nabla^{TM}, \nabla^{L}, g, d, \tau)$ *is a modular form over* SL(2, **Z**) *with the weight* $2k$.

Remark. Suppose M is a $(4k-1)$-dimensional spin manifold. If $p_1(M) = 0$ and $c_3(E, g, d) = 0$, then $\widetilde{Q}(\nabla^{TM}, g, d, \tau)$ is a modular form over SL(2, **Z**) with the weight $2k$.

Direct computation show

$$\begin{aligned}
\widetilde{Q}(\nabla^{TM}, \nabla^{L}, g, d, \tau) &= \left[\widehat{A}(TM, \nabla^{TM}) \exp\left(\frac{c}{2}\right) \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d)\right]^{4k-1} \\
&\quad + q\Big\{\widehat{A}(TM, \nabla^{TM}) \exp\left(\frac{c}{2}\right) \cdot \big[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M})\, \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) \\
&\qquad + \mathrm{ch}(2 \wedge^2 \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \otimes (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \\
&\qquad + \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}})\, \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) + \mathrm{ch}(\triangle(E) \\
&\qquad \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)\big]\Big\}^{4k-1} \\
&\quad + q^2\Big\{\widehat{A}(TM, \nabla^{TM}) \exp\left(\frac{c}{2}\right) \\
&\qquad \cdot \big[\mathrm{ch}(S^2\widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M}) \cdot \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) \\
&\qquad + \mathrm{ch}(\wedge^2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \otimes \wedge^2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} + 2 \wedge^4 \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \\
&\qquad - 2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \otimes \wedge^3\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} + 2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \\
&\qquad \otimes \wedge^2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \otimes \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \otimes \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \\
&\qquad + \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} + \wedge^2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}})\, \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) \\
&\qquad + \mathrm{ch}(\widetilde{T_{\mathbf{C}}M} \otimes (2 \wedge^2 \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \\
&\qquad \otimes (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) + \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \\
&\qquad \cdot \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d)) + \mathrm{ch}(2 \wedge^2 \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \\
&\qquad - (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \otimes (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) + \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \\
&\qquad \cdot \mathrm{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d) \\
&\qquad + \mathrm{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d) \\
&\qquad + \mathrm{ch}(\triangle(E) \otimes (\wedge^2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2\widetilde{E_{\mathbf{C}}} + 2 \wedge^4 \widetilde{E_{\mathbf{C}}} - 2\widetilde{E_{\mathbf{C}}}
\end{aligned}$$

$$\begin{aligned}&\otimes\wedge^3\widetilde{E_{\mathbf{C}}}+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\\&\otimes\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}\\&+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d)\Big]\Big\}^{4k-1}+\cdots.\end{aligned}\tag{3.63}$$

When $\dim M=7$, then $\widetilde{Q}(\nabla^{TM},\nabla^L,g,d,\tau)$ is a modular form over $\mathrm{SL}(2,\mathbf{Z})$ with the weight 4 and $\widetilde{Q}(\nabla^{TM},\nabla^L,g,d,\tau)=\lambda E_4(\tau)$. When $\dim M=11$, then we obtain $\widetilde{Q}(\nabla^{TM},\nabla^L,g,d,\tau)$ is a modular form over $\mathrm{SL}(2,\mathbf{Z})$ with the weight 6 and $\widetilde{Q}(\nabla^{TM},\nabla^L,g,d,\tau)=\lambda E_6(\tau)$. When $\dim M=15$, then $\widetilde{Q}(\nabla^{TM},\nabla^L,g,d,\tau)$ is a modular form over $\mathrm{SL}(2,\mathbf{Z})$ with the weight 8 and $\widetilde{Q}(\nabla^{TM},\nabla^L,g,d,\tau)=\lambda E_4(\tau)^2$. When $\dim M=19$, then $\widetilde{Q}(\nabla^{TM},\nabla^L,g,d,\tau)$ is a modular form over $\mathrm{SL}(2,\mathbf{Z})$ with the weight 10 and $\widetilde{Q}(\nabla^{TM},\nabla^L,g,d,\tau)=\lambda E_4(\tau)E_6(\tau)$. When $\dim M=23$, then $\widetilde{Q}(\nabla^{TM},\nabla^L,g,d,\tau)$ is a modular form over $\mathrm{SL}(2,\mathbf{Z})$ with the weight 12 and $\widetilde{Q}(\nabla^{TM},\nabla^L,g,d,\tau)=\lambda_1E_4(\tau)^3+\lambda_2E_6(\tau)^2$. So, we get the following theorem.

Theorem 3.13. *Let* $\dim M=7$. *If* $3p_1(L)-p_1(M)=0$ *and* $c_3(E,g,d)=0$, *we have*

$$\begin{aligned}&\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\Big[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}+2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\&\qquad\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\&\qquad+\mathrm{ch}(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)\Big]\Big\}^{(7)}\\&=240\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\Big\}^{(7)},\end{aligned}\tag{3.64}$$

$$\begin{aligned}&\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\Big[\mathrm{ch}(S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\&\qquad+2\wedge^4\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\wedge^3\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\&\qquad\otimes\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\&\qquad+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\widetilde{T_{\mathbf{C}}M}\otimes(2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\&\qquad\quad\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}))\cdot\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\&\qquad+\mathrm{ch}(2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\&\qquad\cdot\mathrm{ch}(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)\\&\qquad+\mathrm{ch}(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)\\&\qquad+\mathrm{ch}(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}+2\wedge^4\widetilde{E_{\mathbf{C}}}-2\widetilde{E_{\mathbf{C}}}\otimes\wedge^3\widetilde{E_{\mathbf{C}}}\end{aligned}$$

$$+ 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}} + \wedge^2 \widetilde{E_{\mathbf{C}}}), g, d)\Big]\Big\}^{(7)}$$

$$= 2160\Big\{\widehat{A}(TM, \nabla^{TM}) \exp\Big(\frac{c}{2}\Big) \operatorname{ch}(\triangle(E), g^{\triangle(E)}, d)\Big\}^{(7)}. \tag{3.65}$$

Corollary 3.14. *Suppose M is a 7-dimensional* spin^c *manifold without boundary. If $c_3(E, g, d) = 0$, then*

$$\operatorname{Ind}(T^c \otimes ((\widetilde{T_{\mathbf{C}}M} + 2 \wedge^2 \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \otimes (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) + \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \otimes (\triangle(E), g^{\triangle(E)}, d) + (\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d))) \equiv 0 \mod 240\mathbf{Z}, \tag{3.66}$$

$$\begin{aligned}\operatorname{Ind}(T \otimes ((&S^2\widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} + \wedge^2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \otimes \wedge^2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} + 2 \wedge^4 \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \\ &- 2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \otimes \wedge^3\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} + 2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \otimes \wedge^2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \\ &\otimes \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \otimes \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} + \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} + \wedge^2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} + \widetilde{T_{\mathbf{C}}M} \\ &\otimes (2 \wedge^2 \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \otimes (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) + \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}})) \\ &\otimes (\triangle(E), g^{\triangle(E)}, d) + (2 \wedge^2 \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \otimes (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) + \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \\ &\otimes (\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d) + (\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} \\ &- \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d) + (\triangle(E) \otimes (\wedge^2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2\widetilde{E_{\mathbf{C}}} + 2 \wedge^4 \widetilde{E_{\mathbf{C}}} \\ &- 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^3\widetilde{E_{\mathbf{C}}} + 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2\widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \\ &+ \widetilde{E_{\mathbf{C}}} + \wedge^2\widetilde{E_{\mathbf{C}}}), g, d))) \equiv 0 \mod 2160\mathbf{Z}.\end{aligned} \tag{3.67}$$

Theorem 3.15. *Let* $\dim M = 11$. *If* $3p_1(L) - p_1(M) = 0$ *and* $c_3(E, g, d) = 0$, *we have*

$$\begin{aligned}\Big\{\widehat{A}(TM, \nabla^{TM}) \exp\Big(\frac{c}{2}\Big)\Big[&\operatorname{ch}(\widetilde{T_{\mathbf{C}}M} + 2 \wedge^2 \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \otimes (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) + \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \\ &\cdot \operatorname{ch}(\triangle(E), g^{\triangle(E)}, d) + \operatorname{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \\ &\otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)\Big]\Big\}^{(11)} \\ = -504\Big\{\widehat{A}(TM, \nabla^{TM}) &\exp\Big(\frac{c}{2}\Big) \operatorname{ch}(\triangle(E), g^{\triangle(E)}, d)\Big\}^{(11)},\end{aligned} \tag{3.68}$$

$$\begin{aligned}\Big\{\widehat{A}(TM, \nabla^{TM}) \exp\Big(\frac{c}{2}\Big)\Big[&\operatorname{ch}(S^2\widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} + \wedge^2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \otimes \wedge^2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \\ &+ 2 \wedge^4 \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - 2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \otimes \wedge^3\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} + 2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \\ &\otimes \wedge^2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \otimes \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \otimes \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} + \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \\ &+ \wedge^2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} + \widetilde{T_{\mathbf{C}}M} \otimes (2 \wedge^2 \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}})\end{aligned}$$

$$
\begin{aligned}
&\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}))\cdot\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\
&+\mathrm{ch}(2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\
&\cdot\mathrm{ch}(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)\\
&+\mathrm{ch}(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)\\
&+\mathrm{ch}(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}+2\wedge^4\widetilde{E_{\mathbf{C}}}-2\widetilde{E_{\mathbf{C}}}\otimes\wedge^3\widetilde{E_{\mathbf{C}}}\\
&\qquad+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}\\
&\qquad+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d)\Big]\Big\}^{(11)}\\
=\ &-16632\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\right\}^{(11)}. \qquad (3.69)
\end{aligned}
$$

Corollary 3.16. *Let M be an 11-dimensional* spin^c *manifold without boundary. If $c_3(E,g,d)=0$, then*

$$
\begin{aligned}
&\mathrm{Ind}(T^c\otimes((\widetilde{T_{\mathbf{C}}M}+2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\
&\quad\otimes(\triangle(E),g^{\triangle(E)},d)+(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\\
&\qquad\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)))\equiv 0 \mod 504\mathbf{Z}, \qquad (3.70)\\
&\mathrm{Ind}(T\otimes((S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+2\wedge^4\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\
&\quad-2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\wedge^3\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\
&\quad\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\widetilde{T_{\mathbf{C}}M}\\
&\quad\otimes(2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}))\\
&\quad\otimes(\triangle(E),g^{\triangle(E)},d)+(2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\
&\quad\otimes(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)+(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\\
&\qquad\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)+(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}+2\wedge^4\widetilde{E_{\mathbf{C}}}-2\widetilde{E_{\mathbf{C}}}\\
&\qquad\otimes\wedge^3\widetilde{E_{\mathbf{C}}}+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}\\
&\qquad+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d)))\equiv 0 \mod 16632\mathbf{Z}. \qquad (3.71)
\end{aligned}
$$

Theorem 3.17. *Let* $\dim M=15$. *If* $3p_1(L)-p_1(M)=0$ *and* $c_3(E,g,d)=0$, *we have*

$$
\begin{aligned}
&\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\Big[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}+2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\right.\\
&\qquad\cdot\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)+\mathrm{ch}(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\\
&\qquad\left.\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)\Big]\right\}^{(15)}\\
&=480\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\right\}^{(15)}, \qquad (3.72)
\end{aligned}
$$

$$\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\big[\mathrm{ch}(S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}$$
$$+2\wedge^4\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\wedge^3\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}$$
$$\otimes\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}$$
$$+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\widetilde{T_{\mathbf{C}}M}\otimes(2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})$$
$$\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}))\cdot\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)$$
$$+\mathrm{ch}(2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})$$
$$\cdot\mathrm{ch}(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)$$
$$+\mathrm{ch}(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)$$
$$+\mathrm{ch}(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}+2\wedge^4\widetilde{E_{\mathbf{C}}}-2\widetilde{E_{\mathbf{C}}}\otimes\wedge^3\widetilde{E_{\mathbf{C}}}$$
$$+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}$$
$$+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d)\big]\Big\}^{(15)}$$
$$=61920\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\Big\}^{(15)}. \tag{3.73}$$

Corollary 3.18. *Let M be a 15-dimensional* spin^c *manifold without boundary. If $c_3(E,g,d)=0$, then*

$$\mathrm{Ind}(T^c\otimes((\widetilde{T_{\mathbf{C}}M}+2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})$$
$$\otimes(\triangle(E),g^{\triangle(E)},d)+(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}$$
$$\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)))\equiv 0 \mod 480\mathbf{Z}, \tag{3.74}$$
$$\mathrm{Ind}(T\otimes((S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+2\wedge^4\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}$$
$$-2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\wedge^3\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}$$
$$\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\widetilde{T_{\mathbf{C}}M}$$
$$\otimes(2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}))$$
$$\otimes(\triangle(E),g^{\triangle(E)},d)+(2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})$$
$$\otimes(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)+(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}$$
$$-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)+(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}+2\wedge^4\widetilde{E_{\mathbf{C}}}$$
$$-2\widetilde{E_{\mathbf{C}}}\otimes\wedge^3\widetilde{E_{\mathbf{C}}}+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}$$
$$+\widetilde{E_{\mathbf{C}}}+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d)))\equiv 0 \mod 61920\mathbf{Z}. \tag{3.75}$$

Theorem 3.19. *Let* $\dim M=19$. *If* $3p_1(L)-p_1(M)=0$ *and* $c_3(E,g,d)=0$*, we have*

$$\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\big[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}+2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})$$

$$
\begin{aligned}
&\cdot \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) + \mathrm{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \\
&\quad \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)\Big]\Big\}^{(19)} \\
= {}& -264\left\{\widehat{A}(TM, \nabla^{TM}) \exp\left(\frac{c}{2}\right) \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d)\right\}^{(19)},
\end{aligned}
\tag{3.76}
$$

$$
\begin{aligned}
&\left\{\widehat{A}(TM, \nabla^{TM}) \exp\left(\frac{c}{2}\right)\Big[\mathrm{ch}(S^2\widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} + \wedge^2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \otimes \wedge^2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}\right. \\
&\quad + 2 \wedge^4 \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - 2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \otimes \wedge^3\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} + 2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \\
&\quad \otimes \wedge^2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \otimes \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \otimes \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} + \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \\
&\quad + \wedge^2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} + \widetilde{T_{\mathbf{C}}M} \otimes (2 \wedge^2 \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \\
&\quad \otimes (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) + \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}})) \cdot \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) \\
&\quad + \mathrm{ch}(2 \wedge^2 \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \otimes (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) + \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \\
&\quad \cdot \mathrm{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d) \\
&\quad + \mathrm{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d) \\
&\quad + \mathrm{ch}(\triangle(E) \otimes (\wedge^2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2\widetilde{E_{\mathbf{C}}} + 2 \wedge^4 \widetilde{E_{\mathbf{C}}} - 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^3\widetilde{E_{\mathbf{C}}} \\
&\quad + 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2\widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}} \\
&\quad \left. + \wedge^2\widetilde{E_{\mathbf{C}}}), g, d)\Big]\right\}^{(19)} \\
= {}& -117288\left\{\widehat{A}(TM, \nabla^{TM}) \exp\left(\frac{c}{2}\right) \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d)\right\}^{(19)}.
\end{aligned}
\tag{3.77}
$$

Corollary 3.20. *Let M be a 19-dimensional* spin^c *manifold without boundary. If $c_3(E, g, d) = 0$, then*

$$
\begin{aligned}
&\mathrm{Ind}(T^c \otimes ((\widetilde{T_{\mathbf{C}}M} + 2 \wedge^2 \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \otimes (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) + \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \\
&\quad \otimes (\triangle(E), g^{\triangle(E)}, d) + (\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \\
&\quad \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d))) \equiv 0 \mod 264\mathbf{Z},
\end{aligned}
\tag{3.78}
$$

$$
\begin{aligned}
&\mathrm{Ind}(T \otimes ((S^2\widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} + \wedge^2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \otimes \wedge^2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} + 2 \wedge^4 \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \\
&\quad - 2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \otimes \wedge^3\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} + 2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \otimes \wedge^2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \\
&\quad \otimes \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} \otimes \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} + \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} + \wedge^2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} + \widetilde{T_{\mathbf{C}}M} \\
&\quad \otimes (2 \wedge^2 \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \otimes (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) + \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}})) \\
&\quad \otimes (\triangle(E), g^{\triangle(E)}, d) + (2 \wedge^2 \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \otimes (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) + \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \\
&\quad \otimes (\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d) + (\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} \\
&\quad - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d) + (\triangle(E) \otimes (\wedge^2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2\widetilde{E_{\mathbf{C}}} + 2 \wedge^4 \widetilde{E_{\mathbf{C}}}
\end{aligned}
$$

$$-2\widetilde{E_{\mathbf{C}}}\otimes\wedge^3\widetilde{E_{\mathbf{C}}}+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}$$
$$+\widetilde{E_{\mathbf{C}}}+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d)))\equiv 0 \mod 117288\mathbf{Z}. \tag{3.79}$$

Theorem 3.21. *Let* $\dim M=23$. *If* $3p_1(L)-p_1(M)=0$ *and* $c_3(E,g,d)=0$, *we have*

$$\begin{aligned}
&\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\Big[\mathrm{ch}(S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\
&\quad+2\wedge^4\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\wedge^3\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\
&\quad\otimes\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\
&\quad+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\widetilde{T_{\mathbf{C}}M}\otimes(2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\
&\quad\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}))\cdot\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\
&\quad+\mathrm{ch}(2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\
&\quad\cdot\mathrm{ch}(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)\\
&\quad+\mathrm{ch}(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)\\
&\quad+\mathrm{ch}(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}+2\wedge^4\widetilde{E_{\mathbf{C}}}-2\widetilde{E_{\mathbf{C}}}\otimes\wedge^3\widetilde{E_{\mathbf{C}}}\\
&\quad+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}\\
&\quad+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d)\Big]\Big\}^{(23)}\\
&=\Big\{196560[\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)]\\
&\quad-24\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\Big[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}+2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\
&\quad\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\
&\quad+\mathrm{ch}(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\\
&\quad\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)\Big]\Big\}\Big\}^{(23)}.
\end{aligned}\tag{3.80}$$

Corollary 3.22. *Let* M *be a 23-dimensional* spin^c *manifold without boundary. If* $c_3(E,g,d)=0$, *then*

$$\begin{aligned}
&\mathrm{Ind}(T\otimes((S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+2\wedge^4\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\
&\quad-2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\wedge^3\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\
&\quad\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\widetilde{T_{\mathbf{C}}M}\\
&\quad\otimes(2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}))\\
&\quad\otimes(\triangle(E),g^{\triangle(E)},d)+(2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\
&\quad\otimes(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)+(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}\\
&\quad-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)+(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}+2\wedge^4\widetilde{E_{\mathbf{C}}}
\end{aligned}$$

$$- 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^3 \widetilde{E_{\mathbf{C}}} + 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}}$$
$$+ \widetilde{E_{\mathbf{C}}} + \wedge^2 \widetilde{E_{\mathbf{C}}}), g, d))) \equiv 0 \mod 24\mathbf{Z}. \tag{3.81}$$

Let $\Theta^*(T_{\mathbf{C}}M, L_{\mathbf{R}} \otimes \mathbf{C})$ be the virtual complex vector bundle over M defined by

$$\Theta^*(T_{\mathbf{C}}M, L_{\mathbf{R}} \otimes \mathbf{C}) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbf{C}}M}) \otimes \bigotimes_{m=1}^{\infty} \wedge_{-q^m}(\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}).$$

Let $\dim M = 4k+1$ and $y = -\frac{\sqrt{-1}}{2\pi}c$. Set

$$\bar{Q}(\nabla^{TM}, \nabla^L, g, d, \tau) = \left\{ \widehat{A}(TM, \nabla^{TM}) \exp\left(\frac{c}{2}\right) \mathrm{ch}(\Theta^*(T_{\mathbf{C}}M, L_{\mathbf{R}} \otimes \mathbf{C})) \cdot \mathrm{ch}(Q(E), g^{Q(E)}, d, \tau) \right\}^{(4k+1)}. \tag{3.82}$$

Then

$$\bar{Q}(M, L, \tau) = \left\{ \left(\prod_{j=1}^{2k+1} \frac{x_j \theta'(0,\tau)}{\theta(x_j,\tau)} \right) \frac{\sqrt{-1}\theta(y,\tau)}{\theta_1(0,\tau)\theta_2(0,\tau)\theta_3(0,\tau)} \right\}^{(4p)}, \tag{3.83}$$

and

$$\bar{Q}(\nabla^{TM}, \nabla^L, g, d, \tau) = (\bar{Q}(M, L, \tau) \cdot \mathrm{ch}(Q(E), g^{Q(E)}, d, \tau))^{(4k+1)}. \tag{3.84}$$

Let p_1 denote the first Pontryagin class. By (2.13)–(2.17), we have $\bar{Q}(\nabla^{TM}, \nabla^L, g, d, \tau + 1) = \bar{Q}(\nabla^{TM}, \nabla^L, g, d, \tau)$ and $\bar{Q}(\nabla^{TM}, \nabla^L, g, d, -\frac{1}{\tau}) = \tau^{2k}\bar{Q}(\nabla^{TM}, \nabla^L, g, d, \tau)$ if $p_1(L) - p_1(M) = 0$ and $c_3(E_{\mathbf{C}}, g, d) = 0$. So we get the following theorem.

Theorem 3.23. *Let* $\dim M = 4k+1$. *If* $p_1(L) - p_1(M) = 0$ *and* $c_3(E, g, d) = 0$, *then* $\bar{Q}(\nabla^{TM}, \nabla^L, g, d, \tau)$ *is a modular form over* $\mathrm{SL}(2, \mathbf{Z})$ *with the weight* $2k$.

By Theorem 3.23, similar to Theorem 3.13 and Corollary 3.22, we can get the following theorems.

Theorem 3.24. *Let* $\dim M = 7$. *If* $p_1(L) - p_1(M) = 0$ *and* $c_3(E, g, d) = 0$, *we have*

$$\Big\{ \widehat{A}(TM, \nabla^{TM}) \exp\left(\frac{c}{2}\right) \big[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M} - \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) + \mathrm{ch}(\triangle(E) \otimes (2\wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d) \big] \Big\}^{(7)}$$
$$= 240 \left\{ \widehat{A}(TM, \nabla^{TM}) \exp\left(\frac{c}{2}\right) \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) \right\}^{(7)}, \tag{3.85}$$

$$\Big\{ \widehat{A}(TM, \nabla^{TM}) \exp\left(\frac{c}{2}\right) \big[\mathrm{ch}(S^2\widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - 2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} + \widetilde{T_{\mathbf{C}}M}$$

$$\otimes \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}})\,\mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) + \mathrm{ch}(\widetilde{T_{\mathbf{C}}M} - \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}})$$
$$\cdot\,\mathrm{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)$$
$$+ \mathrm{ch}(\triangle(E) \otimes (\wedge^2 \widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} + 2 \wedge^4 \widetilde{E_{\mathbf{C}}} - 2\widetilde{E_{\mathbf{C}}}$$
$$\otimes \wedge^3 \widetilde{E_{\mathbf{C}}} + 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}}$$
$$\left. + \widetilde{E_{\mathbf{C}}} + \wedge^2 \widetilde{E_{\mathbf{C}}}), g, d)\big]\right\}^{(7)}$$
$$= 2160 \left\{ \widehat{A}(TM, \nabla^{TM}) \exp\left(\frac{c}{2}\right) \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) \right\}^{(7)}. \tag{3.86}$$

Corollary 3.25. *Suppose M is a 7-dimensional* spin^c *manifold without boundary. If $c_3(E, g, d) = 0$, then*

$$\begin{aligned} &\mathrm{Ind}(T^c \otimes ((\widetilde{T_{\mathbf{C}}M} - \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \otimes (\triangle(E), g^{\triangle(E)}, d) \\ &\qquad + (\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d))) \equiv 0 \mod 240\mathbf{Z}, \end{aligned} \tag{3.87}$$

$$\begin{aligned} &\mathrm{Ind}(T \otimes ((S^2 \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - 2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} + \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \\ &\quad \otimes (\triangle(E), g^{\triangle(E)}, d) + (\widetilde{T_{\mathbf{C}}M} - \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \otimes (\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} \\ &\quad - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d) + (\triangle(E) \otimes (\wedge^2 \widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} \\ &\quad + 2 \wedge^4 \widetilde{E_{\mathbf{C}}} - 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^3 \widetilde{E_{\mathbf{C}}} + 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \\ &\quad \otimes \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}} + \wedge^2 \widetilde{E_{\mathbf{C}}}), g, d))) \equiv 0 \mod 2160\mathbf{Z}. \end{aligned} \tag{3.88}$$

Theorem 3.26. *Let* $\dim M = 11$. *If $p_1(L) - p_1(M) = 0$ and $c_3(E, g, d) = 0$, we have*

$$\begin{aligned} &\left\{ \widehat{A}(TM, \nabla^{TM}) \exp\left(\frac{c}{2}\right) \big[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M} - \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}})\,\mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) \right. \\ &\qquad \left. + \mathrm{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)\big] \right\}^{(11)} \\ &= -504 \left\{ \widehat{A}(TM, \nabla^{TM}) \exp\left(\frac{c}{2}\right) \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) \right\}^{(11)}, \end{aligned} \tag{3.89}$$

$$\left\{ \widehat{A}(TM, \nabla^{TM}) \exp\left(\frac{c}{2}\right) \big[\mathrm{ch}(S^2 \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - 2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} + \widetilde{T_{\mathbf{C}}M} \right.$$
$$\otimes \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}})\,\mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) + \mathrm{ch}(\widetilde{T_{\mathbf{C}}M} - \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}})$$
$$\cdot\,\mathrm{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)$$
$$+ \mathrm{ch}(\triangle(E) \otimes (\wedge^2 \widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} + 2 \wedge^4 \widetilde{E_{\mathbf{C}}} - 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^3 \widetilde{E_{\mathbf{C}}}$$

$$
\begin{aligned}
&+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}\\
&+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d)\Big]\Big\}^{(11)}
\end{aligned}
$$

$$
=-16632\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\right\}^{(11)}. \tag{3.90}
$$

Corollary 3.27. *Let M be an 11-dimensional* spin^c *manifold without boundary. If $c_3(E,g,d)=0$, then*

$$
\begin{aligned}
&\mathrm{Ind}(T^c\otimes((\widetilde{T_{\mathbf{C}}M}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\triangle(E),g^{\triangle(E)},d)\\
&\quad+(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)))\equiv 0 \mod 504\mathbf{Z},
\end{aligned} \tag{3.91}
$$

$$
\begin{aligned}
&\mathrm{Ind}(T\otimes((S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\
&\quad\otimes(\triangle(E),g^{\triangle(E)},d)+(\widetilde{T_{\mathbf{C}}M}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}\\
&\quad-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)+(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}\\
&\quad+2\wedge^4\widetilde{E_{\mathbf{C}}}-2\widetilde{E_{\mathbf{C}}}\otimes\wedge^3\widetilde{E_{\mathbf{C}}}+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\\
&\quad\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d)))\equiv 0 \mod 16632\mathbf{Z}.
\end{aligned} \tag{3.92}
$$

Theorem 3.28. *Let* $\dim M=15$. *If $p_1(L)-p_1(M)=0$ and $c_3(E,g,d)=0$, we have*

$$
\begin{aligned}
&\Big\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\Big[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\
&\quad+\mathrm{ch}(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)\Big]\Big\}^{(15)}\\
&=480\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\right\}^{(15)},
\end{aligned} \tag{3.93}
$$

$$
\begin{aligned}
&\Big\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\Big[\mathrm{ch}(S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\widetilde{T_{\mathbf{C}}M}\\
&\quad\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)+\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\
&\quad\cdot\mathrm{ch}(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)\\
&\quad+\mathrm{ch}(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}+2\wedge^4\widetilde{E_{\mathbf{C}}}-2\widetilde{E_{\mathbf{C}}}\otimes\wedge^3\widetilde{E_{\mathbf{C}}}\\
&\quad+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}\\
&\quad+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d)\Big]\Big\}^{(15)}\\
&=6192\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\right\}^{(15)}.
\end{aligned} \tag{3.94}
$$

Corollary 3.29. *Let* M *be a* 15*-dimensional* spin^c *manifold without boundary. If* $c_3(E,g,d)=0$*, then*

$$\begin{aligned}\mathrm{Ind}(T^c\otimes((\widetilde{T_{\mathbf{C}}M}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\triangle(E),g^{\triangle(E)},d)\\ +(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)))\equiv 0\quad \mathrm{mod}\,480\mathbf{Z},\end{aligned}\tag{3.95}$$

$$\begin{aligned}\mathrm{Ind}(T\otimes((S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\ \otimes(\triangle(E),g^{\triangle(E)},d)+(\widetilde{T_{\mathbf{C}}M}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}\\ -\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)+(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}\\ +2\wedge^4\widetilde{E_{\mathbf{C}}}-2\widetilde{E_{\mathbf{C}}}\otimes\wedge^3\widetilde{E_{\mathbf{C}}}+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\\ \otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d)))\equiv 0\quad \mathrm{mod}\,6192\mathbf{Z}.\end{aligned}\tag{3.96}$$

Theorem 3.30. *Let* $\dim M=19$*. If* $p_1(L)-p_1(M)=0$ *and* $c_3(E,g,d)=0$*, we have*

$$\begin{aligned}&\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\big[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\ &\qquad+\mathrm{ch}(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)\big]\Big\}^{(19)}\\ &=-264\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\Big\}^{(19)},\end{aligned}\tag{3.97}$$

$$\begin{aligned}&\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\big[\mathrm{ch}(S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\widetilde{T_{\mathbf{C}}M}\\ &\qquad\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)+\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\ &\qquad\cdot\mathrm{ch}(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)\\ &\qquad+\mathrm{ch}(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}+2\wedge^4\widetilde{E_{\mathbf{C}}}-2\widetilde{E_{\mathbf{C}}}\otimes\wedge^3\widetilde{E_{\mathbf{C}}}\\ &\qquad+2\widetilde{E_{\mathbf{C}}}\otimes\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}\\ &\qquad+\wedge^2\widetilde{E_{\mathbf{C}}}),g,d)\big]\Big\}^{(19)}\\ &=-117288\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\Big\}^{(19)}.\end{aligned}\tag{3.98}$$

Corollary 3.31. *Let* M *be a* 19*-dimensional* spin^c *manifold without boundary. If* $c_3(E,g,d)=0$*, then*

$$\begin{aligned}\mathrm{Ind}(T^c\otimes((\widetilde{T_{\mathbf{C}}M}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\triangle(E),g^{\triangle(E)},d)\\ +(\triangle(E)\otimes(2\wedge^2\widetilde{E_{\mathbf{C}}}-\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)))\equiv 0\quad \mathrm{mod}\,264\mathbf{Z},\end{aligned}\tag{3.99}$$

$$\mathrm{Ind}(T\otimes((S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})$$

$$\otimes (\triangle(E), g^{\triangle(E)}, d) + (\widetilde{T_{\mathbf{C}}M} - \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \otimes (\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d) + (\triangle(E) \otimes (\wedge^2 \widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} + 2 \wedge^4 \widetilde{E_{\mathbf{C}}} - 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^3 \widetilde{E_{\mathbf{C}}} + 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}} + \wedge^2 \widetilde{E_{\mathbf{C}}}), g, d))) \equiv 0 \mod 117288\mathbf{Z}. \tag{3.100}$$

Theorem 3.32. *Let* $\dim M = 23$. *If* $p_1(L) - p_1(M) = 0$ *and* $c_3(E, g, d) = 0$, *we have*

$$\begin{aligned}
&\Big\{\widehat{A}(TM, \nabla^{TM}) \exp\Big(\frac{c}{2}\Big)\big[\mathrm{ch}(S^2\widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - 2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} + \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \,\mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) + \mathrm{ch}(\widetilde{T_{\mathbf{C}}M} - \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \\
&\quad \cdot \mathrm{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d) \\
&\quad + \mathrm{ch}(\triangle(E) \otimes (\wedge^2 \widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} + 2 \wedge^4 \widetilde{E_{\mathbf{C}}} - 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^3 \widetilde{E_{\mathbf{C}}} + 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}} + \wedge^2 \widetilde{E_{\mathbf{C}}}), g, d)\big]\Big\}^{(23)} \\
&= \Big\{196560\Big[\widehat{A}(TM, \nabla^{TM}) \exp\Big(\frac{c}{2}\Big) \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d)\Big] \\
&\quad - 24\Big\{\widehat{A}(TM, \nabla^{TM}) \exp\Big(\frac{c}{2}\Big)\big[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M} - \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \cdot \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) \\
&\quad + \mathrm{ch}(\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)\big]\Big\}\Big\}^{(23)}.
\end{aligned} \tag{3.101}$$

Corollary 3.33. *Let* M *be a 23-dimensional* spin^c *manifold without boundary. If* $c_3(E, g, d) = 0$, *then*

$$\mathrm{Ind}(T \otimes ((S^2\widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - 2\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} + \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \otimes (\triangle(E), g^{\triangle(E)}, d) + (\widetilde{T_{\mathbf{C}}M} - \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \otimes (\triangle(E) \otimes (2 \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d) + (\triangle(E) \otimes (\wedge^2 \widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} + 2 \wedge^4 \widetilde{E_{\mathbf{C}}} - 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^3 \widetilde{E_{\mathbf{C}}} + 2\widetilde{E_{\mathbf{C}}} \otimes \wedge^2 \widetilde{E_{\mathbf{C}}} - \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}} + \wedge^2 \widetilde{E_{\mathbf{C}}}), g, d))) \equiv 0 \mod 24\mathbf{Z}. \tag{3.102}$$

4. Some modular forms and Witten genus over $\Gamma^0(2), \Gamma_0(2), \Gamma_\theta$

In this section, we will construct some modular forms over $\Gamma^0(2)$, $\Gamma_0(2)$, Γ_θ in odd dimensions. Furthermore, we calculate and derive some new cancellation formulas for odd-dimensional spin manifolds and odd-dimensional spin^c manifolds, respectively.

Meanwhile, we will derive some divisibility results for the index of the Toeplitz operators on spin manifolds and spin^c manifolds.

4.1. Some modular forms and Witten genus over $\Gamma^0(2), \Gamma_0(2), \Gamma_\theta$ in spin manifolds

Let M be a $(4k-1)$-dimensional spin manifold. By (3.1)–(3.3) and (3.10)–(3.12), we can construct the following forms:

$$\begin{aligned} Q_1(\nabla^{TM}, g, d, \tau) = \{\widehat{A}(TM, \nabla^{TM})\,\mathrm{ch}([\triangle(M) \otimes \Theta_1(T_{\mathbf{C}}M) + 2^{2k}\Theta_2(T_{\mathbf{C}}M) \\ + 2^{2k}\Theta_3(T_{\mathbf{C}}M)]) \\ \cdot\,\mathrm{ch}(Q_1(E), g^{Q_1(E)}, d, \tau)\}^{(4k-1)}, \end{aligned} \tag{4.1}$$

$$\begin{aligned} Q_2(\nabla^{TM}, g, d, \tau) = \{\widehat{A}(TM, \nabla^{TM})\,\mathrm{ch}([\triangle(M) \otimes \Theta_1(T_{\mathbf{C}}M) + 2^{2k}\Theta_2(T_{\mathbf{C}}M) \\ + 2^{2k}\Theta_3(T_{\mathbf{C}}M)]) \\ \cdot\,\mathrm{ch}(Q_2(E), g^{Q_2(E)}, d, \tau)\}^{(4k-1)}, \end{aligned} \tag{4.2}$$

$$\begin{aligned} Q_3(\nabla^{TM}, g, d, \tau) = \{\widehat{A}(TM, \nabla^{TM})\,\mathrm{ch}([\triangle(M) \otimes \Theta_1(T_{\mathbf{C}}M) + 2^{2k}\Theta_2(T_{\mathbf{C}}M) \\ + 2^{2k}\Theta_3(T_{\mathbf{C}}M)]) \\ \cdot\,\mathrm{ch}(Q_3(E), g^{Q_3(E)}, d, \tau)\}^{(4k-1)}. \end{aligned} \tag{4.3}$$

Following [10], we defined $\mathrm{ch}(Q_j(E), g^{Q_j(E)}, d, \tau)$ for $j = 1, 2, 3$ as follows:

$$\mathrm{ch}(Q_j(E), \nabla_0^{Q_j(E)}, \tau) - \mathrm{ch}(Q_j(E), \nabla_1^{Q_j(E)}, \tau) = d\,\mathrm{ch}(Q_j(E), g^{Q_j(E)}, d, \tau), \tag{4.4}$$

where

$$\mathrm{ch}(Q_1(E), g^{Q_1(E)}, d, \tau) = -\frac{2^{\frac{N}{2}}}{8\pi^2}\int_0^1 \mathrm{Tr}\Big[g^{-1}dg\,\frac{\theta_1'(R_u/(4\pi^2), \tau)}{\theta_1(R_u/(4\pi^2), \tau)}\Big]du, \tag{4.5}$$

and for $j = 2, 3$,

$$\mathrm{ch}(Q_j(E), g^{Q_j(E)}, d, \tau) = -\frac{1}{8\pi^2}\int_0^1 \mathrm{Tr}\Big[g^{-1}dg\,\frac{\theta_j'(R_u/(4\pi^2), \tau)}{\theta_j(R_u/(4\pi^2), \tau)}\Big]du. \tag{4.6}$$

By [10, Proposition 2.2], we have if $c_3(E, g, d) = 0$, then for any integer $k \geq 1$ and $j = 1, 2, 3$, $\mathrm{ch}(Q_j(E), g^{Q_j(E)}, d, \tau)^{(4k-1)}$ are modular forms of weight $2k$ over $\Gamma_0(2)$, $\Gamma^0(2)$ and Γ_θ, respectively. By [10, Proposition 2.4, Theorem 2.6], we understand that if $c_3(E, g, d) = 0$, then for any integer $k \geq 1$ and $j = 1, 2, 3$, $Q_1(\nabla^{TM}, g, d, \tau)^{(4k-1)}$, $Q_2(\nabla^{TM}, g, d, \tau)^{(4k-1)}$ and $Q_3(\nabla^{TM}, g, d, \tau)^{(4k-1)}$ are modular forms of weight $2k$ over $\Gamma_0(2)$, $\Gamma^0(2)$ and Γ_θ, respectively.

Let p_1 denote the first Pontryagin class. If ω is a differential form over M, we denote by ω^{4k-1} its top degree component. Our main results include the following theorem.

Theorem 4.1. *If* $c_3(E,g,d)=0$, *then*

$$\begin{aligned}\{\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M)+2^{k+1})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\}^{(4k-1)}\\ =2^{\frac{N}{2}+k}\sum_{s=1}^{[\frac{k}{2}]}2^{-6s}h_s,\end{aligned} \tag{4.7}$$

where each h_s, $1\le s\le\left[\frac{k}{2}\right]$, *is a canonical integral linear combination of*

$$\begin{aligned}&\{\widehat{A}(TM)\,\mathrm{ch}(\triangle(M))\widetilde{\mathrm{ch}}(B_\alpha^1(T_{\mathbf{C}}M,E_{\mathbf{C}}))+2^{2k}\,\widehat{A}(TM)\widetilde{\mathrm{ch}}(B_\alpha^2(T_{\mathbf{C}}M,E_{\mathbf{C}}))\\&\quad+2^{2k}\,\widehat{A}(TM)\widetilde{\mathrm{ch}}(B_\alpha^3(T_{\mathbf{C}}M,E_{\mathbf{C}}))\}^{(4k-1)},\quad 0\le\alpha\le s,\end{aligned}$$

and h_1, h_2 *are given by* (4.16) *and* (4.17).

Proof. Let $\{\pm 2\pi\sqrt{-1}x_j \mid 1\le j\le k\}$ be the Chern roots of $T_{\mathbf{C}}M$. We have

$$\begin{aligned}&Q_1(\nabla^{TM},g,d,\tau)\\&=\Bigg\{\prod_{j=1}^{2k}\frac{2x_j\theta'(0,\tau)}{\theta(x_j,\tau)}\left(\prod_{j=1}^{2k}\frac{\theta_1(x_j,\tau)}{\theta_1(0,\tau)}+\prod_{j=1}^{2k}\frac{\theta_2(x_j,\tau)}{\theta_2(0,\tau)}+\prod_{j=1}^{2k}\frac{\theta_3(x_j,\tau)}{\theta_3(0,\tau)}\right)\\&\qquad\cdot\mathrm{ch}(Q_1(E),g^{Q_1(E)},d,\tau)\Bigg\}^{(4k-1)},\end{aligned} \tag{4.8}$$

$$\begin{aligned}&Q_2(\nabla^{TM},g,d,\tau)\\&=\Bigg\{\prod_{j=1}^{2k}\frac{2x_j\theta'(0,\tau)}{\theta(x_j,\tau)}\left(\prod_{j=1}^{2k}\frac{\theta_1(x_j,\tau)}{\theta_1(0,\tau)}+\prod_{j=1}^{2k}\frac{\theta_2(x_j,\tau)}{\theta_2(0,\tau)}+\prod_{j=1}^{2k}\frac{\theta_3(x_j,\tau)}{\theta_3(0,\tau)}\right)\\&\qquad\cdot\mathrm{ch}(Q_2(E),g^{Q_1(E)},d,\tau)\Bigg\}^{(4k-1)}.\end{aligned} \tag{4.9}$$

Moreover, we can show by direct computations that

$$\begin{aligned}Q_1(\nabla^{TM},g,d,\tau)=&\,[\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M))\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\&+2^{2k+1}\,\widehat{A}(TM,\nabla^{TM})\cdot\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)]^{(4k-1)}\\&+q\{\widehat{A}(TM,\nabla^{TM})\cdot[\mathrm{ch}(\triangle(M)\otimes 2\widetilde{T_{\mathbf{C}}M})\cdot\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\&\qquad+\mathrm{ch}(\triangle(M))\,\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)]+2^{2k+1}\\&\quad\cdot\widehat{A}(TM,\nabla^{TM})\cdot[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\&\qquad+\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)]\}^{(4k-1)}\\&+q^2\{\widehat{A}(TM,\nabla^{TM})[\mathrm{ch}(\triangle(M)\otimes(2\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}\\&\qquad\otimes\widetilde{T_{\mathbf{C}}M}+S^2\widetilde{T_{\mathbf{C}}M}))\end{aligned}$$

$$
\begin{aligned}
&\cdot \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) + \mathrm{ch}(\triangle(M) \otimes 2\widetilde{T_{\mathbf{C}}M})\\
&\cdot \mathrm{ch}(\triangle(E) \otimes \widetilde{E_{\mathbf{C}}}, g, d) + \mathrm{ch}(\triangle(M))\\
&\cdot \mathrm{ch}(\triangle(E) \otimes (\wedge^2 \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)] + 2^{2k+1}\\
&\cdot \widehat{A}(TM, \nabla^{TM})[\mathrm{ch}(\wedge^4 \widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M}\\
&\otimes \widetilde{T_{\mathbf{C}}M} + S^2 \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M})\\
&\cdot \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) + \mathrm{ch}(\widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M})\\
&\cdot \mathrm{ch}(\triangle(E) \otimes \widetilde{E_{\mathbf{C}}}, g, d) + \mathrm{ch}(\triangle(E)\\
&\otimes (\wedge^2 \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)]\}^{(4k-1)} + \cdots ,
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
Q_2(\nabla^{TM}, g, d, \tau) = {}& -q^{\frac{1}{2}}[\widehat{A}(TM, \nabla^{TM})\, \mathrm{ch}(\triangle(M))\, \mathrm{ch}(\widetilde{E_{\mathbf{C}}}, g, d)\\
&+ 2^{2k+1} \widehat{A}(TM, \nabla^{TM}) \cdot \mathrm{ch}(\widetilde{E_{\mathbf{C}}}, g, d)]^{(4k-1)}\\
&+ q[\widehat{A}(TM, \nabla^{TM}) \cdot \mathrm{ch}(\triangle(M))\, \mathrm{ch}(\wedge^2 \widetilde{E_{\mathbf{C}}}, g, d)\\
&+ 2^{2k+1} \widehat{A}(TM, \nabla^{TM}) \cdot \mathrm{ch}(\wedge^2 \widetilde{E_{\mathbf{C}}}, g, d)]^{(4k-1)}\\
&- q^{\frac{3}{2}}\{\widehat{A}(TM, \nabla^{TM}) \cdot [\mathrm{ch}(\triangle(M) \otimes 2\widetilde{T_{\mathbf{C}}M})\, \mathrm{ch}(\widetilde{E_{\mathbf{C}}}, g, d)\\
&+ \mathrm{ch}\, \triangle(M)\, \mathrm{ch}(\widetilde{E_{\mathbf{C}}} + \wedge^3 \widetilde{E_{\mathbf{C}}}, g, d)]\\
&+ 2^{2k+1} \widehat{A}(TM, \nabla^{TM})[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M})\, \mathrm{ch}(\widetilde{E_{\mathbf{C}}}, g, d)\\
&+ \mathrm{ch}(\widetilde{E_{\mathbf{C}}} + \wedge^3 \widetilde{E_{\mathbf{C}}}, g, d)]\}^{(4k-1)}\\
&+ q^2\{\widehat{A}(TM, \nabla^{TM}) \cdot [\mathrm{ch}(\triangle(M) \otimes 2\widetilde{T_{\mathbf{C}}M}) \cdot \mathrm{ch}(\wedge^2 \widetilde{E_{\mathbf{C}}}, g, d)\\
&+ \mathrm{ch}\, \triangle(M) \cdot \mathrm{ch}(\wedge^4 \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}}, g, d)]\\
&+ 2^{2k+1} \widehat{A}(TM, \nabla^{TM})\\
&\cdot [\mathrm{ch}(\widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M})\, \mathrm{ch}(\wedge^2 \widetilde{E_{\mathbf{C}}}, g, d)\\
&+ \mathrm{ch}(\wedge^4 \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}} \otimes \widetilde{E_{\mathbf{C}}}, g, d)]\}^{(4k-1)} + \cdots ,
\end{aligned}
\tag{4.11}
$$

and we can represent $Q_2(\nabla^{TM}, g, d, \tau)$ as

$$
\begin{aligned}
Q_2(\nabla^{TM}, g, d, \tau) = {}& \widehat{A}(TM, \nabla^{TM})\, \mathrm{ch}(\triangle(M)) \widetilde{\mathrm{ch}}(B_0^1(T_{\mathbf{C}}M, E_{\mathbf{C}}))\\
&+ 2^{2k} \widehat{A}(TM, \nabla^{TM}) \widetilde{\mathrm{ch}}(B_0^2(T_{\mathbf{C}}M, E_{\mathbf{C}}))\\
&+ 2^{2k} \widehat{A}(TM, \nabla^{TM}) \widetilde{\mathrm{ch}}(B_0^3(T_{\mathbf{C}}M, E_{\mathbf{C}}))\\
&+ [\widehat{A}(TM, \nabla^{TM})\, \mathrm{ch}(\triangle(M)) \widetilde{\mathrm{ch}}(B_1^1(T_{\mathbf{C}}M, E_{\mathbf{C}}))\\
&+ 2^{2k} \widehat{A}(TM, \nabla^{TM}) \widetilde{\mathrm{ch}}(B_1^2(T_{\mathbf{C}}M, E_{\mathbf{C}}))\\
&+ 2^{2k} \widehat{A}(TM, \nabla^{TM}) \widetilde{\mathrm{ch}}(B_1^3(T_{\mathbf{C}}M, E_{\mathbf{C}}))] q^{\frac{1}{2}}\\
&+ [\widehat{A}(TM, \nabla^{TM})\, \mathrm{ch}(\triangle(M)) \widetilde{\mathrm{ch}}(B_2^1(T_{\mathbf{C}}M, E_{\mathbf{C}}))\\
&+ 2^{2k} \widehat{A}(TM, \nabla^{TM}) \widetilde{\mathrm{ch}}(B_2^2(T_{\mathbf{C}}M, E_{\mathbf{C}}))
\end{aligned}
$$

$$
\begin{aligned}
&+ 2^{2k}\widehat{A}(TM,\nabla^{TM})\widetilde{\mathrm{ch}}(B_2^3(T_{\mathbf{C}}M,E_{\mathbf{C}}))]q\\
&+ [\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M))\widetilde{\mathrm{ch}}(B_3^1(T_{\mathbf{C}}M,E_{\mathbf{C}}))\\
&+ 2^{2k}\widehat{A}(TM,\nabla^{TM})\widetilde{\mathrm{ch}}(B_3^2(T_{\mathbf{C}}M,E_{\mathbf{C}}))\\
&+ 2^{2k}\widehat{A}(TM,\nabla^{TM})\widetilde{\mathrm{ch}}(B_3^3(T_{\mathbf{C}}M,E_{\mathbf{C}}))]q^{\frac{3}{2}}+\cdots. \qquad (4.12)
\end{aligned}
$$

Let $P_1(\tau)=Q_1(\nabla^{TM},g,d,\tau)^{4k-1}$ and $P_2(\tau)=Q_2(\nabla^{TM},g,d,\tau)^{4k-1}$. Similarly to the computations in [13] and by [10, (2.26)] and the condition $c_3(E,g,d)=0$, we have

$$
P_1\left(-\frac{1}{\tau}\right)=2^{\frac{N}{2}}\tau^{2k}P_2(\tau). \qquad (4.13)
$$

Observe that at any point $x\in M$, up to the volume form determined by the metric on T_xM, both $P_i(\tau),\ i=1,2$, can be viewed as a power series of $q^{\frac{1}{2}}$ with real Fourier coefficients. By Lemma 2.2, we have

$$
P_2(\tau)=h_0(8\delta_2)^k+h_1(8\delta_2)^{k-2}\varepsilon_2+\cdots+h_{[\frac{k}{2}]}(8\delta_2)^{k-2[\frac{k}{2}]}\varepsilon_2^{[\frac{k}{2}]}, \qquad (4.14)
$$

where each $h_s,\ 0\le s\le\left[\frac{k}{2}\right]$, is a real multiple of the volume form at x. By (2.19), (4.13) and (4.14), we get

$$
P_1(\tau)=2^{\frac{N}{2}}\left[h_0(8\delta_1)^k+h_1(8\delta_1)^{k-2}\varepsilon_1+\cdots+h_{[\frac{k}{2}]}(8\delta_1)^{k-2[\frac{k}{2}]}\varepsilon_1^{[\frac{k}{2}]}\right]. \qquad (4.15)
$$

By comparing the constant term in (4.15), we get (4.7). By comparing the coefficients of $q^{\frac{j}{2}},\ j\ge 0$ on both sides of (4.14), we can use the induction method to prove that $h_0=0$ and each $h_s,\ 1\le s\le\left[\frac{k}{2}\right]$, can be expressed through a canonical integral linear combination of

$$
\begin{aligned}
&\{\widehat{A}(TM)\,\mathrm{ch}(\triangle(M))\widetilde{\mathrm{ch}}(B_\alpha^1(T_{\mathbf{C}}M,E_{\mathbf{C}}))+2^{2k}\widehat{A}(TM)\widetilde{\mathrm{ch}}(B_\alpha^2(T_{\mathbf{C}}M,E_{\mathbf{C}}))\\
&\quad+2^{2k}\widehat{A}(TM)\widetilde{\mathrm{ch}}(B_\alpha^3(T_{\mathbf{C}}M,E_{\mathbf{C}}))\}^{(4k-1)},\quad 0\le\alpha\le s.
\end{aligned}
$$

By (4.11), (4.12) and comparing the coefficient of $q^{\frac{1}{2}}$ of (4.14), we get

$$
\begin{aligned}
h_1&=(-1)^{k-2}[\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M))\widetilde{\mathrm{ch}}(B_1^1(T_{\mathbf{C}}M,E_{\mathbf{C}}))\\
&\qquad+2^{2k}\widehat{A}(TM,\nabla^{TM})\widetilde{\mathrm{ch}}(B_1^2(T_{\mathbf{C}}M,E_{\mathbf{C}}))\\
&\qquad+2^{2k}\widehat{A}(TM,\nabla^{TM})\widetilde{\mathrm{ch}}(B_1^3(T_{\mathbf{C}}M,E_{\mathbf{C}}))]^{(4k-1)}\\
&=(-1)^{k-1}[\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M))\,\mathrm{ch}(\widetilde{E_{\mathbf{C}}},g,d)\\
&\qquad+2^{2k+1}\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\widetilde{E_{\mathbf{C}}},g,d)]^{(4k-1)}. \qquad (4.16)
\end{aligned}
$$

By (4.11), (4.12) and comparing the coefficient of q of (4.14), we get

$$\begin{aligned}
h_2 = (-1)^{k-4}[&\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M))\widetilde{\mathrm{ch}}(B_2^1(T_{\mathbf{C}}M,E_{\mathbf{C}}))\\
&+2^{2k}\widehat{A}(TM,\nabla^{TM})\widetilde{\mathrm{ch}}(B_2^2(T_{\mathbf{C}}M,E_{\mathbf{C}}))\\
&+2^{2k}\widehat{A}(TM,\nabla^{TM})\widetilde{\mathrm{ch}}(B_2^3(T_{\mathbf{C}}M,E_{\mathbf{C}}))]^{(4k-1)}\\
&-[8-24(k-2)(-1)^d]h_1\\
=(-1)^{k-4}[&\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M))\,\mathrm{ch}(\wedge^2\widetilde{E_{\mathbf{C}}},g,d)\\
&+2^{2k+1}\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\wedge^2\widetilde{E_{\mathbf{C}}},g,d)]^{(4k-1)}\\
&-[24(k-2)+8(-1)^{k-1}][\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M))\,\mathrm{ch}(\widetilde{E_{\mathbf{C}}},g,d)\\
&+2^{2k+1}\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\widetilde{E_{\mathbf{C}}},g,d)]^{(4k-1)}. \qquad (4.17)
\end{aligned}$$

The proof is completed. ∎

Corollary 4.2. *Let M be a $(4k-1)$-dimensional spin manifold. If $c_3(E,g,d)=0$, then*

$$\mathrm{Ind}(T\otimes(\triangle(M)+2^{k+1})\otimes(\triangle(E),g^{\triangle(E)},d)) = -2^{\frac{N}{2}+k}\sum_{s=0}^{[\frac{k}{2}]}2^{-6s}h_s, \qquad (4.18)$$

where each h_s, $1\le s\le\left[\frac{k}{2}\right]$, is a canonical integral linear combination of

$$\begin{aligned}
\mathrm{Ind}(T\otimes(\triangle(M)\otimes(B_\alpha^1(T_{\mathbf{C}}M,E_{\mathbf{C}}))+2^{2k}\otimes(B_\alpha^2(T_{\mathbf{C}}M,E_{\mathbf{C}}))\\
+2^{2k}\otimes(B_\alpha^3(T_{\mathbf{C}}M,E_{\mathbf{C}})))). \qquad (4.19)
\end{aligned}$$

Corollary 4.3. *Let M be a $(4k-1)$-dimensional spin manifold and $c_3(E,g,d)=0$. If k is even, then*

$$\mathrm{Ind}(T\otimes(\triangle(M)+2^{k+1})\otimes(\triangle(E),g^{\triangle(E)},d))\equiv 0 \quad \left(\mathrm{mod}\,2^{\frac{N}{2}-2k}\right), \qquad (4.20)$$

If k is odd, then

$$\mathrm{Ind}(T\otimes(\triangle(M)+2^{k+1})\otimes(\triangle(E),g^{\triangle(E)},d))\equiv 0 \quad \left(\mathrm{mod}\,2^{\frac{N}{2}+3-2k}\right). \qquad (4.21)$$

Corollary 4.4. *Let M be an 11-dimensional spin manifold. If $c_3(E,g,d)=0$, then*

$$\begin{aligned}
&\{\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M)+2^7)\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\}^{(11)}\\
&\quad=2^{\frac{N}{2}-3}\{\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M)+2^7)\,\mathrm{ch}(\widetilde{E_{\mathbf{C}}},g,d)\}^{(11)}. \qquad (4.22)
\end{aligned}$$

Corollary 4.5. *Let M be a 15-dimensional spin manifold. If $c_3(E,g,d)=0$, then*

$$\begin{aligned}
&\{\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M)+2^9)\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\}^{(15)}\\
&\quad=-13\times2^{\frac{N}{2}-5}\{\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M)+2^9)\,\mathrm{ch}(\widetilde{E_{\mathbf{C}}},g,d)\}^{(15)}\\
&\qquad+2^{\frac{N}{2}-8}\{\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M)+2^9)\,\mathrm{ch}(\wedge^2\widetilde{E_{\mathbf{C}}},g,d)\}^{(15)}. \qquad (4.23)
\end{aligned}$$

By [4, (5.22)], we have for $1 \le s \le \left[\frac{k}{2}\right]$,

$$(8\delta_1)^{k-2s}\varepsilon_1^s = 2^{k-6s}[1 + (24k - 64s)q + (288k^2 - 1536ks + 2048s^2 + 512s - 264k)q^2 + O(q^3)]. \tag{4.24}$$

By (4.10), we are comparing the coefficients of q in (4.15). Then we get the following theorem.

Theorem 4.6. *If $c_3(E,g,d) = 0$, then*

$$\begin{aligned}
&\{\widehat{A}(TM,\nabla^{TM})[\mathrm{ch}(\triangle(M)\otimes 2\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d) + \mathrm{ch}(\triangle(M))\\
&\quad\cdot \mathrm{ch}(\triangle(E)\otimes \widetilde{E_{\mathbf{C}}},g,d)] + 2^{2k+1}\widehat{A}(TM,\nabla^{TM})\\
&\quad\cdot [\mathrm{ch}(\widetilde{T_{\mathbf{C}}M} + \wedge^2\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d) + \mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)]\\
&\quad - 24k[\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M)+2^{2k+1})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)]\}^{(4k-1)}\\
&= -2^{\frac{N}{2}+k+6}\sum_{s=1}^{[\frac{k}{2}]} s2^{-6s}h_s.
\end{aligned} \tag{4.25}$$

Corollary 4.7. *Let M be a $(4k-1)$-dimensional spin manifold and $c_3(E,g,d) = 0$. If k is even, then*

$$\begin{aligned}
&\mathrm{Ind}(T\otimes(\triangle(M)\otimes(2\widetilde{T_{\mathbf{C}}M}\otimes(\triangle(E),g^{\triangle(E)},d) + (\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)) + 2^{2k+1}\\
&\quad\otimes((\widetilde{T_{\mathbf{C}}M} + \wedge^2\widetilde{T_{\mathbf{C}}M})\otimes(\triangle(E),g^{\triangle(E)},d) + (\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d))))\\
&\quad - 24k\,\mathrm{Ind}(T\otimes(\triangle(M)+2^{2k+1})\otimes(\triangle(E),g^{\triangle(E)},d))\\
&\equiv 0 \pmod{2^{\frac{N}{2}-2k+6}},
\end{aligned} \tag{4.26}$$

If k is odd, then

$$\begin{aligned}
&\mathrm{Ind}(T\otimes(\triangle(M)\otimes(2\widetilde{T_{\mathbf{C}}M}\otimes(\triangle(E),g^{\triangle(E)},d) + (\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)) + 2^{2k+1}\\
&\quad\otimes((\widetilde{T_{\mathbf{C}}M} + \wedge^2\widetilde{T_{\mathbf{C}}M})\otimes(\triangle(E),g^{\triangle(E)},d) + (\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d))))\\
&\quad - 24k\,\mathrm{Ind}(T\otimes(\triangle(M)+2^{2k+1})\otimes(\triangle(E),g^{\triangle(E)},d))\\
&\equiv 0 \pmod{2^{\frac{N}{2}-2k+9}}.
\end{aligned} \tag{4.27}$$

Corollary 4.8. *Let M be an 11-dimensional spin manifold. If $c_3(E,g,d) = 0$, then*

$$\begin{aligned}
&\{\widehat{A}(TM,\nabla^{TM})[\mathrm{ch}(\triangle(M)\otimes 2\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d) + \mathrm{ch}(\triangle(M))\\
&\quad\cdot \mathrm{ch}(\triangle(E)\otimes \widetilde{E_{\mathbf{C}}},g,d)] + 2^{7}\widehat{A}(TM,\nabla^{TM})\\
&\quad\cdot [\mathrm{ch}(\widetilde{T_{\mathbf{C}}M} + \wedge^2\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d) + \mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)]\\
&\quad - 72[\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M)+2^{7})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)]\}^{(11)}\\
&= -2^{\frac{N}{2}+3}\{\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M)+2^{7})\,\mathrm{ch}(\widetilde{E_{\mathbf{C}}},g,d)\}^{(11)}.
\end{aligned} \tag{4.28}$$

Corollary 4.9. *Let M be a 15-dimensional spin manifold. If $c_3(E,g,d)=0$, then*

$$\begin{aligned}
&\{\widehat{A}(TM,\nabla^{TM})[\mathrm{ch}(\triangle(M)\otimes 2\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)+\mathrm{ch}(\triangle(M))\\
&\qquad\cdot\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)]+2^9\widehat{A}(TM,\nabla^{TM})\\
&\qquad\cdot[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)+\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)]\\
&\qquad-96[\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M)+2^9)\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)]\}^{(15)}\\
&=9\times 2^{\frac{N}{2}+2}\{\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M)+2^9)\,\mathrm{ch}(\widetilde{E_{\mathbf{C}}},g,d)\}^{(15)}\\
&\qquad-2^{\frac{N}{2}-1}\{\widehat{A}(TM,\nabla^{TM})\,\mathrm{ch}(\triangle(M)+2^9)\,\mathrm{ch}(\wedge^2\widetilde{E_{\mathbf{C}}},g,d)\}^{(15)}.
\end{aligned}\tag{4.29}$$

By (4.10), we compare the coefficients of q^2 in (4.15). Then we get the following theorem.

Theorem 4.10. *If $c_3(E,g,d)=0$, then*

$$\begin{aligned}
&\{\widehat{A}(TM,\nabla^{TM})[\mathrm{ch}(\triangle(M)\otimes(2\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{T_{\mathbf{C}}M}+S^2\widetilde{T_{\mathbf{C}}M}))\\
&\qquad\cdot\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)+\mathrm{ch}(\triangle(M)\otimes 2\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)\\
&\qquad+\mathrm{ch}(\triangle(M))\cdot\mathrm{ch}(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)]+2^{2k+1}\widehat{A}(TM,\nabla^{TM})\\
&\qquad\cdot[\mathrm{ch}(\wedge^4\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{T_{\mathbf{C}}M}+S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M})\\
&\qquad\cdot\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)+\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)\\
&\qquad+\mathrm{ch}(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)]\\
&\qquad+(288k^2+72k)[\widehat{A}(TM,\nabla^{TM})\cdot\mathrm{ch}(\triangle(M)+2^{2k+1})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)]\\
&\qquad-(24k-8)\{\widehat{A}(TM,\nabla^{TM})\cdot[\mathrm{ch}(\triangle(M)\otimes 2\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\
&\qquad\qquad+\mathrm{ch}(\triangle(M))\,\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)]\\
&\qquad\qquad+2^{2k+1}\widehat{A}(TM,\nabla^{TM})[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\
&\qquad\qquad+\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)]\}\}^{(4k-1)}\\
&=2^{\frac{N}{2}+k+11}\sum_{s=1}^{[\frac{k}{2}]}s^2 2^{-6s}h_s.
\end{aligned}\tag{4.30}$$

Corollary 4.11. *Let M be a $(4k-1)$-dimensional spin manifold and $c_3(E,g,d)=0$. If k is even, then*

$$\begin{aligned}
&\mathrm{Ind}(T\otimes(\triangle(M)\otimes((2\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{T_{\mathbf{C}}M}+S^2\widetilde{T_{\mathbf{C}}M})\\
&\qquad\otimes(\triangle(E),g^{\triangle(E)},d)+2\widetilde{T_{\mathbf{C}}M}\otimes(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)\\
&\qquad+(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d))+2^{2k+1}\\
&\qquad\otimes((\wedge^4\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{T_{\mathbf{C}}M}+S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M})\\
&\qquad\otimes(\triangle(E),g^{\triangle(E)},d)+(\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{T_{\mathbf{C}}M})\otimes(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)
\end{aligned}$$

$$
\begin{aligned}
&\quad + (\triangle(E) \otimes (\wedge^2 \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)))) + (288k^2 + 72k) \\
&\quad \cdot \operatorname{Ind}(T \otimes (\triangle(M) + 2^{2k+1}) \otimes (\triangle(E), g^{\triangle(E)}, d)) - (24k - 8) \\
&\quad \cdot \operatorname{Ind}(T \otimes (\triangle(M) \otimes (2\widetilde{T_{\mathbf{C}}M} \otimes (\triangle(E), g^{\triangle(E)}, d) + (\triangle(E) \otimes \widetilde{E_{\mathbf{C}}}, g, d)) \\
&\qquad + 2^{2k+1} \otimes ((\widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M}) \otimes (\triangle(E), g^{\triangle(E)}, d) \\
&\qquad + (\triangle(E) \otimes \widetilde{E_{\mathbf{C}}}, g, d)))) \\
&\equiv 0 \pmod{2^{\frac{N}{2} - 2k + 11}}.
\end{aligned} \tag{4.31}
$$

If k is odd, then

$$
\begin{aligned}
&\operatorname{Ind}(T \otimes (\triangle(M) \otimes ((2\widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M} + S^2 \widetilde{T_{\mathbf{C}}M}) \\
&\qquad \otimes (\triangle(E), g^{\triangle(E)}, d) + 2\widetilde{T_{\mathbf{C}}M} \otimes (\triangle(E) \otimes \widetilde{E_{\mathbf{C}}}, g, d) \\
&\qquad + (\triangle(E) \otimes (\wedge^2 \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)) + 2^{2k+1} \\
&\quad \otimes ((\wedge^4 \widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M} + S^2 \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M}) \\
&\quad \otimes (\triangle(E), g^{\triangle(E)}, d) + (\widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M}) \otimes (\triangle(E) \otimes \widetilde{E_{\mathbf{C}}}, g, d) \\
&\quad + (\triangle(E) \otimes (\wedge^2 \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)))) + (288k^2 + 72k) \\
&\quad \cdot \operatorname{Ind}(T \otimes (\triangle(M) + 2^{2k+1}) \otimes (\triangle(E), g^{\triangle(E)}, d)) - (24k - 8) \\
&\quad \cdot \operatorname{Ind}(T \otimes (\triangle(M) \otimes (2\widetilde{T_{\mathbf{C}}M} \otimes (\triangle(E), g^{\triangle(E)}, d) + (\triangle(E) \otimes \widetilde{E_{\mathbf{C}}}, g, d)) \\
&\qquad + 2^{2k+1} \otimes ((\widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M}) \otimes (\triangle(E), g^{\triangle(E)}, d) \\
&\qquad + (\triangle(E) \otimes \widetilde{E_{\mathbf{C}}}, g, d)))) \\
&\equiv 0 \pmod{2^{\frac{N}{2} - 2k + 14}}.
\end{aligned} \tag{4.32}
$$

Corollary 4.12. *Let M be an 11-dimensional spin manifold. If $c_3(E, g, d) = 0$, then*

$$
\begin{aligned}
&\{\widehat{A}(TM, \nabla^{TM})[\operatorname{ch}(\triangle(M) \otimes (2\widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M} + S^2 \widetilde{T_{\mathbf{C}}M})) \\
&\qquad \cdot \operatorname{ch}(\triangle(E), g^{\triangle(E)}, d) + \operatorname{ch}(\triangle(M) \otimes 2\widetilde{T_{\mathbf{C}}M}) \operatorname{ch}(\triangle(E) \otimes \widetilde{E_{\mathbf{C}}}, g, d) \\
&\qquad + \operatorname{ch}(\triangle(M)) \operatorname{ch}(\triangle(E) \otimes (\wedge^2 \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)] + 2^7 \widehat{A}(TM, \nabla^{TM}) \\
&\quad \cdot [\operatorname{ch}(\wedge^4 \widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M} + S^2 \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M}) \\
&\quad \cdot \operatorname{ch}(\triangle(E), g^{\triangle(E)}, d) + \operatorname{ch}(\widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M}) \operatorname{ch}(\triangle(E) \otimes \widetilde{E_{\mathbf{C}}}, g, d) \\
&\quad + \operatorname{ch}(\triangle(E) \otimes (\wedge^2 \widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)] \\
&\quad + 2808[\widehat{A}(TM, \nabla^{TM}) \operatorname{ch}(\triangle(M) + 2^7) \operatorname{ch}(\triangle(E), g^{\triangle(E)}, d)] \\
&\quad - 64\{\widehat{A}(TM, \nabla^{TM})[\operatorname{ch}(\triangle(M) \otimes 2\widetilde{T_{\mathbf{C}}M}) \operatorname{ch}(\triangle(E), g^{\triangle(E)}, d) \\
&\qquad + \operatorname{ch}(\triangle(M)) \operatorname{ch}(\triangle(E) \otimes \widetilde{E_{\mathbf{C}}}, g, d)] \\
&\qquad + 2^7 \widehat{A}(TM, \nabla^{TM})[\operatorname{ch}(\widetilde{T_{\mathbf{C}}M} + \wedge^2 \widetilde{T_{\mathbf{C}}M}) \cdot \operatorname{ch}(\triangle(E), g^{\triangle(E)}, d) \\
&\qquad + \operatorname{ch}(\triangle(E) \otimes \widetilde{E_{\mathbf{C}}}, g, d)]\}\}^{(11)} \\
&= 2^{\frac{N}{2} + 8}\{\widehat{A}(TM, \nabla^{TM}) \operatorname{ch}(\triangle(M) + 2^9) \operatorname{ch}(\widetilde{E_{\mathbf{C}}}, g, d)\}^{(11)}.
\end{aligned} \tag{4.33}
$$

Corollary 4.13. *Let M be an 15-dimensional spin manifold. If $c_3(E, g, d) = 0$, then*

$$
\begin{aligned}
&\{\widehat{A}(TM, \nabla^{TM})[\mathrm{ch}(\triangle(M) \otimes (2\widetilde{T_{\mathbf{C}}M} + \wedge^2\widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M} + S^2\widetilde{T_{\mathbf{C}}M}))\\
&\qquad \cdot \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) + \mathrm{ch}(\triangle(M) \otimes 2\widetilde{T_{\mathbf{C}}M}) \cdot \mathrm{ch}(\triangle(E) \otimes \widetilde{E_{\mathbf{C}}}, g, d)\\
&\qquad + \mathrm{ch}(\triangle(M))\,\mathrm{ch}(\triangle(E) \otimes (\wedge^2\widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)] + 2^9\widehat{A}(TM, \nabla^{TM})\\
&\qquad \cdot [\mathrm{ch}(\wedge^4\widetilde{T_{\mathbf{C}}M} + \wedge^2\widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} \otimes \widetilde{T_{\mathbf{C}}M} + S^2\widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M})\\
&\qquad \cdot \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) + \mathrm{ch}(\widetilde{T_{\mathbf{C}}M} + \wedge^2\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E) \otimes \widetilde{E_{\mathbf{C}}}, g, d)\\
&\qquad + \mathrm{ch}(\triangle(E) \otimes (\wedge^2\widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)]\\
&\qquad + 4896[\widehat{A}(TM, \nabla^{TM})\,\mathrm{ch}(\triangle(M) + 2^9)\,\mathrm{ch}(\triangle(E), g^{\triangle(E)}, d)]\\
&\qquad - 88\{\widehat{A}(TM, \nabla^{TM})[\mathrm{ch}(\triangle(M) \otimes 2\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E), g^{\triangle(E)}, d)\\
&\qquad\qquad + \mathrm{ch}(\triangle(M))\,\mathrm{ch}(\triangle(E) \otimes \widetilde{E_{\mathbf{C}}}, g, d)] + 2^9\widehat{A}(TM, \nabla^{TM})\\
&\qquad \cdot [\mathrm{ch}(\widetilde{T_{\mathbf{C}}M} + \wedge^2\widetilde{T_{\mathbf{C}}M})\,\mathrm{ch}(\triangle(E), g^{\triangle(E)}, d)\\
&\qquad + \mathrm{ch}(\triangle(E) \otimes \widetilde{E_{\mathbf{C}}}, g, d)]\}\}^{(15)}\\
&= -7 \times 2^{\frac{N}{2}+8}\{\widehat{A}(TM, \nabla^{TM})\,\mathrm{ch}(\triangle(M) + 2^9)\,\mathrm{ch}(\widetilde{E_{\mathbf{C}}}, g, d)\}^{(15)}\\
&\qquad + 2^{\frac{N}{2}+5}\{\widehat{A}(TM, \nabla^{TM})\,\mathrm{ch}(\triangle(M) + 2^9)\,\mathrm{ch}(\wedge^2\widetilde{E_{\mathbf{C}}}, g, d)\}^{(15)}.
\end{aligned}
\tag{4.34}
$$

4.2. Some modular forms and Witten genus over $\Gamma^0(2), \Gamma_0(2), \Gamma_\theta$ in spin^c manifolds

Let M be closed oriented spin^c manifold and L be the complex line bundle associated to the given spin^c structure on M. Denote by $c = c_1(L)$ the first Chern class of L. Also, we use $L_{\mathbf{R}}$ for the notation of L, when it is viewed as an oriented real plane bundle. Let $\Theta(T_{\mathbf{C}}M, L_{\mathbf{R}} \otimes \mathbf{C})$ be the virtual complex vector bundle over M defined by

$$
\begin{aligned}
\Theta(T_{\mathbf{C}}M, L_{\mathbf{R}} \otimes \mathbf{C}) &= \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbf{C}}M}) \otimes \bigotimes_{m=1}^{\infty} \wedge_{q^m}(\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}})\\
&\qquad \otimes \bigotimes_{r=1}^{\infty} \wedge_{-q^{r-\frac{1}{2}}}(\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \otimes \bigotimes_{s=1}^{\infty} \wedge_{q^{s-\frac{1}{2}}}(\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}).
\end{aligned}
$$

Let $\dim M = 4k - 1$ and $y = -\frac{\sqrt{-1}}{2\pi}c$. Set

$$
\widetilde{Q}_1(\nabla^{TM}, \nabla^L, g, d, \tau) = \left\{\widehat{A}(TM, \nabla^{TM}) \exp\left(\frac{c}{2}\right) \mathrm{ch}(\Theta(T_{\mathbf{C}}M, L_{\mathbf{R}} \otimes \mathbf{C})) \cdot \mathrm{ch}(Q_1(E), g^{Q_1(E)}, d, \tau)\right\}^{(4k-1)}. \tag{4.35}
$$

$$
\widetilde{Q}_2(\nabla^{TM}, \nabla^L, g, d, \tau) = \left\{\widehat{A}(TM, \nabla^{TM}) \exp\left(\frac{c}{2}\right) \mathrm{ch}(\Theta(T_{\mathbf{C}}M, L_{\mathbf{R}} \otimes \mathbf{C})) \cdot \mathrm{ch}(Q_2(E), g^{Q_2(E)}, d, \tau)\right\}^{(4k-1)}. \tag{4.36}
$$

$$\widetilde{Q}_3(\nabla^{TM},\nabla^L,g,d,\tau)=\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\operatorname{ch}(\Theta(T_{\mathbf{C}}M,L_{\mathbf{R}}\otimes\mathbf{C}))\\ \cdot\operatorname{ch}(Q_3(E),g^{Q_3(E)},d,\tau)\Big\}^{(4k-1)}. \tag{4.37}$$

Then

$$\widetilde{Q}_i(\nabla^{TM},\nabla^L,g,d,\tau)=\Bigg\{\Bigg(\prod_{j=1}^{2k-1}\frac{x_j\theta'(0,\tau)}{\theta(x_j,\tau)}\Bigg(\frac{\theta_1(y,\tau)}{\theta_1(0,\tau)}\frac{\theta_2(y,\tau)}{\theta_2(0,\tau)}\frac{\theta_3(y,\tau)}{\theta_3(0,\tau)}\Bigg)\Bigg)\\ \cdot\operatorname{ch}(Q_i(E),g^{Q_i(E)},d,\tau)\Bigg\}^{(4k-1)},\quad 1\le i\le 3. \tag{4.38}$$

Let $\widetilde{P}_1(\tau)=\widetilde{Q}_1(\nabla^{TM},\nabla^L,g,d,\tau)^{4k-1}$, $\widetilde{P}_2(\tau)=\widetilde{Q}_2(\nabla^{TM},\nabla^L,g,d,\tau)^{4k-1}$. By (2.13)–(2.17) and $3p_1(L)-p_1(M)=0$, then $\widetilde{P}_1(\tau)$ is a modular form of weight $2k$ over $\Gamma_0(2)$, where $\widetilde{P}_2(\tau)$ is a modular form of weight $2k$ over $\Gamma^0(2)$. Moreover, the following identity holds:

$$\widetilde{P}_1\Big(-\frac{1}{\tau}\Big)=2^{\frac{N}{2}}\tau^{2k}\widetilde{P}_2(\tau). \tag{4.39}$$

We can show by direct computations that

$$\widetilde{Q}_1(\nabla^{TM},\nabla^L,g,d,\tau)=\Big[\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\operatorname{ch}(\triangle(E),g^{\triangle(E)},d)\Big]^{(4k-1)}\\ +q\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\\ \cdot\big[\operatorname{ch}(\widetilde{T_{\mathbf{C}}M}+2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\ +\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\operatorname{ch}(\triangle(E),g^{\triangle(E)},d)\\ +\operatorname{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)\big]\Big\}^{(4k-1)}\\ +q^2\Big\{\widehat{A}(TM,\nabla^{TM})\cdot\exp\Big(\frac{c}{2}\Big)\\ \cdot\big[\operatorname{ch}(S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}+(2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\ \otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\ \otimes\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+2\wedge^4\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\ \otimes\wedge^3\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\ -\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\ +\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\operatorname{ch}(\triangle(E),g^{\triangle(E)},d)\\ +\operatorname{ch}(\widetilde{T_{\mathbf{C}}M}+2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})$$

$$\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\,\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)$$
$$+\,\mathrm{ch}(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)]\Big\}^{(4k-1)}+\cdots, \tag{4.40}$$

$$\begin{aligned}\widetilde{Q}_2(\nabla^{TM},\nabla^L,g,d,\tau)=&-q^{\frac12}\left[\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\widetilde{E_{\mathbf{C}}},g,d)\right]^{(4k-1)}\\&+q\left[\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\wedge^2\widetilde{E_{\mathbf{C}}},g,d)\right]^{(4k-1)}\\&-q^{\frac32}\Big\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\\&\quad\cdot[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}+2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\&\qquad+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\,\mathrm{ch}(\widetilde{E_{\mathbf{C}}},g,d)\\&\qquad+\mathrm{ch}(\widetilde{E_{\mathbf{C}}}+\wedge^3\widetilde{E_{\mathbf{C}}},g,d)]\Big\}^{(4k-1)}\\&+q^2\Big\{\widehat{A}(TM,\nabla^{TM})\cdot\exp\left(\frac{c}{2}\right)\\&\quad\cdot\big[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}+2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\&\qquad+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\,\mathrm{ch}(\wedge^2\widetilde{E_{\mathbf{C}}},g,d)\\&\qquad+\mathrm{ch}(\wedge^4\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}},g,d)\big]\Big\}^{(4k-1)}+\cdots.\end{aligned} \tag{4.41}$$

Playing the same game as in the proof of Theorem 4.1, we obtain the following theorem.

Theorem 4.14. *If $c_3(E,g,d)=0$ and $3p_1(L)-p_1(M)=0$, then*

$$\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\right\}^{(4k-1)}=2^{\frac{N}{2}+k}\sum_{s=1}^{[\frac{k}{2}]}2^{-6s}h_s, \tag{4.42}$$

where each h_s, $1\le s\le\left[\frac{k}{2}\right]$, is a canonical integral linear combination of

$$\left\{\widehat{A}(TM)\exp\left(\frac{c}{2}\right)\widetilde{\mathrm{ch}}(B_\alpha(T_{\mathbf{C}}M,L_{\boldsymbol{R}}\otimes\mathbf{C},E_{\mathbf{C}}))\right\}^{(4k-1)},\quad 0\le\alpha\le s.$$

Corollary 4.15. *Let M be a $(4k-1)$-dimensional* spin^c *manifold. If $c_3(E,g,d)=0$ and $3p_1(L)-p_1(M)=0$, then*

$$\mathrm{Ind}(T^c\otimes(\triangle(E),g^{\triangle(E)},d))=-2^{\frac{N}{2}+k}\sum_{s=0}^{[\frac{k}{2}]}2^{-6s}h_s, \tag{4.43}$$

where each h_s, $1 \le s \le \left[\frac{k}{2}\right]$, *is a canonical integral linear combination of*

$$\mathrm{Ind}(T^c \otimes (\widetilde{B}_\alpha(T_{\mathbf{C}}M, L_{\boldsymbol{R}} \otimes \mathbf{C}, E_{\mathbf{C}}))), \tag{4.44}$$

and T^c *is the* spin^c *Toeplitz operator.*

Corollary 4.16. *Let* M *be an* 11*-dimensional* spin^c *manifold. If* $c_3(E, g, d) = 0$ *and* $3p_1(L) - p_1(M) = 0$*, then*

$$\begin{aligned}&\left\{\widehat{A}(TM, \nabla^{TM}) \exp\left(\frac{c}{2}\right) \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d)\right\}^{(11)}\\ &\quad = 2^{\frac{N}{2}-3}\left\{\widehat{A}(TM, \nabla^{TM}) \exp\left(\frac{c}{2}\right) \mathrm{ch}(\widetilde{E_{\mathbf{C}}}, g, d)\right\}^{(11)}.\end{aligned} \tag{4.45}$$

Corollary 4.17. *Let* M *be a* 15*-dimensional* spin^c *manifold. If* $c_3(E, g, d) = 0$ *and* $3p_1(L) - p_1(M) = 0$*, then*

$$\begin{aligned}&\left\{\widehat{A}(TM, \nabla^{TM}) \exp\left(\frac{c}{2}\right) \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d)\right\}^{(15)}\\ &\quad = -13 \times 2^{\frac{N}{2}-5}\left\{\widehat{A}(TM, \nabla^{TM}) \exp\left(\frac{c}{2}\right) \mathrm{ch}(\widetilde{E_{\mathbf{C}}}, g, d)\right\}^{(15)}\\ &\qquad + 2^{\frac{N}{2}-8}\left\{\widehat{A}(TM, \nabla^{TM}) \exp\left(\frac{c}{2}\right) \mathrm{ch}(\wedge^2\widetilde{E_{\mathbf{C}}}, g, d)\right\}^{(15)}.\end{aligned} \tag{4.46}$$

By comparing the coefficients of q, q^2 in (4.59), we obtain the following theorem.

Theorem 4.18. *If* $c_3(E, g, d) = 0$ *and* $3p_1(L) - p_1(M) = 0$*, then*

$$\begin{aligned}&\Bigg\{\widehat{A}(TM, \nabla^{TM}) \exp\left(\frac{c}{2}\right)[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M} + 2\wedge^2 \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}} - (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) \otimes (\widetilde{L_{\mathbf{R}} \otimes \mathbf{C}}) + \widetilde{L_{\mathbf{R}} \otimes \mathbf{C}})\\ &\qquad \cdot \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d) + \mathrm{ch}(\triangle(E) \otimes \widetilde{E_{\mathbf{C}}}, g, d)]\\ &\quad - 24k\left[\widehat{A}(TM, \nabla^{TM}) \exp\left(\frac{c}{2}\right) \mathrm{ch}(\triangle(E), g^{\triangle(E)}, d)\right]\Bigg\}^{(4k-1)}\\ &= -2^{\frac{N}{2}+k+6}\sum_{s=1}^{[\frac{k}{2}]} s 2^{-6s} h_s.\end{aligned} \tag{4.47}$$

Corollary 4.19. *Let M be a $(4k-1)$-dimensional* spin^c *manifold. If $c_3(E,g,d)=0$ and $3p_1(L)-p_1(M)=0$, then*

$$\begin{aligned}&\mathrm{Ind}(T^c\otimes((\widetilde{T_{\mathbf{C}}M}+2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\&\qquad\otimes(\triangle(E),g^{\triangle(E)},d)+(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)))\\&\qquad-24k\,\mathrm{Ind}(T^c\otimes(\triangle(E),g^{\triangle(E)},d))\\&=2^{\frac{N}{2}+k+6}\sum_{s=1}^{[\frac{k}{2}]}s2^{-6s}h_s.\end{aligned}\tag{4.48}$$

Corollary 4.20. *Let M be an 11-dimensional* spin^c *manifold. If $c_3(E,g,d)=0$ and $3p_1(L)-p_1(M)=0$, then*

$$\begin{aligned}&\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}+2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\&\qquad\cdot\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)+\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)]\\&\qquad-72\Big[\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\Big]\Big\}^{(11)}\\&=-2^{\frac{N}{2}+3}\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\mathrm{ch}(\widetilde{E_{\mathbf{C}}},g,d)\Big\}^{(11)}.\end{aligned}\tag{4.49}$$

Corollary 4.21. *Let M be a 15-dimensional* spin^c *manifold. If $c_3(E,g,d)=0$ and $3p_1(L)-p_1(M)=0$, then*

$$\begin{aligned}&\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}+2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\&\qquad\cdot\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)+\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)]\\&\qquad-96\Big[\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\Big]\Big\}^{(15)}\\&=9\times2^{\frac{N}{2}+2}\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\mathrm{ch}(\widetilde{E_{\mathbf{C}}},g,d)\Big\}^{(15)}\\&\qquad-2^{\frac{N}{2}-1}\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\mathrm{ch}(\wedge^2\widetilde{E_{\mathbf{C}}},g,d)\Big\}^{(15)}.\end{aligned}\tag{4.50}$$

Theorem 4.22. *If $c_3(E,g,d)=0$ and $3p_1(L)-p_1(M)=0$, then*

$$\begin{aligned}&\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)[\mathrm{ch}(S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}+(2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\&\qquad+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\&\qquad\otimes\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+2\wedge^4\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\wedge^3\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + 2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}} - \widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\
&\qquad\qquad + \widetilde{L_{\mathbf{R}}\otimes\mathbf{C}} + \wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\,\mathrm{ch}(\triangle(E), g^{\triangle(E)}, d)\\
&\qquad\quad + \mathrm{ch}(\widetilde{T_{\mathbf{C}}M} + 2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}} - (\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\
&\qquad\qquad + \widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\,\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}}, g, d)\\
&\qquad\quad + \mathrm{ch}(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)]\\
&+ (288k^2 + 72k)\cdot\left[\widehat{A}(TM, \nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\triangle(E), g^{\triangle(E)}, d)\right] - (24k-8)\\
&\cdot\left\{\widehat{A}(TM, \nabla^{TM})\cdot\exp\left(\frac{c}{2}\right)[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M} + 2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}} - (\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\right.\\
&\qquad\qquad\qquad + \widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\,\mathrm{ch}(\triangle(E), g^{\triangle(E)}, d)\\
&\qquad\qquad\qquad \left.\left. + \mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}}, g, d)]\right\}\right\}^{(4k-1)}\\
&= 2^{\frac{N}{2}+k+11}\sum_{s=1}^{[\frac{k}{2}]} s^2 2^{-6s} h_s. \qquad (4.51)
\end{aligned}
$$

Corollary 4.23. *Let M be an $(4k-1)$-dimensional* spin^c *manifold. If $c_3(E, g, d) = 0$ and $3p_1(L) - p_1(M) = 0$, then*

$$
\begin{aligned}
&\mathrm{Ind}(T^c\otimes((S^2\widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} + (2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}} - (\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}) + \widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\
&\qquad\otimes\widetilde{T_{\mathbf{C}}M} + \wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}} + 2\wedge^4\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}} - 2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\
&\qquad\otimes\wedge^3\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}} + 2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}} - \widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\
&\qquad + \widetilde{L_{\mathbf{R}}\otimes\mathbf{C}} + \wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\triangle(E), g^{\triangle(E)}, d)\\
&\qquad + (\widetilde{T_{\mathbf{C}}M} + 2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}} - (\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}) + \widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\
&\qquad\otimes(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}}, g, d) + (\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}), g, d)))\\
&\qquad + (288k^2 + 72k)\,\mathrm{Ind}(T^c\otimes(\triangle(E), g^{\triangle(E)}, d)) - (24k-8)\\
&\qquad\cdot\mathrm{Ind}(T^c\otimes((\widetilde{T_{\mathbf{C}}M} + 2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}} - (\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}) + \widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\
&\qquad\qquad\qquad\otimes(\triangle(E), g^{\triangle(E)}, d) + (\triangle(E)\otimes\widetilde{E_{\mathbf{C}}}, g, d)))\\
&= -2^{\frac{N}{2}+k+11}\sum_{s=1}^{[\frac{k}{2}]} s^2 2^{-6s} h_s. \qquad (4.52)
\end{aligned}
$$

Corollary 4.24. *Let M be an 11-dimensional* spin^c *manifold. If $c_3(E,g,d)=0$ and $3p_1(L)-p_1(M)=0$, then*

$$
\begin{aligned}
&\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)[\mathrm{ch}(S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}+(2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\right.\\
&\qquad\qquad+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\
&\qquad\qquad\otimes\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+2\wedge^4\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\
&\qquad\qquad\otimes\wedge^3\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\
&\qquad\qquad\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\cdot\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\
&\qquad\quad+\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}+2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\
&\qquad\qquad+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\,\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)\\
&\qquad\quad+\mathrm{ch}(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)]\\
&\quad+2808\left[\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\cdot\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\right]-64\\
&\quad\cdot\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}+2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\right.\\
&\qquad\qquad+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\cdot\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\
&\qquad\qquad\left.\left.+\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)]\right\}\right\}^{(11)}\\
&=2^{\frac{N}{2}+8}\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\widetilde{E_{\mathbf{C}}},g,d)\right\}^{(11)}.
\end{aligned}
\tag{4.53}
$$

Corollary 4.25. *Let M be a 15-dimensional* spin^c *manifold. If $c_3(E,g,d)=0$ and $3p_1(L)-p_1(M)=0$, then*

$$
\begin{aligned}
&\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)[\mathrm{ch}(S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}+(2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\right.\\
&\qquad\qquad+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\
&\qquad\qquad\otimes\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+2\wedge^4\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\wedge^3\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\
&\qquad\qquad+2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\
&\qquad\qquad+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\
&\qquad\quad+\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}+2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\
&\qquad\qquad+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\cdot\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)\\
&\qquad\quad+\mathrm{ch}(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)]\\
&\quad+4896[\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\cdot\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)]-88
\end{aligned}
$$

$$\cdot\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}+2\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\right.$$
$$+\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\cdot\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)$$
$$\left.+\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)]\Big\}\right\}^{(15)}$$
$$=-7\times 2^{\frac{N}{2}+8}\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\widetilde{E_{\mathbf{C}}},g,d)\right\}^{(15)}$$
$$+2^{\frac{N}{2}+5}\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\wedge^2\widetilde{E_{\mathbf{C}}},g,d)\right\}^{(15)}. \tag{4.54}$$

Let $\dim M=4k+1$ and $y=-\frac{\sqrt{-1}}{2\pi}c$. Set

$$\bar{Q}_1(\nabla^{TM},\nabla^L,g,d,\tau)=\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\Theta^*(T_{\mathbf{C}}M,L_{\mathbf{R}}\otimes\mathbf{C}))\right.$$
$$\left.\cdot\,\mathrm{ch}(Q_1(E),g^{Q_1(E)},d,\tau)\right\}^{(4k+1)}, \tag{4.55}$$
$$\bar{Q}_2(\nabla^{TM},\nabla^L,g,d,\tau)=\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\Theta^*(T_{\mathbf{C}}M,L_{\mathbf{R}}\otimes\mathbf{C}))\right.$$
$$\left.\cdot\,\mathrm{ch}(Q_2(E),g^{Q_2(E)},d,\tau)\right\}^{4k+1}, \tag{4.56}$$
$$\bar{Q}_3(\nabla^{TM},\nabla^L,g,d,\tau)=\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\Theta^*(T_{\mathbf{C}}M,L_{\mathbf{R}}\otimes\mathbf{C}))\right.$$
$$\left.\cdot\,\mathrm{ch}(Q_3(E),g^{Q_3(E)},d,\tau)\right\}^{(4k+1)}, \tag{4.57}$$

where

$$\Theta^*(T_{\mathbf{C}}M,L_{\mathbf{R}}\otimes\mathbf{C})=\bigotimes_{n=1}^{\infty}S_{q^n}(\widetilde{T_{\mathbf{C}}M})\otimes\bigotimes_{m=1}^{\infty}\wedge_{-q^m}(\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})$$

is the virtual complex vector bundle over M.
Then

$$\bar{Q}_i(\nabla^{TM},\nabla^L,g,d,\tau)=\left\{\left\{\left(\prod_{j=1}^{2k+1}\frac{x_j\theta'(0,\tau)}{\theta(x_j,\tau)}\right)\frac{\sqrt{-1}\theta(y,\tau)}{\theta_1(0,\tau)\theta_2(0,\tau)\theta_3(0,\tau)}\right\}\right.$$
$$\left.\cdot\,\mathrm{ch}(Q_i(E),g^{Q(E)},d,\tau)\right\}^{(4k+1)},\quad 1\le i\le 3. \tag{4.58}$$

Let $\bar{P}_1(\tau)=\bar{Q}_i(\nabla^{TM},\nabla^L,g,d,\tau)^{4k+1}$, $\bar{P}_2(\tau)=\bar{Q}_i(\nabla^{TM},\nabla^L,g,d,\tau)^{4k+1}$. By (2.13)–(2.17) and $p_1(L)-p_1(M)=0$, then $\bar{P}_1(\tau)$ is a modular form of weight $2k$ over $\Gamma_0(2)$, where $\overline{P}_2(\tau)$ is a modular form of weight $2k$ over $\Gamma^0(2)$. Moreover, the following identity holds:

$$\overline{P}_1\left(-\frac{1}{\tau}\right) = 2^{\frac{N}{2}}\tau^{2k}\,\overline{P}_2(\tau). \tag{4.59}$$

We can show by direct computations that

$$\begin{aligned}
&\bar{Q}_1(\nabla^{TM}, \nabla^L, g, d, \tau)\\
&\quad= \left[\widehat{A}(TM, \nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\triangle(E), g^{\triangle(E)}, d)\right]^{(4k+1)}\\
&\qquad + q\Big\{\widehat{A}(TM, \nabla^{TM})\exp\left(\frac{c}{2}\right)[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M} - \widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\cdot\mathrm{ch}(\triangle(E), g^{\triangle(E)}, d)\\
&\qquad\qquad + \mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}}, g, d)]\Big\}^{(4k+1)}\\
&\qquad + q^2\Big\{\widehat{A}(TM, \nabla^{TM})\exp\left(\frac{c}{2}\right)[\mathrm{ch}(S^2\widetilde{T_{\mathbf{C}}M} + \widetilde{T_{\mathbf{C}}M} + \wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}} - \widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\\
&\qquad\qquad + \widetilde{T_{\mathbf{C}}M}\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\cdot\mathrm{ch}(\triangle(E), g^{\triangle(E)}, d)\\
&\qquad\qquad + \mathrm{ch}(\widetilde{T_{\mathbf{C}}M} - \widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}\cdot\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}}, g, d)\\
&\qquad\qquad + \mathrm{ch}(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}\\
&\qquad\qquad + \widetilde{E_{\mathbf{C}}}), g, d))]\Big\}^{(4k+1)} + \cdots,
\end{aligned} \tag{4.60}$$

$$\begin{aligned}
&\bar{Q}_2(\nabla^{TM}, \nabla^L, g, d, \tau)\\
&\quad= -q^{\frac{1}{2}}\left[\widehat{A}(TM, \nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\widetilde{E_{\mathbf{C}}}, g, d)\right]^{(4k+1)}\\
&\qquad + q\left[\widehat{A}(TM, \nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\wedge^2\widetilde{E_{\mathbf{C}}}, g, d)\right]^{(4k+1)}\\
&\qquad - q^{\frac{3}{2}}\Big\{\widehat{A}(TM, \nabla^{TM})\exp\left(\frac{c}{2}\right)[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M} - \widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\cdot\mathrm{ch}(\widetilde{E_{\mathbf{C}}}, g, d)\\
&\qquad\qquad + \mathrm{ch}(\widetilde{E_{\mathbf{C}}} + \wedge^3\widetilde{E_{\mathbf{C}}}, g, d)]\Big\}^{(4k+1)}\\
&\qquad + q^2\Big\{\widehat{A}(TM, \nabla^{TM})\exp\left(\frac{c}{2}\right)[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M} - \widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\cdot\mathrm{ch}(\wedge^2\widetilde{E_{\mathbf{C}}}, g, d)\\
&\qquad\qquad + \mathrm{ch}(\wedge^4\widetilde{E_{\mathbf{C}}} + \widetilde{E_{\mathbf{C}}}\otimes\widetilde{E_{\mathbf{C}}}, g, d)]\Big\}^{(4k+1)} + \cdots.
\end{aligned} \tag{4.61}$$

Playing the same game as in the proof of Theorem 4.1, we obtain the following theorem.

Theorem 4.26. *If $c_3(E,g,d)=0$ and $p_1(L)-p_1(M)=0$, then*

$$\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\right\}^{(4k+1)}=2^{\frac{N}{2}+k}\sum_{s=1}^{[\frac{k}{2}]}2^{-6s}h_s,\tag{4.62}$$

where each h_s, $1\le s\le\left[\frac{k}{2}\right]$, is a canonical integral linear combination of

$$\left\{\widehat{A}(TM)\exp\left(\frac{c}{2}\right)\widetilde{\mathrm{ch}}(B_\alpha(T_{\mathbf{C}}M,L_{\boldsymbol{R}}\otimes\mathbf{C},E_{\mathbf{C}}))\right\}^{(4k+1)},\quad 0\le\alpha\le s.$$

Corollary 4.27. *Let M be a $(4k+1)$-dimensional* spin^c *manifold. If $c_3(E,g,d)=0$ and $p_1(L)-p_1(M)=0$, then*

$$\mathrm{Ind}(T^c\otimes(\triangle(E),g^{\triangle(E)},d))=-2^{\frac{N}{2}+k}\sum_{s=0}^{[\frac{k}{2}]}2^{-6s}h_s,\tag{4.63}$$

where each h_s, $1\le s\le\left[\frac{k}{2}\right]$, is a canonical integral linear combination of

$$\mathrm{Ind}(T^c\otimes(\overline{B}_\alpha(T_{\mathbf{C}}M,L_{\boldsymbol{R}}\otimes\mathbf{C},E_{\mathbf{C}}))).\tag{4.64}$$

Corollary 4.28. *Let M be a 13-dimensional* spin^c *manifold. If $c_3(E,g,d)=0$ and $p_1(L)-p_1(M)=0$, then*

$$\begin{aligned}&\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\right\}^{(13)}\\&\quad=2^{\frac{N}{2}-3}\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\widetilde{E_{\mathbf{C}}},g,d)\right\}^{(13)}.\end{aligned}\tag{4.65}$$

Corollary 4.29. *Let M be a 17-dimensional* spin^c *manifold. If $c_3(E,g,d)=0$ and $p_1(E)+p_1(L)-p_1(M)=0$, then*

$$\begin{aligned}&\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\right\}^{(17)}\\&\quad=-13\times2^{\frac{N}{2}-5}\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\widetilde{E_{\mathbf{C}}},g,d)\right\}^{(17)}\\&\qquad+2^{\frac{N}{2}-8}\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\wedge^2\widetilde{E_{\mathbf{C}}},g,d)\right\}^{(17)}.\end{aligned}\tag{4.66}$$

By comparing the coefficients of q, q^2 in (4.59), we obtain the following theorem.

Theorem 4.30. *If* $c_3(E,g,d)=0$ *and* $p_1(L)-p_1(M)=0$*, then*

$$\begin{aligned}\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)&[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\&+\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)]\\&-24k\Big[\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\Big]\Big\}^{(4k+1)}\\&=-2^{\frac{N}{2}+k+6}\sum_{s=1}^{[\frac{k}{2}]}s2^{-6s}h_s.\end{aligned}\tag{4.67}$$

Corollary 4.31. *Let* M *be a* $(4k+1)$*-dimensional* spin^c *manifold. If* $c_3(E,g,d)=0$ *and* $p_1(L)-p_1(M)=0$*, then*

$$\begin{aligned}&\mathrm{Ind}(T^c\otimes((\widetilde{T_{\mathbf{C}}M}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\triangle(E),g^{\triangle(E)},d)+(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)))\\&\qquad-24k\,\mathrm{Ind}(T^c\otimes(\triangle(E),g^{\triangle(E)},d))\\&=2^{\frac{N}{2}+k+6}\sum_{s=1}^{[\frac{k}{2}]}s2^{-6s}h_s.\end{aligned}\tag{4.68}$$

Corollary 4.32. *Let* M *be a* 13*-dimensional* spin^c *manifold. If* $c_3(E,g,d)=0$ *and* $p_1(L)-p_1(M)=0$*, then*

$$\begin{aligned}\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)&[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\&+\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)]\\&-72\Big[\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\Big]\Big\}^{(13)}\\&=-2^{\frac{N}{2}+3}\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\,\mathrm{ch}(\widetilde{E_{\mathbf{C}}},g,d)\Big\}^{(13)}.\end{aligned}\tag{4.69}$$

Corollary 4.33. *Let* M *be a* 17*-dimensional* spin^c *manifold. If* $c_3(E,g,d)=0$ *and* $p_1(L)-p_1(M)=0$*, then*

$$\begin{aligned}\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)&[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\&+\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)]\\&-96\Big[\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\Big]\Big\}^{(17)}\\&=9\times2^{\frac{N}{2}+2}\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\,\mathrm{ch}(\widetilde{E_{\mathbf{C}}},g,d)\Big\}^{(17)}\\&\quad-2^{\frac{N}{2}-1}\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\,\mathrm{ch}(\wedge^2\widetilde{E_{\mathbf{C}}},g,d)\Big\}^{(17)}.\end{aligned}\tag{4.70}$$

Theorem 4.34. *If* $c_3(E,g,d)=0$ *and* $p_1(L)-p_1(M)=0$, *then*

$$\begin{aligned}\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)&[\mathrm{ch}((S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\widetilde{T_{\mathbf{C}}M}\\ &\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)+\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\ &\cdot\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)+\mathrm{ch}(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d))]\\ &+(288k^2+72k)\Big[\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\Big]-(24k-8)\\ &\cdot\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\cdot\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\\ &+\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)]\Big\}\Big\}^{(4k-1)}\\ &=2^{\frac{N}{2}+k+11}\sum_{s=1}^{[\frac{k}{2}]}s^2 2^{-6s}h_s.\end{aligned}\tag{4.71}$$

Corollary 4.35. *Let* M *be a* $(4k+1)$*-dimensional* spin^c *manifold. If* $c_3(E,g,d)=0$ *and* $p_1(L)-p_1(M)=0$, *then*

$$\begin{aligned}&\mathrm{Ind}(T^c\otimes((S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\widetilde{T_{\mathbf{C}}M}\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\ &\quad\otimes(\triangle(E),g^{\triangle(E)},d)+(\widetilde{T_{\mathbf{C}}M}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)\\ &\quad+(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d)))\\ &\quad+(288k^2+72k)\,\mathrm{Ind}(T^c\otimes(\triangle(E),g^{\triangle(E)},d))\\ &\quad-(24k-8)\,\mathrm{Ind}(T^c\otimes((\widetilde{T_{\mathbf{C}}M}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\otimes(\triangle(E),g^{\triangle(E)},d)\\ &\qquad+(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)))\\ &=-2^{\frac{N}{2}+k+11}\sum_{s=1}^{[\frac{k}{2}]}s^2 2^{-6s}h_s.\end{aligned}\tag{4.72}$$

Corollary 4.36. *Let* M *be a* 13*-dimensional* spin^c *manifold. If* $c_3(E,g,d)=0$ *and* $p_1(L)-p_1(M)=0$, *then*

$$\begin{aligned}\Big\{\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)&[\mathrm{ch}((S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\widetilde{T_{\mathbf{C}}M}\\ &\otimes\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)+\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\\ &\cdot\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)+\mathrm{ch}(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d))]\\ &+2808\Big[\widehat{A}(TM,\nabla^{TM})\exp\Big(\frac{c}{2}\Big)\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\Big]\end{aligned}$$

$$- 64\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\cdot\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\right.$$
$$\left.+\,\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)]\right\}\Bigg\}^{(13)}$$
$$= 2^{\frac{N}{2}+8}\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\widetilde{E_{\mathbf{C}}},g,d)\right\}^{(13)}. \tag{4.73}$$

Corollary 4.37. *Let M be a 17-dimensional* spin^c *manifold. If $c_3(E,g,d)=0$ and $p_1(L)-p_1(M)=0$, then*

$$\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)[\mathrm{ch}((S^2\widetilde{T_{\mathbf{C}}M}+\widetilde{T_{\mathbf{C}}M}+\wedge^2\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}}+\widetilde{T_{\mathbf{C}}M}\right.$$
$$\otimes\,\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\,\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)+\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})$$
$$\cdot\,\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)+\mathrm{ch}(\triangle(E)\otimes(\wedge^2\widetilde{E_{\mathbf{C}}}+\widetilde{E_{\mathbf{C}}}),g,d))]$$
$$+\,4896\left[\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\right]$$
$$- 88\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)[\mathrm{ch}(\widetilde{T_{\mathbf{C}}M}-\widetilde{L_{\mathbf{R}}\otimes\mathbf{C}})\cdot\mathrm{ch}(\triangle(E),g^{\triangle(E)},d)\right.$$
$$\left.+\,\mathrm{ch}(\triangle(E)\otimes\widetilde{E_{\mathbf{C}}},g,d)]\right\}\Bigg\}^{(17)}$$
$$= -7\times 2^{\frac{N}{2}+8}\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\widetilde{E_{\mathbf{C}}},g,d)\right\}^{(17)}$$
$$+\,2^{\frac{N}{2}+5}\left\{\widehat{A}(TM,\nabla^{TM})\exp\left(\frac{c}{2}\right)\mathrm{ch}(\wedge^2\widetilde{E_{\mathbf{C}}},g,d)\right\}^{(17)}. \tag{4.74}$$

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