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Invariants and automorphisms for slice regular functions

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Abstract. Let A be one of the following Clifford algebras: $\mathbb{R}_2 \cong \mathbb{H}$ or $\mathbb{R}_3$. For the algebra A, the automorphism group $\mathrm{Aut}(A)$ and its invariants are well known. In this paper, we will describe the invariants of the automorphism group of the algebra of slice regular functions over A.

1. Introduction

The theory of slice regular functions was introduced by Gentili and Struppa in two seminal papers in 2006 [20] and in [21]: they used the fact that $\forall I \in \mathbb{S}_{\mathbb{H}} = \{J \in \mathbb{H} \mid J^2 = -1\}$ the real subalgebra $\mathbb{C}_I$ generated by 1 and I is isomorphic to $\mathbb{C}$ and they decomposed the algebra $\mathbb{H}$ into a "book-structure" via these complex "slices":

$$\mathbb{H} = \bigcup_{I \in \mathbb{S}_{\mathbb{H}}} \mathbb{C}_I.$$

On an open set $\Omega \subset \mathbb{H}$, they defined a differentiable function $f: \Omega \to \mathbb{H}$ to be (Cullen or) slice regular if, for each $I \in \mathbb{S}$, the restriction of f to $\Omega_I = \Omega \cap \mathbb{C}_I$ is a holomorphic function from Ω_I to $\mathbb{H}$, both endowed with the complex structure defined by left multiplication with I. This definition covers all functions given by convergent power series of the form

$$\sum_{n \in \mathbb{N}_0} q^n a_n$$

with $\{a_n\}_{n \in \mathbb{N}_0} \subset \mathbb{H}$.

Later on, the approach introduced by Ghiloni and Perotti in 2011 [22], for an alternative $*$-algebra A over $\mathbb{R}$, makes use of the complexified algebra $A \otimes_{\mathbb{R}} \mathbb{C}$ denoted by $A_{\mathbb{C}}$.

Let us denote its elements as $a + \iota b$, where $a, b \in A$ and ι is to be considered as the imaginary unit of $\mathbb{C}$.

For any slice regular function, and for any $I \in \mathbb{S}_{\mathbb{H}}$, the restriction $f: \mathbb{C}_I \to \mathbb{H}$ can be lifted through the map $\phi_I: \mathbb{H}_{\mathbb{C}} \to \mathbb{H}$, $\phi_I(a + \iota b) := a + Ib$ and it turns out that the lift does not depend on I. In other words, there exists a holomorphic function $F: \mathbb{C} \cong \mathbb{R}_{\mathbb{C}} \to \mathbb{H}_{\mathbb{C}}$

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which makes the following diagram commutative for all $I \in \mathbb{S}_{\mathbb{H}}$:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F} & \mathbb{H}_{\mathbb{C}} \\ {\scriptstyle \phi_I}\downarrow & & \downarrow{\scriptstyle \phi_I} \\ \mathbb{H} & \xrightarrow{f} & \mathbb{H} \end{array}$$

After the first definitions were given, the theory of slice regular functions knew a big development, see, among the others, the following references [1, 4–13, 19].

Non-commutative associative division algebras admit many automorphisms, because $x \mapsto y^{-1}xy$ is an automorphism for every invertible element y.

The "essential" properties of a number or a function should not be changed by automorphisms.

In the case of the algebras under consideration here, there is an *antiinvolution* $x \mapsto \bar{x}$ which commutes with all automorphisms. As a consequence, $\mathrm{N}(x) = x\bar{x}$ and $\mathrm{Tr}(x) = x + \bar{x}$ are *invariant* under automorphisms. In fact, for $A = \mathbb{H}$, we have the equivalence

$$\mathrm{N}(z) = \mathrm{N}(w) \quad \text{and} \quad \mathrm{Tr}(z) = \mathrm{Tr}(w) \iff \exists\, \phi \in \mathrm{Aut}(A) : \phi(z) = w.$$

(Here, ϕ is an automorphism of A as an $\mathbb{R}$-algebra.)

This raises the question whether a similar correspondence holds not only for the elements in the algebra, but also for slice-regular functions of this algebra.

As it turns out, essentially this is true, but only via the associated stem functions and up to a condition on the multiplicity with which values in the center of $A_{\mathbb{C}}$ are assumed. To state the latter condition, in Section 6.2, we introduce the notion of a "*central divisor*" cdiv.

More precisely, the conjugation $x \mapsto \bar{x}$ on the algebra A defines a conjugation on the space of slice-regular functions which allows the definition of Tr and N as before.

This conjugation corresponds to a conjugation on the associated space of stem functions $F : \mathbb{C} \to A \otimes_{\mathbb{R}} \mathbb{C}$ defined as

$$F^c : z \mapsto \overline{(F(z))},$$

where $\overline{(q \otimes w)}$ is defined as $\bar{q} \otimes (w)$ (with $q \in A$, $w \in \mathbb{C}$). Again, conjugation induces Tr and N as

$$\mathrm{Tr}(F) : z \mapsto (F + F^c)(z), \quad \mathrm{N}(F) : z \mapsto (F(z))(F^c(z)).$$

With these definitions, the correspondence between regular functions and stem functions is compatible with conjugation. As a consequence, if F is the stem function for f, then F^c is the stem function for f^c. Moreover, $\mathrm{N}(F)$, respectively, $\mathrm{Tr}(F)$, are the stem functions for $\mathrm{N}(f)$, respectively, $\mathrm{Tr}(f)$.

As it turns out, essentially, $\mathrm{N}(F)$, $\mathrm{Tr}(F)$, and $\mathrm{cdiv}(F)$ (or, equivalently, $\mathrm{N}(f)$, $\mathrm{Tr}(f)$, and $\mathrm{cdiv}(f)$) characterize F (equivalently, f) up to replacing F with $z \mapsto \phi(z)(F(z))$ for some holomorphic map ϕ from $\mathbb{C}$ to the automorphism group of the complex algebra $A \otimes_{\mathbb{R}} \mathbb{C}$.

Here, cdiv is an additional invariant which we introduce in Section 6.2 for slice regular functions which are not slice preserving.

In this paper, we describe the group of automorphisms of the algebra of slice regular functions with values in $\mathbb{H}$ and $\mathbb{R}_3 \cong \mathbb{H} \oplus \mathbb{H}$.[1]

Our main theorem is the following.

Theorem 1.1. *Let $\mathbb{H}$ denote the algebra of quaternions, $\mathbb{H}_{\mathbb{C}} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$, $G = \mathrm{Aut}(\mathbb{H}) \cong \mathrm{SO}(3, \mathbb{R})$, $G_{\mathbb{C}} = \mathrm{Aut}(\mathbb{H}_{\mathbb{C}}) \cong \mathrm{SO}(3, \mathbb{C})$. Let $D \subset \mathbb{C}$ be a symmetric domain and let $\Omega_D \subset \mathbb{H}$ denote the corresponding axially symmetric domain.*

Let $f, h : \Omega_D \to \mathbb{H}$ be slice regular functions and let $F, H : D \to \mathbb{H}_{\mathbb{C}}$ denote the corresponding stem functions.

(a) *Assume that neither f nor h are slice preserving. Then, the following are equivalent:*

(i) *f and h have the same invariants* cdiv, Tr, N,

(ii) *F and H have the same invariants* cdiv, Tr, N,

(iii) $\mathrm{cdiv}(F) = \mathrm{cdiv}(H)$ *and for every $z \in D$, there exists $\alpha \in \mathrm{Aut}(\mathbb{H}_{\mathbb{C}}) = G_{\mathbb{C}}$ such that $F(z) = \alpha(H(z))$,*

(iv) *there is a holomorphic map $\phi : D \to G_{\mathbb{C}}$ such that*

$$F(z) = \phi(z)(H(z)) \quad \forall z \in D,$$

(v) *there is a holomorphic map $\alpha : D \to \mathbb{H}^*_{\mathbb{C}}$ such that*

$$F(z) = \alpha(z)^{-1} \cdot H(z) \cdot \alpha(z).$$

(b) *Assume that f is slice preserving. Then, the following are equivalent:*

(i) *$f = h$,*

(ii) *$F = H$,*

(iii) *for every $z \in D$, there exists an element $\alpha \in \mathrm{Aut}(\mathbb{H}_{\mathbb{C}}) = G_{\mathbb{C}}$ such that $F(z) = \alpha(H(z))$,*

(iv) *there is a holomorphic map $\phi : D \to G_{\mathbb{C}}$ such that $F(z) = \phi(z)(H(z)) \forall z \in D$,*

(v) *there is a holomorphic map $\alpha : D \to \mathbb{H}^*_{\mathbb{C}}$ such that*

$$F(z) = \alpha(z)^{-1} \cdot H(z) \cdot \alpha(z).$$

Remark. The notion of a "central divisor" is defined only if the function is not slice preserving. This is similar to the ordinary complex situation where the divisor of a holomorphic function is defined only if it is not constantly zero.

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[1] In a forthcoming paper, we will investigate other algebras as well.

Theorem 1.1 is proved in Section 18.

We also derive a corresponding result for the Clifford algebra $\mathbb{R}_3$ (which as $\mathbb{R}$-algebra is isomorphic to $\mathbb{H} \oplus \mathbb{H}$) (Theorem 8.1).

Quaternions can be used to describe orthogonal complex structures (OCS) on 4-dimensional Euclidean space, since their imaginary units parametrize automorphisms of $\mathbb{R}^4 \cong \mathbb{C}^2$. An injective slice regular function on a symmetric slice domain of $\mathbb{H}$ minus the reals defines a new OCS via the push-forward of the standard one. It would be very interesting in the authors' opinion to understand if slice regular functions that are in the same orbit, by the action of automorphisms as explained in this paper, induce the same OCS or isomorphic OCS [18].

1.1. Related work

In [2], Altavilla, de Fabritiis investigated equivalence relations for *semi-regular* functions.

These semi-regular functions are locally $*$-quotient of slice regular functions and correspond to meromorphic functions in complex analysis.

Semi-regular functions have the advantage that they are always invertible unless they are zero-divisors. This eases the use of linear algebra.

In [2], it is proved that for any two semi-regular functions f and g the following properties are equivalent.

- $\mathrm{Tr}(f) = \mathrm{Tr}(g)$ and $\mathrm{N}(f) = \mathrm{N}(g)$.
- There is a semi-regular function h with $h * f * h^{-*} = g$.
- The "Sylvester-operator" $S_{f,-g} : x \mapsto f * x - x * g$ is not invertible.

In comparison, we obtain a stronger conclusion (namely, conjugation by a *regular* function instead by a function which is only semi-regular), but for this we need an additional assumption, namely, equality not only of trace Tr and norm N, but also of the "central divisor" cdiv which we introduce in Section 6.2. See Section 4 for an instructive example.

2. Preparations

Here, we collect basic facts and notions needed for our main result. First, we discuss conjugation, norm and trace, then types of domains, then slice regular functions and stem functions, followed by investigating conjugation, norm and trace for function algebras.

2.1. Conjugation, norm and trace

Let A be an alternative $\mathbb{R}$-algebra with 1, and let $x \mapsto \bar{x}$ be an *antiinvolution*, i.e., an $\mathbb{R}$-linear map such that $\overline{xy} = (\bar{y}) \cdot (\bar{x})$ and $\overline{(\bar{x})} = x$ for all $x, y \in A$. (An $\mathbb{R}$-algebra with an antiinvolution is often called $*$-algebra.)[2]

[2]In this article, we are mainly concerned with the algebra of quaternions and $\mathbb{R}_3 \cong \mathbb{H} \oplus \mathbb{H}$, but we would also like to prepare for a future article on other $\mathbb{R}$-algebras.

Definition 2.1. Given an $\mathbb{R}$-algebra A with antiinvolution $x \mapsto \bar{x}$, we define

$$\text{Trace: } \operatorname{Tr}(x) = x + \bar{x},$$
$$\text{Norm: } \operatorname{N}(x) = x\bar{x}.$$

Consider

$$C = \{x \in A : x = \bar{x}\}.$$

We assume that C is *central* and associates with all other elements, i.e.,

$$\forall c \in C, x, y \in A : cx = xc \quad \text{and} \quad c(xy) = (cx)y.$$

It is easy to verify that C is a subalgebra (under these assumptions, i.e., if C is assumed to be central).

Lemma 2.2. *Under the above assumptions, the following properties hold:*

(1) $\forall x \in \mathbb{R} : x = \bar{x}$,

(2) $\forall x \in A : \operatorname{N}(x), \operatorname{Tr}(x) \in C = \{y \in A : y = \bar{y}\}$,

(3) $\forall x \in A : x\bar{x} = \bar{x}x$,

(4) $\forall x \in A : \operatorname{N}(x) = \operatorname{N}(\bar{x})$,

(5) $\forall x, y \in A : \operatorname{N}(xy) = \operatorname{N}(x)\operatorname{N}(y)$.

Proof. (1) Observe that

$$\bar{1} = 1 \cdot \bar{1} \implies 1 = \overline{1 \cdot \bar{1}} = 1 \cdot \bar{1} = \bar{1}.$$

By $\mathbb{R}$-linearity of the antiinvolution, this yields $\forall x \in \mathbb{R} : x = \bar{x}$.

(2) This follows from

$$\overline{(\operatorname{Tr}(x))} = \overline{x + \bar{x}} = \bar{x} + x = \operatorname{Tr}(x)$$

and

$$\overline{(\operatorname{N}(x))} = \overline{(x\bar{x})} = \overline{(\bar{x})}\bar{x} = x\bar{x} = \operatorname{N}(x).$$

(3) $x + \bar{x}$ is central; hence, $x(x + \bar{x}) = (x + \bar{x})x$ which implies $x^2 + x\bar{x} = x^2 + \bar{x}x$, and consequently, $x\bar{x} = \bar{x}x$.

(4) $\operatorname{N}(\bar{x}) = (\bar{x})\overline{(\bar{x})} = (\bar{x})x = x(\bar{x}) = \operatorname{N}(x)$.

(5) If A is associative, we argue as follows:

$$\begin{aligned} \operatorname{N}(xy) &= (xy)\overline{(xy)} = xy(\bar{y}\bar{x}) \\ &= x(y\bar{y})\bar{x} = x\bar{x}(y\bar{y}) = \operatorname{N}(x)\operatorname{N}(y). \end{aligned}$$

For the general case, we observe that $x, \bar{x}, y, \bar{y}$ are all contained in the C-algebra A_0 generated by x and y (note that $x + \bar{x}, y + \bar{y} \in C$). Artin's theorem (see [27, Theorem 3.1]) implies that A_0 is associative. Thus, all the calculations in the above sequence of equations take place *within* an associative algebra, namely, A_0 and the proof is therefore still valid, even if A itself is not associative, but only alternative. ■

2.2. Relation with notions in linear algebra and number theory

The norm and trace as considered here for the algebra of quaternions are closely related to notions in linear algebra and number theory.

The algebra $\mathbb{H}$ is a *central simple* $\mathbb{R}$-algebra with $\mathbb{H}\otimes_{\mathbb{R}}\mathbb{C} \cong \mathrm{Mat}(2\times 2, \mathbb{C})$.

In the theory of central simple algebras (see, e.g., [24, Chapter 29]), one considers *reduced traces* Trd and *reduced norms* Nrd defined as $\mathrm{Tr}(\phi(x))$, respectively, $\det(\phi(x))$ for $x \in \mathbb{H}$, where ϕ denotes the embedding of $\mathbb{H}$ into $\mathrm{Mat}(2\times 2, \mathbb{C})$ via the natural embedding $\mathbb{H} \subset \mathbb{H}\otimes_{\mathbb{R}}\mathbb{C}$ composed with an isomorphism $\mathbb{H}\otimes_{\mathbb{R}}\mathbb{C} \cong \mathrm{Mat}(2\times 2, \mathbb{C})$.

There is also a connection with notions of norm and trace in algebraic number theory.

If $A = \mathbb{C}$ and $x \mapsto x^c$ is complex conjugation, then Tr and N defined as here agree with the number-theoretical notions Tr and N for the Galois field extension $\mathbb{C}/\mathbb{R}$.

2.3. Stem functions

A *stem function* is a (usually holomorphic) function F defined on a *symmetric domain*[3] D in $\mathbb{C}$ with values in $A_{\mathbb{C}} = A\otimes_{\mathbb{R}}\mathbb{C}$ satisfying $\overline{F(z)} = F(\bar{z})$ where we use the *complex conjugation* on $A\otimes_{\mathbb{R}}\mathbb{C}$.

For a stem function F, we define $(F^c)(z) = (F(z))^c$, i.e., we apply quaternionic conjugation pointwise. Thus, we obtain a *conjugation* on the algebra of stem functions defined on a domain $D \subset \mathbb{C}$. As before, we define norm and trace and obtain

$$\mathrm{N}(F)(z) = (FF^c)(z) = (F(z))(F^c(z)) = (F(z))(F(z))^c = \mathrm{N}(F(z))$$

and

$$(\mathrm{Tr}\, F)(z) = (F + F^c)(z) = F(z) + (F(z))^c = \mathrm{Tr}(F(z)).$$

Globally defined slice regular functions are given as globally convergent power series

$$f(q) = \sum_{k=0}^{+\infty} q^k a_k.$$

In this case,

$$(f^c)(q) = \sum_{k=0}^{+\infty} q^k a_k^c.$$

The space of slice regular functions on an axially symmetric domain forms an associative $\mathbb{R}$-algebra with the $*$-product as multiplication.

Hence,

$$(\mathrm{Tr}\, f) = f + f^c \quad \text{and} \quad \mathrm{N}(f) = f * f^c.$$

Immediately from the construction, we obtain the following proposition.

[3] $D \subset \mathbb{C}$ is called *symmetric* iff $z \in D \iff \bar{z} \in D$.

Proposition 2.3. *Let f be a slice function and F its associated stem function. Then,* $\mathrm{N}(F)$*,* $\mathrm{Tr}(F)$ *and* F^c *are the stem functions associated to* $\mathrm{N}(f)$*,* $\mathrm{Tr}(f)$*, and* f^c*.*

Remark. In general, we have $f^c(q) \neq \overline{f(q)}$. Only for real points $q \in \mathbb{R}$ we have $f^c(q) = \overline{f(q)}$, and consequently, $(\mathrm{N}\, f)(q) = \mathrm{N}(f(q))$ and $(\mathrm{Tr}\, f)(q) = \mathrm{Tr}(f(q))$.

2.4. Compatibility

Recall that conjugation for slice regular functions is *not* just pointwise conjugation of the function values.

Therefore, in general,

$$(\mathrm{Tr}\, f)(q) \neq \mathrm{Tr}(f(q)), \quad (\mathrm{N}\, f)(q) \neq \mathrm{N}(f(q)).$$

Let B be a $\mathbb{R}$-sub algebra of an $\mathbb{R}$ algebra A equipped with an antiinvolution which preserves B.

Then, for $x \in B$, the notions $\mathrm{Tr}(x)$ and $\mathrm{N}(x)$ defined with respect to this antiinvolution are the same regardless whether we regard x as an element of B or as an element of A.

As a consequence, we have the following:

- for $x \in \mathbb{H}$ the notions $\mathrm{N}(x)$, $\mathrm{Tr}(x)$ agree independent of whether we consider x in $\mathbb{H}$ or in $\mathbb{H}_{\mathbb{C}}$,
- for an element $x \in \mathbb{H}$ the notions $\mathrm{N}(x)$, $\mathrm{Tr}(x)$ agree whether we regard x in $\mathbb{H}$ or as a constant slice regular function with value x.

2.5. Other notions

For $x \in \mathbb{H}$, the term $\frac{1}{2}\mathrm{Tr}(x)$ is often called *real part* of x, sometimes denoted by x_0, and $N(x)$ is also called the *symmetrization* of x and denoted by x^s.

3. An example

Warning. The conditions of the main Theorem 1.1 do *not* imply that, for any $q \in \mathbb{H}$, we can find an element $\alpha \in \mathrm{Aut}(\mathbb{H})$ such that $f(q) = \alpha(g(q))$.

Example 3.1. Let $f(q) = I$, and let $g(q) = \cos(q)I + \sin(q)J$; i.e.,

$$g(q) = \sum_{k=0}^{+\infty} \left(q^{2k} \frac{(-1)^k}{(2k)!} I + q^{2k+1} \frac{(-1)^k}{(2k+1)!} J \right),$$

where I, J are any two orthogonal imaginary units in $\mathbb{H}$ with $K = IJ$.

We recall the classical identities

$$\sin(it) = i \sinh(t), \quad \cos(it) = \cosh(t) \quad \forall t \in \mathbb{R}.$$

For $t \in \mathbb{R}$, we deduce that

$$\begin{aligned} g(tJ) &= \cos(tJ)I + \sin(tJ)J \\ &= \cosh(t)I + \sinh(t) \underbrace{J^2}_{=-1} \\ &= \cosh(t)I - \sinh(t). \end{aligned}$$

In particular, the "real part" $\frac{1}{2}\,\mathrm{Tr}(g(tJ))$ is non-zero if $\sinh(t) \neq 0$, i.e., if $t \neq 0$. Hence, for $t \in \mathbb{R}^*$, we have

$$\mathrm{Tr}(g(tJ)) \neq 0 = \mathrm{Tr}(I) = \mathrm{Tr}(f(tJ)).$$

Since $\mathrm{Tr}(\phi(q)) = \mathrm{Tr}(q)$ for every $q \in \mathbb{H}$ and every automorphism ϕ of $\mathbb{H}$, it follows that there is no automorphism of $\mathbb{H}$ mapping $g(tJ)$ to $f(tJ) = I$.

On the other hand, we have

$$\begin{aligned} g^c(q) &= -\cos(q)I - \sin(q)J \\ &\Longrightarrow \\ (g * g^c)(q) &= -(\cos(q)I + \sin(q)J) * (\cos(q)I + \sin(q)J) \\ &= -\Big(\cos^2(q) \underbrace{I^2}_{=-1} + \sin(q)^2 \underbrace{J^2}_{=-1} + \sin(q)\cos(q) \underbrace{(IJ + JI)}_{=0}\Big) \\ &= \Big(\underbrace{\cos^2(q) + \sin^2(q)}_{=1}\Big) = 1 = f * f^c \end{aligned}$$

and

$$\begin{aligned} g(q) + g^c(q) &= \underbrace{\cos(q)I + \sin(q)J}_{g(q)} + \underbrace{-\cos(q)I - \sin(q)J}_{g^c(q)} = 0 \\ &= I + (-I) = f(q) + f^c(q) \quad \forall q \in \mathbb{H}. \end{aligned}$$

Thus, $\mathrm{Tr}(f) = \mathrm{Tr}(g) = 0$ and $\mathrm{N}(f) = \mathrm{N}(g) = 1$. The stem functions $F, G : \mathbb{C} \to \mathbb{H}\otimes\mathbb{C}$ associated to f and g are

$$F(z) = 1\otimes I, \quad G(z) = \cos(z)\otimes I + \sin(z)\otimes J.$$

Since $sin^2(z) + \cos^2(z) = 1\ \forall z \in \mathbb{C}$, both F and G avoid the center $\mathbb{R}\otimes\mathbb{C}$ of $\mathbb{H}\otimes\mathbb{C}$. Hence, the *central divisors* $\mathrm{cdiv}(F)$ and $\mathrm{cdiv}(G)$ (as defined in Section 6.2) are both empty. Therefore, we have verified that in this example

$$\mathrm{Tr}(f) = \mathrm{Tr}(g), \quad \mathrm{N}(f) = \mathrm{N}(g), \quad \mathrm{cdiv}(f) = \mathrm{cdiv}(g),$$

but for some $q \in \mathbb{H}$ there is no $\phi \in \mathrm{Aut}(\mathbb{H})$ with

$$f(q) = \phi(g(q)).$$

On the other hand, our main result implies that there exists a holomorphic map $\phi : \mathbb{C} \to \mathrm{Aut}(\mathbb{H}_{\mathbb{C}})$ such that the corresponding stem functions F and G satisfy $F(z) = \phi(z)(G(z))$.

References [2, 3] pointed out that there exists a slice regular function h such that $f * h = h * g$. This implies that

$$h^{-*} * f * h = g,$$

where h is slice regular, but h^{-*} is possibly only semi-regular.

In contrast, our result implies the existence of such a *slice regular* function h which is *invertible* in the sense that h^{-*} is likewise slice regular, and not only semi-regular.

To give an explicit example, let

$$H(z) = \cos(z/2) - K \sin(z/2).$$

Then,

$$\begin{aligned} H(z)^{-1} \cdot F(z) \cdot H(z) &= \left(\cos\left(\frac{z}{2}\right) + K \sin\left(\frac{z}{2}\right)\right) \cdot I \cdot \left(\cos\left(\frac{z}{2}\right) - K \sin\left(\frac{z}{2}\right)\right) \\ &= I \cos z + J \sin z = G(z). \end{aligned}$$

4. Another example

Consider the stem functions

$$\begin{aligned} F(z) &= I + zJ + \frac{1}{2} z^2 K, \\ G(z) &= \left(1 + \frac{1}{2} z^2\right) I. \end{aligned}$$

We have

$$\begin{aligned} \mathrm{Tr}(F) &= \mathrm{Tr}(G) = 0, \\ \mathrm{N}(F) &= \mathrm{N}(G) = 1 + z^2 + \frac{1}{4} z^4, \\ \mathrm{cdiv}(F) = \{\,\}, \quad \mathrm{cdiv}(G) &= 1\{\sqrt{2}i\} + 1\{-\sqrt{2}i\}, \end{aligned}$$

where F and G are not equivalent in our sense because $\mathrm{cdiv}(F) \neq \mathrm{cdiv}(G)$. Therefore, there does not exist a stem function $H : \mathbb{C} \to \mathbb{H}_{\mathbb{C}}^*$ such that both H and H^{-1} are holomorphic and

$$F = H^{-1} \cdot G \cdot H.$$

In contrast, Altavilla and de Fabritiis do not need a condition on cdiv for their results in [2, 3]. Hence, their results imply that there exists a *meromorphic* stem function H with

$$F = H^{-1} \cdot G \cdot H.$$

Indeed, an explicit calculation shows that

$$H(z) = I\left(2 + \frac{1}{2}z^2\right) + Jz + \frac{1}{2}z^2 K, \quad H^{-1}(z) = -\frac{I(2 + \frac{1}{2}z^2) + Jz + \frac{1}{2}z^2 K}{\frac{1}{2}z^4 + 3z^2 + 4}$$

is such a function.

5. More preparations

5.1. The Clifford algebra $\mathbb{R}_3$

The *Clifford algebra* $\mathbb{R}_3$ may be realized as the associative $\mathbb{R}$-algebra generated by e_1, e_2, and e_3 with the relations

$$e_{jk} + e_{kj} = -2\delta_{jk}, \quad j, k \in \{1, 2, 3\},$$

where δ_{jk} is the Kronecker symbol and $e_{jk} = e_j \cdot e_k$; i.e.,

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

It contains the idempotents ω_+, ω_- satisfying $\omega_+\omega_- = 0$, namely,

$$\omega_+ = \frac{1}{2}(e_1e_2e_3 + 1), \quad \omega_- = \frac{1}{2}(e_1e_2e_3 - 1).$$

This yields a direct sum decomposition of $\mathbb{R}_3$

$$\mathbb{R}_3 = \omega_+\mathbb{H} \oplus \omega_-\mathbb{H},$$

where $\mathbb{H} \cong \mathbb{R}_2$ is embedded in $\mathbb{R}_3$ as the subalgebra generated by e_1, e_2.

See, e.g., [17] and [23, Chapter V] for more details on Clifford algebras.

5.2. Invariants for $\mathbb{R}_3$

For more clarity in this paragraph, we use N_A, respectively, Tr_A in order to denote the norm and trace for a given algebra A.

The Clifford algebra $\mathbb{R}_3$ is isomorphic (as $\mathbb{R}$-algebra with an anti-involution which we call conjugation) to $\mathbb{H} \oplus \mathbb{H}$.

As a consequence, we have

$$\mathrm{N}_{\mathbb{R}_3}(q_1, q_2) = \mathrm{N}_{\mathbb{H}\oplus\mathbb{H}}(q_1, q_2) = (\mathrm{N}_{\mathbb{H}}(q_1), \mathrm{N}_{\mathbb{H}}(q_2)),$$
$$\mathrm{Tr}_{\mathbb{R}_3}(q_1, q_2) = \mathrm{Tr}_{\mathbb{H}\oplus\mathbb{H}}(q_1, q_2) = (\mathrm{Tr}_{\mathbb{H}}(q_1), \mathrm{Tr}_{\mathbb{H}}(q_2))$$

for $q_1, q_2 \in \mathbb{H} \oplus \mathbb{H} \cong \mathbb{R}_3$.

Similarly for the complexified algebra $\mathbb{R}_3 \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{H}_{\mathbb{C}} \oplus \mathbb{H}_{\mathbb{C}}$ and the corresponding function algebras.

Caveat. Since $(z, w) \mapsto (w, z)$ is an automorphism of $\mathbb{H} \oplus \mathbb{H}$ (see Section 11), these functions N, Tr are not invariants. They are only invariants up to changing the order of the two components; i.e., they are invariants for the connected component $\mathrm{Aut}^0(\mathbb{H} \oplus \mathbb{H})$ which equals $\mathrm{Aut}(\mathbb{H}) \times \mathrm{Aut}(\mathbb{H})$ acting component-wise on $A = \mathbb{H} \oplus \mathbb{H}$.

5.3. Slice preserving functions

Proposition 5.1. *Let $A = \mathbb{H}$, and let $f : A \to A$ be a slice regular function with stem function $F : \mathbb{C} \to A_{\mathbb{C}}$. Then, the following are equivalent:*

(1) $f = f^c$,

(2) $F = F^c$,

(3) $F(\mathbb{C}) \subset \mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} \subset A \otimes_{\mathbb{R}} \mathbb{C} = A_{\mathbb{C}}$,

(4) $f(\mathbb{C}_I) \subset \mathbb{C}_I$ *for all* $I \in \mathbb{S} = \{q \in A : q^2 = -1\}$ (with $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$).

Definition 5.2. If one (hence all) of these properties are fulfilled, f is called "*slice preserving*".

Proof. These equivalences are well known. (iv) $\iff$ (iii) follows from representation formula (i) $\iff$ (ii) $\iff$ (iii) by construction of $(\ )^c$. ∎

5.4. Slice regular functions for $A = \mathbb{R}_3$

In [22], *slice regular functions* are considered for real alternative algebras, e.g., Clifford algebras. For generalities on Clifford algebras, see, e.g., [23, Chapter V].

A *quadratic cone* $Q_A \subset A$ is defined such that every $q \in Q_A$ is contained in the image of some $\mathbb{R}$-algebra homomorphism from $\mathbb{C}$ to A. While $Q_A = A$ for $A = \mathbb{H}$, we have $Q_A \neq A$ for the Clifford algebra $A = \mathbb{R}_3 \cong \mathbb{H} \oplus \mathbb{H}$. More precisely, (see [22])

$$\begin{aligned} Q_A &= \mathbb{R} \cup \{x \in A : \mathrm{Tr}(x), \mathrm{N}(x) \in \mathbb{R}, 4\,\mathrm{N}(x) > \mathrm{Tr}(x)^2\} \\ &= \{x \in A : \mathrm{Tr}(x), \mathrm{N}(x) \in \mathbb{R}\} \\ &\cong \{(q_1, q_2) \in \mathbb{H} \oplus \mathbb{H} : \mathrm{Tr}_{\mathbb{H}}(q_1) = \mathrm{Tr}_{\mathbb{H}}(q_2), \mathrm{N}_{\mathbb{H}}(q_1) = \mathrm{N}_{\mathbb{H}}(q_2)\}. \end{aligned}$$

(For $A = \mathbb{R}_3$, the condition $4\,\mathrm{N}(x) > \mathrm{Tr}(x)^2$ is automatically satisfied for every element $x \in A \setminus \mathbb{R}$ with $\mathrm{N}(x), \mathrm{Tr}(x) \in \mathbb{R}$.)

Given a symmetric domain $D \subset \mathbb{C}$, there is the "*circularization*"

$$\Omega_D = \{(x + yH_1, x + yH_2) : x, y \in \mathbb{R}, H_1, H_2 \in \mathbb{S}_{\mathbb{H}}, x + yi \in D\}$$

(with $\mathbb{S}_{\mathbb{H}} = \{J \in \mathbb{H} : J^2 = -1\}$). Evidently, $\Omega_D \subset Q_A$.

But due to the special direct sum nature of $\mathbb{R}_3$, we may regard a larger set, namely,

$$W_D = \{(x_1 + y_1 H_1, x_2 + y_2 H_2) : H_j \in \mathbb{S}_{\mathbb{H}}, x_j, y_j \in \mathbb{R}, x_j + y_j i \in D\ \forall j \in \{1, 2\}\}. \tag{5.1}$$

Note that $\Omega_D = Q_A \cap W_D$. Thus, Ω_D relates to W_D like the quadratic cone Q_A to the whole algebra $A = \mathbb{R}_3$.

Slice regular functions. As defined in [22] they are certain functions defined on the cone Q_A. It is shown in [22] that these *slice regular functions* correspond to *holomorphic stem functions*, i.e., holomorphic functions $F : D \to A_{\mathbb{C}}$ such that $\overline{F(z)} = F(\bar{z})\ \forall z \in \mathbb{C}$.

We remark that there is a natural way to extend a function $f : \Omega_D \to \mathbb{R}_3$ to a function $\tilde{f} : W_D \to \mathbb{R}_3$.

Namely, we use the decomposition $\mathbb{R}_3 \cong \mathbb{H} \oplus \mathbb{H}$ to represent f as $f = (f_1, f_2)$ with $f_i : \Omega_D \to \mathbb{H}$ and define

$$\begin{aligned}&\tilde{f}(x_1 + y_1 H_1, x_2 + y_2 H_2)\\ &\quad = (f_1(x_1 + y_1 H_1, x_1 + y_1 H_2), f_2(x_2 + y_2 H_1, x_2 + y_2 H_2)).\end{aligned}$$

In this way, the algebra of slice regular functions on Ω_D can be identified with a certain algebra of functions on W_D.

Let us now discuss the special case of globally defined functions (i.e., $D = \mathbb{C}$, $\Omega_D = Q_A$, and $W_D = A = \mathbb{R}_3$.)

In this case, stem functions are holomorphic functions with values in $A_{\mathbb{C}}$ which are defined on the whole of $\mathbb{C}$. Such a holomorphic function may be defined by a convergent power series $\sum_{k=0}^{+\infty} z^k a_k$ with $a_k \in A$. (A priori, $a_k \in A_{\mathbb{C}}$ for a holomorphic function $F : \mathbb{C} \to A_{\mathbb{C}}$, but the condition $\overline{F(z)} = F(\bar{z})$ ensures that $a_k \in A$.)

Such a power series may also be regarded as power series in a variable in A; in this case, it defines a function $f : A \to A$. (It is easily show that it is again globally convergent.)

Thus, there are bijective correspondences between the following classes of functions.

- *Slice regular functions* $f : Q_A \to A$ as defined in [22].
- *Entire functions*, i.e., functions $f : A \to A$ which are defined by globally convergent power series $\sum_{k=0}^{+\infty} q^k a_k$ $(a_k \in A)$.
- *Stem functions*, i.e., holomorphic functions $F : \mathbb{C} \to A_{\mathbb{C}}$ with $\overline{F(z)} = F(\bar{z})$.

In particular, slice regular functions $f : Q_A \to A$ extend naturally to functions $f : A \to A$ defined by globally convergent power series.

Thus, for the special case of the Clifford algebra $\mathbb{R}_3$, there is no need to restrict the domain of definition to (subdomains of) the quadratic cone Q_A.

Let D, Ω_D, W_D be as in the preceding subsection.

We described above how to associate a function $\tilde{f} : W_D \to A \cong \mathbb{H} \oplus \mathbb{H}$ to a given function $f : \Omega_D \to A$.

6. Central divisors

6.1. Divisors for vector-valued functions

Normally, *divisors* are defined for holomorphic functions with values in $\mathbb{C}$. Here, we extend this notion to holomorphic maps from Riemann surfaces to higher-dimensional complex vector spaces.

Definition 6.1. Let $f : X \to V = \mathbb{C}^n$ be a holomorphic map from a Riemann surface X to a complex vector space $V = \mathbb{C}^n$. Assume that f is not identically zero.

The *divisor* of f is the divisor corresponding to the pull back of the ideal sheaf of the origin; i.e., for $f = (f_1, \dots, f_n)$, $f_i : X \to \mathbb{C}$, we have $\mathrm{div}(f) = \sum_{p\in X} m_p\{p\}$, where m_p denotes the minimum of the multiplicities $\mathrm{mult}_p(f_i)$.

6.2. Central divisor for stem functions

In [11, Definition 3.1], we introduced the notion of a *slice divisor*. Here, we will need a different notion of divisor.

Namely, we need a notion of divisor which measures how far the function is from being *slice-preserving*. This we call "*central divisor*". Here, we define it for the quaternion case.

Definition 6.2. Let $Z \cong \mathbb{C}$ be the center of $\mathbb{H}_\mathbb{C}$, $D \subset \mathbb{C}$ a symmetric domain, and $F : D \to A_\mathbb{C}$ a holomorphic map. Assume $F(D) \not\subset Z$.

The *central divisor* $\mathrm{cdiv}(F)$ is defined as the divisor (in the sense of Definition 6.1) of the map from D to $\mathbb{H}_\mathbb{C}/Z$ induced by F.

Let $(1, i, j, k)$ be the standard basis of $\mathbb{H}$, and let $W\otimes_\mathbb{R}\mathbb{C}$ denote the complex vector subspace of $\mathbb{H}_\mathbb{C}$ generated by $i\otimes 1$, $j\otimes 1$, $k\otimes 1$.

Then, $\mathbb{H}_\mathbb{C}$ may be represented as the vector space direct sum $\mathbb{H}_\mathbb{C} = Z \oplus W\otimes_\mathbb{R}\mathbb{C}$, where $Z \cong \mathbb{C}$ denotes the center. And we can decompose $F : D \to \mathbb{H}_\mathbb{C}$ as $F = (F', F'') : D \to Z \times (W\otimes_\mathbb{R}\mathbb{C})$ and the central divisor $\mathrm{cdiv}(F)$ equals $\sum_{p\in D} n_p\{p\}$, where n_p denotes the vanishing order of F'' at p.

Let $F = (F', F'')$ be a stem function for a slice function f. Then, f is *slice-preserving* if $F'' \equiv 0$ (Definition 5.2).

Hence, the assumption that f is not slice preserving implies that F'' does not vanish identically and we therefore may define $\mathrm{cdiv}(F)$ as above.

Example 6.3. Consider $F : \mathbb{C} \to \mathbb{H}\otimes_\mathbb{R}\mathbb{C}$ defined as

$$F(z) = 1\otimes z + i\otimes z^2(z-1) + j\otimes z^3(z-1)^2.$$

Then,

$$\mathrm{cdiv}(F) = 2\cdot\{0\} + 1\cdot\{1\}.$$

Caveat. These central divisors do *not* satisfy the usual functoriality.

Example 6.4. Let

$$F(z) = 1 + iz, \quad G(z) = 1 + j(1+z).$$

Then, $\mathrm{cdiv}(F) = 1\cdot\{0\}$ and $\mathrm{cdiv}(G) = 1\cdot\{-1\}$, but

$$\mathrm{cdiv}(FG) = \mathrm{cdiv}(1 + iz + j(z+1) + k(z+1)z)$$

is empty. Thus,

$$\mathrm{cdiv}(FG) \neq \mathrm{cdiv}(F) + \mathrm{cdiv}(G).$$

6.3. Central divisor for slice functions

We simply define the *central divisor* for a given slice regular function as the central divisor of the corresponding stem function (as defined in Definition 6.2).

6.4. Central divisors for $\mathbb{R}_3$

The *central divisor* cdiv of a slice regular function $f : \mathbb{R}_3 \to \mathbb{R}_3$ is defined via the stem function F. Given a stem function $F : D \to A_{\mathbb{C}} \cong \mathbb{H}_{\mathbb{C}} \oplus \mathbb{H}_{\mathbb{C}}$, we define

$$\operatorname{cdiv}(F) = (\operatorname{cdiv} F_1, \operatorname{cdiv} F_2),$$

where F_1, F_2 are the components of the stem function F with respect to the decomposition of $\mathbb{R}_3$ into a direct sum of two copies of $\mathbb{H}$.

What we need is to treat $\mathbb{R}_3$ consequently as product; i.e., we regard slice regular functions from (a domain in) $\mathbb{R}_3$ to $\mathbb{R}_3$ as a pair of the quaternionic slice regular functions and *define* the "central divisor" of a slice regular function $f = (f_1; f_2)$ as

$$\operatorname{cdiv}(f) = \operatorname{cdiv}(f_1; f_2) = (\operatorname{cdiv}(f_1), \operatorname{cdiv}(f_2)).$$

7. Strategy

Let A be a $\mathbb{R}$-algebra, $A_{\mathbb{C}} = A \otimes_{\mathbb{R}} \mathbb{C}$ its complexification and $G_{\mathbb{C}}$ the automorphism group of the complex algebra $A_{\mathbb{C}}$.

For certain[4] holomorphic maps $F, H : \mathbb{C} \to A_{\mathbb{C}}$, we want to show the following statement.

If, for every $x \in \mathbb{C}$, there exists an element $g \in G_{\mathbb{C}}$ such that $F(x) = g(H(x))$, then there exists a holomorphic map $\phi : \mathbb{C} \to G_{\mathbb{C}}$ with $F(x) = \phi(x)(H(x))\ \forall x$.

This amounts to find a section for a certain projection map, namely, $\pi : V \to \mathbb{C}$ with

$$V = \{(x, g) \in \mathbb{C} \times G_{\mathbb{C}} : F(x) = g(H(x))\}$$

and

$$\pi(x, g) = x.$$

First, we discuss the locus where F and H assume zero as value.

Outside this zero locus D, i.e., restricted to $V_0 = \pi^{-1}(\mathbb{C} \setminus D)$, the map π has some nice properties if $A \cong \mathbb{H}$:

(1) all π-fibers are of the same dimension, homogeneous, and moreover, biholomorphic to a Lie subgroup of $G_{\mathbb{C}}$,

(2) there are everywhere local holomorphic sections.

[4] "Stem functions".

Moreover, for $A = \mathbb{H}$, there is a Zariski open subset Ω in $\mathbb{C}$ on which π is a locally holomorphically trivial fiber bundle with $\mathbb{C}^*$ as fiber.

We show that there exists a global holomorphic section on Ω.

We will use strongly that the generic isotropy group (i.e., $\mathbb{C}^*$) is both commutative and one-dimensional.

8. Main result for $\mathbb{R}_3$

Theorem 8.1. *Let $D \subset \mathbb{C}$ be a symmetric domain, $\Omega_D \subset \mathbb{R}_3$ as defined above* (see (5.1)), *$f, h : \Omega_D \to \mathbb{R}_3$ slice regular functions as defined above.*

Let $F, H : D \to A_{\mathbb{C}}$ be the associated stem functions.

Let $G_{\mathbb{C}}$ denote the connected component of the neutral element of the group of $\mathbb{C}$-algebra automorphisms of $A_{\mathbb{C}} = \mathbb{R}_3 \otimes_{\mathbb{R}} \mathbb{C}$, i.e., $G_{\mathbb{C}} = \mathrm{Aut}(\mathbb{H}_{\mathbb{C}}) \times \mathrm{Aut}(\mathbb{H}_{\mathbb{C}})$.

Let N, Tr, *and* cdiv *be defined as in Section 5.2. Then, the following are equivalent.*

(1) *There exists a holomorphic map $\phi : D \to G_{\mathbb{C}}$ with*

$$F(z) = \phi(z)(H(z)) \quad \forall z \in D.$$

(2) *There exists a holomorphic map $\alpha : D \to A_{\mathbb{C}}{}^* \cong \mathbb{H}_{\mathbb{C}}^* \times \mathbb{H}_{\mathbb{C}}^*$ with*

$$F(z) = \alpha(z)(H(z))(\alpha(z))^{-1} \quad \forall z \in D.$$

(3) $\mathrm{cdiv}(F) = \mathrm{cdiv}(H)$, $\mathrm{Tr}(F) = \mathrm{Tr}(H)$, *and* $\mathrm{N}(F) = \mathrm{N}(H)$.

Proof. Using $A \cong \mathbb{H} \oplus \mathbb{H}$ and $G_{\mathbb{C}}^0 \cong \mathrm{Aut}(\mathbb{H}_{\mathbb{C}}) \times \mathrm{Aut}(\mathbb{H}_{\mathbb{C}})$, the equivalence of the existence of such a map $\phi : D \to G_{\mathbb{C}}$ with the condition

$$\mathrm{cdiv}(F) = \mathrm{cdiv}(H), \quad \mathrm{Tr}(F) = \mathrm{Tr}(H), \quad \mathrm{N}(F) = \mathrm{N}(H)$$

follows from the respective result for quaternions (Theorem 1.1, proved in Section 18). ■

9. Local equivalence

Assumptions 9.1. Let G be a connected complex Lie group acting holomorphically on a complex manifold X such that all the orbits have the same dimension d.

Let

$$F, H : \Delta = \{z \in \mathbb{C} : |z| < 1\} \to X$$

be holomorphic maps such that for every $z \in \Delta$ there exists an element $g \in G$ (depending on z, not necessarily unique) with $F(z) = g \cdot H(z)$.

Let

$$V = \big\{(z, g) \in \Delta \times G : F(z) = g \cdot H(z)\big\}.$$

Lemma 9.2. *Under the above assumption 9.1, for every $t \in \Delta$ there is a biholomorphic map between the fiber*

$$V_t = \{(z, g) \in V : z = t\}$$

and the isotropy group

$$G_{F(t)} = \{g \in G : g \cdot F(t) = F(t)\}.$$

Proof. Fix $t \in \Delta$ and choose a point $(t, g_0) \in V_t$. Note that

$$F(t) = g_0 \cdot H(t)$$

because $(t, g_0) \in V_t \subset V$.

We define a map

$$\zeta : G_{F(t)} \xrightarrow{\sim} V_t$$

as

$$\zeta(g) = (t, g \cdot g_0).$$

We claim that this is a biholomorphic map from $G_{F(t)}$ to V_t. First, we verify that $\zeta(g) \in V_t$. Indeed, $F(t) = g_0 \cdot H(t)$ (as seen above) and $g \cdot F(t) = F(t)$ because $g \in G_{F(t)}$.

Combining these facts implies that

$$F(t) = g \cdot \underbrace{g_0 \cdot H(t)}_{F(t)} \implies \zeta(g) = (t, g g_0) \in V,$$

where ζ is obviously injective. Let us check surjectivity. Let $(t, p) \in V_t$. Then, $F(t) = p \cdot H(t)$. Recall that

$$F(t) = g_0 \cdot H(t).$$

It follows that

$$(p \cdot g_0^{-1})F(t) = (p \cdot g_0^{-1})g_0 \cdot H(t) = pH(t) = F(t).$$

Hence,

$$p \cdot g_0^{-1} \in G_{F(t)}.$$

We claim that

$$\zeta(p \cdot g_0^{-1}) = (t, p).$$

Indeed,

$$\zeta(p \cdot g_0^{-1}) = (t, (p \cdot g_0^{-1}) \cdot g_0) = (t, p).$$

Therefore,

$$\zeta : G_{F(t)} \xrightarrow{\sim} V_t$$

is biholomorphic. ∎

Lemma 9.3. *Under the above Assumption 9.1, V is smooth.*

Proof. Consider

$$Z := \{(z,x,x) : z \in \Delta, x \in X\}.$$

Note that Z is a submanifold of $\Delta \times X \times X$.

Define

$$\phi : \Delta \times G \to \Delta \times X \times X, \quad \phi(z,g) = (z, F(z), g \cdot H(z))$$

and observe that $V = \phi^{-1}(Z)$.

In order to verify the smoothness of V, it suffices to show that $D\phi$ has everywhere the same rank.

Let $(x,g) \in \Delta \times G$ and consider

$$D\phi_{(x,g)} : T_{(x,g)}(\Delta \times G) \to T_{\phi(x,g)}(\Delta \times X \times X).$$

We observe that

$$T_{(x,g)}(\Delta \times G) \cong (T_x\Delta) \times \mathrm{Lie}(G).$$

From $\phi(z,g) = (z, F(z), g(H(z)))$, we infer that

$$(v,w) \in \ker D\phi_{(x,g)} \subset (T_x\Delta) \times \mathrm{Lie}(G) \iff v = 0, \quad w \in \ker D\zeta \text{ for } \zeta : g \mapsto g(H(z)).$$

Here, $\zeta : G \to X$ is the orbit map $g \mapsto g(H(z))$. Standard theory of transformation groups implies that $w \in \ker D\zeta$ iff $w \in T_g G \cong \mathrm{Lie}(G)$ is contained in the Lie subalgebra of the isotropy group of the G-action at $g(H(z))$.

By assumption, all the G-orbits in X have the same dimension d, Hence, every isotropy group is of dimension $\dim(G) - d$. It follows that

$$\dim \ker D\phi_{z,g} = \dim(G) - d \quad \forall z, g.$$

Thus, ϕ is a map of constant rank and $V = \phi^{-1}(Z)$ is smooth. ∎

Proposition 9.4. *Under the above assumptions, there exist* $0 < r < 1$ *and a holomorphic map*

$$\phi : \Delta_r = \{z : |z| < r\} \to G$$

such that

$$F(z) = \phi(z)(H(z)) \quad \forall z \in \Delta_r.$$

Proof. Let $g_0 \in G$ such that $F(0) = g_0(H(0))$. We may replace $H(z)$ by the function $\widetilde{H}(z) = g_0(H(z))$. In this way, we see that there is no loss of generality in assuming $F(0) = H(0)$.

We may replace G by its universal covering and therefore assume that G is simply-connected. Then, we can use the fact that every simply-connected complex Lie group is a Stein manifold (see [25]). Hence, we may assume that G is a Stein manifold.

Recall that

$$V = \{(z,g) \in \Delta \times G : F(z) = g(H(z))\}.$$

Let $\pi : V \to \Delta$ be the natural projection and $V_t = \pi^{-1}(t)$ (for $t \in \Delta$). Due to Lemma 9.2, there is a bijective map from V_t to the isotropy group of the G-action at $F(t)$. Since all the G-orbits are assumed to have the same dimension d, each isotropy group has the dimension $\dim(G) - d$. Therefore, all the fibers of $\pi : V \to \Delta$ have the same dimension (namely, $\dim(G) - d$).

Due to Lemma 9.3, the complex space V is smooth.

We consider the relative symmetric product $S^m_\Delta V$; i.e., $S^m_\Delta V$ is the quotient of

$$\{(z; v_1, \dots, v_m) \in \Delta \times V^m : \pi(v_i) = z, \ \forall i\}$$

under the natural action of the symmetric group S_m permuting the components of V^m.

V may be embedded into $S^m_\Delta V$ diagonally as

$$V \ni v = (z, g) \overset{\delta}{\mapsto} [(z; v, \dots, v)] \in S^m_\Delta V.$$

Note that V is Stein as a closed analytic subspace of the Stein manifold $\Delta \times G$. It follows that

$$\left\{(z; v_1, \dots, v_m) \in \Delta \times V^m : \pi(v_i) = z \ \forall i\right\}$$

is likewise Stein. Thus, $S^m_\Delta V$ is the quotient of a Stein space by a finite group (namely, S_m). Hence, $S^m_\Delta V$ is Stein and therefore admits an embedding $j : S^m_\Delta V \hookrightarrow \mathbb{C}^N$ for some $N \in \mathbb{N}$. Recall that $S^m_\Delta V$ is a quotient of a subspace of $\Delta \times V^m$ by a S_m-action which is trivial on the first factor Δ. The natural projection from $\Delta \times V^m$ onto its first factor thus yields a natural map $p : S^m_\Delta V \to \Delta$.

Now, we define an embedding $\xi : S^m_\Delta V \to \Delta \times \mathbb{C}^N$ as

$$\xi : w \mapsto (p(w), j(w)).$$

We obtain a commutative diagram

$$\begin{array}{ccccc}
V & \xrightarrow{\ \delta\ } & S^m_\Delta V & \xrightarrow{\ \xi\ } & \Delta \times \mathbb{C}^N \\
{\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle \mathrm{pr}_1} \\
\Delta & = & \Delta & = & \Delta
\end{array}$$

Here,

$$\mathrm{pr}_1 : \Delta \times \mathbb{C}^N \to \Delta$$

is the projection to the first factor.

Because V is smooth (see Lemma 9.3), the "tubular neighbourhood theorem" (see [16, Theorem 3.3.3]) implies the existence of a holomorphic retraction of an open neighborhood W of $Y = \xi(\delta(V))$ in $\Delta \times \mathbb{C}^N$ onto Y.

This can be done in a relative way, over Δ, cf. [15, Lemma 3.3], [16, Theorem 3.3.4].

Thus, there is a holomorphic map $\rho : W \to Y$ with $\rho|_Y = \mathrm{id}_Y$ and $\mathrm{pr}_1(w) = \mathrm{pr}_1(\rho(w))$ for all $w \in W$

$$\begin{array}{ccccccc} \xi(\delta(V)) & = & Y & \underset{\rho}{\overset{\hookrightarrow}{\longleftarrow}} & W & \overset{\text{open}}{\hookrightarrow} & \Delta \times \mathbb{C}^N \\ & & & \searrow & \downarrow & \swarrow & \\ & & & & \Delta & & \end{array}$$

Let C be a (local) smooth complex curve in V through the point $(0, e)$ which is not contained in the fiber $V_0 = \pi^{-1}(0)$. Now, $C \cap V_0$ is a discrete subset of C. Hence, by shrinking C, we may assume that with

$$C \cap V_0 = \{(0, e)\}.$$

Now, $\pi|_C : C \to \Delta$ is a non-constant holomorphic map between one-dimensional complex manifolds.

By one-dimensional complex analysis, after appropriately shrinking Δ and C, the projection map from C onto Δ is a finite ramified covering of some degree $m \in \mathbb{N}$.

Thus, we obtain a local multisection; i.e., there is an open neighborhood Δ' of 0 in Δ and a closed analytic subset $C' \subset \pi^{-1}(\Delta')$ such that $\pi|_{C'} : C' \to \Delta'$ is a finite ramified covering. Let m denote the degree of $\pi|_{C'}$.

To each point $t \in \Delta'$ we associate the finite space $\pi^{-1}(t) \cap C'$. This is a finite subspace of $\pi^{-1}(t) \subset V$ of degree m which is mapped by π into one single point of Δ. Such a finite subset corresponds to a point in the relative symmetric product $S^m_\Delta V$. Therefore, our multisection yields a section $s : \Delta' \to S^m_\Delta V$.

We recall that we assumed $C \cap V_0 = \{(0, e)\}$. Hence,

$$s(0) = \delta(0, e) \implies \xi(s(0)) \in Y \subset W.$$

Now, W is open, and $(\xi \circ s)(0) \in W$. Let $\Delta'' = (\xi \circ s)^{-1}(W)$.

We compose $\xi \circ s : \Delta'' \to W$ with the holomorphic retraction

$$\rho : W \to Y = \xi(\delta(V)).$$

Since $\xi \circ \delta : V \to Y$ is an isomorphism, there is a unique holomorphic map $\sigma : \Delta'' \to V$ satisfying

$$\xi \circ \delta \circ \sigma = \rho \circ \xi \circ s,$$

namely, $\sigma = (\xi \circ \delta)^{-1} \circ \rho \circ \xi \circ s$.

By construction, this map σ is a section of $\pi : V \to \Delta$ on Δ'', i.e., a holomorphic map $\sigma : \Delta'' \to V$ with $\pi \circ \sigma(z) = z \ \forall z \in \Delta''$.

Recall that V is defined as

$$V = \big\{(z, g) \in \Delta \times G : F(z) = g(H(z))\big\}.$$

Therefore, a section σ is given as

$$\sigma : z \mapsto (z, \phi(z)),$$

where $\phi : \Delta'' \to G$ is a map which fulfills

$$F(z) = \phi(z)(H(z)) \quad \forall z \in \Delta''.$$

This yields the statement of the proposition if we choose $r \in (0, 1)$ such that $\Delta_r \subset \Delta''$. ■

Remark. It is essential to assume that all the orbits have the same dimension. For example, let $G = \mathbb{C}^*$ act on $X = \mathbb{C}$ as usual, let $F(z) = z^2$ and $H(z) = z^3$. Then, $\forall z$, $\exists \lambda \in G$: $\lambda z^3 = z^2$, but there is no holomorphic function $\phi : \Delta \to G$ with $\phi(z)z^3 = z^2$.

10. Automorphisms of quaternions

10.1. Automorphisms of $\mathbb{H}$

We will need the following classical result (see, e.g., [14, Section 6.8]). It is based on the fact that $\mathbb{H}$ is a central simple $\mathbb{R}$-algebra and the Skolem–Noether theorem (see, e.g., [24, Section 29]) which states that every automorphism of a central simple algebra is inner.

Proposition 10.1. *Every ring automorphism of the* $\mathbb{R}$*-algebra of quaternions* $\mathbb{H}$ *is already an* $\mathbb{R}$*-algebra automorphism (thus,* $\mathbb{R}$*-linear and continuous) and preserves the scalar product on* $\mathbb{H}$ *defined as*

$$< x, y >= \frac{1}{2}(x\bar{y} + y\bar{x}).$$

Let G *be the group of ring automorphisms of* $\mathbb{H}$.

Then, G *is isomorphic to* $\mathrm{SO}(3, \mathbb{R})$*, acting trivially on the center* $\mathbb{R}$ *and by the standard action of* $\mathrm{SO}(3, \mathbb{R})$ *on* $\mathbb{R}^3$*, the orthogonal complement* W *of* $\mathbb{R}$*, if we identify* W *with* $\mathbb{R}^3$ *using the standard basis* i, j, k *of* W.

For $q \in \mathbb{H}$*, let* G_q *be its isotropy group, i.e.,* $G_q = \{g \in G = \mathrm{Aut}(\mathbb{H}) : g(q) = q\}$. *Then,* $G = G_q$*, if* q *is in the center* $\mathbb{R}$ *of* $\mathbb{H}$ *and* $G_q \cong S^1 \cong \mathrm{SO}(2, \mathbb{R})$ *for* $q \notin \mathbb{R}$.

Remark. We have $G_{\mathbb{C}} \cong \mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{SO}(3, \mathbb{C})$ acting on a three-dimensional vector space preserving a non-degenerate bilinear form. This representation must be isomorphic to the adjoint representation of $\mathrm{PSL}_2(\mathbb{C})$ on its Lie algebra $\mathrm{Lie}(\mathrm{PSL}_2(\mathbb{C}))$ which preserves the *Killing form* $\mathrm{Kill}(x, y) = \mathrm{Trace}(\mathrm{ad}(x)\mathrm{ad}(y))$. Let $v \in \mathrm{Lie}(\mathrm{PSL}_2(\mathbb{C})) \setminus \{0\}$. Then, $\mathrm{Kill}(v, v) = 0$ if the one-parameter subgroup $\exp(v\mathbb{C})$ is unipotent, while $\exp(v\mathbb{C}) \cong \mathbb{C}^*$ if $\mathrm{Kill}(v, v) \neq 0$. The isotropy group of $G_{\mathbb{C}}$ at $v \in \mathrm{Lie}(G_{\mathbb{C}})$ is the centralizer of $\exp(v\mathbb{C})$. For $G_{\mathbb{C}} = \mathrm{PSL}_2(\mathbb{C})$ and $v \neq 0$, this is always $\exp(v\mathbb{C})$ itself. Hence, the isotropy group of $G_{\mathbb{C}}$ acting on $\mathrm{Lie}(G_{\mathbb{C}})$ at an element $v \neq 0$ is isomorphic to $\mathbb{C}^*$ if

$$\mathrm{Kill}(v, v) \neq 0$$

and isomorphic to $\mathbb{C}$ if $Kill(v, v) = 0$. Similarly for $G_{\mathbb{C}}$ acting on $W \otimes_{\mathbb{R}} \mathbb{C}$, replacing $Kill(,)$ by $B(,)$.

As a consequence, one obtains the following corollary.

Corollary 10.2. *Let $p, q \in \mathbb{H} \setminus \{0\}$. Then, the following conditions are equivalent.*

- $\mathrm{N}(p) = \mathrm{N}(q)$ *and* $\mathrm{Tr}(p) = \mathrm{Tr}(q)$,
- *There is an $\mathbb{R}$-algebra automorphism ϕ of $\mathbb{H}$ such that $\phi(p) = q$.*

11. Automorphisms of $\mathbb{R}_3 = \mathbb{H} \oplus \mathbb{H}$

$\mathbb{R}_3$ denotes the Clifford algebra associated to $\mathbb{R}^3$ endowed with a positive definite quadratic form. As an $\mathbb{R}$-algebra, $\mathbb{R}_3 \cong \mathbb{H} \oplus \mathbb{H}$. This direct sum structure allows an easy determination of the automorphism group of $\mathbb{R}_3$.

11.1. Algebraic preparation

Lemma 11.1. *Let R be a (possibly non-commutative) ring with 1 and without zero-divisors. Then, every ring automorphism of $A = R \oplus R$ is either of the form*

$$(x, y) \mapsto (\phi(x), \psi(y)), \quad \phi, \psi \in \mathrm{Aut}(R)$$

or

$$(x, y) \mapsto (\psi(y), \phi(x)), \quad \phi, \psi \in \mathrm{Aut}(R)$$

Proof. Since R has no zero-divisors, the only idempotents are 0 and 1, i.e.,

$$\{x \in R : x^2 = x\} = \{0, 1\}.$$

As a consequence, we have

$$I = \left\{z = (x, y) \in R \times R : z^2 = z\right\} = \{0, 1\} \times \{0, 1\}.$$

Every ring automorphism α of A must stabilize I and fix $0 = (0, 0)$ and $1 = (1, 1)$. Hence, either α fixes $(0, 1)$ and $(1, 0)$ or

$$(0, 1) \xrightarrow{\alpha} (1, 0) \xrightarrow{\alpha} (0, 1).$$

In the first case, α stabilizes both $(0, 1)A = \{0\} \times R$ and $(1, 0)A = R \times \{0\}$, which implies that $\alpha(x, y) = (\phi(x), \psi(y))$ for some $\phi, \psi \in \mathrm{Aut}(R)$.

Similarly, in the second case,

$$(x, y) \mapsto (\psi(y), \phi(x))$$

for some $\phi, \psi \in \mathrm{Aut}(R)$. ■

11.2. Description of Aut($\mathbb{R}_3$)

As a consequence of Lemma 11.1 and the description of the automorphisms of $\mathbb{H}$, we obtain the following proposition.

Proposition 11.2. *The automorphism group of the Clifford algebra $\mathbb{R}_3 \cong \mathbb{H} \oplus \mathbb{H}$ is generated by $(z, w) \mapsto (w, z)$ and $\mathrm{SO}(3, \mathbb{R}) \times \mathrm{SO}(3, \mathbb{R})$ where $\mathrm{SO}(3, \mathbb{R})$ acts as the group of all orientation preserving orthogonal linear transformations of the imaginary parts of the factors of the product $\mathbb{H} \times \mathbb{H}$.*

The automorphism group of the complex algebra $\mathbb{R}_3 \otimes \mathbb{C}$ is generated by $(z, w) \mapsto (w, z)$ and $\mathrm{SO}(3, \mathbb{C}) \times \mathrm{SO}(3, \mathbb{C})$.

11.3. Isotropy groups

Using the above description of the full automorphism group and the product structure $\mathbb{R}_3 \cong \mathbb{H} \oplus \mathbb{H}$, it is easy to determine the isotropy groups of an element (q_1, q_2).

- The isotropy group contains all automorphisms of the form

$$\phi : (x_1, x_2) \mapsto (\phi_1(x_1), \phi_2(x_2))$$

with

$$\phi_i \in \mathrm{Aut}(\mathbb{H}) \quad \text{and} \quad \phi_i(q_i) = q_i \quad (i \in \{1, 2\}).$$

- If there is an automorphism $\alpha \in \mathrm{Aut}(\mathbb{H})$ with $\alpha(q_1) = q_2$, then the isotropy group contains in addition all automorphisms of the form $\phi : (x_1, x_2) \mapsto (\phi_1(x_2), \phi_2(x_1))$ with $\phi_i \in \mathrm{Aut}(\mathbb{H})$ and

$$(q_1, q_2) = (\phi_1(q_2), \phi_2(q_1)).$$

In view of $\alpha(q_1) = q_2$, the above equation is equivalent to

$$(\alpha \circ \phi_1)(q_2) = \alpha(q_1) = q_2,$$
$$(\alpha^{-1} \circ \phi_2)(q_1) = \alpha^{-1}(q_2) = q_1.$$

Therefore, $\alpha \circ \phi_1$ must be in the isotropy group of q_2 and $\alpha^{-1} \circ \phi_2$ in the isotropy group of q_1.

12. Orbits in the complexified algebra

The proposition below is principally applied to the situation, where

$$A \cong \mathbb{H} \quad \text{and} \quad A = \mathbb{R} \oplus V$$

as vector space, V being the subspace of totally imaginary elements.

In fact, we have seen that in this case $\mathrm{Aut}(A)$ acts trivially on $\mathbb{R}$ and by orthogonal transformations on V such that every sphere centered at the origin in V is one orbit.

Proposition 12.1. *Let $V = \mathbb{R}^n$, and let G be a connected real Lie group acting by orthogonal linear transformations on V such that the unit sphere $S = \{v \in \mathbb{R}^n : ||v|| = 1\}$ is a G-orbit.*

Let $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$*, and let* B *denote the* $\mathbb{C}$*-bilinear form on* $V_{\mathbb{C}}$ *extending the standard Euclidean scalar product on* $V = \mathbb{R}^n$*.*

Let $G_{\mathbb{C}}$ *be the smallest complex Lie subgroup of* $GL(V_{\mathbb{C}})$ *containing* G*. Then, the* $G_{\mathbb{C}}$*-orbits in* $V_{\mathbb{C}}$ *are the following:*

- $H_\lambda = \{v \in V_{\mathbb{C}} : B(v, v) = \lambda\}$ *for* $\lambda \in \mathbb{C}^*$*,*
- $H_0 = \{v \in V_{\mathbb{C}} : B(v, v) = 0\} \setminus \{0\}$*,*
- $\{0\}$.

Proof. First, we observe that $B(\ , \ )$ is invariant under the $G_{\mathbb{C}}$-action because G acts by orthogonal transformations.

Since G acts transitively on S and G acts linearly on V, the G-orbits in V are precisely $\{0\}$ and the spheres $S_r = \{v \in V : ||v|| = r\}$ $(r > 0)$.

Let $v = u + \iota w \in V_{\mathbb{C}}$, $u, w \in V$. If $v \neq 0$, then $(u, w) \neq (0, 0)$. If $u \neq 0$, then the projection onto the real part yields a map from the G-orbit through v onto a sphere in $\mathbb{R}^n$, which implies that the real tangent space of Gv contains at least $n - 1$ $\mathbb{C}$-linearly independent tangent vectors. It follows that the $G_{\mathbb{C}}$-orbit through v is of complex dimension $\geq n - 1$.

On the other hand, the complex hypersurfaces

$$C_\lambda = \{v \in V_{\mathbb{C}} : B(v, v) = \lambda\}$$

are irreducible, complex $(n - 1)$-dimensional and $G_{\mathbb{C}}$-invariant. This implies the assertion. ∎

Corollary 12.2. *Let* A *be a finite-dimensional* $\mathbb{R}$*-algebra with* $\mathbb{R}$ *as center. Let* $A = \mathbb{R} \oplus V$ *as vector space and let* G *be a real Lie group acting trivially on* $\mathbb{R}$ *and by orthogonal linear transformations on* V*. Assume that* G *acts transitively on the unit sphere of* V*.*

Let $G_{\mathbb{C}}$ *be the smallest complex Lie subgroup of* $GL(A \otimes_{\mathbb{R}} \mathbb{C})$ *containing* G*. Then, all the* $G_{\mathbb{C}}$*-orbits in* $(V \otimes_{\mathbb{R}} \mathbb{C}) \setminus \{0\}$ *are complex hypersurfaces. In particular, they all have the same dimension.*

Corollary 12.3. *Under the assumptions of Corollary 12.2 for every point* $q \in (V \otimes_{\mathbb{R}} \mathbb{C}) \setminus \{0\}$*, the isotropy group* I *of the* $G_{\mathbb{C}}$*-action at* q *satisfies*

$$\dim_{\mathbb{C}} I + \dim_{\mathbb{R}} V - 1 = \dim_{\mathbb{C}}(G_{\mathbb{C}}).$$

In particular, we have $\dim_{\mathbb{C}}(I) = 1$ *for* $A \cong \mathbb{H}$.

Proof. The $G_{\mathbb{C}}$-orbit through q is a complex hypersurface (Corollary 12.2). Hence, its complex dimension equals $\dim_{\mathbb{C}}(V \otimes_{\mathbb{R}} \mathbb{C}) - 1$, and therefore,

$$\dim_{\mathbb{C}}(G_{\mathbb{C}}) = \dim_{\mathbb{C}} I + \dim_{\mathbb{C}}(V \otimes_{\mathbb{R}} \mathbb{C}) - 1.$$

Now, $\dim_{\mathbb{R}}(V) = \dim_{\mathbb{C}}(V \otimes_{\mathbb{R}} \mathbb{C})$. This yields the assertion. ∎

Corollary 12.4. *Under the same assumptions, there is a Zariski open subset, namely,* $\Omega \subset A \otimes_{\mathbb{R}} \mathbb{C} \overset{\zeta}{\sim} \mathbb{C} \oplus (V \otimes_{\mathbb{R}} \mathbb{C})$ *defined as* $\Omega = \{q : \zeta(q) = (x, v), B(v, v) \neq 0\}$ *such that all the isotropy groups of* $G_{\mathbb{C}}$ *at points in* Ω *are conjugate.*

Proof. By assumption, $A = \mathbb{R} \oplus V$ as vector space, and correspondingly,

$$A \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \underbrace{(V \otimes_{\mathbb{R}} \mathbb{C})}_{V_{\mathbb{C}}}.$$

With respect to this vector space sum decomposition, we define

$$\Omega = \{r + v : r \in \mathbb{C}, v \in V_{\mathbb{C}}, B(v, v) \neq 0\}.$$

Due to Proposition 12.1, the $G_{\mathbb{C}}$-orbit through a given point $x = r + v \in \Omega$ is

$$\{r + w, w \in V_{\mathbb{C}}, B(w, w) = B(v, v)\}.$$

The isotropy groups in the same orbit are clearly conjugate. Let $x = r + v, y = s + w \in \Omega$ (with $r, s \in \mathbb{C}$, $v, w \in V_{\mathbb{C}}$) be in different orbits. Choose $\lambda \in \mathbb{C}^*$ such that $B(w, w) = B(\lambda v, \lambda v)$. Now, the isotropy group at $r + v$ agrees with the isotropy group at $\lambda(r + v) = \lambda r + \lambda v$ because the group action is linear. Since $G_{\mathbb{C}}$ acts trivially on the center $\mathbb{C}$ of $A \otimes_{\mathbb{R}} \mathbb{C}$, the isotropy groups at $\lambda r + \lambda v$ and $s + \lambda v$ coincide. Finally, due to $B(w, w) = B(\lambda v, \lambda v)$ the elements $s + \lambda v$ and $s + w$ are in the same $G_{\mathbb{C}}$-orbit. Consequently, their isotropy groups are conjugate. Combined, these facts imply that the isotropy groups at $r + v$ and $s + w$ are conjugate. ∎

Corollary 12.5. *Let* $A = \mathbb{H}$ *and* $A_{\mathbb{C}} = A \otimes_{\mathbb{R}} \mathbb{C}$ *and let* $p, q \in A_{\mathbb{C}} \setminus \{0\}$. *Then, the following properties are equivalent:*

(1) $\mathrm{Tr}(p) = \mathrm{Tr}(q)$ *and* $\mathrm{N}(p) = \mathrm{N}(q)$,

(2) *there is an automorphism* $\phi \in \mathrm{Aut}(A_{\mathbb{C}})$ *such that* $\phi(p) = q$.

Proof. By construction, we have $\mathrm{N}(x) = x(x^c) = B(x, x)$ for all $x \in A_{\mathbb{C}}$ and $x \mapsto \frac{1}{2} tr(x)$ equals the projection of x to $\mathbb{C}$ with respect to the direct sum decomposition

$$A_{\mathbb{C}} = \mathbb{C} \oplus W.$$

Let $p = p' + p''$, $q = q' + q''$ with $p', q' \in \mathbb{C}$, $p'', q'' \in W$. Since $\mathrm{Aut}(A_{\mathbb{C}})$ acts trivially on $\mathbb{C}$ and by linear, B-preserving transformations on W, (ii) implies that $p' = q'$ and $B(p'', p'') = B(q'', q'')$ which in turn implies $\mathrm{Tr}(p) = \mathrm{Tr}(q)$, $\mathrm{N}(p) = \mathrm{N}(q)$.

Conversely,

$$(\mathrm{Tr}(p) = \mathrm{Tr}(q)) \wedge (\mathrm{N}(p) = \mathrm{N}(q)) \implies (p' = q') \wedge (B(p'', p'') = B(q'', q'')),$$

and the latter condition implies $\exists \phi \in \mathrm{Aut}(A_{\mathbb{C}}) : \phi(p) = q$ due to Proposition 12.1. ∎

13. Constructing an auxiliary vector field

Proposition 13.1. *Let G be a complex Lie group acting holomorphically on a complex manifold X. Assume that all isotropy groups are connected and one-dimensional.*

Let Z be a non-compact Riemann surface. Let $C, F : Z \to X$ be holomorphic maps, and let

$$V = \{(g,t) \in G \times Z : g \cdot C(t) = F(t)\}.$$

Let $\pi : V \to Z$ be the natural projection map $\pi(g,t) = t$.

Then, there is an integrable holomorphic vector field $\underline{X}$ on V such that for every $t \in Z$ the associated flow (one-parameter-group) stabilizes $V_t = \pi^{-1}(\{t\})$ and coincides with the action of the isotropy group

$$G_{F(t)} = \{g \in G : g \cdot F(t) = F(t)\}$$

on V_t (with $g : (h,t) \mapsto (gh,t)$).

Proof. We define a line bundle on Z by associating to each point $p \in Z$ the Lie algebra of

$$\{g \in G : g \cdot C(p) = C(p)\}.$$

On a non-compact Riemann surface, every holomorphic line bundle is trivial. Thus, we obtain $\underline{X}$ as any nowhere vanishing section of this trivial holomorphic line bundle. ∎

14. Reduction to the case Tr = 0

Lemma 14.1. *Let $A_{\mathbb{C}}$ be a $\mathbb{C}$-algebra with an antiinvolution $(\ )^c$ such that $Z = \{x \in A_{\mathbb{C}} : x = x^c\}$ is central in $A_{\mathbb{C}}$. Let $G_{\mathbb{C}}$ be a complex Lie group acting by $\mathbb{C}$-algebra homomorphisms on $A_{\mathbb{C}}$ such that the Z is fixed pointwise. Let D be a domain in $\mathbb{C}$. Let $F, H : D \to A_{\mathbb{C}}$ be holomorphic maps. Assume that* $\mathrm{N}(F) = \mathrm{N}(H)$ *and* $\mathrm{Tr}(F) = \mathrm{Tr}(H)$ *(with* $\mathrm{Tr}(F) = F + F^c$ *and* $\mathrm{N}(F) = FF^c$*).*

Define $\widehat{F} = \frac{1}{2}(F - F^c)$ and $\widehat{H} = \frac{1}{2}(H - H^c)$. Then, the following statements hold.

(1) $\mathrm{Tr}(\widehat{F}) = 0 = \mathrm{Tr}(\widehat{H})$.

(2) $\mathrm{N}(\widehat{F}) = \mathrm{N}(\widehat{H})$.

(3) *There exists a holomorphic map $\phi : D \to G_{\mathbb{C}}$ with $F = \phi(H)$ if and only if there exists a holomorphic map $\phi : D \to G_{\mathbb{C}}$ with $\widehat{F} = \phi(\widehat{H})$.*

Proof. (1) We have

$$\mathrm{Tr}(\widehat{F}) = \frac{1}{2}\mathrm{Tr}(F - F^c) = \frac{1}{2}(F + F^c - (F^c + F)) = 0$$

and similarly, $\mathrm{Tr}(\widehat{H}) = 0$.

(2) Let $F_0 = \frac{1}{2}\operatorname{Tr}(F)$ and $H_0 = \frac{1}{2}\operatorname{Tr}(H)$. Then, $F_0^c = F_0$ and $H_0^c = H_0$. Observe that F_0 and H_0 are central in $A_{\mathbb{C}}$.

By construction,

$$F = F_0 + \widehat{F}, \quad H = H_0 + \widehat{H}, \quad \widehat{F}^c = -\widehat{F}, \quad \widehat{H}^c = -\widehat{H}.$$

We obtain

$$\begin{aligned}
\mathrm{N}(F) &= \mathrm{N}(F_0 + \widehat{F}) = (F_0 + \widehat{F}) \cdot (F_0^c + (\widehat{F})^c) \\
&= (F_0 + \widehat{F}) \cdot (F_0 - \widehat{F}) \\
&= F_0^2 - F_0 \widehat{F} + \widehat{F} F_0 - (\widehat{F})^2 \\
&= F_0^2 - \widehat{F} F_0 + \widehat{F} F_0 - (\widehat{F})^2 \text{ because } F_0 \text{ is central} \\
&= F_0^2 - (\widehat{F})^2 \\
&= \frac{1}{4}(\operatorname{Tr}(F))^2 + \mathrm{N}(\widehat{F}).
\end{aligned}$$

Similarly, we obtain

$$\mathrm{N}(H) = \frac{1}{4}(\operatorname{Tr}(H))^2 + \mathrm{N}(\widehat{H}).$$

In combination with $\mathrm{N}(F) = \mathrm{N}(H)$ and $\operatorname{Tr}(F) = \operatorname{Tr}(H)$, this yields $\mathrm{N}(\widehat{F}) = \mathrm{N}(\widehat{H})$.

(3) Note that the image of $\operatorname{Tr} : A_{\mathbb{C}} \to A_{\mathbb{C}}$ is contained in the center of $A_{\mathbb{C}}$ which is pointwise stabilized by the $G_{\mathbb{C}}$-action. It follows that for every holomorphic map $\phi : D \to A_{\mathbb{C}}$, we have

$$\begin{aligned}
F(z) &= \phi(z)(H(z)) \quad \forall z \in D \\
F_0(z) + \widehat{F}(z) &= \underbrace{\phi(z)(H_0(z))}_{=H_0(z)=F_0(z)} + \phi(z)(\widehat{H}(z)) \\
\widehat{F}(z) &= \phi(z)(\widehat{H}(z)).
\end{aligned}$$

■

15. Vanishing orders

Proposition 15.1. *Let X be a non-compact Riemann surface and V a vector space and let $f, g : X \to V$ be holomorphic maps which do not vanish identically.*

Assume that f, g have the same divisor (as defined in Definition 6.1). Then, there exist a holomorphic function $h : X \to \mathbb{C}$ and holomorphic maps $\tilde{f}, \tilde{g} : X \to V \setminus \{0\}$ such that $f = h\tilde{f}$, $g = h\tilde{g}$.

Proof. Recall that on a non-compact Riemann surface every divisor is a *principal* divisor, i.e., the divisor of a holomorphic function.

We choose a holomorphic function h on X with

$$\operatorname{div}(h) = \operatorname{div}(g) = \operatorname{div}(f)$$

and define $\tilde{f} = f/h$, $\tilde{g} = g/h$. ■

Lemma 15.2. *Let X be a Riemann surface, V a vector space, and $f : X \to V$, $\phi : X \to GL(V)$ holomorphic maps. Assume that f is not vanishing identically.*

Define $g(z) = \phi(z)(f(z))$. Then, f and g have the same divisor.

Proof. Let $\operatorname{div}(f) = \sum_p m_p\{p\}$. Then, for every $p \in X$ and $i \in \{1, \dots, n\}$, the germ of f_i at p is a divisible by $z_p^{m_p}$, where z_p is a local coordinate with $z_p(p) = 0$. Since $\phi(p)$ is linear, the components g_i likewise have germs at p which are divisible by $z_p^{m_p}$. Hence, $\operatorname{div}(g) \geq \operatorname{div}(f)$.

The same arguments show that also $\operatorname{div}(f) \geq \operatorname{div}(g)$, since

$$f(z) = \tilde{\phi}(z)(g(z))$$

for

$$\tilde{\phi}(z) = (\phi(z))^{-1}.$$

Thus, $\operatorname{div}(f) = \operatorname{div}(g)$. ■

16. Some cohomology

We need the lemma below as a preparation for the proof of Proposition 16.2.

Lemma 16.1. *Let X be a Riemann surface, $D \subset X$ a discrete subset and $\tau_0 : P \to X \setminus D$ an unramified $2:1$-covering. Then, τ_0 extends to a ramified covering $\tau : X' \to X$; i.e., we have a commutative diagram:*

$$\begin{array}{ccc} P & \hookrightarrow & X' \\ \downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \tau} \\ X \setminus D & \hookrightarrow & X \end{array}$$

Proof. Let $q \in D$, and let $\psi : U \to \Delta$ be a coordinate chart with $\psi(q) = 0$, $U \cap D = \{q\}$.

An unramified $2:1$ covering over $\Delta^* = \Delta \setminus \{0\}$ is either a product or given by a group homomorphism $\rho : \pi_1(\Delta^*) \to \mathbb{Z}/2\mathbb{Z}$. There is only one non-trivial group homomorphism from $\mathbb{Z} \cong \pi_1(\Delta^*)$ to $\mathbb{Z}/2\mathbb{Z}$.

Hence, either τ_0 restricts on $U^* = U \setminus \{q\}$ to a direct product $U^* \times \mathbb{Z}/2\mathbb{Z}$ (in which case τ_0 trivially extends through q), or we can identify the restriction to U^* as the only non-trivial $2:1$-covering, which can be realized as $z \mapsto z^2$ as a map from Δ^* to Δ^*. Thus, we obtain the following commutative diagram:

$$\begin{array}{ccc} \tau^{-1}(U^*) & \xrightarrow{\sim} & \Delta^* \\ \downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle z \mapsto z^2} \\ U^* & \xrightarrow{\psi} & \Delta^* \end{array}$$

Now, the covering map $z \mapsto z^2$ obviously extends from an unramified covering $\Delta^* \to \Delta^*$ to a ramified covering $\Delta \to \Delta$.

Performing this procedure around every point of D, we obtain the desired extension. ∎

Proposition 16.2. *Let X be a non-compact Riemann surface, D a discrete subset, $\mathcal{S}$ a sheaf which is locally isomorphic to $\underline{\mathbb{Z}}$ (the sheaf of locally constant $\mathbb{Z}$-valued functions) on $X \setminus D$. Let*

$$0 \to \mathcal{S} \to \mathcal{O}_X \to \mathcal{A} \to 0 \tag{16.1}$$

be a short exact sequence of $\mathcal{O}_X$ module sheaves. Then, $H^1(X, \mathcal{A}) = \{0\}$ and $H^2(X, \mathcal{S}) = \{0\}$.

Proof. X is Stein; hence, $H^k(X, \mathcal{O}) = 0$ for $k > 0$. Therefore, the long exact cohomology sequence associated to the above sequence of sheaves (16.1) implies

$$H^1(X, \mathcal{A}) \cong H^2(X, \mathcal{S}).$$

Let $\mathcal{S}_0$ be the sheaf on X defined by

$$\mathcal{S}_0(U) = \begin{cases} \{0\} & \text{if } U \cap D \neq \{\}, \\ \mathcal{S}(U) & \text{if } U \cap D = \{\}. \end{cases}$$

Then, we have a short exact sequence

$$0 \to \mathcal{S}_0 \to \mathcal{S} \to \mathcal{F} \to 0, \tag{16.2}$$

where $\mathcal{F}$ is a skyscraper sheaf supported on D. Thus, $H^k(X, \mathcal{F}) = 0$ for $k > 0$. Consequently, the long exact cohomology sequence associated to (16.2) implies

$$H^2(X, \mathcal{S}) \cong H^2(X, \mathcal{S}_0).$$

By construction, $\mathcal{S}_0$ is a sheaf on $X \setminus D$ locally isomorphic to $\underline{\mathbb{Z}}$. $(\mathbb{Z}, +)$ admits only one non-trivial automorphism, namely, $n \mapsto -n$. Hence, we may regard $\mathcal{S}_0$ as the sheaf of sections in a locally trivial fiber bundle $B \to X \setminus D$ with structure group $\mathbb{Z}/2\mathbb{Z}$ and typical fiber $\mathbb{Z}$. We consider the sub-bundle P with typical fiber $\{+1, -1\}$. Such a bundle is an unramified 2:1-covering $\tau_0 : P \to X \setminus D$. It extends to a ramified 2:1-covering $\tau : X' \to X$ by adding one point above each point of D.

We consider the natural sheaf homomorphism $\underline{\mathbb{Z}}_X \to \tau_* \underline{\mathbb{Z}}_{X'}$ given by $f \mapsto f \circ \tau$. We define a sheaf homomorphism α from $\tau_* \underline{\mathbb{Z}}_{X'}$ to $\mathcal{S}_0$ as follows: given a $\mathbb{Z}$-valued function f on $\{-1, +1\}$, let

$$d = |f(1) - f(-1)|,$$

and associate

$$\alpha(f) \stackrel{\text{def}}{=} \begin{cases} d & \text{if } f(1) \geq f(-1), \\ -d & \text{if } f(1) < f(-1). \end{cases}$$

This yields a short exact sequence of sheaves

$$0 \to \underline{\mathbb{Z}}_X \to \tau_* \underline{\mathbb{Z}}_{X'} \xrightarrow{\alpha} \mathcal{S}_0 \to 0.$$

We note that outside D the covering map τ is locally biholomorphic, while for a point $p \in D$, the preimage of a small neighborhood of p in X will be a small neighborhood of $\tau^{-1}(p)$ in X' and will be contractible, since X' is a complex space.

It follows that the higher direct image sheaves $R^q \tau_* \underline{\mathbb{Z}}$ $(q > 0)$ are trivial. We consider the Leray spectral sequence for the sheaf $\underline{\mathbb{Z}}$ on X' and the map $\tau : X' \to X$:

$$H^{p+q}(X', \underline{\mathbb{Z}}) = E^{p+q} \Leftarrow E_2^{pq} = H^p(X, R^q \tau_* \underline{\mathbb{Z}}).$$

Since $R^q \tau_* \underline{\mathbb{Z}} = 0 \,\forall q > 0$, this spectral sequence degenerates. Therefore,

$$H^k(X, \tau_* \underline{\mathbb{Z}}) \cong H^k(X', \underline{\mathbb{Z}}) \cong H^k(X', \mathbb{Z}).$$

Since X' and X are non-compact Riemann surfaces, we have $H^k(X', \mathbb{Z}) = H^k(X, \mathbb{Z}) = 0$ for $k \geq 2$, and therefore,

$$\cdots \to \underbrace{H^2(X', \mathbb{Z})}_{=0} \to H^2(X, \mathcal{S}_0) \to \underbrace{H^3(X, \mathbb{Z})}_{=0} \to \cdots$$

Therefore,

$$H^1(X, \mathcal{A}) \cong H^2(X, \mathcal{S}) \cong H^2(X, \mathcal{S}_0) = \{0\}. \qquad \blacksquare$$

17. Automorphisms and conjugacy

For $\mathbb{H}$ and $\mathbb{H}_{\mathbb{C}}$, every $\mathbb{R}$-(respectively, $\mathbb{C}$-) algebra automorphism ϕ is *inner*, i.e., given by conjugation with an element: $\exists q : \phi : x \mapsto qxq^{-1}$. Conversely, for every $q \in \mathbb{H}$ (respectively, $q \in \mathbb{H}_{\mathbb{C}}$) conjugation by q defines an automorphism. This automorphism is trivial if and only if q is in the center.

Therefore, we have a short exact sequence of Lie groups.

$$1 \to \mathbb{C}^* \to \mathbb{H}_{\mathbb{C}}^* \xrightarrow{\zeta} G_{\mathbb{C}} \to 1 \tag{17.1}$$

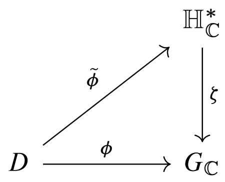
with $\zeta(g) : x \mapsto gxg^{-1}$.

In particular, $G_{\mathbb{C}} \cong \mathbb{H}_{\mathbb{C}}^* / \mathbb{C}^*$, which implies that $\mathbb{H}_{\mathbb{C}}^*$ is a $\mathbb{C}^*$-principal bundle over $G_{\mathbb{C}}$ (see, e.g., [26, Proposition 13.25]).

Proposition 17.1. *Let D be an open subset in $\mathbb{C}$ and let $\phi : D \to G_{\mathbb{C}}$ be a holomorphic map. Then, there exists a holomorphic map $\tilde{\phi} : D \to \mathbb{H}_{\mathbb{C}}^*$ with $\phi = \zeta \circ \tilde{\phi}$.*

Proof. Due to (17.1), we have a $\mathbb{C}^*$-principal bundle on $G_{\mathbb{C}}$ which we may pull back via ϕ to obtain a $\mathbb{C}^*$-principal bundle on D. But D is a (not necessarily connected) Stein Riemann surface, and therefore, every $\mathbb{C}^*$-principal bundle on D is holomorphically trivial.

Thus, it admits a section. Such a section corresponds to a lifting as follows:



■

18. Proof of the main theorem

We are now in a position to prove our main Theorem 1.1.

First, we prove the most difficult part of Theorem 1.1, which is the following.

Theorem 18.1. *Let $A = \mathbb{H}$ and $A_{\mathbb{C}} = A \otimes_{\mathbb{R}} \mathbb{C}$.*

Let $D \subset \mathbb{C}$ be a domain which is invariant under conjugation.

Let $F, H : D \to A_{\mathbb{C}} \setminus \{0\}$ be holomorphic maps such that $\overline{F(\bar{z})} = F(z)$, $\overline{H(\bar{z})} = H(z)$, $\mathrm{Tr}(F) = \mathrm{Tr}(H) = 0$, $\mathrm{N}(F) = \mathrm{N}(H)$.[5] *Then, there exists a holomorphic map $\phi : D \to G_{\mathbb{C}}$ such that*

$$\phi(z)(F(z)) = H(z) \ \forall z \in D.$$

Proof. Throughout the proof, we will use the fact that D is a non-compact Riemann surface.

Note that we assume $\mathrm{Tr}(F) = \mathrm{Tr}(H) = 0$. It follows that the images $F(D)$, $H(D)$ are contained in $W \otimes_{\mathbb{R}} \mathbb{C}$.

Thus, we may regard F and H as holomorphic maps from D to $(W \otimes_{\mathbb{R}} \mathbb{C}) \setminus \{0\} = (W \otimes_{\mathbb{R}} \mathbb{C}) \cap (A_{\mathbb{C}} \setminus \{0\})$.

From our assumption on F and H, we deduce that for every $t \in D$ there is an element $g \in G_{\mathbb{C}}$ with $H(t) = g(F(t))$ (Corollary 12.5).

The case where both F and H are constant is trivial. Hence, we may assume that at least one of the two maps is not constant. Without loss of generality, F is not constant.

We define

$$V = \{(\alpha, z) \in G_{\mathbb{C}} \times D : F(z) = \alpha H(z)\}.$$

The isotropy groups for the $G_{\mathbb{C}}$-action on $(W \otimes_{\mathbb{R}} \mathbb{C}) \setminus \{0\}$ have all the same dimension (namely, 1 if $A = \mathbb{H}$) due to Corollary 12.3.

Therefore, we may apply Proposition 9.4. It follows that for every $p \in D$ there is an open neighborhood U of p in D and a holomorphic map $\psi : U \to G_{\mathbb{C}}$ with

$$F(z) = \psi(z) H(z) \quad \forall z \in U.$$

[5] We need no condition on cdiv because the assumptions $\mathrm{Tr}(F) = \mathrm{Tr}(H) = 0$ and $F(D), H(D) \subset A_{\mathbb{C}} \setminus \{0\}$ *imply* that $\mathrm{cdiv}(F)$ and $\mathrm{cdiv}(H)$ are empty.

Thanks to Proposition 13.1, we obtain a vector field which gives us a one-parameter group acting transitively on the fibers of $V \to D$; i.e., there is an action

$$\mu : (\mathbb{C}, +) \times V \to V.$$

We define $\mathcal{A}$ as the sheaf on D of holomorphic maps ξ from D to $G_{\mathbb{C}}$ such that $\xi(t)$ is contained in the isotropy group

$$G_{F(t)} = \{g \in G : g(F(t)) = F(t)\}$$

for every t.

Due to Proposition 13.1, we may identify $\mathcal{A}$ with a sheaf of fiber-preserving automorphisms; namely, for an open subset $U \subset D$, we define $\mathcal{A}(U)$ as the set of biholomorphic self maps of $\pi^{-1}(U)$ which may be written as

$$\eta_\zeta : x \mapsto \mu(\zeta(\pi(x)), x)$$

for some holomorphic function $\zeta \in \mathcal{O}(U)$.

The map associating η_ζ to ζ defines a morphism of sheaves on D. Let $\mathcal{S}$ denote its kernel. Then, we have a short exact sequence of sheaves

$$0 \to \mathcal{S} \to \mathcal{O} \xrightarrow{\zeta \mapsto \eta_\zeta} \mathcal{A} \to 0,$$

where $\mathcal{S}$ may be regarded as the sheaf of those holomorphic functions ζ for which $\mu(\zeta(t))$ equals the identity map on the fiber $\pi^{-1}(t)$.

Generically, the isotropy group of $G_{\mathbb{C}}$ at a point in $A_{\mathbb{C}}$ is isomorphic to $\mathbb{C}^*$. Elsewhere, the isotropy group is $\mathbb{C}$. Hence, the sheaf $\mathcal{S}$ is locally isomorphic to $\underline{\mathbb{Z}}$ on an open subset Ω of D and trivial on the complement $D \setminus \Omega$.

Next, we want to show that $H^1(D, \mathcal{A}) = \{0\}$.

If Ω is empty, then $\mathcal{S}$ is trivial, and consequently, $\mathcal{A} \cong \mathcal{O}$. This implies $H^1(D, \mathcal{A}) = \{0\}$ because D is Stein.

Thus, we may assume that Ω is not empty. Then, Ω is Zariski open and dense and its complement $D \setminus \Omega$ is discrete. Then, $H^1(D, \mathcal{A}) = 0$ follows from Proposition 16.2.

We choose an open cover $(U_i)_i$ of D with holomorphic maps $\psi_i : U_i \to G_{\mathbb{C}}$ such that

$$F(z) = \psi_i(z)H(z) \quad \forall z \in U_i.$$

(This is possible thanks to Proposition 9.4.)

On each intersection $U_{ij} = U_i \cap U_j$, we have

$$F(z) = \psi_i(z)H(z) \quad \text{and} \quad F(z) = \psi_j(z)H(z),$$

implying that

$$\psi_{ji}(z) \stackrel{\text{def}}{=} \psi_j(z) \circ (\psi_i(z))^{-1}$$

is contained in the isotropy group at $F(z)$.

Thus, ψ_{ij} defines a 1-cocycle of $\mathcal{A}$. Since $H^1(D, \mathcal{A}) = 0$, this cocycle is a coboundary; i.e., there are $\phi_i \in \Gamma(U_i, \mathcal{A})$ with $\psi_{ij} = \phi_i(\phi_j)^{-1}$.

Therefore,

$$\begin{aligned} \psi_i(z)(\psi_j(z))^{-1} &= \psi_{ij}(z) = \phi_i(z)(\phi_j(z))^{-1} \\ \implies \psi_i(z) &= \phi_i(z)(\phi_j(z))^{-1}(\psi_j(z)) \\ \implies (\phi_i(z))^{-1}\psi_i(z) &= (\phi_j(z))^{-1}(\psi_j(z)). \end{aligned}$$

It follows that

$$\phi(z) = (\phi_i(z))^{-1}\psi_i(z)$$

is well defined. Since $F(z) = \psi_i(z)H(z)$ and $\phi_i(z)$ is in the isotropy group of $G_{\mathbb{C}}$ at $F(z)$, we infer

$$F(z) = \phi(z)H(z).$$

This completes the proof. ∎

Proof of Theorem 1.1. First, we deal with the case where neither f nor h is slice preserving.

We proceed as follows:

$$\begin{array}{ccccc} \text{(i)} & \Longleftrightarrow & \text{(ii)} & \Longleftrightarrow & \text{(iii)} \\ & & \Downarrow & \nearrow & \\ \text{(v)} & \Longleftrightarrow & \text{(iv)} & & \end{array}$$

(ii) $\Longrightarrow$ (iv):

By assumption, we have $\mathrm{Tr}(F) = \mathrm{Tr}(H)$. Define

$$\hat{F} = \frac{1}{2}(F - F^c), \quad \hat{H} = \frac{1}{2}(H - H^c).$$

Evidently, $\mathrm{Tr}(\hat{H}) = \mathrm{Tr}(\hat{F}) = 0$. Moreover, $\mathrm{N}(F) = \mathrm{N}(H)$ in combination with Lemma 14.1 implies that $\mathrm{N}(\hat{F}) = \mathrm{N}(\hat{H})$.

With respect to the decomposition $A_{\mathbb{C}} = Z \oplus W \otimes_{\mathbb{R}} \mathbb{C}$ the map $\hat{F}$, respectively, $\hat{H}$ is just the second component of F, respectively, H. By the definition of the central divisor (introduced in Section 6.2) this implies that $\mathrm{cdiv}(F) = \mathrm{cdiv}(\hat{F})$ and $\mathrm{cdiv}(H) = \mathrm{cdiv}(\hat{H})$.

Since $\mathrm{cdiv}(F) = \mathrm{cdiv}(H)$, it follows that there are holomorphic maps $\tilde{F}, \tilde{H} : D \to A_{\mathbb{C}} \setminus \{0\}$ and $\lambda : D \to \mathbb{C}$ such that

$$\hat{F} = \lambda \tilde{F}, \quad \hat{H} = \lambda \tilde{H}$$

(Proposition 15.1). Observe that

$$\begin{aligned} 0 = \mathrm{Tr}(\hat{F}) = \lambda\, \mathrm{Tr}(\tilde{F}), \quad \mathrm{N}(\hat{F}) = \lambda^2 N(\tilde{F}), \\ 0 = \mathrm{Tr}(\hat{H}) = \lambda\, \mathrm{Tr}(\tilde{H}), \quad \mathrm{N}(\hat{H}) = \lambda^2 N(\tilde{H}), \end{aligned}$$

and that λ does not vanish identically because f and h are not slice preserving. Hence, $\mathrm{Tr}(\widetilde{F}) = 0 = \mathrm{Tr}(\widetilde{H})$ and $\mathrm{N}(\widetilde{F}) = \mathrm{N}(\widetilde{H})$, and Theorem 18.1 implies that there is a holomorphic map $\phi : D \to G_{\mathbb{C}}$ such that

$$\phi(z)(\widetilde{F}(z)) = \widetilde{H}(z) \quad \forall z \in D,$$

which in turn implies

$$\phi(z)(\widehat{F}(z)) = \widehat{H}(z) \quad \forall z \in D$$

because $G_{\mathbb{C}}$ acts linearly, $\widehat{F} = h\widetilde{F}$ and $\widehat{H} = h\widetilde{H}$. Finally,

$$\phi(z)(F(z)) = H(z) \quad \forall z \in D$$

follows via Lemma 14.1.

(iv) $\Longrightarrow$ (iii): the implication (iv) $\Longrightarrow$ $\mathrm{cdiv}(F) = \mathrm{cdiv}(H)$ is due to Lemma 15.2, the other assertion is obvious.

For (iii) $\Longleftrightarrow$ (ii), see Corollary 12.5.

For (i) $\Longleftrightarrow$ (ii), see Proposition 2.3 and Section 6.2.

(v) $\Longrightarrow$ (iv): every $\alpha \in \mathbb{H}_{\mathbb{C}}^*$ defines an automorphism of $\mathbb{H}_{\mathbb{C}}$ via $q \mapsto \alpha q \alpha^{-1}$. This defines a natural map from $\mathbb{H}_{\mathbb{C}}^*$ to $G_{\mathbb{C}}$. Composition with this map yields the desired implication.

(iv) $\Longrightarrow$ (v) follows from Proposition 17.1.

This finishes the proof for case (a). Let us deal now with the case (b); i.e., we assume that f is slice preserving.

(i) $\Longleftrightarrow$ (ii) is due to the correspondence between slice regular functions and stem functions.

(ii) $\Longrightarrow$ (v) $\Longrightarrow$ (iv) $\Longrightarrow$ (iii) is trivial.

(iii) $\Longrightarrow$ (ii): because f is slice preserving, all the values of F are contained in the center $Z \cong \mathbb{C}$ of $\mathbb{H}_{\mathbb{C}}$. But $G_{\mathbb{C}}$ fixes the center pointwise. Hence, the statement is proved. ∎

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