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Serre's theorem for coherent sheaves via Auslander's techniques

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Abstract. For an abelian category and a distinguished object with a graded endomorphism ring a necessary and sufficient criterion is given so that the category is equivalent to the abelian quotient of the category of finitely presented graded modules modulo the Serre subcategory of finite length modules. A particular example is the category of coherent sheaves on a projective variety, following a theorem of Serre from 1955. The proof uses Auslander's theory of coherent functors, and there are no noetherian assumptions. A theorem of Lenzing for representations of hereditary algebras is given as an application.

1. Introduction

Serre's theorem relating coherent sheaves on a projective variety to graded modules over some appropriate graded ring is a cornerstone of non-commutative geometry, for instance, by the non-commutative analogue of this theorem due to Artin and Zhang [1, 21]. In this note, we revisit the theory, motivated by work of Lenzing which introduces for any finite dimensional hereditary algebra of infinite representation type a category of coherent sheaves [15]. The corresponding graded ring is the preprojective algebra [7], which appeared already in work of Gel'fand and Ponomarev [9] as well as in work of Dlab and Ringel [8]. The novel aspect of the present work is a new formulation and proof of the analogue of Serre's theorem which is based on Auslander's theory of coherent functors [3]. In particular, we do not impose any noetherian assumptions. This is necessary because the preprojective algebra is noetherian only when the corresponding hereditary algebra is of tame representation type [6, 7].

2. An analogue of Serre's theorem

We formulate the analogue of Serre's theorem in the setting of Krull–Schmidt categories. Let $\mathcal{C}$ be an additive category that is *Krull–Schmidt*. Thus, any object X admits a finite decomposition

$$X = \bigoplus_i X_i$$

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such that each X_i is indecomposable with a local endomorphism ring. The *radical* $\operatorname{Rad}\mathcal{C}$ of $\mathcal{C}$ is the ideal given by subgroups

$$\operatorname{Rad}(X,Y) \subseteq \operatorname{Hom}(X,Y) \quad (X,Y \in \mathcal{C})$$

consisting of all morphisms $X \to Y$ such that for any pair of decompositions $X = \bigoplus_i X_i$ and $Y = \bigoplus_j Y_j$ into indecomposable objects no component $X_i \to Y_j$ is an isomorphism. Let $\operatorname{Rad}^0 \mathcal{C}$ denote the ideal of all morphisms in $\mathcal{C}$ and for $n \geq 0$, we set $\operatorname{Rad}^{n+1} \mathcal{C} = (\operatorname{Rad}\mathcal{C})(\operatorname{Rad}^n \mathcal{C})$. We say that the *length* of a morphism $X \to Y$ is bounded by n if all morphisms $X' \to Y$ from $\operatorname{Rad}^n \mathcal{C}$ factor through $X \to Y$.

For a $\mathbb{Z}$-graded ring $A = \bigoplus_{n\in\mathbb{Z}} A_n$, we consider the category $\operatorname{GrMod} A$ of $\mathbb{Z}$-graded right A-modules. Let $\operatorname{grmod} A$ denote the full subcategory of finitely presented modules and $\operatorname{grproj} A$ denote the full subcategory of finitely generated projective modules. We write $\operatorname{grmod}_0 A$ for the full subcategory of all finite length modules and always assume $\operatorname{grmod}_0 A \subseteq \operatorname{grmod} A$; this is automatic if the ring is right noetherian.

We say that a graded ring A is *right coherent* if $\operatorname{grmod} A$ is an abelian category, and A is *semiperfect* if $\operatorname{grproj} A$ is Krull–Schmidt. These properties hold, for instance, when A is a right noetherian algebra over a field such that each homogeneous component A_n is finite dimensional; see [2, 11] for the Krull–Schmidt property.

Theorem 2.1. *Let $(\mathcal{A}, C, \sigma)$ be a triple consisting of an abelian category $\mathcal{A}$, a distinguished object C, and an equivalence $\sigma\colon \mathcal{A} \xrightarrow{\sim} \mathcal{A}$. Suppose that* $\operatorname{Hom}(C, \sigma^n C) = 0$ *for all $n < 0$ and that the graded ring*

$$A = \bigoplus_{n\geq 0} \operatorname{Hom}(C, \sigma^n C)$$

is right coherent and semiperfect with $\operatorname{grmod}_0 A \subseteq \operatorname{grmod} A$. *Then, the assignment*

$$X \mapsto \Gamma_*(X) := \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}(C, \sigma^n X) \quad (X \in \mathcal{A})$$

admits a partial left adjoint $T\colon \operatorname{grmod} A \to \mathcal{A}$, which is right exact and determined by $T(A) = C$. The functor T is exact and induces an equivalence

$$(\operatorname{grmod} A)/(\operatorname{grmod}_0 A) \xrightarrow{\sim} \mathcal{A}$$

if and only if the full subcategory $\mathcal{C} \subseteq \mathcal{A}$ consisting of all direct summands of finite direct sums of objects $\sigma^n C$ with $n \in \mathbb{Z}$ satisfies the following statements.

(A1) *Every object $X \in \mathcal{A}$ admits an epimorphism $C \to X$ with $C \in \mathcal{C}$, which can be chosen in* $\operatorname{Rad}\mathcal{C}$ *when $X \in \mathcal{C}$.*

(A2) *A morphism in $\mathcal{C}$ has finite length if it is an epimorphism in $\mathcal{A}$.*

Fix a triple $(\mathcal{A}, C, \sigma)$ as above. The algebra $A = \bigoplus_{n\geq 0} A_n$ is the *orbit algebra* of C with multiplication given by $xy = (\sigma^q x) \circ y$ for $x \in A_p$ and $y \in A_q$. The assignment

$X \mapsto \Gamma_*(X)$ yields an equivalence $\mathscr{C} \xrightarrow{\sim} \operatorname{grproj} A$ and $T\colon \operatorname{grmod} A \to \mathscr{A}$ is the right exact functor extending

$$\operatorname{grproj} A \xrightarrow{\sim} \mathscr{C} \hookrightarrow \mathscr{A}.$$

Thus, we have the following diagram:

$$\begin{array}{ccc} \operatorname{grmod} A & \hookrightarrow & \operatorname{GrMod} A \\ & {\scriptstyle T}\searrow \quad \nearrow{\scriptstyle \Gamma_*} & \\ & \mathscr{A} & \end{array}$$

where Γ_* corestricts to a right adjoint of T if and only if $\mathscr{C} \subseteq \mathscr{A}$ is contravariantly finite so that $\Gamma_*(X)$ is a finitely presented A-module for each $X \in \mathscr{A}$; see Remark 3.7. Nonetheless, we have for all $X \in \operatorname{grmod} A$ and $Y \in \mathscr{A}$ the adjointness isomorphism

$$\operatorname{Hom}_{\mathscr{A}}(T(X), Y) \cong \operatorname{Hom}_A(X, \Gamma_*(Y)).$$

The conditions (A1)–(A2) express the 'ampleness' of the pair (C, σ). The prototypical example is a theorem of Serre [21], and we refer to [1,18] for non-commutative analogues.

Example 2.2 (Serre). Consider the projective variety $\mathbb{P}^r(K)$ given by an algebraically closed field K and an integer $r \geq 1$. Let $\operatorname{coh} \mathbb{P}^r(K)$ denote the category of coherent sheaves on $\mathbb{P}^r(K)$. The assignment

$$\mathcal{F} \mapsto \Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \operatorname{Hom}(\mathcal{O}, \mathcal{F}(n))$$

yields a functor $\operatorname{coh} \mathbb{P}^r(K) \to \operatorname{grmod} A$ to the category of graded modules over the orbit algebra of the structure sheaf

$$A = \bigoplus_{n \geq 0} \operatorname{Hom}(\mathcal{O}, \mathcal{O}(n)) \cong K[t_0, \ldots, t_r].$$

Taking a module M to its associated sheaf $\widetilde{M}$ provides a left adjoint functor, which is exact and annihilates all modules of finite length. In [21, no. 65], it is shown that the canonical morphism $\widetilde{\Gamma_*(\mathcal{F})} \to \mathcal{F}$ is an isomorphism, while kernel and cokernel of the canonical morphism $M \to \Gamma_*(\widetilde{M})$ are of finite length. An equivalent statement is that $M \mapsto \widetilde{M}$ induces an equivalence

$$(\operatorname{grmod} A)/(\operatorname{grmod}_0 A) \xrightarrow{\sim} \operatorname{coh} \mathbb{X}.$$

Let us comment on the notions 'finite length' and 'torsion' for graded modules because it is common to work modulo the category of torsion modules in the context of Serre's theorem.

Remark 2.3. Let $A = \bigoplus_{n \geq 0} A_n$ be a graded ring, and suppose that the ring $A_0 = A/A_{\geq 1}$ is semisimple. Then, a graded A-module M has finite length if and only if M is finitely generated and torsion, so $M_n = 0$ for $n \gg 0$. To see this, fix a finitely generated module M and consider the sequence

$$\cdots \twoheadrightarrow M/M_{\geq 2} \twoheadrightarrow M/M_{\geq 1} \twoheadrightarrow M/M_{\geq 0} \twoheadrightarrow M/M_{\geq -1} \twoheadrightarrow \cdots,$$

which stabilises in the right direction since M is finitely generated. It stabilises in both directions if and only if M has finite length, since each subquotient has finite length (using the assumption $\operatorname{grmod}_0 A \subseteq \operatorname{grmod} A$ so that each subquotient is finitely generated over A_0).

3. A relative Auslander formula

The proof of the analogue of Serre's theorem is based on a relative version of Auslander's formula; it is established in this section and may be of independent interest. For instance, the Gabriel–Popescu theorem for Grothendieck categories is another result that can be explained in terms of this Auslander formula.

Let $\mathcal{C}$ be an additive category. Following Auslander [3] an additive functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ into the category of abelian groups is called *finitely presented* or *coherent* if it admits a presentation

$$\operatorname{Hom}(-,X) \to \operatorname{Hom}(-,Y) \to F \to 0. \tag{3.1}$$

In that case, $X \to Y$ is called the *presenting morphism*. Let $\operatorname{mod}\mathcal{C}$ denote the category of finitely presented functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$. The assignment $X \mapsto \operatorname{Hom}(-,X)$ yields the fully faithful *Yoneda functor* $\mathcal{C} \to \operatorname{mod}\mathcal{C}$. We collect some basic facts from [3] which will be used throughout without further reference. These properties reflect the fact that $\mathcal{C} \to \operatorname{mod}\mathcal{C}$ is nothing but the completion of $\mathcal{C}$ under finite colimits.

A morphism $X \to Y$ is a *weak kernel* of a morphism $Y \to Z$ if the induced sequence $\operatorname{Hom}(C,X) \to \operatorname{Hom}(C,Y) \to \operatorname{Hom}(C,Z)$ is exact for all objects C.

Lemma 3.1. *For an additive category $\mathcal{C}$ we have the following.*

(1) *The category* $\operatorname{mod}\mathcal{C}$ *is additive and every morphism has a cokernel; it is abelian iff every morphism in $\mathcal{C}$ has a weak kernel.*

(2) *The Yoneda functor $\mathcal{C} \to$* $\operatorname{mod}\mathcal{C}$ *admits an exact left adjoint if $\mathcal{C}$ is abelian.*

(3) *An additive functor $\mathcal{C} \to \mathcal{A}$ into a category with cokernels extends to a right exact functor* $\operatorname{mod}\mathcal{C} \to \mathcal{A}$.

(4) *An additive functor $f\colon \mathcal{C} \to \mathcal{D}$ extends to a right exact functor $f^*\colon$* $\operatorname{mod}\mathcal{C} \to \operatorname{mod}\mathcal{D}$*, and f^* is fully faithful if and only if f is fully faithful.*

Proof. See [3] or [12, Sections 2.1–2.3]. ∎

Example 3.2. For a graded ring A the inclusion $\operatorname{grproj} A \to \operatorname{grmod} A$ induces an equivalence $\operatorname{mod}(\operatorname{grproj} A) \xrightarrow{\sim} \operatorname{grmod} A$.

Let $\mathcal{A}$ be an abelian category. A functor $F \in \operatorname{mod} \mathcal{A}$ with presentation (3.1) is called *effaceable* if it belongs to the kernel of the left adjoint of the Yoneda functor $\mathcal{A} \to \operatorname{mod} \mathcal{A}$. An equivalent condition is that the presenting morphism $\phi\colon X \to Y$ is an epimorphism because the left adjoint preserves cokernels, so it sends $F = \operatorname{Coker} \operatorname{Hom}(-, \phi)$ to $\operatorname{Coker} \phi$. We write $\operatorname{eff} \mathcal{A}$ for the full subcategory of effaceable functors and note that it is a Serre subcategory of $\operatorname{mod} \mathcal{A}$.

Proposition 3.3 (Auslander). *The right exact functor* $\operatorname{mod} \mathcal{A} \to \mathcal{A}$ *extending the identity* $\mathcal{A} \to \mathcal{A}$ *is exact and induces an equivalence*

$$(\operatorname{mod} \mathcal{A})/(\operatorname{eff} \mathcal{A}) \xrightarrow{\sim} \mathcal{A}. \qquad \blacksquare$$

This result from [3] is also known as *Auslander's formula* [16]. We need the following relative version. Let $i\colon \mathcal{C} \to \mathcal{A}$ denote the inclusion of a full additive subcategory. We view the induced functor $i^*\colon \operatorname{mod} \mathcal{C} \to \operatorname{mod} \mathcal{A}$ as an inclusion and set

$$\operatorname{eff}(\mathcal{A}, \mathcal{C}) := \operatorname{mod} \mathcal{C} \cap \operatorname{eff} \mathcal{A}.$$

The subcategory $\mathcal{C}$ *generates* $\mathcal{A}$ if every object $X \in \mathcal{A}$ admits an epimorphism $C \to X$ with $C \in \mathcal{C}$.

Proposition 3.4. *Let* $\mathcal{A}$ *be an abelian category and* $\mathcal{C} \subseteq \mathcal{A}$ *a full additive subcategory such that* $\operatorname{mod} \mathcal{C}$ *is abelian. Then,* $\mathcal{C}$ *generates* $\mathcal{A}$ *if and only if the right exact functor* $\operatorname{mod} \mathcal{C} \to \mathcal{A}$ *extending the inclusion* $\mathcal{C} \to \mathcal{A}$ *is exact and induces an equivalence*

$$(\operatorname{mod} \mathcal{C})/(\operatorname{eff}(\mathcal{A}, \mathcal{C})) \xrightarrow{\sim} \mathcal{A}.$$

Proof. We write the right exact functor $\operatorname{mod} \mathcal{C} \to \mathcal{A}$ as composite

$$\operatorname{mod} \mathcal{C} \xrightarrow{i^*} \operatorname{mod} \mathcal{A} \twoheadrightarrow \mathcal{A}$$

with i^* induced by the inclusion $i\colon \mathcal{C} \to \mathcal{A}$. The functor i^* is fully faithful, and it is exact when any weak kernel sequence in $\mathcal{C}$ is exact in $\mathcal{A}$. The latter property follows when $\mathcal{C}$ generates $\mathcal{A}$. Moreover, the condition that $\mathcal{C}$ generates is equivalent to the property of $\operatorname{mod} \mathcal{C} \to \mathcal{A}$ to be essentially surjective.

Now, suppose that $\mathcal{C}$ generates $\mathcal{A}$. Then, the exact functor i^* induces a functor

$$(\operatorname{mod} \mathcal{C})/(\operatorname{eff}(\mathcal{A}, \mathcal{C})) \to (\operatorname{mod} \mathcal{A})/(\operatorname{eff} \mathcal{A})$$

by the definition of $\operatorname{eff}(\mathcal{A}, \mathcal{C})$. The morphisms in the localised categories are computed via a calculus of fractions. From this, it follows that the functor is fully faithful when the following cofinality condition holds: Every $F \in \operatorname{mod} \mathcal{A}$ admits a morphism $F' \to F$ such that $F' \in \operatorname{mod} \mathcal{C}$ and the image under $\operatorname{mod} \mathcal{A} \twoheadrightarrow \mathcal{A}$ is an isomorphism; cf. [12, Lemma 1.2.5]. For this cofinality, see Lemma 3.5 below, and it remains to compose this functor with the equivalence $(\operatorname{mod} \mathcal{A})/(\operatorname{eff} \mathcal{A}) \xrightarrow{\sim} \mathcal{A}$. $\blacksquare$

Lemma 3.5. *Let $\mathcal{A}$ be an abelian category and $\mathcal{C} \subseteq \mathcal{A}$ a full additive subcategory such that* $\operatorname{mod} \mathcal{C}$ *is abelian and $\mathcal{C}$ generates $\mathcal{A}$. Then, every $F \in \operatorname{mod} \mathcal{A}$ admits a morphism $F' \to F$ such that $F' \in \operatorname{mod} \mathcal{C}$ and the image under* $\operatorname{mod} \mathcal{A} \twoheadrightarrow \mathcal{A}$ *is an isomorphism.*

Proof. Let $X \to Y$ be the morphism presenting F. We find objects C_i, D_i in $\mathcal{C}$ and morphisms such that the following diagram in $\mathcal{A}$ commutes and has exact rows:

$$\begin{array}{ccccccc} C_1 & \xrightarrow{\gamma} & C_0 & \longrightarrow & X & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ D_1 & \xrightarrow{\delta} & D_0 & \longrightarrow & Y & \longrightarrow & 0 \end{array}$$

This induces the following commutative diagram with exact rows in $\operatorname{mod} \mathcal{A}$:

$$\begin{array}{ccccccc} \operatorname{Coker}\operatorname{Hom}(-,\gamma) & \longrightarrow & \operatorname{Coker}\operatorname{Hom}(-,\delta) & \longrightarrow & F' & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \operatorname{Hom}(-,X) & \longrightarrow & \operatorname{Hom}(-,Y) & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

The top row lies in $\operatorname{mod} \mathcal{C}$. By construction, the vertical morphisms on the left and in the middle are mapped to isomorphisms under $\operatorname{mod} \mathcal{A} \twoheadrightarrow \mathcal{A}$. It follows that $F' \to F$ has the desired properties. ∎

Remark 3.6. One may think of the relative Auslander formula as a variation of Popescu–Gabriel theorem [19] which says, for a Grothendieck category $\mathcal{A}$ and a generator G that the left adjoint of $\operatorname{Hom}(G,-)\colon \mathcal{A} \to \operatorname{Mod} A$ with $A = \operatorname{End}(G)$ induces an equivalence

$$(\operatorname{Mod} A)/\mathcal{L} \xrightarrow{\sim} \mathcal{A}$$

for some appropriate localising subcategory $\mathcal{L} \subseteq \operatorname{Mod} A$. In fact, we may take for $\mathcal{C} \subseteq \mathcal{A}$ the full subcategory all coproducts of copies of G. Then, the colimit preserving composite $\operatorname{Mod} A \to \operatorname{mod} \mathcal{C} \to \mathcal{A}$ that identifies A with G induces equivalences

$$(\operatorname{Mod} A)/\mathcal{L} \xrightarrow{\sim} (\operatorname{mod} \mathcal{C})/(\operatorname{eff}(\mathcal{A},\mathcal{C})) \xrightarrow{\sim} \mathcal{A}.$$

Remark 3.7. For a full additive subcategory $\mathcal{C} \subseteq \mathcal{A}$ the right exact functor $\operatorname{mod} \mathcal{C} \to \mathcal{A}$ admits a right adjoint if and only if the subcategory $\mathcal{C}$ is *contravariantly finite*, so each object $X \in \mathcal{A}$ admits a morphism $\pi\colon C \to X$ such that $C \in \mathcal{C}$ and each morphism $C' \to X$ with $C' \in \mathcal{C}$ factors through π. This condition means that $\operatorname{Hom}(-,X)|_{\mathcal{C}}$ belongs to $\operatorname{mod} \mathcal{C}$ for all $X \in \mathcal{A}$, so $X \mapsto \operatorname{Hom}(-,X)|_{\mathcal{C}}$ provides the right adjoint.

4. Serre's theorem via Auslander's techniques

We use the relative Auslander formula from Proposition 3.4 to prove Theorem 2.1. We need some preparations and fix an additive category $\mathcal{C}$ that is Krull–Schmidt. Let $\operatorname{mod}_0 \mathcal{C}$

denote the category of all additive functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ that are of finite length. For an object $X \in \mathcal{C}$ set

$$S_X = \mathrm{Hom}(-, X)/\,\mathrm{Rad}(-, X).$$

We note that S_X is simple when $\mathrm{End}(X)$ is local, and

$$S_X \cong \bigoplus_i S_{X_i}$$

for any finite decomposition $X = \bigoplus_i X_i$.

Given an indecomposable object X, a radical morphism $X' \to X$ is called *right almost split* if all radical morphisms terminating at X factor through $X' \to X$. Simple functors and their connection to almost split morphisms are discussed in great detail in [4, Chapter II]. In our context, the following is needed.

Lemma 4.1. *Let $\mathcal{C}$ be a Krull–Schmidt category. Then,* $\mathrm{mod}_0\,\mathcal{C} \subseteq \mathrm{mod}\,\mathcal{C}$ *holds if and only if every indecomposable object in $\mathcal{C}$ admits a right almost split morphism.*

Proof. A functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ is simple if and only if $F \cong S_X$ for some indecomposable object $X \in \mathcal{C}$; see [4, Proposition II.1.8]. Here, one uses that $\mathcal{C}$ is Krull–Schmidt. On the other hand, for an indecomposable object $X \in \mathcal{C}$ there exists a right almost split morphism $X' \to X$ if and only if S_X belongs to $\mathrm{mod}\,\mathcal{C}$, because a right almost split morphism $X' \to X$ amounts to an epimorphism $\mathrm{Hom}(-, X') \to \mathrm{Rad}(-, X)$. ∎

Lemma 4.2. *Let $\mathcal{C}$ be a Krull–Schmidt category such that* $\mathrm{mod}\,\mathcal{C}$ *is abelian and* $\mathrm{mod}_0\,\mathcal{C} \subseteq \mathrm{mod}\,\mathcal{C}$. *A functor $F \in \mathrm{mod}\,\mathcal{C}$ with presentation* (3.1) *belongs to* $\mathrm{mod}_0\,\mathcal{C}$ *if and only if the morphism $X \to Y$ in $\mathcal{C}$ has finite length.*

Proof. We write $\mathrm{rad}\, F$ for the intersection of all maximal subobjects of F and set

$$\mathrm{rad}^{n+1} F = \mathrm{rad}(\mathrm{rad}^n F) \quad \text{for all } n \geq 0.$$

For $H_X = \mathrm{Hom}(-, X)$, observe that $\mathrm{rad}^n H_X = \mathrm{Rad}^n(-, X)$ for all $n \geq 0$; see [4, Proposition II.1.8]. Thus, the presentation of F induces an epimorphism $S_Y \to F/(\mathrm{rad}\, F)$. Then, the radical filtration

$$\cdots \subseteq \mathrm{rad}^2 F \subseteq \mathrm{rad}^1 F \subseteq \mathrm{rad}^0 F = F$$

lies in $\mathrm{mod}\,\mathcal{C}$ and each subquotient

$$(\mathrm{rad}^n F)/(\mathrm{rad}^{n+1} F)$$

has finite length. It follows that F has finite length if and only if $\mathrm{rad}^n F = 0$ for $n \gg 0$. The presenting morphism $X \to Y$ has by definition length at most n if and only if the epimorphism $H_Y \to F$ factors through $H_Y \to H_Y/(\mathrm{rad}^n H_Y)$. On the other hand, $H_Y \to F$ maps $\mathrm{rad}^n H_Y$ onto $\mathrm{rad}^n F$, and therefore, it factors through $H_Y \to H_Y/(\mathrm{rad}^n H_Y)$ if and only if $\mathrm{rad}^n F = 0$. ∎

Lemma 4.3. *Let $\mathcal{A}$ be an abelian category and $\mathcal{C} \subseteq \mathcal{A}$ a full additive subcategory. Suppose that* $\operatorname{mod}\mathcal{C}$ *is abelian and that* $\operatorname{mod}_0\mathcal{C} \subseteq \operatorname{mod}\mathcal{C}$. *Then, the right exact functor* $\operatorname{mod}\mathcal{C} \to \mathcal{A}$ *extending the inclusion* $\mathcal{C} \to \mathcal{A}$ *is exact and induces an equivalence*

$$(\operatorname{mod}\mathcal{C})/(\operatorname{mod}_0\mathcal{C}) \xrightarrow{\sim} \mathcal{A}$$

if and only if $\mathcal{C}$ *generates* $\mathcal{A}$ *and* $\operatorname{mod}_0\mathcal{C} = \operatorname{eff}(\mathcal{A}, \mathcal{C})$.

Proof. The assertion is an immediate consequence of Proposition 3.4. ∎

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. We recall the equivalences

$$\mathcal{C} \xrightarrow{\sim} \operatorname{grproj} A \quad \text{and} \quad \operatorname{mod}\mathcal{C} \xrightarrow{\sim} \operatorname{grmod} A.$$

Our assumption $\operatorname{grmod}_0 A \subseteq \operatorname{grmod} A$ implies that every indecomposable object in $\mathcal{C}$ admits a right almost split morphism; see Lemma 4.1.

Now, we apply Lemma 4.3 and need to show that conditions (A1)–(A2) in Theorem 2.1 are equivalent to the conditions in Lemma 4.3. This is clear for the first pair of conditions expressing the fact that $\mathcal{C}$ is generating $\mathcal{A}$. It follows from Lemma 4.2 that (A2) holds if and only if

$$\operatorname{eff}(\mathcal{A}, \mathcal{C}) = \operatorname{mod}\mathcal{C} \cap \operatorname{eff}\mathcal{A} \subseteq \operatorname{mod}_0\mathcal{C}.$$

For the other inclusion, it suffices that all simple objects in $\operatorname{mod}\mathcal{C}$ belong to $\operatorname{eff}(\mathcal{A}, \mathcal{C})$. This means that for every indecomposable $X \in \mathcal{C}$ the right almost split morphism $X' \to X$ is an epimorphism. But this is precisely the extra condition in (A1) that there is an epimorphism $X' \to X$ in $\operatorname{Rad}\mathcal{C}$. ∎

5. Hereditary algebras

Let K be a field and Λ be a hereditary finite dimensional K-algebra. This means that $\operatorname{Ext}^n_\Lambda(-,-) = 0$ for all $n > 1$. We consider the category $\operatorname{mod}\Lambda$ of finitely presented Λ-modules. This category admits a canonical decomposition. For an additive category $\mathcal{C}$ we use the notation $\mathcal{C} = \bigvee_i \mathcal{C}_i$ and call this a *decomposition* when each $\mathcal{C}_i$ is a full additive subcategory such that each object in $\mathcal{C}$ can be written as a coproduct $\coprod_i C_i$ with $C_i \in \mathcal{C}_i$ for all i, and $\mathcal{C}_i \cap \big(\bigvee_{j \neq i} \mathcal{C}_j\big) = 0$ for all i.

Assume that Λ is connected and of infinite representation type. Then, there is a decomposition

$$\operatorname{mod}\Lambda = \mathcal{P} \vee \mathcal{R} \vee \mathcal{I},$$

where $\mathcal{P}$ denotes the full subcategory of *preprojective* Λ-modules, $\mathcal{R}$ denotes the full subcategory of *regular* Λ-modules, and $\mathcal{I}$ denotes the full subcategory of *postinjective*

Λ-modules [5, VIII]. Because $\mathcal{A} = \operatorname{mod}\Lambda$ is a hereditary abelian category we have a decomposition of the bounded derived category

$$\mathbf{D}^{\mathrm{b}}(\mathcal{A}) = \bigvee_{n\in\mathbb{Z}} \mathcal{A}[n],$$

where $\mathcal{A}[n]$ denotes the full subcategory of complexes with cohomology concentrated in degree $-n$. Note that this category has Serre duality which extends the Auslander–Reiten duality for $\mathcal{A}$. Thus, there is a functor $\tau\colon \mathbf{D}^{\mathrm{b}}(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}(\mathcal{A})$ such that for all objects X, Y there is a natural isomorphism

$$D\operatorname{Hom}(X, Y[1]) \cong \operatorname{Hom}(Y, \tau X),$$

where $D = \operatorname{Hom}_K(-, K)$; see [12] for details.

In [15], Lenzing proposes a geometric approach and introduces the following full additive subcategory:

$$\mathcal{H} := \mathcal{I}[-1] \vee \mathcal{P}[0] \vee \mathcal{R}[0] \subseteq \mathbf{D}^{\mathrm{b}}(\operatorname{mod}\Lambda).$$

He shows that $\mathcal{H}$ is a hereditary abelian category with Serre duality given by Auslander–Reiten translate $\tau\colon \mathcal{H} \xrightarrow{\sim} \mathcal{H}$. One way of seeing this is that the torsion pair $(\mathcal{I}, \mathcal{P} \vee \mathcal{R})$ for $\mathcal{A}$ induces a t-structure on $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ such that $\mathcal{H}$ identifies with its heart [10]. Moreover, $\Lambda \in \mathcal{P}[0]$ is a tilting object and $\mathbf{R}\operatorname{Hom}(\Lambda, -)$ provides a triangle equivalence

$$\mathbf{D}^{\mathrm{b}}(\mathcal{H}) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}(\operatorname{mod}\Lambda).$$

The *preprojective algebra* of Λ is the orbit algebra

$$\Pi := \bigoplus_{n\geq 0} \operatorname{Hom}(\Lambda, \tau^{-n}\Lambda),$$

and the following theorem is implicit in Lenzing's geometric treatment of hereditary algebras; cf. [15, Theorem 4.10]. We deduce this from Theorem 2.1. Note that Π is noetherian if and only if Λ is of tame type [6, 7].

Theorem 5.1. *For a connected hereditary algebra Λ of infinite representation type, the assignment*

$$X \mapsto \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}(\Lambda, \tau^{-n}X)$$

induces an equivalence

$$\mathcal{H} \xrightarrow{\sim} (\operatorname{grmod}\Pi)/(\operatorname{grmod}_0\Pi).$$

Proof. We set $\sigma = \tau^{-}$ and the distinguished object is $C = \Lambda$. We need to check the conditions in Theorem 2.1 for the triple $(\mathcal{H}, C, \sigma)$ and note that $\mathcal{C} = \mathcal{I}[-1] \vee \mathcal{P}[0]$. The preprojective algebra Π is semiperfect since each homogeneous component is finite dimensional over a field. The algebra is right coherent because the category $\mathcal{C}$ has kernels.

Here, one uses that $\operatorname{mod}\mathcal{C} \xrightarrow{\sim} \operatorname{grmod}\Pi$. Serre duality for $\mathcal{H}$ provides for each indecomposable $X \in \mathcal{C}$ an almost split sequence $0 \to \tau X \to X' \to X \to 0$ [20, Theorem I.3.3]. By the definition of an almost split sequence, the morphism $X' \to X$ is right almost split. The condition (A1) is clear because we have for each regular module $X \in \mathcal{R}[0]$ an epimorphism $X' \to X$ from a projective module $X' \in \mathcal{P}[0]$. For an indecomposable object $X \in \mathcal{C}$ one uses the right-hand morphism $X' \to X$ from the corresponding almost split sequence. To check (A2) consider an epimorphism $Y \to Z$ in $\mathcal{C}$ which yields an exact sequence $0 \to X \to Y \to Z \to 0$. We may apply a power of τ and assume that the sequence lies in $\mathcal{I}[-1]$. Consider the induced exact sequence

$$0 \to \operatorname{Hom}(-, X) \to \operatorname{Hom}(-, Y) \to \operatorname{Hom}(-, Z) \to \operatorname{Ext}^1(-, X)$$

in $\operatorname{mod}\mathcal{C}$. We need to show that all morphisms $Z' \to Z$ from $\operatorname{Rad}^n \mathcal{C}$ factor through $Y \to Z$ for $n \gg 0$. For this, it suffices to show $\operatorname{Rad}^n \mathcal{C}$ annihilates the functor $\operatorname{Ext}^1(-, X)|_{\mathcal{I}[-1]}$ for $n \gg 0$, because $\operatorname{Hom}(P, Z) = 0$ for all $P \in \mathcal{P}[0]$. We have

$$D \operatorname{Hom}(\tau^- X, -) \cong \operatorname{Ext}^1(-, X)$$

and from this, the claim follows, because for any postinjective Λ-module M we have that $\operatorname{Rad}^n(M, -) = 0$ for $n \gg 0$ in $\operatorname{mod}\Lambda$. ∎

When passing to derived categories, we have the following immediate consequence; see also [17, Corollary 5.4].

Corollary 5.2. *The quotient functor* $\operatorname{grmod}\Pi \twoheadrightarrow \mathcal{H}$ *induces a triangle equivalence*

$$\mathbf{D}^{\mathrm{b}}((\operatorname{grmod}\Pi)/(\operatorname{grmod}_0 \Pi)) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}(\mathcal{H}) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}(\operatorname{mod}\Lambda).$$ ∎

The following example connects the results of Serre and Lenzing; see [15, Proposition 6.3].

Example 5.3. Consider the Kronecker algebra $\Lambda = \left[\begin{smallmatrix} K & 0 \\ K^2 & K \end{smallmatrix}\right]$. In that case, $\mathcal{H}$ identifies with the category of coherent sheaves on the projective line $\mathbb{P}^1$. More precisely, the category $\operatorname{coh}\mathbb{P}^1$ admits a tilting object $T = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ such that $\operatorname{End}(T) \cong \Lambda$. Then, the functor $\mathbf{R}\operatorname{Hom}(T, -)$ yields a triangle equivalence

$$\mathbf{D}^{\mathrm{b}}(\operatorname{coh}\mathbb{P}^1) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}(\operatorname{mod}\Lambda),$$

which restricts to an equivalence $\operatorname{coh}\mathbb{P}^1 \xrightarrow{\sim} \mathcal{H}$. Note that this identifies the twist $\mathcal{F} \mapsto \mathcal{F}(2)$ with $X \mapsto \sigma X$. We have

$$R := K[x, y] \cong \bigoplus_{n \geq 0} \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(n)),$$

and therefore,

$$\Pi \cong \bigoplus_{n \geq 0} \left[\begin{smallmatrix} R_{2n} & R_{2n-1} \\ R_{2n+1} & R_{2n} \end{smallmatrix}\right].$$

The assignment

$$\bigoplus_{n\in\mathbb{Z}} M_n \mapsto \bigoplus_{n\in\mathbb{Z}}(M_{2n} \oplus M_{2n-1})$$

provides an equivalence grmod $R \xrightarrow{\sim}$ grmod Π which identifies the equivalence from Serre's theorem in Example 2.2 with the equivalence in Theorem 5.1, though their twisting objects are different ($\mathcal{O}_{\mathbb{P}^1}$ in coh $\mathbb{P}^1$ versus Λ in $\mathcal{H}$).

The example suggests that there are other and arguably better choices for a distinguished object $C \in \mathcal{H}$ and a twist $\sigma\colon \mathcal{H} \xrightarrow{\sim} \mathcal{H}$ (different from $C = \Lambda$ and $\sigma = \tau^-$), in particular, when Λ is a tame hereditary algebra; see [13, 14] for a detailed discussion.

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