Veronese subalgebras and Veronese morphisms for a class of Yang–Baxter algebras

Tatiana Gateva-Ivanova

Abstract. We study d-Veronese subalgebras $\mathcal{A}^{(d)}$ of Yang–Baxter algebras $\mathcal{A}_X = \mathcal{A}(\mathbf{k}, X, r)$ related to finite nondegenerate involutive set-theoretic solutions (X, r) of the Yang–Baxter equation, where $\mathbf{k}$ is a field and $d \geq 2$ is an integer. We find an explicit presentation of the d-Veronese $\mathcal{A}^{(d)}$ in terms of one-generators and quadratic relations. We introduce the notion of *a d-Veronese solution* (Y, r_Y), canonically associated to (X, r) and use its Yang–Baxter algebra $\mathcal{A}_Y = \mathcal{A}(\mathbf{k}, Y, r_Y)$ to define a Veronese morphism $v_{n,d} : \mathcal{A}_Y \to \mathcal{A}_X$. We prove that the image of $v_{n,d}$ is the d-Veronese subalgebra $\mathcal{A}^{(d)}$ and find explicitly a minimal set of generators for its kernel. The results agree with their classical analogues in the commutative case. We show that the Yang–Baxter algebra $\mathcal{A}(\mathbf{k}, X, r)$ is a PBW algebra if and only if (X, r) is a square-free solution. In this case, the d-Veronese $A^{(d)}$ is also a PBW algebra.

1. Introduction

It was established in the last three decades that solutions of the linear braid or Yang–Baxter equation (YBE)

$$r^{12}r^{23}r^{12} = r^{23}r^{12}r^{23}$$

on a vector space of the form $V^{\otimes 3}$ lead to remarkable algebraic structures. Here $r : V \otimes V \to V \otimes V$, $r^{12} = r \otimes \mathrm{id}$, $r^{23} = \mathrm{id} \otimes r$ is a linear automorphism and structures include coquasitriangular bialgebras $A(r)$, their quantum group (Hopf algebra) quotients, quantum planes and associated objects, at least in the case of specific standard solutions; see [25, 32]. On the other hand, the variety of all solutions on vector spaces of a given dimension has remained rather elusive in any degree of generality. It was proposed by V.G. Drinfeld [6], to consider the same equations in the category of sets, and in this setting numerous results were found. It is clear that a set-theoretic solution extends to a linear one, but more important than this is that set-theoretic solutions lead to their own remarkable algebraic and combinatoric structures, only somewhat analogous to quantum group constructions. In the present paper, we continue our systematic study of set-theoretic solutions based on the associated quadratic algebras and monoids that they generate.

The study of noncommutative algebras defined by quadratic relations as examples of quantum noncommutative spaces has received considerable impetus from the seminal work of Faddeev, Reshetikhin, and Takhtajan [8], where the authors considered general deformations of quantum groups and spaces arising from an R-matrix, and from Manin's programme for noncommutative geometry [27]. The quadratic algebras related to set-theoretic solutions of the Yang–Baxter equation studied here can be considered as special quantum noncommutative spaces important for both noncommutative algebra and non-commutative algebraic geometry, as they provide a rich source of examples of interesting associative algebras and noncommutative spaces some of which are Artin–Schelter regular algebras. Our work is motivated by the relevance of those algebras for noncommutative geometry, especially in relation to the theory of quantum groups, and inspired by the interpretation of morphisms between noncommutative algebras as "maps between non-commutative spaces". In [18, 21] and the present paper, we consider noncommutative analogues of the Veronese and Segre embeddings, two fundamental maps that play pivotal roles not only in classical algebraic geometry but also in applications to other fields of mathematics.

In this paper, "a solution of YBE", or shortly, "a solution" means "a nondegenerate involutive set-theoretic solution of YBE"; see Definition 2.5.

The Yang–Baxter algebras $\mathcal{A}_X = \mathcal{A}(\mathbf{k}, X, r)$ related to solutions (X, r) of finite order n will play a central role in the paper. It was proven in [15,22] that the quadratic algebra $\mathcal{A}_X$ of every finite solution (X, r) of YBE has remarkable algebraic, homological, and com-binatorial properties. In general, the algebra $\mathcal{A}_X$ is noncommutative and in most cases it is not even a Poincaré–Birkhoff–Witt (PBW) algebra, but it preserves various good prop-erties of the commutative polynomial ring $\mathbf{k}[x_1, \ldots, x_n]$: $\mathcal{A}_X$ has finite global dimension and polynomial growth, and it is Cohen–Macaulay, Koszul, and a Noetherian domain.

There are close relations between various combinatorial properties of the solu-tion (X, r) and the properties of the corresponding Yang–Baxter algebra $\mathcal{A}_X$; see, for example, [12, 13, 15, 17, 20, 33, 34]. In the special case when (X, r) is *a finite nondegener-ate involutive square-free quadratic set* whose quadratic algebra $\mathcal{A}_X = \mathcal{A}(\mathbf{k}, X, r)$ has a **k**-basis of PBW type, the conditions "$\mathcal{A}$ is an Artin–Schelter regular algebra" and "(X, r) *is a solution of YBE*" *are equivalent*; see details in Section 3. The study of Artin–Schelter regular algebras is a central problem for noncommutative algebraic geometry.

A first stage of noncommutative geometry on quadratic algebras $\mathcal{A}_X = \mathcal{A}(\mathbf{k}, X, r)$ was proposed in [20, Section 6], where the quantum spaces under investigation are Yang–Baxter algebras $\mathcal{A}(\mathbf{k}, X, r)$ associated to multipermutation (square-free) solutions of level two. In [2], a class of quadratic PBW algebras called "noncommutative projective spaces" were investigated, and analogues of Veronese and Segre morphisms between noncom-mutative projective spaces were introduced and studied. It is natural to formulate similar problems for the class of Yang–Baxter algebras $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ related to finite solutions (X, r), but to find reasonable solutions of these problems is a nontrivial task. In contrast to [2], where the "noncommutative projective spaces" under investigation have almost commutative quadratic relations which form Gröbner bases, and the main results follow

naturally from the theory of noncommutative Gröbner bases, the Yang–Baxter algebras $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ have complicated quadratic relations, which in most cases do not form Gröbner bases. These relations remain complicated even when $\mathcal{A}$ is a PBW algebra, so we need more sophisticated arguments and techniques; see, for example, [18].

In the present paper, we consider the following problems.

Problems 1.1. Suppose (X, r) is a finite solution of YBE with $|X| = n$, and $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ is its Yang–Baxter algebra.

(1) Find necessary and sufficient conditions on (X, r) such that there exists an enumeration $X = \{x_1, \ldots, x_n\}$, so that $\mathcal{A}$ is a PBW algebra with a set of PBW generators $x_1, \ldots, x_n$.

(2) Let $d \geq 2$ be an integer. Find a presentation of the d-Veronese subalgebra $\mathcal{A}^{(d)}$ of its Yang–Baxter algebra $\mathcal{A}$ in terms of one-generators and quadratic relations.

(3) Introduce analogues of Veronese maps for the class of Yang–Baxter algebras of finite solutions of YBE.

(4) Answer questions (2) and (3) in the special case when (X, r) is a square-free solution.

Our main results are Theorems 3.8, 4.13, and 5.4 which solve completely problems (1), (2), and (3). We give a complete answer to (4) in Section 6.

The paper is organised as follows. In Section 2, we recall basic definitions and facts used throughout the paper. In Section 3, we consider the Yang–Baxter algebra $\mathcal{A}_X = \mathcal{A}(\mathbf{k}, X, r)$ of a finite nondegenerate solution (X, r). We fix the main settings and conventions and recall some of the most important properties of the Yang–Baxter algebras $\mathcal{A}_X$ used throughout the paper. The main result of the section is Theorem 3.8 which shows that the Yang–Baxter algebra $\mathcal{A}(\mathbf{k}, X, r)$ is PBW with respect to some proper enumeration of X iff the solution (X, r) is square-free. Proposition 3.9 gives more information on a special case of PBW quadratic algebras. A natural open question arises at end the section; see Question 3.10. In Section 4, we study the d-Veronese subalgebra $\mathcal{A}^{(d)}$ of $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$. We use the fact that the algebra $\mathcal{A}$ and its Veronese subalgebras are intimately connected with the braided monoid $S(X, r)$. To solve the main problem, we introduce successively three finite isomorphic solutions associated naturally to (X, r) and involved in the proof of our results. The first and the most natural of the three is *the monomial d-Veronese solution (S_d, r_d) associated with (X, r)*. It is a finite solution induced from the graded braided monoid (S, r_S) which depends only on the map r and on d. The monomial d-Veronese solution is intimately connected with the d-Veronese subalgebra $\mathcal{A}^{(d)}$ and its quadratic relations, but it is not convenient for an explicit description of the relations. This solution is needed to define *the normalised d-Veronese solution* $(\mathcal{N}_d, \rho_d)$ isomorphic to (S_d, r_d); see Definition 4.10. The solution $(\mathcal{N}_d, \rho_d)$ is central for the proof of the main result (Theorem 4.13). In Section 5, we introduce and study analogues of Veronese maps between Yang–Baxter algebras of finite solutions and prove

Theorem 5.4. In Section 6, we consider two special cases of solutions. We pay special attention to Yang–Baxter algebras $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ of square-free solutions (X, r) and their Veronese subalgebras. In this case, $\mathcal{A}$ is a binomial skew polynomial ring and the set of ordered monomials (terms) in n variables forms an explicit $\mathbf{k}$-basis of $\mathcal{A}$. Theorem 4.13 implies a more precise result in this case: Corollary 6.3 shows that the d-Veronese $A^{(d)}$ is a PBW algebra, where the terms of length d ordered lexicographically are its PBW generators and its relations given explicitly form a quadratic Gröbner basis. An important result in this section is Theorem 6.4 which shows that if (X, r) is a finite square-free solution and $d \geq 2$ is an integer, then the monomial d-Veronese solution (S_d, r_d) is square-free if and only if (X, r) is a trivial solution. This implies that the notion of Veronese morphisms introduced for the class of Yang–Baxter algebras of finite solutions cannot be restricted to the subclass of Yang–Baxter algebras associated to finite square-free solutions. In Section 7, we present two examples which illustrate the results of the paper.

2. Preliminaries

Let X be a non-empty set, and let $\mathbf{k}$ be a field. We denote by $\langle X \rangle$ the free monoid generated by X, where the unit is the empty word denoted by 1, and by $\mathbf{k}\langle X \rangle$ the unital free associative $\mathbf{k}$-algebra generated by X. For a non-empty set $F \subseteq \mathbf{k}\langle X \rangle$, (F) denotes the two-sided ideal of $\mathbf{k}\langle X \rangle$ generated by F. When the set X is finite, with $|X| = n$, and ordered, we write $X = \{x_1, \ldots, x_n\}$ and fix the degree-lexicographic order $<$ on $\langle X \rangle$, where $x_1 < \cdots < x_n$. As usual, $\mathbb{N}$ denotes the set of all positive integers, and $\mathbb{N}_0$ is the set of all non-negative integers.

We shall consider associative graded $\mathbf{k}$-algebras. Suppose $A = \bigoplus_{m \in \mathbb{N}_0} A_m$ is a graded $\mathbf{k}$-algebra such that $A_0 = \mathbf{k}$, $A_p A_q \subseteq A_{p+q}$, $p, q \in \mathbb{N}_0$ and such that A is finitely generated by elements of positive degree. Recall that its Hilbert function is $h_A(m) = \dim A_m$ and its Hilbert series is the formal series $H_A(t) = \sum_{m \in \mathbb{N}_0} h_A(m) t^m$. In particular, the algebra $\mathbf{k}[X]$ of commutative polynomials satisfies

$$h_{\mathbf{k}[X]}(d) = \binom{n + d - 1}{d} = \binom{n + d - 1}{n - 1} \quad \text{and} \quad H_{\mathbf{k}[X]} = \frac{1}{(1 - t)^n}. \tag{2.1}$$

We shall use the *natural grading by length* on the free associative algebra $\mathbf{k}\langle X \rangle$. For $m \geq 1$, X^m will denote the set of all words of length m in $\langle X \rangle$, where the length of $u = x_{i_1} \cdots x_{i_m} \in X^m$ will be denoted by $|u| = m$. Then

$$\langle X \rangle = \bigsqcup_{m \in \mathbb{N}_0} X^m, \quad X^0 = \{1\} \quad \text{and} \quad X^k X^m \subseteq X^{k+m},$$

so the free monoid $\langle X \rangle$ is naturally *graded by length*.

Similarly, the free associative algebra $\mathbf{k}\langle X \rangle$ is also graded by length:

$$\mathbf{k}\langle X \rangle = \bigoplus_{m \in \mathbb{N}_0} \mathbf{k}\langle X \rangle_m, \quad \text{where } \mathbf{k}\langle X \rangle_m = \mathbf{k} X^m.$$

A polynomial $f \in \mathbf{k}\langle X \rangle$ is *homogeneous of degree m* if $f \in \mathbf{k}X^m$. We denote by

$$\mathcal{T} = \mathcal{T}(X) := \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \langle X \rangle \mid \alpha_i \in \mathbb{N}_0, \, i \in \{0, \ldots, n\}\} \qquad (2.2)$$

the set of ordered monomials (terms) in $\langle X \rangle$ and by

$$\mathcal{T}_d = \mathcal{T}(X)_d := \left\{ x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathcal{T} \mid \sum_{i=1}^{n} \alpha_i = d \right\}$$

the set of ordered monomials of length d.

2.1. Gröbner bases for ideals in the free associative algebra

We shall remind some basics of noncommutative Gröbner bases theory which we use throughout in the paper. In this subsection $X = \{x_1, \ldots, x_n\}$, we fix the degree-lexicographic order $<$ on the free monoid $\langle X \rangle$ extending $x_1 < x_2 < \cdots < x_n$ (we refer to it as "deg-lex ordering"). Suppose $f \in \mathbf{k}\langle X \rangle$ is a nonzero polynomial. Its leading monomial with respect to the deg-lex order $<$ will be denoted by $\mathbf{LM}(f)$. One has $\mathbf{LM}(f) = u$ if $f = cu + \sum_{1 \le i \le m} c_i u_i$, where $c, c_i \in \mathbf{k}$, $c \ne 0$, and $u > u_i$ in $\langle X \rangle$ for all $i \in \{1, \ldots, m\}$. Given a set $F \subseteq \mathbf{k}\langle X \rangle$ of noncommutative polynomials, we consider the set of leading monomials $\mathbf{LM}(F) = \{\mathbf{LM}(f) \mid f \in F\}$. A monomial $u \in \langle X \rangle$ is *normal modulo F* if it does not contain any of the monomials $\mathbf{LM}(f)$ as a subword. The set of all normal monomials modulo F is denoted by $N(F)$.

Let I be a two-sided graded ideal in $K\langle X \rangle$ and let $I_m = I \cap \mathbf{k}X^m$. We shall assume that *I is generated by homogeneous polynomials of degree ≥ 2* and $I = \bigoplus_{m \ge 2} I_m$. Then the quotient algebra $A = \mathbf{k}\langle X \rangle / I$ is finitely generated and inherits its grading $A = \bigoplus_{m \in \mathbb{N}_0} A_m$ from $\mathbf{k}\langle X \rangle$. We shall work with the so-called *normal $\mathbf{k}$-basis of A*. We say that a monomial $u \in \langle X \rangle$ is *normal modulo I* if it is normal modulo $\mathbf{LM}(I)$. We set $N(I) := N(\mathbf{LM}(I))$. In particular, the free monoid $\langle X \rangle$ splits as a disjoint union

$$\langle X \rangle = N(I) \sqcup \mathbf{LM}(I). \qquad (2.3)$$

The free associative algebra $\mathbf{k}\langle X \rangle$ splits as a direct sum of $\mathbf{k}$-vector subspaces

$$\mathbf{k}\langle X \rangle \simeq \mathrm{Span}_{\mathbf{k}} \, N(I) \oplus I,$$

and there is an isomorphism of vector spaces $A \simeq \mathrm{Span}_{\mathbf{k}} \, N(I)$.

It follows that every $f \in \mathbf{k}\langle X \rangle$ can be written uniquely as $f = f_0 + h$, where $f_0 \in \mathbf{k}N(I)$ and $h \in I$. The element f_0 is called *the normal form of f (modulo I)* and denoted by $\mathrm{Nor}(f)$. We define

$$N(I)_m = \{u \in N(I) \mid u \text{ has length } m\}.$$

In particular, $N(I)_1 = X, N(I)_0 = 1$. Then $A_m \simeq \mathrm{Span}_{\mathbf{k}} \, N(I)_m$ for every $m \in \mathbb{N}_0$.

A subset $G \subseteq I$ of monic polynomials is a *Gröbner basis* of I (with respect to the order $<$) if

(1) G generates I as a two-sided ideal, and

(2) for every $f \in I$ there exists $g \in G$ such that $\mathbf{LM}(g)$ is a subword of $\mathbf{LM}(f)$, that is, $\mathbf{LM}(f) = a\mathbf{LM}(g)b$ for some $a, b \in \langle X \rangle$.

A Gröbner basis G of I is *reduced* if (i) the set $G \setminus \{f\}$ is not a Gröbner basis of I, whenever $f \in G$; (ii) each $f \in G$ is a linear combination of normal monomials modulo $G \setminus \{f\}$.

It is well known that every ideal I of $\mathbf{k}\langle X \rangle$ has a unique reduced Gröbner basis $G_0 = G_0(I)$ with respect to $<$. However, G_0 may be infinite. For more details, we refer the reader to [23, 28, 29].

The set of leading monomials of the reduced Gröbner basis $G_0 = G_0(I)$

$$W = \{\mathbf{LM}(f) \mid f \in G_0(I)\} \tag{2.4}$$

is *the set of obstructions* for $A = \mathbf{k}\langle X \rangle / I$, in the sense of Anick [1]. There are equalities of sets $N(I) = N(G_0) = N(W)$. We shall use the set of obstructions for the proof of Theorem 3.8.

Bergman's Diamond Lemma [5, Theorem 1.2] implies the following.

Remark 2.1. Let $G \subset \mathbf{k}\langle X \rangle$ be a set of noncommutative polynomials. Let $I = (G)$ and let $A = \mathbf{k}\langle X \rangle / I$. Then the following conditions are equivalent.

(1) The set G is a Gröbner basis of I.

(2) Every element $f \in \mathbf{k}\langle X \rangle$ has a unique normal form modulo G, denoted by $\mathrm{Nor}_G(f)$.

(3) There is an equality $N(G) = N(I)$, so there is an isomorphism of vector spaces

$$\mathbf{k}\langle X \rangle \simeq I \oplus \mathbf{k}N(G).$$

(4) The image of $N(G)$ in A is a $\mathbf{k}$-basis of A. In this case, A can be identified with the $\mathbf{k}$-vector space $\mathbf{k}N(G)$, made a $\mathbf{k}$-algebra by the multiplication $a \bullet b := \mathrm{Nor}(ab)$.

In this paper, we focus on a class of quadratic finitely presented algebras A associated with set-theoretic nondegenerate involutive solutions (X, r) of finite order n. Following Yuri Manin [26], we call them *Yang–Baxter algebras*.

2.2. Quadratic algebras

A quadratic algebra is an associative graded algebra $A = \bigoplus_{i \geq 0} A_i$ over a ground field $\mathbf{k}$ determined by a vector space of generators $V = A_1$ and a subspace of homogeneous quadratic relations $R = R(A) \subset V \otimes V$. We assume that A is finitely generated, so $\dim A_1 < \infty$. Thus $A = T(V)/(R)$ inherits its grading from the tensor algebra $T(V)$.

Following the classical tradition (and a recent trend), we take a combinatorial approach to study A. The properties of A will be read off a presentation $A = \mathbf{k}\langle X \rangle / (\mathfrak{R})$, where by convention X is a fixed finite set of generators of degree 1, $|X| = n$, and $(\mathfrak{R})$ is the

two-sided ideal of relations, generated by a *finite* set $\mathfrak{R}$ of homogeneous polynomials of degree 2. In particular, $A_1 = V = \mathrm{Span}_{\mathbf{k}} X$.

Definition 2.2. A quadratic algebra A is *a PBW type algebra* or shortly *a PBW algebra* if there exists an enumeration $X = \{x_1, \ldots, x_n\}$ of X such that the quadratic relations $\mathfrak{R}$ form a (noncommutative) Gröbner basis with respect to the deg-lex order $<$ on $\langle X \rangle$. In this case, the set of normal monomials (mod $\mathfrak{R}$) forms a **k**-basis of A called a *PBW basis* and $x_1, \ldots, x_n$ (taken exactly with this enumeration) are called *PBW generators of A*.

PBW algebras were introduced by Priddy [31]. The *PBW basis* is a generalisation of the classical PBW basis for the universal enveloping of a finite-dimensional Lie algebra. PBW algebras form an important class of Koszul algebras. The interested reader can find information on quadratic algebras and, in particular, on Koszul algebras and PBW algebras in [30]. A special class of PBW algebras important for this paper are the *binomial skew polynomial rings* introduced and studied by the author in [10, 11].

Definition 2.3 ([10, 11]). A *binomial skew polynomial ring* is a quadratic algebra $A = \mathbf{k}\langle x_1, \ldots, x_n\rangle/(\mathfrak{R}_0)$ with precisely $\binom{n}{2}$ defining relations

$$\mathfrak{R}_0 = \{\varphi_{ji} = x_j x_i - c_{ij} x_{i'} x_{j'} \mid 1 \le i < j \le n\} \tag{2.5}$$

such that

(a) for every pair i, j, $1 \le i < j \le n$, the relation $x_j x_i - c_{ij} x_{i'} x_{j'} \in \mathfrak{R}_0$ satisfies $j > i', i' < j', c_{ij} \in \mathbf{k}^{\times}$;

(b) every ordered monomial $x_i x_j$, with $1 \le i < j \le n$ occurs (as a second term) in some relation in $\mathfrak{R}_0$;

(c) $\mathfrak{R}_0$ is the *reduced Gröbner basis* of the two-sided ideal $(\mathfrak{R}_0)$, with respect to the deg-lex order $<$ on $\langle X \rangle$; or equivalently,

(c′) the set of terms $\mathcal{T} = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \langle X \rangle \mid \alpha_i \in \mathbb{N}_0, i \in \{1, \ldots, n\}\}$ projects to a **k**-basis of A.

The equivalence of (c) and (c′) follows from the Diamond Lemma; see Remark 2.1.

It is clear that each binomial skew polynomial ring A is a PBW algebra with a set of PBW generators $x_1, \ldots, x_n$. It was proven in [22] that A defines via its relations a square-free solution of the Yang–Baxter equation. Conversely, if (X, r) is a finite square-free solution, then there exists an enumeration $X = \{x_1, x_2, \ldots, x_n\}$ such that the Yang–Baxter algebra $\mathcal{A}(\mathbf{k}, X, r)$ is a binomial skew polynomial ring; this follows from results of Rump [33] (see also details in [12]).

Example 2.4. Let $A = \mathbf{k}\langle x_1, x_2, x_3, x_4\rangle/(\mathfrak{R}_0)$, where

$$\mathfrak{R}_0 = \{x_4 x_2 - x_1 x_3, \; x_4 x_1 - x_2 x_3, \; x_3 x_2 - x_1 x_4, \; x_3 x_1 - x_2 x_4,$$
$$x_4 x_3 - x_3 x_4, \; x_2 x_1 - x_1 x_2\}.$$

The algebra A is a binomial skew polynomial ring. It is a PBW algebra with PBW generators $X = \{x_1, x_2, x_3, x_4\}$. The relations of A define in a natural way a solution of YBE.

2.3. Set-theoretic solutions of the Yang–Baxter equation and their Yang–Baxter algebras

The notion of *a quadratic set* was introduced in [12], see also [19], as a set-theoretic analogue of a quadratic algebra.

Definition 2.5 ([12]). Let X be a non-empty set (possibly infinite), and let $r : X \times X \to X \times X$ be a bijective map. In this case, we use notation (X, r) and refer to it as *a quadratic set*. The image of (x, y) under r is written as

$$r(x, y) = ({}^x y, x^y).$$

This formula defines a "left action" $\mathcal{L} : X \times X \to X$ and a "right action" $\mathcal{R} : X \times X \to X$ on X as: $\mathcal{L}_x(y) = {}^x y$, $\mathcal{R}_y(x) = x^y$ for all $x, y \in X$. (i) (X, r) is *nondegenerate*, if the maps $\mathcal{L}_x$ and $\mathcal{R}_x$ are bijective for each $x \in X$. (ii) (X, r) is *involutive* if $r^2 = \mathrm{id}_{X \times X}$. (iii) (X, r) is *square-free* if $r(x, x) = (x, x)$ for all $x \in X$. (iv) (X, r) is *a set-theoretic solution of the Yang–Baxter equation* (YBE) if the braid relation

$$r^{12} r^{23} r^{12} = r^{23} r^{12} r^{23}$$

holds in $X \times X \times X$, where $r^{12} = r \times \mathrm{id}_X$, and $r^{23} = \mathrm{id}_X \times r$. In this case, we refer to (X, r) also as *a braided set*. (v) A braided set (X, r) with r involutive is called *a symmetric set*. (vi) A nondegenerate symmetric set is called simply *a solution*.

We say that (X, r) is *the trivial solution* on X if $r(x, y) = (y, x)$ for all $x, y \in X$.

Remark 2.6 ([7]). Let (X, r) be a quadratic set. Then r obeys the YBE, that is, (X, r) is a braided set, iff the following three conditions hold for all $x, y, z \in X$:

l1: ${}^x({}^y z) = {}^{{}^x y}({}^{x^y} z),$ **r1:** $(x^y)^z = (x^{{}^y z})^{y^z},$ **lr3:** $({}^x y)^{({}^{x^y} z)} = {}^{(x^{{}^y z})}(y^z).$

The map r is involutive iff

$$\textbf{inv:} \quad {}^{x^y}(x^y) = x, \quad \text{and} \quad ({}^x y)^{x^y} = y.$$

Convention 2.7. In this paper, "*a solution of YBE*" or simply "*a solution*" means "*a nondegenerate symmetric set*" (X, r), where X is a set of arbitrary cardinality.

As a notational tool, we shall identify the sets $X^{\times m}$ of ordered m-tuples, $m \geq 2$, and X^m, the set of all monomials of length m in the free monoid $\langle X \rangle$. We shall use also notation $\cdot r(x, y) := xy$. Sometimes for simplicity we shall write $r(xy)$ instead of $r(x, y)$.

Definition 2.8 ([12, 19]). To each quadratic set (X, r) we associate canonically algebraic objects generated by X and with quadratic relations $\Re = \Re(r)$ naturally determined as

$$xy = y'x' \in \Re(r) \quad \text{iff } r(x, y) = (y', x') \quad \text{and} \quad (x, y) \neq (y', x') \quad \text{hold in } X \times X.$$

The monoid $S = S(X, r) = \langle X; \Re(r) \rangle$ with a set of generators X and a set of defining relations $\Re(r)$ is called *the monoid associated with* (X, r). The *group* $G = G(X, r) = G_X$ *associated with* (X, r) is defined analogously. For an arbitrary fixed field $\mathbf{k}$, *the* $\mathbf{k}$-*algebra associated with* (X, r) is defined as

$$\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r) = \mathbf{k}\langle X \rangle / (\Re_{\mathcal{A}}) \simeq \mathbf{k}\langle X; \Re(r) \rangle,$$
$$\text{where } \Re_{\mathcal{A}} = \{xy - y'x' \mid xy = y'x' \in \Re(r)\}.$$

Clearly, $\mathcal{A}$ is a quadratic algebra generated by X and with defining relations $\Re_{\mathcal{A}}$, which is isomorphic to the monoid algebra $\mathbf{k}S(X, r)$.

When (X, r) is a solution of YBE, the algebra $\mathcal{A}$ is called *a Yang–Baxter algebra* [26], or shortly *a YB algebra*.

Suppose (X, r) is a finite quadratic set. Then $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ is *a connected graded* $\mathbf{k}$-algebra (naturally graded by length), $\mathcal{A} = \bigoplus_{i \geq 0} \mathcal{A}_i$, where $\mathcal{A}_0 = \mathbf{k}$, and each graded component $\mathcal{A}_i$ is finite-dimensional. Moreover, the associated monoid $S = S(X, r)$ *is naturally graded by length*:

$$S = \bigsqcup_{i \geq 0} S_i, \quad \text{where } S_0 = 1, \ S_1 = X, \ S_i = \{u \in S \mid |u| = i\}, \ S_i.S_j \subseteq S_{i+j}. \quad (2.6)$$

In the sequel, by "*a graded monoid S*", we shall mean that S is generated by $S_1 = X$ and graded by length. The grading of S induces a canonical grading of its monoid algebra $\mathbf{k}S(X, r)$. The isomorphism $\mathcal{A} \cong \mathbf{k}S(X, r)$ agrees with the canonical gradings, so there is an isomorphism of vector spaces $\mathcal{A}_m \cong \mathrm{Span}_{\mathbf{k}} S_m$.

If (X, r) is a nondegenerate involutive quadratic set of finite order $|X| = n$, then, by [14, Proposition 2.3], the set $\Re$ consists of precisely $\binom{n}{2}$ quadratic relations. In this case, the associated algebra $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ satisfies $\dim \mathcal{A}_2 = \binom{n+1}{2}$.

Definition-Notation 2.9 ([17]). Suppose (X, r) is an involutive quadratic set. Then the cyclic group $\langle r \rangle = \{1, r\}$ acts on the set X^2 and splits it into disjoint r-orbits $\{xy, r(xy)\}$, where $xy \in X^2$. An r-orbit $\{xy, r(xy)\}$ is *nontrivial* if $xy \neq r(xy)$. The element $xy \in X^2$ is *an* r-*fixed point* if $r(xy) = xy$. The set of r-fixed points in X^2 will be denoted by $\mathcal{F}(X, r)$:

$$\mathcal{F}(X, r) = \{xy \in X^2 \mid r(xy) = xy\}. \quad (2.7)$$

The following useful corollary is a consequence of [17, Lemma 3.7].

Corollary 2.10. *Let* (X, r) *be a solution of YBE of finite order* $|X| = n$, *and let* $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ *be its Yang–Baxter algebra.* (1) *There are exactly n fixed points* $\mathcal{F} = \mathcal{F}(X, r) =$

$\{x_1 y_1, \ldots, x_n y_n\} \subset X^2$, so $|\mathcal{F}(X, r)| = |X| = n$. *In the special case, when (X, r) is a square-free solution, one has $\mathcal{F}(X, r) = \Delta_2 = \{xx \mid x \in X\}$, the diagonal of X^2. (2) The number of nontrivial r-orbits is exactly $\binom{n}{2}$. Each such an orbit has two distinct elements: xy and $r(xy)$, where $xy, r(xy) \in X^2$. (3) The set X^2 splits into $\binom{n+1}{2}$ r-orbits. For $xy, zt \in X^2$, there is an equality $xy = zt$ in $\mathcal{A}$ iff $zt \in \{xy, r(xy)\}$. (4) In particular, $\mathcal{A}$ has exactly $\binom{n}{2}$ defining relations (each relation corresponds to a nontrivial r-orbit).*

Remark 2.11 ([13]). Let (X, r) be an involutive quadratic set, and let $S = S(X, r)$ be the associated monoid.

(i) By definition, two monomials $w, w' \in \langle X \rangle$ are equal in S iff w can be transformed to w' by a finite sequence of replacements each of the form

$$axyb \longrightarrow ar(xy)b, \quad \text{where } x, y \in X, \ a, b \in \langle X \rangle.$$

Clearly, every such replacement preserves monomial length, which therefore descends to $S(X, r)$. Furthermore, replacements coming from the defining relations are possible only on monomials of length ≥ 2, hence $X \subset S(X, r)$ is an inclusion. For monomials of length 2, $xy = zt$ holds in $S(X, r)$ iff $zt \in \{xy, r(xy)\}$.

(ii) It is convenient for each $m \geq 2$ to refer to the subgroup $D_m = D_m(r)$ of the symmetric group $\mathrm{Sym}(X^m)$ generated concretely by the maps

$$r^{ii+1} : X^m \longrightarrow X^m, \quad r^{ii+1} = \mathrm{id}_{X^{i-1}} \times r \times \mathrm{id}_{X^{m-i-1}}, \quad i = 1, \ldots, m-1. \quad (2.8)$$

One can also consider the free groups

$$\mathcal{D}_m(r) = {}_{\mathrm{gr}}\langle r^{ii+1} \mid i = 1, \ldots, m-1 \rangle,$$

where the r^{ii+1} are treated as abstract symbols, as well as various quotients depending on the further type of r of interest. These free groups and their quotients act on X^m via the actual maps r^{ii+1}, so that the image of $\mathcal{D}_m(r)$ in $\mathrm{Sym}(X^m)$ is $D_m(r)$. In particular, $D_2(r) = \langle r \rangle \subset \mathrm{Sym}(X^2)$ is the cyclic group generated by r. It follows straightforwardly from part (i) that $w, w' \in \langle X \rangle$ are equal as words in $S(X, r)$ iff they have the same length, say m, and belong to the same orbit $\mathcal{O}_{\mathcal{D}_m}$ of $\mathcal{D}_m(r)$ in X^m. In this case, the equality $w = w'$ holds in $S(X, r)$ and in the algebra $\mathcal{A}(\mathbf{k}, X, r)$.

An effective part of our combinatorial approach is the exploration of the action of the group $\mathcal{D}_2(r) = \langle r \rangle$ on X^2 and the properties of the corresponding orbits. In the literature, a $\mathcal{D}_2(r)$-orbit $\mathcal{O}$ in X^2 is often called "*an r-orbit*", and we shall use this terminology.

In notation and assumption as above, let (X, r) be a finite quadratic set with $S = S(X, r)$ graded by length. Then the order of the graded component S_m equals the number of $\mathcal{D}_m(r)$-orbits in X^m.

3. The Yang–Baxter algebra $\mathcal{A}(\mathbf{k}, X, r)$ of a finite nondegenerate symmetric set (X, r)

It was proven through the years that the Yang–Baxter algebras $\mathcal{A}(\mathbf{k}, X, r)$ corresponding to finite nondegenerate symmetric sets have remarkable algebraic and homological properties. They are noncommutative, but have many of the "good" properties of the commutative polynomial ring $\mathbf{k}[x_1, \ldots, x_n]$; see Remarks 3.1 and 3.5 and Theorem 3.8. This motivates us to look for more analogues coming from commutative algebra and algebraic geometry.

3.1. Basic facts about the YB algebras $\mathcal{A}(\mathbf{k}, X, r)$ of finite solutions (X, r)

The following remark observes the importance of finite square-free solutions and their close relations to Artin–Schelter regularity. The results are extracted from [13, 15, 22, 33].

Remark 3.1. Suppose (X, r) is a square-free nondegenerate and involutive *quadratic set* of order n. Let $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ be the associated quadratic algebra. The following conditions are equivalent.

(1) $\mathcal{A}$ is an Artin–Schelter regular PBW algebra.

(2) (X, r) is a solution of YBE.

(3) There exists an enumeration $X = \{x_1, x_2, \ldots, x_n\}$ such that $\mathcal{A}$ is a binomial skew polynomial algebra.

The implication (1) $\Rightarrow$ (3) follows from [15, Theorem 1.2]. (3) $\Rightarrow$ (1) is proven in [13, Theorem B] (see also [22]). (3) $\Rightarrow$ (2) is proven in [22, Theorem 1.1]. The implication (2) $\Rightarrow$ (3) was conjectured by the author and proven by Rump; see [33, Theorem 1].

Remark 3.2. Note that among all Yang–Baxter algebras $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ of finite solutions studied in this paper, the only PBW algebras are those corresponding to square-free solutions (X, r). This follows from Theorem 3.8 which will be proven in the next subsection.

Artin–Schelter regular algebras (or shortly AS regular algebras) were introduced and studied first in [3]. The study of AS regular algebras, their classification, and finding new classes of such algebras is one of the central problems in noncommutative geometry.

Convention 3.3. Let (X, r) be a finite solution of YBE of order n, and let $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ be the associated Yang–Baxter algebra.

(a) If (X, r) is square-free, we fix an enumeration such that $X = \{x_1, \ldots, x_n\}$ is a set of PBW generators of $\mathcal{A}$. In this case, $\mathcal{A}$ is a binomial skew polynomial ring; see Definition 2.3.

(b) If (X, r) is not square-free, we fix an arbitrary enumeration $X = \{x_1, \ldots, x_n\}$ on X.

In each of the cases (a) and (b), we extend the fixed enumeration on X to the deg-lex order $<$ on $\langle X \rangle$. By convention, the Yang–Baxter algebra $\mathcal{A} = \mathcal{A}_X = \mathcal{A}(\mathbf{k}, X, r)$ is presented as

$$\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r) = \mathbf{k}\langle X \rangle / (\mathfrak{R}_{\mathcal{A}}) \simeq \mathbf{k}\langle X; \mathfrak{R}(r) \rangle,$$
$$\text{where } \mathfrak{R}_{\mathcal{A}} = \{xy - y'x' \mid xy > y'x', \text{ and } r(xy) = y'x'\}. \tag{3.1}$$

Consider the two-sided ideal $I = (\mathfrak{R}_{\mathcal{A}})$ of $\mathbf{k}\langle X \rangle$, let $G = G(I)$ be the unique reduced Gröbner basis of I with respect to $<$. Here we do not need an explicit description of the reduced Gröbner basis G of I, but we need some details. In the case (a), one has $G = \mathfrak{R}_{\mathcal{A}}$. It follows from Remark 3.2 that in the case (b) the set of relations $\mathfrak{R}_{\mathcal{A}}$ is not a Gröbner basis of I, but $\mathfrak{R}_{\mathcal{A}} \subsetneqq G$. Moreover, the shape of the relations $\mathfrak{R}_{\mathcal{A}}$ and standard techniques from noncommutative Gröbner bases theory imply that the Gröbner basis G is finite, or countably infinite, and consists of homogeneous binomials $f_j = u_j - v_j$, where $\mathbf{LM}(f_j) = u_j > v_j$, and $u_j, v_j \in X^m$, for some $m \geq 2$. The set of all normal monomials modulo I is denoted by $\mathcal{N}$. As we mentioned above, $\mathcal{N} = \mathcal{N}(I) = \mathcal{N}(G)$. An element $f \in \mathbf{k}\langle X \rangle$ is in normal form (modulo I) if $f \in \mathrm{Span}_{\mathbf{k}} \mathcal{N}$. The free monoid $\langle X \rangle$ splits as a disjoint union $\langle X \rangle = \mathcal{N} \sqcup \mathbf{LM}(I)$. The free associative algebra $\mathbf{k}\langle X \rangle$ splits as a direct sum of $\mathbf{k}$-vector subspaces $\mathbf{k}\langle X \rangle \simeq \mathrm{Span}_{\mathbf{k}} \mathcal{N} \oplus I$, and there is an isomorphism of vector spaces $\mathcal{A} \simeq \mathrm{Span}_{\mathbf{k}} \mathcal{N}$. As usual, we denote

$$\mathcal{N}_d = \{u \in \mathcal{N} \mid u \text{ has length } d\}. \tag{3.2}$$

Then $\mathcal{A}_d \simeq \mathrm{Span}_{\mathbf{k}} \mathcal{N}_d$ for every $d \in \mathbb{N}_0$. According to Corollary 3.6, $\dim \mathcal{A}_d = |\mathcal{N}_d| = \binom{n+d-1}{d} \ \forall d \geq 0$. Note that since the set of relations $\mathfrak{R}_{\mathcal{A}}$ is a finite set of homogeneous polynomials, the elements of the reduced Gröbner basis $G = G(I)$ of degree $\leq d$ can be found effectively (using the standard strategy for constructing a Gröbner basis), and therefore the set of normal monomials $\mathcal{N}_d$ can be found inductively for $d = 1, 2, 3, \ldots$ (Here we do not need an explicit description of the reduced Gröbner basis G of I.) It follows from Bergman's Diamond Lemma [5, Theorem 1.2] that if we consider the space $\mathbf{k}\mathcal{N}$ endowed with multiplication defined by

$$f \bullet g := \mathrm{Nor}(fg) \quad \text{for every } f, g \in \mathbf{k}\mathcal{N},$$

then $(\mathbf{k}\mathcal{N}, \bullet)$ has a well-defined structure of a graded algebra, and there is an isomorphism of graded algebras

$$\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r) \cong (\mathbf{k}\mathcal{N}, \bullet), \quad \text{so } \mathcal{A} = \bigoplus_{d \in \mathbb{N}_0} A_d \cong \bigoplus_{d \in \mathbb{N}_0} \mathbf{k}\mathcal{N}_d.$$

By convention, we shall often identify the algebra $\mathcal{A}$ with $(\mathbf{k}\mathcal{N}, \bullet)$. In the case (a) when (X, r) is square-free, the set of normal monomials is exactly $\mathcal{T}$ (the set of ordered terms in X), so $\mathcal{A}$ is identified with $(\mathbf{k}\mathcal{T}, \bullet)$ and $S(X, r)$ is identified with $(\mathcal{T}, \bullet)$.

We shall recall more properties of the Yang–Baxter algebras which will be used in the sequel, but first we need the following lemma which is involved in our interpretation of [22, Theorem 1.3] as presented in Remark 3.5.

Lemma 3.4. *Every nondegenerate involutive quadratic set (X, r) satisfies the following condition (*Ore condition*).*

> *Given $a, b \in X$, there exist unique $c, d \in X$ such that $r(ca) = db$.*
>
> *Furthermore, if $a = b$, then $c = d$.*
>
> <div align="right">(3.3)</div>

In particular, r is 2-cancellative.

Proof. Let (X, r) be a nondegenerate involutive quadratic set (not necessarily finite). Let $a, b \in X$. We have to find unique pair c, d such that $r(c, a) = (d, b)$. By the nondegeneracy, there is unique $c \in X$ such that $c^a = b$. Let $d = {}^c a$, then $r(c, a) = ({}^c a, c^a) = (d, b)$, as desired. It also follows from the nondegeneracy that the pair c, d with this property is unique. Assume now that $a = b$. The equality $r(c, a) = (d, a)$ implies $({}^c a, c^a) = (d, a)$, so $c^a = a$. But r is involutive, thus $(c, a) = r(d, a) = ({}^d a, d^a)$, and therefore $d^a = a$. It follows that $c^a = d^a$, and, by the nondegeneracy, $c = d$.   ∎

The following facts are a compilation of results from [22] and are true for every finite solution of YBE.

Remark 3.5. Suppose (X, r) is a finite solution of YBE of order n, $X = \{x_1, \ldots, x_n\}$. Let $S = S(X, r)$ be the associated Yang–Baxter monoid, and let $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ be the associated Yang–Baxter algebra. Then the following conditions hold.

(1) (A modified version of [22, Theorem 1.3])
   The monoid S is a semigroup of I-type, that is, there is a bijective map $v : \mathcal{U} \mapsto S$, where $\mathcal{U}$ is the free n-generated abelian monoid $\mathcal{U} = [u_1, \ldots, u_n]$ such that $v(1) = 1$, and such that

$$\{v(u_1 a), \ldots, v(u_n a)\} = \{x_1 v(a), \ldots, x_n v(a)\} \quad \text{for all } a \in \mathcal{U}.$$

(2) The Hilbert series of $\mathcal{A}$ is $H_{\mathcal{A}}(t) = 1/(1 - t)^n$.

(3) (a) $\mathcal{A}$ has finite global dimension and polynomial growth; (b) $\mathcal{A}$ is Koszul; (c) $\mathcal{A}$ is left and right Noetherian; (d) $\mathcal{A}$ satisfies the Auslander condition and is Cohen–Macaulay; and (e) A is finite over its centre [22, Theorem 1.4].

(4) $\mathcal{A}$ is a domain, and in particular the monoid S is cancellative [22, Corollary 1.5].

For convenience of the reader we shall make a brief observation. Note first that the hypothesis of Remark 3.5 is satisfied by arbitrary finite solution of YBE (X, r) which is not necessarily square-free, and, in general, the algebra $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ is not a binomial skew polynomial ring, or equivalently, $\mathcal{A}$ is not a PBW algebra.

Next observe that part (1) of Remark 3.5 is a modification of the original second part of [22, Theorem 1.3] which states (in our terminology): "Suppose that (X, r) is a finite solution of YBE of order n which satisfies the condition (3.3). Then the monoid $S(X, r)$ is of I-type". However, under the hypothesis of Remark 3.5, Lemma 3.4 implies the necessary condition (3.3).

The following corollary is straightforward from Remark 3.5 (1) and will be used throughout the paper.

Corollary 3.6. *In notation and conventions as above. Let (X, r) be a solution of YBE of finite order n. Then for every integer $d \geq 1$, there are equalities*

$$\dim \mathcal{A}_d = \binom{n + d - 1}{d} = \binom{n + d - 1}{n - 1} = |\mathcal{N}_d|. \tag{3.4}$$

3.2. Every finite solution (X, r) whose Yang–Baxter algebra $\mathcal{A}(k, X, r)$ is PBW is square-free

In this subsection, we give an answer to Problems 1.1 (1).

Suppose (X, r) is a finite solution of YBE whose Yang–Baxter algebra $A = A(\mathbf{k}, X, r)$ is PBW, where $X = \{x_1, x_2, \dots, x_n\}$ is a set of PBW generators. Then $\mathcal{A} = \mathbf{k}\langle X \rangle / (\mathfrak{R}_{\mathcal{A}})$, where the set of (quadratic) defining relations $\mathfrak{R}_{\mathcal{A}}$ of $\mathcal{A}$ coincides with the reduced Gröbner basis of the ideal $(\mathfrak{R}_{\mathcal{A}})$ modulo the deg-lex order on $\langle X \rangle$. The cardinality of $\mathfrak{R}_{\mathcal{A}}$ is exactly $\binom{n}{2}$; see Corollary 2.10. Recall that the set of leading monomials

$$W = \{\mathbf{LM}(f) \mid f \in \mathfrak{R}_{\mathcal{A}}\} \tag{3.5}$$

is called *the set of obstructions* for $\mathcal{A}$, in the sense of Anick [1].

Lemma 3.7. *Suppose (X, r) is a solution of YBE of order n and that its Yang–Baxter algebra $\mathcal{A} = \mathcal{A}(k, X, r)$ is PBW, where $X = \{x_1, x_2, \dots, x_n\}$ is a set of PBW generators. Then there exists a permutation*

$$y_1 = x_{s_1}, y_2 = x_{s_2}, \dots, y_n = x_{s_n} \quad \text{of } x_1, x_2, \dots, x_n$$

such that the following conditions hold.

(1) *The set of obstructions $W = \{\mathbf{LM}(f) \mid f \in \mathfrak{R}_{\mathcal{A}}\}$ consists of $\binom{n}{2}$ monomials given below*

$$W = \{y_j y_i \mid 1 \leq i < j \leq n\}. \tag{3.6}$$

(2) *The normal $\mathbf{k}$-basis of $\mathcal{A}$ modulo $I = (\mathfrak{R}_{\mathcal{A}})$ is the set*

$$\mathcal{N} = \{y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_n^{\alpha_n} \mid \alpha_i \geq 0 \text{ for } 1 \leq i \leq n\}. \tag{3.7}$$

Proof. Let W be the set of obstructions defined via (3.5), and let A_W be *the associated monomial algebra* defined as

$$A_W := \mathbf{k}\langle X \rangle / (W). \tag{3.8}$$

It is well known that a word $u \in \langle X \rangle$ is normal modulo $I = (\Re_{\mathcal{A}})$ iff u is normal modulo the set of obstructions W. Therefore, the two algebras $\mathcal{A}$ and A_W share the same normal **k**-basis $\mathcal{N} = \mathcal{N}(I) = \mathcal{N}(W)$ and their Hilbert series are equal. By Remark 3.5 part (2), the Hilbert series of $\mathcal{A}$ is $H_{\mathcal{A}}(t) = 1/(1-t)^n$; therefore,

$$H_{A_W}(t) = H_{\mathcal{A}}(t) = 1/(1-t)^n. \tag{3.9}$$

Thus the Hilbert series of A_W satisfies condition (5) of [15, Theorem 3.7, p. 2163], and it follows from the theorem that there exists a permutation $y_1 = x_{s_1}, y_2 = x_{s_2}, \ldots, y_n = x_{s_n}$ of the generators $x_1, x_2, \ldots, x_n$ such that the set of obstructions W satisfies (3.6). The Diamond Lemma [5] and the explicit description of the obstruction set W given in (3.6) imply that the set of normal words $\mathcal{N} = \mathcal{N}(I) = \mathcal{N}(W)$ is described in (3.7). ∎

Observe that if the permutation given in the lemma is not trivial, then there is an inversion, that is, a pair i, j with $i < j$ and $y_j < y_i$.

Theorem 3.8. *Suppose (X, r) is a solution of YBE of order n, and $\mathcal{A} = \mathcal{A}(k, X, r)$ is its Yang–Baxter algebra. Then $\mathcal{A}$ is a PBW algebra with a set of PBW generators $X = \{x_1, x_2, \ldots, x_n\}$ (enumerated properly) if and only if (X, r) is a square-free solution.*

Proof. If (X, r) is square-free, then there exists an enumeration $X = \{x_1, \ldots, x_n\}$, so that $\mathcal{A}$ is a binomial skew polynomial ring in the sense of [11], and therefore $\mathcal{A}$ is PBW. This was a conjecture of the author which was proven later by Rump [33, Theorem 1].

Assume now that (X, r) is a finite solution of order n whose Yang–Baxter algebra $\mathcal{A} = \mathcal{A}(k, X, r)$ is PBW, where $X = \{x_1, x_2, \ldots, x_n\}$ is a set of PBW generators. We have to show that (X, r) is square-free, that is, $r(x, x) = (x, x)$ for all $x \in X$.

It follows from our assumptions that in the presentation $\mathcal{A} = k\langle X \rangle/(\Re_{\mathcal{A}})$ the set of (quadratic) defining relations $\Re_{\mathcal{A}}$ of $\mathcal{A}$ is the reduced Gröbner basis of the ideal $(\Re_{\mathcal{A}})$ modulo the deg-lex order on $\langle X \rangle$. By Lemma 3.7, there exists a permutation $y_1 = x_{s_1}, y_2 = x_{s_2}, \ldots, y_n = x_{s_n}$ of $x_1, x_2, \ldots, x_n$ such that the obstruction set $W = \{\mathbf{LM}(f) \mid f \in \Re_{\mathcal{A}}\}$ satisfies (3.6), and the set of normal monomials $\mathcal{N}$ described in (3.7) is a PBW basis of $\mathcal{A}$.

We use some properties of (X, r) and the relations of $\mathcal{A}$ listed below.

(i) The solution (X, r) is 2-cancellative. This follows from Lemma 3.4. (ii) There are exactly n fixed points $xy \in X^2$ with $r(x, y) = (x, y)$ and the set $\Re_{\mathcal{A}}$ consists of exactly $\binom{n}{2}$ relations. This follows from Corollary 2.10. (iii) Every monomial of the shape $y_j y_i$, $1 \le i < j \le n$ is the leading monomial of some polynomial $\varphi_{ji} \in \Re_{\mathcal{A}}$. (It is possible that $y_j < y_i$ for some $j > i$.)

Therefore, the algebra $\mathcal{A}$ has a presentation

$$\mathcal{A} = k\langle x_1, \ldots, x_n \rangle/(\Re_{\mathcal{A}})$$

with precisely $\binom{n}{2}$ defining relations

$$\Re_{\mathcal{A}} = \{\varphi_{ji} = y_j y_i - u_{ij} \mid 1 \le i < j \le n\} \tag{3.10}$$

such that

(1) For every pair i, j, $1 \leq i < j \leq n$, the monomial u_{ij} satisfies $u_{ij} = y_{i'} y_{j'}$, where $i' \leq j'$ and $y_j > y_{i'}$ (since $\mathbf{LM}(\varphi_{ji}) = y_j y_i > y_{i'} y_{j'}$, and since (X, r) is 2-cancellative).

(2) Each monomial $y_i y_j$ with $1 \leq i \leq j \leq n$ occurs at most once in $\mathfrak{R}_{\mathcal{A}}$ (since r is a bijective map).

(3) $\mathfrak{R}_{\mathcal{A}}$ is the *reduced Gröbner basis* of the two-sided ideal $(\mathfrak{R}_{\mathcal{A}})$, with respect to the degree-lexicographic order $<$ on $\langle X \rangle$.

In terms of the relations $\mathfrak{R}_{\mathcal{A}}$, our claim that $r(x, x) = (x, x)$, for all $x \in X$, is equivalent to

$$u_{ij} \neq xx, \quad \text{where } x \in X \text{ and } 1 \leq i < j \leq n. \tag{3.11}$$

So far we know that (X, r) has exactly n fixed points, and each monomial $y_j y_i$, $1 \leq i < j \leq n$ is not a fixed point. Therefore, it will be enough to show that a monomial $y_i y_j$, with $1 \leq i < j \leq n$, cannot be a fixed point.

Assume on the contrary that $r(y_i, y_j) = (y_i, y_j)$ for some $1 \leq i < j \leq n$. We claim that in this case $\mathfrak{R}_{\mathcal{A}}$ contains two relations of the shape

$$y_p y_q - y_j y_j, \quad \text{where } p > q, \; y_p > y_j. \tag{3.12a}$$

$$y_s y_t - y_i y_i, \quad \text{where } s > t, \; y_s > y_i. \tag{3.12b}$$

Consider the increasing chain of left ideals of $\mathcal{A}$

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k \subseteq \cdots,$$

where for $k \geq 1$, I_k is the left ideal

$$I_k = {}_{\mathcal{A}}(y_i y_j, y_i y_j^2, \ldots, y_i y_j^k).$$

By [22, Theorem 1.4], see also Remark 3.5 (3), the algebra $\mathcal{A}$ is left Noetherian, hence there exists $k > 1$ such that $I_{k-1} = I_k = I_{k+1} = \cdots$, and therefore $y_i y_j^k \in I_{k-1}$. This implies

$$w \bullet (y_i y_j^c) = y_i y_j^k \in \mathcal{N} \quad \text{for some } c, \; 1 \leq c \leq k - 1,$$

$$\text{and some } w \in \mathcal{N}, |w| = k - c. \tag{3.13}$$

It follows from (3.13) that the monomial $v_0 = y_i y_j^k$ can be obtained from the monomial $w(y_i y_j^c)$ by applying a finite sequence of replacements (reductions) in $\langle X \rangle$. More precisely, there exists a sequence of monomials

$$v_0 = y_i y_j^k, v_1, \ldots, v_{t-1}, \quad v_t = w(y_i y_j^c) \in \langle X \rangle$$

and replacements

$$v_t \rightarrow v_{t-1} \rightarrow \cdots \rightarrow v_1 \rightarrow v_0 = y_i y_j^k \in \mathcal{N}, \tag{3.14}$$

where each replacement comes from some quadratic relation $f_{pq} = y_p y_q - u_{qp}$ in (3.10) and has the shape

$$a[y_p y_q]b \to a(u_{qp})b, \quad \text{where } n \geq p > q \geq 1, \, a, b \in \langle X \rangle.$$

We have assumed that $y_i y_j$ is a fixed point, so it cannot occur in a relation in (3.10). Thus the rightmost replacement in (3.14) is of the form

$$u_1 = y_i y_j \cdots y_j [y_p y_q] \cdots y_j \to y_i y_j \cdots y_j (u_{qp}) \cdots y_j = y_i y_j \cdots y_j (y_j y_j) \cdots y_j = v_0,$$

where p, q is a pair with $1 \leq q < p \leq n$, $u_{qp} = y_j y_j$ and $y_p > y_j$. In other words, the set $\mathfrak{R}_\mathcal{A}$ contains a relation of type (a) $y_p y_q - x_j x_j$ where $p > q$, $y_q > y_j$.

Analogous argument proves the existence of a relation of type (b) in (3.12). This time we consider an increasing chain of right ideals $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k \subseteq \cdots$, where I_k is the right ideal $I_k = (y_i y_j, y_i^2 y_j, \ldots, y_i^k y_j)_\mathcal{A}$ and apply the right Noetherian property of $\mathcal{A}$.

Consider now the subset of fixed points

$$\mathcal{F}_0(X, r) = \{y_i y_j \in X^2 \text{ such that } i < j \text{ and } r(y_i, y_j) = (y_i, y_j)\},$$

which by our assumption is not empty. Then $\mathcal{F}_0(X, r)$ has cardinality $m \geq 1$ and $\mathfrak{R}_\mathcal{A}$ contains at least $m + 1$ (distinct) relations of the type

$$y_p y_q - xx, \quad \text{where } x \in X, \, p > q \text{ and } y_p > x. \tag{3.15}$$

The set $\mathcal{N}_2$ of normal monomials of length 2 contains $\binom{n}{2}$ elements of the shape $y_s y_t$, $1 \leq s < t \leq n$, and we have assumed that m of them are fixed. Then there are $\binom{n}{2} - m$ distinct monomials $y_i y_j \in \mathcal{N}_2$, $1 \leq i < j \leq n$ which are not fixed. Each of these monomials occurs in exactly one relation

$$y_s y_t - y_i y_j, \quad \text{where } r(y_s, y_t) = (y_i, y_j), \, s > t, \, y_s > y_i.$$

Thus $\mathfrak{R}_\mathcal{A}$ contains $\binom{n}{2} - m$ distinct square-free relations and at least $m + 1$ relations which contain squares as in (3.15). Therefore, the set of relations has cardinality

$$|\mathfrak{R}_\mathcal{A}| \geq \binom{n}{2} - m + m + 1 > \binom{n}{2}$$

which is a contradiction.

We have shown that a monomial $y_i y_j$ with $1 \leq i < j \leq n$ cannot be a fixed point, and therefore it occurs in a relation in $\mathfrak{R}_\mathcal{A}$. But (X, r) has exactly n fixed points, so these are the elements of the diagonal of X^2, $x_i x_i$, $1 \leq i \leq n$. It follows that (X, r) is square-free. ∎

Proposition 3.9. *Let (X, r) be a finite nondegenerate involutive quadratic set, and let $\mathcal{A} = \mathcal{A}(k, X, r) = k\langle X \rangle / (\mathfrak{R}_\mathcal{A})$ be its quadratic algebra. Assume that there is an enumeration $X = \{x_1, x_2, \ldots, x_n\}$ of X such that the set*

$$\mathcal{N} = \{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \mid \alpha_i \geq 0 \text{ for } 1 \leq i \leq n\}$$

*is a normal **k**-basis of $\mathcal{A}$ modulo the ideal $I = (\mathfrak{R}_{\mathcal{A}})$. Then $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ is a PBW algebra, where $X = \{x_1, x_2, \ldots, x_n\}$ is a set of PBW generators of $\mathcal{A}$ and the set of relations $\mathcal{R}_0$ is a quadratic Gröbner basis of the two-sided ideal $(\mathfrak{R}_{\mathcal{A}})$. The following conditions are equivalent.*

(1) *The algebra $\mathcal{A}$ is left and right Noetherian.*

(2) *The quadratic set (X, r) is square-free.*

(3) *(X, r) is a solution of YBE.*

(4) *$\mathcal{A}$ is a binomial skew polynomial ring in the sense of* [11].

Proof. The quadratic set (X, r) and the relations of $\mathcal{A}$ satisfy conditions similar to those listed in the proof of Theorem 3.8. More precisely: (i) (X, r) is 2-cancellative. This follows from Lemma 3.4. (ii) There are exactly n fixed points $xy \in X^2$ with $r(x, y) = (x, y)$. This follows from Corollary 2.10. (iii) It follows from the hypothesis that every monomial of the shape $x_j x_i$, $1 \leq i < j \leq n$ is not in the normal **k**-basis $\mathcal{N}$, and therefore it is the highest monomial of some polynomial $\varphi_{ji} \in \mathfrak{R}_{\mathcal{A}}$. According to [14, Proposition 2.3], if (X, r) is a nondegenerate involutive quadratic set of order n, then the set $\mathfrak{R}_{\mathcal{A}}$ consists of exactly $\binom{n}{2}$ relations. Therefore, the algebra $\mathcal{A}$ has a presentation

$$\mathcal{A} = \mathbf{k}\langle x_1, \ldots, x_n \rangle / (\mathfrak{R}_{\mathcal{A}})$$

with precisely $\binom{n}{2}$ defining relations

$$\mathfrak{R}_{\mathcal{A}} = \{\varphi_{ji} = x_j x_i - x_{i'} x_{j'} \mid 1 \leq i < j \leq n\} \tag{3.16}$$

such that

(a) For every pair i, j, $1 \leq i < j \leq n$, one has $i' \leq j'$, and $j > i'$ (since $\mathbf{LM}(\varphi_{ji}) = x_j x_i > x_{i'} x_{j'}$, and since (X, r) is 2-cancellative).

(b) Each ordered monomial (term) of length 2 occurs at most once in $\mathfrak{R}_{\mathcal{A}}$ (since r is a bijective map).

(c) $\mathfrak{R}_{\mathcal{A}}$ is the *reduced Gröbner basis* of the two-sided ideal $(\mathfrak{R}_{\mathcal{A}})$, with respect to the deg-lex order $<$ on $\langle X \rangle$, or equivalently the overlaps $x_k x_j x_i$ with $k > j > i$ do not give rise to new relations in $\mathcal{A}$.

We shall prove now the equivalence of the conditions (1) through (4). (1) $\Rightarrow$ (2). The proof is analogous to the proof of Theorem 3.8. It is enough to show that a monomial $x_i x_j$ with $1 \leq i < j \leq n$ cannot be a fixed point. Assuming the contrary, and applying an argument similar to the proof of Theorem 3.8, in which we involve the left and right Noetherian properties of $\mathcal{A}$, we get a contradiction. Thus every monomial $x_i x_j$ with $1 \leq i < j \leq n$ occurs in a relation in $\mathfrak{R}_{\mathcal{A}}$. At the same time, the monomials $x_j x_i$ with $1 \leq i < j \leq n$ are also involved in the relations $\mathfrak{R}_{\mathcal{A}}$, hence they are not fixed points. But (X, r) has exactly n fixed points, so these are the elements of the diagonal of X^2, $x_i x_i$, $1 \leq i \leq n$. It follows that (X, r) is square-free. (2) $\Rightarrow$ (4). If (X, r) is square-free, then the relations $\mathfrak{R}_{\mathcal{A}}$ given

in (3.16) are exactly the defining relations of a binomial skew polynomial ring. Moreover, by the hypothesis of the proposition, condition (d′) in Definition 2.3 is in force; therefore, all conditions in Definition 2.3 hold, and $\mathcal{A}$ is a skew polynomial ring with binomial relations in the sense of [11]. The implication (4) $\Rightarrow$ (3) follows from [22, Theorem 1.1]. The implication (3) $\Rightarrow$ (1) follows from [22, Theorem 1.4]; see also Remark 3.5 (3). ∎

We end the section with the following general question.

Question 3.10. (1) Suppose (X, r) is a finite nondegenerate involutive solution of order n. Determine conditions on (X, r) which imply that the Yang–Baxter algebra $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r) = \mathbf{k}\langle X \rangle / (\mathfrak{R}_{\mathcal{A}})$ is standard finitely presented (or shortly, an s.f.p. algebra) with respect to some appropriate enumeration $X = \{x_1, \ldots, x_n\}$ of X. In other words, the two-sided ideal $I = (\mathfrak{R}_{\mathcal{A}})$ of $\mathbf{k}\langle X \rangle$ has a finite reduced Gröbner basis with respect to the deg-lex order $<$ on $\langle X \rangle$. (2) Classify all solutions whose Yang–Baxter algebras are s.f.p. (3) Find examples of finite nondegenerate involutive solutions (X, r) whose Yang–Baxter algebra is not s.f.p. with respect to any enumeration of X, or prove that such solutions do not exist.

The Yang–Baxter algebra $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ of the solution (X, r) in Example 7.2 is not PBW, but it is standard finitely presented with respect to any enumeration of X.

4. The d-Veronese subalgebra $\mathcal{A}^{(d)}$ of the Yang–Baxter algebra $\mathcal{A}(\mathbf{k}, X, r)$, its generators and relations

In this section, (X, r) is a finite solution of YBE, and $d \geq 2$ is an integer. We shall study the d-Veronese subalgebras $\mathcal{A}^{(d)}$ of the Yang–Baxter algebra $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$. This is an algebraic construction which mirrors the Veronese embedding. Results on Veronese subalgebras of noncommutative graded algebras appeared first in [9] and [4]. Our main reference here is [30, Section 3.2]. We shall prove Theorem 4.13 which presents the d-Veronese subalgebra $\mathcal{A}^{(d)}$ in terms of generators and quadratic relations.

4.1. Veronese subalgebras of graded algebras

We recall first some basic definitions and facts about Veronese subalgebras of general graded algebras.

Definition 4.1. Let $A = \bigoplus_{m \in \mathbb{N}_0} A_m$ be a graded $\mathbf{k}$-algebra. For any integer $d \geq 1$, the d-*Veronese subalgebra* of A is the graded algebra

$$A^{(d)} = \bigoplus_{m \in \mathbb{N}_0} A_{md}.$$

By definition, the algebra $A^{(d)}$ is a subalgebra of A. However, the embedding is not a graded algebra morphism. The Hilbert function of $A^{(d)}$ satisfies

$$h_{A^{(d)}}(m) = \dim(A^{(d)})_m = \dim(A_{md}) = h_A(md).$$

It follows from [30, Proposition 2.2, Chapter 3] that if A is a one-generated, quadratic, Koszul algebra, then its Veronese subalgebras are also one-generated, quadratic, and Koszul.

Corollary 4.2. *Let (X, r) be a solution of order n, let $\mathcal{A} = \mathcal{A}(k, X, r)$ be its Yang–Baxter algebra, and let $d \geq 2$ be an integer. Then the d-Veronese subalgebra $\mathcal{A}^{(d)}$ is one-generated, quadratic, and Koszul.*

Proof. If (X, r) is a solution of order n, then, by definition, the Yang–Baxter algebra $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ is one-generated and quadratic. Moreover, $\mathcal{A}$ is Koszul; see Remark 3.5. It follows straightforwardly from [30, Proposition 2.2, Chapter 3] that $\mathcal{A}^{(d)}$ is one-generated, quadratic, and Koszul. ∎

We shall prove in the next section that $\mathcal{A}^{(d)}$ is a left and a right Noetherian domain.

In the assumptions of Corollary 4.2, it is clear that the d-Veronese subalgebra $\mathcal{A}^{(d)}$ satisfies

$$\mathcal{A}^{(d)} = \bigoplus_{m \in \mathbb{N}_0} A_{md} \cong \bigoplus_{m \in \mathbb{N}_0} \mathbf{k}\mathcal{N}_{md}. \tag{4.1}$$

Moreover, the normal monomials $w \in \mathcal{N}_d$ of length d are degree one generators of $A^{(d)}$, and by Corollary 3.6 there are equalities

$$|\mathcal{N}_d| = \dim \mathcal{A}_d = \binom{n + d - 1}{d}.$$

We set

$$N = \binom{n + d - 1}{d} \tag{4.2}$$

and order the elements of $\mathcal{N}_d$ lexicographically:

$$\mathcal{N}_d := \{w_1 < w_2 < \cdots < w_N\}. \tag{4.3}$$

The d-Veronese $\mathcal{A}^{(d)}$ is a quadratic algebra with one-generators $w_1, w_2, \ldots, w_N$. We shall find a minimal set of quadratic relations for $\mathcal{A}^{(d)}$, each of which is a linear combination of products $w_i w_j$ for some $i, j \in \{1, \ldots, N\}$. The relations are intimately connected with the properties of the Yang–Baxter monoid $S(X, r)$. As a first step, we shall introduce a finite nondegenerate symmetric set (S_d, r_d) induced in a natural way by the braided monoid $S(X, r)$.

4.2. The braided monoid $S = S(X, r) = (S, r_S)$ of a braided set (X, r)

Matched pairs of monoids, M3-monoids, and braided monoids in a most general setting were studied in [19], where the interested reader can find the necessary definitions and the original results. Here we extract only some facts which will be used in the paper.

Fact 4.3 ([19, Theorems 3.6 and 3.14]). Let (X, r) be a braided set, and let $S = S(X, r)$ be its Yang–Baxter monoid. Then

(1) The left and the right actions $^{()}\!\circ : X \times X \to X$, and $\circ^{()} : X \times X \to X$ defined via r can be extended in a unique way to a left and a right action

$$^{()}\!\circ : S \times S \longrightarrow S, \quad (a, b) \mapsto {}^a b, \quad \text{and} \quad \circ^{()} : S \times S \longrightarrow S, \quad (a, b) \mapsto a^b$$

which make S *a strong graded* **M3**-*monoid*. In particular, the following equalities hold in S for all $a, b, u, v \in S$.

$$
\begin{array}{llll}
\text{ML 0:} & {}^a 1 = 1, \quad {}^1 u = u; & \text{MR 0:} & 1^u = 1, \quad a^1 = a \\
\text{ML 1:} & {}^{(ab)} u = {}^a({}^b u), & \text{MR 1:} & a^{(uv)} = (a^u)^v \\
\text{ML 2:} & {}^a(u.v) = ({}^a u)({}^{a^u} v), & \text{MR 2:} & (a.b)^u = (a^{{}^b u})(b^u) \\
\text{M 3:} & {}^u v u^v = uv.
\end{array}
\tag{4.4}
$$

These actions define a bijective map

$$r_S : S \times S \longrightarrow S \times S, \quad r_S(u, v) := ({}^u v, u^v)$$

which obeys the Yang–Baxter equation, so (S, r_S) is *a braided monoid*. In particular, (S, r_S) is a set-theoretic solution of YBE, and the associated bijective map r_S restricts to r.

(2) The following conditions hold.

(a) (S, r_S) is *a graded braided monoid*, that is, the actions agree with the grading (by length) of S:

$$|^a u| = |u| = |u^a| \quad \forall a, u \in S. \tag{4.5}$$

(b) (S, r_S) is nondegenerate iff (X, r) is nondegenerate.

(c) (S, r_S) is involutive iff (X, r) is involutive.

(d) (S, r_S) is square-free iff (X, r) is a trivial solution.

Let (X, r) be a nondegenerate symmetric set, and let (S, r_S) be the associated graded braided monoid, where we consider the natural grading by length given in (2.6):

$$S = \bigsqcup_{d \in \mathbb{N}_0} S_d, \quad S_0 = \{1\}, \quad S_1 = X, \quad \text{and} \quad S_k S_m \subseteq S_{k+m}.$$

Each of the graded components S_d, $d \geq 1$, is r_S-invariant. Consider the restriction $r_d = (r_S)_{|S_d \times S_d}$, where r_d is the map $r_d : S_d \times S_d \to S_d \times S_d$.

Corollary 4.4. *Let (X, r) be a solution of YBE. Then the following conditions hold.*

(1) *For every positive integer $d \geq 1$, (S_d, r_d) is a solution of YBE (a nondegenerate symmetric set). Moreover, if (X, r) is of finite order n, then (S_d, r_d) is a finite solution of YBE of order*

$$|S_d| = \binom{n + d - 1}{d} = N. \tag{4.6}$$

(2) *The number of fixed points is $|\mathcal{F}(S_d, r_d)| = N$.*

Definition 4.5. We call (S_d, r_d) *the monomial d-Veronese solution associated with* (X, r).

The monomial d-Veronese solution (S_d, r_d) depends only on the map r and on the integer d; it is invariant with respect to the enumeration of X. Although (S_d, r_d) is intimately connected with the d-Veronese subalgebra $\mathcal{A}^{(d)}$ and its quadratic relations, this solution is not convenient for an explicit description of the relations. Its rich structure inherited from the braiding in (S, r_S) is used in the proof of Theorem 6.4. The solution (S_d, r_d) induces in a natural way an isomorphic solution $(\mathcal{N}_d, \rho_d)$, and the fact that $\mathcal{N}_d$ is ordered lexicographically makes this solution convenient for our description of the relations of $\mathcal{A}^{(d)}$. Note that the set $\mathcal{N}_d$, as a subset of the set of normal monomials $\mathcal{N}$, depends on the initial enumeration of X. We shall construct $(\mathcal{N}_d, \rho_d)$ below.

Remark 4.6. The d-Veronese solution (S_d, r_d) is closely related to a recently introduced notion of cabling for involutive solutions (see [24]), where the "cabling" operations on solutions are used to obtain new results on decomposability of solutions and a conceptual interpretation of the Dehornoy class. In fact, for every $d \geq 2$, the so-called "d-cabled solution" r^d is a very particular subsolution of the d-Veronese solution (S_d, r_d).

Remark 4.7. Given the monomials $a = a_1 a_2 \cdots a_p \in X^p$ and $b = b_1 b_2 \cdots b_q \in X^q$, we can find effectively the monomials $^a b \in X^q$ and $a^b \in X^p$. Indeed, as in [19], we use condition (4.4) to extend the left and the right actions inductively:

$$^c(b_1 b_2 \cdots b_q) = (^c b_1)(^{c^{b_1}} b_2) \cdots (^{(c^{(b_1 \cdots b_{q-1})})} b_q) \quad \text{for all } c \in X$$
$$^{(a_1 a_2 \cdots a_p)} b = {}^{a_1}(^{(a_2 \cdots a_p)} b). \tag{4.7}$$

We proceed similarly with the right action.

Lemma 4.8. *Notation as in Remark 2.11. Suppose $a, a_1 \in X^p, a_1 \in \mathcal{O}_{\mathcal{D}_p}(a)$, and $b, b_1 \in X^q, b_1 \in \mathcal{O}_{\mathcal{D}_q}(b)$,*

(1) *The following are equalities of words in the free monoid $\langle X \rangle$:*

$$\text{Nor}(^{a_1} b_1) = \text{Nor}(^a b), \quad \text{Nor}(a_1{}^{b_1}) = \text{Nor}(a^b). \tag{4.8}$$

In particular, if $a, a_1 \in X^p$ and $b, b_1 \in X^q$, the equalities $a = a_1$ in S and $b = b_1$ in S imply that $^{a_1} b_1 = {}^a b$ and $a_1^{b_1} = a^b$ hold in S.

(2) *The following are equalities in the monoid S:*

$$ab = {}^a b a^b = \mathrm{Nor}({}^a b)\,\mathrm{Nor}(a^b). \tag{4.9}$$

Proof. By Remark 2.11, there is an equality $a = a_1$ in S iff $a_1 \in \mathcal{O}_{\mathcal{D}_p}(a)$; in this case, $\mathcal{O}_{\mathcal{D}_p}(a) = \mathcal{O}_{\mathcal{D}_p}(a_1)$. At the same time, $a = a_1$ in S iff $\mathrm{Nor}(a_1) = \mathrm{Nor}(a)$ as words in X^p; in particular, $\mathrm{Nor}(a) \in \mathcal{O}_{\mathcal{D}_p}(a)$. Similarly, $b_1 = b$ in S iff $b_1 \in \mathcal{O}_{\mathcal{D}_q}(b)$, and in this case $\mathrm{Nor}(b) = \mathrm{Nor}(b_1) \in \mathcal{O}_{\mathcal{D}_q}(b)$. Part (1) follows from the properties of the actions in (S, r_S) studied in [19, Proposition 3.11]. (2) (S, r_S) is an M3-braided monoid (see Fact 4.3), so condition M3 implies the first equality in (4.9). Now (4.8) implies the second equality in (4.9). ∎

Definition-Notation 4.9. In notation and conventions as above. Let $d \geq 1$ be an integer. Suppose (X, r) is a solution of order n, $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ is the associated Yang–Baxter algebra, and (S, r_S) is the associated braided monoid. By convention, we identify $\mathcal{A}$ with $(\mathbf{k}\mathcal{N}, \bullet)$ and S with $(\mathcal{N}, \bullet)$. Define a left "action" and a right "action" on $\mathcal{N}_d$ as follows:

$$\begin{aligned}
\triangleright &: \mathcal{N}_d \times \mathcal{N}_d \longrightarrow \mathcal{N}_d, \quad a \triangleright b := \mathrm{Nor}({}^a b) \in \mathcal{N}_d \quad \forall a, b \in \mathcal{N}_d \\
\triangleleft &: \mathcal{N}_d \times \mathcal{N}_d \longrightarrow \mathcal{N}_d, \quad a \triangleleft b := \mathrm{Nor}(a^b) \in \mathcal{N}_d \quad \forall a, b \in \mathcal{N}_d.
\end{aligned} \tag{4.10}$$

It follows from Lemma 4.8 (1) that the two actions are well defined.
Define the map

$$\rho_d : \mathcal{N}_d \times \mathcal{N}_d \longrightarrow \mathcal{N}_d \times \mathcal{N}_d, \quad \rho_d(a, b) := (a \triangleright b, a \triangleleft b). \tag{4.11}$$

For simplicity of notation (when there is no ambiguity), we shall often write $(\mathcal{N}_d, \rho)$, where $\rho = \rho_d$.

Definition 4.10. We call $(\mathcal{N}_d, \rho_d)$ *the normalised d-Veronese solution associated with* (X, r).

Proposition 4.11. *In assumption and notation as above.*

(1) *Let $\rho_d : \mathcal{N}_d \times \mathcal{N}_d \to \mathcal{N}_d \times \mathcal{N}_d$ be the map defined as $\rho_d(a, b) = (a \triangleright b, a \triangleleft b)$. Then $(\mathcal{N}_d, \rho_d)$ is a solution of YBE of order $|\mathcal{N}_d| = \binom{n+d-1}{d} = N$.*

(2) *$(\mathcal{N}_d, \rho_d)$ and (S_d, r_d) are isomorphic solutions of YBE.*

Proof. (1) By Corollary 4.4, (S_d, r_d) is a nondegenerate symmetric set, that is, a solution of YBE. Thus, by Remark 2.6, the left and the right actions associated with (S_d, r_d) satisfy conditions **l1**, **r1**, **lr3**, and **inv**. Consider the actions $\triangleright$ and $\triangleleft$ on $\mathcal{N}_d$, given in Definition-Notation 4.9. It follows from (4.10) and Lemma 4.8 that these actions also satisfy **l1**, **r1**, **lr3**, and **inv**. Therefore, by Remark 2.6 again, ρ_d obeys YBE and is involutive, so $(\mathcal{N}_d, \rho_d)$ is a symmetric set. Moreover, the nondegeneracy of (S_d, r_d) implies that $(\mathcal{N}_d, \rho_d)$ is nondegenerate. By Corollary 3.4, there are equalities $|\mathcal{N}_d| = |S_d| = \binom{n+d-1}{d} = N$.

(2) We shall prove that the map $\mathrm{Nor} : S_d \to \mathcal{N}_d, u \mapsto \mathrm{Nor}(u)$ is an isomorphism of solutions. It is clear that the map is bijective. We have to show that Nor is a homomorphism of solutions, that is,

$$(\mathrm{Nor} \times \mathrm{Nor}) \circ r_d = \rho_d \circ (\mathrm{Nor} \times \mathrm{Nor}). \tag{4.12}$$

Let $(u, v) \in S_d \times S_d$, then the equalities $u = \mathrm{Nor}(u)$ and $v = \mathrm{Nor}(v)$ hold in S_d, so

$$\mathrm{Nor}(^u v) = \mathrm{Nor}(^{\mathrm{Nor}(u)}\mathrm{Nor}(v)), \quad \mathrm{Nor}(u^v) = \mathrm{Nor}(\mathrm{Nor}(u)^{\mathrm{Nor}(v)})$$

which together with (4.10) imply

$$(\mathrm{Nor} \times \mathrm{Nor}) \circ r_d(u, v) = \mathrm{Nor} \times \mathrm{Nor}(^u v, u^v) = (\mathrm{Nor}(^u v), \mathrm{Nor}(u^v))$$
$$= (\mathrm{Nor}(u) \rhd \mathrm{Nor}(v), \mathrm{Nor}(u) \lhd \mathrm{Nor}(v)) = \rho_d(\mathrm{Nor}(u), \mathrm{Nor}(v)).$$

This proves (4.12). ∎

Let us consider the monomials in $\mathcal{N}_d$ are ordered lexicographically, $\mathcal{N}_d := \{w_1 < w_2 < \cdots < w_N\}$ (see (4.3)), and we shall use this order throughout the paper.

Proposition 4.12. *In assumption and notation as above. Let $(\mathcal{N}_d, \rho_d)$ be the normalised d-Veronese solution; see Definition 4.10. Then the Yang–Baxter algebra $B = \mathcal{A}(\mathbf{k}, \mathcal{N}_d, \rho_d)$ is generated by the set $\mathcal{N}_d$ and has $\binom{N}{2}$ quadratic defining relations given below:*

$$\mathfrak{R} = \{f_{ji} = w_j w_i - w_{i'} w_{j'} \mid 1 \leq i, j \leq n, \text{ where } \rho_d(w_j, w_i) = (w_{i'}, w_{j'}),$$
$$\text{and } w_j > w_{i'} \text{ holds in } \langle X \rangle\}. \tag{4.13}$$

Moreover,

  (i) *There is a 1-to-1 correspondence between the set of relations $\mathfrak{R}$ and the set of nontrivial ρ_d-orbits in $\mathcal{N}_d \times \mathcal{N}_d$.*

  (ii) *For every pair $(a, b) \in (\mathcal{N}_d \times \mathcal{N}_d) \setminus \mathcal{F}(\mathcal{N}_d, \rho_d)$, the monomial ab occurs exactly once in $\mathfrak{R}$.*

  (iii) *Every relation f_{ji} has leading monomial $\mathbf{LM}(f_{ji}) = w_j w_i$.*

Proof. For simplicity of notation we set $\rho_d = \rho$. It is clear that there is a 1-to-1 correspondence between the set of relations of the algebra B and the set of nontrivial orbits of the map ρ.

By definition, each nontrivial relation of the Yang–Baxter algebra B corresponds to a nontrivial orbit of ρ, and vice versa. Say

$$\mathcal{O} = \{(w_j, w_i), \rho(w_j, w_i) = (w_{i'}, w_{j'})\} = \{(w_{i'}, w_{j'}), \rho(w_{i'}, w_{j'}) = (w_j, w_i)\},$$

and without loss of generality we may assume that the relation is

$$w_j w_i - w_{i'} w_{j'}, \quad \text{where } w_j w_i > w_{i'} w_{j'}.$$

By Lemma 4.8 (2), the equality $w_j w_i = w_{i'} w_{j'}$ holds in S. The monoid $S = S(X, r)$ is cancellative (see Remark 3.5), hence an assumption that $w_j = w_{i'}$ would imply $w_i = w_{j'}$, a contradiction. Therefore, $w_j > w_{i'}$.

Conversely, if $\cdot\rho(w_j, w_i) = w_{i'} w_{j'}$ and $w_j > w_{i'}$, then f_{ji} is a (nontrivial) relation of the algebra $B = B(\mathbf{k}, \mathcal{N}_d, \rho_d)$. Clearly, $w_j > w_{i'}$ implies $w_j w_i > w_{i'} w_{j'}$ in $\langle X \rangle$, so $\mathbf{LM}(f_{ji}) = w_j w_i$, and the number of relations g_{ji} is exactly $\binom{N}{2}$. ∎

4.3. The d-Veronese $\mathcal{A}^{(d)}$ presented in terms of generators and relations

According to Convention 3.3 and using the notation as above, the following result describes the d-Veronese $\mathcal{A}^{(d)}$ of the Yang–Baxter algebra $\mathcal{A}$ in terms of one-generators and quadratic relations.

Theorem 4.13. *Let $d \geq 2$ be an integer. Let (X, r) be a finite solution of YBE, where $X = \{x_1, \ldots, x_n\}$, let $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ be its Yang–Baxter algebra, and let $(\mathcal{N}_d, \rho)$ be the normalised d-Veronese solution from Definition 4.10, where $\mathcal{N}_d = \{w_1, \ldots, w_N\}$ is the set of normal monomials of length d ordered lexicographically.*

The d-Veronese subalgebra $\mathcal{A}^{(d)} \subseteq \mathcal{A}$ is a quadratic algebra with $N = \binom{n+d-1}{d}$ one-generators, namely the set $\mathcal{N}_d$ of normal monomials of length d, subject to $N^2 - \binom{n+2d-1}{n-1}$ linearly independent quadratic relations $\mathcal{R}$ described below.

(1) *The relations $\mathcal{R}$ split into two disjoint subsets $\mathcal{R} = \mathcal{R}_a \cup \mathcal{R}_b$, as follows.*

  (a) *The set $\mathcal{R}_a$ contains $\binom{N}{2}$ relations corresponding to the nontrivial ρ-orbits:*

$$\mathcal{R}_a = \{f_{ji} = w_j w_i - w_{i'} w_{j'} \mid 1 \leq i, j \leq n, \text{ where } \rho(w_j, w_i) = (w_{i'}, w_{j'}),$$
$$\text{and } w_j > w_{i'} \text{ holds in } \langle X \rangle\}. \tag{4.14}$$

   Each monomial $w_i w_j$ such that (w_i, w_j) is in a nontrivial ρ-orbit occurs exactly once in $\mathcal{R}_a$. Every relation f_{ji} has leading monomial $\mathbf{LM}(f_{ji}) = w_j w_i$.

  (b) *The set $\mathcal{R}_b$ contains $\binom{N+1}{2} - \binom{n+2d-1}{n-1}$ relations:*

$$\mathcal{R}_b = \{g_{ij} = w_i w_j - w_{i_0} w_{j_0} \mid 1 \leq i, j \leq n, \text{ where } \cdot\rho(w_i, w_j) \geq w_i w_j, w_{i_0}, w_{j_0} \in \mathcal{N}_d,$$
$$\text{and } w_i w_j > \mathrm{Nor}(w_i w_j) = w_{i_0} w_{j_0} \in \mathcal{N}_{2d}$$
$$\text{is the normal form of } w_i w_j\}. \tag{4.15}$$

   In particular, $\mathbf{LM}(g_{ij}) = w_i w_j > w_{i_0} w_{j_0}$.

(2) *The d-Veronese subalgebra $\mathcal{A}^{(d)}$ has a second set of linearly independent quadratic relations, $\mathcal{R}_1$, which splits into two disjoint subsets $\mathcal{R}_1 = \mathcal{R}_{1a} \cup \mathcal{R}_b$ as follows.*

  (a) *The set $\mathcal{R}_{1a}$ is a reduced version of $\mathcal{R}_a$ and contains exactly $\binom{N}{2}$ relations:*

$$\mathcal{R}_{1a} = \{g_{ji} = w_j w_i - w_{i''} w_{j''} \mid 1 \leq i, j \leq n, \text{ where } \cdot\rho(w_j, w_i) < w_j w_i,$$
$$\text{and } w_{i''} w_{j''} = \mathrm{Nor}(w_j w_i) \in \mathcal{N}_{2d}\}. \tag{4.16}$$

In particular, $\mathbf{LM}(g_{ji}) = w_j w_i > w_{i''} w_{j''} \in \mathcal{N}_{2d}$.

(b) *The set $\mathcal{R}_b$ is given in* (4.15).

(3) *The two sets of relations $\mathcal{R}$ and $\mathcal{R}_1$ are equivalent:* $\mathcal{R} \Leftrightarrow \mathcal{R}_1$.

Proof. We start with a general observation. By Convention 3.3, we identify the algebra $\mathcal{A}$ with $(\mathbf{k}\mathcal{N}, \bullet)$. We know that the d-Veronese subalgebra $\mathcal{A}^{(d)}$ is one-generated and quadratic; see Corollary 4.2. By (4.1),

$$\mathcal{A}^{(d)} = \bigoplus_{m \in \mathbb{N}_0} \mathcal{A}_{md} \cong \bigoplus_{m \in \mathbb{N}_0} \mathbf{k}\mathcal{N}_{md}.$$

So $\mathcal{A}_1^{(d)} = \mathbf{k}\mathcal{N}_d$ and the ordered monomials $w \in \mathcal{N}_d$ of length d are degree one generators of $\mathcal{A}^{(d)}$. There are equalities

$$\dim \mathcal{A}_d = |\mathcal{N}_d| = \binom{n+d-1}{d} = N.$$

Moreover,

$$\dim(\mathcal{A}^{(d)})_2 = \dim(\mathcal{A}_{2d}) = \dim(\mathbf{k}\mathcal{N}_{2d}) = |\mathcal{N}_{2d}| = \binom{n+2d-1}{n-1}.$$

We want to find a finite presentation in terms of generators and relations

$$\mathcal{A}^{(d)} = \mathbf{k}\langle w_1, \dots, w_N \rangle / (R),$$

where the two-sided (graded) ideal $I = (R)$ is generated by linearly independent homogeneous relations R of degree 2 in the variables w_i, so $I_2 = \mathrm{Span}_{\mathbf{k}} R$.

(1) We compare dimensions to find the number of quadratic linearly independent relations for the d-Veronese $\mathcal{A}^{(d)}$. The equality of vectors spaces

$$\mathbf{k}\langle w_1, \dots, w_N \rangle = I \oplus \left(\bigoplus_{m \in \mathbb{N}_0} \mathbf{k}\mathcal{N}_{md} \right)$$

implies an equality for the graded components

$$(\mathbf{k}\langle w_1, \dots, w_N \rangle)_2 = I_2 \oplus \mathbf{k}\mathcal{N}_{2d}.$$

Hence if R is a set of linearly independent quadratic relations defining $\mathcal{A}^{(d)}$, that is, R generates the ideal of relations $I = (R)$, there must be an equality $|R| + \dim \mathcal{A}_2^{(d)} = N^2$, so

$$|R| = N^2 - \binom{n+2d-1}{n-1}. \tag{4.17}$$

(2) We shall prove that the set of quadratic polynomials $\mathcal{R} = \mathcal{R}_a \cup \mathcal{R}_b$ given above consists of relations of $\mathcal{A}^{(d)}$; it has order $|\mathcal{R}| = N^2 - \binom{n+2d-1}{n-1}$ and is linearly independent.

(a) Observe that there is a 1-to-1 correspondence between the polynomials $f_{ji} \in \mathcal{R}_a$ and the set on nontrivial ρ-orbits in $\mathcal{N}_d \times \mathcal{N}_d$, and therefore $\mathcal{R}_a$ has exactly $\binom{N}{2}$ elements. The ρ-orbits in $\mathcal{N}_d \times \mathcal{N}_d$ are disjoint, and therefore every monomial $w_i w_j$, with $1 \leq i, j \leq N$, such that (w_i, w_j) is in a nontrivial ρ-orbit occurs exactly once in some polynomial $f \in \mathcal{R}_a$. We claim that $\mathcal{R}_a$ consists of relations of $\mathcal{A}^{(d)}$. Consider an element $f_{ji} = w_j w_i - w_{i'} w_{j'} \in \mathcal{R}_a$. It is obvious that f_{ji} is not identically zero in $\mathbf{k}\langle w_1, \ldots, w_N \rangle$. We have to show that $w_j w_i - w_{i'} w_{j'} = 0$, or equivalently, $w_j w_i = w_{i'} w_{j'}$ holds in $\mathcal{A}^{(d)}$. But $\mathcal{A}^{(d)}$ is a subalgebra of the Yang–Baxter algebra $\mathcal{A}$ which is isomorphic to the monoid algebra $\mathbf{k}S$. Thus it will be enough to prove that

$$w_j w_i = w_{i'} w_{j'} \quad \text{is an equality in } S. \tag{4.18}$$

Note that $\mathcal{N}$ is a subset of $\langle X \rangle$ and $a = b$ in $\mathcal{N}$ is equivalent to $a, b \in \mathcal{N}$ and $a = b$ as words in $\langle X \rangle$. Clearly, each equality of words in $\langle X \rangle$ holds also in S. By assumption,

$$\rho(w_j, w_i) = (w_{i'}, w_{j'}) \quad \text{holds in } \mathcal{N}_d \times \mathcal{N}_d. \tag{4.19}$$

By Definition-Notation 4.9 (see (4.10) and (4.11)), one has

$$\rho(w_j, w_i) = (\mathrm{Nor}(^{w_j} w_i), \mathrm{Nor}(w_j^{w_i})), \quad \text{in } \mathcal{N}_d \times \mathcal{N}_d, \tag{4.20}$$

and comparing (4.19) with (4.20), we obtain that

$$\mathrm{Nor}(^{w_j} w_i) = w_{i'} \quad \text{and} \quad \mathrm{Nor}(w_j^{w_i}) = w_{j'}$$
$$\text{are equalities of words in } \mathcal{N}_d \subset X^d. \tag{4.21}$$

The equality $u = \mathrm{Nor}(u)$ (the normal form of u, modulo the ideal I) holds in S and in $\mathcal{A}$ for every $u \in \langle X \rangle$; therefore, the following are equalities in S:

$$\mathrm{Nor}(^{w_j} w_i) = {}^{w_j} w_i, \quad \mathrm{Nor}(w_j^{w_i}) = w_j^{w_i}$$
$$(\mathrm{Nor}(^{w_j} w_i))(\mathrm{Nor}(w_j^{w_i})) = (^{w_j} w_i)(w_j^{w_i}). \tag{4.22}$$

Now (4.21) and (4.22) imply that

$$w_{i'} w_{j'} = (^{w_j} w_i)(w_j^{w_i}) \quad \text{holds in } S. \tag{4.23}$$

But S is an M3-braided monoid, so by condition (4.4) M3, the following is an equality in S:

$$w_j w_i = (^{w_j} w_i)(w_j^{w_i}). \tag{4.24}$$

This together with (4.23) imply the desired equality $w_j w_i = w_{i'} w_{j'}$ in S. It follows that $f_{ji} = w_j w_i - w_{i'} w_{j'}$ is identically 0 in $\mathcal{A}$ and therefore in $\mathcal{A}^{(d)}$. Clearly, for $f_{ji} = w_j w_i - w_{i'} w_{j'}$, the inequality $w_j > w_{i'}$ implies that $w_j w_i > w_{i'} w_{j'}$ as elements of $\langle X \rangle$, so the leading monomial of every relation $f_{ji} \in \mathcal{R}_a$ is $\mathbf{LM}(f_{ji}) = w_j w_i$.

(b) Next we consider the elements $g_{ij} = w_i w_j - w_{i_0} w_{j_0} \in \mathcal{R}_b$. These are homogeneous polynomials of degree $2d$. It follows from their description that $w_i w_j > \text{Nor}(w_i w_j) = w_{i_0} w_{j_0}$, so their leading monomials satisfy $\mathbf{LM}(g_{ij}) = w_i w_j$.

Moreover, the description of $\mathcal{R}_b$ implies that there is a 1-to-1 correspondence between the polynomials in $\mathcal{R}_b$ and the set of all ρ-orbits $\mathcal{O}$ *which do not contain elements* $(a, b) \in \mathcal{N}_d \times \mathcal{N}_d$ *such that* ab *is in normal form, that is,* $ab \in \mathcal{N}_{2d}$.

If $\mathcal{O}$ is such an orbit, then $\mathcal{O} = \{(w_i, w_j), \rho(w_i, w_j)\}$, where $\cdot\rho(w_i, w_j) \geq w_i w_j$ and $w_i w_j$ is not in normal form. (In particular, $\mathcal{O}$ can be a one-element orbit.) Clearly, $w_i w_j = \text{Nor}(w_i w_j)$ is an identity in $\mathcal{A}$ (and in $(\mathbf{k}\mathcal{N}, \bullet)$). The normal form $\text{Nor}(w_i w_j)$ is a monomial of length $2d$, so it can be written as a product $\text{Nor}(w_i w_j) = w_{i_0} w_{j_0}$, where $w_{i_0}, w_{j_0} \in \mathcal{N}_d$. It follows that

$$g_{ij} = w_i w_j - w_{i_0} w_{j_0} = 0$$

is an identity in $\mathcal{A}$ (and in $(\mathbf{k}\mathcal{N}, \bullet)$). Conversely, it follows from the description of $\mathcal{R}_b$ that each relation $g_{ij} = w_i w_j - w_{i_0} w_{j_0} \in \mathcal{R}_b$ determines uniquely the ρ-orbit $\mathcal{O} = \{(w_i, w_j), \rho(w_i, w_j)\}$ with the above properties. Note that each $(a, b) \in \mathcal{N}_d \times \mathcal{N}_d$ such that $ab \in \mathcal{N}_{2d}$ belongs to exactly one orbit, so the number of such orbits equals the cardinality

$$|\mathcal{N}_{2d}| = \binom{n + 2d - 1}{n - 1}.$$

The number of all ρ-orbits in $\mathcal{N}_d \times \mathcal{N}_d$ (including the one-element orbits) is $\binom{N+1}{2}$. Therefore, for the (distinct) elements $g_{ij} = w_i w_j - w_{i_0} w_{j_0} \in \mathcal{R}_b$, the number of disjoint ρ-orbits which "produce" distinct leading monomials $w_i w_j$ is exactly

$$\binom{N + 1}{2} - \binom{n + 2d - 1}{n - 1} = |\mathcal{R}_b|. \tag{4.25}$$

The sets $\mathcal{R}_a$ and $\mathcal{R}_b$ are disjoint, since $\{\mathbf{LM}(f) \mid f \in \mathcal{R}_a\} \cap \{\mathbf{LM}(g) \mid g \in \mathcal{R}_b\} = \emptyset$. Therefore, there are equalities:

$$|\mathcal{R}| = |\mathcal{R}_a| + |\mathcal{R}_b| = \binom{N}{2} + \left(\binom{N + 1}{2} - \binom{n + 2d - 1}{n - 1}\right)$$

$$= N^2 - \binom{n + 2d - 1}{n - 1}, \tag{4.26}$$

so the set $\mathcal{R}$ has exactly the desired number of relations given in (4.17). It remains to show that $\mathcal{R}$ consists of linearly independent elements of $\mathbf{k}\langle X \rangle$.

Lemma 4.14. *Under the hypothesis of Theorem 4.13, the set of polynomials $\mathcal{R} \subset k\langle X \rangle$ is linearly independent.*

Proof. It is well known that the set of all words in $\langle X \rangle$ forms a basis of $\mathbf{k}\langle X \rangle$ (considered as a vector space); in particular, every finite set of distinct words in $\langle X \rangle$ is linearly independent. All words occurring in $\mathcal{R}$ are elements of X^{2d}, but some of them occur in more than one relation, for example, every $w_i w_j$, which is not a fixed point of ρ but is the leading monomial of $g_{ij} \in \mathcal{R}_b$, occurs also as a second term of a polynomial $f_{pq} = w_p w_q - w_i w_j \in \mathcal{R}_a$, where $\cdot \rho(w_i, w_j) = w_p w_q > w_i w_j$. We shall prove the lemma in three steps.

(1) The set of polynomials $\mathcal{R}_a \subset \mathbf{k}\langle X \rangle$ is linearly independent.

Notice that the polynomials in $\mathcal{R}_a$ are in 1-to-1 correspondence with the nontrivial ρ-orbits in $\mathcal{N}_d \times \mathcal{N}_d$: each polynomial $f_{ji} = w_j w_i - w_{i'} w_{j'} \in \mathcal{R}_a$ is formed out of the two monomials in the nontrivial ρ-orbit $\{(w_j, w_i), (w_{i'}, w_{j'}) = \rho(w_j, w_i)\}$. But the ρ-orbits are disjoint, hence each monomial $\cdot(a, b)$ with $(a, b) \neq \rho(a, b)$ occurs exactly once in $\mathcal{R}_a$. Present each $f \in \mathcal{R}_a$ as $f = u_f - v_f$, where $u_f = \mathbf{LM}(f)$. Then a linear relation

$$0 = \sum_{f \in \mathcal{R}_a} \alpha_f f = \sum_{f \in \mathcal{R}_a} \alpha_f (u_f - v_f), \quad \text{where all } \alpha_f \in \mathbf{k}$$

involves only pairwise distinct monomials in X^{2d}, and therefore it must be trivial: $\alpha_f = 0 \; \forall f \in \mathcal{R}_a$. It follows that $\mathcal{R}_a$ is linearly independent.

(2) The set $\mathcal{R}_b \subset \mathbf{k}\langle X \rangle$ is linearly independent.

Assume the contrary. Present each element of $\mathcal{R}_b$ as $g = u_g - v_g \in \mathcal{R}_b$, where $u_g - \mathbf{LM}(g)$. Then there exists a nontrivial linear relation for the elements of $\mathcal{R}_b$:

$$\sum_{g \in \mathcal{R}_b} \beta_g g = \sum_{g \in \mathcal{R}_b} \beta_g (u_g - v_g) = 0 \quad \text{with } \beta_g \in \mathbf{k}. \tag{4.27}$$

Let g_{ij} be the polynomial with $\beta_{ij} = \beta_{g_{ij}} \neq 0$ whose leading monomial is the highest among all leading monomials of polynomials $g \in \mathcal{R}_b$, with $\beta_g \neq 0$. So we have

$$\mathbf{LM}(g_{ij}) = w_i w_j > \mathbf{LM}(g) \quad \text{for all } g \in \mathcal{R}_b, \; g \neq g_{ij}, \text{ where } \beta_g \neq 0.$$

We use (4.27) to yield the following equality in $\mathbf{k}\langle X \rangle$:

$$w_i w_j = w_{i_0} w_{j_0} - \sum_{g \in \mathcal{R}_b, \mathbf{LM}(g) < w_i w_j} \frac{\beta_g}{\beta_{ij}} (u_g - v_g).$$

Observe that the right-hand side of this equality is a linear combination of monomials strictly smaller than $w_i w_j$, which is impossible. It follows that the set $\mathcal{R}_b \subset \mathbf{k}\langle X \rangle$ is linearly independent.

(3) The set $\mathcal{R} \subset \mathbf{k}\langle X \rangle$ is linearly independent. For simplicity of notation (as before), we present every $f \in \mathcal{R}_a$ and every $g \in \mathcal{R}_b$ as

$$f = u_f - v_f, \quad \text{where } u_f = \mathbf{LM}(f) > v_f, \quad g = u_g - v_g, \quad \text{where } u_g = \mathbf{LM}(g) > v_g.$$

Assume the polynomials in $\mathcal{R}$ satisfy a linear relation

$$\sum_{f \in \mathcal{R}_a} \alpha_f f + \sum_{g \in \mathcal{R}_b} \beta_g g = 0, \quad \text{where for all } f \in \mathcal{R}_a, \, g \in \mathcal{R}_b, \, \alpha_f, \beta_g \in \mathbf{k}. \quad (4.28)$$

This gives the following equality in the free associative algebra $\mathbf{k}\langle X \rangle$:

$$S_1 = \sum_{f \in \mathcal{R}_a} \alpha_f u_f = \sum_{f \in \mathcal{R}_a} \alpha_f v_f - \sum_{g \in \mathcal{R}_b} \beta_g g = S_2. \quad (4.29)$$

The element $S_1 = \sum_{f \in \mathcal{R}_a} \alpha_f u_f$ is in the space $U = \text{Span } B_1$, where $B_1 = \{\mathbf{LM}(f) \mid f \in \mathcal{R}_a\}$ is linearly independent.

The element

$$S_2 = \sum_{f \in \mathcal{R}_a} \alpha_f v_f - \sum_{g \in \mathcal{R}_b} \beta_g g$$

on the right-hand side of the equality is in the space $V = \text{Span } B$, where

$$B = \{v_f \mid f \in \mathcal{R}_a\} \cup \{u_g, v_g \mid g \in \mathcal{R}_b\}.$$

Take a subset $B_2 \subset B$ which forms a basis of V. Note that $B_1 \cap B = \emptyset$, hence $B_1 \cap B_2 = \emptyset$. Moreover, each of the sets B_1 and B_2 consists of pairwise distinct and therefore linearly independent monomials, and it is easy to show that $U \cap V = 0$. Thus the equality (4.29) with $S_1 = S_2 \in U \cap V = 0$ implies a linear relation

$$S_1 = \sum_{f \in \mathcal{R}_a} \alpha_f u_f = 0,$$

for the set of leading monomials $u_f = \mathbf{LM}(f)$, $f \in \mathcal{R}_a$ which are pairwise distinct and therefore linearly independent. It follows that $\alpha_f = 0$ for all $f \in \mathcal{R}_a$. This together with (4.28) implies the linear relation

$$\sum_{g \in \mathcal{R}_b} \beta_g g = 0,$$

and since by (2) $\mathcal{R}_b$ is linearly independent, we get again $\beta_g = 0 \; \forall \, g \in \mathcal{R}_b$. It follows that the linear relation (4.28) must be trivial, and therefore $\mathcal{R}$ is a linearly independent set of polynomials. ∎

It is now easy to see that $\mathcal{R}$ is a set of defining relation for the d-Veronese subalgebra $\mathcal{A}^{(d)}$.

We know that $\mathcal{A}^{(d)}$ is a quadratic algebra whose one-generators are the monomials $w_1, \ldots, w_N$, that is, its ideal of relations I is generated by homogeneous polynomials of degree 2 in the w_i's. Consider the ideal $J = (\mathcal{R})$ of the free associative algebra $\mathbf{k}\langle w_1, \ldots, w_N \rangle$. We have proven that each element of $\mathcal{R}$ is a relation of $\mathcal{A}^{(d)}$; therefore,

$J \subseteq I$. To show that J is the ideal of relations of $\mathcal{A}^{(d)}$, it will be enough to verify that there is an isomorphism of vector spaces:

$$(\mathcal{R})_2 \oplus (\mathcal{A}^{(d)})_2 = (\mathbf{k}\langle w_1, \ldots, w_N\rangle)_2,$$

or equivalently,

$$\dim \mathrm{Span}_{\mathbf{k}} \mathcal{R} + \dim(\mathcal{A}^{(d)})_2 = \dim(\mathbf{k}\langle w_1, \ldots, w_N\rangle)_2.$$

We have shown that $\mathcal{R}$ is linearly independent, so $\dim \mathrm{Span}_{\mathbf{k}} \mathcal{R} = |\mathcal{R}| = N^2 - \binom{n+2d-1}{n-1}$; see (4.26). On the other hand, $\dim(\mathcal{A}^{(d)})_2 = \dim \mathcal{A}_{2d} = \binom{n+2d-1}{n-1}$; see Corollary 3.6. Therefore,

$$\dim \mathrm{Span}_{\mathbf{k}} \mathcal{R} + \dim(\mathcal{A}^{(d)})_2 = N^2 - \binom{n+2d-1}{n-1} + \binom{n+2d-1}{n-1}$$
$$= N^2 = \dim(\mathbf{k}\langle w_1, \ldots, w_N\rangle)_2,$$

as desired. It follows that $\mathcal{R}$ is a set of defining relations for the Veronese subalgebra $\mathcal{A}^{(d)}$. We have proven part (1) of the theorem.

Analogous argument proves part (2). Note that the polynomials of $\mathcal{R}_{1a}$ are reduced from $\mathcal{R}_a$ using $\mathcal{R}_b$. It is not difficult to prove the equivalence $\mathcal{R} \Leftrightarrow \mathcal{R}_1$.   ∎

5. Veronese maps

In this section, we shall introduce an analogue of Veronese maps between quantum spaces (Yang–Baxter algebras) associated to finite solutions of YBE. We keep the notation and all conventions from the previous sections. As usual, (X, r) is a finite solution of order n, $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ is the associated Yang–Baxter algebra, where we fix an enumeration, $X = \{x_1, \ldots, x_n\}$ as in Convention 3.3, $d \geq 2$ is an integer, $N = \binom{n+d-1}{d}$, and $\mathcal{N}_d = \{w_1 < w_2 < \cdots < w_N\}$ is the set of all normal monomials of length d in X^d ordered lexicographically, as in (4.3).

5.1. The d-Veronese solution of YBE associated to a finite solution (X, r)

We have shown that the braided monoid (S, r_S) associated to (X, r) induces the normalised d-Veronese solution $(\mathcal{N}_d, \rho_d)$ of order $N = \binom{n+d-1}{d}$; see Definition 4.10. We shall use this construction to introduce the notion of *a d-Veronese solution of YBE associated to (X, r)*, denoted by (Y, r_Y).

Definition-Notation 5.1. In notation as above. Let (X, r) be a finite solution, $X = \{x_1, \ldots, x_n\}$, let $\mathcal{N}_d = \{w_1 < w_2 < \cdots < w_N\}$ be the set of normal monomials of length d, and let $(\mathcal{N}_d, \rho) = (\mathcal{N}_d, \rho_d)$ be the normalised d-Veronese solution. Let $Y =$

$\{y_1, y_2, \ldots, y_N\}$ be an abstract set and consider the quadratic set (Y, r_Y), where the map $r_Y : Y \times Y \to Y \times Y$ is defined as

$$r_Y(y_j, y_i) := (y_{i'}, y_{j'}) \quad \text{iff} \quad \rho(w_j, w_i) = (w_{i'}, w_{j'}), \quad 1 \le i, j, i', j' \le n. \quad (5.1)$$

It is straightforward that (Y, r_Y) is a solution of YBE (a nondegenerate symmetric set) of order N isomorphic to $(\mathcal{N}_d, \rho_d)$. We shall refer to it as *the d-Veronese solution of YBE associated to (X, r)*.

By Corollary 2.10, the set $Y \times Y$ splits into $\binom{N}{2}$ two-element r_Y-orbits and N one-element r_Y-orbits.

As usual, we consider the degree-lexicographic ordering on the free monoid $\langle Y_N \rangle$ extending $y_1 < y_2 < \cdots < y_N$. The Yang–Baxter algebra $\mathcal{A}_Y = \mathcal{A}(\mathbf{k}, Y, r_Y) \simeq \mathbf{k}\langle Y; \mathcal{R}_Y \rangle$ has exactly $\binom{N}{2}$ quadratic relations which can be written explicitly as

$$\mathcal{R}_Y = \{\gamma_{ji} = y_j y_i - y_{i'} y_{j'} \mid 1 \le i, j \le N, r_Y(y_j, y_i) = (y_{i'}, y_{j'}),$$
$$\text{where } y_j y_i > y_{i'} y_{j'} \text{ holds in } \langle Y_N \rangle\}. \quad (5.2)$$

Each relation corresponds to a nontrivial r_Y-orbit. The leading monomials satisfy **LM**$(\gamma_{ji}) = y_j y_i > y_{i'} y_{j'}$.

5.2. The Veronese map $v_{n,d}$ and its kernel

We recall that in algebraic geometry a d-Veronese morphism means an algebra homomorphism $B \to A$ with image $A^{(d)}$, where B is a graded algebra of the same type as A.

Lemma 5.2. *In notation as above. Let (X, r) be a solution of order n, $\mathcal{A}_X = \mathcal{A}(\mathbf{k}, X, r)$, let $d \ge 2$ be an integer, and let $N = \binom{n+d-1}{d}$. Suppose (Y, r_Y) is the associated d-Veronese solution, $Y = \{y_1, \ldots, y_N\}$, and $\mathcal{A}_Y = \mathcal{A}(\mathbf{k}, Y, r_Y)$ is the corresponding Yang–Baxter algebra.*

The assignment

$$y_1 \mapsto w_1, y_2 \mapsto w_2, \ldots, y_N \mapsto w_N$$

extends to an algebra homomorphism $v_{n,d} : \mathcal{A}_Y \to \mathcal{A}_X$. The image of the map $v_{n,d}$ is the d-Veronese subalgebra $\mathcal{A}_X^{(d)}$.

Proof. Naturally we set $v_{n,d}(y_{i_1} \cdots y_{i_p}) := w_{i_1} \cdots w_{i_p}$ for all words $y_{i_1} \cdots y_{i_p} \in \langle Y \rangle$ and then extend this map linearly. Note that for each polynomial $\gamma_{ji} \in \mathcal{R}_Y$, one has

$$v_{n,d}(\gamma_{ji}) = f_{ji} \in \mathcal{R}_a,$$

where the set $\mathcal{R}_a$ is a part of the relations of $\mathcal{A}_X^{(d)}$ given in (4.14). Indeed, let $\gamma_{ji} \in \mathcal{R}_Y$, so $\gamma_{ji} = y_j y_i - y_{i'} y_{j'}$, where $(y_{i'}, y_{j'}) = r_Y(y_j, y_i)$, and $y_j y_i > y_{i'} y_{j'}$; see (5.1). Then

$$v_{n,d}(\gamma_{ji}) = v_{n,d}(y_j y_i - y_{i'} y_{j'})$$
$$= w_j w_i - w_{i'} w_{j'}, \quad \text{where } (w_{i'}, w_{j'}) = \rho(w_j, w_i), \text{ and } w_j w_i > w_{i'} w_{j'}$$
$$= f_{ji} \in \mathcal{R}_a.$$

We have shown that f_{ji} equals identically 0 in $\mathcal{A}_X$, so the map $v_{n,d}$ agrees with the relations of the algebra $\mathcal{A}_X$. It follows that $v_{n,d} : \mathcal{A}_Y \to \mathcal{A}_X$ is a well-defined homomorphism of algebras.

The image of $v_{n,d}$ is the subalgebra of $\mathcal{A}_X$ generated by the normal monomials $\mathcal{N}_d$, which by Theorem 4.13 is exactly the d-Veronese subalgebra $\mathcal{A}_X^{(d)}$. ∎

Definition 5.3. We call the map $v_{n,d}$ from Lemma 5.2 *the (n, d)-Veronese map.*

Theorem 5.4. *In assumption and notation as above. Let (X, r) be a solution of order n, with $X = \{x_1, \ldots, x_n\}$, let $\mathcal{A}_X = \mathcal{A}(k, X, r)$ be its Yang–Baxter algebra. Let $d \geq 2$ be an integer, $N = \binom{n+d-1}{d}$, and suppose that (Y, r_Y) is the associated d-Veronese solution of YBE with corresponding Yang–Baxter algebra $\mathcal{A}_Y = \mathcal{A}(k, Y, r_Y)$. Let $v_{n,d} : \mathcal{A}_Y \to \mathcal{A}_X$ be the (n, d)-Veronese map (homomorphism of algebras) extending the assignment*

$$y_1 \mapsto w_1, y_2 \mapsto w_2, \ldots, y_N \mapsto w_N.$$

Then the following conditions hold.

(1) *The image of $v_{n,d}$ is the d-Veronese subalgebra $\mathcal{A}_X^{(d)}$ of $\mathcal{A}_X$.*

(2) *The kernel $\mathfrak{K} := \ker(v_{n,d})$ of the Veronese map is generated by the set of $\binom{N+1}{2} - \binom{n+2d-1}{n-1}$ linearly independent quadratic binomials:*

$$\mathcal{R}(\mathfrak{K}) = \{\gamma_{ij} = y_i y_j - y_{i_0} y_{j_0} \mid 1 \leq i, j \leq n, \text{ where } g_{ij} = w_i w_j - w_{i_0} w_{j_0} \in \mathcal{R}_b\}. \quad (5.3)$$

In particular, the leading monomial of each γ_{ij} satisfies

$$\mathbf{LM}(\gamma_{ij}) = y_i y_j > y_{i_0} y_{j_0}.$$

Proof. Part (1) follows from Lemma 5.2.

Part (2). We have to verify that the set $\mathcal{R}(\mathfrak{K})$ generates $\mathfrak{K}$.

By direct computation, one shows that for every $\gamma_{ij} \in \mathcal{R}(\mathfrak{K})$ one has

$$v_{n,d}(\gamma_{ij})(y_1, \ldots, y_N) = g_{ij}(w_1, \ldots, w_n) \in \mathcal{R}_b;$$

in fact, $v_{n,d}$ induces a 1-to-1 map $\mathcal{R}(\mathfrak{K}) \to \mathcal{R}_b$. It follows that

$$|\mathcal{R}(\mathfrak{K})| = |\mathcal{R}_b| = \binom{N+1}{2} - \binom{n+2d-1}{n-1}. \quad (5.4)$$

Moreover, $v_{n,d}(\mathcal{R}(\mathfrak{K})) = \mathcal{R}_b$, the set of relations of the d-Veronese $\mathcal{A}_X^{(d)}$ given in (4.15), so $\mathcal{R}(\mathfrak{K}) \subset \mathfrak{K}$.

The Yang–Baxter algebra $\mathcal{A}_Y$ is a quadratic algebra with N generators and $\binom{N}{2}$ defining quadratic relations which are linearly independent, so

$$\dim(\mathcal{A}_Y)_2 = N^2 - \binom{N}{2} = \binom{N+1}{2}.$$

By the first isomorphism theorem $(\mathcal{A}_Y/\mathcal{K})_2 \cong (\mathcal{A}_X^{(d)})_2 = (\mathcal{A}_X)_{2d}$, hence

$$\dim(\mathcal{A}_Y)_2 = \dim(\mathcal{K})_2 + \dim(\mathcal{A}_X)_{2d}.$$

We know that $\dim(\mathcal{A}_X)_{2d} = |\mathcal{N}_{2d}| = \binom{n+2d-1}{n-1}$, hence

$$\binom{N+1}{2} = \dim(\mathcal{K})_2 + \binom{n+2d-1}{n-1}.$$

This together with (5.4) implies that

$$\dim(\mathcal{K})_2 = \binom{N+1}{2} - \binom{n+2d-1}{n-1} = |\mathcal{R}(\mathcal{K})|.$$

The set $\mathcal{R}(\mathcal{K})$ is linearly independent, since $v_{n,d}(\mathcal{R}(\mathcal{K})) = \mathcal{R}_b$, and by Lemma 4.14 the set $\mathcal{R}_b$ is linearly independent. This together with the equality $|\mathcal{R}(\mathcal{K})| = \dim(\mathcal{K})_2$ implies that the set $\mathcal{R}(\mathcal{K})$ is a basis of the graded component $\mathcal{K}_2$, so $\mathcal{K}_2 = \mathbf{k}\mathcal{R}(\mathcal{K})$. But the ideal $\mathcal{K}$ is generated by homogeneous polynomials of degree 2, and therefore

$$\mathcal{K} = (\mathcal{K}_2) = (\mathcal{R}(\mathcal{K})). \tag{5.5}$$

We have proven that $\mathcal{R}(\mathcal{K})$ is a minimal set of generators for the kernel $\mathcal{K}$. ∎

Corollary 5.5. *Let (X, r) be a solution of order n, let $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ be its Yang–Baxter algebra, and let $d \geq 2$ be an integer. Then the d-Veronese subalgebra $\mathcal{A}^{(d)}$ is a left and a right Noetherian domain.*

Proof. The d-Veronese $\mathcal{A}^{(d)}$ is a subalgebra of $\mathcal{A}$ which is a domain (see Remark 3.5), and therefore $\mathcal{A}^{(d)}$ is a domain. By Theorem 5.4, $A^{(d)}$ is a homomorphic image of the Yang–Baxter algebra $\mathcal{A}_Y = \mathcal{A}(\mathbf{k}, Y, r_Y)$, where (Y, r_Y) is the d-Veronese solution associated with (X, r). The algebra $\mathcal{A}_Y$ is Noetherian, since (Y, r_Y) is a finite solution of YBE (see Remark 3.5), so $\mathcal{A}^{(d)}$ is a left and a right Noetherian domain. ∎

Remark 5.6. In [21], we study the Yang–Baxter algebras $\mathcal{A}(\mathbf{k}, X, r)$ of *finite left nondegenerate idempotent solutions* (X, r). In contrast with the results of the present paper, [21, Theorem 3.5], proves that every Yang–Baxter algebra $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ of a finite left nondegenerate idempotent solution is PBW. Moreover, in this case for each integer d, the d-Veronese subalgebra $\mathcal{A}^{(d)}$ is also PBW and can be identified with the YB algebra of the d-Veronese solution (S_d, r_d); see [21, Theorems 6.2 and 6.8]. In this case, for every integer $d \geq 2$, the Veronese map $v_{n,d}$ is injective, while in the involutive case it has a large kernel.

6. Special cases

6.1. Veronese subalgebras of the Yang–Baxter algebra of a square-free solution

In this subsection, (X, r) is a finite square-free solution of YBE of order n, and $d \geq 2$ is an integer. We keep the conventions and notation from the previous sections. We apply

Remark 3.1 and fix an appropriate enumeration $X = \{x_1, \ldots, x_n\}$ such that the Yang–Baxter algebra $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ is a binomial skew polynomial ring. More precisely, $\mathcal{A}$ is a PBW algebra $\mathcal{A} = \mathbf{k}\langle x_1, \ldots, x_n \rangle / (\mathfrak{R}_{\mathcal{A}})$, where

$$\mathfrak{R}_{\mathcal{A}} = \{\varphi_{ji} = x_j x_i - x_{i'} x_{j'} \mid 1 \leq i < j \leq n\} \tag{6.1}$$

is such that for every pair i, j, $1 \leq i < j \leq n$, the relation $\varphi_{ji} = x_j x_i - x_{i'} x_{j'} \in \mathfrak{R}_{\mathcal{A}}$ satisfies $j > i'$, $i' < j'$ and every term $x_i x_j$, $1 \leq i < j \leq n$ occurs in some relation in $\mathfrak{R}_{\mathcal{A}}$. In particular,

$$\mathbf{LM}(\varphi_{ji}) = x_j x_i, \quad 1 \leq i < j \leq n. \tag{6.2}$$

The set $\mathfrak{R}_{\mathcal{A}}$ is a quadratic Gröbner basis of the ideal $I = (\mathfrak{R}_{\mathcal{A}})$ with respect to the degree-lexicographic ordering $<$ on $\langle X \rangle$. It follows from the shape of the elements of the Gröbner basis $\mathfrak{R}_{\mathcal{A}}$ and (6.2) that the set $\mathcal{N} = \mathcal{N}(I)$ of normal monomials modulo $I = (\mathfrak{R}_{\mathcal{A}})$ coincides with the set $\mathcal{T}$ of ordered monomials (terms) in X:

$$\mathcal{N} = \mathcal{T} = \mathcal{T}(X) = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \langle X \rangle \mid \alpha_i \in \mathbb{N}_0, \ i \in \{0, \ldots, n\}\}. \tag{6.3}$$

All definitions, notation, and results from Sections 4 and 5 are valid, but they can be rephrased in more explicit terms replacing the abstract sets $\mathcal{N} = \mathcal{N}(I)$, $\mathcal{N}_d$, and $\mathcal{N}_{2d}$, respectively, with the explicit set of ordered monomials $\mathcal{T} = \mathcal{T}(X)$, $\mathcal{T}_d$, and $\mathcal{T}_{2d}$. In this case, we consider the space $\mathbf{k}\mathcal{T}$ endowed with multiplication defined by $f \bullet g := \mathrm{Nor}_{\mathfrak{R}_{\mathcal{A}}}(fg)$ for every $f, g \in \mathbf{k}\mathcal{T}$. Then there is an isomorphism of graded algebras

$$\mathcal{A} = \mathcal{A}(\mathbf{k}, X_n, r) \cong (\mathbf{k}\mathcal{T}, \bullet), \tag{6.4}$$

and we identify the PBW algebra $\mathcal{A}$ with $(\mathbf{k}\mathcal{T}, \bullet)$. Similarly, the Yang–Baxter monoid $S(X, r)$ is identified with $(\mathcal{T}, \bullet)$.

We order the elements of $\mathcal{T}_d$ lexicographically, so

$$\mathcal{T}_d = \{w_1 = (x_1)^d < w_2 = (x_1)^{d-1} x_2 < \cdots < w_N = (x_n)^d\},$$

$$\text{where } N = \binom{n + d - 1}{d}. \tag{6.5}$$

The normalised d-Veronese solution (see Definition 4.10) is denoted by $(\mathcal{T}_d, \rho) = (\mathcal{T}_d, \rho_d)$. The d-Veronese subalgebra $\mathcal{A}^{(d)}$ is a quadratic algebra (one)-generated by $w_1, w_2, \ldots, w_N$.

Now [30, Proposition 4.3, Chapter 4] imply the following corollary.

Corollary 6.1. *In notation as above, the d-Veronese $\mathcal{A}^{(d)}$ is a quadratic PBW algebra with PBW generators the terms of length d, $w_1, w_2, \ldots, w_N$, ordered lexicographically; see (6.5).*

For the class of finite square-free solutions (X, r), Theorem 4.13 and especially the description of the set $\mathcal{R}_b$ become more precise.

Remark 6.2. If $w_i = x_{i_1} \cdots x_{i_d}, w_j = x_{j_1} \cdots x_{j_d} \in \mathcal{T}_d$, the product $w_i w_j$ is the leading monomial of an element $g_{ij} \in \mathcal{R}_b$ if and only if $i_d > j_1$ and $\cdot \rho(w_i, w_j) \geq w_i w_j$.

Corollary 6.3. *Let (X, r) be a finite square-free solution of order n, let $X = \{x_1, \ldots, x_n\}$ be enumerated so that the algebra $\mathcal{A} = \mathcal{A}(\mathbf{k}, X_n, r)$ is a binomial skew polynomial ring, let $d \geq 2$ be an integer, and $N = \binom{n+d-1}{d}$. Let $(\mathcal{T}_d, \rho)$ be the normalised d-Veronese solution.*

The d-Veronese subalgebra $\mathcal{A}^{(d)} \subseteq \mathcal{A}$ is a quadratic PBW algebra

$$A^{(d)} \cong \mathbf{k}\langle w_1, \ldots, w_N \rangle/(\mathcal{R}),$$

with PBW generators $\mathcal{T}_d = \{w_1, \ldots, w_N\}$, and $N^2 - \binom{n+2d-1}{n-1}$ linearly independent quadratic relations $\mathcal{R}$. The relations $\mathcal{R}$ split into two disjoint subsets $\mathcal{R} = \mathcal{R}_a \cup \mathcal{R}_b$, described below.

(1) *The set $\mathcal{R}_a$ contains $\binom{N}{2}$ relations corresponding to the nontrivial ρ-orbits in $\mathcal{T}_d \times \mathcal{T}_d$:*

$$\mathcal{R}_a = \{f_{ji} = w_j w_i - w_{i'} w_{j'} \mid 1 \leq i, j \leq n, \text{ where } \rho(w_j, w_i) = (w_{i'}, w_{j'}),$$
$$\text{and } w_j > w_{i'} \text{ holds in } \langle X \rangle\}. \tag{6.6}$$

Each monomial $w_i w_j$ such that (w_i, w_j) is in a nontrivial ρ-orbit occurs exactly once in $\mathcal{R}_a$. Every relation f_{ji} has leading monomial $\mathbf{LM}(f_{ji}) = w_j w_i$.

(2) *The set $\mathcal{R}_b$ contains $\binom{N+1}{2} - \binom{n+2d-1}{n-1}$ relations*

$$\mathcal{R}_b = \{g_{ij} = w_i w_j - w_{i_0} w_{j_0} \mid 1 \leq i, j \leq n, \text{ where } w_i = x_{i_1} \cdots x_{i_d},$$
$$w_j = x_{j_1} \cdots x_{j_d} \in \mathcal{T}_d, i_d > j_1$$
$$\text{and } \cdot \rho(w_i, w_j) \geq w_i w_j, w_{i_0}, w_{j_0} \in \mathcal{T}_d$$
$$\text{are such that } \mathrm{Nor}(w_i w_j) = w_{i_0} w_{j_0} \in \mathcal{T}_{2d}\}. \tag{6.7}$$

In particular, $\mathbf{LM}(g_{ij}) = w_i w_j > w_{i_0} w_{j_0}$.

The relations $\mathcal{R}$ form a Gröbner basis of the ideal $(\mathcal{R})$ of the free associative algebra $\mathbf{k}\langle w_1, \ldots, w_n \rangle$.

For a square-free solution (X, r) and $(\mathcal{T}_d, \rho_d)$ as above, the d-Veronese solution (Y, r_Y), associated to (X, r), is defined in Definition-Notation 5.1. One has $Y = \{y_1, y_2, \ldots, y_N\}$, and the map $r_Y : Y \times Y \to Y \times Y$ is determined by

$$r_Y(y_j, y_i) := (y_{i'}, y_{j'}) \quad \text{iff} \quad \rho(w_j, w_i) = (w_{i'}, w_{j'}), \quad 1 \leq i, j, i', j' \leq n. \tag{6.8}$$

By definition, (Y, r_Y) is isomorphic to the solution $(\mathcal{T}_d, \rho_d)$. Its Yang–Baxter algebra $\mathcal{A}_Y = \mathcal{A}(\mathbf{k}, Y, r_Y)$ is needed to define the (n, d) Veronese homomorphism $v_{n,d} : \mathcal{A}_Y \to \mathcal{A}_X$ extending the assignment

$$y_1 \mapsto w_1, y_2 \mapsto w_2, \ldots, y_N \mapsto w_N.$$

Theorem 5.4 shows that the image of $v_{n,d}$ is the d-Veronese subalgebra $\mathcal{A}^{(d)}$ and determines a minimal set of generators of its kernel.

The finite square-free solutions (X, r) form an important subclass of the class of all finite solutions; see, for example, [15]. Moreover, Theorem 3.8 shows that the Yang–Baxter algebra $\mathcal{A}(\mathbf{k}, X, r)$ of a finite solution (X, r) is a PBW algebra if and only if (X, r) is square-free. So it is natural to ask *"can we define an analogue of Veronese morphisms between Yang–Baxter algebras of square-free solutions?"* We shall prove that it is not possible to restrict the definition of Veronese maps introduced for Yang–Baxter algebras of finite solutions to the subclass of Yang–Baxter algebras of finite square-free solutions. Indeed, if we assume that (X, r) is square-free, then the algebra A_Y involved in the definition of the map $v_{n,d}$ is associated with the d-Veronese solution (Y, r_Y), which, in general, is not square-free; see Corollary 6.5.

To prove the following result, we work with the monomial d-Veronese solution (S_d, r_d) keeping in mind that it has special "hidden" properties induced by the braided monoid (S, r_S).

Theorem 6.4. *Let $d \geq 2$ be an integer. Suppose (X, r) is a finite square-free solution of order $n \geq 2$, (S, r_S) is the associated braided monoid, and (S_d, r_d) is the monomial d-Veronese solution induced by (S, r_S); see Definition 4.5. Then (S_d, r_d) is a square-free solution if and only if (X, r) is a trivial solution.*

Proof. Assume that (S_d, r_d) is a square-free solution. We shall prove that (X, r) is a trivial solution.

Observe that if (Z, r_Z) is a solution, then (i) (Z, r_Z) is square-free if and only if

$$^z z = z \quad \text{for all } z \in Z$$

and (ii) (Z, r_Z) is the trivial solution if and only if

$$^y x = x \quad \text{for all } x, y \in Z.$$

Let $x, y \in X, x \neq y$ and consider the monomial $a = x^{d-1} y \in S_d$. Our assumption that (S_d, r_d) is square-free implies that $^a a = a$ holds in S_d, and therefore in S. Now Remark 2.11 implies the words a and $^a a$ (considered as elements of X^d) belong to the orbit $\mathcal{O} = \mathcal{O}_{\mathcal{D}_m}(a)$ of $a = x^{d-1} y$ in X^d. We analyse the orbit $\mathcal{O} = \mathcal{O}(x^{d-1} y)$ to find that it contains two types of elements:

$$u = (^{x^{d-1}} y) b, \quad \text{where } b = (x^{d-1})^y \in X^{d-1}, \tag{6.9}$$

and

$$v = x^i c, \quad \text{where } 1 \leq i \leq d - 1 \text{ and } c \in X^{d-i}. \tag{6.10}$$

A reader who is familiar with the techniques and properties of square-free solutions such as "cyclic conditions" and condition "lri" may compute that $b = (x^{d-1})^y = (x^y)^{d-1}$ and

$c = ({}^{x^{d-i-1}}y)(x^y)^{d-i-1}$, but these details are not used in our proof. We use condition ML2 (see (4.4)) to yield the following equality in S:

$$^a a = {}^{(x^{d-1}y)}(x^{d-1}y) = ({}^{x^{d-1}y}x)({}^{(x^{d-1}y)^x}x)\cdots({}^{(x^{d-1}y)^{x^{d-1}}}y) = \omega. \qquad (6.11)$$

The assumption $^a a = a$ implies that word ω, considered as an element of X^d, is in the orbit $\mathcal{O}$ of a, and therefore two cases are possible.

Case 1. The following is an equality of words in X^d:

$$\omega = ({}^{x^{d-1}y}x)({}^{(x^{d-1}y)^x}x)\cdots({}^{(x^{d-1}y)^{x^{d-1}}}y) = ({}^{x^{d-1}}y)b, \quad b \in X^{d-1}.$$

Then there is an equality of elements of X:

$$^{(x^{d-1}y)}x = {}^{x^{d-1}}y. \qquad (6.12)$$

Now we use condition ML1 (see (4.4)) to obtain

$$^{(x^{d-1}y)}x = {}^{(x^{d-1})}({}^y x)$$

which together with (6.12) gives

$$^{(x^{d-1})}({}^y x) = {}^{(x^{d-1})}y. \qquad (6.13)$$

The nondegeneracy implies that $^y x = y$. At the same time $^y y = y$, since (X, r) is square-free, and using the nondegeneracy again one gets $x = y$, a contradiction. It follows that Case 1 is impossible, whenever $x \neq y$.

Case 2. The following is an equality of words in X^d:

$$\omega = ({}^{x^{d-1}y}x)({}^{(x^{d-1}y)^x}x)\cdots({}^{(x^{d-1}y)^{x^{k-1}}}y) = x^i c, \quad \text{where } 1 \leq i \leq d-1, c \in X^{d-i}.$$

Then

$$^{(x^{d-1}y)}x = x. \qquad (6.14)$$

At the same time, the equality $^x x = x$ and condition ML1 imply $^{x^{d-1}}x = x$, which together with (6.14) and ML1 (again) gives

$$^{x^{d-1}}x = {}^{(x^{d-1}y)}x = {}^{x^{d-1}}({}^y x).$$

Thus, by the nondegeneracy, $^y x = x$. We have shown that $^y x = x$ for all $x, y \in X$, $y \neq x$. But (X, r) is square-free, so $^y y = y$ for all $y \in X$. It follows that $^y x = x$ holds for all $x, y \in X$, and therefore (X, r) is the trivial solution. ∎

By construction, the (abstract) d-Veronese solution (Y, r_Y) associated to (X, r) is isomorphic to the normalised d-Veronese solution $(\mathcal{T}_d, \rho_d)$, and therefore it is isomorphic to the solution (S_d, r_d). Theorem 6.4 implies straightforwardly the following corollary.

Corollary 6.5. *Let $d \geq 2$ be an integer, suppose (X, r) is a square-free solution of finite order. Then the d-Veronese solution (Y, r_Y) is square-free if and only if (X, r) is a trivial solution.*

Remark 6.6. It follows from Corollary 6.5 that the notion of Veronese morphisms introduced for the class of Yang–Baxter algebras of finite solutions of YBE cannot be restricted to the subclass of Yang–Baxter algebras associated to finite square-free solutions.

6.2. Involutive permutation solutions

Recall that a symmetric set (X, r) is *an involutive permutation solution* of Lyubashenko (or shortly *a permutation solution*) if there exists a permutation $f \in \text{Sym}(X)$ such that $r(x, y) = (f(y), f^{-1}(x))$. In this case, we shall write (X, f, r); see [6] and [16, p. 691].

Proposition 6.7. *Suppose (X, f, r) is an involutive permutation solution of finite order n defined as $r(x, y) = (f(y), f^{-1}(x))$, where f is a permutation of X, and let A be the associated Yang–Baxter algebra.*

(1) *For every integer $d \geq 2$, the monomial d-Veronese solution (S_d, r_d) is an involutive permutation solution.*

(2) *If the permutation f has order m, then for every integer d divisible by m the d-Veronese subalgebra $A^{(d)}$ of A is a quotient of the commutative polynomial ring $k[y_1, y_2, \ldots, y_N]$, where $N = \binom{n+d-1}{d}$.*

Proof. (1) Let $q \geq 2$ be an integer. The condition ML1 in (4.4) implies that

$$^a t = f^q(t) \quad \text{and} \quad t^a = f^{-q}(t) = (f^{-1})^q(t)$$

$$\text{for all monomials } a \in S_q \text{ and all } t \in X. \tag{6.15}$$

Moreover, since S is a graded braided monoid, the monomials a, $^b a$, and a^b have the same length; therefore,

$$^a t = {}^{ab} t = f^q(t), \quad t^a = t^{ba} = f^{-q}(t) \quad \text{for all } a \in S_q \, b \in S \text{ and all } t \in X. \tag{6.16}$$

It follows then from (4.4) ML2 that S acts on itself (on the left and on the right) as automorphisms. In particular, for $a, t_1 t_2 \cdots t_d \in S_d$, one has

$$^a(t_1 t_2 \cdots t_d) = (^a t_1)(^a t_2) \cdots (^a t_d) = f^d(t_1) f^d(t_2) \cdots f^d(t_d).$$
$$(t_1 t_2 \cdots t_d)^a = (t_1{}^a)(t_2{}^a) \cdots (t_d{}^a) = f^{-d}(t_1) f^{-d}(t_2) \cdots f^{-d}(t_d). \tag{6.17}$$

Therefore, (S_d, r_d) is a permutation solution, (S_d, f_d, r_d), where the permutation $f_d \in \text{Sym}(S_d)$ is expressed as $f_d(t_2 t_2 \cdots t_d) := f^d(t_1) f^d(t_2) \cdots f^d(t_d)$. One has $f_d^{-1}(t_2 t_2 \cdots t_d) := f^{-d}(t_1) f^{-d}(t_2) \cdots f^{-d}(t_d)$.

(2) Assume now that $d = km$ for some integer $k \geq 1$, then $f^d = \mathrm{id}_X$. It will be enough to prove that the monomial d-Veronese solution (S_d, r_d) is the trivial solution. It follows from (6.17) that if $a \in S_d$, then

$$^a(t_1 t_2 \cdots t_d) = t_1 t_2 \cdots t_d, \quad \text{where } t_i \in X, \ 1 \leq i \leq n. \tag{6.18}$$

This implies $^a b = b$ for all $a, b \in S_d$. Similarly, $a^b = a$ for all $a, b \in S_d$. It follows that (S_d, r_d) is the trivial solution. But the associated d-Veronese solution (Y, r_Y) is isomorphic to (S_d, r_d), hence (Y, r_Y) is also a trivial solution, and therefore its Yang–Baxter algebra $\mathcal{A}(\mathbf{k}, Y, r_Y)$ is the commutative polynomial ring $\mathbf{k}[y_1, y_2, \ldots, y_N]$. It follows from Theorem 5.4 that the d-Veronese subalgebra $\mathcal{A}^{(d)}$ is isomorphic to the quotient $\mathbf{k}[y_1, y_2, \ldots, y_N]/(\mathfrak{R})$, where $\mathfrak{R}$ is the kernel of the Veronese map $v_{n,d}$. ∎

Question 6.8. In notation as above. Suppose (X, f, r) is an involutive permutation solution of finite order n, where f is a permutation of X of order m, and let $\mathcal{A}$ be the associated Yang–Baxter algebra. By Proposition 6.7 part (2), for every integer d divisible by m, the d-Veronese subalgebra $\mathcal{A}^{(d)}$ of $\mathcal{A}$ is a quotient of the commutative polynomial ring $\mathbf{k}[y_1, y_2, \ldots, y_N]$, where $N = \binom{n+d-1}{d}$. *Is it true that in this case $\mathcal{A}^{(d)}$ is a PBW algebra?*

Example 7.2 gives an affirmative answer in a very particular case.

Remark 6.9. In [21, Section 5], we consider finite permutation idempotent solutions (X, r_f) and their Yang–Baxter algebras $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r_f)$. It is proven that in this case all Yang–Baxter algebras $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r_f)$ are PBW, and any two such algebras are isomorphic. Moreover, for every integer d, the Veronese subalgebra $\mathcal{A}^{(d)}$ is PBW, and there is an isomorphism $\mathcal{A}^{(d)} \simeq \mathcal{A}$.

7. Examples

We shall present two examples which illustrate the results of the paper. We use the notation of the previous sections.

Example 7.1. Let $n = 3$, consider the solution (X, r), where

$$X = \{x_1, x_2, x_3\},$$
$$r(x_3, x_1) = (x_2, x_3), \quad r(x_2, x_3) = (x_3, x_1),$$
$$r(x_3, x_2) = (x_1, x_3), \quad r(x_1, x_3) = (x_3, x_2),$$
$$r(x_2, x_1) = (x_1, x_2), \quad r(x_1, x_2) = (x_2, x_1),$$
$$r(x_i, x_i) = (x_i, x_i), \quad 1 \leq i \leq 3.$$

Then

$$A(\mathbf{k}, X, r) = \mathbf{k}\langle X \rangle/(\mathfrak{R}_{\mathcal{A}}), \quad \text{where } \mathfrak{R}_{\mathcal{A}} = \{x_3 x_2 - x_1 x_3, \ x_3 x_1 - x_2 x_3, \ x_2 x_1 - x_1 x_2\}.$$

The algebra $A = A(\mathbf{k}, X, r)$ is a PBW algebra with PBW generators $X = \{x_1, x_2, x_3\}$; in fact, it is a binomial skew polynomial algebra.

We first give an explicit presentation of the 2-Veronese $\mathcal{A}^{(2)}$ in terms of generators and quadratic relations. In this case, $N = \binom{3+1}{2} = 6$ and the 2-Veronese subalgebra $\mathcal{A}^{(2)}$ is generated by $\mathcal{T}_2$, the terms of length 2 in $\mathbf{k}\langle x_1, x_2, x_3 \rangle$. These are all normal (modulo $\mathfrak{R}_{\mathcal{A}}$) monomials of length 2 ordered lexicographically:

$$\mathcal{T}_2 = \{w_1 = x_1 x_1, \, w_2 = x_1 x_2, \, w_3 = x_1 x_3, \, w_4 = x_2 x_2, \, w_5 = x_2 x_3, \, w_6 = x_3 x_3\}. \quad (7.1)$$

Determine the normalised 2-Veronese solution $(\mathcal{T}_2, \rho_2) = (\mathcal{T}_2, \rho)$, where $\rho(a, b) = (\mathrm{Nor}(^a b), \mathrm{Nor}(a^b))$. An explicit description of ρ is given below:

$$
\begin{aligned}
(x_3 x_3, w_i) &\longleftrightarrow (w_i, x_3 x_3), & 1 &\leq i \leq 5, \\
(x_2 x_3, x_2 x_3) &\longleftrightarrow (x_1 x_3, x_1 x_3), & (x_2 x_3, x_2 x_2) &\longleftrightarrow (x_1 x_1, x_2 x_3), \\
(x_2 x_3, x_1 x_2) &\longleftrightarrow (x_1 x_2, x_2 x_3), & (x_2 x_3, x_1 x_1) &\longleftrightarrow (x_2 x_2, x_2 x_3), \\
(x_2 x_2, x_1 x_3) &\longleftrightarrow (x_1 x_3, x_1 x_1), & (x_2 x_2, x_1 x_2) &\longleftrightarrow (x_1 x_2, x_2 x_2), \\
(x_2 x_2, x_1 x_1) &\longleftrightarrow (x_1 x_1, x_2 x_2), & (x_1 x_3, x_2 x_2) &\longleftrightarrow (x_1 x_1, x_1 x_3), \\
(x_1 x_3, x_1 x_2) &\longleftrightarrow (x_1 x_2, x_1 x_3), & (x_1 x_2, x_1 x_1) &\longleftrightarrow (x_1 x_1, x_1 x_2).
\end{aligned}
\quad (7.2)
$$

The fixed points $\mathcal{F} = \mathcal{F}(\mathcal{T}_2, \rho_2)$ are the monomials ab determined by the one-element orbits of ρ, where one has $(a, b) = (^a b, a^b)$. There are exactly six fixed points:

$$
\begin{aligned}
\mathcal{F} = \{ & w_1 w_1 = (x_1 x_1)(x_1 x_1) \in \mathcal{T}_4, \; w_4 w_4 = (x_2 x_2)(x_2 x_2) \in \mathcal{T}_4, \\
& w_6 w_6 = (x_3 x_3)(x_3 x_3) \in \mathcal{T}_4, \; w_2 w_2 = (x_1 x_2)(x_1 x_2) \notin \mathcal{T}_4, \\
& w_3 w_5 = (x_1 x_3)(x_2 x_3) \notin \mathcal{T}_4, \; w_5 w_3 = (x_2 x_3)(x_1 x_3) \notin \mathcal{T}_4 \}.
\end{aligned}
\quad (7.3)
$$

There are exactly $15 = \binom{N}{2}$ nontrivial ρ-orbits in $\mathcal{T}_2 \times \mathcal{T}_2$ determined by (7.2). These orbits imply the following equalities in $\mathcal{A}^{(2)}$:

$$
\begin{aligned}
(x_3 x_3) w_i &= w_i (x_3 x_3) \in \mathcal{T}_4, & 1 &\leq i \leq 5, \\
(x_2 x_3)(x_2 x_3) &= (x_1 x_3)(x_1 x_3) \notin \mathcal{T}_4, & (x_2 x_3)(x_2 x_2) &= (x_1 x_1)(x_2 x_3) \in \mathcal{T}_4, \\
(x_2 x_3)(x_1 x_2) &= (x_1 x_2)(x_2 x_3) \in \mathcal{T}_4, & (x_2 x_3)(x_1 x_1) &= (x_2 x_2, x_2 x_3) \in \mathcal{T}_4, \\
(x_2 x_2)(x_1 x_3) &= (x_1 x_3)(x_1 x_1) \notin \mathcal{T}_4, & (x_2 x_2)(x_1 x_2) &= (x_1 x_2)(x_2 x_2) \in \mathcal{T}_4, \\
(x_2 x_2)(x_1 x_1) &= (x_1 x_1)(x_2 x_2) \in \mathcal{T}_4, & (x_1 x_3)(x_2 x_2) &= (x_1 x_1)(x_1 x_3) \in \mathcal{T}_4, \\
(x_1 x_3)(x_1 x_2) &= (x_1 x_2)(x_1 x_3) \notin \mathcal{T}_4, & (x_1 x_2)(x_1 x_1) &= (x_1 x_1)(x_1 x_2) \in \mathcal{T}_4.
\end{aligned}
\quad (7.4)
$$

Note that for every pair $(w_i, w_j) \in \mathcal{T}_2 \times \mathcal{T}_2 \setminus \mathcal{F}$, the monomial $w_i w_j$ occurs exactly once in (7.4).

Six additional quadratic relations of $\mathcal{A}^{(2)}$ arise from (7.3) and (7.4), and the obvious equality $a = \mathrm{Nor}(a) \in \mathcal{T}$, which hold in $\mathcal{A}^{(2)}$ for every $a \in X^2$. In this case, we simply

pick up all monomials which occur in (7.3) or (7.4) but are not in $\mathcal{T}_4$ and equalise each of them with its normal form. This way we get the six relations which determine $\mathcal{R}_b$:

$$
\begin{aligned}
(x_1x_2)(x_1x_2) &= (x_1x_1)(x_2x_2), & (x_1x_3)(x_2x_3) &= (x_1x_1)(x_3x_3), \\
(x_2x_3)(x_1x_3) &= (x_2x_2)(x_3x_3), & (x_1x_3)(x_1x_3) &= (x_1x_2)(x_3x_3), & \text{(7.5)} \\
(x_2x_2)(x_1x_3) &= (x_1x_2)(x_2x_3), & (x_1x_2)(x_1x_3) &= (x_1x_1)(x_2x_3).
\end{aligned}
$$

The 2-Veronese algebra $\mathcal{A}^{(2)}$ has 6 generators $w_1, \ldots, w_6$ written explicitly in (7.1), and a set of 21 relations presented as a disjoint union $\mathcal{R} = \mathcal{R}_a \cup \mathcal{R}_b$ described below.

(1) The relations $\mathcal{R}_a$ are:

$$
\begin{aligned}
& w_6w_i - w_iw_6, \ w_iw_6 \in \mathcal{T}_4, & & 1 \leq i \leq 5, \\
& w_5w_5 - w_3w_3, \ w_3w_3 \notin \mathcal{T}_4, & & w_5w_4 - w_1w_5, \ w_1w_5 \in \mathcal{T}_4, \\
& w_5w_2 - w_2w_5, \ w_2w_5 \in \mathcal{T}_4, & & w_5w_1 - w_4w_5, \ w_4w_5 \in \mathcal{T}_4, \\
& w_4w_3 - w_3w_1, \ w_3w_1 \notin \mathcal{T}_4, & & w_4w_2 - w_2w_4, \ w_2w_4 \in \mathcal{T}_4, & \text{(7.6)} \\
& w_4w_1 - w_1w_4, \ w_1w_4 \in \mathcal{T}_4, & & w_3w_4 - w_1w_3, \ w_1w_3 \in \mathcal{T}_4, \\
& w_3w_2 - w_2w_3, \ w_2w_3 \notin \mathcal{T}_4, & & w_2w_1 - w_1w_2, \ w_1w_2 \in \mathcal{T}_4.
\end{aligned}
$$

(2) The relations $\mathcal{R}_b$ are:

$$
\begin{aligned}
& w_2w_2 - w_1w_4, & & w_3w_5 - w_1w_6, & & w_5w_3 - w_4w_6, \\
& w_3w_3 - w_2w_6, & & w_3w_1 - w_2w_5, & & w_2w_3 - w_1w_5.
\end{aligned} \quad \text{(7.7)}
$$

The elements of $\mathcal{R}_b$ correspond to the generators of the kernel of the Veronese map.

(1a) The relations $\mathcal{R}_{a1}$ are:

$$
\begin{aligned}
& w_6w_i - w_iw_6, \ w_iw_6 \in \mathcal{T}_4, & & 1 \leq i \leq 5, \\
& w_5w_5 - w_2w_6, \ w_2w_6 \in \mathcal{T}_4, & & w_5w_4 - w_1w_5, \ w_1w_5 \in \mathcal{T}_4, \\
& w_5w_2 - w_2w_5, \ w_2w_5 \in \mathcal{T}_4, & & w_5w_1 - w_4w_5, \ w_4w_5 \in \mathcal{T}_4, \\
& w_4w_3 - w_2w_5, \ w_2w_5 \in \mathcal{T}_4, & & w_4w_2 - w_2w_4, \ \in \mathcal{T}_4, & \text{(7.8)} \\
& w_4w_1 - w_1w_4, \ w_1w_4 \in \mathcal{T}_4, & & w_3w_4 - w_1w_3, \ w_1w_3 \in \mathcal{T}_4, \\
& w_3w_2 - w_1w_5, \ w_1w_5 \in \mathcal{T}_4, & & w_2w_1 = w_1w_2, \ w_1w_2 \in \mathcal{T}_4.
\end{aligned}
$$

Thus the 2-Veronese $\mathcal{A}^{(2)}$ of the algebra $\mathcal{A}$ is a quadratic algebra presented as

$$
\mathcal{A}^{(2)} \simeq \mathbf{k}\langle w_1, \ldots, w_6 \rangle / (\mathcal{R}) \simeq \mathbf{k}\langle w_1, \ldots, w_6 \rangle / (\mathcal{R}_1),
$$

where $\mathcal{R} = \mathcal{R}_a \cup \mathcal{R}_b$ and $\mathcal{R}_1 = \mathcal{R}_{a1} \cup \mathcal{R}_b$.

The 2-Veronese subalgebra $\mathcal{A}^{(2)}$ in our example is a PBW algebra.

The associated 2-Veronese solution of YBE (Y, r_Y) can be found straightforwardly: one has $Y = \{y_1, y_2, y_3, y_4, y_5, y_6\}$ and $r_Y(y_i, y_j) = (y_k, y_l)$ iff $\rho_2(w_i, w_j) = (w_k, w_l)$,

$1 \leq i, j, k, l \leq 6$. The solution (Y, r_Y) is nondegenerate and involutive, but it is not a square-free solution. The corresponding Yang–Baxter algebra is

$$\mathcal{A}_Y = (\mathbf{k}, Y, r_Y) = \mathbf{k}\langle y_1, y_2, y_3, y_4, y_5, y_6 \rangle / (\mathfrak{R}),$$

where $\mathfrak{R}$ is the set of quadratic relations given below:

$$
\begin{array}{lll}
y_6 y_i - y_i y_6, & 1 \leq i \leq 5, & y_5 y_5 - y_3 y_3, \\
y_5 y_4 - y_1 y_5, & y_5 y_1 - y_4 y_5, & y_5 y_2 - y_2 y_5, \\
y_4 y_3 - y_3 y_1, & y_4 y_2 - y_2 y_4, & y_4 y_1 - y_1 y_4, \\
y_3 y_4 - y_1 y_3, & y_3 y_2 - y_2 y_3, & y_2 y_1 = y_1 y_2.
\end{array}
\tag{7.9}
$$

Note that $\mathfrak{R}$ is not a Gröbner basis of the ideal $(\mathfrak{R})$ (with respect to the degree-lexicographic ordering on $\langle Y \rangle$). For example, the overlap $y_5 y_4 y_3$ implies the new relation $y_5 y_3 y_1 - y_1 y_5 y_3$ which is in the ideal $(\mathfrak{R})$ but cannot be reduced using (7.9). There are more such overlaps.

The Veronese map

$$v_{n,2} : \mathcal{A}_Y \rightarrow \mathcal{A}_X$$

is the algebra homomorphism extending the assignment $y_1 \mapsto w_1, y_2 \mapsto w_2, \ldots, y_6 \mapsto w_6$. Its image is the 2-Veronese, $\mathcal{A}^{(2)}$. The kernel $\mathfrak{K}$ of the map $v_{n,2}$ is generated by the set $\mathfrak{R}_1$ of polynomials given below:

$$
\begin{array}{lll}
y_2 y_2 - y_1 y_4, & y_3 y_5 - y_1 y_6, & y_5 y_3 - y_4 y_6, \\
y_3 y_3 - y_2 y_6, & y_4 y_3 - y_2 y_5, & y_2 y_3 - y_1 y_5.
\end{array}
\tag{7.10}
$$

Denote by J the two-sided ideal $J = (\mathfrak{R} \cup \mathfrak{R}_1)$ of $\mathbf{k}\langle Y \rangle$. A direct computation shows that the set $\mathfrak{R} \cup \mathfrak{R}_1$ is a (quadratic) Gröbner basis of J. The 2-Veronese subalgebra $\mathcal{A}^{(2)}$ of $\mathcal{A} = \mathcal{A}_X$ is isomorphic to the quotient $\mathbf{k}\langle Y \rangle / J$, hence it is a PBW algebra.

The following example is an illustration of Proposition 6.7.

Example 7.2. Let $n = 2$, consider the solution (X, r), where

$$
\begin{aligned}
X &= \{x_1, x_2\}, \\
r(x_2, x_2) &= (x_1, x_1), & r(x_1, x_1) &= (x_2, x_2), \\
r(x_2, x_1) &= (x_2, x_1), & r(x_1, x_2) &= (x_1, x_2).
\end{aligned}
$$

This is a permutation solution (X, f, r), where f is the transposition $f = (x_1 x_2)$. One has

$$A(\mathbf{k}, X, r) = \mathbf{k}\langle x_1, x_2 \rangle / (\mathfrak{R}_\mathcal{A}), \quad \text{where } \mathfrak{R}_\mathcal{A} = \{x_2 x_2 - x_1 x_1\}.$$

The set $\mathfrak{R}_\mathcal{A}$ is not a Gröbner basis of $I = (\mathfrak{R}_\mathcal{A})$ with respect to the deg-lex ordering induced by any of the choices $x_1 < x_2$, or $x_2 < x_1$. We keep the convention $x_1 < x_2$ and

apply standard computation to find that the reduced Gröbner basis of I (with respect to the deg-lex ordering) is

$$G = \{f_1 = x_2 x_2 - x_1 x_1, \ f_2 = x_2 x_1 x_1 - x_1 x_1 x_2\}.$$

Then

$$\mathcal{N} = \mathcal{N}(I) = \{x_1^\alpha (x_2 x_1)^\beta x_2^\varepsilon \mid \varepsilon \in \{0,1\} \text{ and } \alpha, \beta \in \mathbb{N}_0\}.$$

It is easy to find an explicit presentation of the 2-Veronese $\mathcal{A}^{(2)}$ in terms of one-generators and quadratic relations. The normal monomials of length 2

$$\mathcal{N}_2 = \{w_1 = x_1 x_1, w_2 = x_1 x_2, w_3 = x_2 x_1\}$$

form a set of one-generators of $\mathcal{A}^{(2)}$.

One has

$$\mathcal{N}_4 = \{x_1^4, \ x_1^3 x_2, \ x_1^2 x_2 x_1, \ x_1 x_2 x_1 x_2, \ x_2 x_1 x_2 x_1\}.$$

Next we determine the normalised d-Veronese solution $(\mathcal{N}_2, \rho)$, where $\rho(a, b) = (\mathrm{Nor}(^a b), \mathrm{Nor}(a^b))$. One has

$$\begin{aligned}
^{(x_i x_j)} x_k = x_k, \quad & x_k^{(x_i x_j)} = x_k \qquad \text{for all } i, j, k \in \{1, 2\} \\
^{w_i} w_j = w_j, \quad & w_j^{w_i} = w_j \qquad \forall i, j \in \{1, 2, 3\}.
\end{aligned}$$

Thus $(\mathcal{N}_2, \rho)$ is the trivial solution on the set $\mathcal{N}_2$:

$$\rho(w_j, w_i) = (w_i, w_j), \quad 1 \le i, \ j \le 3.$$

In this case, the three fixed points are normal monomials:

$$\begin{aligned}
\mathcal{F} = \{ & w_1 w_1 = (x_1 x_1)(x_1 x_1) \in \mathcal{N}_4, \ w_2 w_2 = (x_1 x_2)(x_1 x_2) \in \mathcal{N}_4, \\
& w_3 w_3 = (x_2 x_1)(x_2 x_1) \in \mathcal{N}_4 \}.
\end{aligned}$$

The set of relations is $\mathcal{R} = \mathcal{R}_a \cup \mathcal{R}_b$. Here $\mathcal{R}_a$ consists of the relations:

$$\begin{aligned}
g_{32} &= w_3 w_2 - w_2 w_3, \quad \text{equivalently,} \quad (x_2 x_1)(x_1 x_2) = (x_1 x_2)(x_2 x_1) \notin \mathcal{N}_4, \\
g_{31} &= w_3 w_1 - w_1 w_3, \quad \text{equivalently,} \quad (x_2 x_1)(x_1 x_1) = (x_1 x_1)(x_2 x_1) \in \mathcal{N}_4, \quad (7.11) \\
g_{21} &= w_2 w_1 - w_1 w_2, \quad \text{equivalently,} \quad (x_1 x_2)(x_1 x_1) = (x_1 x_1)(x_1 x_2) \in \mathcal{N}_4.
\end{aligned}$$

The unique word not in normal form which participates in $\mathcal{R}_a$ is $w_2 w_3 = (x_1 x_2)(x_2 x_1) \notin \mathcal{N}_4$, the second term of the relation g_{32} in (7.11). By finding its normal form, we shall determine the unique relation in $\mathcal{R}_b$. One has

$$\mathrm{Nor}(w_2 w_3) = \mathrm{Nor}(x_1 x_2)(x_2 x_1) = \mathrm{Nor}(x_1 (x_2 x_2) x_1) = x_1 x_1 x_1 x_1 = w_1 w_1,$$

hence

$$\mathcal{R}_b = \{g_{23} = w_2 w_3 - w_1 w_1\}. \tag{7.12}$$

It follows that

$$\mathcal{A}^{(2)} \simeq \mathbf{k}\langle w_1, w_2, w_3\rangle/(\mathcal{R}), \quad \text{where } \mathcal{R} = \{w_3w_2 - w_2w_3, \ w_3w_1 - w_1w_3,$$
$$w_2w_1 - w_1w_2, \ w_2w_3 - w_1w_1\}.$$

In our notation, the second set $\mathcal{R}_1$ consisting of equivalent relation is:

$$\mathcal{R}_1 = \{w_3w_2 - w_1w_1, \ w_3w_1 - w_1w_3, \ w_2w_1 - w_1w_2, \ w_2w_3 - w_1w_1\},$$

and $\mathcal{A}^{(2)} \simeq \mathbf{k}\langle w_1, w_2, w_3\rangle/(\mathcal{R}_1)$. It is easy to see that the set $\mathcal{R}$ is a (minimal) Gröbner basis of the two-sided ideal $I = (\mathcal{R})$ of $\mathbf{k}\langle w_1, w_2w_3\rangle$, with respect to the degree-lexicographic order on $\langle w_1, w_2, w_3\rangle$, while the set $\mathcal{R}_1$ is the reduced Gröbner basis of the ideal I. Thus the 2-Veronese subalgebra $\mathcal{A}^{(2)}$ in this example is a PBW algebra. As expected, the 2-Veronese $\mathcal{A}^{(2)}$ is a commutative algebra isomorphic to $\mathbf{k}[w_1, w_2, w_3]/(w_2w_3 - w_1w_1)$.

The associated 2-Veronese solution of YBE (Y, r_Y) is the trivial solution on the set $Y = \{y_1, y_2, y_3\}$, $r_Y(y_j, y_i) = (y_i, y_j)$ for all $1 \leq i, j \leq 3$. The corresponding Yang–Baxter algebra $\mathcal{A}_Y$ is

$$\mathcal{A}_Y = (\mathbf{k}, Y, r_Y) = \mathbf{k}\langle y_1, y_2, y_3\rangle/(y_3y_2 - y_2y_3, \ y_3y_1 - y_1y_3, \ y_2y_1 - y_1y_2)$$
$$\simeq \mathbf{k}[y_1, y_2, y_3].$$

Obviously, $\mathcal{A}_Y$ is PBW. The Veronese map

$$v_{n,2} : \mathcal{A}_Y \rightarrow \mathcal{A}_X$$

is the algebra homomorphism extending the assignment $y_1 \mapsto w_1$, $y_2 \mapsto w_2$, $y_3 \mapsto w_3$. Its image is the d-Veronese $\mathcal{A}^{(2)}$, and its kernel $\mathcal{K}$ is generated by the polynomial $y_2y_3 - y_1y_1$.

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Tatiana Gateva-Ivanova

Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany; Department of Mathematics and Science, American University in Bulgaria, 1 Georgy Izmirliev Sq., 2700 Blagoevgrad, Bulgaria; tatyana@aubg.edu