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**Algebraic Geometry.** – *Hurwitz moduli varieties parameterizing pointed covers of an algebraic curve with a fixed monodromy group*, by Vassil Kanev, accepted on 11 August 2025.

Abstract. – Given a smooth, projective curve $Y$, a point $y_0 \in Y$, a positive integer $n$, and a transitive subgroup $G$ of the symmetric group $S_d$, we study smooth, proper families, parameterized by algebraic varieties, of pointed degree $d$ covers of $(Y, y_0)$, $(X, x_0) \to (Y, y_0)$, branched in $n$ points of $Y \setminus y_0$, whose monodromy group equals $G$. We construct a Hurwitz space $H$, an algebraic variety whose points are in bijective correspondence with the equivalence classes of pointed covers of $(Y, y_0)$ of this type. We construct explicitly a family parameterized by $H$, whose fibers belong to the corresponding equivalence classes, and prove that it is universal. We use classical tools of algebraic topology and of complex algebraic geometry.

Keywords. – cover of a curve, monodromy group, family of covers, Hurwitz space, moduli space.

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## 1. Introduction

Fulton studied in [10] smooth, proper families of covers of $\mathbb{P}^1$ of degree $d \geq 3$ simply branched in $n \geq 2d - 2$ points, parameterized by schemes over $\mathbb{Z}$. In particular, over $\mathbb{C}$ he constructed a universal family of such covers, parameterized by the Hurwitz space $H^{d,n}$, a variety whose points correspond bijectively to the equivalence classes of covers of $\mathbb{P}^1$ of the above type.

This paper is concerned with generalizing Fulton's results to degree $d$ covers of an arbitrary smooth, irreducible, projective curve $Y$, branched in $n$ points, whose monodromy group is a fixed transitive subgroup $G$ of $S_d$. When $Y = \mathbb{P}^1$, this problem was first studied by Fried in [9] in connection with the Inverse Galois Problem. An obstruction to the existence of a universal family of covers of $Y$ of this type is the presence of nontrivial covering automorphisms of $f : X \to Y$. This happens if the centralizer of $G$ in $S_d$ is nontrivial. This obstacle may be avoided by working with pointed covers of $(Y, y_0)$, where $y_0$ is a marked point of $Y$. Namely, we consider pairs $(f : X \to Y, x_0)$ where $X$ is a smooth, irreducible, projective curve, $f$ is a cover branched in a set $D \subset Y$ of $n$ points, $y_0 \notin D$, $x_0 \in f^{-1}(y_0)$, and there is a numeration of $f^{-1}(y_0)$ such that the monodromy group obtained from the action of $\pi_1(Y \setminus D, y_0)$ on $f^{-1}(y_0)$ equals $G$. We construct an algebraic variety $H$, a Hurwitz space, whose points correspond bijectively to the equivalence classes of pointed covers of $(Y, y_0)$ of the above type.

We study smooth, proper families of pointed covers of $(Y, y_0)$ parameterized by algebraic varieties. Our main result is the construction of a universal family of degree $d$ pointed covers of $(Y, y_0)$, branched in $n$ points, with monodromy group $G \subset S_d$, parameterized by the Hurwitz space $H$. The construction is explicit and uses the universal family of pointed Galois covers of $(Y, y_0)$ with Galois group isomorphic to $G$ constructed in [22].

The covers $f : X \to Y$ with restricted monodromy group and the related Galois covers $p : C \to Y$ yield polarized abelian varieties isogenous to abelian subvarieties of the Jacobian variety $J(X)$ [4, 6, 17, 26]. Given a smooth, proper family of such covers, one obtains a morphism of its parameter variety to a certain moduli space of polarized abelian varieties, which may be studied by means of the associated variation of the Hodge structures of weight one [3, 13, 14]. This approach to the moduli of polarized abelian varieties was used in [18, 19] where the unirationality of $\mathcal{A}_3(1, 1, d)$ and $\mathcal{A}_3(1, d, d)$ with $2 \le d \le 4$ was proved by means of families of simply ramified covers of elliptic curves branched in 6 points. The Prym–Tyurin morphism from a Hurwitz space to $\mathcal{A}_6$ was used in [1] to prove that every sufficiently general principally polarized abelian variety of dimension 6 is isomorphic to a Prym–Tyurin variety of a cover of $\mathbb{P}^1$ of degree 27, branched in 24 points, with monodromy group the Weyl group $W(E_6) \subset S_{27}$.

Emsalem studied in [7] the moduli properties of the Hurwitz spaces of pointed degree $d$ covers of $(Y, y_0)$, where $Y = \mathbb{P}^1$, with a fixed monodromy group $G \subset S_d$. He worked with families, which are not proper, whose fibers are étale covers of open subsets of $\mathbb{P}^1$, complements of $\le n$ points, and whose parameter varieties are étale covers of open subsets of $(\mathbb{P}^1 \setminus y_0)^{(n)}_*$. He constructed universal families of this type parameterized by the corresponding Hurwitz spaces [7, Thm. 3′].

Section 2 contains generalities on degree $d$ covers of a curve $Y$ with a fixed monodromy group $G \subset S_d$. We associate with every cover some data called monodromy invariant by which one parameterizes the equivalence classes of covers of $Y$ of the above type.

Section 3 of the paper contains generalities on smooth, proper families of degree $d$ covers of a curve $Y$ with a fixed monodromy group $G \subset S_d$. We discuss their connection with the smooth, proper families of Galois covers of $Y$ with Galois group isomorphic to $G$.

In Section 4, given a curve $Y$, a point $y_0 \in Y$, a positive integer $n$, and a transitive subgroup $G \subset S_d$, we endow with a structure of an algebraic variety the set $H$ which parameterizes the equivalence classes of pointed degree $d$ covers of $(Y, y_0)$, branched in $n$ points, with monodromy group $G$. We construct explicitly in Theorem 4.13 a smooth, proper family of pointed covers of $(Y, y_0)$ of this type $(\phi : \mathcal{X} \to Y \times H, \xi : H \to \mathcal{X})$, parameterized by $H$, whose fiber over every $h \in H$ belongs to the equivalence class of covers corresponding to $h$. We give the explicit form of $\phi$ locally at the ramification points in analytic coordinates (Proposition 4.6 and Remark 4.14). We prove that the constructed family is universal in Theorem 4.17. We mention that a key ingredient of the proof is the use of the criterion for extending morphisms from [21]. We prove in Proposition 4.18 that, after performing an étale base change, every smooth, proper family of pointed degree $d$ covers of $(Y, y_0)$ with monodromy group $G \subset S_d$ is isomorphic to the quotient by the isotropy subgroup $H = G(1)$, of a smooth, proper family of pointed $G$-covers of $(Y, y_0)$. We give in Theorem 4.22 a variant of the main theorem in which the pointed covers of $(Y, y_0)$ have local monodromies at the branch points in a set of marked conjugacy classes of $G$.

*Notation and conventions.* We assume the base field is $\mathbb{C}$. Algebraic varieties are reduced, separated, possibly reducible schemes of finite type; *points* are closed points. Fiber products and pullbacks are those defined in the category of schemes over $\mathbb{C}$. A cover $f : X \to Z$ of algebraic varieties is a finite, surjective morphism. If $Y$ is a smooth, projective, irreducible curve, we denote by $Y^{(n)}$ the $n$-th symmetric power of $Y$ and by $Y^{(n)}_*$ the subset of $Y^{(n)}$ consisting of effective divisors of $Y$ of degree $n$ without multiple points. We identify such divisors with their support. If $\Omega \subset Y^{(n)}$, then $\Omega_* = \Omega \cap Y^{(n)}_*$. Given an algebraic variety $(X, \mathcal{O}_X)$, the canonically associated complex space is denoted by $(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$ [28]. Its topological space is denoted by $|X^{\mathrm{an}}|$. We assume that the homotopy of paths leaves the end points fixed [24, Ch. 2, §2]. The set of paths homotopic to $\alpha$ is denoted by $[\alpha]$. The product of the paths $\alpha$ and $\beta$ is denoted by $\alpha \cdot \beta$ and it equals the path $\gamma$, where $\gamma(t) = \alpha(2t)$ if $t \in [0, \frac{1}{2}]$, $\gamma(t) = \beta(2t-1)$ if $t \in [\frac{1}{2}, 1]$. Given a covering space $p : M \to N$ of the topological space $N$, the map $p$ is called topological covering map. Lifting a path $\alpha$ of $N$ from an initial point $z \in M$, the terminal point is denoted by $z\alpha$.

## 2. Parameterization of covers with a fixed monodromy group

2.1. Throughout the paper, $Y$ is a smooth, projective, irreducible curve of genus $g \geq 0$, $n$ is a positive integer, $G$ is a finite group, which can be generated by $2g+n-1$ elements, $\Lambda$ is a finite set, $|\Lambda| = d$, on which $G$ acts on the right faithfully and transitively, we choose an element $\lambda_0 \in \Lambda$ as a marked element, and $G(\lambda_0) = \{g \in G \mid \lambda_0 g = \lambda_0\}$. We identify $G$ with its image in the symmetric group $S(\Lambda) \cong S_d$, where $S(\Lambda)$ acts on the right.

DEFINITION 2.2. We call a finite, surjective morphism $f : X \to Y$ a *cover* of $Y$ if $X$ is a smooth, projective, irreducible curve. We say that two covers $f : X \to Y$ and $f_1 : X_1 \to Y$ are equivalent if there exists a covering isomorphism $h : X \to X_1$, i.e., one such that $f_1 \circ h = f$. A $G$-cover of $Y$ is a cover $p : C \to Y$ as above, such that $G$ acts faithfully on the left on $C$ by automorphisms of $C$, $p$ is $G$-invariant, and $\bar{p} : C/G \to Y$ is an isomorphism. Two $G$-covers of $Y$, $p : C \to Y$ and $p_1 : C_1 \to Y$, are called $G$-equivalent if there exists a covering isomorphism $h : C \to C_1$ which is $G$-equivariant.

2.3. A necessary condition for two covers $f : X \to Y$ and $f_1 : X_1 \to Y$ to be equivalent is that they have the same branch locus $D$. Let $y_0 \in Y \setminus D$. Let $Y^{\mathrm{an}}$ be the Riemann surface associated with $Y$. Then $\pi_1(Y^{\mathrm{an}} \setminus D, y_0)$ acts on the right on $f^{-1}(y_0)$ by $x \cdot [\alpha] = x\alpha \ \forall x \in f^{-1}(y_0)$ and $\forall [\alpha] \in \pi_1(Y^{\mathrm{an}} \setminus D, y_0)$, and similarly it acts on the right on $f_1^{-1}(y_0)$. Then $f$ is equivalent to $f_1$ if and only if there exists a $\pi_1(Y^{\mathrm{an}} \setminus D, y_0)$-equivariant bijection $\mu : f^{-1}(y_0) \to f_1^{-1}(y_0)$. In fact, let $X' = f^{-1}(Y \setminus D)$, $X'_1 = f_1^{-1}(Y \setminus D)$. Then such a bijection yields a covering biholomorphic map $X'^{\mathrm{an}} \to X_1'^{\mathrm{an}}$ over $Y \setminus D$, which extends to a covering biholomorphic map $X^{\mathrm{an}} \to X_1^{\mathrm{an}}$ over $Y^{\mathrm{an}}$. Furthermore, this map equals $h^{\mathrm{an}}$, where $h : X \to X_1$ is a covering isomorphism over $Y$. Abusing notation, we will write $\pi_1(Y \setminus D, y_0)$ instead of $\pi_1(Y^{\mathrm{an}} \setminus D, y_0)$.

DEFINITION 2.4. Let $y_0 \in Y$. Let $f : X \to Y$ be a cover such that $y_0$ does not belong to the branch locus of $f$ and let $x_0 \in f^{-1}(y_0)$. We call $(f : X \to Y, x_0)$ a *pointed cover* of $(Y, y_0)$. Two pointed covers $(f : X \to Y, x_0)$ and $(f_1 : X_1 \to Y, x'_0)$ of $(Y, y_0)$ are called equivalent if there exists an isomorphism $h : X \to X_1$ such that $f_1 \circ h = f$ and $h(x_0) = x'_0$. A pointed $G$-cover of $(Y, y_0)$ is a pointed cover $(p : C \to Y, z_0)$ as above such that $p : C \to Y$ is a $G$-cover. Two pointed $G$-covers of $(Y, y_0)$, $(p : C \to Y, z_0)$ and $(p_1 : C_1 \to Y, z'_0)$ are called $G$-equivalent if there exists a $G$-equivariant isomorphism $h : C \to C_1$ such that $p = p_1 \circ h$ and $h(z_0) = z'_0$.

2.5. The isomorphism $h$ of Definition 2.4 is unique. In fact, if $h_1$ is a second one, then $\varphi = h_1^{-1} \circ h : X \to X$ is a covering automorphism over $Y$, such that $\varphi(x_0) = x_0$. Let $D$ be the branch locus of $f$ and let $X' = X \setminus f^{-1}(D)$. Then $f^{\mathrm{an}}|_{X'^{\mathrm{an}}} : X'^{\mathrm{an}} \to Y^{\mathrm{an}} \setminus D$ is a topological covering map, and $X'^{\mathrm{an}}$ is connected, so $\varphi^{\mathrm{an}}|_{X'^{\mathrm{an}}}$ is the identity map, since $\varphi^{\mathrm{an}}(x_0) = x_0$. Therefore, $h = h_1$.

2.6. Let $D = \{b_1, \dots, b_n\} \subset Y \setminus y_0$. Let $\overline{U}_1, \dots, \overline{U}_n$ be embedded closed disks in $Y \setminus y_0$ which are disjoint and such that $b_i \in U_i \ \forall i$, where $U_i$ is the interior of $\overline{U}_i$. For every $i = 1, \dots, n$, let us choose a path $\eta_i : I \to Y \setminus \bigcup_{j=1}^{n} U_j$ such that $\eta_i(0) = y_0$, $\eta_i(1) \in \partial\overline{U}_i$, and let $\gamma_i : I \to Y \setminus D$ be the closed path which starts at $y_0$, travels along $\eta_i$, then makes a counterclockwise loop along $\partial\overline{U}_i$, and returns back to $y_0$ along $\eta_i^-$.

Let $f : X \to Y$ be a $d$-sheeted cover of $Y$ unbranched at $y_0$. Let $X' = X \setminus f^{-1}(D)$, $f' = f|_{X'}$. The branch locus of $f$ equals $D$ if and only if $f' : X' \to Y \setminus D$ is unramified and the monodromy $m : \pi_1(Y \setminus D, y_0) \to S(f^{-1}(y_0))$ defined by $x \cdot m([\alpha]) = x\alpha$ $\forall x \in f^{-1}(y_0)$ and $\forall [\alpha] \in \pi_1(Y \setminus D, y_0)$ satisfies the condition

$$m\big([\gamma_1]\big) \neq 1, \ldots, m\big([\gamma_n]\big) \neq 1. \tag{1}$$

Let $m : \pi_1(Y \setminus D, y_0) \to S_d$ be a homomorphism and let $G = \mathrm{Im}(m)$. Suppose that $m$ satisfies condition (1). If one chooses in a different way $\bar{U}_1, \ldots, \bar{U}_n$ and $\eta_1, \ldots, \eta_n$ so as to satisfy the above conditions, then for every $i \in [1, n]$ the new element $[\gamma_i']$ belongs to the conjugacy class of $[\gamma_i]$ in $\pi_1(Y \setminus D, y_0)$, so $m([\gamma_i'])$ belongs to the conjugacy class of $m([\gamma_i])$ in $G$. Therefore, condition (1) for $m$ is independent of the choice of $\bar{U}_1, \ldots, \bar{U}_n$ and that of $\eta_1, \ldots, \eta_n$.

Lemma 2.7. *Let $G \subset S(\Lambda)$ be as in §2.1. Let $N_{S(\Lambda)}(G) = \{\sigma \in S(\Lambda) \mid \sigma G \sigma^{-1} = G\}$. Let $f : X \to Y$ and $f_1 : X_1 \to Y$ be two covers with the same branch locus $D$. Let $y_0 \in Y \setminus D$. Suppose that there are bijections $\varepsilon : \Lambda \to f^{-1}(y_0)$, $\varepsilon_1 : \Lambda \to f_1^{-1}(y_0)$, and epimorphisms $m : \pi_1(Y \setminus D, y_0) \to G$, $m_1 : \pi_1(Y \setminus D, y_0) \to G$, such that $\varepsilon(\lambda m([\alpha])) = \varepsilon(\lambda)\alpha$ and $\varepsilon_1(\lambda m_1([\alpha])) = \varepsilon_1(\lambda)\alpha$ $\forall \lambda \in \Lambda$ and $\forall [\alpha] \in \pi_1(Y \setminus D, y_0)$. Then $f$ is equivalent to $f_1$ if and only if there exists a $\sigma \in N_{S(\Lambda)}(G)$ such that $m_1 = \sigma m \sigma^{-1}$. The covering isomorphism $h : X \to X_1$ is unique if and only if the centralizer of $G$ in $S(\Lambda)$ is trivial, $Z_{S(\Lambda)}(G) = \{1\}$.*

Proof. The covers $f : X \to Y$ and $f_1 : X_1 \to Y$ are equivalent if and only if there exists a $\pi_1(Y \setminus D, y_0)$-equivariant bijection $\mu : f^{-1}(y_0) \to f_1^{-1}(y_0)$ (cf. §2.3). Let $\mu$ be such a bijection. Let $s : \Lambda \to \Lambda$ be the bijection, which makes the following diagram commutative:

$$\begin{array}{ccc} \Lambda & \xrightarrow{\ \varepsilon\ } & f^{-1}(y_0) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \mu} \\ \Lambda & \xrightarrow{\ \varepsilon_1\ } & f_1^{-1}(y_0). \end{array} \tag{2}$$

Let $\sigma \in S(\Lambda)$ satisfy $s(\lambda) = \lambda\sigma^{-1}$ $\forall \lambda \in \Lambda$. Let $\lambda \in \Lambda$ and let $x = \varepsilon(\lambda) \in f^{-1}(y_0)$. For all $[\alpha] \in \pi_1(Y \setminus D, y_0)$, one has $\mu(x\alpha) = \mu(x)\alpha$ and

$$\mu(x\alpha) = \mu\big(\varepsilon(\lambda)\alpha\big) = \mu\big(\varepsilon\big(\lambda m\big([\alpha]\big)\big)\big) = \varepsilon_1\big(\lambda m\big([\alpha]\big)\sigma^{-1}\big),$$
$$\mu(x)\alpha = \mu\big(\varepsilon(\lambda)\big)\alpha = \varepsilon_1(\lambda\sigma^{-1})\alpha = \varepsilon_1\big(\lambda\sigma^{-1} m_1\big([\alpha]\big)\big).$$

This shows that $m_1([\alpha]) = \sigma m([\alpha])\sigma^{-1}$ $\forall [\alpha] \in \pi_1(Y \setminus D, y_0)$. In particular, $\sigma \in N_{S(\Lambda)}(G)$ since $m$ and $m_1$ are epimorphisms. We conclude that $m_1 = \sigma m \sigma^{-1}$. Vice versa, suppose that $m_1 = \sigma m \sigma^{-1}$ for some $\sigma \in N_{S(\Lambda)}(G)$. Let $s : \Lambda \to \Lambda$ be the bijection

$\varepsilon(\lambda) = \lambda\sigma^{-1}$ and let $\mu : f^{-1}(y_0) \to f_1^{-1}(y_0)$ be the bijection which makes diagram (2) commutative. Then $\mu(x\alpha) = \mu(x)\alpha\ \forall x \in f^{-1}(y_0)$ and $\forall[\alpha] \in \pi_1(Y \setminus D, y_0)$; therefore, $f : X \to Y$ is equivalent to $f_1 : X \to Y$.

Suppose $f$ is equivalent to $f_1$. The covering isomorphism $h : X \to X_1$ is unique if and only if every covering automorphism $\varphi : X \to X$ equals the identity. Consider the diagram (2) with $f_1 = f$, $\varepsilon_1 = \varepsilon$, and $m_1 = m$. Let $\sigma \in S(\Lambda)$ satisfy $\mu(\varepsilon(\lambda)) = \varepsilon(\lambda\sigma^{-1})$. Then $\mu(x\alpha) = \mu(x)\alpha\ \forall x \in f^{-1}(y_0)$ and $\forall[\alpha] \in \pi_1(Y \setminus D, y_0)$ if and only if $\sigma m([\alpha])\sigma^{-1} = m([\alpha])\ \forall[\alpha] \in \pi_1(Y \setminus D, y_0)$. Since $\mathrm{Im}(m) = G$, this holds if and only if $\sigma \in Z_{S(\Lambda)}(G)$. Therefore, $\mu = \mathrm{id}$, equivalently, $\varphi = \mathrm{id}_X$, if and only if $Z_{S(\Lambda)}(G) = \{1\}$. ∎

Proposition 2.8. *In the setup of §2.1, let $y_0 \in Y$, let $D = \{b_1, \dots, b_n\} \subset Y \setminus y_0$, and let $(f : X \to Y, x_0)$ be a pointed cover of $(Y, y_0)$ of degree $d$ branched in $D$. The following conditions are equivalent:*

(i) *There is a bijection $\varepsilon : \Lambda \to f^{-1}(y_0)$ and $m : \pi_1(Y \setminus D, y_0) \to G$, an epimorphism, such that*

$$\varepsilon\big(\lambda m([\alpha])\big) = \varepsilon(\lambda)\alpha \quad \forall\lambda \in \Lambda,\ \forall[\alpha] \in \pi_1(Y \setminus D, y_0),$$
$$\varepsilon(\lambda_0) = x_0. \tag{3}$$

(ii) *Let $X' = X \setminus f^{-1}(D)$, $f' = f|_{X'}$ and let $\Gamma_{x_0} = f'_*\pi_1(X', x_0)$. There is an epimorphism $m : \pi_1(Y \setminus D, y_0) \to G$ such that $\Gamma_{x_0} = m^{-1}(G(\lambda_0))$ where $G(\lambda_0)$ is the isotropy group of $\lambda_0$.*

*Let $N(\lambda_0) = \{\sigma \in N_{S(\Lambda)}(G) \mid \lambda_0\sigma = \lambda_0\}$. Let $(f : X \to Y, x_0)$ and $(f_1 : X_1 \to Y, x_0')$ be two pointed covers of $(Y, y_0)$ branched in $D$ which satisfy condition* (i) *with $(\varepsilon, m)$ and $(\varepsilon_1, m_1)$, respectively. They are equivalent if and only if there exists a $\sigma \in N(\lambda_0)$ such that $m_1 = \sigma m\sigma^{-1}$. Furthermore, $\sigma \in N(\lambda_0)$ with this property is unique.*

Proof. The conditions (i) and (ii) are equivalent since the map $\pi_1(Y \setminus D, y_0) \to f^{-1}(y_0)$ defined by $[\alpha] \mapsto x_0\alpha$ induces a $\pi_1(Y \setminus D, y_0)$-equivariant bijection between the set of right cosets ${}_{\Gamma_{x_0}} \setminus \pi_1(Y \setminus D, y_0)$ and $f^{-1}(y_0)$, and $m$ induces a $\pi_1(Y \setminus D, y_0)$-equivariant bijection between ${}_{\Gamma_{x_0}} \setminus \pi_1(Y \setminus D, y_0)$ and ${}_{G(\lambda_0)} \setminus G \cong \Lambda$. Under these bijections, $\lambda_0 \mapsto G(\lambda_0) \mapsto \Gamma_{\lambda_0} \mapsto x_0$.

The pointed covers $(f : X \to Y, x_0)$ and $(f_1 : X_1 \to Y, x_0')$ are equivalent if and only if there exists a $\pi_1(Y \setminus D, y_0)$-equivariant bijection $\mu : f^{-1}(y_0) \to f_1^{-1}(y_0)$ such that $\mu(x_0) = x_0'$. By Lemma 2.7, this is equivalent to the existence of a $\sigma \in N_{S(\Lambda)}(G)$ such that $m_1 = \sigma m\sigma^{-1}$ and $\lambda_0\sigma^{-1} = \lambda_0$, i.e., $\sigma \in N(\lambda_0)$. In order to prove the uniqueness of such a $\sigma$, it suffices to consider the case $m_1 = m$ and prove that if $\sigma \in N(\lambda_0)$ satisfies $m = \sigma m\sigma^{-1}$, then $\sigma = \mathrm{id}_\Lambda$. In fact, let $g \in G = \mathrm{Im}(m)$; then $g = \sigma g\sigma^{-1}$ and $\lambda_0 g = \lambda_0(\sigma g\sigma^{-1}) = (\lambda_0 g)\sigma^{-1}$. Since $\Lambda = \lambda_0 G$, this shows that $\lambda\sigma = \lambda\ \forall\lambda \in \Lambda$. ∎

2.9. The set

$$H_n^G(Y,y_0)=\big\{(D,m)\mid D\in(Y\setminus y_0)_*^{(n)}, m:\pi_1(Y\setminus D,y_0)\to G \text{ is an epimorphism which satisfies condition (1)}\big\} \tag{4}$$

is in bijective correspondence with the set of $G$-equivalence classes of pointed $G$-covers of $(Y,y_0)$ branched in $n$ points $\{[p:C\to Y,z_0]\}$ (cf. [20,22]). Given a pointed $G$-cover $(p:C\to Y,z_0)$ of $(Y,y_0)$, $p(z_0)=y_0$, the associated pair $(D,m)$, its monodromy invariant, consists of the branch locus $D$ of $p$ and of the homomorphism $m:\pi_1(Y\setminus D,y_0)\to G$ defined as follows. Let $C'=p^{-1}(Y\setminus D)$. For every loop $\alpha:I\to Y\setminus D$ based at $y_0$ let $z_0\alpha$ be the end point of the lifting of $\alpha$ in $C'$ with initial point $z_0$. Let $z_0\alpha=gz_0$, $g\in G$. Then $m([\alpha])=g$.

Definition 2.10. In the setup of §2.1, let $N(\lambda_0)=\{\sigma\in N_{S(\Lambda)}(G)\mid\lambda_0\sigma=\lambda_0\}$. Consider the left action of $N(\lambda_0)$ on $H_n^G(Y,y_0)$ defined by $\sigma*(D,m)=(D,\sigma m\sigma^{-1})$. We denote by $H_{n,\lambda_0}^{\Lambda,G}(Y,y_0)$ the quotient set

$$H_{n,\lambda_0}^{\Lambda,G}(Y,y_0)=H_n^G(Y,y_0)/N(\lambda_0).$$

Definition 2.11. Let $y_0\in Y$. A pointed cover $(f:X\to Y,x_0)$ of $(Y,y_0)$ which satisfies the conditions of Proposition 2.8 (i) for some $D\in(Y\setminus y_0)_*^{(n)}$ and a pair $(\varepsilon,m)$ is called *pointed* $(\Lambda,G)$*-cover* of $(Y,y_0)$ branched in $n$ points. The pair

$$(D,m^{N(\lambda_0)})=\big(D,\{\sigma m\sigma^{-1}\}_{\sigma\in N(\lambda_0)}\big)\in H_{n,\lambda_0}^{\Lambda,G}(Y,y_0)$$

is called its *monodromy invariant*.

Lemma 2.12. *Let $(p:C\to Y,z_0)$ be a pointed $G$-cover of $(Y,y_0)$ with monodromy invariant $(D,m)$ (cf. §2.9). Consider the action of $G$ on $\Lambda\times C$ defined by $g(\lambda,z)=(\lambda g^{-1},gz)$. Let $X=(\Lambda\times C)/G:=\Lambda\times^G C$. Let $\pi:\Lambda\times C\to X$ be the quotient morphism, let $f:X\to Y$ be the morphism defined by $f(\pi(\lambda,z))=p(z)$, and let $x_0=\pi(\lambda_0,z_0)$. Then $(f:X\to Y,x_0)$ is a pointed $(\Lambda,G)$-cover of $(Y,y_0)$ with monodromy invariant $(D,m^{N(\lambda_0)})$.*

Proof. Let $G(\lambda_0)=H$. Let $\rho:C\to C/H$ be the quotient morphism. One has $\Lambda\times^G C\cong C/H$. In fact, let us choose for every $\lambda\in\Lambda$ an element $a_\lambda\in G$ such that $\lambda=\lambda_0a_\lambda$; let $a_{\lambda_0}=1$. The morphisms $C\to\Lambda\times^G C$ and $\Lambda\times C\to C/H$, defined by

$$z\mapsto\pi(\lambda_0,z)\quad\text{and}\quad(\lambda,z)\mapsto\rho(a_\lambda z), \tag{5}$$

are respectively $H$-invariant and $G$-invariant. The induced morphisms $G/H\to\Lambda\times^G C$ and $\Lambda\times^G C\to G/H$ are inverse to each other. The curve $C$ is projective, smooth, and irreducible, so $X \cong C/H$ has the same properties. Consider the map

$$(6) \qquad \varepsilon : \Lambda \to f^{-1}(y_0) \quad \text{defined by} \quad \varepsilon(\lambda) = \pi(\lambda, z_0).$$

It is bijective since $G$ acts on $p^{-1}(y_0)$ transitively and freely. One has $\varepsilon(\lambda_0) = x_0$. Let $\varepsilon(\lambda) = \pi(\lambda, z_0)$ be an arbitrary point of $f^{-1}(y_0)$. Let $\alpha : I \to Y \setminus D$ be a loop based at $y_0$ and let $g = m([\alpha])$. Let $C' = p^{-1}(Y \setminus D)$. Lifting $\alpha$ in $\Lambda \times C$ with initial point $(\lambda, z_0)$, the end point is $(\lambda, z_0\alpha) = (\lambda, gz_0)$. Applying $\pi$, one obtains

$$\varepsilon(\lambda)\alpha = \pi(\lambda, gz_0) = \pi(\lambda g, z_0) = \varepsilon\big(\lambda m\big([\alpha]\big)\big).$$

Let $y \in Y \setminus D$ and let $z \in p^{-1}(y)$. The map $\Lambda \to f^{-1}(y)$ defined by $\lambda \mapsto \pi(\lambda, z)$ is bijective, so $f$ is unbranched at every $y \in Y \setminus D$. The monodromy homomorphism $m = m_{z_0}$ satisfies condition (1), so every $[\gamma_i]$ acts on the fiber $f^{-1}(y_0)$ as a nontrivial permutation, $i = 1, \dots, n$. Therefore, the branch locus of $f : X \to Y$ equals $D$. We see that $(f : X \to Y, x_0)$ satisfies the conditions of Proposition 2.8 (i) with $(\varepsilon, m)$, so its monodromy invariant is $(D, m^{N(\lambda_0)})$. ∎

Proposition 2.13. *The map* $[f : X \to Y, x_0] \mapsto (D, m^{N(\lambda_0)})$ *establishes a bijective correspondence between the set of equivalence classes of pointed* $(\Lambda, G)$*-covers of* $(Y, y_0)$ *branched in n points and the set* $H^{\Lambda,G}_{n,\lambda_0}(Y, y_0)$.

Proof. The map is well defined and injective by Proposition 2.8. It is surjective by Lemma 2.12. ∎

Corollary 2.14. *Let* $(f : X \to Y, x_0)$ *be a pointed cover of* $(Y, y_0)$ *branched in* $D \in (Y \setminus y_0)^{(n)}_*$. *It is a pointed* $(\Lambda, G)$*-cover of* $(Y, y_0)$ *and satisfies condition* (i) *of Proposition* 2.8 *with some pair* $(\varepsilon, m)$ *if and only if it is equivalent to*

$$\big(\Lambda \times^G C, \pi(\lambda_0, z_0)\big) \to (Y, y_0)$$

*where* $(p : C \to Y, z_0)$ *is a pointed* $G$*-cover of* $(Y, y_0)$ *with monodromy invariant* $(D, m)$. *A pointed* $G$*-cover of* $(Y, y_0)$ *has this property if and only if its monodromy invariant equals* $(D, m_1)$ *where* $m_1 = \sigma m \sigma^{-1}$ *for some* $\sigma \in N(\lambda_0)$. *The set of* $G$*-equivalence classes of such pointed* $G$*-covers of* $(Y, y_0)$ *has cardinality* $|N(\lambda_0)|$.

Proof. This follows from Proposition 2.8. ∎

Proposition 2.15. *Let* $D = \{b_1, \dots, b_n\} \subset Y$. *Let* $f : X \to Y$ *be a cover branched in* $D$. *The following conditions are equivalent:*

(i) *There is a point* $y_0 \in Y \setminus D$, *a bijection* $\varepsilon : \Lambda \to f^{-1}(y_0)$, *and an epimorphism* $m : \pi_1(Y \setminus D, y_0) \to G$ *such that*

$$\varepsilon\big(\lambda m\big([\alpha]\big)\big) = \varepsilon(\lambda)\alpha \quad \forall \lambda \in \Lambda,\ \forall [\alpha] \in \pi_1(Y \setminus D, y_0).$$

(ii) *The condition of* (i) *holds for every* $y \in Y \setminus D$ *for some pair* $\varepsilon_1 : \Lambda \to f^{-1}(y)$, $m_1 : \pi_1(Y \setminus D, y) \to G$.

*Suppose that* $f_1 : X_1 \to Y$ *and* $f_2 : X_2 \to Y$ *are two covers branched in* $D$. *Suppose that* $f_1$ *and* $f_2$ *satisfy condition (i) respectively for some* $y_1 \in Y \setminus D$ *and a pair* $(\varepsilon_1, m_1)$ *and for some* $y_2 \in Y \setminus D$ *and a pair* $(\varepsilon_2, m_2)$. *Then* $f_1$ *is equivalent to* $f_2$ *if and only if there is a path* $\gamma : I \to Y \setminus D$, $\gamma(0) = y_1$, $\gamma(1) = y_2$, *and a* $\sigma \in N_{S(\Lambda)}(G)$ *such that*

$$m_2\big([\alpha]\big) = \sigma m_1\big([\gamma \cdot \alpha \cdot \gamma^-]\big)\sigma^{-1} \quad \forall [\alpha] \in \pi_1(Y \setminus D, y_2).$$

Proof. (i)⇒(ii): Let $y_1 \in Y \setminus D$. Let $\gamma : I \to Y \setminus D$ be a path with $\gamma(0) = y_0$, $\gamma(1) = y_1$. Let $\varepsilon_1 : \Lambda \to f^{-1}(y_1)$ be the bijection defined by $\varepsilon_1(\lambda) = \varepsilon(\lambda)\gamma$. Let $m_1 : \pi_1(Y \setminus D, y_1) \to G$ be the epimorphism $m_1 = m^\gamma$ defined by $m_1([\beta]) = m([\gamma \cdot \beta \cdot \gamma^-])$. Then

$$\varepsilon_1\big(\lambda m_1\big([\beta]\big)\big) = \varepsilon\big(\lambda m\big([\gamma \cdot \beta \cdot \gamma^-]\big)\big)\gamma = \varepsilon(\lambda)\gamma \cdot \beta \cdot \gamma^- \cdot \gamma = \varepsilon_1(\lambda)\beta.$$

Given $f_1$ and $f_2$, let $\gamma : I \to Y \setminus D$ be a path such that $\gamma(0) = y_1$, $\gamma(1) = y_2$. Let $\varepsilon_1' : \Lambda \to f_1^{-1}(y_2)$ and $m_1' = m_1^\gamma$ be the pair obtained from $(\varepsilon_1, m_1)$ by $\gamma$ as above. Then by Lemma 2.7, $f_1$ is equivalent to $f_2$ if and only if $m_2 = \sigma m_1^\gamma \sigma^{-1}$ for some $\sigma \in N_{S(\Lambda)}(G)$. ■

2.16. Let $D = \{b_1, \dots, b_n\} \subset Y$. Given two points $y_1, y_2 \in Y \setminus D$ and two homomorphisms $m_1 : \pi_1(Y \setminus D, y_1) \to G$ and $m_2 : \pi_1(Y \setminus D, y_2) \to G$, $G \subset S(\Lambda)$, we write $m_1 \sim_N m_2$ if there is a path $\gamma : I \to Y \setminus D$ with $\gamma(0) = y_1$, $\gamma(1) = y_2$, and an element $\sigma \in N_{S(\Lambda)}(G)$, such that

$$m_2\big([\alpha]\big) = \sigma m_1\big([\gamma \cdot \alpha \cdot \gamma^-]\big)\sigma^{-1} \quad \forall [\alpha] \in \pi_1(Y \setminus D, y_2).$$

This is a relation of equivalence. Given a homomorphism $m : \pi_1(Y \setminus D, y) \to G$, we denote by $[m]$ its equivalence class. It is clear that if $m$ is an epimorphism which satisfies condition (1) relative to the base point $y$, every other homomorphism of $[m]$ has these properties relative to its base point. Suppose a cover $f : X \to Y$ branched in $D$ satisfies condition (i) of Proposition 2.15 with $(\varepsilon, m)$ for some $y_0 \in Y$. Then the set of all possible epimorphisms $m_1 : \pi_1(Y \setminus D, y) \twoheadrightarrow G$, $y \in Y \setminus D$ as in Proposition 2.15 (ii) equals the equivalence class $[m]$ (apply Proposition 2.15 to $f_1 = f = f_2$, $y_1 = y_0$, $y_2 = y$). We denote by $[m]_f$ the equivalence class with respect to $\sim_N$ determined by $f : X \to Y$.

Definition 2.17. In the setup of §2.1, we denote by $H_n^{\Lambda,G}(Y)$ the set

$$H_n^{\Lambda,G}(Y) = \big\{\big(D, [m]\big) \mid D \in Y_*^{(n)}, \text{ where } m : \pi_1(Y \setminus D, y) \to G \text{ is surjective and satisfies condition (1)}\big\}.$$

DEFINITION 2.18. A cover $f : X \to Y$ branched in a set $D$ of $n$ points, which satisfies condition (i) of Proposition 2.15 for some $y_0 \in Y \setminus D$ and some pair $(\varepsilon, m)$, is called $(\Lambda, G)$-cover of $Y$ branched in $n$ points. The pair $(D, [m]_f) \in H_n^{\Lambda,G}(Y)$ is called the monodromy invariant of the cover (cf. §2.16).

PROPOSITION 2.19. *The map $[f : X \to Y] \mapsto (D, [m]_f)$ establishes a bijective correspondence between the set of equivalence classes of $(\Lambda, G)$-covers of $Y$ branched in $n$ points and the set $H_n^{\Lambda,G}(Y)$. If the centralizer of $G$ in $S(\Lambda)$ is trivial, $Z_{S(\Lambda)}(G) = \{1\}$, then the covering isomorphism between every two $(\Lambda, G)$-covers with the same monodromy invariant is unique.*

PROOF. The map is well defined and injective by Proposition 2.15. We claim that it is surjective. Let $(D, [m]) \in H_n^{\Lambda,G}(Y)$ where $m : \pi_1(Y \setminus D, y_0) \twoheadrightarrow G$ with $y_0 \in Y \setminus D$ satisfies condition (1). Let $(p : C \to Y, z_0)$ be a pointed $G$-cover of $(Y, y_0)$ with monodromy invariant $(D, m)$. Then the cover $f : X \to Y$ constructed in Lemma 2.12 is a $(\Lambda, G)$-cover with monodromy invariant $(D, [m])$. The last statement of the proposition was proved in Lemma 2.7. ∎

PROPOSITION 2.20. *Let $f : X \to Y$ be a $(\Lambda, G)$-cover with monodromy invariant $(D, [m]) \in H_n^{\Lambda,G}(Y)$. Then $f$ has a structure of a $(\Lambda, G_1)$-cover if and only if $G_1 = \varphi G \varphi^{-1}$ for some $\varphi \in S(\Lambda)$ and the corresponding monodromy invariant is $(D, \varphi[m]\varphi^{-1}) \in H_n^{\Lambda,G_1}(Y)$.*

PROOF. Let $y_0 \in Y \setminus D$ and let $\varepsilon : \Lambda \xrightarrow{\sim} f^{-1}(y_0)$, $m : \pi_1(Y \setminus D, y_0) \twoheadrightarrow G$ satisfy

$$\varepsilon\big(\lambda m([\alpha])\big) = \varepsilon(\lambda)\alpha \quad \forall \lambda \in \Lambda,\ \forall [\alpha] \in \pi_1(Y \setminus D, y_0).$$

Let $\varphi \in S(\Lambda)$ and let $G_1 = \varphi G \varphi^{-1}$. Let $\varepsilon_1 : \Lambda \to f^{-1}(y_0)$ be the bijection defined by $\varepsilon_1(\lambda) = \varepsilon(\lambda\varphi)$ and let $m_1 = \varphi m \varphi^{-1} : \pi_1(Y \setminus D, y_0) \to G_1$. Then $\forall [\alpha] \in \pi_1(Y \setminus D, y_0)$,

$$\varepsilon_1\big(\lambda m_1([\alpha])\big) = \varepsilon_1\big(\lambda\varphi m([\alpha])\varphi^{-1}\big) = \varepsilon\big(\lambda\varphi m([\alpha])\big) = \varepsilon(\lambda\varphi)\alpha = \varepsilon_1(\lambda)\alpha;$$

hence, $f : X \to Y$ is a $(\Lambda, G_1)$-cover with monodromy invariant $(D, \varphi[m]\varphi^{-1})$. Vice versa, let $\varepsilon_1 : \Lambda \xrightarrow{\sim} f^{-1}(y_0)$ and $m_1 : \pi_1(Y \setminus D, y_0) \twoheadrightarrow G_1$ satisfy

$$\varepsilon_1\big(\lambda m_1([\alpha])\big) = \varepsilon_1(\lambda)\alpha \quad \forall \lambda \in \Lambda,\ \forall [\alpha] \in \pi_1(Y \setminus D, y_0).$$

Let $\varphi \in S(\Lambda)$ satisfy $\varepsilon_1(\lambda) = \varepsilon(\lambda\varphi)$ $\forall \lambda \in \Lambda$. Then

$$\varepsilon\big((\lambda\varphi)\varphi^{-1}m_1([\alpha])\varphi\big) = \varepsilon_1\big(\lambda m_1([\alpha])\big) = \varepsilon_1(\lambda)\alpha = \varepsilon(\lambda\varphi)\alpha = \varepsilon\big((\lambda\varphi)m([\alpha])\big),$$

so $m_1([\alpha]) = \varphi m([\alpha])\varphi^{-1}$ $\forall [\alpha] \in \pi_1(Y \setminus D, y_0)$. This implies that $G_1 = \varphi G \varphi^{-1}$ since $m$ and $m_1$ are epimorphisms by assumption. ∎

PROPOSITION 2.21. *The following two conditions for a cover $f : X \to Y$ are equivalent:*

(i) *$(f : X \to Y, x_0)$ is a pointed cover of $(Y, y_0)$ (cf. Definition 2.4) and $f : X \to Y$ is a $(\Lambda, G)$-cover.*

(ii) *Let $\lambda_0 \in \Lambda$. Then $(f : X \to Y, x_0)$ is a pointed $(\Lambda, G)$-cover of $(Y, y_0)$ according to Definition 2.11.*

PROOF. Let $D$ be the branch locus of $f$. Condition (i) implies that $y_0 \notin D$ and there exists a bijection $\varepsilon : \Lambda \to f^{-1}(y_0)$ and an epimorphism $m : \pi_1(Y \setminus D, y_0) \to G$ such that $\varepsilon(\lambda m([\alpha])) = \varepsilon(\lambda)\alpha$ $\forall \lambda \in \Lambda$ and $\forall [\alpha] \in \pi_1(Y \setminus D, y_0)$. The group $G$ acts transitively on $\Lambda$. One replaces the pair $(\varepsilon, m)$ with the pair $(\varepsilon', m')$, where $\varepsilon'(\lambda) = \varepsilon(\lambda g)$, $m' = gmg^{-1}$ for an appropriate $g \in G$, so that $(\varepsilon', m')$ satisfies the additional condition $\varepsilon'(\lambda_0) = x_0$. ■

REMARK 2.22. Choosing a marked element $\lambda_0 \in \Lambda$ and imposing $\varepsilon(\lambda_0) = x_0$ is a normalizing condition for pointed degree $d$ covers of $(Y, y_0)$ whose monodromy group equals $G$. This serves for defining the monodromy invariant $(D, m^{N(\lambda_0)}) \in H^{\Lambda,G}_{n,\lambda_0}(Y, y_0)$ so that the bijection of Proposition 2.13 holds. The choice of another marked element $\lambda_1 \in \Lambda$ is discussed in Proposition 4.23.

## 3. SMOOTH, PROPER FAMILIES OF COVERS WITH A FIXED MONODROMY GROUP

DEFINITION 3.1. Let $X$ and $S$ be algebraic varieties.

(i) A morphism $f : X \to Y \times S$ is called a smooth, proper family of covers of $Y$ branched in $n$ points if $\pi_2 \circ f : X \to S$ is a proper, smooth morphism, for every $s \in S$ the fiber $X_s$ is irreducible, and $f_s : X_s \to Y$ is a cover branched in $n$ points. Two families of this type $f_1 : X_1 \to Y \times S$ and $f_2 : X_2 \to Y \times S$ are called equivalent if there exists an isomorphism $h : X_1 \to X_2$ such that $f_1 = f_2 \circ h$.

(ii) If, furthermore, $f_s : X_s \to Y$ is a $(\Lambda, G)$-cover, $\forall s \in S$ the morphism $f : X \to Y \times S$ is called a smooth, proper family of $(\Lambda, G)$-covers of $Y$ branched in $n$ points.

Under the assumptions of (i), the morphism $f : X \to Y \times S$ is finite, surjective, and flat [22, Prop. 2.6]. If $S$ is connected, then all covers $f_s : X_s \to Y$ have the same degree $d$, where $d$ is the rank of the locally free sheaf $f_* \mathcal{O}_X$. It is clear that two families are equivalent if and only if there is an $S$-isomorphism $h : X_1 \to X_2$, which is a covering isomorphism over $Y$ $\forall s \in S$, i.e., $f_{1s} = f_{2s} \circ h_s$ $\forall s \in S$.

DEFINITION 3.2. Let $y_0 \in Y$.

(i) A smooth, proper family of pointed covers of $(Y, y_0)$ branched in $n$ points is a pair $(f : X \to Y \times S, \eta : S \to X)$ of morphisms of algebraic varieties such that $f : X \to Y \times S$ satisfies the conditions of Definition 3.1 (i), $\forall s \in S$ the cover $f_s : X_s \to Y \times \{s\}$ is unbranched at $(y_0, s)$, and $\eta(s) \in f^{-1}(y_0, s)$. Two families of this type $(f_1 : X_1 \to Y \times S, \eta_1)$ and $(f_2 : X_2 \to Y \times S, \eta_2)$ are called equivalent if there exists an isomorphism $h : X_1 \to X_2$ such that $f_1 = f_2 \circ h$ and $\eta_2 = h \circ \eta_1$.

(ii) If, furthermore, $f_s : X_s \to Y \times \{s\}$ is a $(\Lambda, G)$-cover, $\forall s \in S$ the pair $(f : X \to Y \times S,\ \eta : S \to X)$ is called a smooth, proper family of pointed $(\Lambda, G)$-covers of $(Y, y_0)$ branched in $n$ points.

Proposition 3.3. *Let $y_0 \in Y$ and let $\lambda_0 \in \Lambda$. Let $(f : X \to Y, \eta : S \to X)$ be a smooth, proper family of pointed $(\Lambda, G)$-covers of $(Y, y_0)$ branched in $n$ points. Then for every $s \in S$ the pointed cover $(f_s : X_s \to Y, \eta(s))$ of $(Y, y_0)$ satisfies the conditions of Definition 2.11 (we identify $Y \times \{s\}$ with $Y$).*

Proof. This follows from Proposition 2.21. ∎

3.4. Let $D = \{b_1, \dots, b_n\} \subset Y \setminus y_0$, and let $\bar{U}_1, \dots, \bar{U}_n$ be disjoint embedded closed disks in $Y \setminus y_0$ such that $b_i \in U_i$ $\forall i$, where $U_i$ is the interior of $\bar{U}_i$. Let $N_D(U_1, \dots, U_n) \subset (Y \setminus y_0)^{(n)}_*$ be the open neighborhood of $D$ in the complex topology (i.e., that of $((Y \setminus y_0)^{(n)}_*)^{\mathrm{an}}$), which consists of $E = \{y_1, \dots, y_n\}$ such that $y_i \in U_i$ for every $i$. The inclusion $Y \setminus \bigcup_{i=1}^n U_i \hookrightarrow Y \setminus D$ is a deformation retract, so for every homomorphism $m : \pi_1(Y \setminus D, y_0) \to G$ and every $E \in N_D(U_1, \dots, U_n)$ there is a unique homomorphism $m(E) : \pi_1(Y \setminus E, y_0) \to G$ such that the following diagram commutes:

$$\begin{array}{ccccc} \pi_1(Y \setminus D, y_0) & \xleftarrow{\cong} & \pi_1(Y \setminus \bigcup_{i=1}^n U_i, y_0) & \xrightarrow{\cong} & \pi_1(Y \setminus E, y_0) \\ & {\scriptstyle m}\searrow & \downarrow & \swarrow{\scriptstyle m(E)} & \\ & & G & & \end{array} \tag{7}$$

Given a path $\gamma : I \to Y \setminus E$, we denote by $[\gamma]_E$ its homotopy class in $Y \setminus E$. We denote by $N_{(D,m)}(U_1, \dots, U_n)$ the subset of $H^G_n(Y, y_0)$

$$N_{(D,m)}(U_1, \dots, U_n) = \big\{\big(E, m(E)\big) \mid E \in N_D(U_1, \dots, U_n)\big\}. \tag{8}$$

The set $H^G_n(Y, y_0)$ has a structure of affine algebraic variety, the map $\delta : H^G_n(Y, y_0) \to (Y \setminus y_0)^{(n)}_*$, defined by $\delta(D, m) = D$, is an étale cover, the sets $N_{(D,m)}(U_1, \dots, U_n)$ form an open sets basis of the topology of $H^G_n(Y, y_0)^{\mathrm{an}}$, and every open set $N_D(U_1, \dots, U_n) \subset (Y \setminus y_0)^{(n)}_*$ is evenly covered with respect to the topological covering map $|\delta^{\mathrm{an}}|$:

$$\delta^{-1}\big(N_D(U_1, \dots, U_n)\big) = \bigsqcup_m N_{(D,m)}(U_1, \dots, U_n) \tag{9}$$

(cf. [20, Sec. 1]).

LEMMA 3.5. *Let $f : X \to Y \times S$ be a smooth, proper family of covers of $Y$ branched in $n$ points. Suppose that $f_{s_0} : X_{s_0} \to Y$ is a $(\Lambda, G)$-cover for some $s_0 \in S$. Then there exists a neighborhood $V \subset |S^{\mathrm{an}}|$ of $s_0$ such that $f_s : X_s \to Y$ is a $(\Lambda, G)$-cover $\forall s \in V$.*

PROOF. Let $D = \{b_1, \dots, b_n\}$ be the branch locus of $f_{s_0}$, let $y_0 \in Y \setminus D$, and let $\varepsilon : \Lambda \to f_{s_0}^{-1}(y_0)$, $m : \pi_1(Y \setminus D, y_0) \twoheadrightarrow G$ satisfy $\varepsilon(\lambda m([\alpha])) = \varepsilon(\lambda)\alpha$ $\forall \lambda \in \Lambda$ and $\forall [\alpha] \in \pi_1(Y \setminus D, y_0)$. Let $B \subset Y \times S$ be the branch locus of $f$. The map $\beta : S \to Y_*^{(n)}$ defined by $\beta(s) = B_s$ is a morphism [22, Prop. 2.6], so $|\beta^{\mathrm{an}}| : |S^{\mathrm{an}}| \to |(Y_*^{(n)})^{\mathrm{an}}|$ is a continuous map. Let $N_D(U_1, \dots, U_n)$ be a neighborhood of $D = \beta(s_0)$ as in §3.4. Let $V_1$ be a neighborhood of $s_0$ in $|S^{\mathrm{an}}|$ such that $\beta(V_1) \subset N_D(U_1, \dots, U_n)$. The restriction $|X^{\mathrm{an}}| \setminus f^{-1}(B) \xrightarrow{|f^{\mathrm{an}}|} |(Y \times S)^{\mathrm{an}}| \setminus B$ is a topological covering map [22, Prop. 2.6] and $(Y \times S)^{\mathrm{an}} \cong Y^{\mathrm{an}} \times S^{\mathrm{an}}$ [28, §1.2]. The complex space $S^{\mathrm{an}}$ is locally connected (cf. [12, Ch. 9, §3 (1)]); therefore, there is an embedded open disk $U \subset Y \setminus \bigcup_{i=1}^{n} \bar{U}_i$, $y_0 \in U$, and a connected neighborhood $V$ of $s_0$, such that $V \subset V_1$, $U \times V \subset Y \times S \setminus B$, and $f^{-1}(U \times V)$ is a disjoint union of connected open sets homeomorphic to $U \times V$. Denote these open sets by $W_\lambda$, $\lambda \in \Lambda$, where $W_\lambda \ni \varepsilon(\lambda)$. For every $s \in V$ we define a bijection $\varepsilon_s : \Lambda \to f^{-1}(y_0, s)$ and an epimorphism $m_s : \pi_1(Y \setminus B_s, y_0) \to G$ by

$$\varepsilon_s(\lambda) = f^{-1}(y_0, s) \cap W_\lambda, \quad m_s = m\big(\beta(s)\big) \tag{10}$$

(cf. §3.4). We claim that $(\varepsilon_s, m_s)$ satisfies condition (i) of Proposition 2.15 for $f_s : X_s \to Y$ (we identify $Y \times \{s\}$ with $Y$). Let $\alpha$ be a loop in $Y \setminus \bigcup_{i=1}^{n} U_i$ based at $y_0$. Let $g = m([\alpha]_D)$. We claim that $\varepsilon_s(\lambda)\alpha = \varepsilon_s(\lambda g)$ $\forall \lambda \in \Lambda$ and $\forall s \in V$. Consider the homotopy $F : [0,1] \times V \to Y^{\mathrm{an}} \times S^{\mathrm{an}} \setminus B$ defined by $F(t, s) = (\alpha(t), s)$. Let $\lambda \in \Lambda$. The restriction $f|_{W_\lambda} : W_\lambda \to U \times V$ is a homeomorphism. Let us denote by $\widetilde{F}_\lambda(0) : \{0\} \times V \to |X^{\mathrm{an}} \setminus f^{-1}(B)|$ the composition $(0, s) \overset{F}{\mapsto} (y_0, s) \mapsto (f|_{W_\lambda})^{-1}(y_0, s) = \varepsilon_s(\lambda)$. By the covering homotopy property (cf. [30, Ch. 2, §3, Thm. 3]), there is a unique continuous lifting $\widetilde{F}_\lambda$ of $F$ which extends $\widetilde{F}_\lambda(0)$

$$\begin{array}{ccc} & & X^{\mathrm{an}} \setminus f^{-1}(B) \\ & \overset{\widetilde{F}_\lambda}{\nearrow} & \downarrow \\ [0,1] \times V & \xrightarrow{F} & Y^{\mathrm{an}} \times S^{\mathrm{an}} \setminus B. \end{array}$$

One has $\widetilde{F}_\lambda(1, s_0) = \varepsilon(\lambda)\alpha = \varepsilon(\lambda g) \in W_{\lambda g}$. The image $\widetilde{F}_\lambda(\{1\} \times V)$ is a connected component of $f^{-1}(\{y_0\} \times V)$, so $\widetilde{F}_\lambda(\{1\} \times V) \subset W_{\lambda g}$. This implies that $\varepsilon_s(\lambda)\alpha = \varepsilon_s(\lambda g)$ $\forall s \in V$. Let $s \in V$. By §3.4, we have

$$g = m\big([\alpha]_D\big) = m\big(\beta(s)\big)\big([\alpha]_{\beta(s)}\big) = m_s\big([\alpha]_{B_s}\big).$$

Every homotopy class of $\pi_1(Y \setminus B_s, y_0)$ may be represented by a loop $\alpha$ in $Y \setminus \bigcup_{i=1}^{n} U_i$ based at $y_0$, so

$$\varepsilon_s\big(\lambda m_s([\alpha]_{B_s})\big) = \varepsilon_s(\lambda)\alpha \quad \forall\lambda \in \Lambda,\ \forall[\alpha] \in \pi_1(Y \setminus B_s, y_0). \qquad \blacksquare$$

Proposition 3.6. *Let $f : X \to Y \times S$ be a smooth , proper family of covers of $Y$ branched in $n$ points. Suppose that $S$ is connected. Suppose that $f_{s_0} : X_{s_0} \to Y$ is a $(\Lambda, G)$-cover for some $s_0 \in S$. Then $f : X \to Y \times S$ is a smooth, proper family of $(\Lambda, G)$-covers of $Y$.*

Proof. Let $d = |\Lambda|$. Every cover $f_s : X_s \to Y$, $s \in S$, is of degree $d$ since $S$ is connected. We have to prove that $f_s : X_s \to Y$ is a $(\Lambda, G)$-cover $\forall s \in S$. For every transitive subgroup $H \subset S(\Lambda)$, let $S^H$ be the set of points $s \in S$ such that $f_s : X_s \to Y$ is a $(\Lambda, H)$-cover. It is clear that $S = \bigcup_H S^H$. By Lemma 3.5, $S^H$ is an open set in $|S^{\mathrm{an}}|$ for every $H$. By Proposition 2.20, given two transitive subgroups $H$ and $K$ of $S(\Lambda)$, the following alternative holds: if $H$ and $K$ are conjugated in $S(\Lambda)$, then $S^H = S^K$; otherwise, $S^H \cap S^K = \emptyset$. The topological space $|S^{\mathrm{an}}|$ is connected (cf. [28, Cor. 2.6]); therefore, $S = S^G$. $\blacksquare$

We denote by $\mathrm{Var}_{\mathbb{C}}$ the category of algebraic varieties and by (Sets) the category of sets.

Definition 3.7. Let $S$ be an algebraic variety. We denote by $\mathcal{H}^{\Lambda,G}_{Y,n}(S)$ the set of equivalence classes $[f : X \to Y \times S]$ of smooth, proper families of $(\Lambda, G)$-covers of $Y$ branched in $n$ points, parameterized by $S$ (cf. Definition 3.1). We denote by $\mathcal{H}^{\Lambda,G}_{(Y,y_0),n}(S)$ the set of equivalence classes $[f : X \to Y \times S, \eta : S \to X]$ of smooth, proper families of pointed $(\Lambda, G)$-covers of $(Y, y_0)$ branched in $n$ points parameterized by $S$ (cf. Definition 3.2).

3.8. Let $u : T \to S$ be a morphism of algebraic varieties. Given a smooth, proper morphism $X \to S$ of reduced, separated schemes of finite type over $\mathbb{C}$, the pullback morphism $X_T = X \times_S T \to T$ is smooth and proper. The scheme $X \times_S T$ is reduced since $T$ is reduced (cf. [25, p. 184]). Hence, $X \times_S T$ is isomorphic to the closed algebraic subvariety of $X \times T$ whose set of points is the set-theoretical fiber product $X(\mathbb{C}) \times_{S(\mathbb{C})} T(\mathbb{C})$.

Let $f : X \to Y \times S$ be a smooth, proper family of covers of $Y$ branched in $n$ points. Let $u : T \to S$ be a morphism of algebraic varieties. Let $X_T \to T$ be the pullback of $\pi_2 \circ f : X \to S$. By the above argument, $X_T = \{(x,t) \mid \pi_2 \circ f(x) = u(t)\} \subset X \times T$. Let $f_T : X_T \to Y \times T$ and $h : X_T \to X$ be the morphisms defined respectively by $f_T(x,t) = (\pi_1 \circ f(x), t)$ and $h(x,t) = x$. One has the following commutative diagram of morphisms:

$$\begin{array}{ccccc} X_T & \xrightarrow{f_T} & Y \times T & \xrightarrow{p_2} & T \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\times u} & & \downarrow{\scriptstyle u} \\ X & \xrightarrow{f} & Y \times S & \xrightarrow{\pi_2} & S \end{array} \tag{11}$$

in which the composed diagram and the right square are Cartesian. Therefore, the left square is Cartesian as well [11, Prop. 4.16].

Given a smooth, proper family of $(\Lambda, G)$-covers $f : X \to Y \times S$ branched in $n$ points, the pullback morphism $f_T : X_T \to Y \times T$ is a smooth, proper family of $(\Lambda, G)$-covers of $Y$ branched in $n$ points. In fact, $\forall t \in T$ the cover $(f_T)_t : (X_T)_t \to Y$ is equivalent to the cover $f_{u(t)} : X_{u(t)} \to Y$, so the conditions of Definition 3.1 are satisfied. Furthermore, the pullbacks of equivalent families are equivalent. This defines a moduli functor $\mathcal{H}^{\Lambda,G}_{Y,n} : \mathrm{Var}_{\mathbb{C}} \to (\mathrm{Sets})$.

Given a smooth, proper family $(f : X \to Y \times S, \eta : S \to X)$ of pointed $(\Lambda, G)$-covers of $(Y, y_0)$ branched in $n$ points and a morphism $u : T \to S$, one defines the pullback family as $(f_T : X_T \to Y \times T, \eta_T : T \to X_T)$, where $\eta_T$ is the morphism defined by $\eta_T(t) = (\eta(u(t)), t)$. This is a smooth, proper family of pointed $(\Lambda, G)$-covers of $(Y, y_0)$. This defines, as above, a moduli functor $\mathcal{H}^{\Lambda,G}_{(Y,y_0),n} : \mathrm{Var}_{\mathbb{C}} \to (\mathrm{Sets})$.

3.9. A morphism $p : \mathcal{C} \to Y \times S$ of algebraic varieties is called a smooth, proper family of $G$-covers of $Y$ branched in $n$ points if

(i) $\pi_2 \circ p : \mathcal{C} \to S$ is a proper, smooth morphism with irreducible fibers;

(ii) $G$ acts on $\mathcal{C}$ on the left by automorphisms, $p : \mathcal{C} \to Y \times S$ is $G$-invariant, and $\forall s \in S$ the action of $G$ on $\mathcal{C}_s$ is faithful, $\bar{p}_s : \mathcal{C}_s/G \to Y \times \{s\}$ is an isomorphism, and $p_s : \mathcal{C}_s \to Y \times \{s\}$ is branched in $n$ points.

Let $y_0 \in Y$. If furthermore $p_s : \mathcal{C}_s \to Y \times \{s\}$ is unbranched at $(y_0, s)$ $\forall s \in S$ and there exists a morphism $\zeta : S \to \mathcal{C}$ such that $p \circ \zeta(s) = (y_0, s)$ $\forall s \in S$, then $(p : \mathcal{C} \to Y \times S, \zeta)$ is called a smooth, proper family of pointed $G$-covers of $(Y, y_0)$ branched in $n$ points (cf. [22, Sec. 3]).

In the next two propositions, we extend the construction of Lemma 2.12 to families.

PROPOSITION 3.10. *Let $p : \mathcal{C} \to Y \times S$ be a smooth, proper family of $G$-covers of $Y$ branched in $n$ points.*

*Consider the left action of $G$ on $\Lambda \times \mathcal{C}$ defined by $g(\lambda, z) = (\lambda g^{-1}, gz)$. Let $X = (\Lambda \times \mathcal{C})/G := \Lambda \times^G \mathcal{C}$, let $\pi : \Lambda \times \mathcal{C} \to X$ be the quotient map, and let $f : X \to Y \times S$ be the map $f(\pi(\lambda, z)) = p(z)$. Then $f : X \to Y \times S$ is a smooth, proper family of $(\Lambda, G)$-covers of $Y$. The branch loci of $p$ and $f$ coincide.*

*Let $X_1 = \mathcal{C}/G(\lambda_0)$, let $\rho : \mathcal{C} \to X_1$ be the quotient map, and let $f_1 : X_1 \to Y \times S$ be the map $f_1(\rho(z)) = p(z)$. Then $f_1 : X_1 \to Y \times S$ is a smooth, proper family of covers of $Y$ equivalent to $f : X \to Y \times S$.*

Proof. The group $G$ acts by covering automorphisms of the finite morphism $\Lambda \times \mathcal{C} \to Y \times S$ defined by $(\lambda, z) \mapsto p(z)$. Therefore, $X = (\Lambda \times \mathcal{C})/G$ has a structure of quotient algebraic variety and the maps $\pi : \Lambda \times \mathcal{C} \to X$ and $f : X \to Y \times S$ are finite morphisms [29, Ch. III, Prop. 19]. The morphism $\pi_2 \circ f : X \to S$ is a composition of a proper and a finite morphism, so it is proper. For every $s \in S$ its scheme-theoretical fiber $X_s$ is isomorphic to the quotient $(\Lambda \times \mathcal{C}_s)/G$ since the formation of quotients by $G$ commutes with base change (cf. [23, Prop. A.7.1.3]). Therefore, $X_s$ is smooth and irreducible. Let $H = G(\lambda_0)$. Similar statements hold for $X_1 = \mathcal{C}/H$, $\rho : \mathcal{C} \to X_1$, and $f_1 : X_1 \to Y \times S$. The two families of covers of $Y$, $f : X \to Y \times S$ and $f_1 : X_1 \to Y \times S$, are equivalent. In fact, the morphisms $\mathcal{C} \to \Lambda \times^G \mathcal{C} = X$ and $\Lambda \times \mathcal{C} \to \mathcal{C}/H = X_1$, defined as in (5), induce morphisms $X_1 \to X$ and $X \to X_1$ over $Y \times S$, given respectively by $\rho(z) \mapsto \pi(\lambda_0, z)$ and $\pi(\lambda, z) = \rho(a_\lambda z)$, which are inverse to each other. We claim that $\pi_2 \circ f$ and $\pi_2 \circ f_1$ are flat morphisms. Let $x \in X_1$, let $\pi_2 \circ f_1(x) = s$, and let $\rho^{-1}(x)$ be the scheme-theoretical fiber of $\rho$. The restriction $\rho_s : \mathcal{C}_s \to (X_1)_s$ is a surjective morphism of smooth, irreducible, projective curves, so $\dim_{\mathbb{C}} H^0(\mathcal{O}_{\rho_s^{-1}(x)}) = |H| = |G|/|\Lambda|$. The schemes $\rho^{-1}(x)$ and $\rho_s^{-1}(x)$ coincide, so $\dim_{\mathbb{C}} H^0(\mathcal{O}_{\rho^{-1}(x)}) = |G|/|\Lambda|$ for every $x \in X_1$. This implies that the coherent sheaf $\rho_* \mathcal{O}_{\mathcal{C}}$ is locally free (cf. [27, Ch. 2, §5, Lem. 1]); therefore, $\rho : \mathcal{C} \to X_1$ is flat. It is moreover faithfully flat since $\rho(\mathcal{C}) = X_1$. By hypothesis, $\pi_2 \circ p : \mathcal{C} \to S$ is flat; therefore, $\pi_2 \circ f_1 : X_1 \to S$ is flat (cf. [25, p. 46]). As we saw above, the scheme-theoretical fiber $(X_1)_s$ over every (closed) point $s \in S$ is smooth. Therefore, $\pi_2 \circ f_1 : X_1 \to S$ is smooth at every closed point of $X_1$ (cf. [2, Ch. VII, Thm. 1.8]). The points of the scheme $X_1$ at which $\pi_2 \circ f_1$ is smooth form an open subset. This open subset contains every closed point of $X_1$; therefore, it coincides with $X_1$ (cf. [11, Prop. 3.35]). The smoothness of $\pi_2 \circ f_1 : X_1 \to S$ implies the smoothness of $\pi_2 \circ f : X \to S$ since there is a $Y \times S$-isomorphism between $X$ and $X_1$. For every $s \in S$ one has $X_s \cong (\Lambda \times \mathcal{C}_s)/G$, so the morphism $f_s : X_s \to Y$ is a $(\Lambda, G)$-cover of $Y$ whose branch locus coincides with that of $p_s : \mathcal{C}_s \to Y$ (cf. Lemma 2.12). Therefore, $f : X \to Y \times S$ is a smooth, proper family of $(\Lambda, G)$-covers of $Y$ branched in $n$ points and it is equivalent to $f_1 : X_1 \to Y \times S$ as we saw above. ■

Proposition 3.11. *Let $y_0 \in Y$. Let $(p : \mathcal{C} \to Y \times S, \zeta)$ be a smooth, proper family of pointed $G$-covers of $(Y, y_0)$ branched in $n$ points.*

*Let $X = \Lambda \times^G \mathcal{C}$, $f : X \to Y \times S$ be as in Proposition 3.10. Let $\eta : S \to X$ be the morphism defined by $\eta(s) = \pi(\lambda_0, \zeta(s))$. Then $(f : X \to Y \times S, \eta)$ is a smooth, proper family of pointed $(\Lambda, G)$-covers of $(Y, y_0)$. The branch loci of $p$ and $f$ coincide.*

*For every $s \in S$, if $(D_s, m_s)$ is the monodromy invariant of $(p_s : \mathcal{C}_s \to Y, \zeta(s))$, then $(D_s, m_s^{N(\lambda_0)})$ is the monodromy invariant of the pointed $(\Lambda, G)$-cover $(f_s : X_s \to Y, \eta(s))$ of $(Y, y_0)$.*

*Let $X_1 = \mathcal{C}/G(\lambda_0)$, $f_1 : X_1 \to Y \times S$ be as in Proposition 3.10. Let $\eta_1 : S \to X_1$ be the morphism defined by $\eta_1 = \rho(\zeta(s))$. Then $(f_1 : X_1 \to Y \times S, \eta_1)$ is a smooth, proper family of pointed covers of $(Y, y_0)$ equivalent to $(f : X \to Y \times S, \eta)$.*

Proof. One has $f(\eta(s)) = p(\zeta(s)) = (y_0, s)$, so $(f : X \to Y \times S, \eta)$ is a smooth, proper family of pointed $(\Lambda, G)$-covers of $(Y, y_0)$. For every $s \in S$ the scheme-theoretical fiber $X_s$ is isomorphic to $(\Lambda \times \mathcal{C}_s)/G$, so the stated relation between the monodromy invariants holds by Lemma 2.12.

One has $f_1(\eta_1(s)) = p(\zeta(s)) = (y_0, s)$ $\forall s \in S$. The $Y \times S$-isomorphism $X_1 \to X$ given by $\rho(z) \mapsto \pi(\lambda_0, z)$ transforms $\eta_1(s)$ in $\eta(s)$ $\forall s \in S$, so the two families of pointed covers of $(Y, y_0)$ are equivalent. ∎

## 4. Universal families of pointed covers with fixed monodromy group

4.1. Given a reduced complex space $X$ and a properly discontinuous group of automorphisms $G$ of $X$, let $p : X \to X/G = Z$ be the quotient map of topological spaces. Cartan defined in [5] a sheaf of complex valued functions $\mathcal{K}$ on $Z$: for every open subset $V$ of $|Z|$, $\mathcal{K}(V) = \mathcal{O}_X(p^{-1}(V))^G$, every stalk $\mathcal{K}_z$ is a local ring, and he proved in [5, Thm. 4] that the $\mathbb{C}$-ringed space $(Z, \mathcal{K})$ is a reduced complex space. Clearly, $(Z, \mathcal{O}_Z)$, $\mathcal{O}_Z = \mathcal{K}$, is the categorical quotient of $(X, \mathcal{O}_X)$ in the category of complex spaces.

Proposition 4.2. *Let $X$ be a normal algebraic variety. Let $G$ be a finite group, which has the property that every orbit is contained in an affine open set. Then $X^{\mathrm{an}}/G$ is biholomorphic to $(X/G)^{\mathrm{an}}$.*

Proof. Consider the composition of morphisms of $\mathbb{C}$-ringed spaces

$$(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \to (X, \mathcal{O}_X) \to (X/G, \mathcal{O}_{X/G}).$$

By the construction of $\mathcal{K} = \mathcal{O}_Z$, it factors as

$$(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \to (Z, \mathcal{O}_Z) \to (X/G, \mathcal{O}_{X/G}).$$

This induces a holomorphic map $X^{\mathrm{an}}/G = Z \to (X/G)^{\mathrm{an}}$ (cf. [28, Thm. 1.1]). The continuous map $|X^{\mathrm{an}}/G| \to |(X/G)^{\mathrm{an}}|$ is a homeomorphism (cf. [22, Lem. 2.5]). The algebraic variety $X/G$ is normal, so $(X/G)^{\mathrm{an}}$ is a normal complex space (cf. [28, Prop. 2.1]). Normality implies maximality (cf. [8, §2.29]); therefore, the holomorphic homeomorphism $X^{\mathrm{an}}/G \to (X/G)^{\mathrm{an}}$ is a biholomorphic map [8, §2.29]. ∎

4.3. Let $p : C \to Y$ be a $G$-cover. Let $b \in Y$ be a branch point. Let $p^{-1}(b) = \{w_1, \dots, w_r\}$. There is an embedded open disk $V \subset Y, b \in V$, such that $p^{-1}(V) = \sqcup_{i=1}^r W_i$ is a disjoint union of connected components $W_i$, $w_i \in W_i$ for $i = 1, \dots, r$, and every $p|_{W_i} : W_i \to V$ is a surjective cyclic analytic covering with Galois group $G(w_i) = St_G(w_i) \cong C_e \subset \mathbb{C}^*$, where $|G| = er$. Consider the left action of $G$ on $\Lambda \times C$ defined by $g(\lambda, z) = (\lambda g^{-1}, gz)$. Let $X = (\Lambda \times C)/G := \Lambda \times^G C$, let $\pi : \Lambda \times C \to X$ be the quotient map, and let $f : X \to Y$ be the map $f(\pi(\lambda, z)) = p(z)$. Let $i$ be an integer, $1 \leq i \leq r$, and let $w = w_i$, $W = W_i$. It is clear that $f^{-1}(V) \cong (\Lambda \times W)/G(w)$. Let $x = \pi(\lambda, w)$. Let $\{\lambda_1, \dots, \lambda_k\} = \lambda G(w)$. Then $(\{\lambda_1, \dots, \lambda_k\} \times W)/G(w)$ is the connected component of $f^{-1}(V)$ which contains $x = \pi(\lambda, w)$. Let $G(\lambda, w) = G(w) \cap G(\lambda)$. Let $|G(\lambda, w)| = q$. Then $e = kq$, the map $(\{\lambda\} \times W)/G(\lambda, w) \to (\{\lambda_1, \dots, \lambda_k\} \times W)/G(w)$ is biholomorphic, and the composition of holomorphic maps

$$W \to \big(\{\lambda\} \times W\big)/G(\lambda, w) \xrightarrow{\sim} \big(\{\lambda_1, \dots, \lambda_k\} \times W\big)/G(w) \hookrightarrow (\Lambda \times W)/G(w) \to W/G(w) \cong V$$

has the following form in local coordinates: $s \mapsto u = s^q \mapsto t = s^e = u^k$. This shows that the ramification index of $f : X \to Y$ at the point $x = \pi(\lambda, w)$ equals $k = |\lambda G(w)|$.

Let $v_0 \in V \setminus \{b\}$, $w_0 \in W$, $p(w_0) = v_0$. Then $f^{-1}(v_0) = \{x_1, \dots, x_d\}$, where $x_i = \pi(\lambda_i, w_0)$. If $\beta : I \to V \setminus \{b\}$ is a simple loop with $\beta(0) = \beta(1) = v_0$, then $w_0\beta = gw_0$, where $g$ is a generator of $G(w)$. One has $x_i\beta = \pi(\lambda_i, w_0\beta) = \pi(\lambda_i, gw_0) = \pi(\lambda_i g, w_0)$. Therefore, the decomposition of the permutation $x_i \mapsto x_i\beta$, $i = 1, \dots, d$, into a product of disjoint cycles, by which one determines the indices of the ramification points over $b$, corresponds to the decomposition of $\Lambda$ into a union of $G(w)$-orbits, as discussed above.

4.4. Let $y_0 \in Y$. Let

$$\big(p : \mathcal{C}(y_0) \to Y \times H_n^G(Y, y_0), \zeta : H_n^G(Y, y_0) \to \mathcal{C}(y_0)\big) \tag{12}$$

be the universal family of pointed $G$-covers of $(Y, y_0)$ branched in $n$ points (cf. [22, Thm. 3.20]). We recall from [22, Sec. 3] the local analytic form of $p$ at the points lying over the branch locus $B = \{(y, (D, m)) \mid y \in D\}$. Let $D = \{b_1, \dots, b_j, \dots, b_n\}$, $y_0 \in Y \setminus D$. Let us choose local analytic coordinates $s_i$ at $b_i$, such that $s_i(b_i) = 0$, $i = 1, \dots, n$. Let $\epsilon \in \mathbb{R}^+, \epsilon \ll 1$ be such that the open sets $U_i = \{y \mid s_i(y) < \epsilon\}$ have disjoint closures $\bar{U}_i$, $i = 1, \dots, n$, and $y_0 \in Y \setminus \bigcup_{i=1}^n \bar{U}_i$. Let $\gamma_1, \dots, \gamma_n$ be closed paths based at $y_0$ as in §2.6. Let $p(w) = (b, (D, m))$, where $b = b_j$. Let $U = U_j$ and let $V = U \times N_{(D,m)}(U_1, \dots, U_n)$ (cf. §3.4). For every $v = (y, (D', m(D'))) \in V$, where $y \in U$, $D' = \{y_1, \dots, y_n\}$, $y_i \in U_i$, let $t_i(v) = s_i(y_i)$ and let $t(v) = s_j(y)$. Let $G(w)$ be the isotropy group of $w$, $|G(w)| = e \geq 2$. Then $w$ has a connected neighborhood $W \subset |\mathcal{C}(y_0)^{\mathrm{an}}|$ which is $G(w)$-invariant, $p(W) = V$, and $gW \cap W = \emptyset$ if $g \in G \setminus G(w)$. Let $E \subset \mathbb{C} \times V$ be the analytic subset defined by $z^e = t - t_j$ and let $p_1 : E \to V$ be the projection map. Then there is a biholomorphic map $\theta : W \to E$, such that $p_1 \circ \theta = p|_W$. The composition $\rho = (z, t_1, \ldots, t_n) \circ \theta : W \to \mathbb{C}^{n+1}$ maps $W$ biholomorphically onto an open subset $\Omega \subset \mathbb{C}^{n+1}$. There exists a primitive character $\chi : G(w) \to \mathbb{C}^*$ such that $\theta$ and $\rho$ are $G(w)$-equivariant with respect to the action of $G(w)$ on $E$ and $\Omega$ defined respectively by $g(z, v) = (\chi(g)z, v)$ and $g(z, z_1, \ldots, z_n) = (\chi(g)z, z_1, \ldots, z_n)$.

We need a simple case of Cartan's theorem [5, Thm. 4].

Lemma 4.5. *Let $H$ be a cyclic group of order $q$, and let $\chi : H \to \mathbb{C}^*$ be a primitive character. Let $\Omega$ be an open subset of $\mathbb{C}^n$ invariant under the action of $H$ on $\mathbb{C}^n$ defined by $h(z_1, z_2, \ldots, z_n) = (\chi(h)z_1, z_2, \ldots, z_n)$. Let $\Omega/H$ be the quotient complex space* [5, Thm. 4] *and let $p : \Omega \to \Omega/H$ be the quotient holomorphic map. Let $\psi : \Omega \to \mathbb{C}^n$ be the map $\psi(z_1, z_2, \ldots, z_n) = (z_1^q, z_2, \ldots, z_n)$. Then $\Omega_1 = \psi(\Omega)$ is an open subset of $\mathbb{C}^n$ and there exists a biholomorphic map $\mu : \Omega_1 \to \Omega/H$ such that $p = \mu \circ \psi$.*

Proof. The holomorphic map $\psi : \Omega \to \mathbb{C}^n$ is open since the map $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^q$ is open. Hence, $\Omega_1 = \psi(\Omega)$ is an open subset of $\mathbb{C}^n$ and $|\Omega_1|$ is homeomorphic to the quotient topological space $|\Omega|/H$. In order to prove that $(\Omega_1, \mathcal{O}_{\Omega_1})$ is isomorphic to the quotient $\mathbb{C}$-ringed space $(\Omega/H, \mathcal{K})$ as defined in [5, §4], it suffices to verify that for every $a \in \Omega$ if $H(a)$ is the isotropy group of $a$ and if $b = \psi(a)$, then

$$(\mathcal{O}_{\Omega,a})^{H(a)} = \psi_a^\sharp(\mathcal{O}_{\Omega_1,b}). \tag{13}$$

Let $a = (a_1, a_2, \ldots, a_n)$. If $a_1 \neq 0$, then $H(a) = \{1\}$, $\psi : \Omega \to \Omega_1$ is locally biholomorphic at $a$, so (13) holds. Let $a_1 = 0$. Then $H(a) = H$. Without loss of generality, we may suppose that $a = (0, 0, \ldots, 0)$. Let the $H$-invariant germ $f \in \mathcal{O}_{\Omega,a}$ be represented by a series $\sum_{\alpha_1,\ldots,\alpha_n \geq 0} a_{\alpha_1\alpha_2\cdots\alpha_n} z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$ which converges absolutely in the polydisk $|z_i| < \varepsilon_i$, $i = 1, \ldots, n$. One has $\alpha_1 = q\beta_1$ for every $\alpha_1$. The series $\sum_{\beta_1,\alpha_2,\ldots,\alpha_n \geq 0} b_{\beta_1\alpha_2\cdots\alpha_n} y_1^{\beta_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$, where $b_{\beta_1\alpha_2\cdots\alpha_n} = a_{q\beta_1\alpha_2\cdots\alpha_n}$, converges absolutely in the polydisk $|y_1| < \varepsilon_1^q$, $|z_i| < \varepsilon_i$, $i = 2, \ldots, n$, and represents a germ $f_1 \in \mathcal{O}_{\Omega_1,b}$ such that $f = \psi_a^\sharp(f_1)$. Hence, (13) holds $\forall a \in \Omega$. ∎

In the next proposition, we use the setup and the notation of §2.1 and §4.4.

Proposition 4.6. *Let $y_0 \in Y$. Let*

$$\big(\varphi : \mathcal{X}(y_0) \to Y \times H_n^G(Y, y_0), \eta : H_n^G(Y, y_0) \to \mathcal{X}(y_0)\big) \tag{14}$$

*be the smooth, proper family of pointed $(\Lambda, G)$-covers of $(Y, y_0)$ obtained from* (12) *as in Proposition 3.11. The variety $\mathcal{X}(y_0)$ is smooth. The branch locus of $\varphi$ is $B$. For every element $(D,m) \in H_n^G(Y,y_0)$ the fiber $(\varphi_{(D,m)} : \mathcal{X}(y_0)_{(D,m)} \to Y, \eta(D,m))$ has monodromy invariant $(D, m^{N(\lambda_0)})$.*

*Let $x = \pi(\lambda, w) \in \mathcal{X}(y_0)$ be a point such that $\varphi(x) = p(w) \in B$. Let $\varphi(x) = (b_j, (D,m))$, where $D = \{b_1, \dots, b_j, \dots, b_n\}$. Let $|\lambda G(w)| = k$. Then there exists a connected, open neighborhood $A$ of $x$ in $|\mathcal{X}(y_0)^{\mathrm{an}}|$, such that*

$$\varphi(A) = U \times N_{(D,m)}(U_1, \dots, U_n) = V$$

*and the following properties hold. Let $E_1 \subset \mathbb{C} \times V$ be the analytic subset defined by $z^k = t - t_j$ and let $\varphi_1 : E_1 \to V$ be the projection map.*

(i) *There exists a biholomorphic map $\theta_1 : A \to E_1$ such that $\varphi_1 \circ \theta_1 = \varphi|_A$.*

(ii) *The composition $\rho_1 = (z, t_1, \dots, t_n) \circ \theta_1 : A \to \mathbb{C}^{n+1}$ maps $A$ biholomorphically onto an open subset of $\mathbb{C}^{n+1}$.*

Proof. One applies Proposition 3.11 to $(p : \mathcal{C}(y_0) \to Y \times H_n^G(Y, y_0), \zeta)$. The composition of the smooth morphisms $\mathcal{X}(y_0) \to H_n^G(Y, y_0) \to \operatorname{Spec} \mathbb{C}$ is smooth, so $\mathcal{X}(y_0)$ is a smooth variety.

It remains to prove the statements about the local analytic form of $\varphi$. According to Proposition 4.2, $\mathcal{X}(y_0)^{\mathrm{an}}$ is biholomorphic to $(\Lambda \times \mathcal{C}(y_0)^{\mathrm{an}})/G$. Let $G(\lambda, w) = G(\lambda) \cap G(w)$. This is a cyclic group of order $q$ and $e = kq$. The open subset $\{\lambda\} \times W \subset \Lambda \times \mathcal{C}(y_0)$ is $G(\lambda, w)$-invariant and for every $g \in G \setminus G(\lambda, w)$ one has $g(\{\lambda\} \times W) \cap (\{\lambda\} \times W) = \emptyset$. Let $A = \pi(\{\lambda\} \times W)$. The open complex subspace $A \subset \mathcal{X}(y_0)^{\mathrm{an}}$ is biholomorphic to $\{\lambda\} \times W/G(\lambda, w)$. Let $H = G(\lambda, w)$ and let $\chi_1 : H \to \mathbb{C}^*$ be the restriction of $\chi : G \to \mathbb{C}^*$ (cf. §4.4). Identifying $\{\lambda\} \times W$ with $W$ the biholomorphic map $\theta : \{\lambda\} \times W \to E$ is $H$-equivariant. Let $\psi_1 : E \to E_1$ be the map $\psi_1(z, v) = (z^q, v)$. The composition $\psi_1 \circ \theta$ is $H$-invariant, so there is a holomorphic map $\theta_1 : A \to E_1$ such that $\psi_1 \circ \theta = \theta_1 \circ \pi|_{\{\lambda\} \times W}$. Let $\Omega = \rho(\{\lambda\} \times W)$, let $\psi : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ be the map $\psi(z, z_1, \dots, z_n) = (z^q, z_1, \dots, z_n)$, and let $\Omega_1 = \psi(\Omega)$. One has the following commutative diagram of holomorphic maps:

$$\begin{array}{ccccc} \{\lambda\} \times W & \xrightarrow{\theta} & E & \xrightarrow{(z,t_1,\dots,t_n)} & \Omega \\ \downarrow \pi & & \downarrow \psi_1 & & \downarrow \psi \\ A & \xrightarrow{\theta_1} & E_1 & \xrightarrow{(z,t_1,\dots,t_n)} & \Omega_1. \end{array} \tag{15}$$

The vertical maps in the right square of (15) are $H$-invariant, the horizontal maps are biholomorphic (cf. [22, Prop. 3.18]), and by Lemma 4.5 $\Omega/H \cong \Omega_1$, therefore, $E_1 \cong E/H$. This implies that $\theta_1 : A \to E_1$ and $\rho_1 : A \to \Omega_1$ are biholomorphic maps. The map $p_1 : E \to V$ equals $\varphi_1 \circ \psi_1$; therefore, the equality $p_1 \circ \theta = p|_{\{\lambda\} \times W}$ implies $\varphi_1 \circ \theta_1 = \varphi|_A$.

We mention that (ii) implies the smoothness of $\mathcal{X}(y_0)$, as well as the smoothness of $\pi_2 \circ \varphi : \mathcal{X}(y_0) \to H_n^G(Y, y_0)$ using [16, Ch. III, Prop. 10.4]. ∎

The next proposition is a particular case of Proposition 3.11.

PROPOSITION 4.7. *Let* $(p : \mathcal{C}(y_0) \to Y \times H_n^G(Y, y_0), \zeta)$ *be as in §4.4. Let* $\rho : \mathcal{C}(y_0) \to \mathcal{C}(y_0)/G(\lambda_0)$ *be the quotient morphism and let* $\eta_1 = \rho \circ \zeta : H_n^G(Y, y_0) \to \mathcal{C}(y_0)/G(\lambda_0)$. *Then* $(\bar{p} : \mathcal{C}(y_0)/G(\lambda_0) \to Y \times H_n^G(Y, y_0), \eta_1)$ *is a family of pointed covers of* $(Y, y_0)$ *equivalent to the family* $(\varphi : \mathcal{X}(y_0) \to Y \times H_n^G(Y, y_0), \eta)$ *defined in Proposition 4.6.*

In Proposition 2.13, we proved that the set $H_{n,\lambda_0}^{\Lambda,G}(Y, y_0)$ is bijective to the set of equivalence classes of pointed $(\Lambda, G)$-covers of $(Y, y_0)$ branched in $n$ points.

PROPOSITION 4.8. *The left action of* $N_{S(\Lambda)}(G)$ *on* $H_n^G(Y, y_0)$ *defined by* $\sigma * (D, m) = (D, \sigma m \sigma^{-1})$ *is an action by covering automorphisms of the étale cover* $\delta : H_n^G(Y, y_0) \to (Y \setminus y_0)_*^{(n)}$, *where* $\delta(D, m) = D$ *(cf. §3.4). The subgroup* $N(\lambda_0) = \{\sigma \in N_{S(\Lambda)}(G) \mid \lambda_0 \sigma = \lambda_0\}$ *acts freely on* $H_n^G(Y, y_0)$ *and the quotient set*

$$H_{n,\lambda_0}^{\Lambda,G}(Y, y_0) = H_n^G(Y, y_0)/N(\lambda_0)$$

*can be endowed with a structure of quotient affine algebraic variety. The quotient map* $\nu : H_n^G(Y, y_0) \to H_{n,\lambda_0}^{\Lambda,G}(Y, y_0)$ *and the map* $\delta_1 : H_{n,\lambda_0}^{\Lambda,G}(Y, y_0) \to (Y \setminus y_0)_*^{(n)}$ *given by* $\delta_1(D, m^{N(\lambda_0)}) = D$ *are étale covers.*

PROOF. The map $\delta : H_n^G(Y, y_0) \to (Y \setminus y_0)_*^{(n)}$ given by $\delta(D, m) = D$ is a topological covering map with respect to the canonical complex topologies; every neighborhood $N_D(U_1, \dots, U_n)$ of $D \in (Y \setminus y_0)_*^{(n)}$ is evenly covered (cf. (9)). It is clear from (8) that for every $\sigma \in N_{S(\Lambda)}(G)$ the map $(E, \mu) \mapsto (E, \sigma \mu \sigma^{-1})$ transforms $N_{(D,m)}(U_1, \dots, U_n)$ into $N_{(D,\sigma m \sigma^{-1})}(U_1, \dots, U_n)$, so $N_{S(\Lambda)}(G)$ acts on $H_n^G(Y, y_0)$ by covering homeomorphism with respect to $|\delta^{an}|$. By [22, Cor. 4.5], this action is by covering automorphisms of the étale cover $\delta : H_n^G(Y, y_0) \to (Y \setminus y_0)_*^{(n)}$. The subgroup $N(\lambda_0)$ acts freely on $H_n^G(Y, y_0)$ (cf. Proposition 2.8), so the quotient set $H_{n,\lambda_0}^{\Lambda,G}(Y, y_0) = H_n^G(Y, y_0)/N(\lambda_0)$ has a structure of a smooth, affine algebraic variety [29, Ch. III, Prop. 18] and the quotient map $\nu$ is an étale cover. One has $\delta = \delta_1 \circ \nu$, so $\delta_1$ is a morphism by the universal property of quotients and it is an étale cover since $\delta$ has this property and $\nu$ is étale. ∎

4.9. The quotient morphism $\nu : H_n^G(Y, y_0) \to H_{n,\lambda_0}^{\Lambda,G}(Y, y_0)$ yields a homeomorphism $|H_n^G(Y, y_0)^{an}|/N(\lambda_0) \xrightarrow{\sim} |H_{n,\lambda_0}^{\Lambda,G}(Y, y_0)^{an}|$ (cf. [22, Lem. 2.5]), so $|H_{n,\lambda_0}^{\Lambda,G}(Y, y_0)^{an}|$ has a neighborhood basis consisting of the open sets

$$N_{(D,m^{N(\lambda_0)})}(U_1, \dots, U_n) = \nu\big(N_{(D,m)}(U_1, \dots, U_n)\big) = \big\{(E, m(E)^{N(\lambda_0)}) \mid E \in N_D(U_1, \dots, U_n)\big\}. \tag{16}$$

For every $\sigma \in N(\lambda_0)$, $\sigma \neq 1$, one has

$$N_{(D,m)}(U_1,\dots,U_n) \cap N_{(D,\sigma m \sigma^{-1})}(U_1,\dots,U_n) = \emptyset,$$

so $|\delta_1^{\mathrm{an}}|$ is a topological covering map and every open subset $N_D(U_1,\dots,U_n) \subset (Y \setminus y_0)_*^{(n)}$ is evenly covered:

$$\delta_1^{-1}\big(N_D(U_1,\dots,U_n)\big) = \bigsqcup_{m^{N(\lambda_0)}} N_{(D,m^{N(\lambda_0)})}(U_1,\dots,U_n). \tag{17}$$

4.10. Let $\mathcal{C}(y_0)' = p^{-1}(Y \times H_n^G(Y,y_0) \setminus B)$. We recall from [22, Sec. 3] that $\mathcal{C}(y_0)'$ is bijective to the set $\{(\Gamma_m[\alpha]_D, D, m)\}$ where $(D,m) \in H_n^G(Y,y_0)$,

$$\Gamma_m = \operatorname{Ker}\big(m : \pi_1(Y \setminus D, y_0) \to G\big),$$

$\alpha$ is a path in $Y \setminus D$ with $\alpha(0) = y_0$, and $[\alpha]_D$ is its homotopy class in $Y \setminus D$. The map $p' = p|_{\mathcal{C}(y_0)'} : \mathcal{C}(y_0)' \to Y \times H_n^G(Y,y_0) \setminus B$, defined by $(\Gamma_m[\alpha]_D, D, m) \mapsto (\alpha(1), (D,m))$, is a topological Galois covering map with respect to the topologies of the associated complex spaces, with a group of Deck transformations isomorphic to $G$, where the action of $G$ is defined as follows: if $g \in G$, $g = m([\sigma]_D)$, then $g(\Gamma_m[\alpha]_D, D, m) = (\Gamma_m[\sigma \cdot \alpha]_D, D, m)$.

In the setup of §2.1, for every $(D,m) \in H_n^G(Y,y_0)$ let $\Gamma_{m,\lambda_0} = m^{-1}(G(\lambda_0)) \subset \pi_1(Y \setminus D, y_0)$. Let $\mathcal{X}(y_0)' = \varphi^{-1}(Y \times H_n^G(Y,y_0) \setminus B)$. By Proposition 4.7, $\mathcal{X}(y_0)'$ is isomorphic to $\mathcal{C}(y_0)'/G(\lambda_0)$ by an isomorphism over $Y \times H_n^G(Y,y_0) \setminus B$. The quotient $\mathcal{C}(y_0)'/G(\lambda_0)$ is bijective to the set $\{(\Gamma_{m,\lambda_0}[\alpha]_D, D, m)\}$ and the quotient morphism $\rho' : \mathcal{C}(y_0)' \to \mathcal{C}(y_0)'/G(\lambda_0)$ is given by $(\Gamma_m[\alpha]_D, D, m) \mapsto (\Gamma_{m,\lambda_0}[\alpha]_D, D, m)$. One has $\mathcal{X}(y_0)'^{\mathrm{an}} \cong \mathcal{C}(y_0)'^{\mathrm{an}}/G(\lambda_0)$ by Proposition 4.2. The map $\varphi' : \mathcal{X}(y_0)' \to Y \times H_n^G(Y,y_0) \setminus B$ may be identified with

$$\big(\Gamma_{m,\lambda_0}[\alpha]_D, D, m\big) \mapsto \big(\alpha(1), (D,m)\big) \tag{18}$$

and it is a topological covering map with respect to the topologies of the associated complex spaces.

Proposition 4.11. *For every $(D,m) \in H_n^G(Y,y_0)$ and every $\sigma \in N(\lambda_0)$ one has $\Gamma_{m,\lambda_0} = \Gamma_{\sigma m \sigma^{-1},\lambda_0}$. Consider the left action of $N(\lambda_0)$ on the set $\mathcal{X}(y_0)'$ defined by*

$$\sigma\big(\Gamma_{m,\lambda_0}[\alpha]_D, D, m\big) = \big(\Gamma_{\sigma m \sigma^{-1},\lambda_0}[\alpha]_D, D, \sigma m \sigma^{-1}\big).$$

*This is an action by covering automorphisms of the composed étale cover*

$$\mathcal{X}(y_0)' \to Y \times H_n^G(Y,y_0) \setminus B \to Y \times (Y \setminus y_0)_*^{(n)} \setminus A, \tag{19}$$

*where $A = \{(y,D) \mid y \in D\}$, and it can be uniquely extended to a left action of $N(\lambda_0)$*

*on $\mathcal{X}(y_0)$ by covering automorphisms of the composed cover*

$$\mathcal{X}(y_0) \xrightarrow{\varphi} Y \times H_n^G(Y, y_0) \xrightarrow{\mathrm{id}\times\delta} Y \times (Y \setminus y_0)_*^{(n)}.$$

*The morphism $\varphi : \mathcal{X}(y_0) \to Y \times H_n^G(Y, y_0)$ is $N(\lambda_0)$-equivariant with respect to the action of $N(\lambda_0)$ on $Y \times H_n^G(Y, y_0)$ defined by $\sigma * (y, (D, m)) = (y, (D, \sigma m \sigma^{-1}))$ (cf. Proposition 4.8).*

Proof. For every $\sigma \in N(\lambda_0)$ one has $\sigma G(\lambda_0)\sigma^{-1} = G(\lambda_0)$, so $\Gamma_{\sigma m \sigma^{-1},\lambda_0} = \Gamma_{m,\lambda_0}$. A basis of open sets of the topology of $\mathcal{C}(y_0)'^{\mathrm{an}}$ was constructed in [22, Sec. 3] as follows. Let $(y, (D, m)) \in Y \times H_n^G(Y, y_0) \setminus B$, $D = \{b_1, \dots, b_n\}$. Let $\bar{U}, \bar{U}_1, \dots, \bar{U}_n$ be disjoint embedded closed disks in $Y$ with interiors $U, U_1, \dots, U_n$, respectively, such that $y \in U$, $\bar{U}_i \subset Y \setminus \{y_0\}$, $b_i \in U_i$, $i = 1, \dots, n$. Let $\alpha : I \to Y \setminus \bigcup_{i=1}^n \bar{U}_i$ be a path such that $\alpha(0) = y_0$, $\alpha(1) = y$. Let

$$\begin{aligned}(20)\quad & N_{(\alpha,D,m)}(U, U_1, \dots, U_n)\\ &= \big\{\big(\Gamma_{m(E)}[\alpha \cdot \tau]_E, E, m(E)\big) \mid E \in N_D(U_1, \dots, U_n),\ \tau : I \to U,\ \tau(0) = y\big\}.\end{aligned}$$

The family of these subsets of $\mathcal{C}(y_0)'$ is a basis of open sets of the topology of $\mathcal{C}(y_0)'^{\mathrm{an}}$. Furthermore (cf. §4.10),

$$p'^{-1}\big(U \times N_{(D,m)}(U_1, \dots, U_n)\big) = \bigsqcup_{j=1}^{|G|} N_{(\alpha_j,D,m)}(U, U_1, \dots, U_n), \tag{21}$$

where $m([\alpha_j \cdot \alpha_i^-]) \neq 1$ for $i \neq j$. Let $h \in G(\lambda_0)$, $h = m([\eta])$, where $\eta$ is a loop in $Y \setminus \bigcup_{i=1}^n \bar{U}_i$. Then

$$hN_{(\alpha,D,m)}(U, U_1, \dots, U_n) = N_{(\eta\cdot\alpha,D,m)}(U, U_1, \dots, U_n).$$

We see that acting on $\mathcal{C}(y_0)'$ the group $G(\lambda_0)$ permutes the open sets on the right-hand side of (21) and the image of $N_{(\alpha,D,m)}(U, U_1, \dots, U_n)$ in $\mathcal{X}(y_0)$ is the set

$$\begin{aligned}(22)\quad & \bar{N}_{(\alpha,D,m)}(U, U_1, \dots, U_n)\\ &= \big\{\big(\Gamma_{m(E),\lambda_0}[\alpha \cdot \tau]_E, E, m(E)\big) \mid E \in N_D(U_1, \dots, U_n),\ \tau : I \to U,\ \tau(0) = y\big\}.\end{aligned}$$

These sets form a basis of open sets of the quotient complex topology of $\mathcal{X}(y_0)' \cong \mathcal{C}(y_0)'/G(\lambda_0)$ and

$$\varphi'^{-1}\big(U \times N_{(D,m)}(U_1, \dots, U_n)\big) = \bigsqcup_{i=1}^{d} \bar{N}_{(\alpha_i,D,m)}(U, U_1, \dots, U_n),$$

where $m([\alpha_j \cdot \alpha_i^-]) \notin G(\lambda_0)$ for $i \neq j$.

Let $\sigma \in N(\lambda_0)$. Then $\sigma \bar{N}_{(\alpha,D,m)}(U, U_1, \dots, U_n) = \bar{N}_{(\alpha,D,\sigma m \sigma^{-1})}(U, U_1, \dots, U_n)$. Therefore, $N(\lambda_0)$ acts on $\mathcal{X}(y_0)'$ by covering homeomorphisms of the composed topological covering

$$\mathcal{X}(y_0)' \xrightarrow{\varphi'} Y \times H_n^G(Y, y_0) \setminus B \to Y \times (Y \setminus y_0)_*^{(n)} \setminus A.$$

By [22, Cor. 4.5], $N(\lambda_0)$ acts on $\mathcal{X}(y_0)'$ by covering automorphisms of the composed étale cover (19). It is clear from (18) that $\varphi'$ is $N(\lambda_0)$-equivariant.

Let $H$ be a connected component of $H_n^G(Y, y_0)$. Let $\mathcal{X}(y_0)'_H = \varphi'^{-1}(Y \times H \setminus B)$. This is an irreducible algebraic variety, quotient of the irreducible variety

$$p'^{-1}(Y \times H \setminus B) = \mathcal{C}(y_0)'_H$$

(cf. [22, §3.16]), and the smooth variety $\mathcal{X}(y_0)_H = \varphi^{-1}(Y \times H)$ is the normalization of $Y \times H$ in the field of rational functions $\mathbb{C}(\mathcal{X}(y_0)'_H)$. Given $\sigma \in N(\lambda_0)$, the map $x \mapsto \sigma x$ defines an isomorphism $\mathcal{X}(y_0)'_H \xrightarrow{\sim} \mathcal{X}(y_0)'_{\sigma * H}$. Passing to normalizations, this isomorphism extends in a unique way to an isomorphism $\mathcal{X}(y_0)_H \xrightarrow{\sim} \mathcal{X}(y_0)_{\sigma * H}$. This defines an action of $N(\lambda_0)$ on $\mathcal{X}(y_0)$ with the required properties. ∎

4.12. The group $N(\lambda_0)$ acts on $\mathcal{X}(y_0)$ by covering automorphisms of the cover $\mathcal{X}(y_0) \to Y \times (Y \setminus y_0)_*^{(n)}$. We denote by $\mathcal{X}(y_0, \lambda_0)$ the quotient algebraic variety $\mathcal{X}(y_0)/N(\lambda_0)$. The $N(\lambda_0)$-equivariant cover $\varphi : \mathcal{X}(y_0) \to Y \times H_n^G(Y, y_0)$ descends to a cover $\phi : \mathcal{X}(y_0, \lambda_0) \to Y \times H_{n,\lambda_0}^{\Lambda,G}(Y, y_0)$. One has the following commutative diagram:

$$\begin{array}{ccc} \mathcal{X}(y_0) & \xrightarrow{\kappa} & \mathcal{X}(y_0, \lambda_0) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\phi} \\ Y \times H_n^G(Y, y_0) & \xrightarrow{\mathrm{id}\times\nu} & Y \times H_{n,\lambda_0}^{\Lambda,G}(Y, y_0), \end{array} \tag{23}$$

where $\kappa$ and $\nu$ are the quotient morphisms with respect to the actions of $N(\lambda_0)$. The morphism $\kappa \circ \eta : H_n^G(Y, y_0) \to \mathcal{X}(y_0, \lambda_0)$ maps $(D, m)$ to $(\Gamma_{m,\lambda_0}[c_{y_0}]_D, D, m^{N(\lambda_0)})$, where $m^{N(\lambda_0)} = \{\sigma m \sigma^{-1} \mid \sigma \in N(\lambda_0)\}$, so $\kappa \circ \eta$ is $N(\lambda_0)$-invariant and can be decomposed as $\xi \circ \nu$, where $\xi : H_{n,\lambda_0}^{\Lambda,G}(Y, y_0) \to \mathcal{X}(y_0, \lambda_0)$ is a morphism with the property that $\phi(\xi(D, m^{N(\lambda_0)})) = (y_0, (D, m^{N(\lambda_0)}))$ $\forall (D, m^{N(\lambda_0)}) \in H_{n,\lambda_0}^{\Lambda,G}(Y, y_0)$.

The closed algebraic subset $B \subset Y \times H_n^G(Y, y_0)$ is $N(\lambda_0)$-invariant. Let $\mathcal{B} = \mathrm{id} \times \nu(B)$ and let $\mathcal{X}(y_0, \lambda_0)' = \mathcal{X}(y_0, \lambda_0) \setminus \phi^{-1}(\mathcal{B})$. Then

$$\mathcal{X}(y_0, \lambda_0)' = \mathcal{X}(y_0)'/N(\lambda_0) \quad \text{and} \quad |\mathcal{X}(y_0, \lambda_0)'^{\mathrm{an}}| = |\mathcal{X}(y_0)'^{\mathrm{an}}|/N(\lambda_0)$$

(cf. [22, Lem. 2.5]). For every element $(y, (D, m^{N(\lambda_0)}))$ of $Y \times H_{n,\lambda_0}^{\Lambda,G}(Y, y_0) \setminus \mathcal{B}$ and every $z = (\Gamma_{m,\lambda_0}[\alpha]_D, D, m^{N(\lambda_0)}) \in \mathcal{X}(y_0, \lambda_0)'$ such that $\phi(z) = (y, (D, m^{N(\lambda_0)}))$, the sets

$$(24)\quad N_{(\alpha,D,m^{N(\lambda_0)})}(U,U_1,\dots,U_n)=\big\{\big(\Gamma_{m(E),\lambda_0}[\alpha\cdot\tau]_E, E, m(E)^{N(\lambda_0)}\big)\mid E\in N_D(U_1,\dots,U_n),\ \tau:I\to U,\ \tau(0)=y\big\}$$

form a basis of neighborhoods of $z$ in the topology of $\mathcal{X}(y_0,\lambda_0)'^{\mathrm{an}}$ (cf. (20) and (22)).

THEOREM 4.13. *The pair*

$$(25)\qquad \big(\phi:\mathcal{X}(y_0,\lambda_0)\to Y\times H^{\Lambda,G}_{n,\lambda_0}(Y,y_0),\ \xi:H^{\Lambda,G}_{n,\lambda_0}(Y,y_0)\to\mathcal{X}(y_0,\lambda_0)\big)$$

*is a smooth, proper family of pointed $(\Lambda,G)$-covers of $(Y,y_0)$ branched in n points. The variety $\mathcal{X}(y_0,\lambda_0)$ is smooth. The branch locus of $\phi$ is $\mathcal{B}$. The fiber of the family over $(D,m^{N(\lambda_0)})\in H^{\Lambda,G}_{n,\lambda_0}(Y,y_0)$ is a pointed $(\Lambda,G)$-cover of $(Y,y_0)$ with monodromy invariant $(D,m^{N(\lambda_0)})$. Every pointed $(\Lambda,G)$-cover of $(Y,y_0)$ branched in n points is equivalent to a unique fiber of the family (25) by a unique covering isomorphism.*

PROOF. The morphism $\varphi:\mathcal{X}(y_0)\to Y\times H^G_n(Y,y_0)$ is $N(\lambda_0)$-equivariant and $N(\lambda_0)$ acts freely on $H^G_n(Y,y_0)$ (cf. Proposition 2.8). Therefore, the action of $N(\lambda_0)$ on the smooth variety $\mathcal{X}(y_0)$ is free, which implies that the quotient algebraic variety $\mathcal{X}(y_0,\lambda_0)$ is smooth. The morphisms $\kappa$ and $\nu$ of (23) are étale covers, so the composition $\pi_2\circ\phi:\mathcal{X}(y_0,\lambda_0)\to H^{\Lambda,G}_{n,\lambda_0}(Y,y_0)$ is smooth, since $\mathcal{X}(y_0)\to H^G_n(Y,y_0)$ is smooth by Proposition 4.6. Furthermore, $\pi_2\circ\phi$ is proper since $\phi$ is finite and $\pi_2$ is proper. Acting by $N(\lambda_0)$ on $\mathcal{X}(y_0)$, every $\sigma\in N(\lambda_0)$ transforms the pointed $(\Lambda,G)$-cover of $(Y,y_0)$, $(\mathcal{X}(y_0)_{(D,m)}\to Y,\eta(D,m))$, with monodromy invariant $(D,m^{N(\lambda_0)})$ (cf. Proposition 4.6) into the equivalent pointed cover $(\mathcal{X}(y_0)_{(D,\sigma m\sigma^{-1})}\to Y,\eta(D,\sigma m\sigma^{-1}))$ of $(Y,y_0)$; therefore, the fiber of the family (25) over $(D,m^{N(\lambda_0)})$ is a pointed cover of $(Y,y_0)$ equivalent to any of these ones. This proves the stated properties of the family (25). The last statement of the theorem follows from Proposition 2.13 and §2.5. ∎

REMARK 4.14. The local analytic form of $\phi:\mathcal{X}(y_0,\lambda_0)\to Y\times H^{\Lambda,G}_{n,\lambda_0}(Y,y_0)$ at the points lying over the branch locus of $\phi$ is the same as that of $\varphi:\mathcal{X}(y_0)\to Y\times H^G_n(Y,y_0)$. In fact, in the commutative diagram (23), $\varphi$ is $N(\lambda_0)$-equivariant and the action of $N(\lambda_0)$ is free. Let $x_1\in\mathcal{X}(y_0,\lambda_0)$ be a point such that $\phi(x_1)$ belongs to $\mathcal{B}$. Let $x\in\mathcal{X}(y_0)$ be a point such that $\kappa(x)=x_1$ and let $\varphi(x)=(b_j,(D,m))$, where $D=\{b_1,\dots,b_j,\dots,b_n\}$ (cf. Proposition 4.6). Acting on $Y\times H^G_n(Y,y_0)$ by

$$\sigma*\big(y,(D,m)\big)=\big(y,(D,\sigma m\sigma^{-1})\big),$$

every $\sigma\in N(\lambda_0)$ transforms $V=U\times N_{(D,m)}(U_1,\dots,U_n)$ into the open set $\sigma V=U\times N_{(D,\sigma m\sigma^{-1})}(U_1,\dots,U_n)$ and $\sigma_1V\cap\sigma_2V=\emptyset$ if $\sigma_1\neq\sigma_2$. Respectively, acting on $\mathcal{X}(y_0)$, the group $N(\lambda_0)$ permutes $\varphi^{-1}(\sigma V)$, $\sigma \in N(\lambda_0)$. If $A$ is the neighborhood of $x$ as in Proposition 4.6, let $A_1 = \kappa(A)$ and let $V_1 = (\mathrm{id}_Y \times \nu)(V) = U \times N_{(D,m^{N(\lambda_0)})}(U_1, \dots, U_n)$. Then one has a commutative diagram of holomorphic maps

$$\begin{array}{ccc} A & \xrightarrow{\kappa^{\mathrm{an}}} & A_1 \\ {\scriptstyle \varphi|_A}\downarrow & & \downarrow{\scriptstyle \phi|_{A_1}} \\ V & \xrightarrow{\mathrm{id}\times\nu^{\mathrm{an}}} & V_1 \end{array}$$

in which the horizontal maps are biholomorphic. Therefore, statements analogous to (i) and (ii) of Proposition 4.6 hold for $\phi|_{A_1} : A_1 \to V_1$.

Proposition 4.15. *Let $(f : X \to Y \times S, \eta : S \to X)$ be a smooth, proper family of pointed $(\Lambda, G)$-covers of $(Y, y_0)$ branched in $n$ points. Let $B \subset Y \times S$ be the branch locus of $f$. For every fiber $(f_s : X_s \to Y, \eta(s))$ let $(\varepsilon_s, m_s)$ be a pair which satisfies the conditions of Proposition 2.8 (i) (cf. Proposition 3.3). For every $s \in S$ let $u(s) = (B_s, m_s^{N(\lambda_0)}) \in H^{\Lambda,G}_{n,\lambda_0}(Y, y_0)$ be the monodromy invariant of $(f_s : X_s \to Y, \eta(s))$. Then $u : S \to H^{\Lambda,G}_{n,\lambda_0}(Y, y_0)$ is a morphism.*

Proof. The map $\beta : S \to (Y \setminus y_0)^{(n)}_* \subset Y^{(n)}$ given by $\beta(s) = B_s$ is a morphism by [22, Prop. 2.6]. The map $u$ fits in the following commutative diagram:

$$\begin{array}{ccc} & & H^{\Lambda,G}_{n,\lambda_0}(Y, y_0) \\ & \overset{u}{\nearrow} & \downarrow{\scriptstyle \delta_1} \\ S & \xrightarrow[\beta]{} & (Y \setminus y_0)^{(n)}_* \end{array}$$

where $\delta_1$ is the étale cover defined by $\delta_1(D, m^{N(\lambda_0)}) = D$ (cf. Proposition 4.8). In order to prove that $u$ is a morphism, it suffices to verify that $u$ is continuous with respect to the topologies of $S^{\mathrm{an}}$ and $H^{\Lambda,G}_{n,\lambda_0}(Y, y_0)^{\mathrm{an}}$ (cf. [22, Cor. 4.5]); i.e., we have to show that for every $s \in S$ and every neighborhood $N$ of $u(s)$ the point $s$ is internal of $u^{-1}(N)$. Let $s_0$ be an arbitrary point of $S$. Let $\beta(s_0) = D = \{b_1, \dots, b_n\}$. Let $(\varepsilon : \Lambda \to f_{s_0}^{-1}(y_0)$, $m : \pi_1(Y \setminus D, y_0) \to G)$ be a pair which satisfies condition (i) of Proposition 2.8 with $\varepsilon(\lambda_0) = \eta(s_0)$. One has $u(s_0) = (D, m^{N(\lambda_0)})$. Let $N_{(D,m^{N(\lambda_0)})}(U_1, \dots, U_n)$ be any of the open sets of the neighborhood basis of $(D, m^{N(\lambda_0)})$ in $|H^{\Lambda,G}_{n,\lambda_0}(Y, y_0)^{\mathrm{an}}|$ contained in $N$ (cf. (16)). In the course of the proof of Lemma 3.5, we showed that there exists a connected neighborhood $V \subset |S^{\mathrm{an}}|$ of $s_0$ such that $\beta(V) \subset N_D(U_1, \dots, U_n)$ and we constructed for every $s \in V$ a pair $(\varepsilon'_s : \Lambda \to f_s^{-1}(y_0), m'_s = m(\beta(s)))$ as in (10). For all $s \in S$ this pair satisfies the conditions of Proposition 2.8 (i). In fact, the first one was verified in the proof of Lemma 3.5. We claim that the second one is satisfied as well: $\varepsilon'_s(\lambda_0) = \eta(s)\ \forall s \in S$. This holds since $\{y_0\} \times V$ is a connected subset of

$U \times V$, so $f^{-1}(\{y_0\} \times V)$ is a disjoint union of $d = |\Lambda|$ connected components $f^{-1}(\{y_0\} \times V) \cap W_\lambda$, $\lambda \in \Lambda$. The map $\eta : S \to f^{-1}(\{y_0\} \times S)$ is continuous with respect to the canonical complex topologies and $\eta(s_0) = \varepsilon(\lambda_0) \in W_{\lambda_0}$. Therefore, $\eta(V)$ is the connected component of $f^{-1}(\{y_0 \times V\})$ contained in $W_{\lambda_0}$. This shows that $\varepsilon_s(\lambda_0) = \eta(s)$ $\forall s \in V$. Using Proposition 2.8, we conclude that $\forall s \in V$ the monodromy invariant $u(s)$ of $(f_s : X_s \to Y, \eta(s))$ equals $(\beta(s), m(\beta(s))^{N(\lambda_0)})$, so $u(s) \in \nu(N_{(D,m)}(U_1, \dots, U_n)) = N_{(D,m^{N(\lambda_0)})}(U_1, \dots, U_n)$ (cf. (16)). Therefore, $V \subset u^{-1}(N)$. ∎

Lemma 4.16. *Let $f : X \to S$ and $g : Z \to S$ be proper, surjective morphisms of algebraic varieties. Let $h : X \to Z$ be a morphism such that $f = g \circ h$. Suppose that for every $s \in S$ the induced morphism of the scheme-theoretical fibers $h_s : X \otimes_S \mathbb{C}(s) \to Z \otimes_S \mathbb{C}(s)$ is an isomorphism. Then $h$ is an isomorphism.*

Proof. The map $h$ is bijective. By [15, Prop. 4.6.7 (i)], every $s \in S$ has an open neighborhood $U$ in $S$ such that $h|_{f^{-1}(U)} : f^{-1}(U) \to g^{-1}(U)$ is a closed embedding. Since $X$ and $Z$ are reduced schemes, $h|_{f^{-1}(U)}$ is an isomorphism. Therefore, $h$ is an isomorphism. ∎

In the next theorem, we assume the setup of §2.1.

Theorem 4.17. *The algebraic variety $H^{\Lambda,G}_{n,\lambda_0}(Y, y_0)$ is a fine moduli variety for the moduli functor $\mathcal{H}^{\Lambda,G}_{(Y,y_0),n}$ of smooth, proper families of pointed $(\Lambda, G)$-covers of $(Y, y_0)$ branched in $n$ points (cf. §3.8). The universal family is (cf. Theorem 4.13)*

$$(26)\qquad \big(\phi : \mathcal{X}(y_0, \lambda_0) \to Y \times H^{\Lambda,G}_{n,\lambda_0}(Y, y_0),\ \xi : H^{\Lambda,G}_{n,\lambda_0}(Y, y_0) \to \mathcal{X}(y_0, \lambda_0)\big).$$

Proof. Let $[f : X \to Y \times S, \eta : S \to X] \in \mathcal{H}^{\Lambda,G}_{(Y,y_0),n}(S)$ and let $B \subset Y \times S$ be the branch locus of $f$. For every fiber $(f_s : X_s \to Y, \eta(s))$ let $(\varepsilon_s, m_s)$ be a pair which satisfies the conditions of Proposition 2.8 (i) (cf. Proposition 3.3). Let $u : S \to H^{\Lambda,G}_{n,\lambda_0}(Y, y_0)$, $u(s) = (B_s, m_s^{N(\lambda_0)})$ be the morphism of Proposition 4.15. We want to prove that $(f : X \to Y \times S, \eta)$ is equivalent to the pullback by $u$ of the family (26). This is the unique morphism with this property since the monodromy invariant classifies the pointed $(\Lambda, G)$-covers of $(Y, y_0)$ up to equivalence. For every $s \in S$ there exists a unique isomorphism $h_s : X_s \to \mathcal{X}(y_0, \lambda_0)_{u(s)}$ such that $\phi_{u(s)} \circ h_s = (\mathrm{id}_Y \times u) \circ f_s$ and $h_s(\eta(s)) = \xi(u(s))$. Let $h : X \to \mathcal{X}(y_0, \lambda_0)$ be the map whose restriction on every $X_s$ equals $h_s$. One obtains the following commutative diagram of maps:

$$(27)\qquad \begin{array}{ccc} X & \xrightarrow{\ h\ } & \mathcal{X}(y_0, \lambda_0) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \phi} \\ Y \times S & \xrightarrow{\ \mathrm{id}\times u\ } & Y \times H^{\Lambda,G}_{n,\lambda_0}(Y, y_0). \end{array}$$

We want to prove that $h$ is a morphism and that (27) is a Cartesian diagram. Let $\mathcal{B} \subset Y \times H^{\Lambda,G}_{n,\lambda_0}(Y, y_0)$ be the branch locus of $\phi$. One has $B = (\mathrm{id}_Y \times u)^{-1}(\mathcal{B})$. Let $X' = X \setminus f^{-1}(B)$, $\mathcal{X}(y_0, \lambda_0)' = \mathcal{X}(y_0, \lambda_0) \setminus \phi^{-1}(\mathcal{B})$, $h' = h|_{X'}$, $f' = f|_{X'}$, $\phi' = \phi|_{\mathcal{X}(y_0,\lambda_0)'}$. Restricting (27) to the complements of the branch loci, one obtains the commutative diagram of maps

$$\begin{array}{ccc} X' & \xrightarrow{\quad h' \quad} & \mathcal{X}(y_0, \lambda_0)' \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle \phi'} \\ Y \times S \setminus B & \xrightarrow{(\mathrm{id}\times u)'} & Y \times H^{\Lambda,G}_{n,\lambda_0}(Y, y_0) \setminus \mathcal{B}. \end{array}$$

We claim that $h'$ is continuous with respect to the topologies of $X'^{\mathrm{an}}$ and $\mathcal{X}(y_0, \lambda_0)'^{\mathrm{an}}$. Let $s \in S$ and let $\varepsilon_s : \Lambda \to f_s^{-1}(y_0)$, $m_s : \pi_1(Y \setminus B_s, y_0) \to G$ be a pair as in Proposition 2.8 (i) with $\varepsilon_s(\lambda_0) = \eta(s)$. One has

$$(f_s')_* \pi_1\big(X_s', \eta(s)\big) = m_s^{-1}\big(G(\lambda_0)\big) = \Gamma_{m_s,\lambda_0}.$$

Let $x \in X_s'$ and let $\gamma : I \to X_s'$ be a path such that $\gamma(0) = \eta(s)$, $\gamma(1) = x$. Then $f' \circ \gamma : I \to Y \setminus B_s$ is a path with initial point $y_0$. Lifting $f' \circ \gamma$ in $\mathcal{X}(y_0, \lambda_0)'_{u(s)}$ with initial point $\xi(u(s)) = (\Gamma_{m_s,\lambda_0}[c_{y_0}]_{B_s}, B_s, m_s^{N(\lambda_0)})$, its terminal point is

$$h'(x) = h_s'(x) = \big(\Gamma_{m_s,\lambda_0}[f' \circ \gamma]_{B_s}, B_s, m_s^{N(\lambda_0)}\big). \tag{28}$$

For every $x_0 \in X'$ and every neighborhood $N$ of $h'(x_0)$ in $\mathcal{X}(y_0, \lambda_0)'^{\mathrm{an}}$ we have to prove that $x_0$ is an internal point of $h'^{-1}(N)$. Let $f'(x_0) = (y, s_0)$, $D = B_{s_0} = \{b_1, \ldots, b_n\}$, $m = m_{s_0} : \pi_1(Y \setminus D, y_0) \to G$. Let $\gamma_0 : I \to X_{s_0}'$ be a path such that $\gamma_0(0) = \eta(s_0)$, $\gamma_0(1) = x_0$, and let $\alpha = f_{s_0}' \circ \gamma_0 : I \to Y \setminus D$. One has $\alpha(0) = y_0$, $\alpha(1) = y$, and $h'(x_0) = (\Gamma_{m,\lambda_0}[\alpha]_D, D, m^{N(\lambda_0)})$. Let $N_{(\alpha,D,m^{N(\lambda_0)})}(U, U_1, \ldots, U_n)$, with $y \in U$, $b_i \in U_i$, $i = 1, \ldots, n$, and $\alpha(I) \subset Y \setminus \bigcup_{i=1}^n \overline{U}_i$, be a neighborhood of $h'(s_0)$ as in (24) contained in $N$. We want to show that there exists a neighborhood $W$ of $x_0$ such that $h'(W) \subset N_{(\alpha,D,m^{N(\lambda_0)})}(U, U_1, \ldots, U_n)$. The complex space $S^{\mathrm{an}}$ is locally connected [12, Ch. 9, §3 (1)], so one may shrink the neighborhood $U$ of $y$ and choose a connected neighborhood $V$ of $s_0$ such that $\beta(V) \subset N_D(U_1, \ldots, U_n)$, $U \times V \subset Y \times S \setminus B$ and $f'^{-1}(U \times V)$ is a disjoint union of connected open sets homeomorphic to $U \times V$. Let $W$ be the connected component of $f'^{-1}(U \times V)$ which contains $x_0$. We claim that

$$h'(W) \subset N_{(\alpha,D,m^{N(\lambda_0)})}(U, U_1, \ldots, U_n).$$

Consider the homotopy

$$F : [0, 1] \times V \to Y \times S \setminus B, \quad F(t, s) = \big(\alpha(t), s\big).$$

By the covering homotopy property of topological covering maps (cf. [30, Ch. 2, §3, Thm. 3]), there is a unique continuous lifting of $F$

$$\begin{array}{ccc} & & X' \\ & \overset{\widetilde{F}}{\nearrow} & \downarrow{\scriptstyle f'} \\ [0,1]\times V & \xrightarrow{F} & Y\times S\setminus B \end{array}$$

such that $\widetilde{F}(0,s)=\eta(s)\ \forall s\in V$. We have $f'\circ\widetilde{F}(0,s)=f'(\eta(s))=(y_0,s)=(\alpha(0),s)$. Let $s\in V$. The path $t\mapsto\widetilde{F}(t,s)$ is a lifting of $\alpha\times\{s\}$ with initial point $\eta(s)$, so $f'(\widetilde{F}(1,s))=(\alpha(1),s)=(y,s)\in U\times V$. If $s=s_0$, then $\widetilde{F}(1,s_0)=\gamma_0(1)=x_0$. This implies that $\widetilde{F}(\{1\}\times V)\subset W$ since $V$ is connected. The map $f'|_W: W\to U\times V$ is a homeomorphism. Let $x\in W$ and let $f'(x)=(z,s)$. We construct a path $\gamma$ in $X'_s$ which connects $\eta(s)$ with $x$ as follows. The path $\widetilde{\alpha}_s(t)=\widetilde{F}(t,s)$, $t\in[0,1]$, has initial point $\eta(s)$ and terminal point $\widetilde{F}(1,s)=w\in W$ such that $f'(w)=(\alpha(1),s)=(y,s)$. Let $\tau: I\to U$ be a path such that $\tau(0)=y$, $\tau(1)=z$. Then $\gamma=\widetilde{\alpha}_s\cdot((f'|_W)^{-1}\circ(\tau\times\{s\}))$ is a path in $X'_s$ which connects $\eta(s)$ with $x$ and $f'_s\circ\gamma=\alpha\cdot\tau$. Let $\beta(s)=E$. We showed in Proposition 4.15 that $m_s^{N(\lambda_0)}=m(E)^{N(\lambda_0)}$, so by (28),

$$h'(x)=\big(\Gamma_{m(E),\lambda_0}[\alpha\cdot\tau]_E, E, m(E)^{N(\lambda_0)}\big).$$

This shows that $h'(W)\subset N_{(\alpha,D,m^{N(\lambda_0)})}(U,U_1,\dots,U_n)$, so $x_0$ is an internal point of $h'^{-1}(N)$. The claim that $h'$ is continuous is proved.

One applies [22, Cor. 4.5] to the commutative diagram

$$\begin{array}{ccc} & & \mathcal{X}(y_0,\lambda_0)' \\ & \overset{h'}{\nearrow} & \downarrow{\scriptstyle \phi'} \\ X' & \xrightarrow[(\mathrm{id}_Y\times u)'\circ f']{} & Y\times H^{\Lambda,G}_{n,\lambda_0}(Y,y_0)\setminus\mathcal{B} \end{array}$$

and concludes that $h': X'\to\mathcal{X}(y_0,\lambda_0)'$ is a morphism.

Let $\phi_S:\mathcal{X}(y_0,\lambda_0)_S\to Y\times S$ be the pullback of $\phi:\mathcal{X}(y_0,\lambda_0)\to Y\times H^{\Lambda,G}_{n,\lambda_0}(Y,y_0)$ by $u: S\to H^{\Lambda,G}_{n,\lambda_0}(Y,y_0)$. The composition $X'\xrightarrow{h'}\mathcal{X}(y_0,\lambda_0)'\hookrightarrow\mathcal{X}(y_0,\lambda_0)$ yields an $S$-morphism $\psi: X'\to\mathcal{X}(y_0,\lambda_0)_S$ which fits in the following commutative diagram of morphisms:

$$\begin{array}{ccccc} X & \hookleftarrow & X' & \xrightarrow{\psi} & \mathcal{X}(y_0,\lambda_0)_S \\ & {\scriptstyle f}\searrow & \downarrow & \swarrow{\scriptstyle \phi_S} & \\ & & Y\times S & & \\ & & \downarrow{\scriptstyle \pi_2} & & \\ & & S. & & \end{array}$$

The graph $\Gamma$ of $\psi$ is contained in the set-theoretical fiber product $X \times_{Y \times S} \mathcal{X}(y_0, \lambda_0)_S$, which is a Zariski closed subset of $X \times \mathcal{X}(y_0, \lambda_0)_S$, so it contains the closure $\overline{\Gamma}$. Therefore, the projection morphism $\overline{\Gamma} \to X$ has finite fibers. Applying [21, Thm. 2], one concludes that $\psi$ can be extended to an $S$-morphism $\widetilde{\psi} : X \to \mathcal{X}(y_0, \lambda_0)_S$. For every $s \in S$ the composition $X_s \xrightarrow{\widetilde{\psi}_s} (\mathcal{X}(y_0, \lambda_0)_S)_s \xrightarrow{\sim} \mathcal{X}(y_0, \lambda_0)_{u(s)}$ is a morphism whose restriction on $X'_s$ coincides with $h'_s$. Hence, this composition equals $h_s$. This implies that $h$ equals the composition of morphisms $X \xrightarrow{\widetilde{\psi}} \mathcal{X}(y_0, \lambda_0)_S \to \mathcal{X}(y_0, \lambda_0)$, so $h : X \to \mathcal{X}(y_0, \lambda_0)$ is a morphism. Furthermore, $\widetilde{\psi}_s$ is an isomorphism for every $s \in S$. Applying Lemma 4.16 to the smooth, proper morphisms $X \to S$ and $\mathcal{X}(y_0, \lambda_0)_S \to S$, one concludes that $\widetilde{\psi} : X \to \mathcal{X}(y_0, \lambda_0)_S$ is an isomorphism, so diagram (27) is Cartesian. One has that $\phi_S \circ \widetilde{\psi} = f$ and $\widetilde{\psi}(\eta(s)) = \psi(\eta(s)) = \xi_S(s)\ \forall s \in S$ since $h(\eta(s)) = \xi(u(s))$ by the construction of the map $h$. Therefore, $(f : X \to Y \times S, \eta)$ is equivalent to the pullback of the family (26) by the morphism $u : S \to H^{\Lambda,G}_{n,\lambda_0}(Y, y_0)$. ■

A pointed cover $(f : X \to Y, x_0)$ of $(Y, y_0)$ is a $(\Lambda, G)$-cover if and only if it is equivalent to $(\Lambda \times^G C \to Y, \pi(\lambda_0, z_0))$ for some pointed $G$-cover $(p : C \to Y, z_0)$ of $(Y, y_0)$ (cf. Corollary 2.14). In the next proposition, applying Theorem 4.17, we extend this to smooth, proper families of pointed covers of $(Y, y_0)$ (cf. also Proposition 3.11).

PROPOSITION 4.18. *Let $y_0 \in Y$. Let $(f : X \to Y \times S, \eta : S \to X)$ be a smooth, proper family of pointed covers of $(Y, y_0)$ branched in $n$ points. Suppose that $S$ is connected and there is a point $s_0 \in S$ such that $(f_{s_0} : X_{s_0} \to Y, \eta(s_0))$ is a $(\Lambda, G)$-cover of $(Y, y_0)$. There is an étale Galois cover $\mu : T \to S$ with Galois group isomorphic to $N(\lambda_0)$ and a smooth, proper family $(p_T : \mathcal{C} \to Y \times T, \zeta_T : T \to \mathcal{C})$ of pointed $G$-covers of $(Y, y_0)$ branched in $n$ points such that the pullback by $\mu$, $(f_T : X_T \to Y \times T, \eta_T : T \to X_T)$, is equivalent to $(\Lambda \times^G \mathcal{C} \to Y \times T, \pi(\lambda_0, \zeta_T) : T \to \Lambda \times^G \mathcal{C})$ (cf. Proposition 3.11). For every $s \in S$ the fibers over the points of $\mu^{-1}(s)$ are the $|N(\lambda_0)|$ pointed $G$-covers of $(Y, y_0)$, nonequivalent to each other, whose quotients by $G(\lambda_0)$ are equivalent to the pointed cover $(X_s \to Y, \eta(s))$ of $(Y, y_0)$ (cf. Corollary 2.14).*

PROOF. By Proposition 3.6, $(f : X \to Y \times S, \eta)$ is a smooth, proper family of pointed $(\Lambda, G)$-covers of $(Y, y_0)$. Let $u : S \to H^{\Lambda,G}_{n,\lambda_0}(Y, y_0)$ and $h : X \to \mathcal{X}(y_0, \lambda_0)$ be the morphisms defined in the proof of Theorem 4.17, which make diagram (27) Cartesian. The quotient morphism $\nu : H^G_n(Y, y_0) \to H^{\Lambda,G}_{n,\lambda_0}(Y, y_0)$ is an étale Galois cover with Galois group $N(\lambda_0)$. Let $T$ be the fiber product, which fits in the Cartesian diagram

$$(29) \qquad \begin{array}{ccc} T & \xrightarrow{\tilde{u}} & H^G_n(Y, y_0) \\ \Big\downarrow{\scriptstyle\mu} & & \Big\downarrow{\scriptstyle\nu} \\ S & \xrightarrow{u} & H^{\Lambda,G}_{n,\lambda_0}(Y, y_0). \end{array}$$

The morphism $\mu : T \to S$ is an étale Galois cover with Galois group $N(\lambda_0)$ (cf. [23, Prop. A.7.1.3]). Let $(f_T : X_T \to Y \times T, \eta_T)$ be the pullback of $(f : X \to Y \times S, \eta)$ by $\mu : T \to S$ (cf. §3.8). One has the following commutative diagram of morphisms in which the two squares are Cartesian:

$$\begin{array}{ccccc} X_T & \xrightarrow{h_\mu} & X & \xrightarrow{h} & \mathcal{X}(y_0, \lambda_0) \\ {\scriptstyle f_T}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \phi} \\ Y \times T & \xrightarrow{\mathrm{id}\times\mu} & Y \times S & \xrightarrow{\mathrm{id}\times u} & Y \times H^{\Lambda,G}_{n,\lambda_0}(Y, y_0). \end{array} \tag{30}$$

Therefore, the composed diagram is Cartesian as well [11, Prop. 4.16]. Let $(\varphi : \mathcal{X}(y_0) \to Y \times H^G_n(Y, y_0), \theta : H^G_n(Y, y_0) \to \mathcal{X}(y_0))$ be the family of pointed $(\Lambda, G)$-covers of $(Y, y_0)$ defined in Proposition 4.6. Let $(f_1 : X_1 \to Y \times T, \eta_1)$ be its pullback by $\tilde{u} : T \to H^G_n(Y, y_0)$. One has the following commutative diagram of morphisms in which the left square is Cartesian and the right one is (23):

$$\begin{array}{ccccc} X_1 & \xrightarrow{h_1} & \mathcal{X}(y_0) & \xrightarrow{\kappa} & \mathcal{X}(y_0, \lambda_0) \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \phi} \\ Y \times T & \xrightarrow{\mathrm{id}\times\tilde{u}} & Y \times H^G_n(Y, y_0) & \xrightarrow{\mathrm{id}\times\nu} & Y \times H^{\Lambda,G}_{n,\lambda_0}(Y, y_0). \end{array} \tag{31}$$

The right square is also Cartesian since the canonical morphism of $\mathcal{X}(y_0)$ into the fiber product is a bijective morphism of smooth algebraic varieties. Therefore, the composed diagram is Cartesian. We have $u \circ \mu = \nu \circ \tilde{u}$ by (29). Comparing (30) with (31), we conclude that there is an isomorphism $q : X_1 \to X_T$ such that $f_T \circ q = f_1$. We claim that $q \circ \eta_1(t) = \eta_T(t)$ $\forall t \in T$. It suffices to check that $\kappa \circ h_1 \circ \eta_1(t) = h \circ h_\mu \circ \eta_T(t)$. One has

$$\kappa\big(h_1(\eta_1(t))\big) = \kappa\big(\theta(\tilde{u}(t))\big) = \xi\big(\nu(\tilde{u}(t))\big)$$

and

$$h\big(h_\mu(\eta_T(t))\big) = h\big(\eta(\mu(t))\big) = \xi\big(u(\mu(t))\big) = \xi\big(\nu(\tilde{u}(t))\big).$$

This shows that $q : X_1 \to X$ defines an equivalence of the families of pointed covers of $(Y, y_0)$, $(f_1 : X_1 \to Y \times T, \eta_1)$ and $(f_T : X_T \to Y \times T, \eta_T)$.

Let $(p_T : \mathcal{C} \to Y \times T, \zeta_T)$ be the pullback of $(p : \mathcal{C}(y_0) \to Y \times H^G_n(Y, y_0), \zeta)$ by $\tilde{u} : T \to H^G_n(Y, y_0)$ [22, §5.2]. One has that $\mathcal{C}$ is reduced and may be identified with the closed subvariety $\mathcal{C} = \{(z, t) \mid \pi_2 \circ p(z) = \tilde{u}(t)\}$ of $\mathcal{C}(y_0) \times T$. The induced action of $G$ is defined by $g(z, t) = (gz, t)$ and $\zeta_T(t) = (\zeta(\tilde{u}(t)), t)$ $\forall t \in T$. Let $j : \mathcal{C} \to \mathcal{C}(y_0)$ be the $G$-equivariant morphism $j(z, t) = z$. Let $H = G(\lambda_0)$ and let $\rho : \mathcal{C}(y_0) \to \mathcal{X}(y_0)$ be the $H$-invariant morphism defined by $\rho(z) = \pi(\lambda_0, z)$. Let

$\rho_1 : \mathcal{C} \to X_1$ be the morphism defined by $\rho_1(z,t) = (\rho(z), t)$. It is $H$-invariant and fits in the following commutative diagram of morphisms:

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{\rho_1} & X_1 & \xrightarrow{f_1} & Y \times T \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle \mathrm{id}\times\tilde{u}} \\ \mathcal{C}(y_0) & \xrightarrow{\rho} & \mathcal{X}(y_0) & \xrightarrow{\varphi} & Y \times H^G_n(Y, y_0). \end{array}$$

One has that $f_1 \circ \rho_1 = p_T$ and $\varphi \circ \rho = p$, so the composed diagram is Cartesian. The right square is Cartesian, so the left square is Cartesian as well [11, Prop. 4.16]. By Proposition 4.7, $\rho$ induces an isomorphism $\overline{\rho} : \mathcal{C}(y_0)/H \xrightarrow{\sim} \mathcal{X}(y_0)$. The formation of quotients by $H$ commutes with base change [23, Prop. A.7.1.3], so $\rho_1$ induces an isomorphism $\overline{\rho}_1 : \mathcal{C}/H \to X_1$ over $Y \times T$. Furthermore, $\rho_1$ transforms the section $\zeta_T : T \to \mathcal{C}$ into $\eta_1 : T \to X_1$. In fact, for every $t \in T$ one has

$$\rho_1\big(\zeta_T(t)\big) = \rho_1\big(\zeta\big(\tilde{u}(t)\big), t\big) = \big(\rho\big(\zeta\big(\tilde{u}(t)\big)\big), t\big) = \big(\theta\big(\tilde{u}(t)\big), t\big) = \eta_1(t).$$

By Proposition 3.11, we conclude that $(f_1 : X_1 \to Y \times T, \eta_1)$ is equivalent to $(\Lambda \times^G \mathcal{C} \to Y \times T, \pi(\lambda_0, \zeta_T))$. This proves the proposition since $(f_T : X_T \to Y \times T, \eta_T)$ is equivalent to $(f_1 : X_1 \to Y \times T, \eta_1)$ as we saw above. The last statement of the proposition follows from diagram (29). ■

4.19. Let $(f : X \to Y, x_0)$ be a pointed $(\Lambda, G)$-cover of $(Y, y_0)$ branched in $D = \{b_1, \dots, b_n\} \subset Y \setminus \{y_0\}$ associated with $\varepsilon : \Lambda \to f^{-1}(y_0)$, $m : \pi_1(Y \setminus D, y_0) \to G$ as in Proposition 2.8 (i). Let $\gamma_1, \dots, \gamma_n$ be loops based at $y_0$ as in §2.4 and let $g_i = m([\gamma_i])$, $i = 1, \dots, n$. Varying $\bar{U}_1, \dots, \bar{U}_n$ and $\eta_1, \dots, \eta_n$, one obtains $\gamma'_1, \dots, \gamma'_n$ and $g'_i = m([\gamma'_i])$ such that $g'_i$ belongs to the conjugacy class of $g_i$ in $G$, $i = 1, \dots, n$ (cf. §2.6). Furthermore, replacing $(f : X \to Y, x_0)$ by an equivalent pointed $(\Lambda, G)$-cover $(f_1 : X_1 \to Y, x'_0)$ of $(Y, y_0)$ results in replacing $(g'_1, \dots, g'_n)$ by $(\sigma g'_1 \sigma^{-1}, \dots, \sigma g'_n \sigma^{-1})$, where $\sigma \in N(\lambda_0)$.

Definition 4.20. Let $O_1, \dots, O_k$ be conjugacy classes of $G$, $O_i \neq O_j$ if $i \neq j$. Let $\underline{n} = n_1 O_1 + \cdots + n_k O_k$ be a formal sum, where $n_i \in \mathbb{N}$. Let $|\underline{n}| = n_1 + \cdots + n_k = n$. We say that a pointed cover $(f : X \to Y, x_0)$ of $(Y, y_0)$ branched in $n$ points is a $(\Lambda, G)$-cover of branching type $\underline{n}$ if there exists a bijection $\varepsilon : \Lambda \to f^{-1}(y_0)$ and an epimorphism $m : \pi_1(Y \setminus D, y_0) \to G$, which satisfy condition (i) of Proposition 2.8, such that

(32) $\qquad n_i$ of the elements $m\big([\gamma_j]\big)$ belong to $O_i \quad$ for $i = 1, \dots, k$.

Notice that the branching type is not uniquely determined by the equivalence class of $f$. It specifies that $(f : X \to Y, x_0)$ is equivalent to $(\Lambda \times^G C \to Y, \pi(\lambda_0, z_0))$ for some pointed $G$-cover $(p : C \to Y, z_0)$ of $(Y, y_0)$ of branching type $\underline{n}$.

4.21. Let $H^G_{\underline{n}}(Y, y_0)$ be the subset of $H^G_n(Y, y_0)$ consisting of the elements $(D, m)$ with $m$ satisfying condition (32). Every nonempty $H^G_{\underline{n}}(Y, y_0)$ is a union of connected components of $H^G_n(Y, y_0)$ and $H^G_n(Y, y_0) = \bigsqcup_{|\underline{n}|=n} H^G_{\underline{n}}(Y, y_0)$. Let $\sigma \in N(\lambda_0)$. Then $\sigma * H^G_{\underline{n}}(Y, y_0) = H^G_{\underline{n}'}(Y, y_0)$, with $\underline{n}' = n_1 O'_1 + \cdots + n_k O'_k$, where $O'_i$ is the conjugacy class $\sigma O_i \sigma^{-1}$ of $G$. Suppose $H^G_{\underline{n}}(Y, y_0) \neq \emptyset$. Let $H^{\Lambda,G}_{\underline{n},\lambda_0}(Y, y_0) = \nu(H^G_{\underline{n}}(Y, y_0))$ (cf. Proposition 4.8). It is a union of connected components of $H^{\Lambda,G}_{n,\lambda_0}(Y, y_0)$. Let us denote by $\phi_{\underline{n}} : \mathcal{X}_{\underline{n}}(y_0, \lambda_0) \to Y \times H^{\Lambda,G}_{\underline{n},\lambda_0}(Y, y_0)$ the restriction of the family

$$\phi : \mathcal{X}(y_0, \lambda_0) \to Y \times H^{\Lambda,G}_{n,\lambda_0}(Y, y_0)$$

and let $\xi_{\underline{n}} : H^{\Lambda,G}_{\underline{n},\lambda_0}(Y, y_0) \to \mathcal{X}_{\underline{n}}(y_0, \lambda_0)$ be the restriction of the morphism

$$\xi : H^{\Lambda,G}_{n,\lambda_0}(Y, y_0) \to \mathcal{X}(y_0, \lambda_0).$$

THEOREM 4.22. *Let $\underline{n} = n_1 O_1 + \cdots + n_k O_k$, $|\underline{n}| = n$, be as in Definition 4.20. Let $(f : X \to Y \times S, \eta : S \to X)$ be a smooth, proper family of pointed covers of $(Y, y_0)$ branched in $n$ points. Suppose that $S$ is connected and there is a point $s_0 \in S$ such that $(f_{s_0} : X_{s_0} \to Y, \eta(s_0))$ is a $(\Lambda, G)$-cover of branching type $\underline{n}$. Then*

(i) *there exists a unique morphism $u : S \to H^{\Lambda,G}_{\underline{n},\lambda_0}(Y, y_0)$ such that $(f : X \to Y \times S, \eta)$ is equivalent to the pullback by $u$ of $(\phi_{\underline{n}} : \mathcal{X}_{\underline{n}}(y_0, \lambda_0) \to Y \times H^{\Lambda,G}_{\underline{n},\lambda_0}(Y, y_0), \xi_{\underline{n}})$;*

(ii) *there exists an étale cover $\mu : T \to S$ and a smooth, proper family of pointed $G$-covers $(p : \mathcal{C} \to Y \times T, \zeta : T \to \mathcal{C})$ of $(Y, y_0)$ of branching type $\underline{n}$, such that the pullback by $\mu$, $(f_T : X_T \to Y \times T, \eta_T)$, is equivalent to $(\Lambda \times^G \mathcal{C} \to Y \times T, \pi(\lambda_0, \zeta_T))$ (cf. Proposition 3.11).*

PROOF. By Proposition 3.6, $(f : X \to Y \times S, \eta)$ is a smooth, proper family of pointed $(\Lambda, G)$-covers of $(Y, y_0)$. Part (i) follows from Theorem 4.17 since the morphism $u : S \to H^{\Lambda,G}_{n,\lambda_0}(Y, y_0)$ of Proposition 4.15 has image contained in a connected component of $H^{\Lambda,G}_{n,\lambda_0}(Y, y_0)$ which is a connected component of $H^{\Lambda,G}_{\underline{n},\lambda_0}(Y, y_0)$.

Part (ii) is proved similarly to Proposition 4.18 replacing (29) by the Cartesian diagram

$$\begin{array}{ccc} T & \xrightarrow{\tilde{u}} & H^G_{\underline{n}}(Y, y_0) \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\nu} \\ S & \xrightarrow{u} & H^{\Lambda,G}_{\underline{n},\lambda_0}(Y, y_0) \end{array} \tag{33}$$

and using the universal family of pointed $G$-covers of $(Y, y_0)$ of branching type $\underline{n}$, $(p_{\underline{n}} : \mathcal{C}_{\underline{n}}(y_0) \to Y \times H^G_{\underline{n}}(Y, y_0), \zeta_{\underline{n}})$ (cf. [22, Thm. 5.8]). ∎

Choosing another $\lambda_1 \in \Lambda$ as a marked element, one obtains a family of pointed $(\Lambda, G)$-covers of $(Y, y_0)$ which is isomorphic to (25).

PROPOSITION 4.23. *Let* $\lambda_1 \in \Lambda$ *and let* $\lambda_1 = \lambda_0 g$, $g \in G$. *Let*

$$(\phi_1 : \mathcal{X}(y_0,\lambda_1) \to Y \times H^{\Lambda,G}_{n,\lambda_1}(Y,y_0), \xi_1 : H^{\Lambda,G}_{n,\lambda_1}(Y,y_0) \to \mathcal{X}(y_0,\lambda_1)) \tag{34}$$

*be the smooth, proper family of pointed* $(\Lambda, G)$*-covers of* $(Y, y_0)$ *branched in n points associated with* $\lambda_1 \in \Lambda$ *(cf. Theorem 4.13). Then there exists an isomorphism*

$$h : \mathcal{X}(y_0,\lambda_1) \to \mathcal{X}(y_0,\lambda_0)$$

*which fits in the following diagram:*

$$\begin{array}{ccc} \mathcal{X}(y_0,\lambda_1) & \xrightarrow{\quad h \quad} & \mathcal{X}(y_0,\lambda_0) \\ {\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \phi} \\ Y \times H^{\Lambda,G}_{n,\lambda_1}(Y,y_0) & \xrightarrow{\mathrm{id}\times u} & Y \times H^{\Lambda,G}_{n,\lambda_0}(Y,y_0), \end{array} \tag{35}$$

*where* $u : H^{\Lambda,G}_{n,\lambda_1}(Y,y_0) \to H^{\Lambda,G}_{n,\lambda_0}(Y,y_0)$ *is an isomorphism given by* $u(D, m_1^{N(\lambda_1)}) = (D, (gm_1g^{-1})^{N(\lambda_0)})$ *and furthermore* $\xi \circ u = h \circ \xi_1$. *Similar statements hold for the universal families of pointed* $(\Lambda, G)$*-covers of* $(Y, y_0)$ *with a fixed branching type* $\underline{n} = n_1 O_1 + \cdots + n_k O_k$.

PROOF. Let $(f : X \to Y, x_0)$ be a pointed $(\Lambda, G)$-cover of $(Y, y_0)$ branched in $D \in (Y \setminus y_0)^{(n)}_*$ and let $(\varepsilon_1 : \Lambda \to f^{-1}(y_0), m_1 : \pi_1(Y \setminus D, y_0) \to G)$ be a pair such that $\varepsilon_1(\lambda m_1([\alpha])) = \varepsilon_1(\lambda)\alpha$ $\forall \lambda \in \Lambda$ and $\forall [\alpha] \in \pi_1(Y \setminus D, y_0)$, and $\varepsilon_1(\lambda_1) = x_0$. Let $\varepsilon : \Lambda \to f^{-1}(y_0)$ be the bijection defined by $\varepsilon(\lambda) = \varepsilon_1(\lambda g)$. Then $\varepsilon(\lambda_0) = x_0$ and $m = gm_1g^{-1}$ satisfies $\varepsilon(\lambda m([\alpha])) = \varepsilon(\lambda)\alpha$ $\forall \lambda \in \Lambda$ and $\forall [\alpha] \in \pi_1(Y \setminus D, y_0)$. Applying this to the fibers of (34), we see that $\forall (D, m_1^{N(\lambda_1)}) \in H^{\Lambda,G}_{n,\lambda_1}(Y,y_0)$ the monodromy invariant relative to $\lambda_0$ of

$$\left(\mathcal{X}(y_0,\lambda_1)_{(D,m_1^{N(\lambda_1)})} \to Y, \xi_1\left(D, m_1^{N(\lambda_1)}\right)\right)$$

is $(D, (gm_1g^{-1})^{N(\lambda_0)})$, so by Proposition 4.15, $u$ is a morphism. By Theorem 4.17, there exists a morphism $h : \mathcal{X}(y_0,\lambda_1) \to \mathcal{X}(y_0,\lambda_0)$ which makes diagram (35) Cartesian and satisfies $h \circ \xi_1 = \xi \circ u$. Replacing $\lambda_0$ with $\lambda_1$, $\lambda_0 = \lambda_1 g^{-1}$, one obtains a morphism

$$u_1 : H^{\Lambda,G}_{n,\lambda_0}(Y,y_0) \to H^{\Lambda,G}_{n,\lambda_1}(Y,y_0)$$

defined by

$$u_1(D, m^{N(\lambda_0)}) = \left(D, (g^{-1}mg)^{N(\lambda_1)}\right),$$

inverse to $u$ and a morphism $h_1 : \mathcal{X}(y_0,\lambda_0) \to \mathcal{X}(y_0,\lambda_1)$. One has $h_1 \circ h = \mathrm{id}_{\mathcal{X}(y_0,\lambda_1)}$ since $h_1 \circ h \circ \xi_1 = \xi_1$ and similarly $h \circ h_1 = \mathrm{id}_{\mathcal{X}(y_0,\lambda_0)}$. ∎

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Vassil Kanev
Institute of Mathematics and Informatics, Bulgarian Academy of Sciences
8 Acad. Georgi Bonchev Str., 1113 Sofia, Bulgaria
kanev@math.bas.bg